Introduction to Mobile Robotics

Robot Motion Planning

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With material from S. LaValle, JC. Latombe, H. Choset et al., W. Burgard

Robot Motion Planning

Contents

Robot Motion Planning

J.-C. Latombe (1991):

"…eminently necessary since, by definition, a robot accomplishes tasks by moving in the real world."

Goals

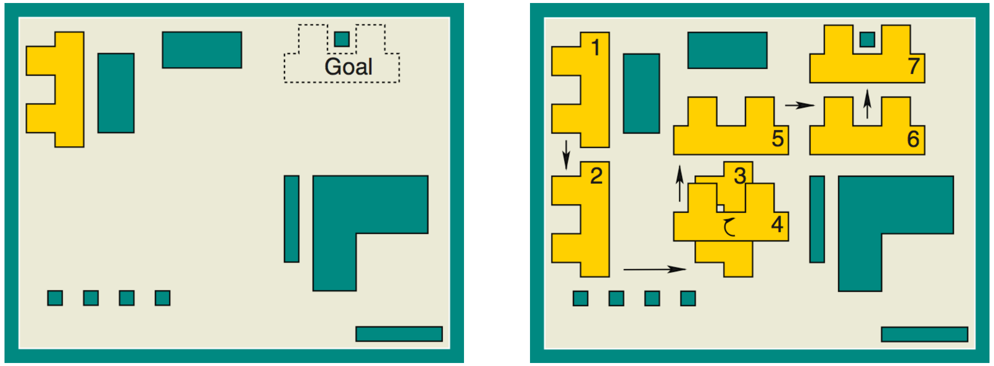
- Collision-free trajectories
- Robot should reach the goal location as fast as possible

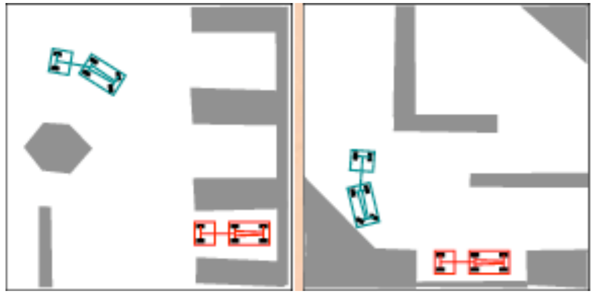
Problem Formulation

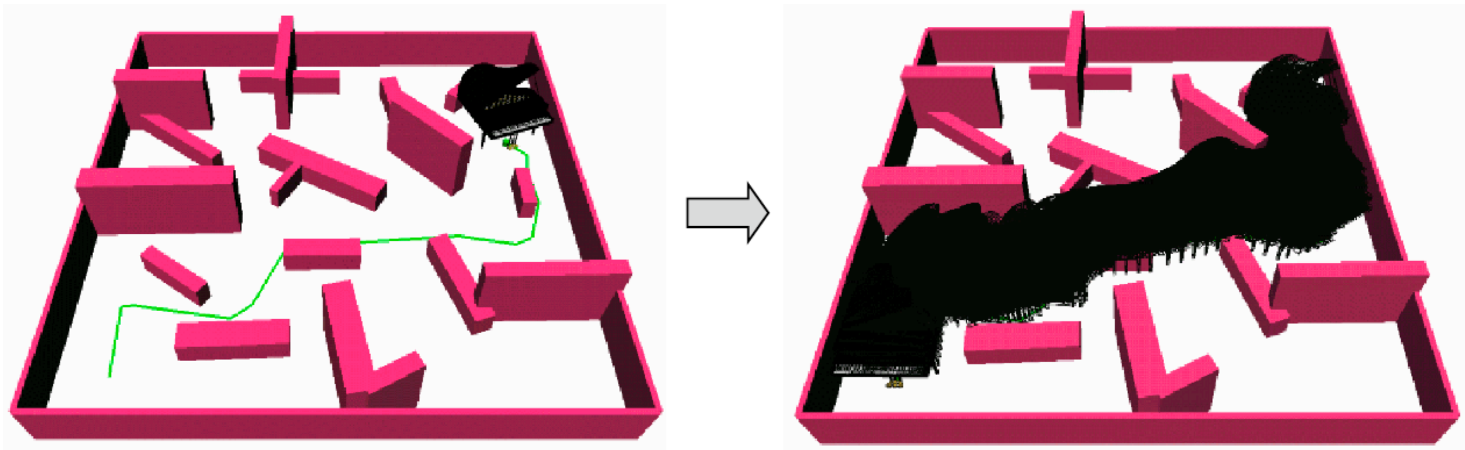
- The **problem of motion planning** can be stated as follows. Given:
  - A **start** pose of the robot
  - A desired **goal** pose
  - A geometric description of the **robot**
  - A geometric description of the **world**

- Find a path that moves the robot gradually from **start** to **goal** while **never touching** any obstacle

Problem Formulation







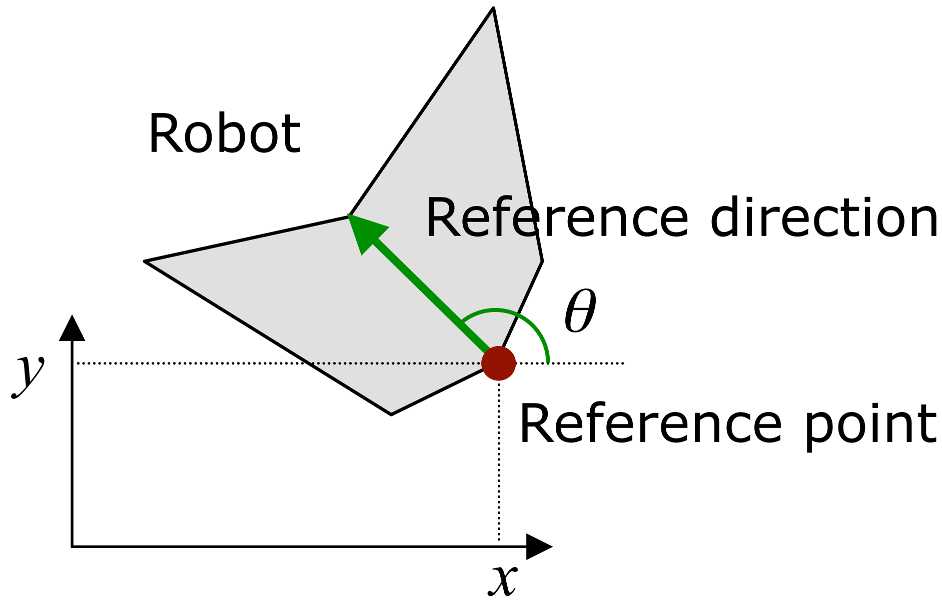
Motion planning is sometimes also called **piano mover's problem**

Configuration Space

- Although the motion planning problem is defined in the regular world, it lives in another space: the **configuration space**
- A robot configuration q is a specification of the positions of all robot points relative to a fixed coordinate system
- Usually a configuration is expressed as a **vector of positions** and **orientations**

Configuration Space

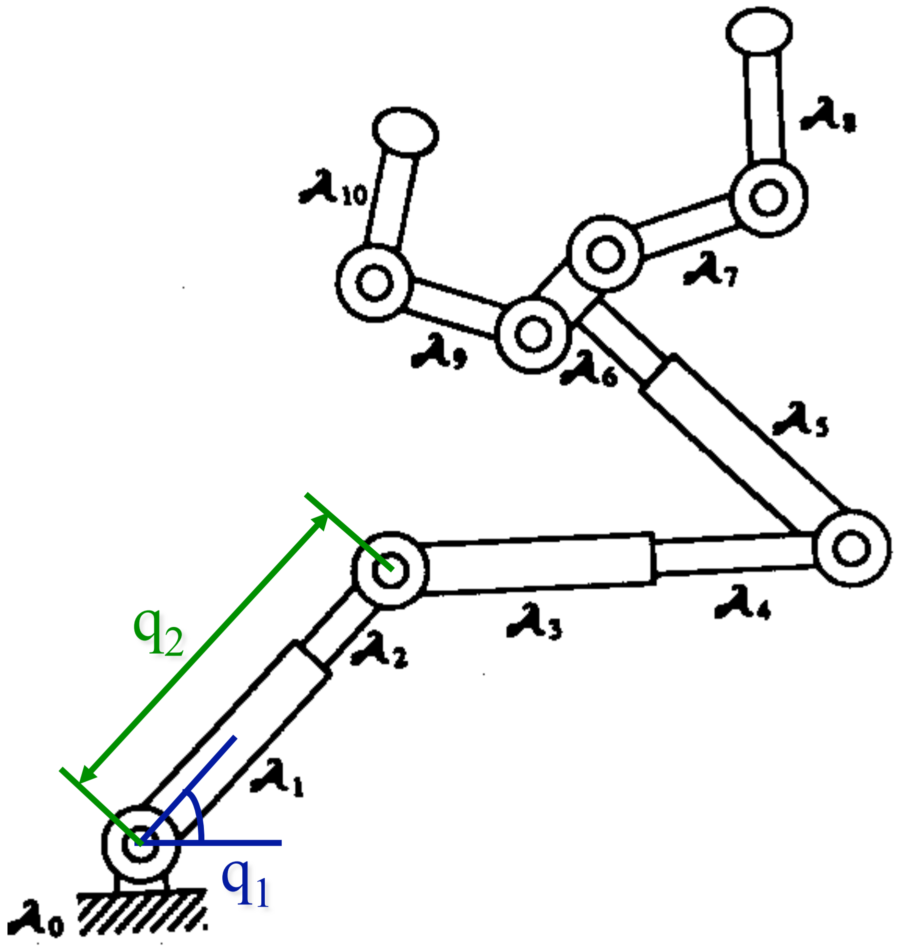
Rigid-body robot example



- 3-parameter representation: $q = (x,y,\theta)$
- In 3D, q would be of the form $(x,y,z,\alpha,\beta,\gamma)$

Configuration Space

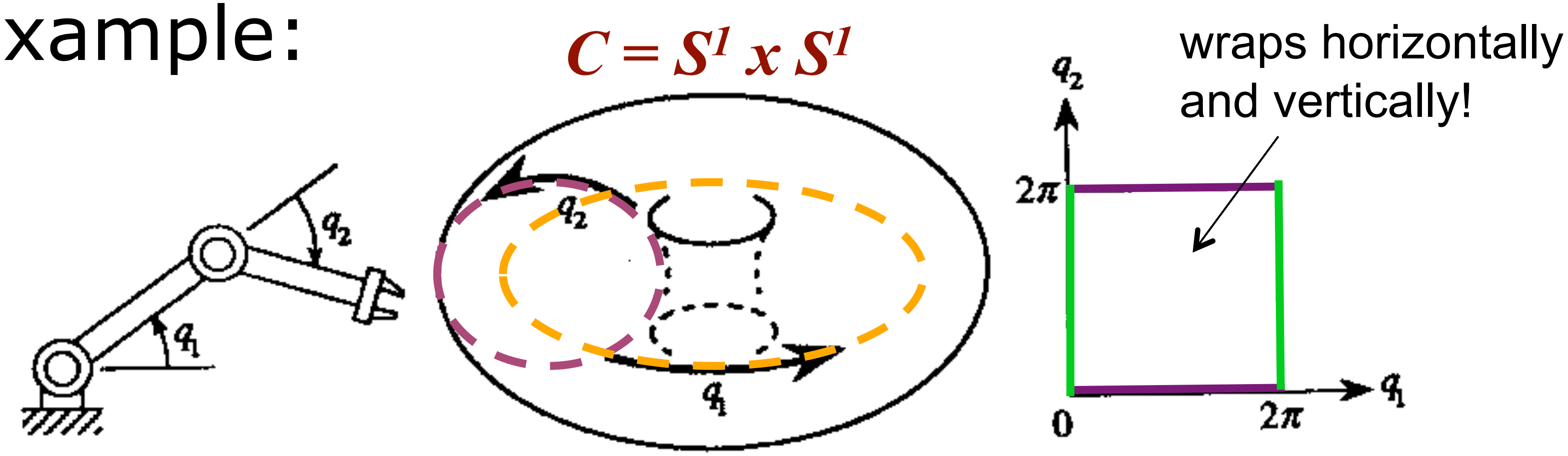
Articulated robot example



$$q = (q_1, q_2, \ldots, q_{10})$$

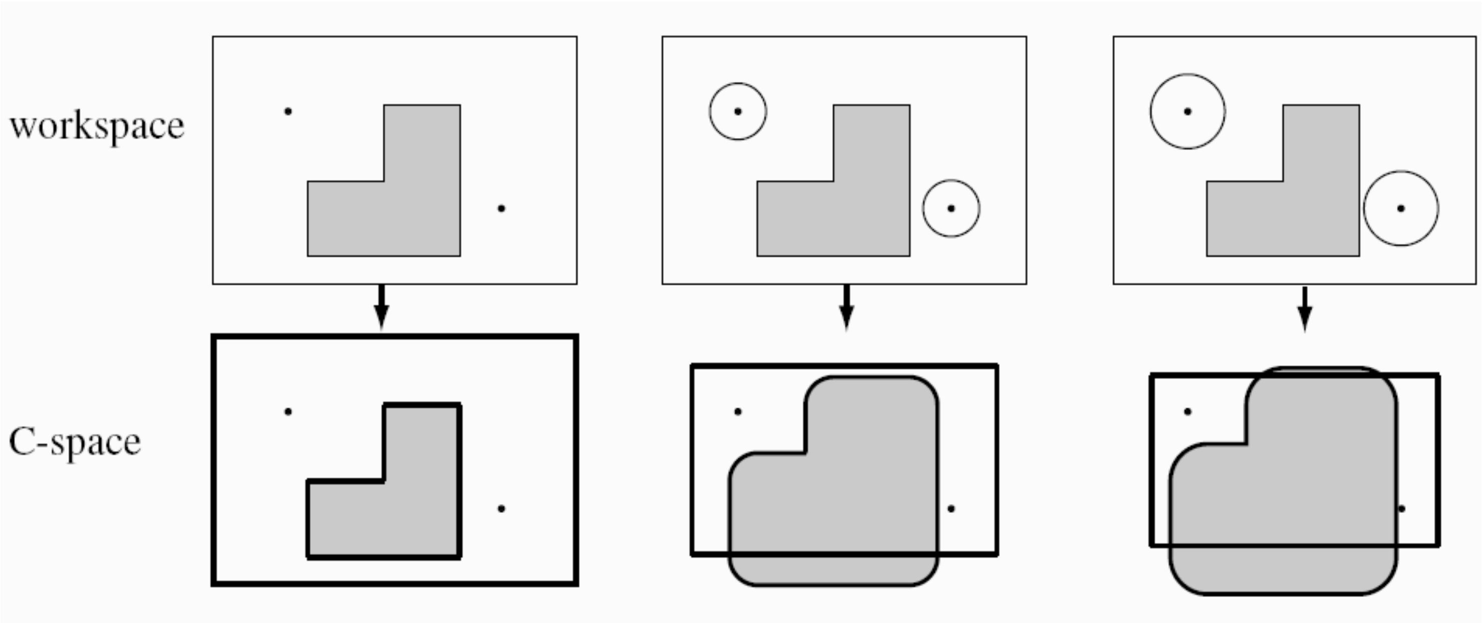
Configuration Space

- The configuration space (C-space) is the **space of all possible configurations**
- The topology of this space is usually **not** that of a Cartesian space
- The C-space is described as a **topological manifold**
- Example:



Configuration Space

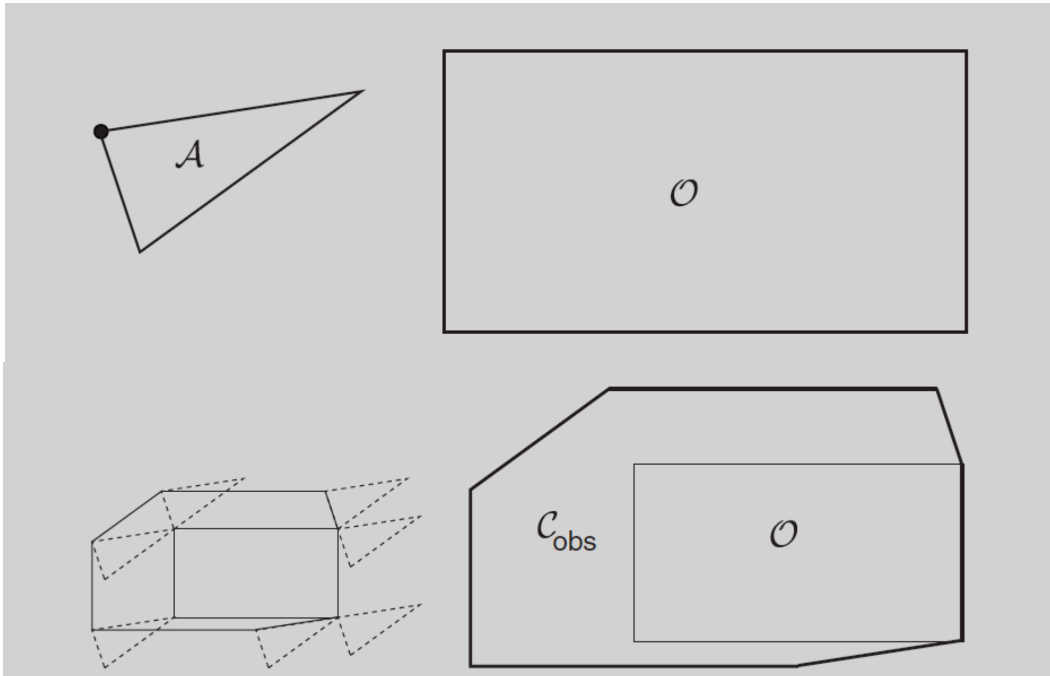
- Example: circular robot



- C-space is obtained by sliding the robot along the edge of the obstacle regions "blowing them up" by the robot radius

Configuration Space

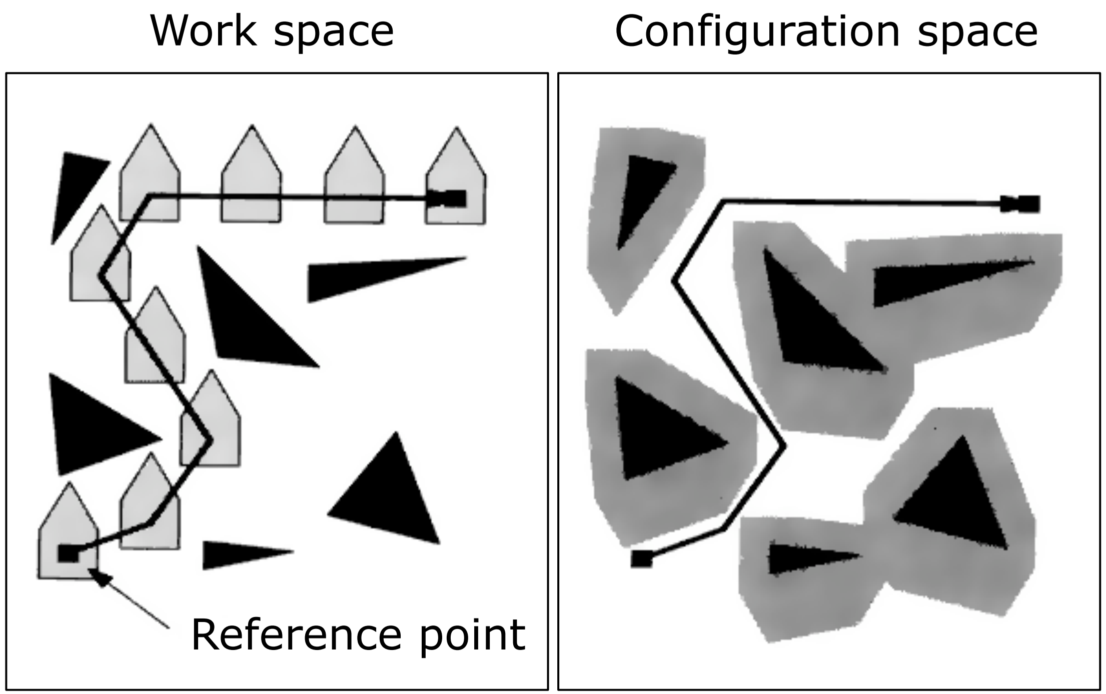
- Example: polygonal robot, translation only



- C-space is obtained by sliding the robot along the edge of the obstacle regions

Configuration Space

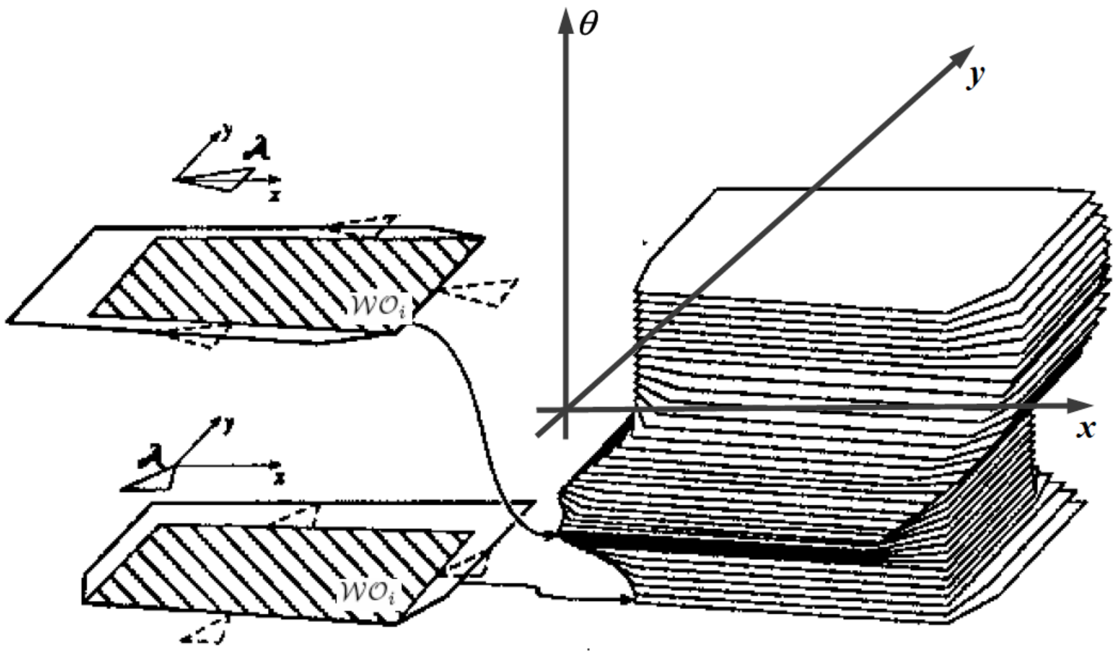
- Example: polygonal robot, translation only



- C-space is obtained by sliding the robot along the edge of the obstacle regions

Configuration Space

- Example: polygonal robot, trans+**rotation**



- C-space is obtained by sliding the robot along the edge of the obstacle regions in all orientations

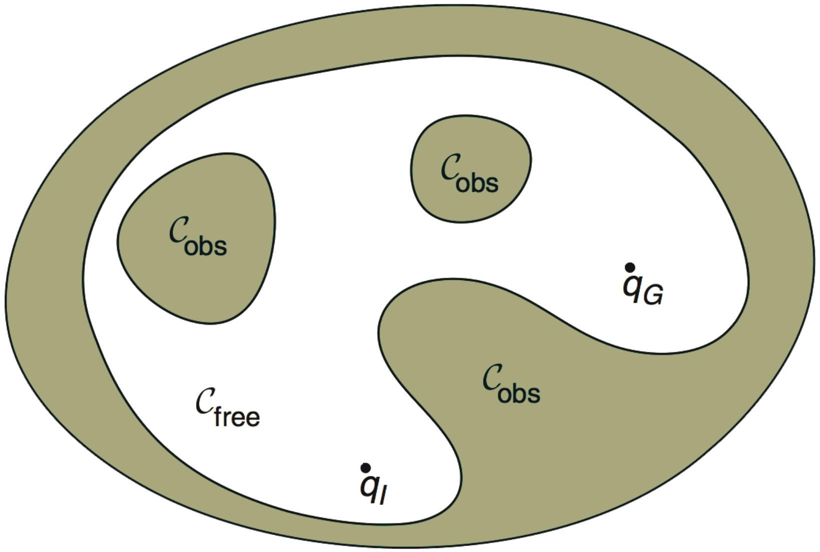
Configuration Space

Free space and **obstacle region**

- With $\mathcal{W} = \mathbb{R}^m$ being the work space, $\mathcal{O} \in \mathcal{W}$ the set of obstacles, $\mathcal{A}(q)$ the robot in configuration $q \in \mathcal{C}$

$$\begin{aligned} \mathcal{C}_{free} &= \{q \in \mathcal{C} \mid \mathcal{A}(q) \cap \mathcal{O} = \emptyset\} \\ \mathcal{C}_{obs} &= \mathcal{C} / \mathcal{C}_{free} \end{aligned}$$

- We further define
  q_I : start configuration
  q_G : goal configuration



Configuration Space

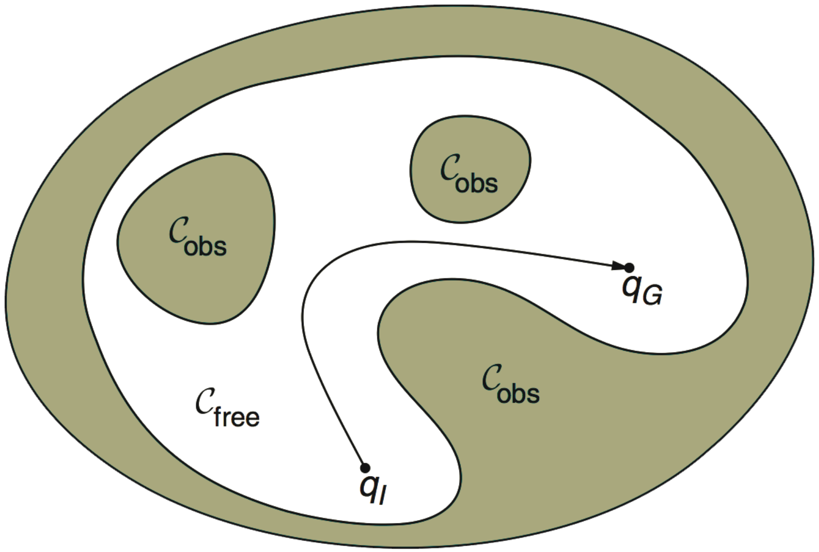
Then, motion planning amounts to

- Finding a continuous path

  $$\tau : [0,1] \rightarrow \mathcal{C}_{free}$$

  with $\tau(0) = q_I \,,\, \tau(1) = q_G$

- Given this setting, we can do planning with the robot being a **point in C-space!**



C-Space Discretizations

- Continuous terrain needs to be **discretized** for path planning
- There are **two general approaches** to discretize C-spaces:
  - **Combinatorial planning**

    Characterizes C_{free} explicitely by capturing the connectivity of C_{free} into a graph and finds solutions using search
  - **Sampling-based planning**

    Uses collision-detection to probe and incrementally search the C-space for solution

