Lecture 5: Predicate logic

CAB203 Discrete Structures

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Outline

Readings

This week:

- Pace: 3.1 to 3.3

Next week:

- Lawson: Chapter 2 relates to lectures 3,5,6

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Logical implication

Sometimes from one true proposition we can derive other true propositions. This is called *logical implication*.

- From $A \wedge B$ we can conclude A
- From $(A \rightarrow B) \wedge A$ we can conclude B

If from A we can conclude B then we write $A \models B$, which is equivalent to saying that $A \rightarrow B$ is a tautology.

Logical implication examples

- $A \land B \models A$
- $A \models A \lor B$
- $(A \rightarrow B) \land A \models B$
- $(A \rightarrow B) \land (B \rightarrow C) \models A \rightarrow C$
- $(A \lor B) \land \neg A \models B$

Finding logical implications

We can find logical implications in two ways:

- Using a truth table
- Using a proof (later)

To show $P \vDash Q$ using a truth table, we check that in every line where $P = T$, we also have $Q = T$.

Logical implication truth table

Lets show that $A \models A \lor B$

A	B	$A \lor B$	
T	T	T	$\leftarrow$
T	F	T	$\leftarrow$
F	T	T	
F	F	F	

We see that whenever A is true, so is $A \lor B$. Notice we only check the rows when the left hand side is True.

Logical implication truth table, version 2

Recall $P \models Q$ is the same as say $P \to Q$ is a tautology.
Lets show that $A \models A \vee B$ this way

A	B	$A \vee B$	$A \to A \vee B$
T	T	T	T
T	F	T	T
F	T	T	T
F	F	F	T

See that $A \to A \vee B$ is True in all cases, so that logical implication is True.

How to use logical implications

Unlike logical equivalence, it is not always safe to make substitutions with logical implications.

- Safe example: substituting an entire formula which is known to be true. We have $A \wedge B \vDash A$ so if $A \wedge B$ *by itself* is true, it can be replaced with A
- From It is cloudy and raining. we can conclude It is raining.

Unsafe use of logical implications

In general we cannot substitute into a formula using logical implications.

- Unsafe example: Use $A \wedge B \models A$ and substitute into $\neg(A \wedge B)$ to get $\neg A$. *This does not work!*
- From NOT (Socrates is human and a teapot.) we *cannot* conclude Socrates is not human

Only substitute using a logical implication if the left hand side is known to be true. This is guaranteed by the rules of proofs (later).

Logical equivalence vs. logical implication

$P \equiv Q$	$P \models Q$
P and Q always have the same truth value	Q is true whenever P is true
Can make substitutions	Can only safely make substitutions when P is true
All rows in truth table are the same	In rows where P are T, Q is also T
$P \leftrightarrow Q$ is a tautology	$P \rightarrow Q$ is a tautology

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Proofs

We can use equivalences and logical implications to derive new true propositions from given true propositions. Suppose:

- Socrates is mortal or Socrates is not human
- Socrates is human

Then we can conclude:

- Socrates is not human or Socrates is mortal (by equivalence)
- Socrates is mortal (by logical implication)

Proofs

Symbolically, we write a *proof*. A proof is a list of formulas. The proof starts with some premises, and every other formula on the list must be:

- Logically equivalent to a formula above it
- Logically implied by a formula above it
- The *AND* of some formulas above it
- Logically implied by the *AND* of some formulas above it

The last rule doesn't add anything new, it just allows us to be a bit lazy.

Proofs and logical implications

A proof produces a new logical implication $P \vDash Q$ where

- P is the *AND* of all the premises
- Q is the last line of the proof

A proof says that, *assuming all the premises are true*, the conclusion is also true.

Proof example

Let's prove our Socrates example, $(M \vee \neg H) \wedge H \models M$:

1	$M \vee \neg H$	premise
2	H	premise
3	$\neg H \vee M$	equivalent to line 1 using $A \vee B \equiv B \vee A$.
4	$\neg\neg H$	equivalent to line 2 using $A \equiv \neg\neg A$.
5	M	logical implication of line 3 *AND* line 4 using $(A \vee B) \wedge \neg A \models B$, with $A = \neg H$ and $B = M$.

Proof example

Let's prove $(A \to B) \wedge \neg B \models \neg A$.

1	$A \to B$	premise
2	$\neg B$	premise
3	$(\neg B) \to (\neg A)$	equivalent to line 1 using $P \to Q \equiv (\neg Q) \to (\neg P)$
4	$\neg A$	logical implication from line 3 *AND* line 2 using $(P \to Q) \wedge P \models Q$ with $P = \neg B$ and $Q = \neg A$.

