CAB203
Tutorial 5

Proofs

1. Use a truth table to decide if $(A \to B) \wedge B$ logically implies A.

Solution:

A	B	$A \to B$	$(A \to B) \wedge B$
T	T	T	T
T	F	F	F
F	T	T	T
F	F	T	F

In the third line we see a problem, which is that $(A \to B) \wedge B$ is true, but A is false. So $(A \to B) \wedge B$ cannot logically imply A.

2. Find a counterexample to show that we cannot use $A \vDash A \vee B$ to substitute into $A \to C$. In particular, find propositions A, B and C so that $A \to C$ is true but $(A \vee B) \to C$ is false.

Solution: Before looking for a counterexample, let's look at what the conditions are. We need to have $A \to C$ be true. This works as long as we don't have $A = T$ and $C = F$. We also need $(A \vee B) \to C$ to be false. So we need $A \vee B$ to be True and C to be false.

From our conditions, we can see immediately that any false C works. Let's use $C =$ Socrates is a teapot. Next, we need $A \vee B$ to be true, but A to be false. We can do this by using any false A, and a true B. Let's use $A =$ Pigs can fly. $B =$ Socrates is human. So we have

$A =$ Pigs can fly.
$B =$ Socrates is human.
$C =$ Socrates is a teapot.

3. Use two logical implications to prove

$$(A \wedge B) \wedge (A \to C) \vDash C$$

Solution:

1	$A \wedge B$	premise
2	$A \to C$	premise
3	A	Logical implication $(p \wedge q) \vDash p$
4	C	Logical implication $(p \to q) \wedge p \vDash q$

4. *(Stretch question)*

   (a) Prove $(A \to B) \vDash A \to (B \vee C)$

Solution:

$A \to B$	premise
$\neg A \vee B$	logical equivalence $p \to q \equiv \neg p \vee q$
$(\neg A \vee B) \vee C$	logical implication $p \vDash p \vee q$
$\neg A \vee (B \vee C)$	logical equivalence $(p \vee q) \vee r \equiv p \vee (q \vee r)$
$A \to (B \vee C)$	logical equivalence $p \to q \equiv \neg p \vee q$

(b) Prove $(A \to B) \wedge (A \vee C) \vDash B \vee C$

Solution: We'll take it for granted that we can rearrange the order and parentheses for $\vee$ and $\wedge$. This can be done formally, but it's just tedious.

$A \to B$	premise
$A \vee C$	premise
$\neg A \vee B$	logical equivalence $p \to q \equiv \neg p \vee q$
$\neg A \vee B \vee C$	logical implication $p \vDash p \vee q$
$A \vee B \vee C$	logical implication $p \vDash p \vee q$
$(\neg A \vee (B \vee C)) \wedge (A \vee (B \vee C))$	ANDing two true statements
$(B \vee C) \vee (A \wedge \neg A)$	logical equivalence $p \vee (q \wedge r) \equiv (p \vee q) \wedge p \vee r)$
$(B \vee C) \vee F$	logical equivalence $p \wedge \neg p \equiv F$
$B \vee C$	logical equivalence $p \vee F \equiv p$

(c) Prove $(A \vee B) \wedge (A \to C) \wedge (B \to D) \vDash C \vee D$.

Solution: We'll take it for granted that we can rearrange the order and parentheses for $\vee$ and $\wedge$. This can be done formally, but it's just tedious.

$A \vee B$	premise
$A \to C$	premise
$B \to D$	premise
$A \to (C \vee D)$	logical implication $p \to q \vDash p \to (q \vee p)$ from question above
$B \to (C \vee D)$	logical implication $p \to q \vDash p \to (q \vee r)$
$(C \vee D) \vee B$	logical implication $(p \to q) \wedge (p \vee r) \vDash q \vee r$ from question above
$(C \vee D) \vee (C \vee D)$	logical implication $(p \to q) \wedge (p \vee r) \vDash q \vee r$
$C \vee D$	logical equivalence $p \vee p \equiv p$

The last logical equivalence isn't mentioned in the slides, but it is trivial to show with a truth table.

Predicate logic

1. Determine the truth value of the following predicates with the values given:

(a) $p(x) = (x^2 = x)$ for $x \in \{0, 1, 2\}$

Solution: $p(0)$ is $0^2 = 0$ which is true.
$p(1)$ is $1^2 = 1$ which is true.
$p(2)$ is $2^2 = 2$ which is false.

(b) $p(x) = (x^2 = 1 \wedge x \geq 0)$ for $x \in \{-1, 0, 1\}$

> **Solution:** $p(-1)$ is $(-1)^2 = 1 \wedge -1 \geq 0$ which is false.
> $p(0)$ is $0^2 = 1 \wedge 0 \geq 0$ which is false.
> $p(1)$ is $1^2 = 1 \wedge 1 \geq 0$ which is true.

