CAB203
Tutorial 4

Set theory

1. Determine whether each element is in the set:

   (a) Is $5 \in \{2x : x \in \mathbb{Z}\}$?

   > **Solution:** No. The only way that we can make $2x = 5$ is for $x = 2.5$ and 2.5 is not an integer.

   (b) Is $10 \in \{x \in \mathbb{Z} : x \text{ divides } 60\}$?

   > **Solution:** Yes. Certainly 10 is an integer, and $60/10 = 6$ so 10 divides 60.

2. Are these subsets?

   (a) Is $\{2, 5\} \subseteq \mathbb{Z}$?

   > **Solution:** Yes. Both 2 and 5 are integers, so $\{2, 5\}$ is a subset of the integers.

   (b) Is $\{2, 3, 6\} \subseteq \{2x : x \in \mathbb{Z}\}$?

   > **Solution:** No. Both 2 and 6 are members of the set on the right, but 3 is not since it is not even.

   (c) Is $\{4x : x \in \mathbb{Z}\} \subseteq \{2x : x \in \mathbb{Z}\}$?

   > **Solution:** Yes. Suppose we have y an element of the left set. Then $y = 4x$ where x is some integer. But we can also write $y = 2(2x)$ where $2x$ is an integer, so y is also in the set on the right. This is true for any y in the left hand set. So every element of the left set is a member of the right set, which makes the left set a subset of the right set.

3. What are these sets?

   (a) $\{1, 3, 5\} \cup \{2, 4, 6\}$

   > **Solution:** We need to have everything in the left set, and everything in the right set. So this is just $\{1, 3, 5, 2, 4, 6\}$. If we like, we can rearrange this into $\{1, 2, 3, 4, 5, 6\}$.

   (b) $\{4, 7, 9\} \cap \{9, 7, 3, 6\}$

   > **Solution:** Going through each element in the set on the left, we see that 4 is not in the set on the right, but 7 and 9 are. So the intersection is $\{7, 9\}$.

(c) $\{1,2,3\} \setminus \{1,2,3,4,5\}$

Solution: Here we have to start with the set on the left and remove everything from the set on the right. So we remove 1,2 and 3. The 4 and 5 are not in the set on the left, so we don't care about them. But we have removed everything, so the result is $\{\} = \emptyset$.

(d) $\mathbb{Z}_{\geq 0} \setminus \{2x : x \in \mathbb{Z}\}$

Solution: Let's start with an element $y \in \mathbb{Z}_{\geq 0}$. There will be two cases. Either $y = 2x$ for some $x \in \mathbb{Z}_{\geq 0}$ or not, in which case we can write $y = 2x - 1$ for some $x \in \mathbb{Z}_{\geq 0}$. In the first case we need to remove y from the final set, since $x \in \mathbb{Z}$ as well, so y is in the right hand set. In the second case, we can write $y = 2x - 1$. So the answer could be written as

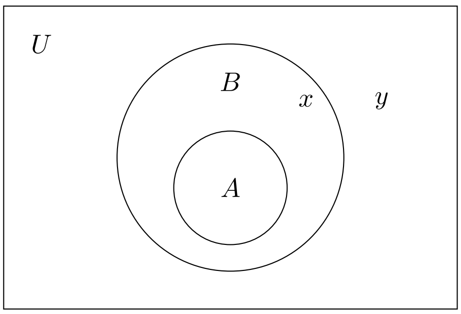
$$\{2x + 1 : x \in \mathbb{Z}_{\geq 0}\}.$$

Or we could recognize that the right hand set is all of the even integers (including things like -2 and 0). Removing these from the left hand set leaves only the odd numbers $\{1, 3, 5, \ldots\}$.

(e) $\overline{\{x : x = 2y\}}$ with universe $U = \mathbb{Z}_{\geq 0}$

Solution: Here, since the universe is $\mathbb{Z}_{\geq 0}$, we understand that x and y are non-negative integers in the setbuilder notation. Pretty clearly, we are looking at the even numbers, but complemented. So the answer will be the odd numbers. The logic is similar to the previous question.

4. Look at the Venn diagram and answer the questions.



(a) Is $B \subseteq A$?

Solution: No. For example x is in B but not in A.

(b) Is $x \in A$?

Solution: No. x is clearly outside the circle for A.

(c) Is $x \in B \setminus A$?

Solution: Yes. x is inside the circle for B, but outside the circle for A.

(d) Is $y \in \overline{B}$?

Solution: Yes. y is outside the circle for B

(e) Is $x \in U$?

Solution: Yes. Everything is in U

(f) Is $x \in \overline{A \cap B}$?