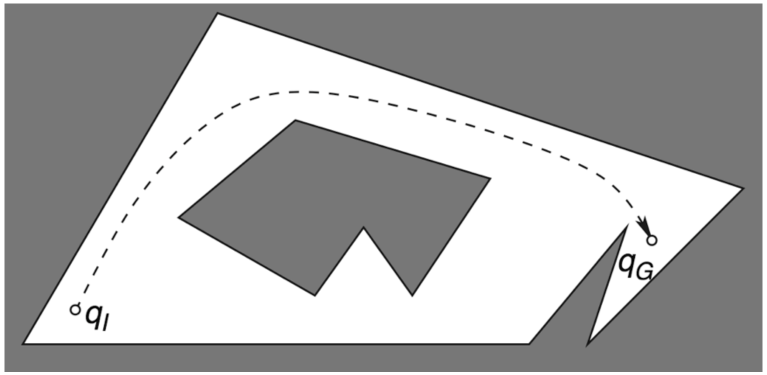
Combinatorial Planning

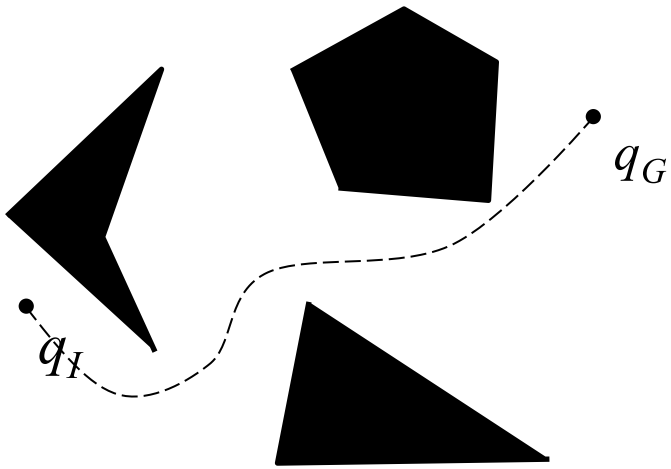
- We will look at four **combinatorial planning techniques**
  - Visibility graphs
  - Voronoi diagrams
  - Exact cell decomposition
  - Approximate cell decomposition

- They all produce a **road map**
  - A **road map** is a **graph in $\boldsymbol{C_{free}}$** in which each vertex is a configuration in C_{free} and each edge is a collision-free path through C_{free}

Combinatorial Planning

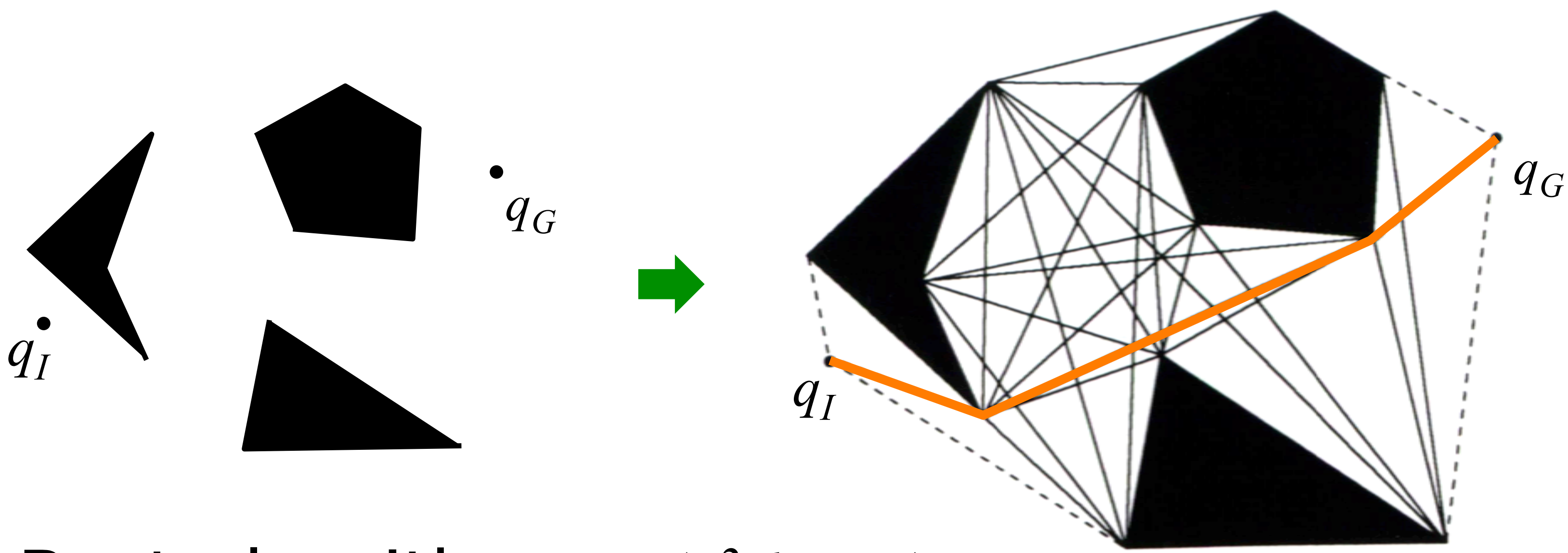
- Without loss of generality, we will consider a problem in $\mathcal{W} = \mathbb{R}^2$ with a **point robot** that cannot rotate. In this case: $\mathcal{C} = \mathbb{R}^2$
- We further assume a **polygonal** world





Visibility Graphs

- **Idea:** construct a path as a polygonal line connecting q_I and q_G through vertices of C_{obs}
- Existence proof for such paths, **optimality**
- One of the earliest path planning methods



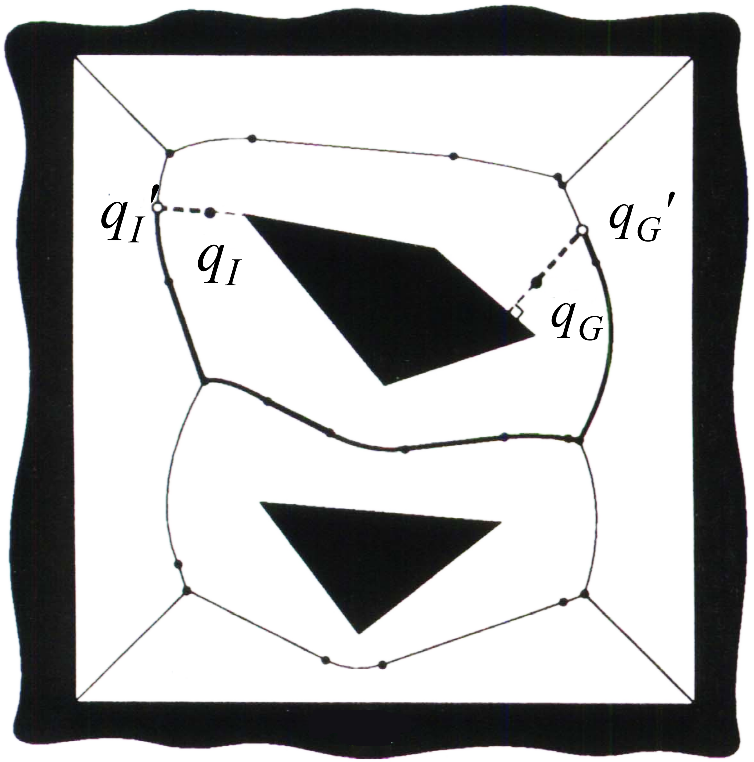
- Best algorithm: $O(n^2 \log n)$

Generalized Voronoi Diagram

- **Defined** to be the set of points q whose cardinality of the set of boundary points of C_{obs} with the same distance to q is greater than 1

- Let us decipher this definition...

- **Informally:** the place with the same **maximal clearance** from all nearest obstacles



Generalized Voronoi Diagram

- **Formally:**

  Let $\beta = \partial \mathcal{C}_{free}$ be the boundary of C_{free}, and $d(p,q)$ the Euclidian distance between p and q. Then, for all q in C_{free}, let

  $$clearance(q) = \min_{p \in \beta} d(p, q)$$

  be the *clearance* of q, and

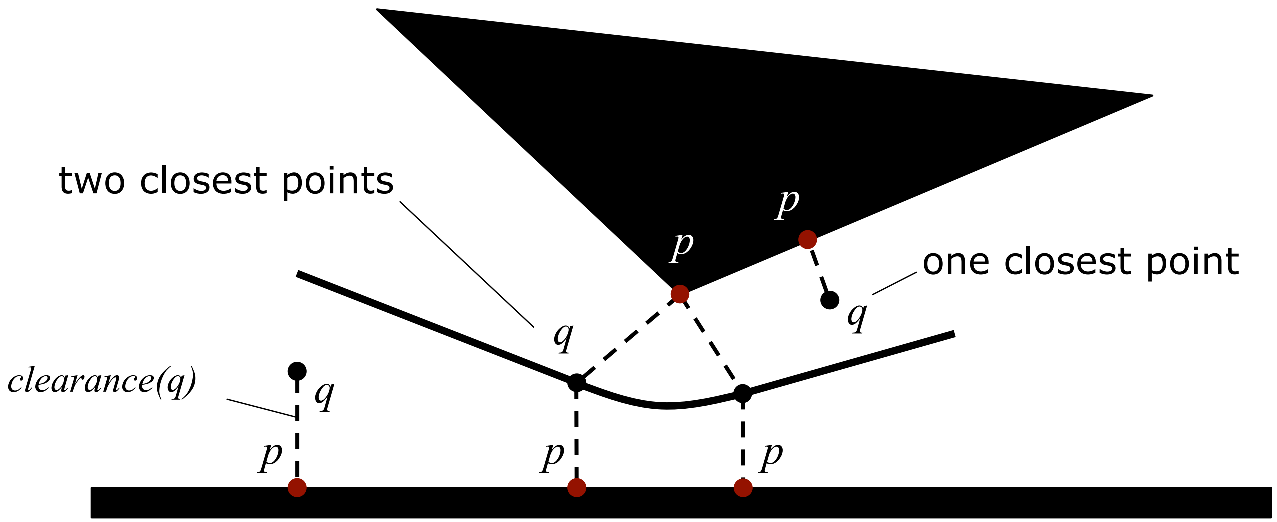
  $$near(q) = \{p \in \beta \mid d(p, q) = clearance(q)\}$$

  the set of "base" points on β with the same clearance to q. The **Voronoi diagram** is then the set of q's with more than one base point p

  $$V(\mathcal{C}_{free}) = \{q \in \mathcal{C}_{free} \mid |near(q)| > 1\}$$

Generalized Voronoi Diagram

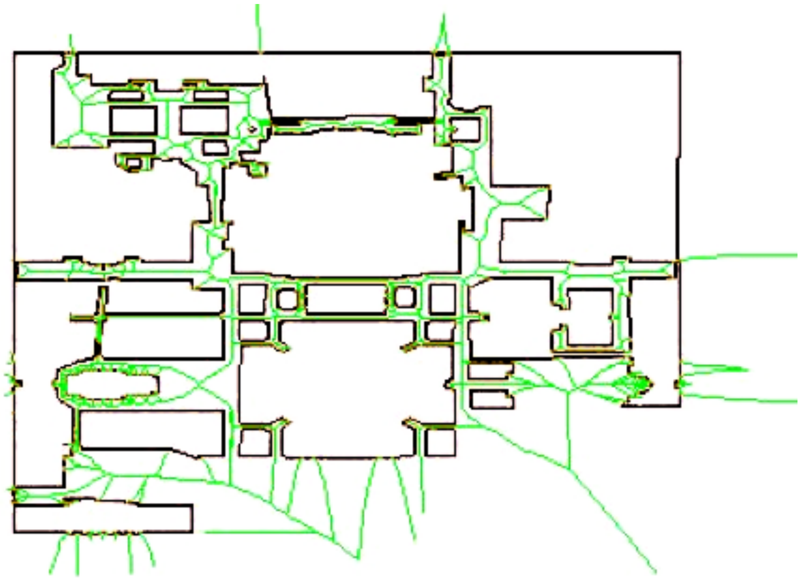
- **Geometrically:**



- For a polygonal C_{obs}, the Voronoi diagram consists of (n) lines and parabolic segments
- Naive algorithm: $O(n^4)$, best: $O(n \log n)$

Voronoi Diagram

- Voronoi diagrams have been well studied for (reactive) **mobile robot** path planning
- Fast methods exist to compute and update the diagram in real-time for low-dim. C's
  - **Pros:** maximize clear-ance is a good idea for an uncertain robot
  - **Cons:** unnatural at-traction to open space, suboptimal paths



- Needs extensions

Exact Cell Decomposition

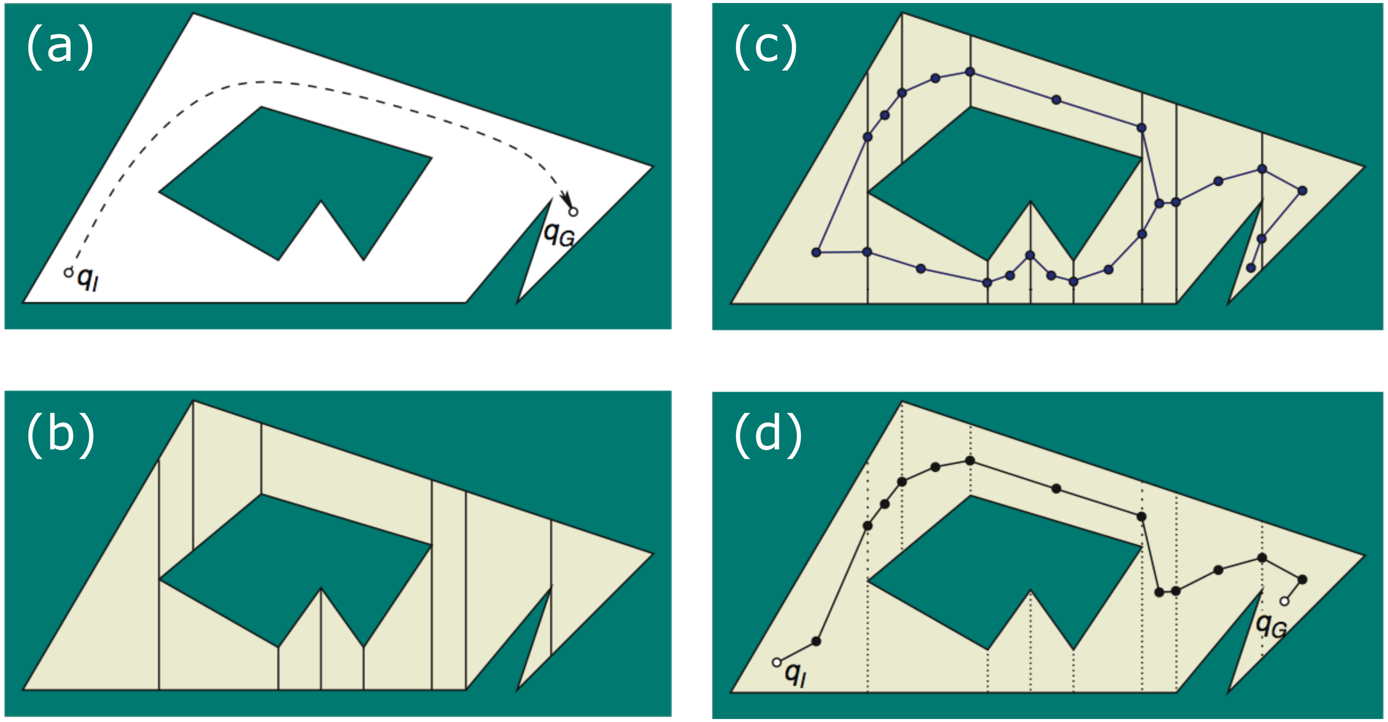
- **Idea:** decompose C_{free} into non-overlapping cells, construct connectivity graph to represent adjacencies, then search

- A popular implementation of this idea:

1. Decompose C_{free} into **trapezoids** with vertical side segments by shooting rays upward and downward from each polygon vertex
2. Place one **vertex** in the interior of every **trapezoid**, pick e.g. the centroid
3. Place one **vertex** in every vertical **segment**
4. Connect the vertices

Exact Cell Decomposition

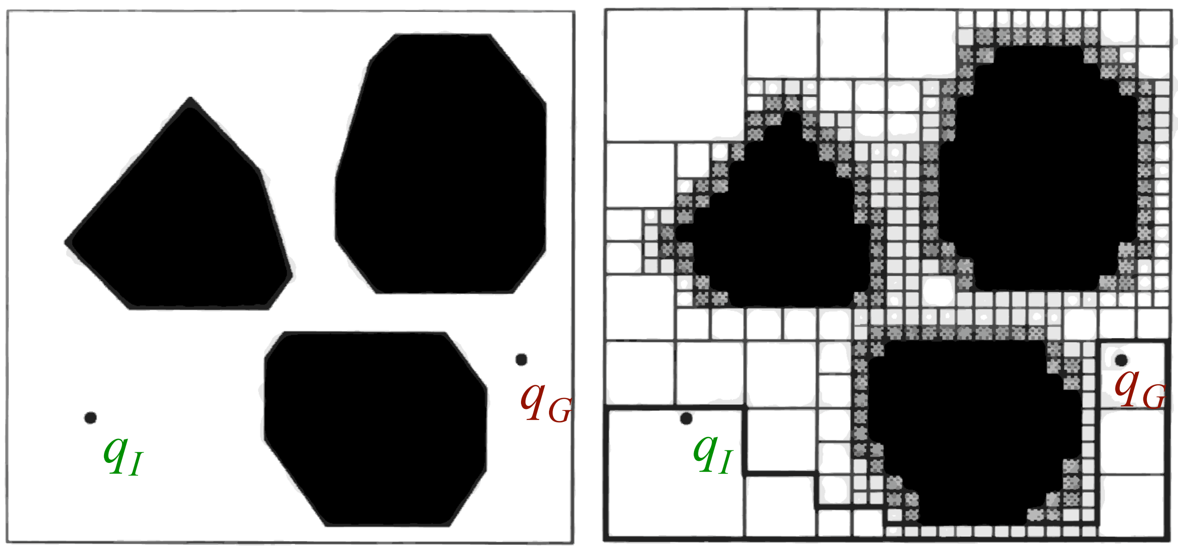
- Trapezoidal decomposition ($\mathcal{C} = \mathbb{R}^3$ max)



- Best known algorithm: $O(n \log n)$ where n is the number of vertices of C_{obs}

Approximate Cell Decomposition

- Exact decomposition methods can be involved and inefficient for complex problems
- Approximate decomposition uses cells with the **same simple predefined shape**



Quadtree decomposition

Approximate Cell Decomposition

- Exact decomposition methods can be involved and inefficient for complex problems
- Approximate decomposition uses cells with the **same simple predefined shape**
- **Pros:**
  - Iterating the **same** simple computations
  - Numerically more **stable**
  - **Simpler** to implement
  - Can be made **complete**

Combinatorial Planning

Wrap Up

- Combinatorial planning techniques are **elegant** and **complete** (they find a solution if it exists, report failure otherwise)
- But: become **quickly intractable** when C-space dimensionality increases (or n resp.)
- Combinatorial **explosion** in terms of **facets** to represent $\mathcal{A}$, $\mathcal{O}$, and $\mathcal{C}_{obs}$, especially when rotations bring in non-linearities and make C a nontrivial manifold

➡ Use **sampling-based planning**
Weaker guarantees but more efficient

Sampling-Based Planning

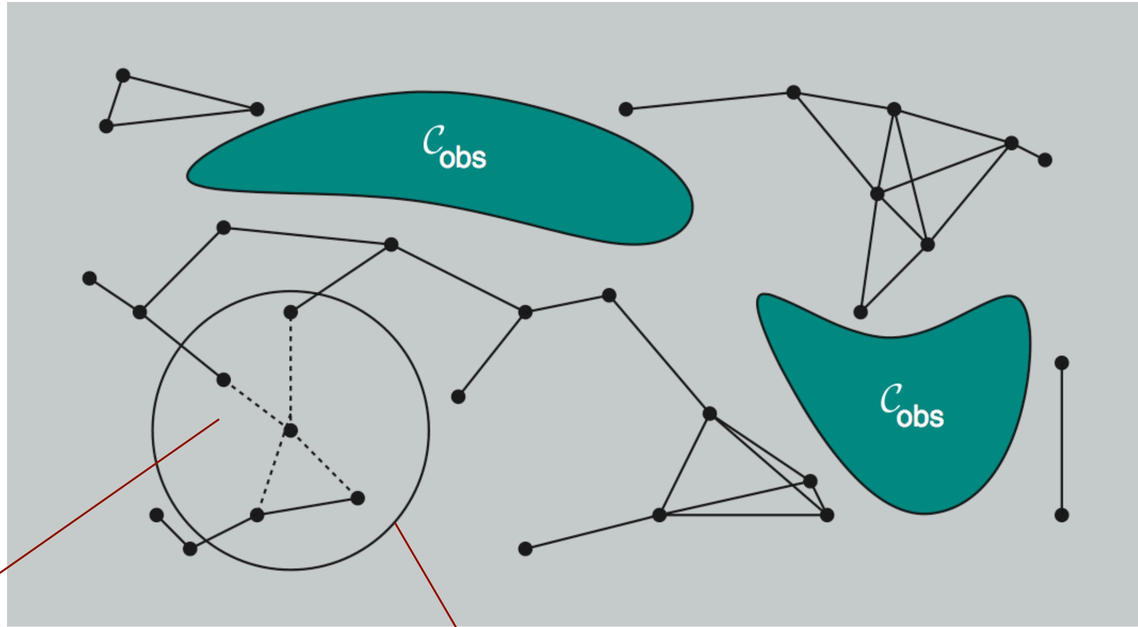
- Abandon the concept of explicitly characterizing C_{free} and C_{obs} and leave the algorithm **in the dark** when exploring C_{free}
- The only light is provided by a **collision-detection algorithm**, that probes C to see whether some configuration lies in C_{free}
- We will have a look at
  - **Probabilistic road maps** (PRM)
    [Kavraki et al., 92]
  - **Rapidly exploring random trees** (RRT)
    [Lavalle and Kuffner, 99]

Probabilistic Road Maps

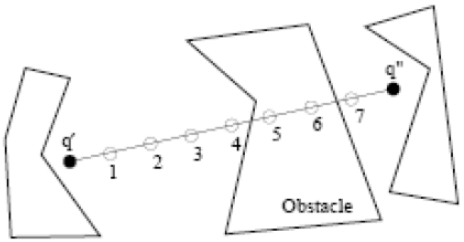
- **Idea:** Take random samples from C, declare them as vertices if in C_{free}, try to connect nearby vertices with local planner
- The **local planner** checks if line-of-sight is collision-free (powerful or simple methods)
- Options for *nearby:* **k-nearest neighbors** or all neighbors within **specified radius**
- Configurations and connections are added to graph until roadmap is **dense enough**

Probabilistic Road Maps

- Example



specified radius



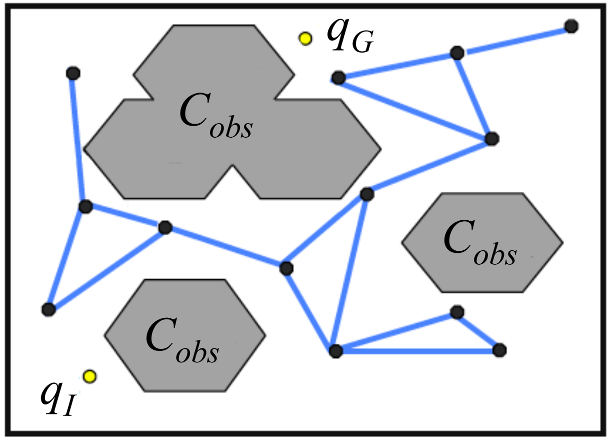
Example local planner

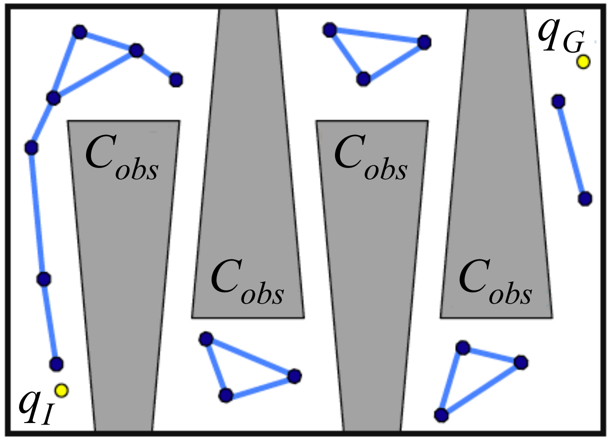
What means "nearby" on a manifold?
Defining a good metric on C is crucial

Probabilistic Road Maps

Good and bad news:

- **Pros:**
  - *Probabilistically complete*
  - Do not construct C-space
  - Apply easily to high-dim. C's
  - PRMs have solved previously unsolved problems
- **Cons:**
  - Do not work well for some problems, narrow passages
  - Not optimal, not complete



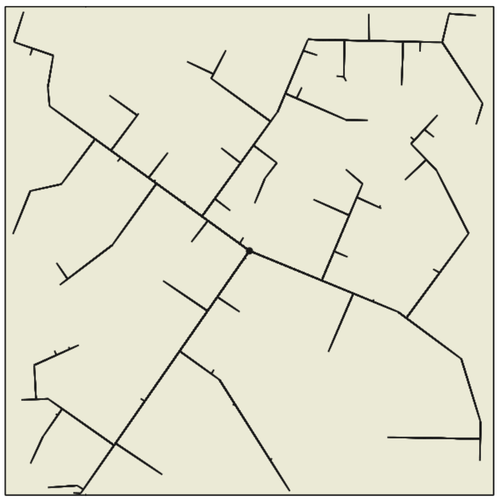


Probabilistic Road Maps

- How to **uniformly sample** C? This is not at all trivial given its topology
- For example over spaces of rotations: Sampling Euler angles gives samples near poles, not uniform over $SO(3)$. Use quaternions!
- However, PRMs are **powerful, popular** and **many extensions** exist: advanced sampling strategies (e.g. near obstacles), PRMs for deformable objects, closed-chain systems, etc.