Longer proof example

Let's prove $(A \to B) \wedge (B \to C) \vDash (A \to C)$.

1	$A \to B$	premise
2	$B \to C$	premise
3	$B \vee \neg A$	equivalent to (1)
4	$(B \vee \neg A) \vee C$	logical implication from (3)
5	$B \vee (\neg A \vee C)$	equivalent to line (4)
6	$\neg B \vee (\neg A \vee C)$	from (2) by similar
7	$(B \vee (\neg A \vee C)) \wedge (\neg B \vee (\neg A \vee C))$	(5) $\wedge$ (6)
8	$(B \wedge \neg B) \vee (\neg A \vee C)$	equivalent to (7)
9	$F \vee (\neg A \vee C)$	equivalent to (8)
10	$\neg A \vee C$	equivalent to (9)
11	$A \to C$	equivalent to (10)

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Parameters and predicates

We can generalise propositions by allowing *parameters*:

- $A(x) = x$ is a cat
- $B(x, y) = x$ and y have the same birthday
- $C(x, y) = x = y + 1$

Parameters allow us to talk more generally about propositions that share a common form and meaning.

A *predicate* is a proposition with one or more variables.

Predicates versus formulas

How are parameters in predicates different from letters in formulas? Compare:

- $A = \neg(p \land q)$
- $B(x) =$ x is a fish

Parameters in predicates can stand in for anything (typically elements of a universe set). Variables in formulas stand in for propositions (which must evaluate to True or False).

Logical connectives and predicates

Just like for propositions, we can form complex predicates out of smaller predicates:

- $A(x) \wedge B(x)$ (understood here that x is the same for both propositions)
- $A(x) \vee B(y)$ (can have different parameters)
- $A(x) \rightarrow B(x)$
- $(A(x) \rightarrow B(y)) \wedge A(x)$

Truth values of predicates

Before we can evaluate the truth of a predicates we need to fill in the parameters with actual values:

- Suppose $A(x)$ is $x^2 = 1$. Then $A(1)$ is True, but $A(2)$ is False
- Suppose $A(x)$ is x is a flower $\rightarrow$ x smells nice. Then $A(\text{rose})$ is true, but $A(\text{raffelesia})$ is false.
- Suppose $A(x)$ is If x is human then x is mortal. We might be tempted to say that $A(x)$ is true, but we can't because we haven't filled in x yet.

Once we have filled in all variables in a predicate, is now a regular proposition and has a truth value.

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Quantifiers

Quantifiers are symbols that allow us to talk about parameters in predicates. Two kinds:

- Existential quantification (there exists)
- Universal quantification (for all)

Quantifiers allow us to form propositions out of predicates without filling in any specific value for parameters.

Existential quantification

The *existential quantifier* has the symbol $\exists$ and means "there exists":

$$\exists x \in \mathbb{Z}\,(x^2 = 4)$$

means There exists an x from $\mathbb{Z}$ such that $x^2 = 4$.

We don't get to learn *which* value of x works, just that *some* value of x works.

Existential examples

- $\exists x \in S\, p(x)$ means There exists an x in S such that $p(x)$ is true.
- $\exists x \in ANIMALS\;\; x$ is a fish means There exists an x in $ANIMALS$ such that x is a fish.
- $\exists x \in ANIMALS\, x$ is a unicorn means There exists some x in $ANIMALS$ such that x is a unicorn.
- $\exists x \in \mathbb{R}\, (x \in \mathbb{Z})$ means There exists some real number x such that x is an integer.

Universal quantification

The *universal quantifier* has the symbol $\forall$ and means "for all":

$$\forall x \in \mathbb{Z}\,(x^2 \geq 0)$$

means For every x in $\mathbb{Z}$, x^2 is non-negative.

Here, we can choose *any* value of x in $\mathbb{Z}$, fill it in to the predicate, and we will get a true statement.

Universal examples

- $\forall x \in S\, p(x)$ means For all x from S, $p(x)$ is true.
- $\forall x \in ANIMALS\ \ (x \text{ is a cat})$ means For all x in $ANIMALS$, x is a cat.
- $\forall x \in \mathbb{Z}\,(x \in \mathbb{R})$ means All integers are real numbers.

Quantifying over sets

When we use a quantifier we have two choices:

- Specify a set explicitly, as in

  $$\forall x \in S\, p(x).$$

  Then we say that we are *quantifying over S*.

- Do not explicitly specify a set, as in

  $$\forall x\, p(x).$$

  Here the set to quantify over must be taken from context. Perhaps x has already been defined to belong to some set, or comes from the current universe.