Solution: Yes. Since A is a subset of B, $A \cap B = A$. x outside A, so it is in $\overline{A} = \overline{A \cap B}$.

5. *(Stretch question)* For some universe U with $A, B \subseteq U$, show:

$$\overline{A \cap B} \quad = \quad \overline{A} \cup \overline{B}$$

Solution: For the first equation, consider $x \in \overline{A \cap B}$. Then x is not in $A \cap B$. This means that either $x \notin A$ or $x \notin B$ (or both). The two cases are similar, so just assume that $x \notin A$. Then $x \in \overline{A}$ and hence $x \in \overline{A} \cup \overline{B}$. We have shown that every x is the LHS is in the RHS, so

$$\overline{A \cap B} \subseteq \overline{A} \cup \overline{B}.$$

Now suppose that $x \in \overline{A} \cup \overline{B}$. Then $x \in \overline{A}$ or $x \in \overline{B}$. Again, the cases are similar, so we'll assume $x \in \overline{A}$ and hence $x \notin A$. Thus $x \notin A \cap B$ as well, so $x \in \overline{A \cap B}$. Again, every x in the RHS is in the LHS and so

$$\overline{A} \cup \overline{B} \subseteq \overline{A \cap B}.$$

Now we if we have two sets S, T with $S \subseteq T$ and $T \subseteq S$ then we must have $S = T$. Thus we obtain

$$\overline{A \cap B} = \overline{A} \cup \overline{B}.$$

6. Write Python programs for the following

(a) Define a function that takes a number x and returns a set containing all non-negative integers less than 100 that are divisible by x

Solution:

```
def f(x):
   return { y for y in range(0, 101) if y % x == 0 }
```

(b) Define a function that implements complements by taking a set S and a universe U and returning the complement of S in U

Solution: This one is much easier than it sounds!

```
def f(S, U):
   return U - S
```

Syllogisms

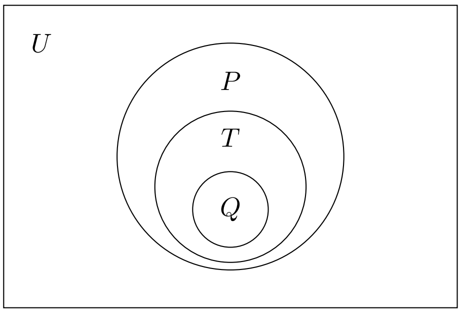
1. Write the syllogism type for the following syllogism. Is it a valid type?:

$$\frac{\begin{array}{l}\text{All trees are plants}\\\text{Pines are trees}\end{array}}{\text{Pines are plants}}$$

Solution: The type is

$$\frac{\begin{array}{l}T \subseteq P\\Q \subseteq T\end{array}}{Q \subseteq P}$$

This is a valid type. The Venn diagram looks something like



2. Is this syllogism type valid? If yes, draw a Venn diagram illustrating the sets and elements. If not give a counter-example:

$$\frac{\begin{array}{l}x \in A\\B \subseteq A\end{array}}{x \in B}$$

Solution: This is not a valid type. For example:

$$\frac{\begin{array}{l}\text{My cat is a mammal}\\\text{All dogs are mammals}\end{array}}{\text{My cat is a dog}}$$

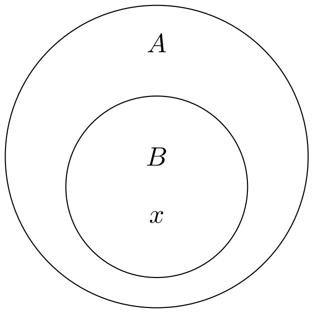
3. Is this syllogism type valid? If yes, draw a Venn diagram illustrating the sets and elements. If not give a counter-example:

$$\frac{\begin{array}{l}x \in B\\B \subseteq A\end{array}}{x \in A}$$

Solution: Yes, this is a valid syllogism. It is equivalent to the form

$$\frac{\begin{array}{l}B \subseteq A\\x \in B\end{array}}{x \in A}$$

which was our very first syllogism type. The Venn diagram is



4. The following syllogism has a false conclusion. Explain two problems with the syllogism:

$$\frac{\begin{array}{l}\text{All mothers are human}\\ \text{Socrates is human}\end{array}}{\text{Socrates is a mother}}$$

Solution: The two problems are that the syllogism type is invalid, and that one of the premises is not true.

The syllogism type is:

$$\frac{\begin{array}{l}B \subseteq A\\ x \in A\end{array}}{x \in B.}$$

We've encountered this type before, in question 2. The only difference is the order of the premises.

The second problem is that there is a false premise. Certainly there are mothers that are not human, such as the mother of a cat.