Rapidly Exploring Random Trees

- **Idea:** aggressively probe and explore the C-space by **expanding incrementally** from an initial configuration q_0
- The explored territory is marked by a **tree rooted at q_0**



45 iterations

2345 iterations

RRTs

- The algorithm: Given C and q_0

Algorithm 1: RRT

$G.\text{init}(q_0)$
repeat
  $q_{rand} \rightarrow \text{RANDOM\_CONFIG}(\mathcal{C})$
  $q_{near} \leftarrow \text{NEAREST}(G, q_{rand})$
  $G.\text{add\_edge}(q_{near}, q_{rand})$
until *condition*

Sample from a **bounded region** centered around q_0

E.g. an axis-aligned relative random translation or random rotation

(but recall sampling over rotation spaces problem)

RRTs

- The algorithm

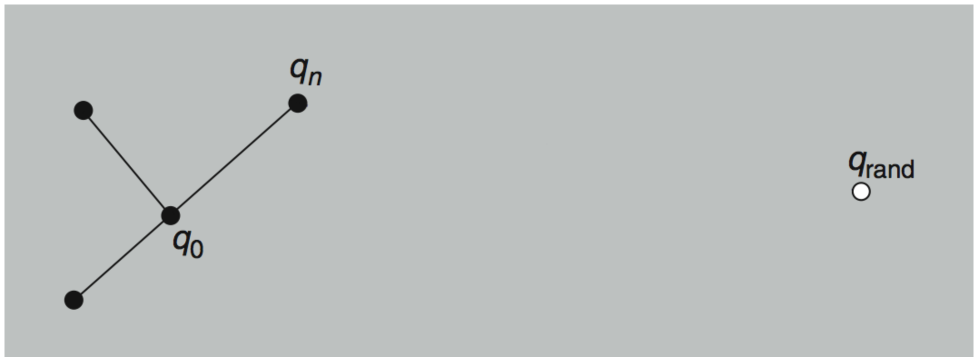
Algorithm 1: RRT

G.init(q_0)
repeat
  $q_{rand} \rightarrow$ RANDOM_CONFIG($\mathcal{C}$)
  $q_{near} \leftarrow$ NEAREST(G, q_{rand})
  G.add_edge(q_{near}, q_{rand})
until *condition*

Finds closest vertex in G using a **distance function**

$$\rho : \mathcal{C} \times \mathcal{C} \longrightarrow [0, \infty)$$

formally a ***metric*** defined on C



RRTs

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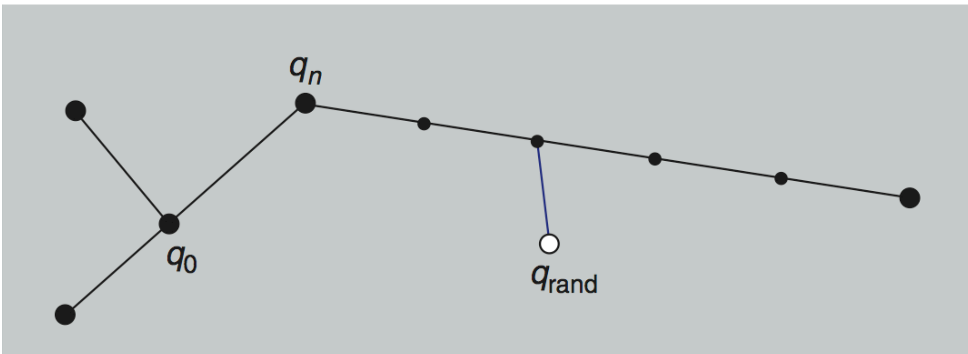


Several stategies to find q_{near} given the closest vertex on G:

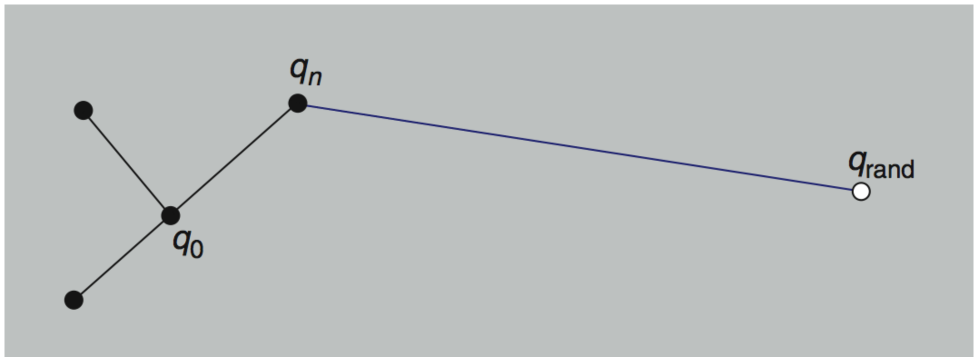
- Take closest vertex
- Check intermediate points at regular intervals and split edge at q_{near}

RRTs

- The algorithm

Algorithm 1: RRT

```
G.init(q_0)
repeat
    q_rand → RANDOM_CONFIG(C)
    q_near ← NEAREST(G, q_rand)
    G.add_edge(q_near, q_rand)
until condition
```



Connect nearest point with random point using a **local planner** that travels from q_{near} to q_{rand}

- No collision: add edge
- Collision: new vertex is q_i, as close as possible to C_{obs}

RRTs

- The algorithm

Algorithm 1: RRT

$G.\text{init}(q_0)$
repeat
  $q_{rand} \rightarrow \text{RANDOM\_CONFIG}(\mathcal{C})$
  $q_{near} \leftarrow \text{NEAREST}(G, q_{rand})$
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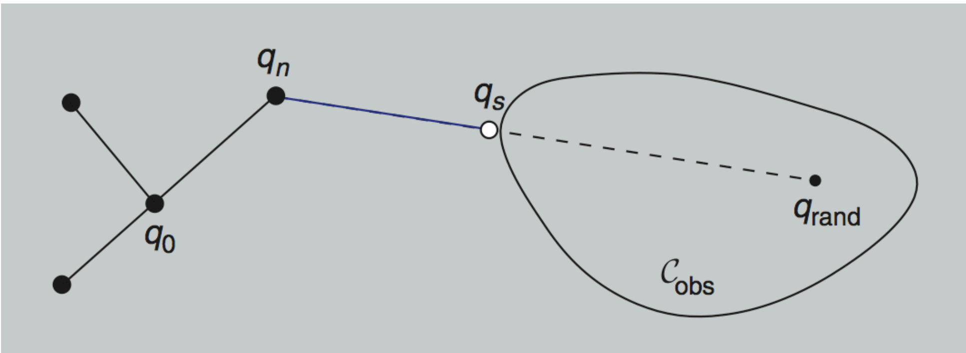
until *condition*



Connect nearest point with random point using a **local planner** that travels from q_{near} to q_{rand}

- No collision: add edge
- Collision: new vertex is q_i, as close as possible to C_{obs}

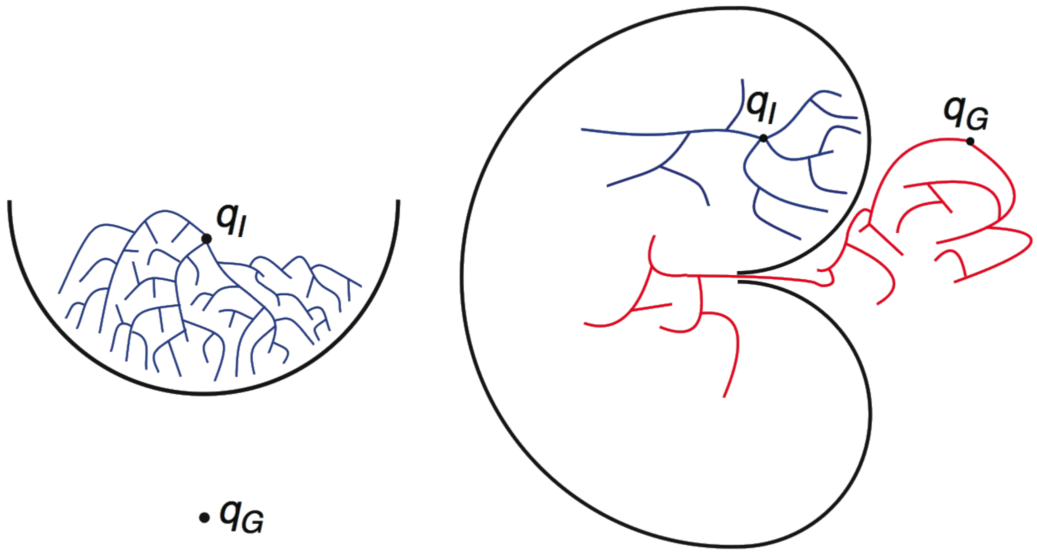
RRTs

- How to perform path planning with RRTs?
  1. Start RRT at q_I
  2. At every, say, 100th iteration, force $q_{rand} = q_G$
  3. If q_G is reached, problem is solved

- Why not picking q_G every time?
- This will fail and waste much effort in running into C_{Obs} instead of exploring the space

RRTs

- However, some problems require more effective methods: **bidirectional search**
- Grow **two** RRTs, one from q_I, one from q_G
- In every other step, try to extend each tree towards the newest vertex of the other tree

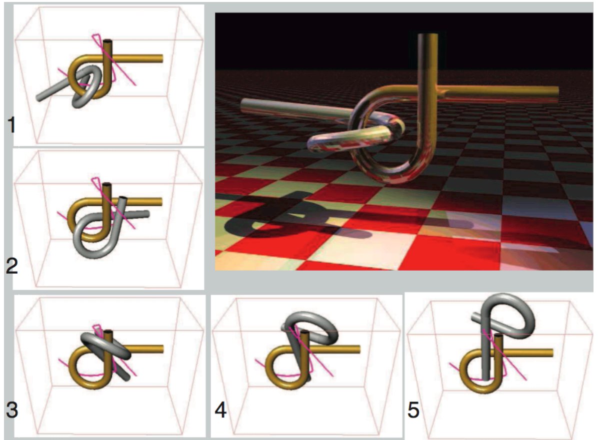


Filling a well

A bug trap

RRTs

- RRTs are popular, many extensions exist: real-time RRTs, anytime RRTs, for dynamic environments etc.
- **Pros:**
  - Balance between greedy search and exploration
  - Easy to implement
- **Cons:**
  - Metric sensivity
  - Unknown rate of convergence



Alpha 1.0 puzzle.
Solved with
bidirectional RRT

From Road Maps to Paths

- All methods discussed so far **construct a road map** (without considering the query pair q_I and q_G)

- Once the investment is made, the **same road map** can be reused for **all** queries (provided world and robot do not change)

  1. **Find** the cell/vertex that contain/is close to q_I and q_G (not needed for visibility graphs)
  2. **Connect** q_I and q_G to the road map
  3. **Search** the road map for a path from q_I to q_G

Sampling-Based Planning

Wrap Up

- Sampling-based planners are **more efficient** in most **practical problems** but offer weaker guarantees
- They are **probabilistically complete**: the probability tends to 1 that a solution is found if one exists (otherwise it may still run forever)
- Performance degrades in problems with **narrow passages**. Subject of active research
- Widely used. Problems with high-dimensional and complex C-spaces are still computationally hard

Potential Field Methods

- All techniques discussed so far aim at capturing the connectivity of C_{free} into a graph
- **Potential Field methods** follow a different idea:

  The robot, represented as a point in C, is modeled as a **particle** under the influence of a **artificial potential field** $\mathbf{U}$

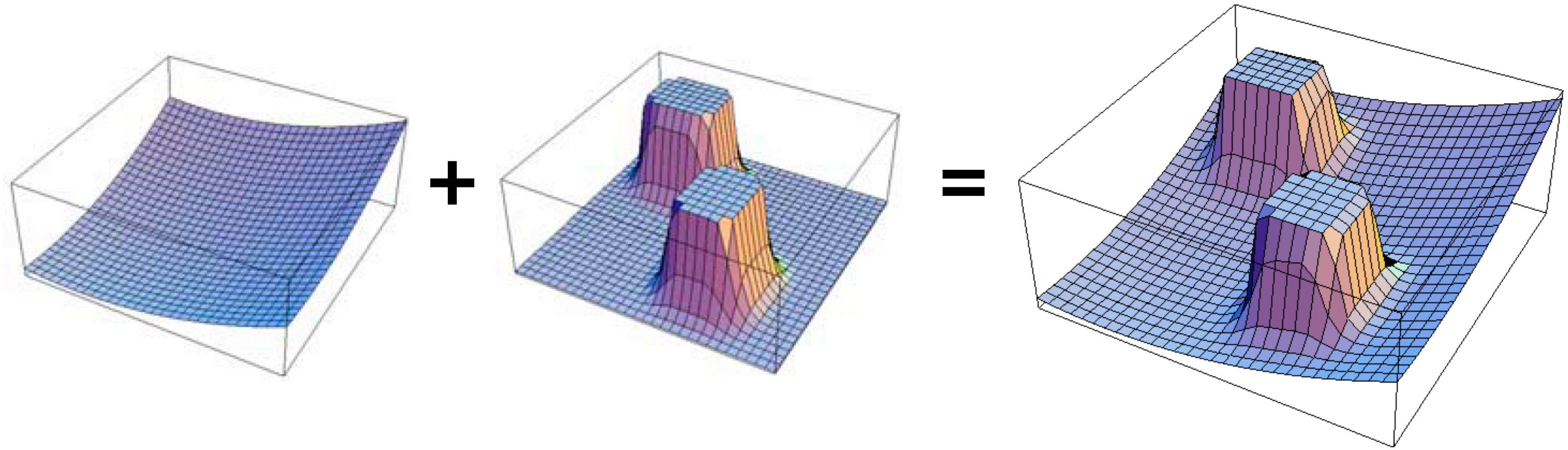
  $\mathbf{U}$ superimposes
  - **Repulsive forces** from obstacles
  - **Attractive force** from goal

Potential Field Methods

- Potential function

$$\mathbf{U}(q) = \mathbf{U}_{att}(q) + \mathbf{U}_{rep}(q)$$

$$\vec{F}(q) = -\vec{\nabla}\mathbf{U}(q)$$



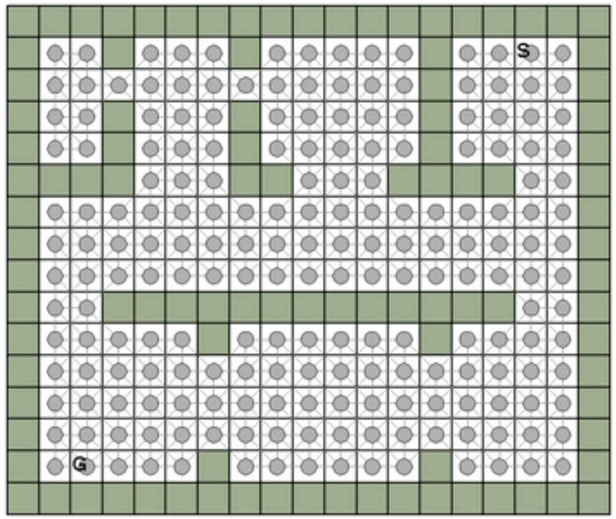
- Simply perform **gradient descent**
- C-pace typically discretized in a grid

Potential Field Methods

- Main problems: robot gets stuck in **local minima**
- Way out: Construct local-minima-free **navigation function** ("NF1"), then do gradient descent (e.g. bushfire from goal)
- The gradient of the potential function defines a **vector field** (similar to a policy) that can be used as **feedback control strategy**, relevant for an uncertain robot
- However, potential fields need to represent C_{free} **explicitely**. This can be too costly.

Robot Motion Planning

- Given a road map, let's do **search**!

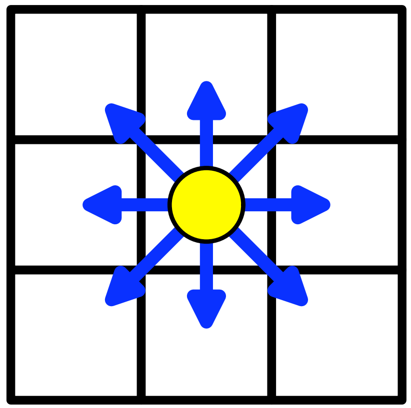


A* Search

- **A*** is one of the most widely-known informed search algorithms with many applications in robotics

- *Where are we?*
  A* is an instance of an **informed algorithm** for the general problem of **search**

- In robotics: planning on a 2D occupancy grid map is a common approach



Search

The problem of **search:** finding a sequence of actions (a *path*) that leads to desirable states (a *goal*)

- **Uninformed search:** besides the problem definition, no further information about the domain ("blind search")
- The only thing one can do is to expand nodes differently
- Example algorithms: breadth-first, uniform-cost, depth-first, bidirectional, etc.

Search

The problem of **search:** finding a sequence of actions (a *path*) that leads to desirable states (a *goal*)

- **Informed search:** further information about the domain through **heuristics**
- Capability to say that a node is "more promising" than another node
- Example algorithms: greedy best-first search, **A***, many variants of A*, D*, etc.

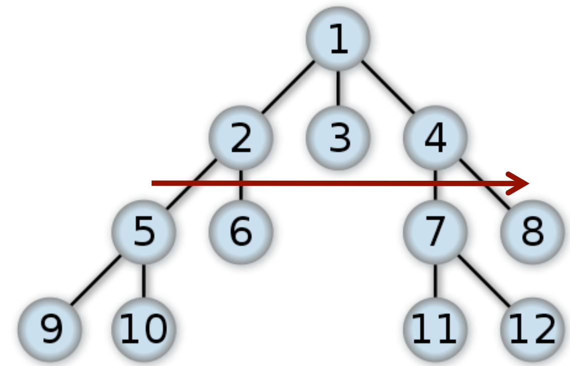
Search

The performance of a search algorithm is measured in four ways:

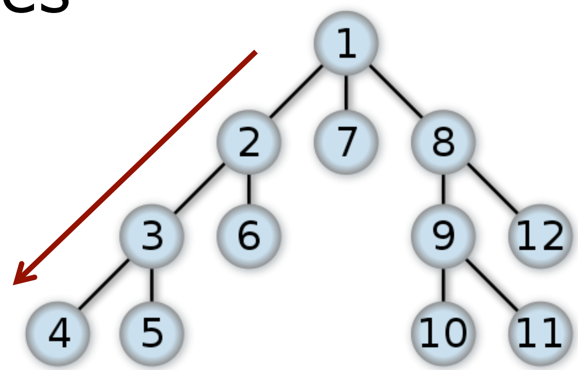
- **Completeness:** does the algorithm find the solution when there is one?
- **Optimality:** is the solution the best one of all possible solutions in terms of path cost?
- **Time complexity:** how long does it take to find a solution?
- **Space complexity:** how much memory is needed to perform the search?

Uninformed Search

- **Breadth-first**
  - Complete
  - Optimal if action costs equal
  - Time and space: $O(b^d)$

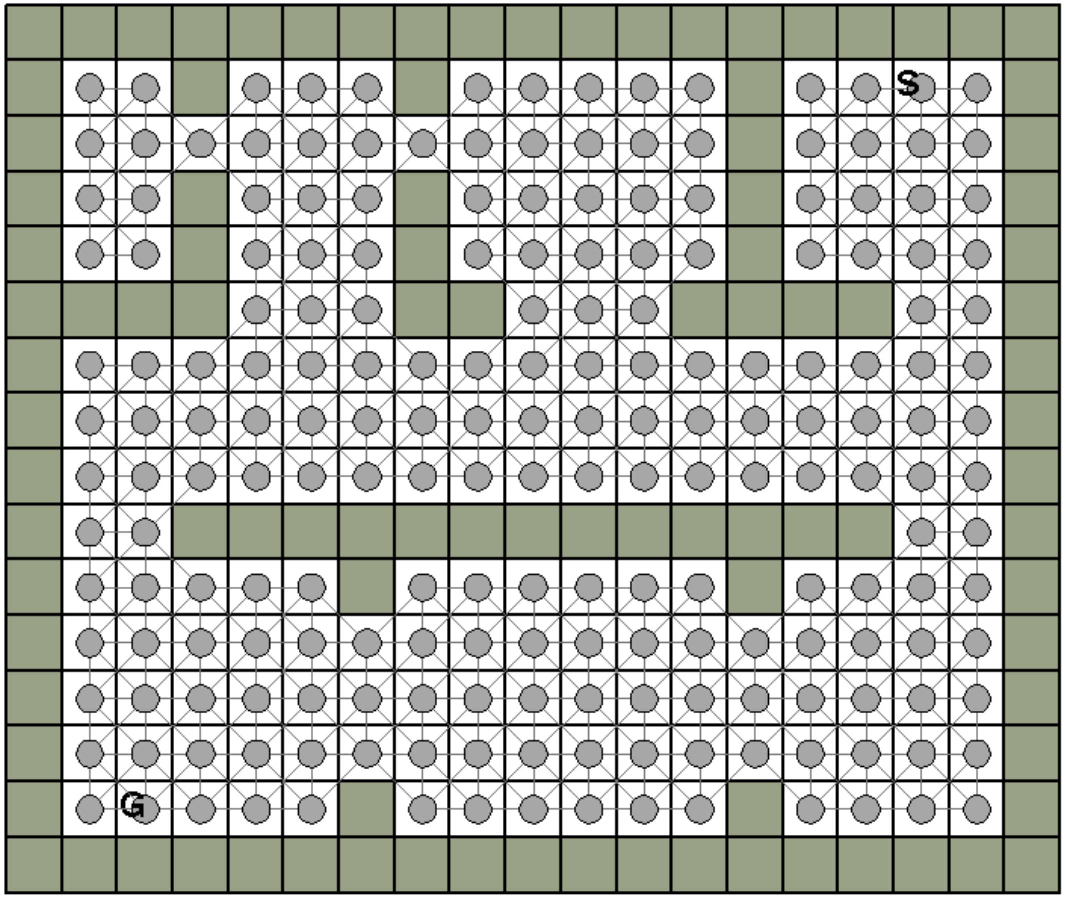


- **Depth-first**
  - Not complete in infinite spaces
  - Not optimal
  - Time: $O(b^m)$
  - Space: $O(bm)$ (can forget explored subtrees)



(b: branching factor, d: goal depth, m: max. tree depth)

Breadth-First Example



Informed Search

- Nodes are selected for expansion based on an **evaluation function** $f(n)$ from the set of generated but not yet explored nodes
- Then select node first with lowest $f(n)$ value

- Key component to every choice of $f(n)$: **Heuristic function** $h(n)$
- Heuristics are most common way to inject domain knowledge and inform search
- Every $h(n)$ is a cost estimate of cheapest path from n to a goal node

Informed Search

- **Greedy Best-First-Search**
  - Simply expands the node closest to the goal

    $$f(n) = h(n)$$

  - Not optimal, not complete, complexity $O(b^m)$

- **A* Search**
  - Combines path cost to n, $g(n)$, and estimated goal distance from n, $h(n)$

    $$f(n) = g(n) + h(n)$$

  - $f(n)$ estimates the cheapest path cost through n
  - If $h(n)$ is *admissible*: **complete** and **optimal**!

Heuristics

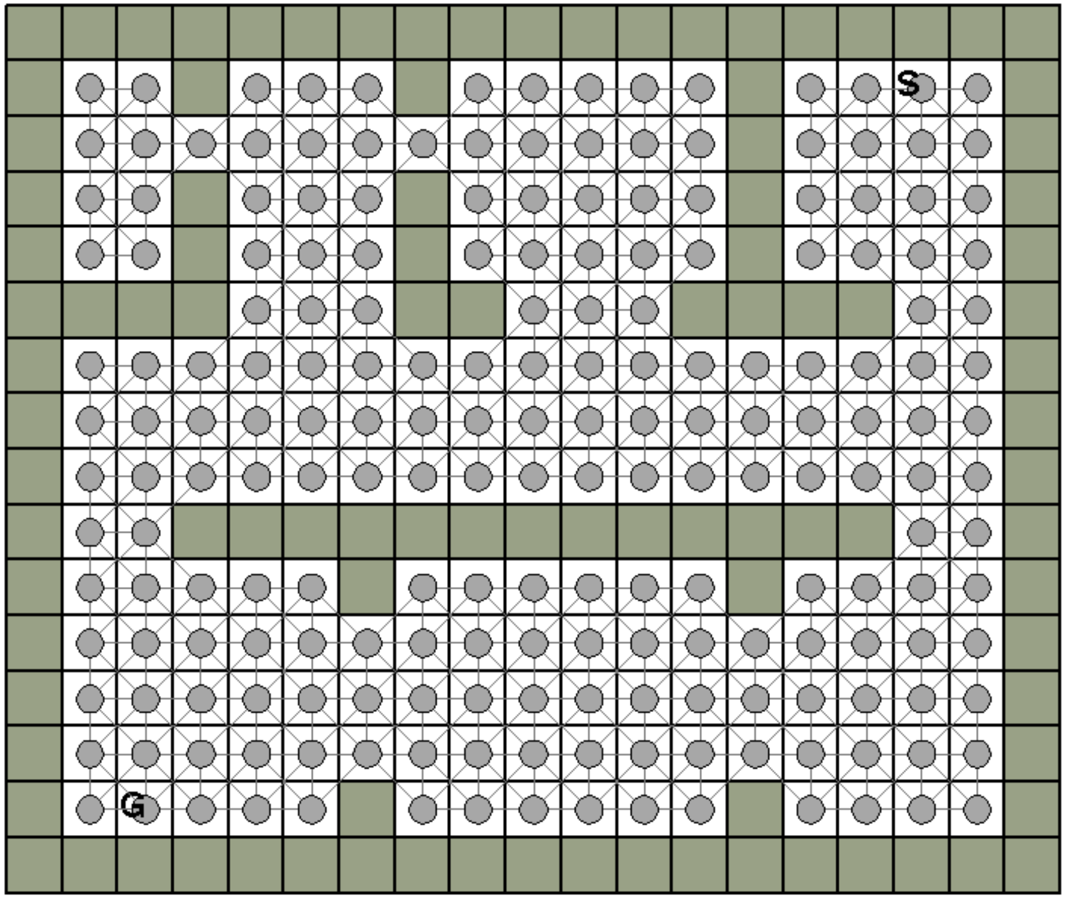
- **Admissible heuristic:**
  - Let $h^*(n)$ be the true cost of the optimal path from n to the goal. Then $h(.)$ is admissible if the following holds for all n:

$$h(n) \leq h^*(n)$$

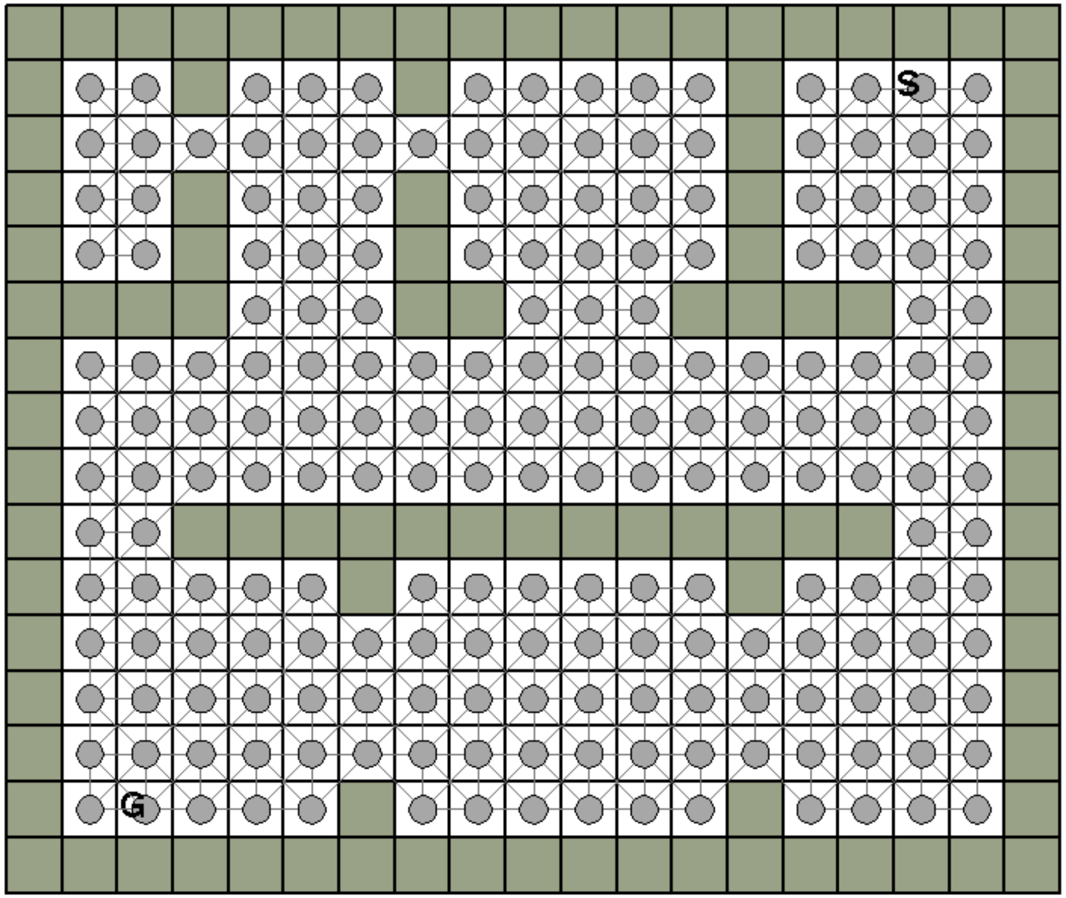
be optimistic, never overestimate the cost

