CAB203

Tutorial 5

Proofs

1. Use a truth table to decide if $(A \to B) \wedge B$ logically implies A.
2. Find a counterexample to show that we cannot use $A \vDash A \vee B$ to substitute into $A \to C$. In particular, find propositions A, B and C so that $A \to C$ is true but $(A \vee B) \to C$ is false.
3. Use two logical implications to prove

$$(A \wedge B) \wedge (A \to C) \vDash C$$

4. *(Stretch question)*
   (a) Prove $(A \to B) \vDash A \to (B \vee C)$
   (b) Prove $(A \to B) \wedge (A \vee C) \vDash B \vee C$
   (c) Prove $(A \vee B) \wedge (A \to C) \wedge (B \to D) \vDash C \vee D$.

Predicate logic

1. Determine the truth value of the following predicates with the values given:
   (a) $p(x) = (x^2 = x)$ for $x \in \{0, 1, 2\}$
   (b) $p(x) = (x^2 = 1 \wedge x \geq 0)$ for $x \in \{-1, 0, 1\}$
2. For each predicate decide whether it is fully quantified and identify any free parameters:
   (a) $\forall x\, p(x, y)$
   (b) $\exists x\, \forall y\, p(x, y)$
   (c) $\exists x, y\, p(x, y, z)$
3. Determine the truth value of the following fully quantified predicates, with universe $\{-1, 0, 1\}$:
   (a) $\exists x\, (x^2 \geq 0 \wedge x \leq 0)$
   (b) $\exists x, y\, (xy \geq 0)$
   (c) $\forall x\, \exists y\, (x + y = 0)$
   (d) $\exists x, \forall y\, (xy = 0)$
4. In each of the following conditions, identify the necessary and sufficient conditions:
   (a) $\forall x \in \mathbb{Z}\, (x \geq 0 \to -x \leq 0)$
   (b) $\forall x \in \mathbb{Z}\, (x^2 = 0 \to x = 0)$
   (c) If x is divisible by 6, then x is divisible by 2.
   (d) $\forall x \in \mathbb{R}\ x \geq 2 \to x^2 \geq 4$
   (e) If you are a mother, then you are biologically female.

5. Use logical equivalences for quantified predicates to rewrite the following statements, then determine their truth value:
   (a) $\neg(\exists x \in \mathbb{Z}\,(x + x^2 \geq 0))$
   (b) $\neg(\forall x \in \mathbb{Z}\,(x^2 + 4x \geq 0))$
6. Find an example that illustrates each of the following logical implications:
   (a) $p(y) \wedge (y \in S) \models \exists y \in S\, p(y)$
   (b) $(\forall x \in S\, p(x)) \wedge (y \in S) \models p(y)$
7. For each of the following quantified predicates, use a logical equivalence to give an equivalent quantified predicate using the opposite quantifier:
   (a) $\forall x \in \mathbb{Z}\ (x^2 \geq 0)$
   (b) $\exists x \in \mathbb{Z}\ (x^2 = 4)$
   (c) $\neg(\forall x \in \mathbb{Z}\ (2x \geq x))$
   (d) $\neg(\exists x \in \mathbb{Z}\ (x^2 = -1))$
8. Use logical implications from the lecture to draw conclusions from the following:
   (a) All cows eat grass and Betsy is a cow.
   (b) $\sqrt{2} \in \mathbb{R} \wedge (\sqrt{2})^2 = 2$
   (c) $(\forall x \in \mathbb{R}\ (\exists y \in \mathbb{R}\ (y^3 = x))) \wedge (2 \in \mathbb{R})$
9. For each of the following, give a formal mathematical statement using quantified predicates:
   (a) x is a factor of y (x, y are natural numbers)
   (b) x is a composite number (x is a natural number)
   (c) For every real number x, the square of x is also a real number.
   (d) For every pair of even numbers x, y, $x + y$ is also even.
   (e) Every natural number can be written as either $2k$ or $2k - 1$ where k is a natural number.

1 Python quantified predicates

1. Write a Python function that takes a set S and a predicate $p(x)$ in one parameter, and outputs whether $\forall x \in S\, p(x)$.
2. Write a Python function which implements the predicate $p(x) = x \equiv 0 \pmod 2$. Use it and the program from above to determine the truth value of $\forall x \in \{0, 2, 4\}\, x \equiv 0 \pmod 2$.
3. Write a Python function that takes a set S and a predicate $p(x)$ in one parameter, and outputs whether $\forall x \in S\, p(x)$.
4. *Stretch question.* Implement the predicate $p(x, y) = x|y$ (i.e. x divides y). Use it and the previous functions to determine whether

$$\exists x \in \{2, 3, 4\}\, \forall y \in \{6, 8, 10\}\, x|y$$

5. *Stretch question.* Do the same as in the last question, but with for loops rather than using your previously defined functions.