Why we quantify over sets

It is good practice to specify a set to quantify over, unless the universe is well understood from the context.

Consider $p = \forall x\,(x \text{ is a cat})$.

- Without any context, we don't know where x should come from
- If the universe is Turkish Vans, then p is true
- If the universe is mammals, then p is false

For this reason, such predicates should be viewed suspiciously unless the universe is clear from the context.

Parameters in predicates

Consider $\exists x\, p(x)$.

- The x is filled in by the $\exists x$, so we can't put in our own value for x.
- We can now assign a truth value. Either there exists an x that makes $p(x)$ true (Then $\exists x\, p(x)$ is true), or there does not (Then $\exists x\, p(x)$ is false).

Free parameters

Consider $A(y) = \exists x \, p(x, y)$.

- The parameter x is quantified over, so we can't fill it in.
- The parameter y is *not* quantified over, so we *can* fill it in.
- The truth value of $A(y)$ depends on the value of the parameter y.
- Here y is called a *free parameter*.
- If there are no free parameters, then the predicate is *fully quantified*.

Truth value of fully quantified predicates

If we are quantifying over a finite set, then we can check all values to determine if a fully quantified predicate is true. Example:

$$\forall x \in \{0, 1\} \, (x^2 = x)$$

is true because $0^2 = 0$ and $1^2 = 1$.

Truth value with existential quantifiers

For existential quantifiers, we just need to find *one* value that works to show something is true:

$$\exists x \in \{0, 1\}\,(x^2 = 1)$$

is true because $1^2 = 1$.

But to show that it is false, we need to check every value:

$$\exists x \in \{0, 1\}\,(x^2 = 2)$$

is false because $0^2 \neq 2$ and $1^2 \neq 2$.

Truth value with universal quantifiers

For universal quantifiers the situation is backwards. We need to check every value to show something is true:

$$\forall x \in \{0, 1\}\, (x^2 = x)$$

is true because $0^2 = 0$ and $1^2 = 1$.

But to show that it is false, we just need to find one value that doesn't work.

$$\forall x \in \{0, 1\}\, (x^2 = 1)$$

is false because $0^2 \neq 1$.

Truth values with existential quantifiers over infinite sets

For infinite sets and existential quantifiers, showing that something is true is the same as for finite sets:

$$\exists x \in \mathbb{Z}\,(x^2 = 1)$$

is true because $1^2 = 1$.

But to show that it is false requires that you check an infinite number of values! We usually get around this by using properties of math:

$$\exists x \in \mathbb{Z}\,(x^2 = -1)$$

is false because for every $x \in \mathbb{Z}$, $x^2 \geq 0$, and hence $x^2 \neq -1$.

The story for universal quantification is similar.

Truth depends

The truth of a fully quantified predicate depends on:

- The set quantified over. Compare:

$$\exists x \in \{0, 1, 2\}\,(x^2 = 1)$$

$$\exists x \in \{0, 2, 4\}\,(x^2 = 1)$$

- Which quantifier you use. Compare:

$$\exists x \in \{0, 1, 2\}\,(x^2 = 1)$$

$$\forall x \in \{0, 1, 2\}\,(x^2 = 1)$$

Multiple quantifications

It is entirely possible to have multiple quantifiers in a predicate, and we can mix types

- $\exists x \in \mathbb{R} \, (\exists y \in \mathbb{R} \, x + y = 0)$ is true
- $\forall x \in \mathbb{R} \, (\exists y \in \mathbb{R} \, x + y = 0)$ is true
- $\forall x \in \mathbb{R} \, (\forall y \in \mathbb{R} \, xy = 1)$ is false
- $\forall x, y, z \in \mathbb{R} \, (\exists c \in \mathbb{R} \, x + y + z = c)$ is true
- $\forall x \, \exists x \, p(x)$ is not a well formed predicate
- $\forall x \, p(x, y)$ has a free parameter y

You can quantify over different sets or the same set, and can combine any number of quantifiers in any order. But at most one quantifier per parameter.

Understanding multiple quantifications

Let's fix the universe as $\{0, 1\}$ and consider:

$$\exists x \, \forall y \, (x + y = 1).$$

This should be understood as:

$$\exists x \, (\forall y \, (x + y = 1)).$$

Order of quantifiers

If there are multiple quantifiers of different types, then the order matters. With universe $\mathbb{Z}$:

$$\forall y \, \exists x \, (x + y = 0)$$

is true. For any y we can use the value $x = -y$. But:

$$\exists x \, \forall y \, (x + y = 0)$$

is false. For any value x we can choose $y = x + 1$ so $x + y = 1 \neq 0$.