- Heuristics are problem-specific. Good ones (admissible, efficient) for **our task** are:
  - **Straight-line distance** $h_{SLD}(n)$ (as with any routing problem)
  - **Octile distance**: Manhattan distance extended to allow diagonal moves
  - Deterministic **Value Iteration**/Dijkstra $h_{VI}(n)$

Greedy Best-First Example



A* with h_{SLD} Example



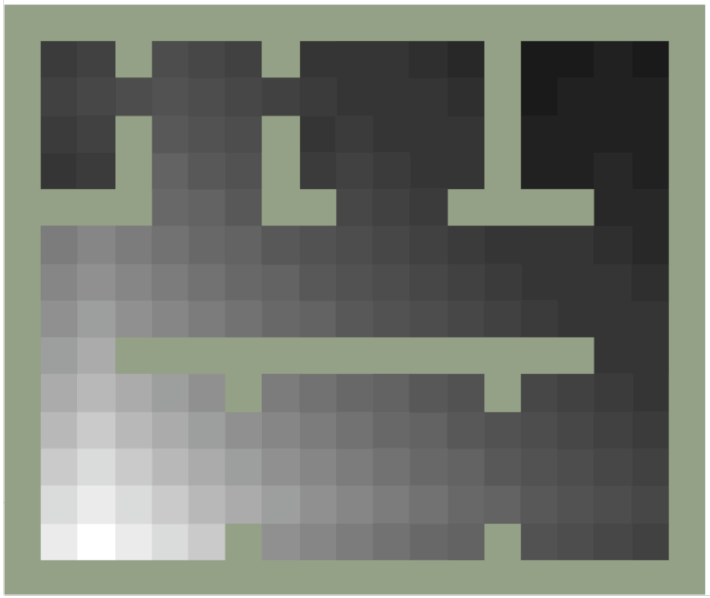
Heuristics for A*

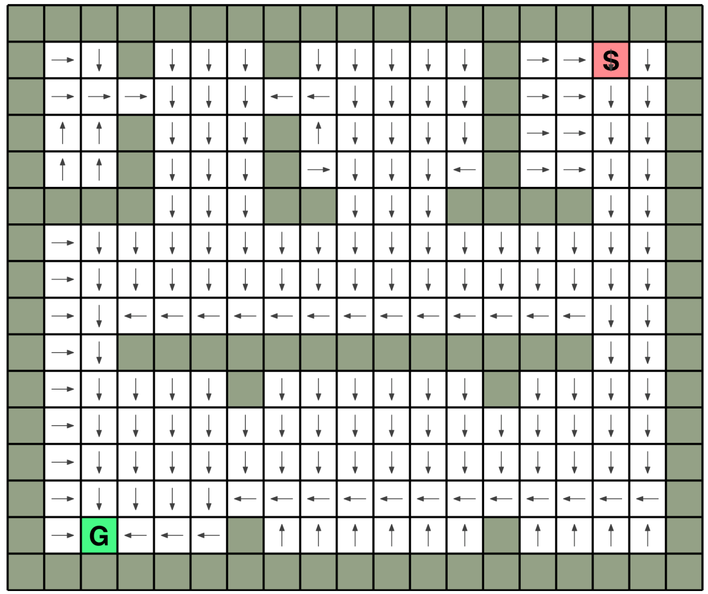
- Deterministic Value Iteration
  - Use Value Iteration for MDPs (later in this course) with rewards -1 and unit discounts
  - Like Dijkstra

- Precompute for dynamic or unknown environments where replanning is likely

Heuristics for A*

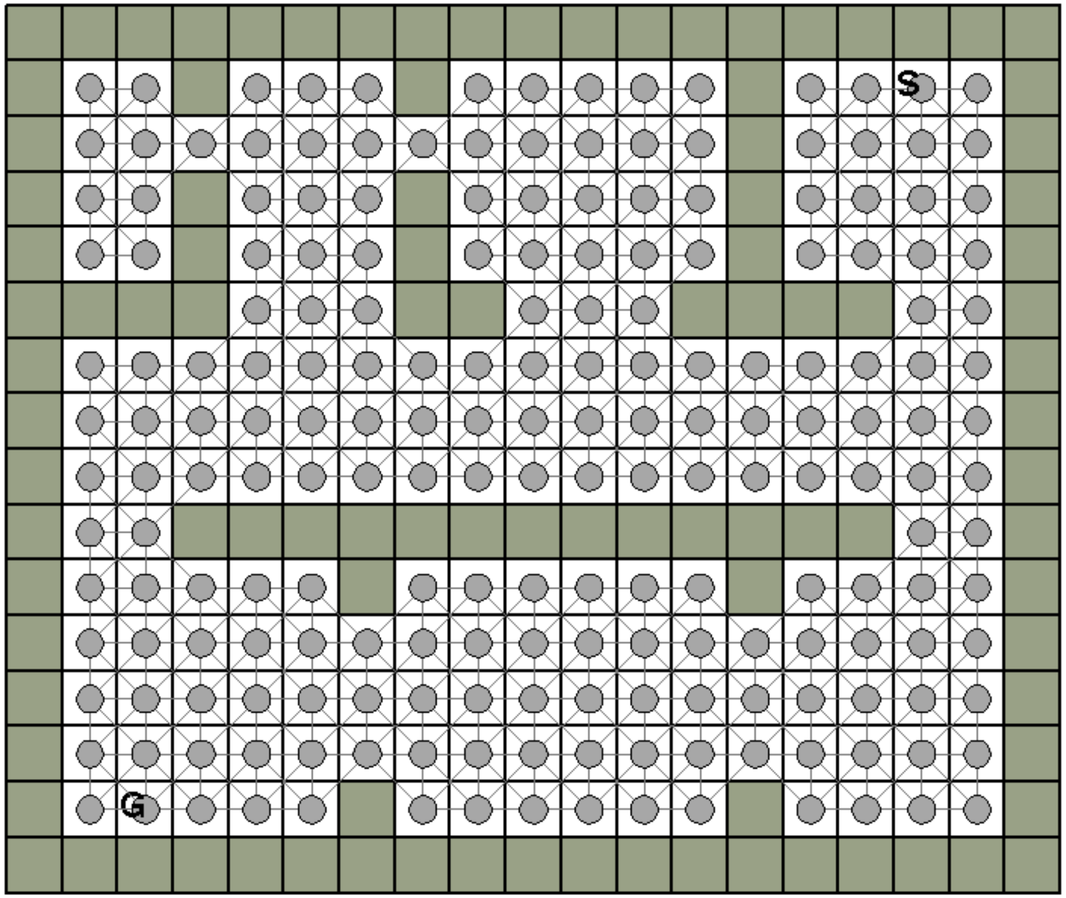
- Deterministic Value Iteration





- Recall vector field from potential functions: allows to implement a **feedback control strategy** for an uncertain robot

A* with h_{VI} Example



Problems with A* on Grids

1. The shortest path is often very **close to obstacles** (cutting corners)
   - Uncertain path execution increases the risk of collisions
   - Uncertainty can come from delocalized robot, imperfect map or poorly modeled dynamic constraints

2. Trajectories **aligned to the grid** structure
   - Path looks unnatural
   - Such paths are longer than the true shortest path in the continuous space

Problems with A* on Grids

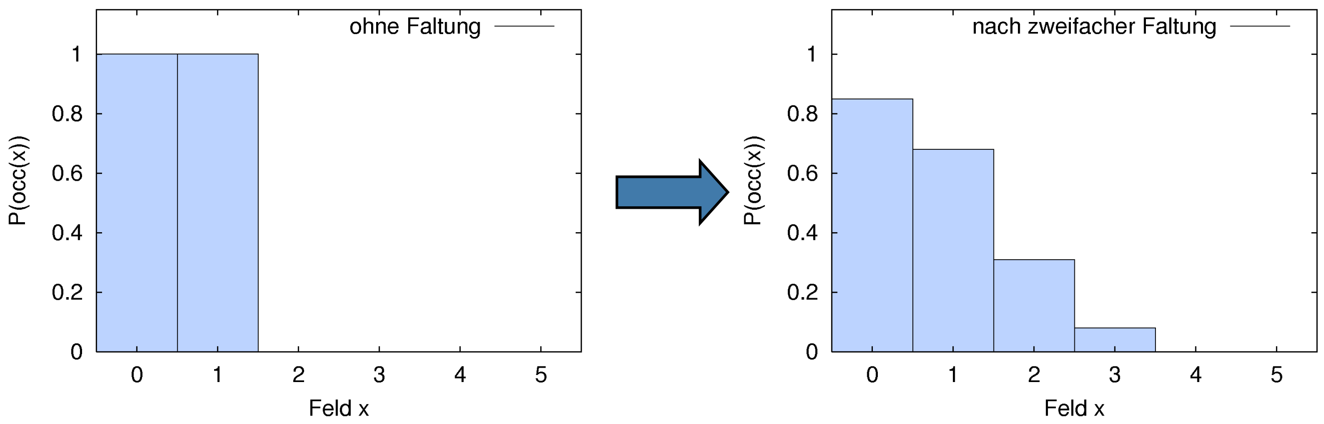
3. When the path turns out to be blocked during traversal, it needs to be **replanned from scratch**
   - In unknown or dynamic environments, this can occur very often
   - Replanning in large state spaces is costly
   - Can we reuse the initial plan?

Let us look at **solutions** to these problems…

Map Smoothing

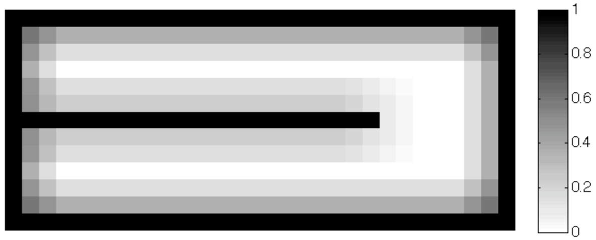
- Given an occupancy grid map
- **Convolution** blurs the map M with kernel k (e.g. a Gaussian kernel)

$$(M * k)[i] = \sum_{j=-\infty}^{\infty} M[j]\, k[i-j]$$



1D example: cells before and after two convolution runs

Map Smoothing



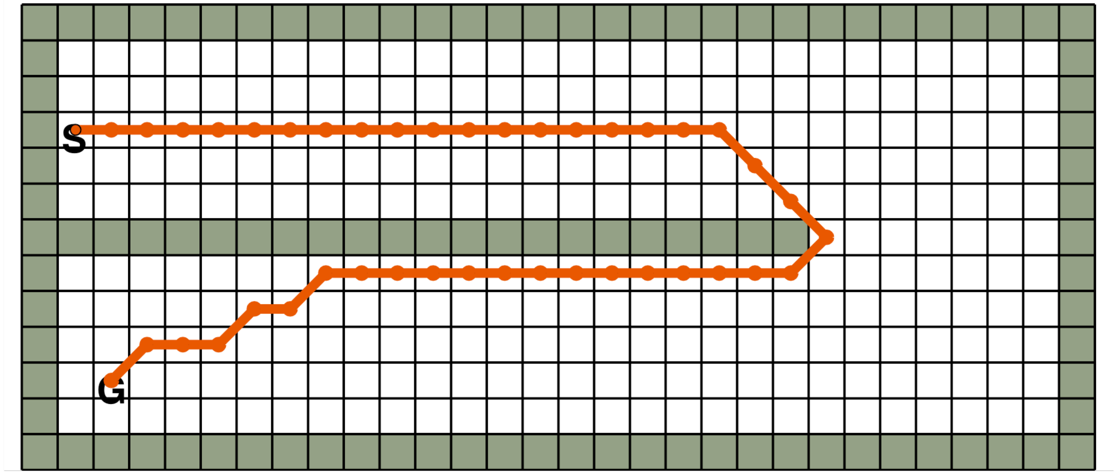
- Leads to above-zero probability areas around obstacles. Obstacles **appear bigger** than in reality
- Perform A* search in **convolved map** with evalution function

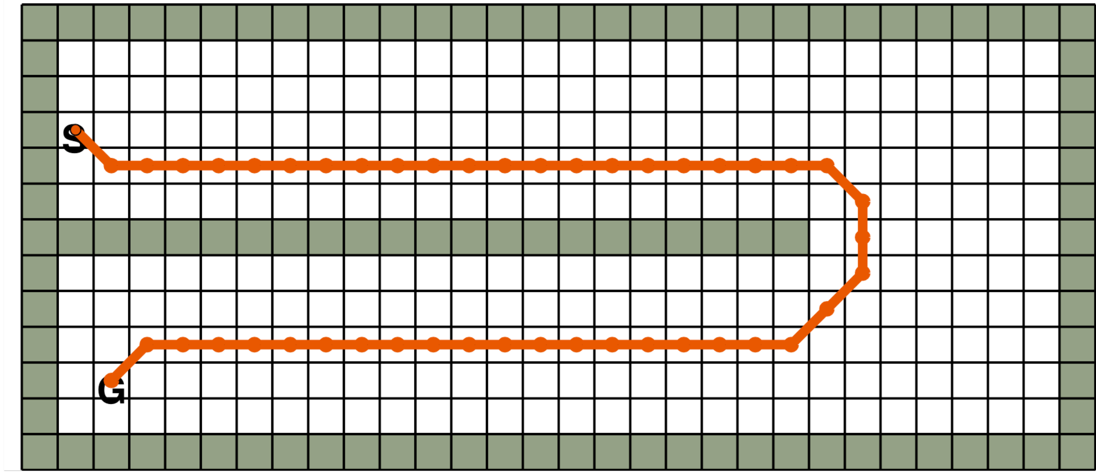
$$f(n) = g(n) \cdot p_{occ}(n) + h(n)$$

$p_{occ}(n)$: occupancy probability of node/cell n

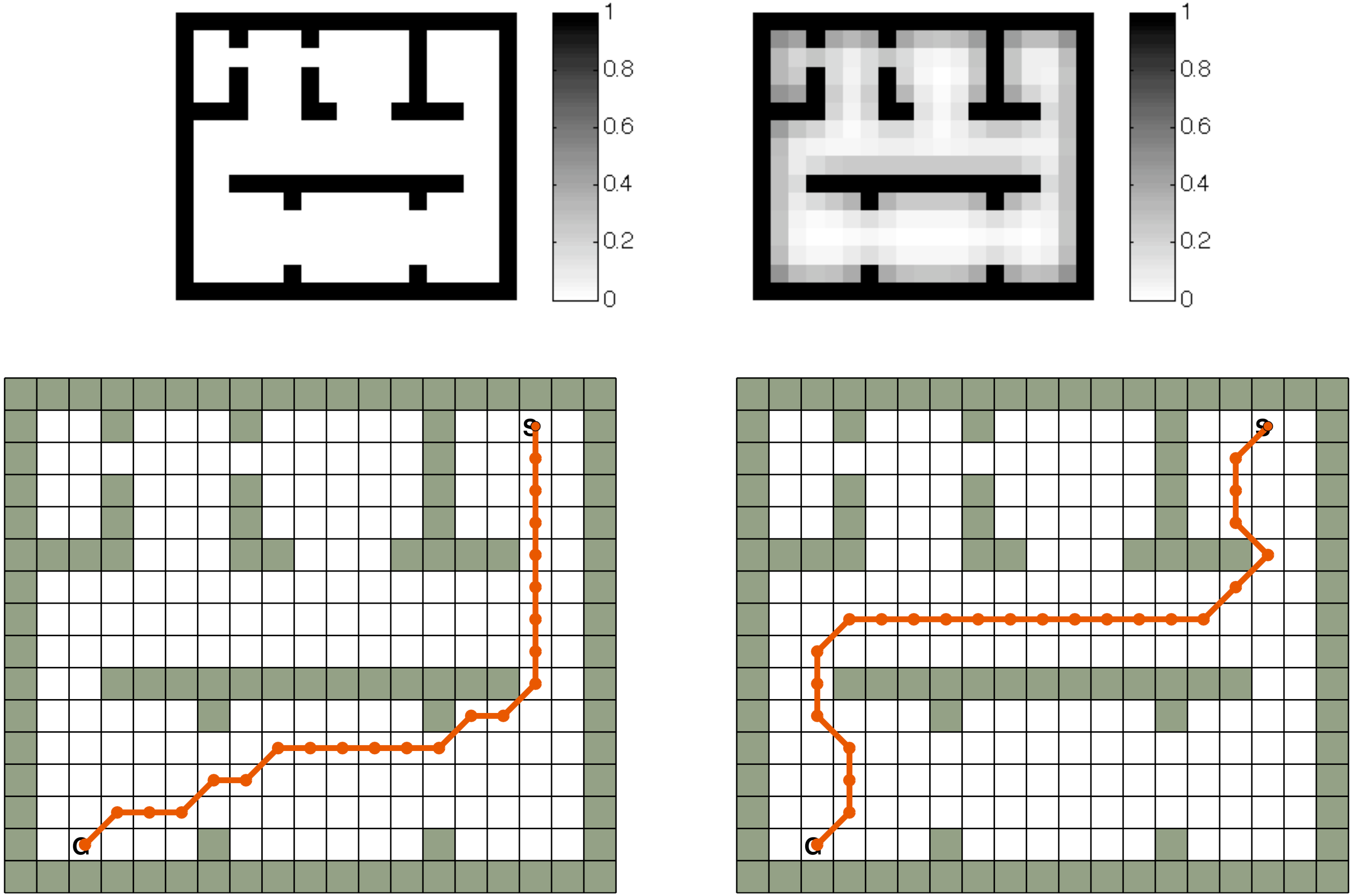
- Could also be a term for cell traversal cost

Map Smoothing



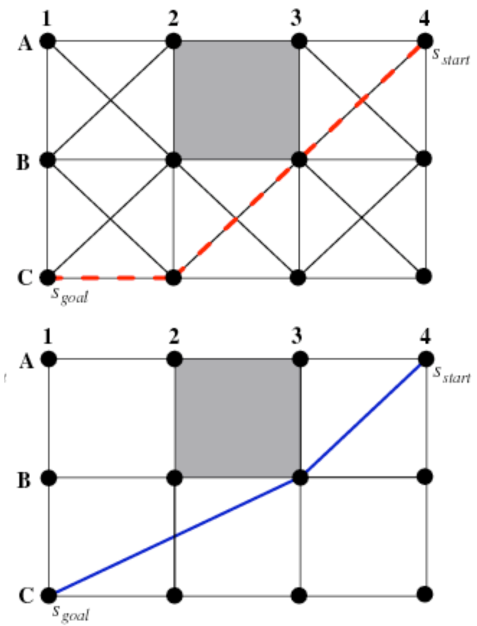


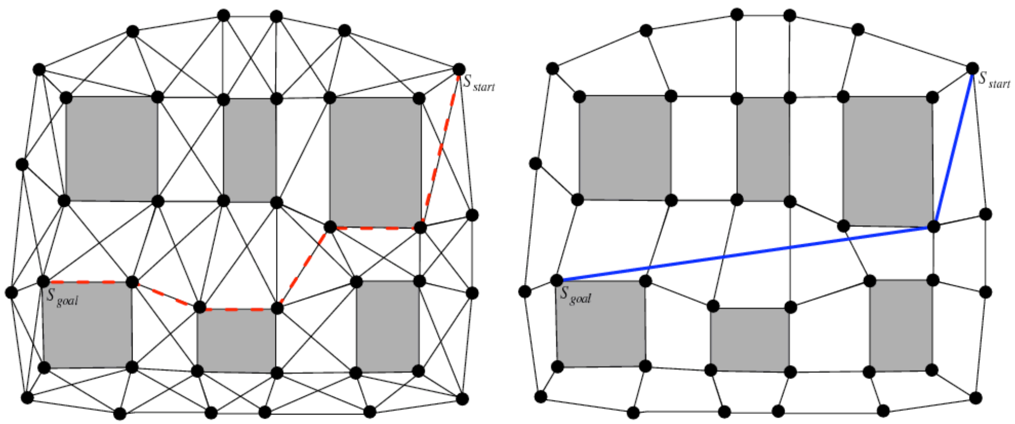
Map Smoothing



Any-Angle A*

- **Problem:** A* search only considers paths that are **constrained to graph edges**
- This can lead to **unnatural**, grid-aligned, and **suboptimal** paths





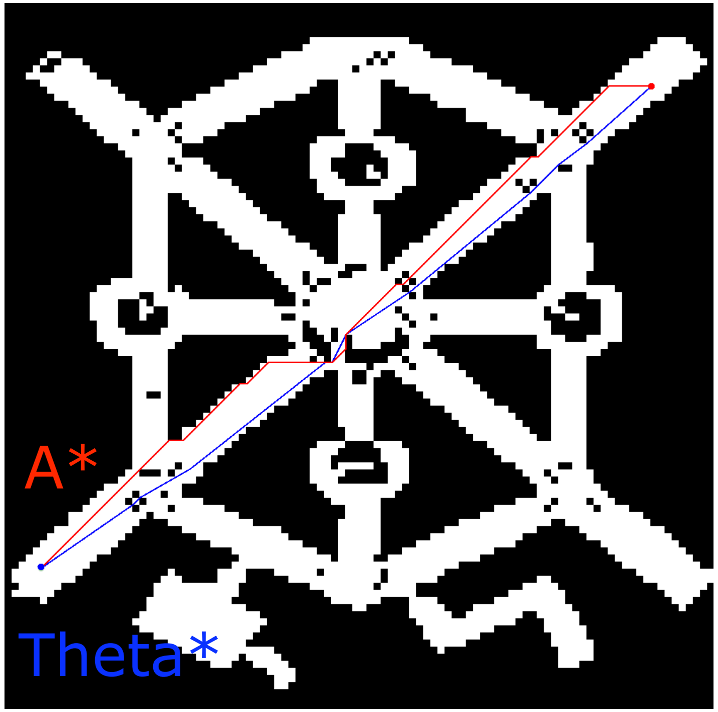
Pictures from *[Nash et al. AAAI'07]*

Any-Angle A*

- Different approaches:
  - **A* on Visibility Graphs**
    Optimal solutions in terms of path length!
  - **A* with post-smoothing**
    Traverse solution and find pairs of nodes with direct line of sight, replace by line segment
  - **Field D*** *[Ferguson and Stentz, JFR'06]*
    Interpolates costs of points not in cell centers. Builds upon D* family, able to efficiently replan
  - **Theta*** *[Nash et al. AAAI'07, AAAI'10]*
    Extension of A*, nodes can have non-neighboring successors based on a line-of-sight test

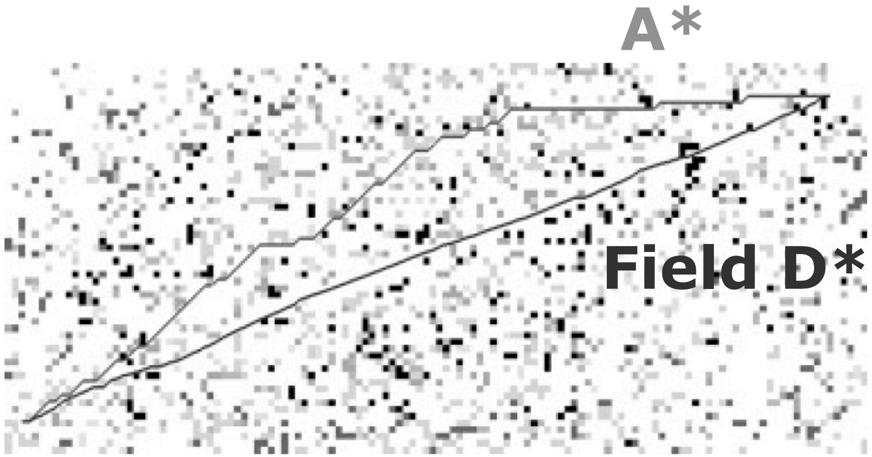
Any-Angle A* Examples

- Theta*



Game environment

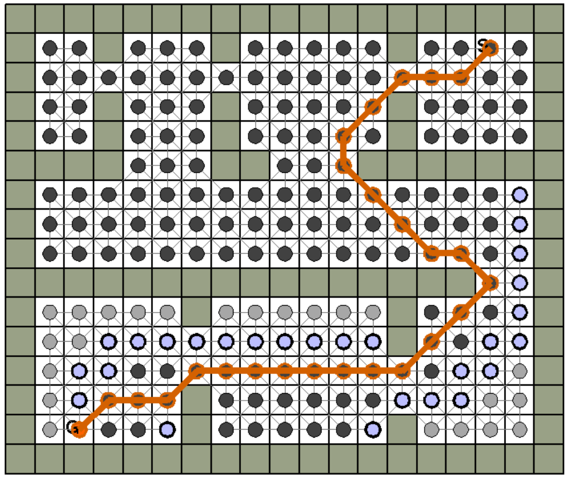
- Field D*



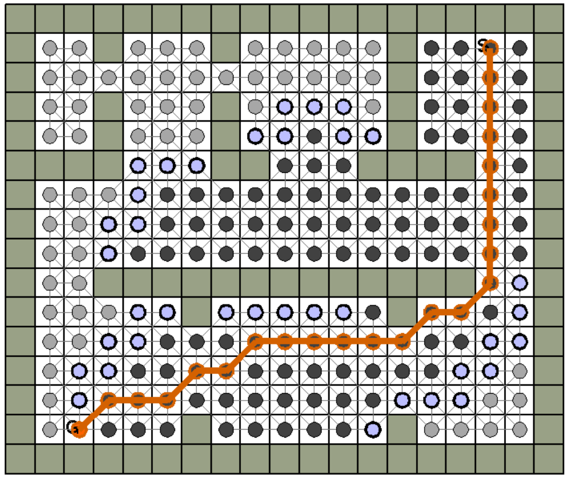
Outdoor environment.
Darker cells have larger **traversal costs**

Any-Angle A* Examples

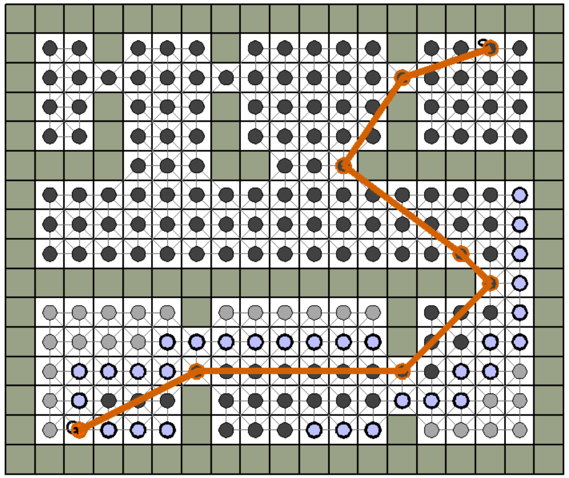
- A* vs. Theta* (*len*: path length, *nhead* = # heading changes)