2. For each predicate decide whether it is fully quantified and identify any free parameters:

(a) $\forall x\, p(x, y)$

> **Solution:** Not fully quantified. y is a free parameter.

(b) $\exists x\, \forall y\, p(x, y)$

> **Solution:** Fully quantified.

(c) $\exists x, y\, p(x, y, z)$

> **Solution:** Not fully quantified. z is a free parameter.

3. Determine the truth value of the following fully quantified predicates, with universe $\{-1, 0, 1\}$:

(a) $\exists x\, (x^2 \geq 0 \wedge x \leq 0)$

> **Solution:** We are using $\exists$, so we just need to find one example. -1 works since $(-1)^2 \geq 0$ and $-1 \leq 0$.

(b) $\exists x, y\, (xy \geq 0)$

> **Solution:** Again, just need to find one x and one y that works. We can use $x = y = 1$ since $1 \cdot 1 \geq 0$.

(c) $\forall x\, \exists y\, (x + y = 0)$

> **Solution:** Let's set $p(x, y)$ to be $x + y = 0$. Then we need figure out the truth values of
>
> $$\exists y\, p(-1, y)$$
> $$\exists y\, p(0, y)$$
> $$\exists y\, p(1, y)$$
>
> These are satisfied by $p(-1, 1)$, $p(0, 0)$ and $p(1, -1)$. Since we made $p(x, y)$ true for all values of x, our original proposition is true.

(d) $\exists x,\, \forall y\, (xy = 0)$

> **Solution:** We need to find a single x which makes $\forall y\, p(x, y)$ true. With a bit of math knowledge, we make the guess that $x = 0$ will work for us. The statement becomes $\forall y\, 0 \cdot y = 0$. We can immediately see that this will work for any value of y in our set. So our original proposition is true.

4. In each of the following conditions, identify the necessary and sufficient conditions:

   (a) $\forall x \in \mathbb{Z}\,(x \geq 0 \rightarrow -x \leq 0)$

   Solution:
   $x \geq 0$ is a sufficient condition for $-x \leq 0$.
   $-x \leq 0$ is a necessary condition for $x \geq 0$.

   (b) $\forall x \in \mathbb{Z}\,(x^2 = 0 \rightarrow x = 0)$

   Solution:
   $x^2 = 0$ is a sufficient condition for $x = 0$.
   $x = 0$ is a necessary condition for $x^2 = 0$.

   (c) If x is divisible by 6, then x is divisible by 2.

   Solution: x is divisible by 6 is sufficient for x to be divisible by 2. x is divisible by 2 is necessary for x to be divisible by 6.

   (d) $\forall x \in \mathbb{R}\; x \geq 2 \rightarrow x^2 \geq 4$

   Solution: $x \geq 2$ is sufficient for $x^2 \geq 4$. $x^2 \geq 4$ is necessary for $x \geq 2$.

   (e) If you are a mother, then you are biologically female.

   Solution: Being a mother is sufficient for being biologically female. Being biologically female is necessary for being a mother.

5. Use logical equivalences for quantified predicates to rewrite the following statements, then determine their truth value:

   (a) $\neg(\exists x \in \mathbb{Z}\,(x + x^2 \geq 0))$

   Solution: Let's use the logical equivalence

   $$\neg(\exists x\, p(x)) \equiv \forall x\, \neg p(x)$$

   with $p(x)$ set to $x + x^2 \geq 0$. Then our original proposition is equal to the right side in the above, and so is logically equivalent to

   $$\forall x \in \mathbb{Z}\,(x + x^2 < 0).$$

   Perhaps we suspect this is false, so we can try a few examples and see if we come up with a counterexample. We find $1 + 1^2 = 2 \geq 0$, so this is enough to prove the statement false.

   (b) $\neg(\forall x \in \mathbb{Z}\,(x^2 + 4x \geq 0))$

   Solution: This time we will use the logical equivalence

   $$\neg(\forall x\, p(x)) \equiv \exists x\, \neg p(x).$$

   Setting $p(x)$ to $x^2 + 4x \geq 0$, we find that our proposition is logically equivalent to

   $$\exists x \in \mathbb{Z}\,(x^2 + 4x < 0).$$

   We can see that this is true by setting $x = -1$ and observing $(-1)^2 - 4 = -3 < 0$.

6. Find an example that illustrates each of the following logical implications:
   (a) $p(y) \wedge (y \in S) \vDash \exists y \in S\, p(y)$

   Solution: Let $p(y)$ be "y has fleas". Let y be my dog and S be the set of dogs. Then the logical statement above becomes:
   "My dog has fleas, and my dog is a dog, so there exists a dog with fleas."

   (b) $(\forall x \in S\, p(x)) \wedge (y \in S) \vDash p(y)$

   Solution: Let $p(x)$ be "x has a mother", y be me, and S be the set of mammals. Then the statement becomes
   "All mammals have mothers, and I am a mammal, so I have a mother."