IF..THEN revisited

The *IF..THEN* logical connective is more intuitive when we use a quantifier. Let $h(x)$ be x is human and $m(x)$ be x is mortal. Then the proposition

$$\forall x \in \mathit{BEINGS}\,(h(x) \to m(x))$$

is closer to our everyday understanding of something like "If you are human then you are mortal." In natural language we often implicitly add a universal quantifier.

But remember, in logic we only care about truth, not meaning, even with quantifiers! For example, with $p =$Tomatoes are blue:

$$\forall x \in \mathit{BEINGS}\,(p \to h(x))$$

is True.

Necessary and sufficient conditions

When $\forall x\,(p(x) \to q(x))$ we say:

- $p(x)$ is a *sufficient condition* for $q(x)$.
- $q(x)$ is a *necessary condition* for $p(x)$.

When $\forall x\, p(x) \leftrightarrow q(x)$ we say

- $p(x)$ is *necessary and sufficient* for $q(x)$.
- $q(x)$ is *necessary and sufficient* for $p(x)$.

We typically don't use this terminology if p or q are trivial, i.e. the truth of p or q does not depend on x.

Necessary and sufficient conditions

Say we had the statement, *squareness is a sufficiant condition for rectangularity*, and we wanted to write it as a quantified conditional statement.
Formally, we could write:

$$\forall x, \text{ if } x \text{ is a square, then } x \text{ is a rectangle.}$$

Or informally, we could write:

If a figure is a square, then it is a rectangle.

Boolean formulas revisited

Suppose $A(x, y)$ is a Boolean formula with x and y some propositions. Then

- A is a tautology means $\forall x, y\, A(x, y)$
- A is a contradiction means $\forall x, y\, \neg A(x, y)$
- A is satisfiable means $\exists x, y\, A(x, y)$

Here the universe is $\{T, F\}$.

We can ask whether a fully quantified Boolean formula is true. This problem is PSPACE-complete, and is believed to be even harder to solve than NP-complete problems.

Outline

Logical equivalence with predicates

Just as for Boolean formulas, we have equivalences for predicates

- All equivalences for Boolean formulas still work
- $\neg(\forall x\, p(x)) \equiv \exists x\, \neg p(x)$:
  It is not the case that all humans are male means There exists some human which is not male
- $\neg(\exists x\, p(x)) \equiv \forall x\, \neg p(x)$:
  It is not the case that there exists a human which lays eggs means All humans do not lay eggs
- $\forall x\, (p(x) \wedge q(x)) \equiv (\forall x\, p(x)) \wedge (\forall x\, q(x))$:
  All humans are mammals and warm-blooded means All humans are mammals and all humans are warm-blooded
- $\exists x\, (p(x) \vee q(x)) \equiv (\exists x\, p(x)) \vee (\exists x\, q(x))$

Logical implication with predicates

We also have logical implications for predicates

- All logical implications for formulas still work
- $(\forall x \in S\, p(x)) \wedge (y \in S) \vDash p(y)$:
  All humans are mortal and Socrates is human implies Socrates is mortal
- $p(x) \wedge (x \in S) \vDash \exists y \in S\, p(y)$:
  Socrates is male and human implies There exists some human who is male
- $(\forall x \in S\, p(x)) \wedge (S \neq \emptyset) \vDash \exists y \in S\, p(y)$:
  All humans are mortal and there exist humans implies There exists a human who is mortal

Expressing problems as predicates

The language of predicates allows us to precisely state problems:

- x is prime $= \neg(\exists y, z \in \mathbb{N}\,((yz = x) \wedge (y \neq 1) \wedge (z \neq 1)))$

We can use logic give an equivalent formulation:

- $\forall y, z \in \mathbb{N}\,((y = 1) \vee (z = 1) \vee (yz \neq x))$

This suggests a simple algorithm:

- Loop through all $y, z \in \mathbb{N}$
- Check if $y = 1$, $z = 1$ or $yz \neq x$

Expressing our problem as a predicate gives explicit conditions that can be translated directly into program logic.

Python predicates

Python doesn't have a special notion of predicates. Any function without side effects that returns True/False can be used like a predicate.
Side effects means changes to the state of the program: it only changes local variables.

```
>>> def p(x,y):
...     return x >= y
...
>>> p(1,2)
False
>>> p(2,1)
True
```

Python doesn't have quantifiers, but you can loop over parameters to determine truth of a fully quantified predicate (see tutorial)