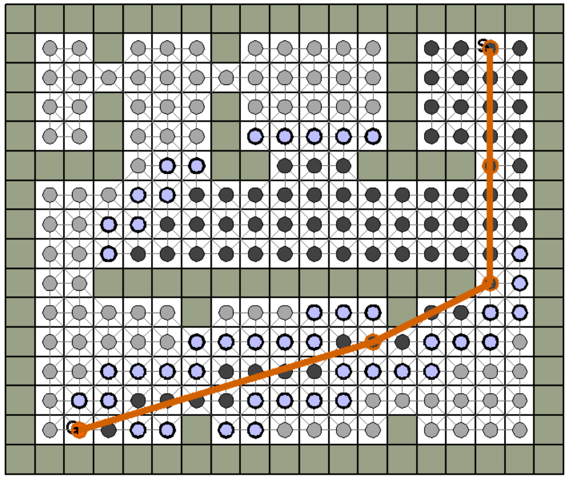
len: 30.0
nhead: 11



len: 24.1
nhead: 9

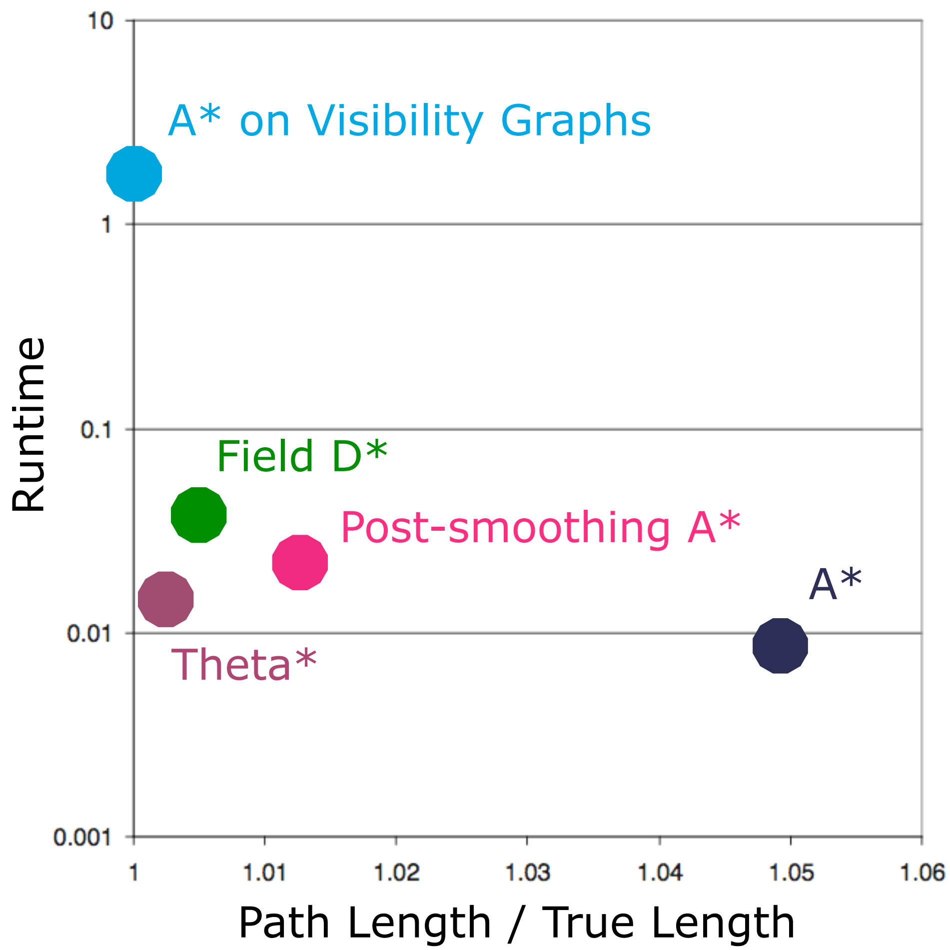


len: 28.9
nhead: 5



len: 22.9
nhead: 2

Any-Angle A* Comparison



[Daniel et al. JAIR'10]

- **A* PS** and **Theta*** provide the best trade off for the problem
- **A* on Visibility Graphs** scales poorly (but is optimal)
- **A* PS** does not always work in nonuniform cost environments. Shortcuts can end up in expensive areas

D* Search

- **Problem:** In unknown, partially known or dynamic environments, the planned path may be blocked and we need to **replan**
- Can this be done efficiently, avoiding to replan the **entire path?**
- **Idea:** Incrementally repair path keeping its modifications local around robot pose
- Several approaches implement this idea:
  - **D*** (Dynamic A*) *[Stentz, ICRA'94, IJCAI'95]*
  - **D* Lite** *[Koenig and Likhachev, AAAI'02]*
  - **Field D*** *[Ferguson and Stentz, JFR'06]*

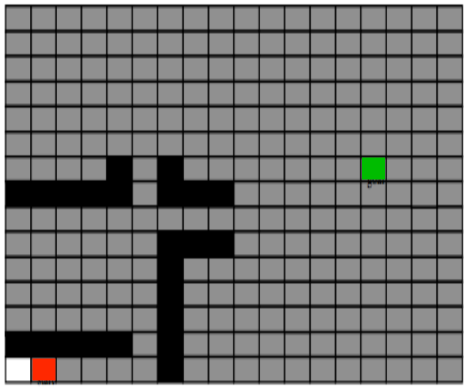
D*/D* Lite

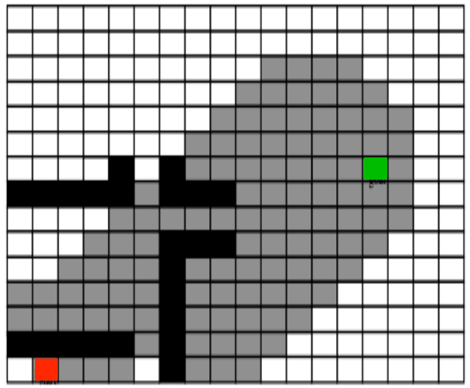
- **Main concepts**
  - **Switched search direction**: search from goal to the current vertex. If a change in edge cost is detected during traversal (around the current robot pose), only few nodes near the goal (=start) need to be updated
  - These nodes are nodes those **goal distances** have changed or not been caculated before AND are **relevant** to recalculate the new shortest path to the goal
  - **Incremental heuristic search** algorithms: able to focus and build upon previous solutions

D* Lite Example

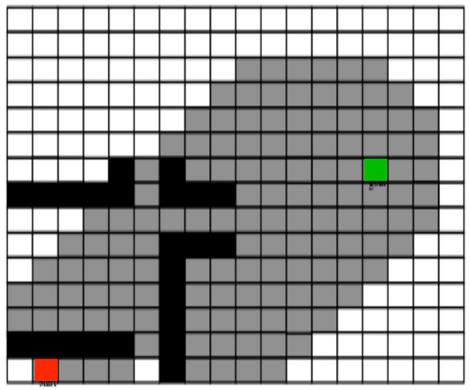
- Situation at start

Breadth-First-Search





A*



D* Lite

Start

Goal

Expanded nodes (goal distance calculated)

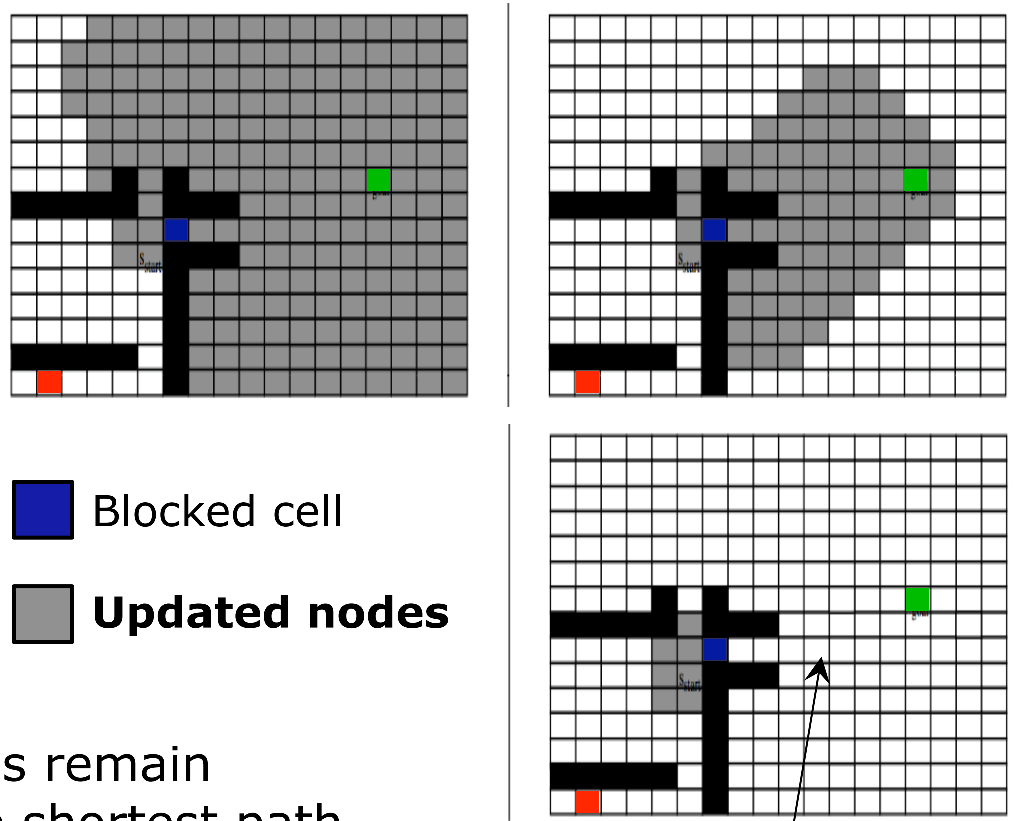
D* Lite Example

- After discovery of blocked cell

Breadth-First-Search

A*

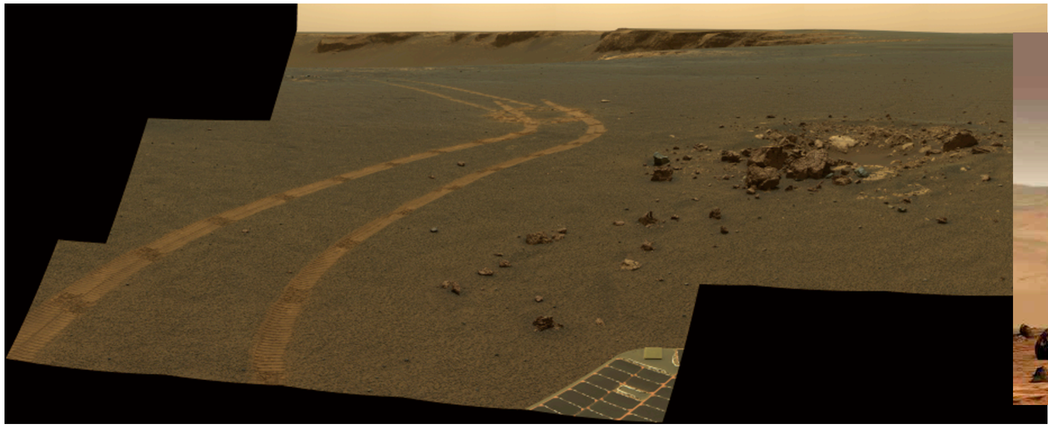
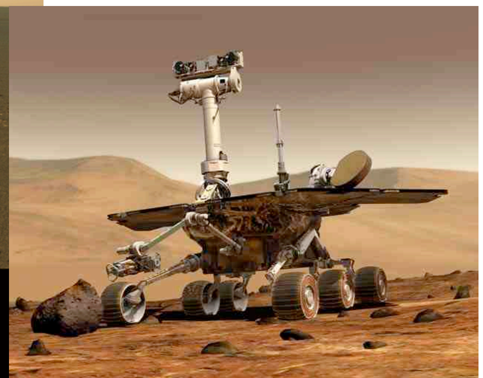
D* Lite



All other nodes remain unaltered, the shortest path can reuse them.

D* Family

- **D* Lite** produces the same paths than D* but is **simpler** and more **efficient**
- D*/D* Lite are **widely used**
- **Field D*** was running on Mars rovers Spirit and Opportunity (retrofitted in yr 3)

Tracks left by a drive executed with Field D*

Still in Dynamic Environments...

- Do we really need to replan the entire path for **each obstacle** on the way?
- What if the robot has to react **quickly** to unforeseen, fast moving obstacles?
  - Even D* Lite can be too slow in such a situation
- Accounting for the **robot shape** (it's not a point)
- Accounting for **kinematic** and **dynamic** vehicle **constraints**, e.g.
  - Decceleration limits,
  - Steering angle limits, etc.

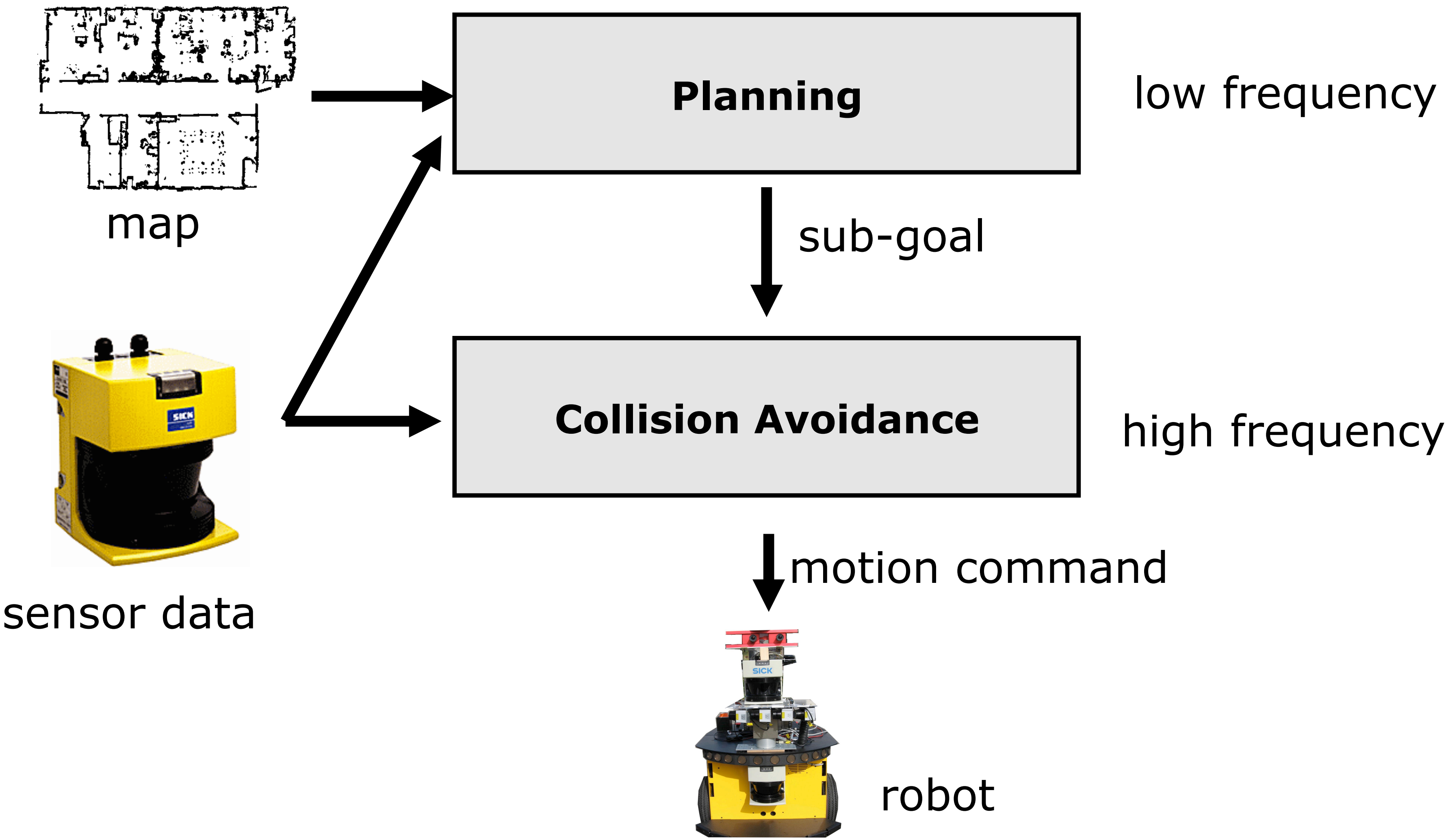
Collision Avoidance

- This can be handled by techniques called **collision avoidance** (obstacle avoidance)
- A well researched subject, different **approaches** exist:
  - Dynamic Window Approaches
    [Simmons, 96], [Fox et al., 97], [Brock & Khatib, 99]
  - Nearness Diagram Navigation
    [Minguez et al., 2001, 2002]
  - Vector-Field-Histogram+
    [Ulrich & Borenstein, 98]
  - Extended Potential Fields
    [Khatib & Chatila, 95]

Collision Avoidance

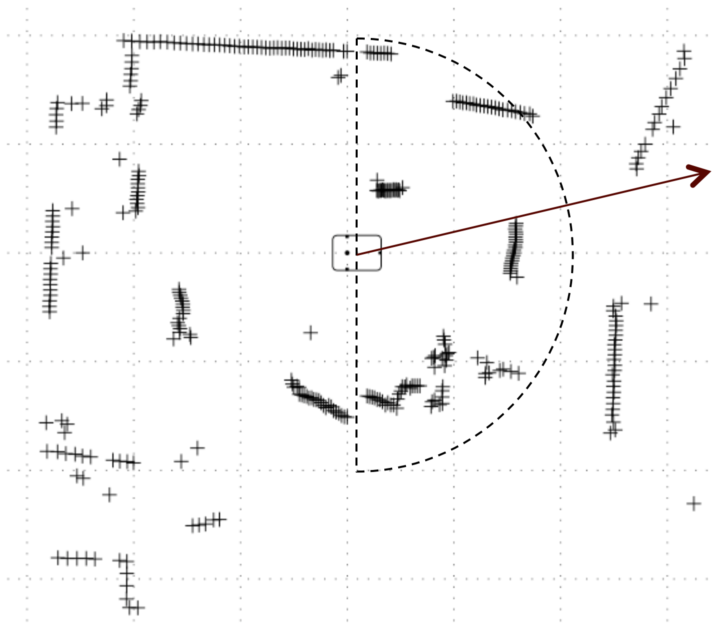
- Integration into general motion planning?
- It is common to subdivide the problem into a global and local planning task:
  - An approximate **global planner** computes paths ignoring the kinematic and dynamic vehicle constraints
  - An accurate **local planner** accounts for the constraints and generates (sets of) feasible local trajectories ("collision avoidance")
- What do we loose? What do we win?

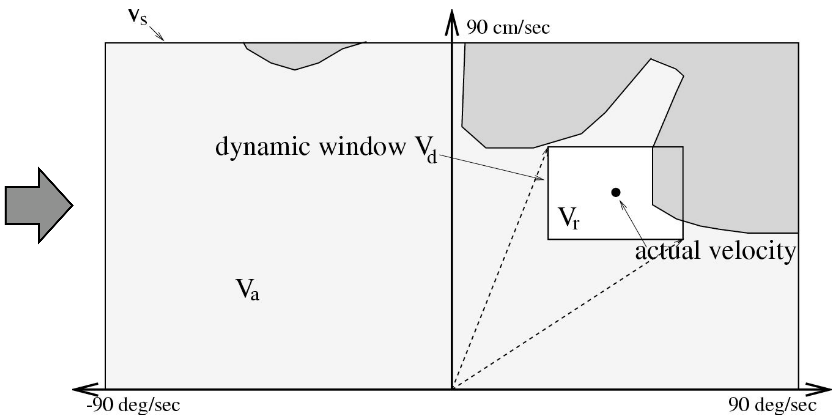
Two-layered Architecture



Dynamic Window Approach

- **Given:** path to goal (a set of via points), range scan of the local vicinity, dynamic constraints
- **Wanted:** collision-free, safe, and fast motion towards the goal





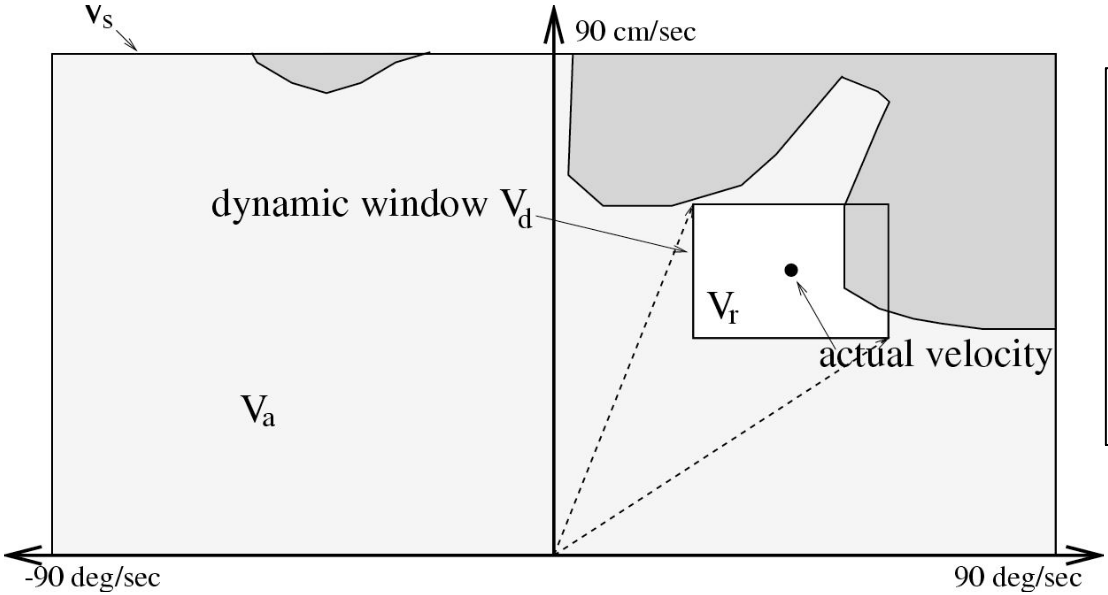
Dynamic Window Approach

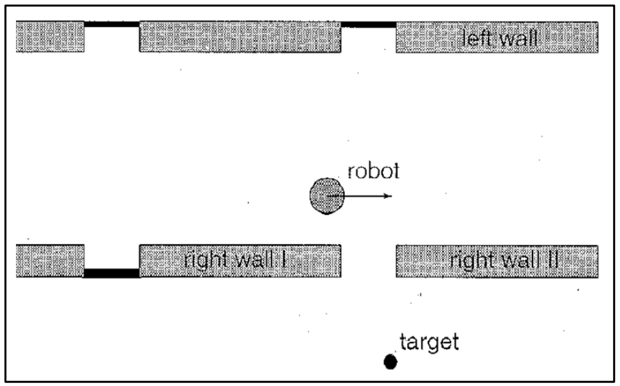
- **Assumption:** robot takes motion commands of the form (v,ω)
- This is saying that the robot moves (instantaneously) on **circular arcs** with radius $r = v / \omega$

- **Question:** which (v,ω)'s are
  - **reasonable**: that bring us to the goal?
  - **admissible**: that are collision-free?
  - **reachable**: under the vehicle constraints?