7. For each of the following quantified predicates, use a logical equivalence to give an equivalent quantified predicate using the opposite quantifier:
   (a) $\forall x \in \mathbb{Z}\ (x^2 \geq 0)$

   Solution: $\neg(\exists x \in \mathbb{Z}\ (x^2 < 0))$

   (b) $\exists x \in \mathbb{Z}\ (x^2 = 4)$

   Solution: $\neg(\forall x \in \mathbb{Z}\ (x^2 \neq 4))$

   (c) $\neg(\forall x \in \mathbb{Z}\ (2x \geq x))$

   Solution: $\exists x \in \mathbb{Z}\ (2x < x)$

   (d) $\neg(\exists x \in \mathbb{Z}\ (x^2 = -1))$

   Solution: $\forall x \in \mathbb{Z}\ (x^2 \neq -1)$

8. Use logical implications from the lecture to draw conclusions from the following:
   (a) All cows eat grass and Betsy is a cow.

   Solution: This can be written like $(\forall x \in C\ (E(x))) \wedge (b \in C)$. We can use a logical implication to derive $E(b)$. In other words, Betsy eats grass.

   (b) $\sqrt{2} \in \mathbb{R} \wedge (\sqrt{2})^2 = 2$

   Solution: $\exists x \in \mathbb{R}\ (x^2 = 2)$

   (c) $(\forall x \in \mathbb{R}\ (\exists y \in \mathbb{R}\ (y^3 = x))) \wedge (2 \in \mathbb{R})$

   Solution: $\exists y \in \mathbb{R}\ (y^3 = 2)$.

9. For each of the following, give a formal mathematical statement using quantified predicates:

(a) x is a factor of y (x, y are natural numbers)

> **Solution:** $\exists z \in \mathbb{N}\ (y = xz \wedge x \neq y \wedge x \neq 1)$

(b) x is a composite number (x is a natural number)

> **Solution:** $\exists y, z \in \mathbb{N}\ (x = yz \wedge y \neq x \wedge y \neq 1)$

(c) For every real number x, the square of x is also a real number.

> **Solution:** $\forall x \in \mathbb{R}\ (x^2 \in \mathbb{R})$

(d) For every pair of even numbers x, y, $x + y$ is also even.

> **Solution:** $\forall x, y \in \mathbb{N}\ ((\exists a, b \in \mathbb{N}\ (x = 2a \wedge y = 2b)) \rightarrow (\exists c \in \mathbb{N}\ (x + y = 2c)))$

(e) Every natural number can be written as either $2k$ or $2k - 1$ where k is a natural number.

> **Solution:** $\forall x \in \mathbb{N}\ (\exists k \in \mathbb{N}\ (x = 2k \vee x = 2k - 1))$

1 Python quantified predicates

1. Write a Python function that takes a set S and a predicate $p(x)$ in one parameter, and outputs whether $\forall x \in S\, p(x)$.

> **Solution:** We loop over all $x \in S$ and if any $p(x)$ returns false then we return false.
>
> ```
> def forall(S, p):
>    for x in S:
>       if not p(x):
>          return False
>    return True
> ```

2. Write a Python function which implements the predicate $p(x) = x \equiv 0 \pmod 2$. Use it and the program from above to determine the truth value of $\forall x \in \{0, 2, 4\}\, x \equiv 0 \pmod 2$.

> **Solution:**
>
> ```
> def p(x):
>    return x % 2 == 0
>
> print(forall({0,2,4}, p))
> ```

3. Write a Python function that takes a set S and a predicate $p(x)$ in one parameter, and outputs whether $\forall x \in S\, p(x)$.

Solution: We loop over all $x \in S$ and if any $p(x)$ returns false then we return false.

```
def exists(S, p):
   for x in S:
      if p(x):
         return True
   return False
```

4. *Stretch question.* Implement the predicate $p(x, y) = x|y$ (i.e. x divides y). Use it and the previous functions to determine whether

$$\exists x \in \{2, 3, 4\}\, \forall y \in \{6, 8, 10\}\, x|y$$

Solution: This looks a bit awkward in Python, but it is much better in more purely functional languages, such as Haskell.

```
def p(x,y):
   return y % x == 0

def inner(x):
   def innerp(y):
      return p(x,y)
   return forall({6,8,10}, innerp)

print(exists({2,3,4}, inner))
```

or with lambdas

```
print(exists({2,3,4}, (lambda x : forall({6,8,10}, lambda y : p(x,y)))))
```

5. *Stretch question.* Do the same as in the last question, but with for loops rather than using your previously defined functions.

Solution:

```
def forallxy():
   for x in {2,3,4}:
      forally = True    # assume ok unless we find a counterexample
      for y in {6,8,10}:
         if y % x != 0:
            forally = False # Found a counterexample
      if forally:
         return True # Found no counterexamples!  this x works
   return False # Tried every x, but didn't find one that works

print(forallxy())
```