DWA Search Space

- 2D velocity search space





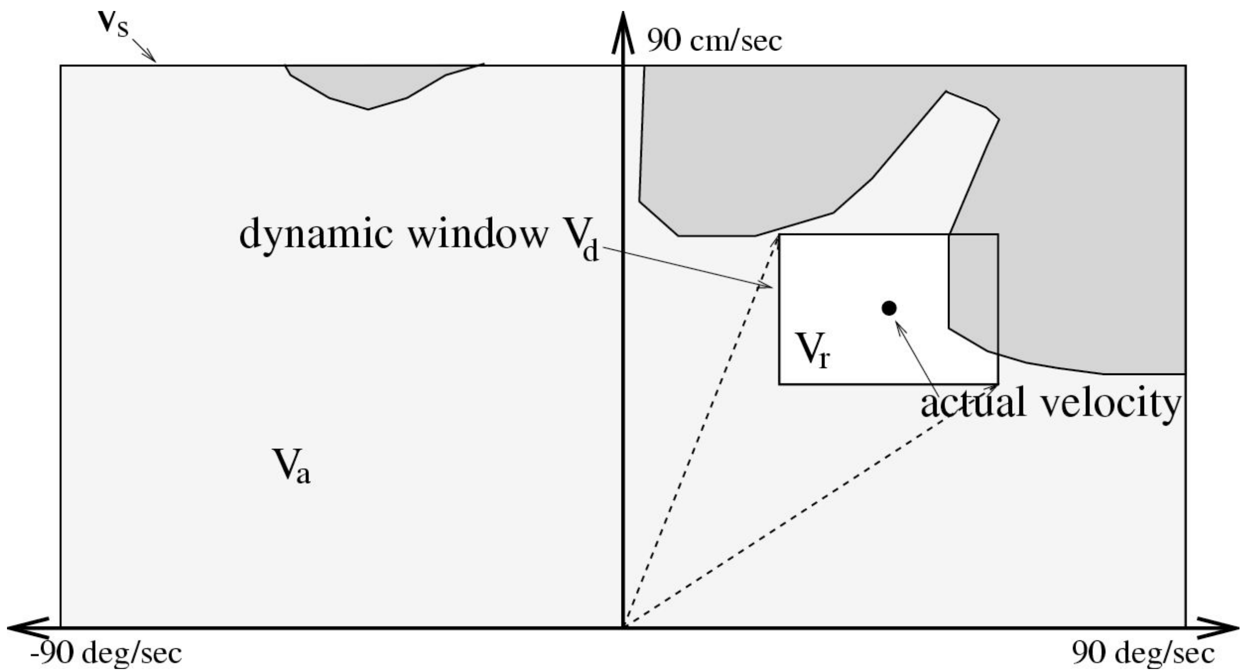
  - V_s = all possible speeds of the robot
  - V_a = obstacle free area
  - V_d = speeds reachable within one time frame given acceleration constraints

$$Space = V_s \cap V_a \cap V_d$$

Reachable Velocities

- Speeds are reachable if

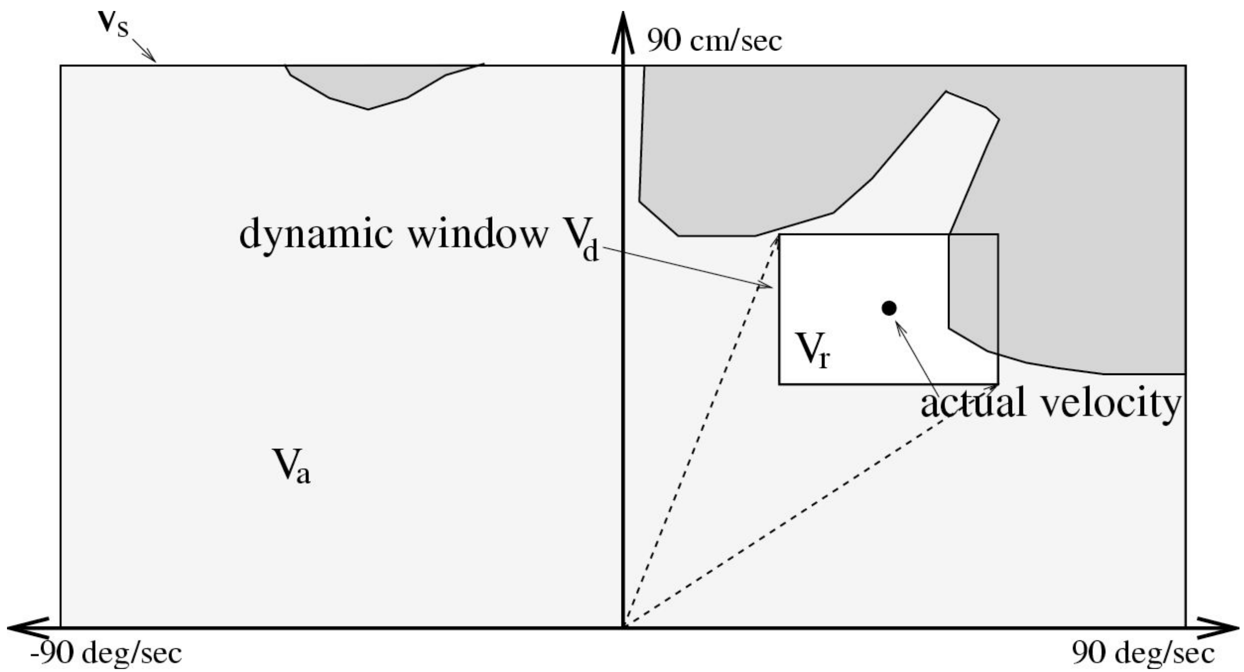
$$V_d = \{(v, \omega) \;\;|\;\; v \in [v - a_{trans}t, v + a_{trans}t] \;\; \wedge$$
$$\omega \in [\omega - a_{rot}t, \omega + a_{rot}t]\}$$



Admissible Velocities

- Speeds are admissible if

$$V_a = \{(v,\omega) \;\mid\; v \le \sqrt{2\text{dist}(v,\omega)a_{trans}} \;\wedge\; \omega \le \sqrt{2\text{dist}(v,\omega)a_{rot}}\}$$



Dynamic Window Approach

- How to choose (v,ω) ?
- Pose the problem as an **optimization problem** of an objective function within the dynamic window, search the maximum
- The objective function is a **heuristic navigation function**
- This function encodes the incentive to minimize the travel time by "driving **fast** and **safe** in the **right direction**"

Dynamic Window Approach

- Heuristic navigation function
- Planning restricted to (v,ω)-space
- Here: assume to have precomputed goal distances from NF1 algorithm

Navigation Function: *[Brock & Khatib, 99]*

$$NF = \alpha \cdot vel + \beta \cdot nf + \gamma \cdot \Delta nf + \delta \cdot goal$$

89

Dynamic Window Approach

- Heuristic navigation function
- Planning restricted to *(v,ω)*-space
- Here: assume to have precomputed goal distances from NF1 algorithm

Navigation Function: *[Brock & Khatib, 99]*

$$NF = \alpha \cdot vel + \beta \cdot nf + \gamma \cdot \Delta nf + \delta \cdot goal$$

Maximizes velocity

Dynamic Window Approach

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Navigation Function: *[Brock & Khatib, 99]*

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Maximizes velocity

Rewards alignment to NF1/A* gradient

Dynamic Window Approach

- Heuristic navigation function
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$$NF = \alpha \cdot vel + \beta \cdot nf + \gamma \cdot \Delta nf + \delta \cdot goal$$

Maximizes velocity

Rewards alignment to NF1/A* gradient

Rewards large advances on NF1/A* path

Dynamic Window Approach

- Heuristic navigation function
- Planning restricted to *(v,ω)*-space
- Here: assume to have precomputed goal distances from NF1 algorithm

Navigation Function: *[Brock &*

Comes in when goal region reached

$$NF = \alpha \cdot vel + \beta \cdot nf + \gamma \cdot \Delta nf + \delta \cdot goal$$

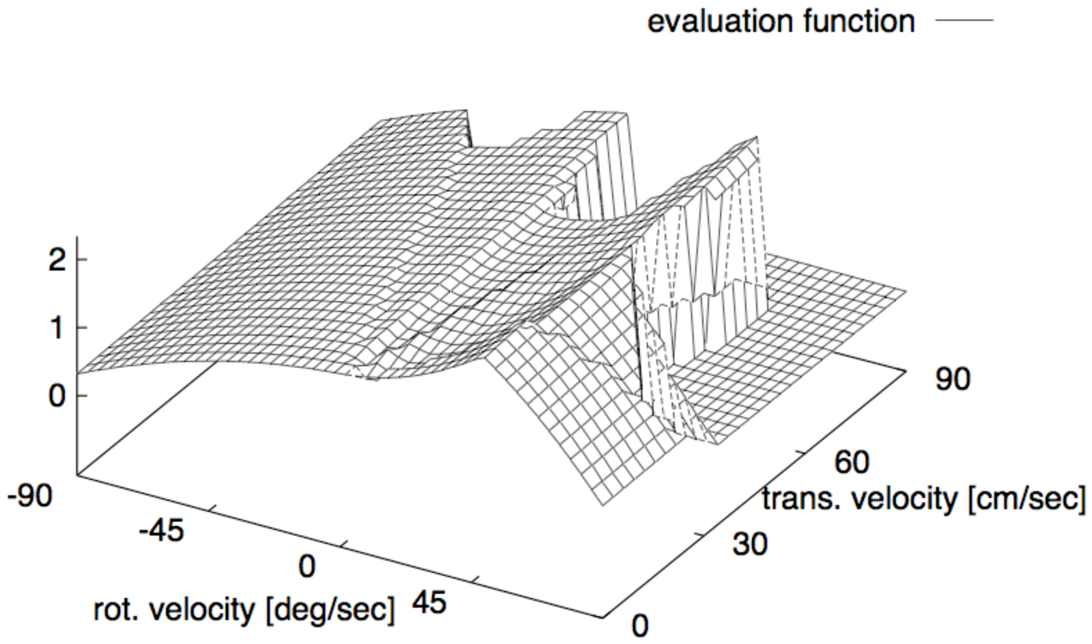
Maximizes velocity

Rewards alignment to NF1/A* gradient

Rewards large advances on NF1/A* path

Dynamic Window Approach

- Navigation function example



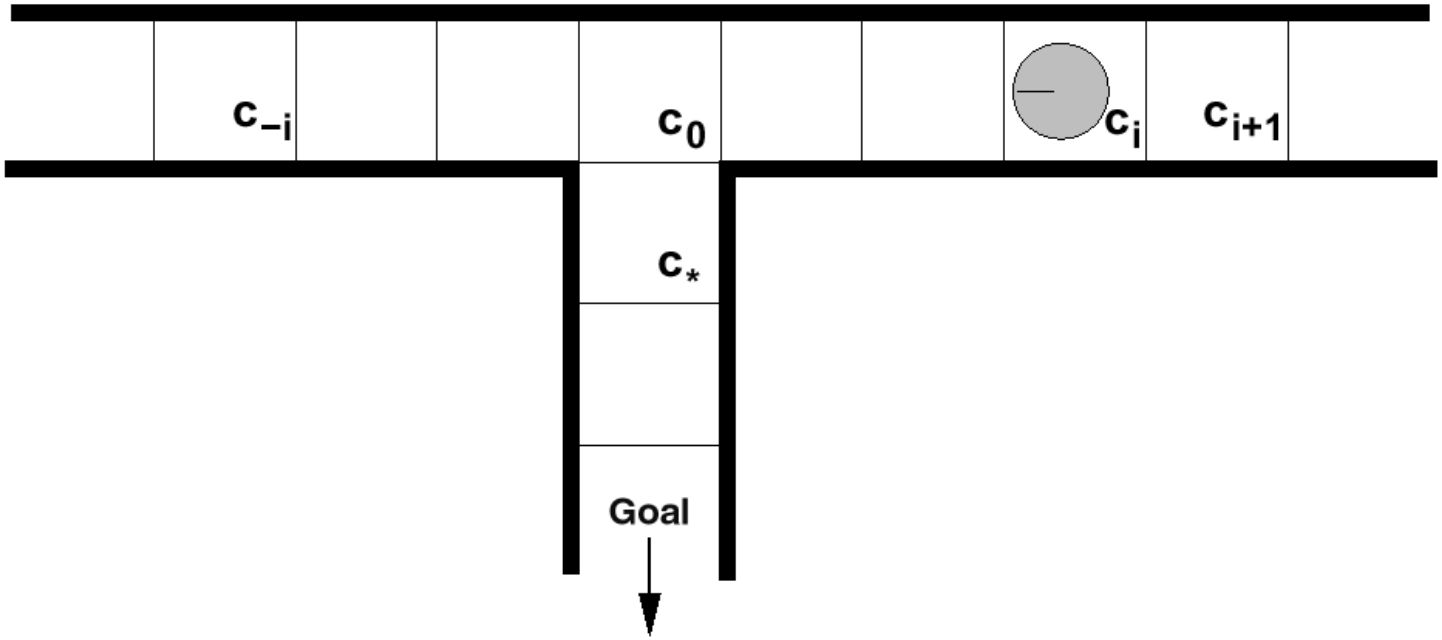
- Now perform search/optimization
- Find maximum

Dynamic Window Approach

- Reacts quickly at low CPU requirements
- Guides a robot on a collision free path
- Successfully used in many real-world scenarios

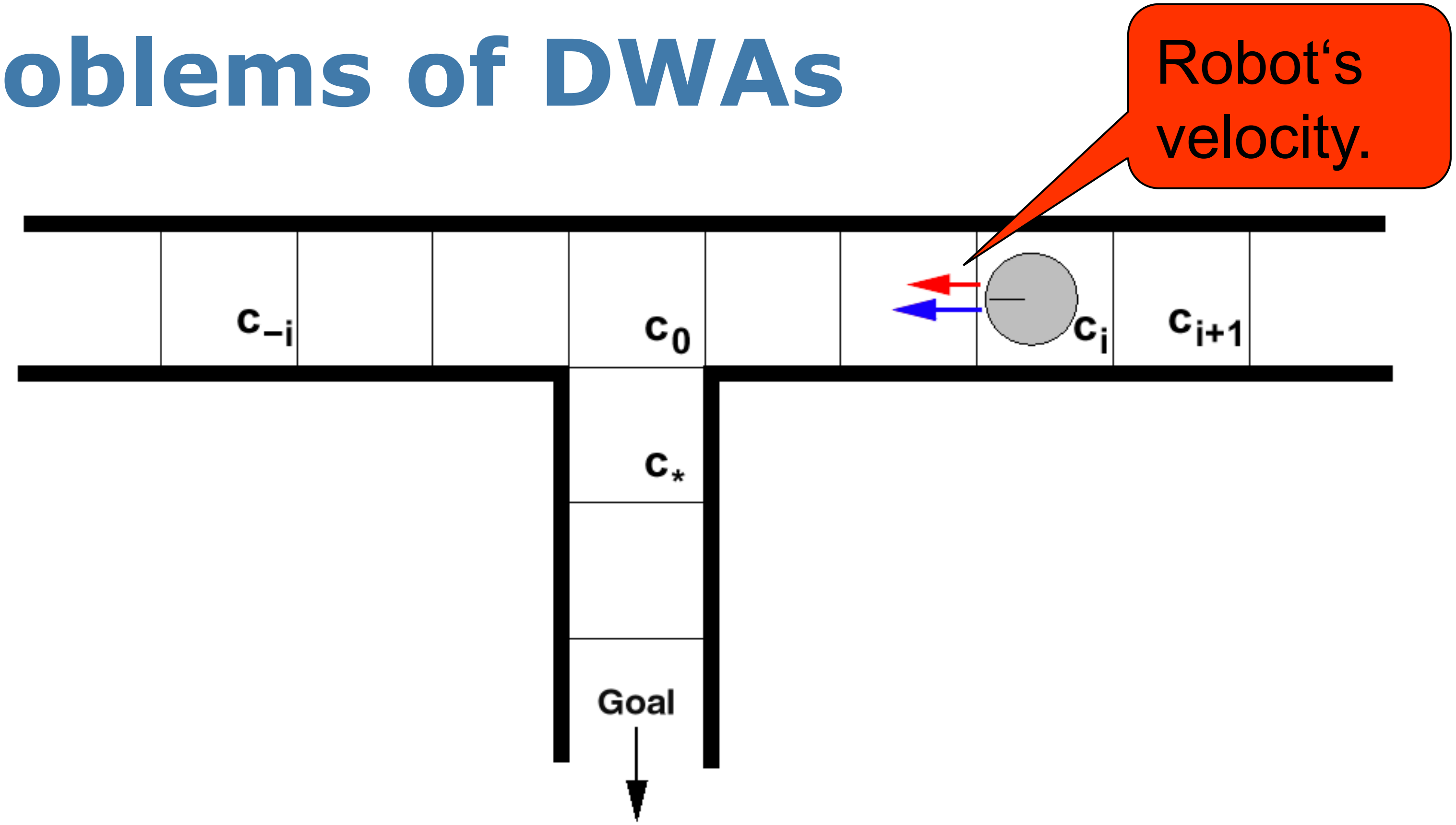
- Resulting trajectories sometimes suboptimal
- Local minima might prevent the robot from reaching the goal location (regular DWA)
- Global DWA with NF1 overcomes this problem

Problems of DWAs



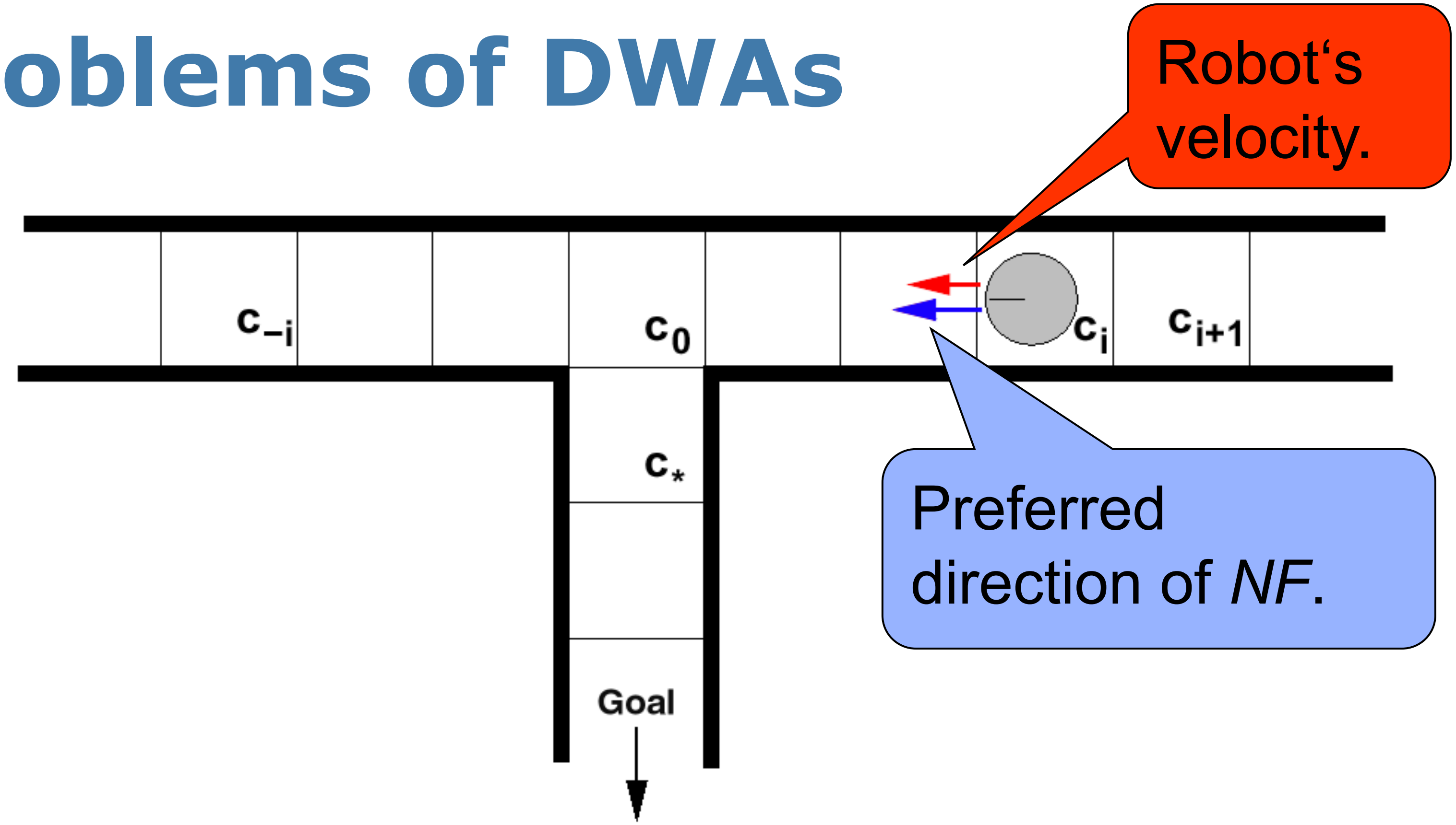
$$NF = \alpha \cdot vel + \beta \cdot nf + \gamma \cdot \Delta nf + \delta \cdot goal$$

Problems of DWAs



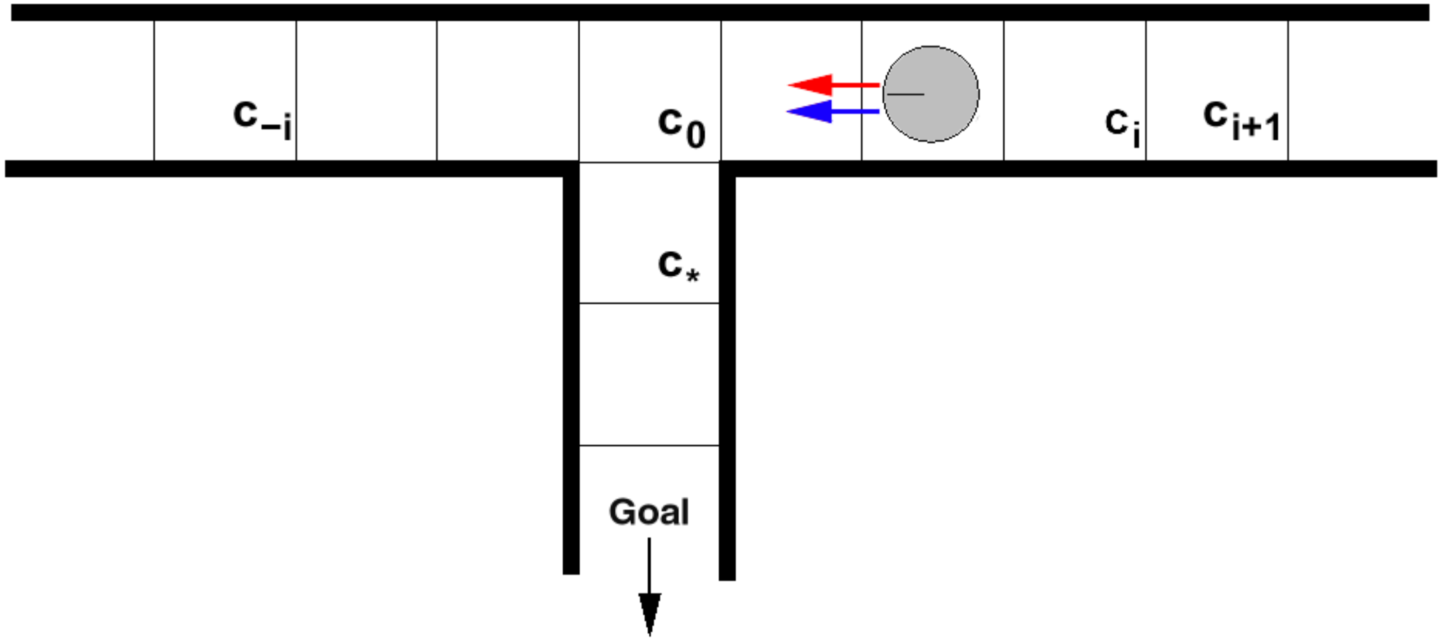
$$NF = \alpha \cdot vel + \beta \cdot nf + \gamma \cdot \Delta nf + \delta \cdot goal$$

Problems of DWAs



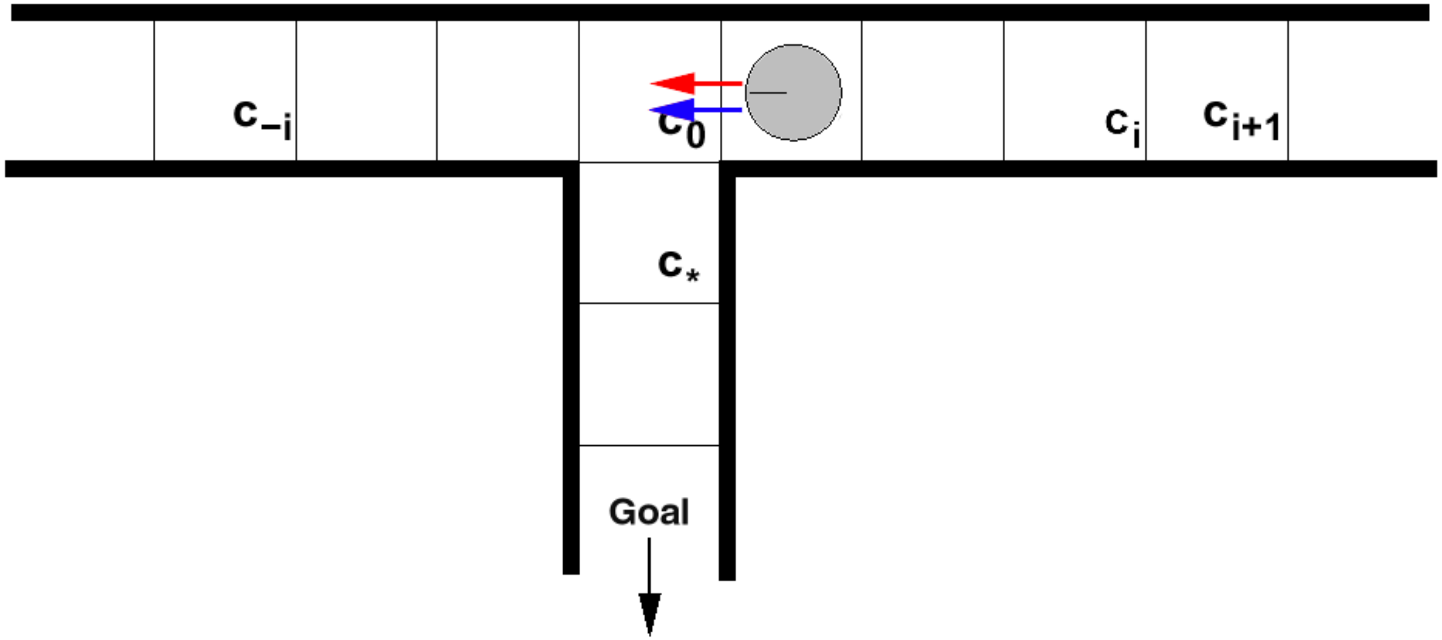
$$NF = \alpha \cdot vel + \beta \cdot nf + \gamma \cdot \Delta nf + \delta \cdot goal$$

Problems of DWAs



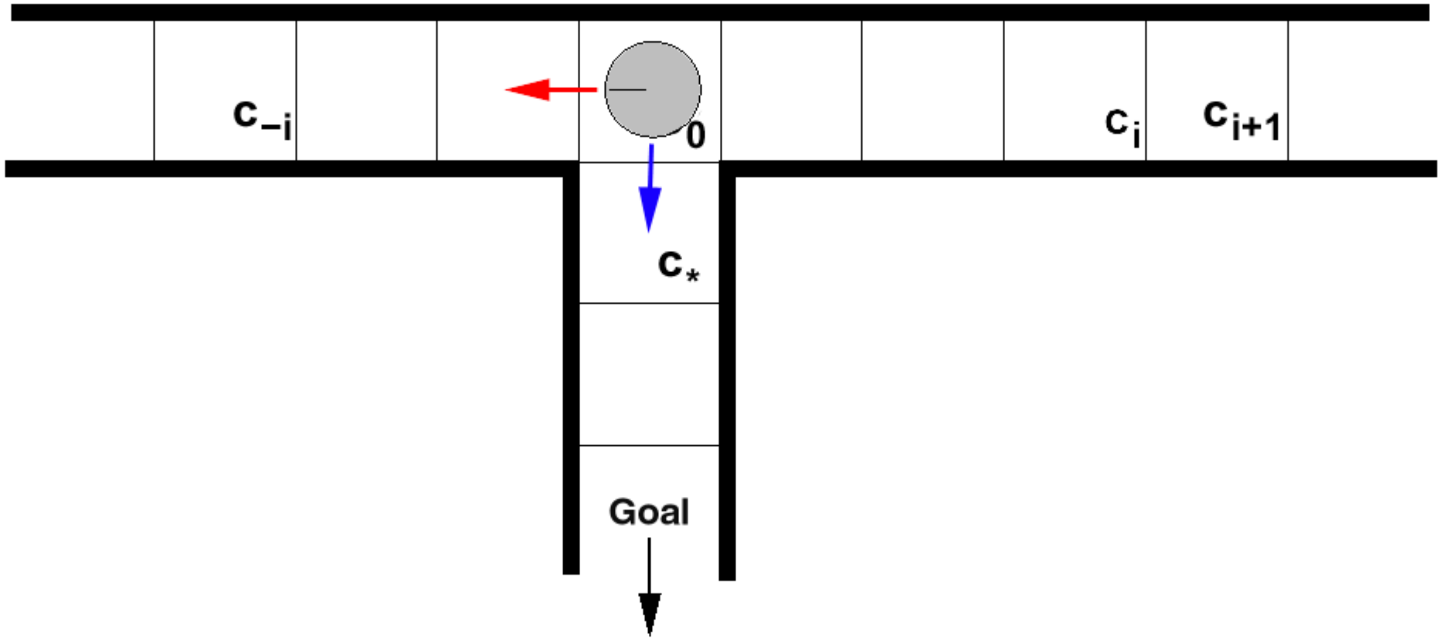
$$NF = \alpha \cdot vel + \beta \cdot nf + \gamma \cdot \Delta nf + \delta \cdot goal$$

Problems of DWAs



$$NF = \alpha \cdot vel + \beta \cdot nf + \gamma \cdot \Delta nf + \delta \cdot goal$$

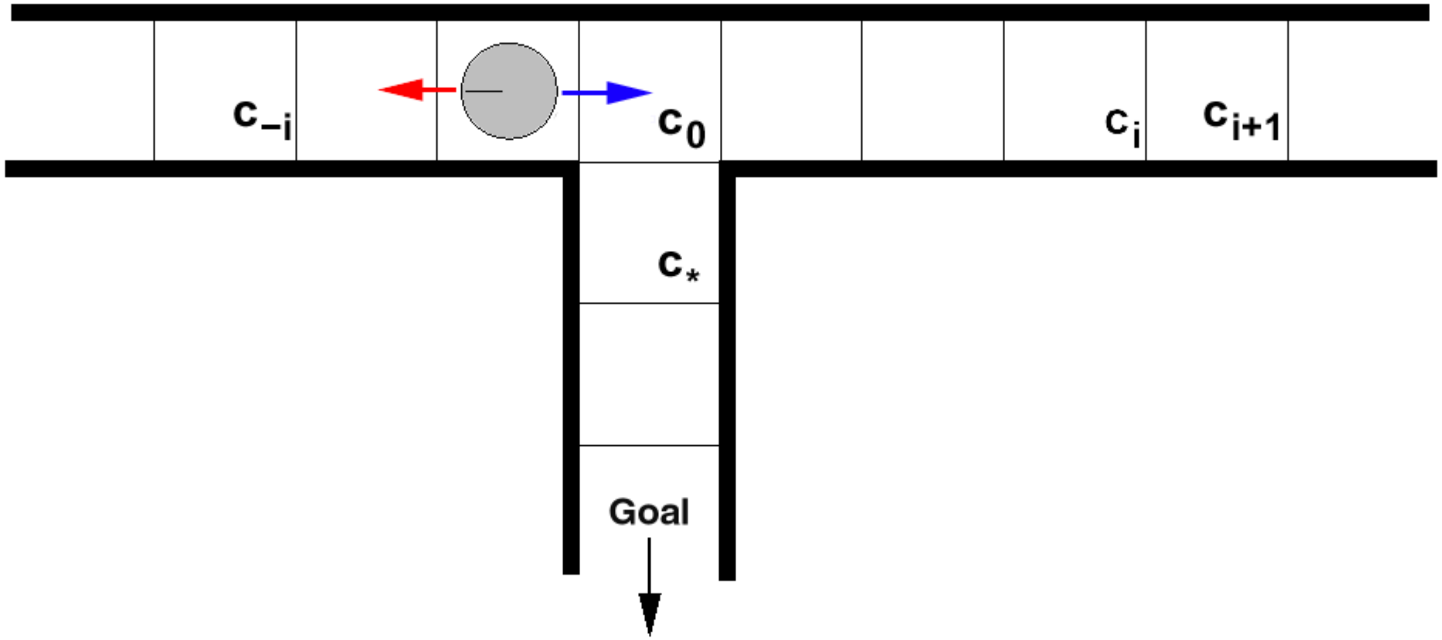
Problems of DWAs



$$NF = \alpha \cdot vel + \beta \cdot nf + \gamma \cdot \Delta nf + \delta \cdot goal$$

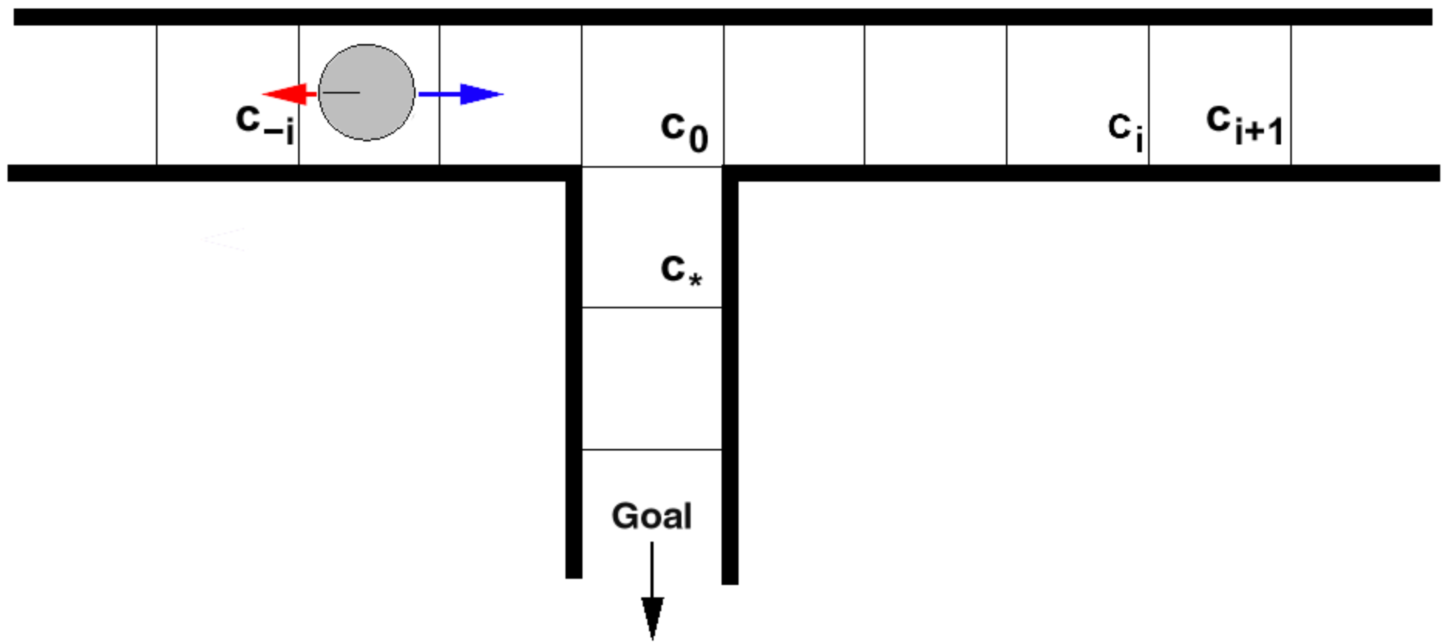
The robot drives too fast at c_0 to enter corridor facing south.

Problems of DWAs



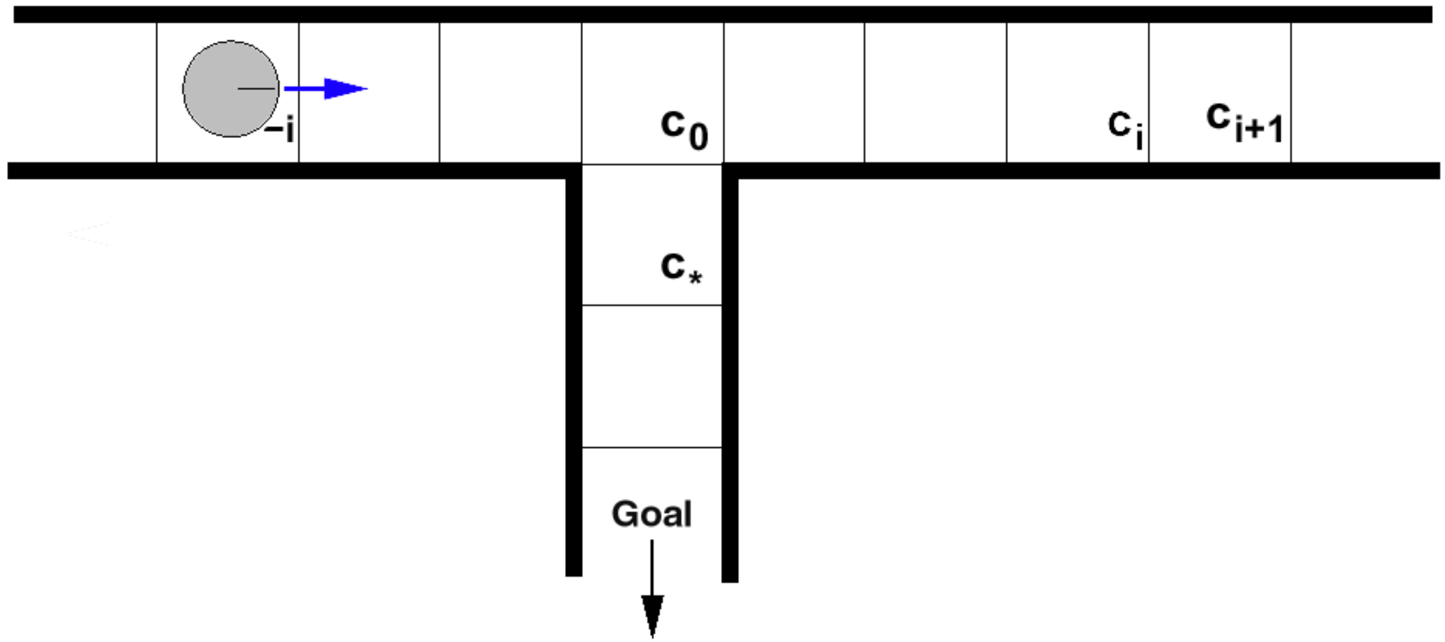
$$NF = \alpha \cdot vel + \beta \cdot nf + \gamma \cdot \Delta nf + \delta \cdot goal$$

Problems of DWAs



$$NF = \alpha \cdot vel + \beta \cdot nf + \gamma \cdot \Delta nf + \delta \cdot goal$$

Problems of DWAs



- Same situation as in the beginning

➔ DWAs have problems to reach the goal

Problems of DWAs

- Typical problem in a real world situation:

- Robot does not slow down early enough to enter the doorway.

Alternative: 5D-Planning

- Plans in the **full** $<x,y,\theta,v,\omega>$-**configuration** space using A*
  - Considers the robot's kinematic constraints

- **Idea:** search in the discretized $<x,y,\theta,v,\omega>$-space

- **Problem:** search space too large to be explored in real-time

- **Solution:** restrict the full search space to "channels"

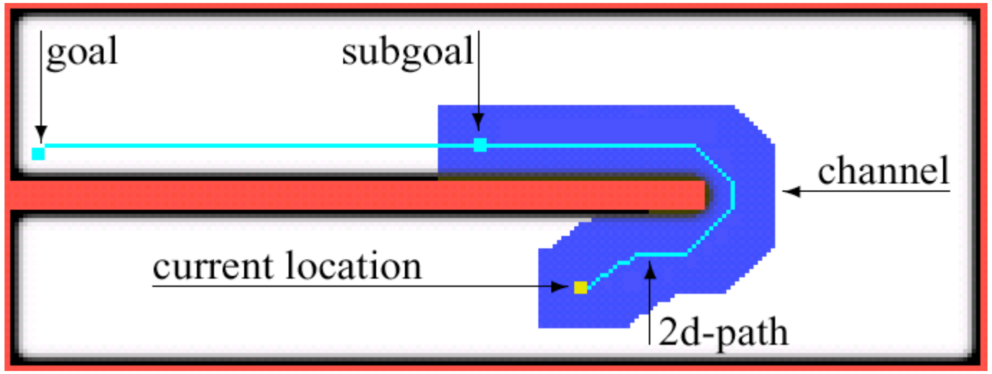
5D-Planning

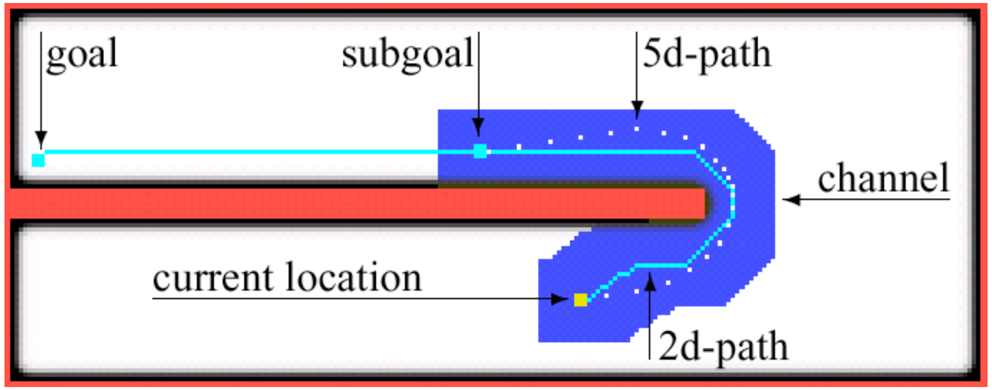
- Use A* to find a trajectory in the **2D** $<x,y>$**-space**

- Choose a **subgoal** lying on the 2D-path within the channel

- Use A^* in the **"channel" 5D-space** to find a sequence of steering commands to reach the subgoal

5D-Planning Example





Summary (1 of 3)

- Motion planning lives in the **C-space**
- **Combinatorial** planning methods scale poorly with C-space dimension and non-linearity but are **complete** and **optimal**
- **Sampling-based** planning methods have weaker guarantees but are more efficient
- They all produce a **road map** that captures the connectivity of the C-space
- For planning on the road map, use **heuristic search** methods such as A*

Summary (2 of 3)

- Deterministic value iteration or Dijkstra yields the **optimal heuristic** for A*. Precompute if on-line replanning is likely
- A* in smoothed grid maps helps to keep the robot **away** from obstacles
- Any-angle A* methods produce **shorter** paths with **fewer** heading changes
- D*/D* Lite **avoids** replanning from **scratch** and finds the (usually few) nodes to be updated for on-line replanning

Summary (3 of 3)

- In highly dynamic environments, reactive **collision avoidance** methods that account for the **kinematic** and **dynamics** vehicle constraints become necessary
- Decoupling into an approximative **global** and an accurate **local** planning problem, integration using a layered architecture
- The Dynamic Window Approach optimizes a navigation function to trade off **feasible**, **reasonable**, and **admissible** motions

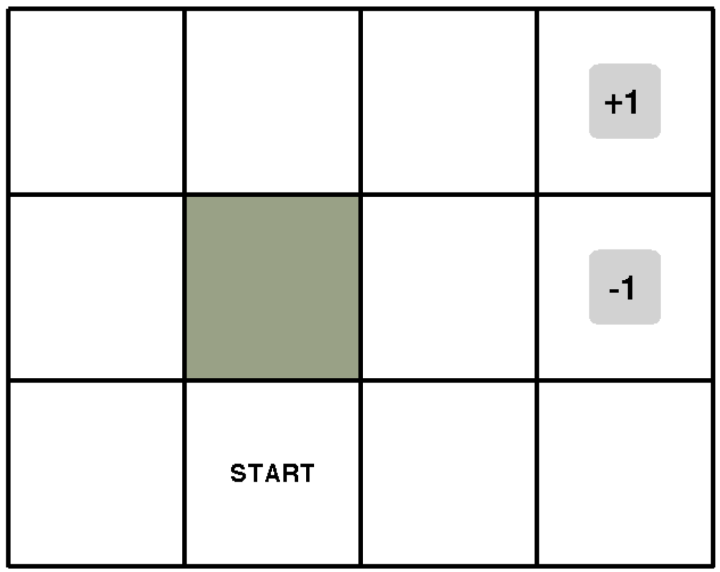
Uncertain Path Execution

- Have you ever **become lost** while trying to follow a **path** (e.g. printed out from Google maps)?
- Problem: **path execution** is inherently **uncertain!**
- Even the best **path** is worthless if the robot is unable to follow it
- Reasons: Underlying trajectoriy controller, DWA, imperfect models of map/dynamics

➔ Instead of a plan, you need a **policy**

Markov Decision Process

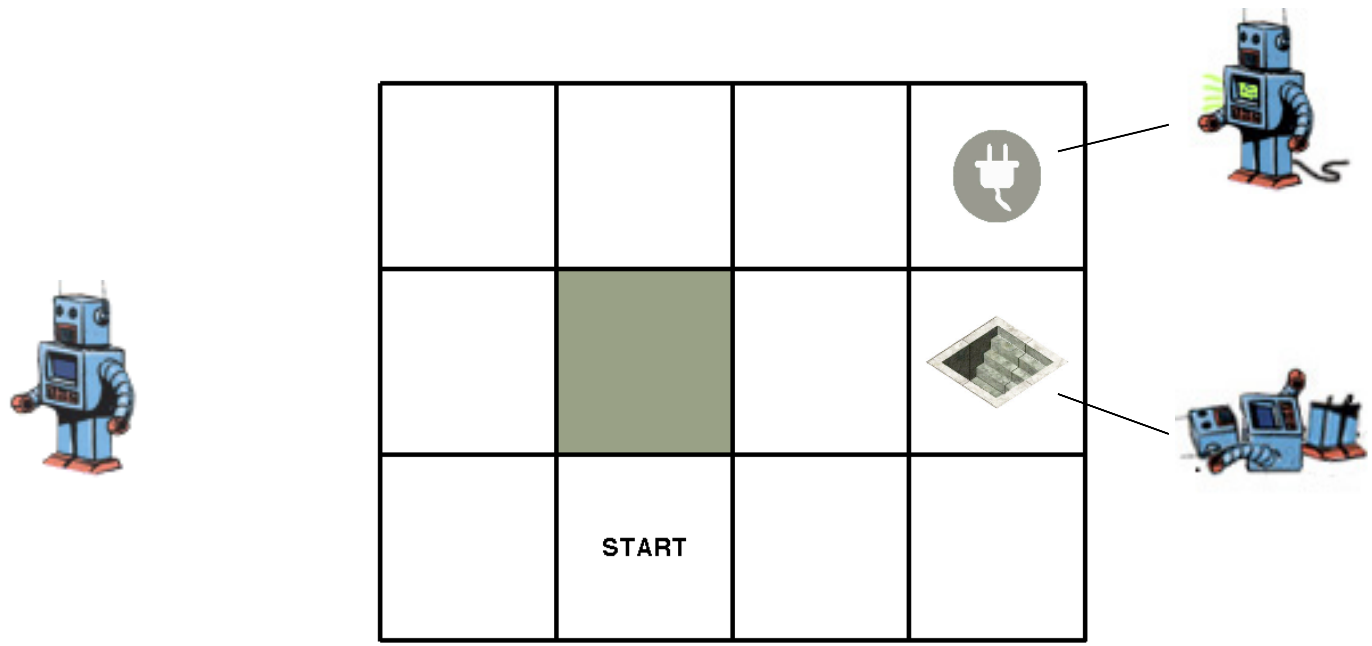
- Consider an agent acting in this environment



- Its mission is to reach the goal marked by +1 avoiding the cell labelled -1

Markov Decision Process

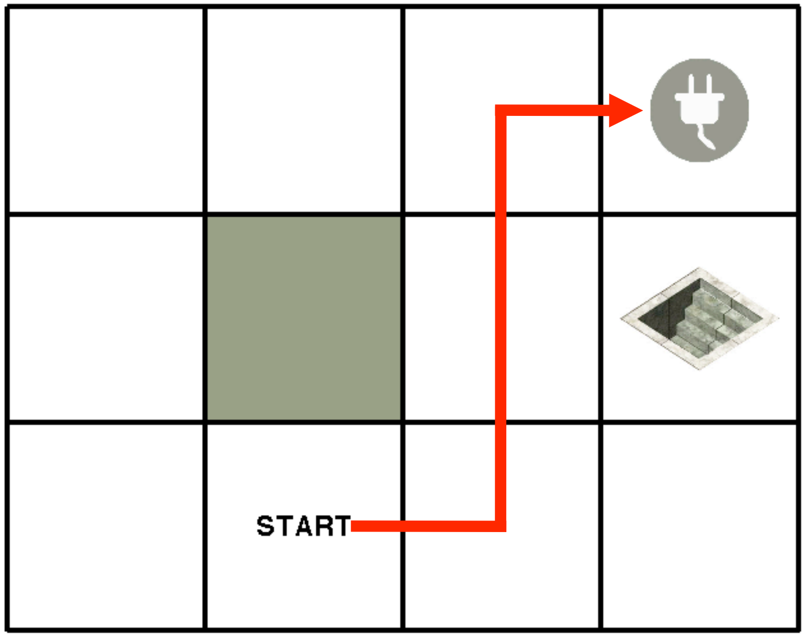
- Consider an agent acting in this environment



- Its mission is to reach the goal marked by +1 avoiding the cell labelled -1

Markov Decision Process

- Easy! Use a search algorithm such as A*



- Best solution (shortest path) is the action sequence *[Right, Up, Up, Right]*

What is the problem?

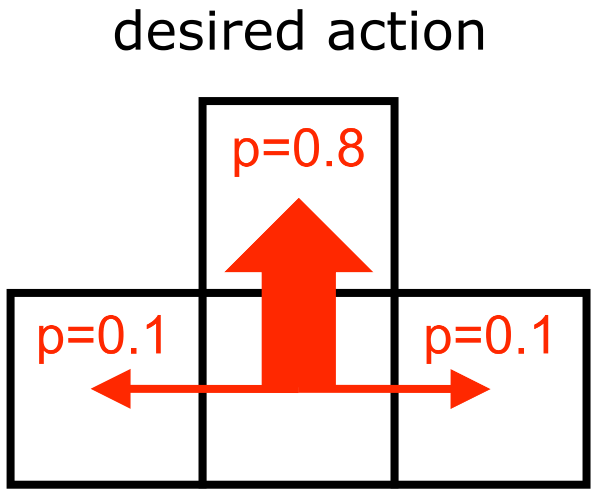
- Consider a non-perfect system in which actions are performed with a **probability less than 1**
- What are the best actions for an agent under this constraint?

- Example: a mobile robot does not *exactly* perform a desired motion
- Example: human navigation

→ Uncertainty about performing actions!

MDP Example

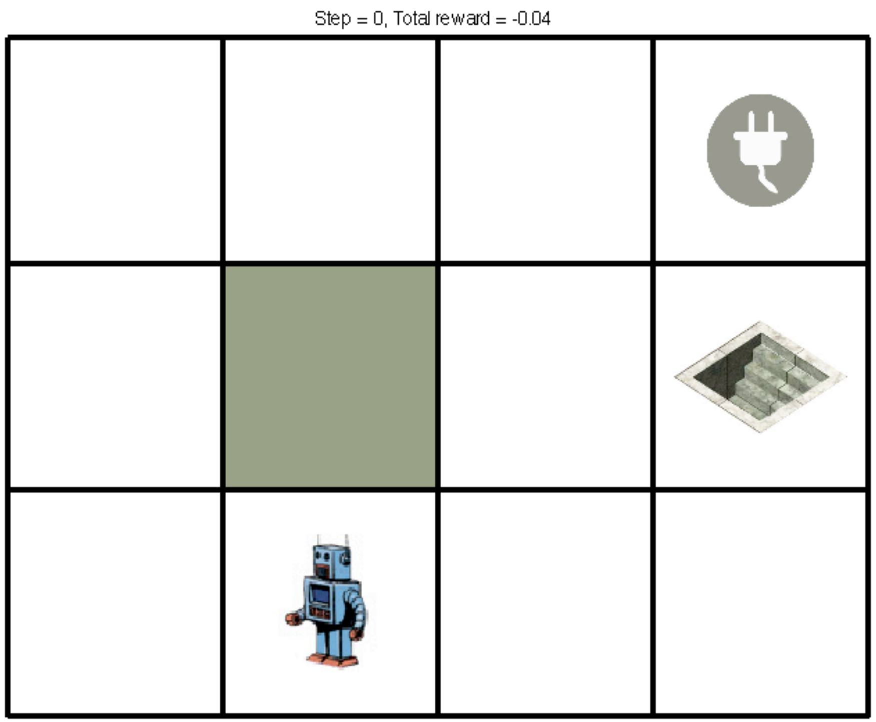
- Consider the **non-deterministic transition model** (N / E / S / W):



- Intended action is executed with p=0.8
- With p=0.1, the agent moves left or right
- Bumping into a wall "reflects" the robot

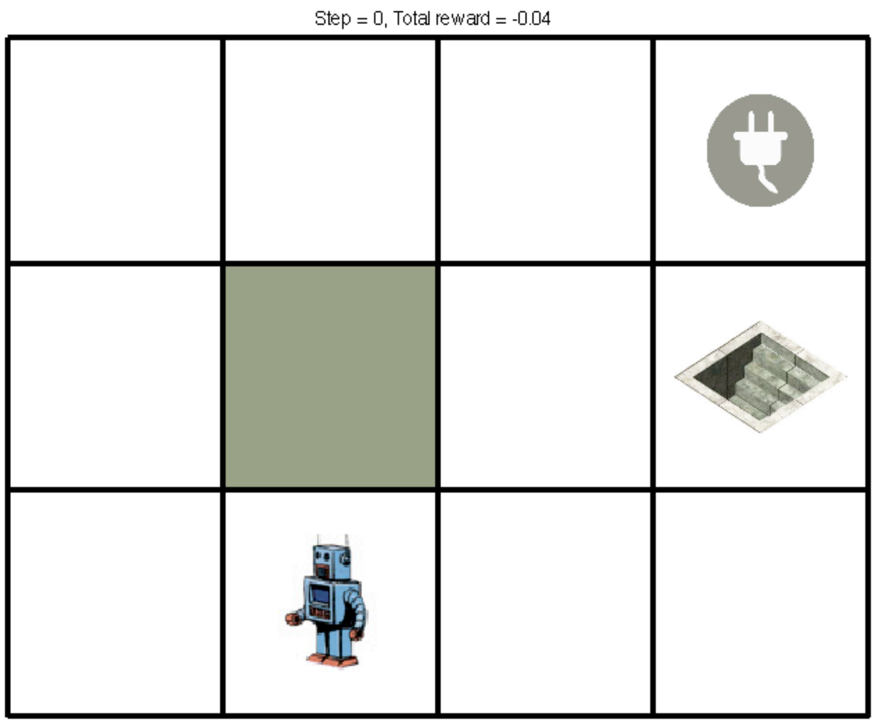
MDP Example

- Executing the **A* plan** in this environment



MDP Example

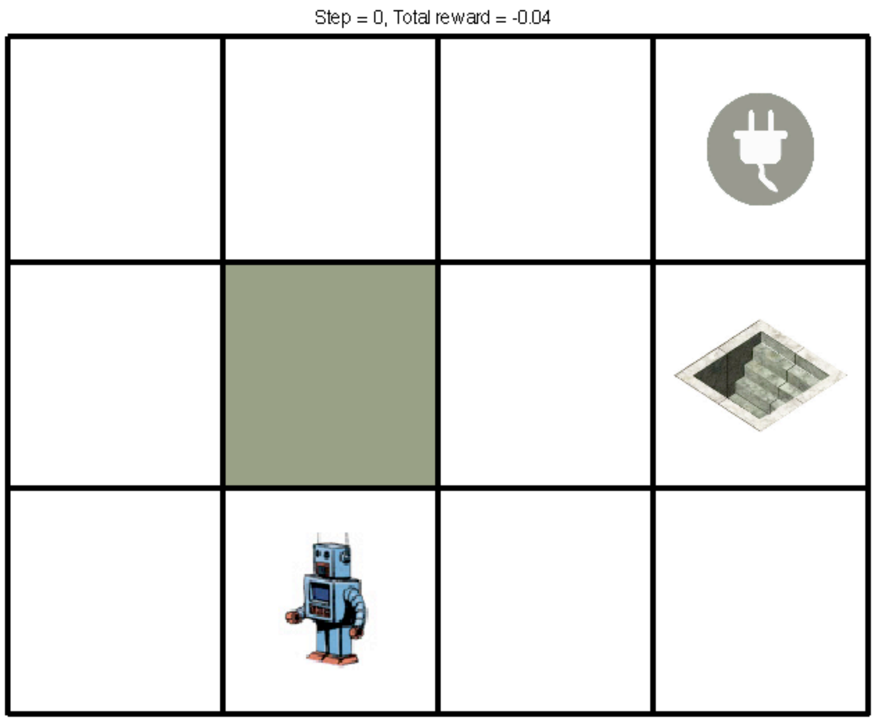
- Executing the **A* plan** in this environment



But: transitions are non-deterministic!

MDP Example

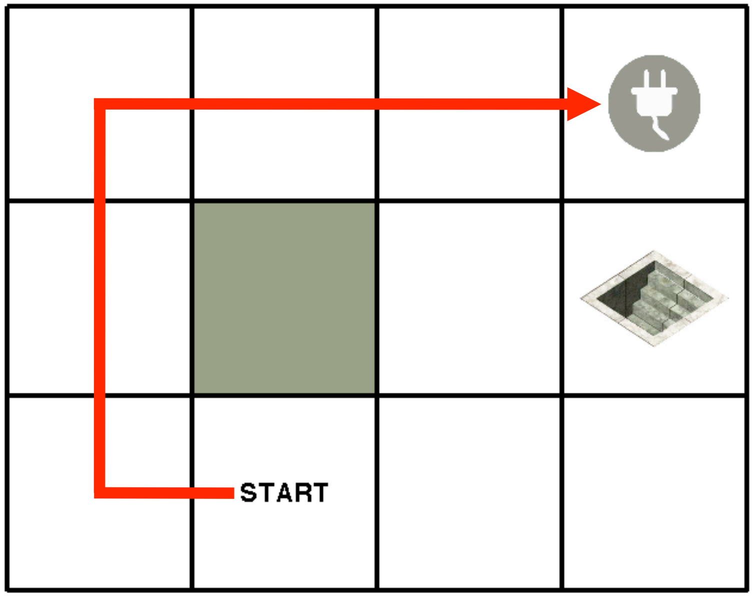
- Executing the **A* plan** in this environment



This will happen sooner or later...

MDP Example

- Use a **longer** path with **lower** probability to end up in cell labelled **-1**



- This path has the **highest overall utility**
- Probability $0.8^6 = 0.2621$

Transition Model

- The probability to reach the next state s' from state s by choosing action a

  $$T(s, a, s')$$

  is called **transition model**

Markov Property:

The transition probabilities from s to s' **depend only on the current state** s and not on the history of earlier states

Reward

- In each state s, the agent receives a **reward** $R(s)$
- The reward may be **positive** or **negative** but must be **bounded**
- This can be generalized to be a function $R(s,a,s')$. Here: consider only $R(s)$, does not change the problem

Reward

- In our example, the reward is **-0.04** in all states (e.g. the cost of motion) except the terminal states (that have rewards **+1**/**-1**)
- A negative reward gives agent an **in-centive to reach the goal quickly**
- Or: "living in this environment is not enjoyable"

-0.04	-0.04	-0.04	+1
-0.04		-0.04	-1
-0.04	-0.04	-0.04	-0.04

MDP Definition

- Given a **sequential decision problem** in a fully observable, stochastic environment with a known Markovian transition model
- Then a **Markov Decision Process** is defined by the components
  - *Set of states:* S
  - *Set of actions:* A
  - *Initial state:* s_0
  - *Transition model:* $T(s, a, s')$
  - *Reward funciton:* $R(s)$

Policy

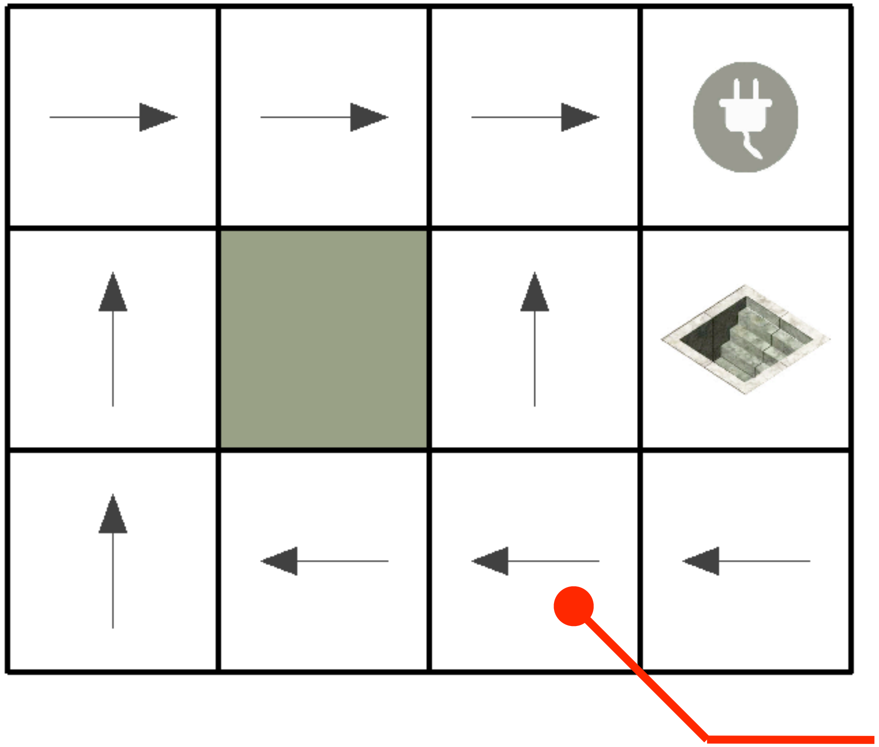
- An MDP solution is called **policy** π
- A policy is a mapping from states to actions

$$policy : States \mapsto Actions$$

- In each state, a policy tells the agent **what to do next**
- Let $\pi(s)$ be the *action* that π specifies for s
- Among the many policies that solve an MDP, the **optimal policy** π* is what we seek. We'll see later what *optimal* means

Policy

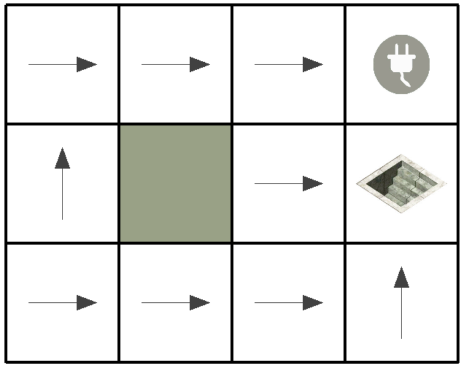
- The optimal policy for our example



Conservative choice
Take long way around as the cost per step of -0.04 is small compared with the penality to fall down the stairs and receive a **-1** reward

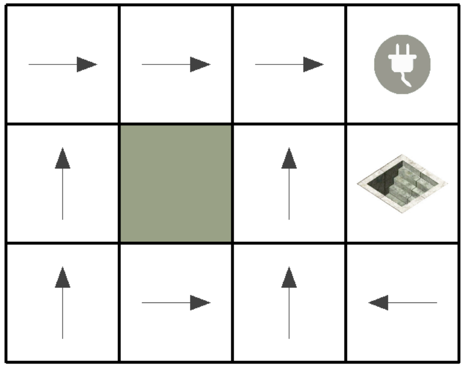
Policy

- When the balance of risk and reward changes, **other policies are optimal**



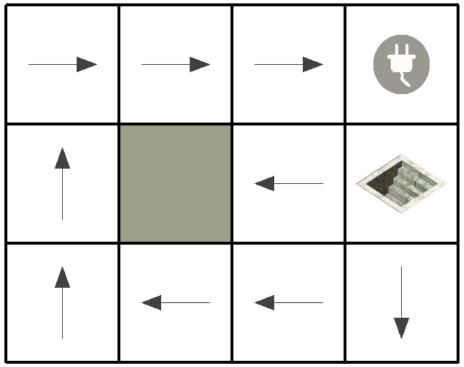
R < -1.63

Leave as soon as possible



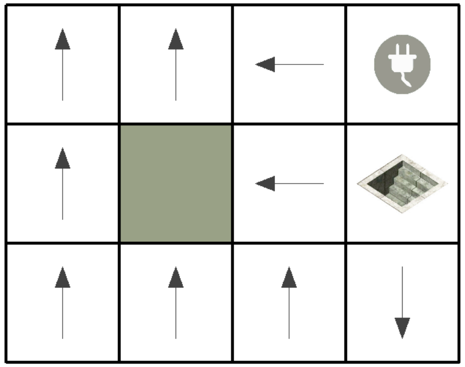
-0.43 < R < -0.09

Take shortcut, minor risks



-0.02 < R < 0

No risks are taken



R > 0

Never leave (inf. #policies)

Utility of a State

- The **utility of a state** $U(s)$ quantifies the **benefit** of a state for the **overall task**
- We first define $U^\pi(s)$ to be the **expected utility of all state sequences that start in** *s* given π

$$U^\pi(s) = E\left[\sum_{t=0}^{\infty} R(s_t) \mid \pi, s_0 = s\right]$$

- $U(s)$ evaluates (and encapsulates) all possible futures **from** s **onwards**

Utility of a State

- With this definition, we can express $U^{\pi}(s)$ as a **function of its next state** s'

$$
\begin{aligned}
U^{\pi}(s) &= E\left[\sum_{t=0}^{\infty} R(s_t) \mid \pi, s_0 = s\right] \\
&= E\left[R(s_0) + R(s_1) + R(s_2) + \ldots \mid \pi, s_0 = s\right] \\
&= E\left[R(s_0) \mid s_0 = s\right] + E\left[R(s_1) + R(s_2) + \ldots \mid \pi\right] \\
&= R(s) + E\left[\sum_{t=0}^{\infty} R(s_t) \mid \pi, s_0 = s'\right] \\
&= R(s) + U^{\pi}(s')
\end{aligned}
$$

Optimal Policy

- The utility of a state allows us to apply the **Maximum Expected Utility principle** to define the optimal policy π*

- The **optimal policy** π* in s chooses the action a that maximizes the expected utility of s (and of s')

$$\pi^*(s) \quad = \quad \underset{a}{\operatorname{argmax}} \; E\left[U^\pi(s)\right]$$

- Expectation taken over all policies

Optimal Policy

- Substituting $U^{\pi}(s)$

$$
\begin{aligned}
\pi^*(s) &= \operatorname*{argmax}_a E\left[U^{\pi}(s)\right] \\
&= \operatorname*{argmax}_a E\left[R(s) + U^{\pi}(s')\right] \\
&= \operatorname*{argmax}_a E\left[R(s)\right] + E\left[U^{\pi}(s')\right] \\
&= \operatorname*{argmax}_a E\left[U(s')\right] \\
&= \operatorname*{argmax}_a \sum_{s'} T(s,a,s')\, U(s')
\end{aligned}
$$

- Recall that $E[X]$ is the weighted average of all possible values that X can take on

Utility of a State

- The **true utility of a state** $U(s)$ is then obtained by application of the optimal policy, i.e. $U^{\pi^*}(s) = U(s)$. We find

$$\begin{aligned}
U(s) &= \max_a E\left[U^{\pi}(s)\right] \\
&= \max_a E\left[R(s) + U^{\pi}(s')\right] \\
&= \max_a E\left[R(s)\right] + E\left[U^{\pi}(s')\right] \\
&= R(s) + \max_a E\left[U(s')\right] \\
&= \underline{R(s) + \max_a \sum_{s'} T(s,a,s')\, U(s')}
\end{aligned}$$

Utility of a State

- This result is noteworthy:

$$U(s) = R(s) + \max_a \sum_{s'} T(s,a,s')\,U(s')$$

  We have found a direct relationship between the **utility of a state** and the **utility of its neighbors**

- The utility of a state is the immediate reward for that state plus the expected utility of the next state, **provided** the agent chooses the **optimal** action

Bellman Equation

$$\boxed{U(s) = R(s) + \max_a \sum_{s'} T(s,a,s')\, U(s')}$$

- For each state there is a Bellman equation to compute its utility
- There are n **states** and n **unknowns**
- Solve the system using Linear Algebra?
- No! The max-operator that chooses the optimal action makes the system nonlinear
- We must go for an **iterative approach**

Discounting

We have made a **simplification** on the way:

- The utility of a state sequence is often defined as the sum of **discounted** rewards

$$U^\pi(s) = E\left[\sum_{t=0}^{\infty} \underline{\gamma^t} R(s_t) \mid \pi, s_0 = s\right]$$

  with $0 \leq \gamma \leq 1$ being the *discount factor*

- Discounting says that **future** rewards are **less significant** than **current** rewards. This is a natural model for many domains

- The other expressions change accordingly

Separability

We have made an **assumption** on the way:

- Not all utility functions (for state sequences) can be used
- The utility function must have the **property of separability** (a.k.a. stationarity), e.g. additive utility functions:

  $$U([s_0 + s_1 + \ldots + s_n]) = R(s_0) + U([s_1 + \ldots + s_n])$$

- Loosely speaking: the preference between two state sequences is unchanged over different start states

Utility of a State

- The **state utilities** for our example

0.812	0.868	0.918	+1
0.762		0.66	-1
0.705	0.655	0.611	0.388

- Note that utilities are higher closer to the goal as fewer steps are needed to reach it

Iterative Computation

Idea:

- The utility is computed iteratively:

$$U_{i+1}(s) \leftarrow R(s) + \max_a \sum_{s'} T(s, a, s')\, U_i(s')$$

- Optimal utility: $U^* = \lim_{t \to \infty} U_t$
- Abort, if change in utility is below a threshold

Dynamic Programming

- The utility function is the basis for **Dynamic Programming**
- Fast solution to compute n-step decision problems
- Naive solution: $O(\,|A|^n\,)$
- Dynamic Programming: $O(\,n\,|A|\,|S|\,)$
- But: what is the correct value of n?
- If the graph has loops: $n \longrightarrow \infty$

The Value Iteration Algorithm

Algorithm 1: Value Iteration

In: An MDP with
– States and action sets S, A,
– Transition model $T(s, a, s')$,
– Reward function $R(s)$,
– Discount factor γ
Out: The utility of all states U

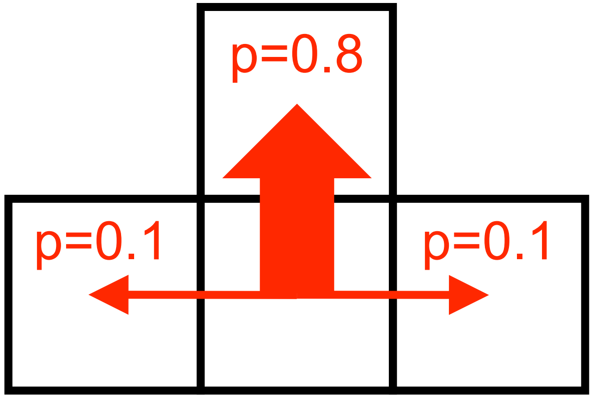
$U' \leftarrow 0$
repeat
    $U \leftarrow U'$
    foreach *state* s *in* S **do**
        $U(s) \leftarrow R(s) + \gamma \max_a \sum_{s'} T(s, a, s') U(s')$
    end
until *close-enough*(U, U')
return U

Value Iteration Example

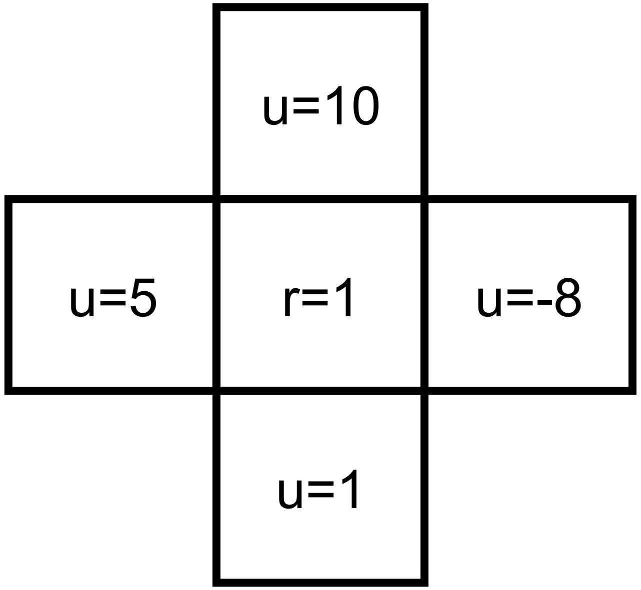
- Calculate utility of the center cell

$$U_{i+1}(s) \leftarrow R(s) + \max_a \sum_{s'} T(s,a,s')\, U_i(s')$$

desired action = *Up*



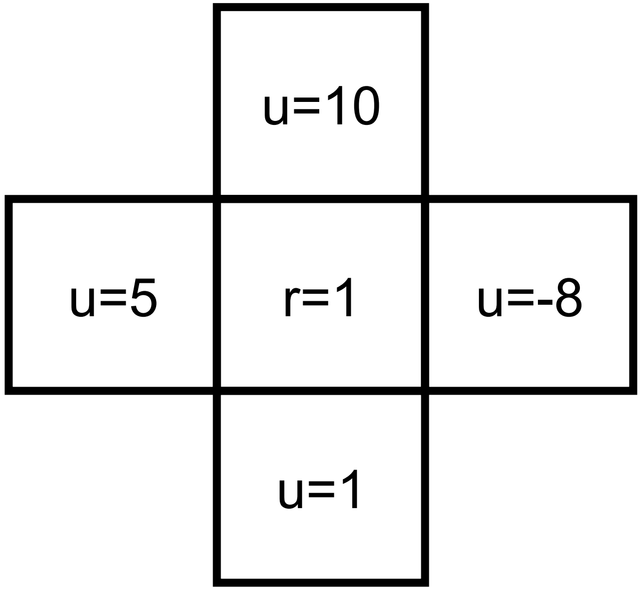
Transition Model



State space
(u=utility, r=reward)

Value Iteration Example

$$U_{i+1}(s) \leftarrow R(s) + \max_a \sum_{s'} T(s,a,s')\, U_i(s')$$



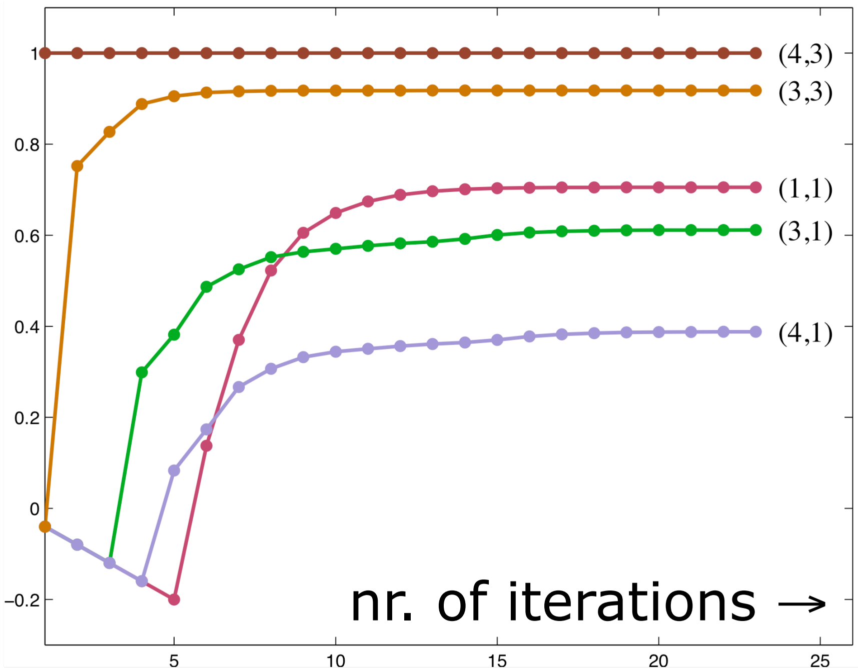
$$\begin{aligned}
&= \mathit{reward} + \mathsf{max}\{ \\
&\quad 0.1 \cdot 1 + 0.8 \cdot 5 + 0.1 \cdot 10 \quad (\leftarrow), \\
&\quad 0.1 \cdot 5 + 0.8 \cdot 10 + 0.1 \cdot -8 \quad (\uparrow), \\
&\quad 0.1 \cdot 10 + 0.8 \cdot -8 + 0.1 \cdot 1 \quad (\rightarrow), \\
&\quad 0.1 \cdot -8 + 0.8 \cdot 1 + 0.1 \cdot 5 \quad (\downarrow)\} \\
&= 1 + \mathsf{max}\{5.1\,(\leftarrow), 7.7\,(\uparrow), \\
&\quad -5.3\,(\rightarrow), 0.5\,(\downarrow)\} \\
&= 1 + 7.7 \\
&= 8.7
\end{aligned}$$

Value Iteration Example

- In our example

0.812	0.868	0.918	+1
0.762		0.66	-1
0.705	0.655	0.611	0.388

(1,1)



- States far from the goal first accumulate negative rewards until a path is found to the goal

Convergence

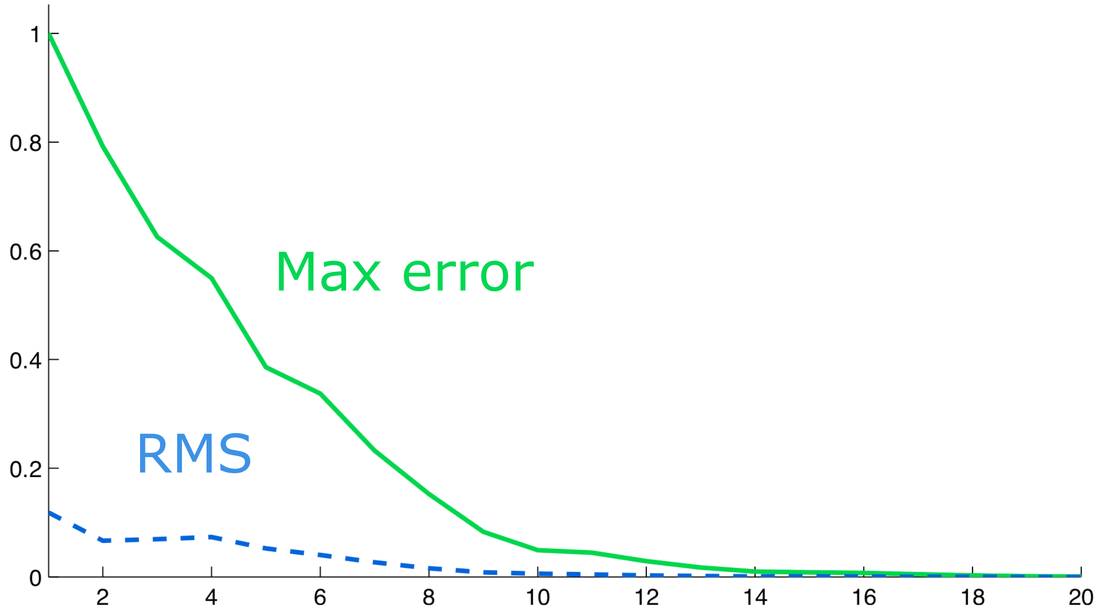
- The condition $close\text{-}enough(U, U')$ in the algorithm can be formulated by

$$RMS = \frac{1}{|S|}\sqrt{\sum_s (U(s) - U'(s))^2}$$

$$RMS(U, U') < \epsilon$$

- Different ways to detect convergence:
  - RMS error: root mean square error
  - Max error: $\| U - U' \| = \max_s |U(s) - U'(s)|$
  - Policy loss

Convergence Example



- What the agent cares about is **policy loss**: How well a policy based on $U_i(s)$ performs
- Policy loss converges much faster (because of the argmax)

Value Iteration

- Value Iteration finds the **optimal solution** to the Markov Decision Problem!
- **Converges** to the **unique solution** of the Bellman equation system for $\gamma < 1$
- Initial values for U' are arbitrary
- Proof involves the concept of *contraction*. $\| B\, U_i - B\, U_i' \| \leq \gamma \, \| U_i - U_i' \|$ with B being the Bellman operator (see textbook)
- VI propagates information through the state space by means of **local updates**

Optimal Policy

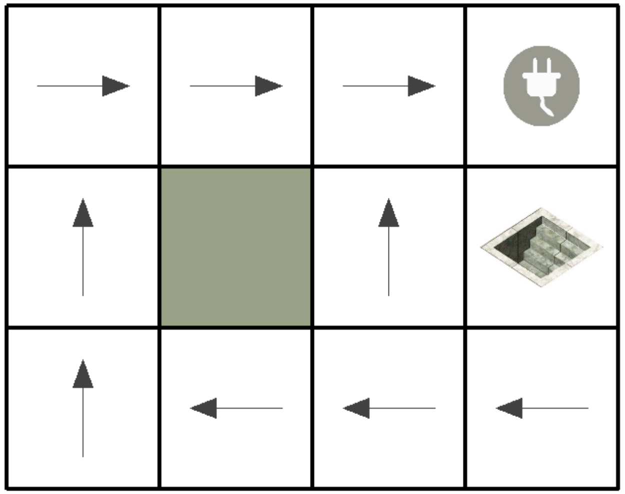
- How to finally compute the **optimal policy**? Can be easily extracted along the way by

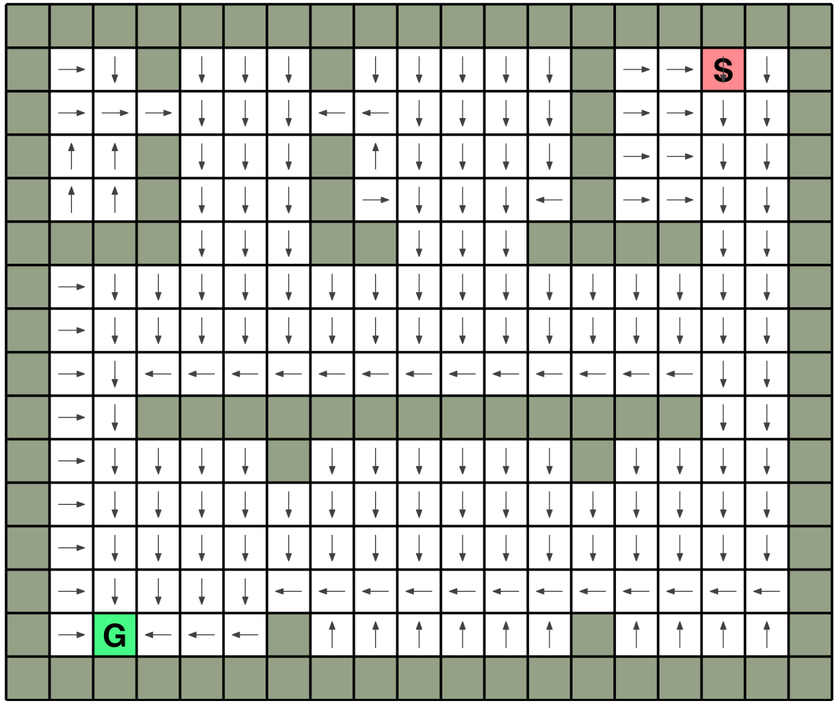
$$\pi^*(s) = \underset{a}{\operatorname{argmax}} \sum_{s'} T(s, a, s')\, U(s')$$

- **Note:** *U(s)* and *R(s)* are quite different quantities. *R(s)* is the **short-term** reward for being in *s*, whereas *U(s)* is the **long-term** reward **from** *s* **onwards**

Optimal Policy

- Examples





Summary

- MDPs describe an uncertain agent with a **stochastic transition model**
- The solution is called **policy** that is a mapping from **states to actions**
- Value Iteration is a instance of dynamic programming, converges for lower-than-one discounts or finite horizons
- A policy allows to implement a **feedback control strategy**, the robot can never become lost anymore

What's missing...?

- Good solutions to **jointly** plan the **path under local constraints** that overcome the decoupling of global and local planning
- Good solutions to implement **feasible feedback** control strategies
- Problem: the **curse of dimensionality**
- AI/planning people and control theory people need to **talk more**
- Hence, the robot motion planning problem is not fully solved yet, but **good** solutions for many **practical** problems exist