A11: PROVING ALGORITHM CORRECTNESS. GRAPHS AND TREES

You can graph human evolution, which is mostly a straight line, but we do get better and change over time, and you can graph technological evolution, which is a line that's going straight up. They are going to intersect each other at some point, and that's happening now.
—Daniel H. Wilson

Course: CS 5002
Fall 2018
Due: Dec 7, 2018, Midnight

OBJECTIVES

After you complete this assignment, you will be comfortable with:

- Several proof techniques, including:
  - Proof by counterexample
  - Proof by induction
  - Proof by loop invariance
- Some basic tree terminology
- Some special trees
- Tree traversals
- Some basic graph terminology
- Graph representations

RELEVANT READING

Rosen:

- Chapter 5.1. Mathematical induction
- Chapter 5.2 Strong Induction and Well-Ordering
- Chapter 5.3 Recursive Definitions and Structural Induction
- Chapter 5.5. Program Correctness
- Chapter 11.1 Introduction to Trees
- Chapter 11.3 Tree Traversals
- Chapter 10.1 Graphs and Graph Models
- Chapter 10.2 Graph Terminology and Special Types of Graphs

NEXT WEEK'S READING

Rosen,

- Chapter 7: Discrete Probability

EXERCISES

Question 1

Let $P(n)$ be the statement that $1^2 + 2^2 + \cdots + n^2 = n(n+1)(2n+1)/6$ for a positive integer n.

(a) What is the statement $P(1)$?

(b) Show that $P(1)$ is true, completing the basis step of the proof.

(c) What is the inductive hypothesis?

(d) What do you need to prove in the inductive step?

(e) Complete the inductive step, identifying where you use the inductive hypothesis.

Points: ______________ out of 6

(a) P(1): 1^2 = 1*2*3/6
(b) left = 1^2 = 1, right = 1*2*3/6 = 1, since left = right, P(1) is true
(c) The inductive hypothesis is that for any arbitrary positive integer k, the statement that 1^2 + 2^2 +... + k^2 = k(k+1)(2k+1)/6 is true
(d) In the inductive step, we need to prove P(k+1) is also true, which is 1^2+...+ k^2 + (k+1)^2 = (k+1)(k+2)(2k+3)/6
(e)Basic step: P(1) is true(as proved in problem(b))
Inductive hypothesis: 1^2 + 2^2 + ...k^2 = k(k+1)(2k+1)/6
Inductive step: left = 1^2 + 2^2 + ... + k^2 + (k+1)^2 = P(k) + (k+1)^2
= k(k+1)(k+2)/6 + (k+1)^2 (use the inductive hypothesis) = (k+1)(k^2 + 7k + 6)/6
= (k+1)(k+2)(2(k+1) + 1)/6, right = (k+1)(k+2)(2(k+1)+1)/6, left = right. Proved!

Question 2

Briefly explain what is wrong with the following "proof":

Theorem: For every positive integer n, if x and y are positive integers with max(x, y) = n, then $x = y$.

Basis step: Suppose that $n = 1$. If max(x, y) = 1, and x and y are positive integers, then we have $x = 1$ and $y = 1$.

The theorem itself is wrong. The inductive step and hypothesis are built on an invalid assumption

The basic step is right because the condition "x and y are positive integers" implies that x>0 and y>0. Given max(x, y) = 1, we have 0<x<=1 and 0<y<=1, which is x=y=1.
The basic step is right because the condition "x and y are positive integers" play a role.

However, the inductive hypothesis is wrong. When max(x, y) = k, x=y is not always true. Because the range of x and y are 0<x<=k and 0<y<=k, and a counter example can be given as x = k, y = 1. Hence the inductive hypothesis is wrong, and the inductive step is wrong. Also the theorem itself is wrong.

Inductive step: Let k be a positive integer. Assume that whenever max$(x, y) = k$ and x and y are positive integers, then $x = y$. Now let max$(x, y) = k + 1$, where x and y are positive integers. Then max$(x - 1, y - 1) = k$, so by inductive hypothesis, $x - 1 = y - 1$. It follows that $x = y$, completing the inductive step.

question3: basic step: when n = 1, left = 1*2 = 2, right = 1*2*3/3 = 2, left = right. Basic step is true.
Inductive hypothesis:assume for any arbitrary positive integer k, 1*2+...+k(k+1) = k(k+1)(k+2)/3
Inductive steps: for k + 1
left = 1*2 +... + k(k+1) + (k+1)(k+2) = k(k+1)(k+2)/3 + (k+1)(k+2) [use the inductive hypothesis]
= (k+1)(k+2)(k+3)/3, right = (k+1)(k+2)(k+3)/3, left = right, proved!
Hence for every positive integer n, we have 1*2 + 2*3 + ...n(n+1) = n(n+1)(n+2)/3

Question 3

Prove that for every positive integer n:

$$1 \cdot 2 + 2 \cdot 3 + \cdot + n(n+1) = n(n+1)(n+2)/3$$

Question 4

Give a recursive algorithm for computing nx whenever n is a positive integer, and x is an integer, using just addition. Please provide pseudocode fro your algorithm.

Algorithm:
use total to represent the result, total = 0 initially
1. when n > 0, add x to total, n = n - 1
2. repeat step 1 until n = 0
pesudo code presented in the right

```
multiply_by_addition(n, x):
    if n == 0:
        return 0
    else:
        return multiply_by_addition(n-1, x) + x
```

Question 5

Use merge sort to sort $b, d, a, f, g, h, z, p, o, k$ into increasing order. Show all steps used by the algorithm.

Step1: bdafghzpok (the initial array)
Step2: bdafg hzpok (divide)
Step3: bd afg hz pok (divide)
Step4: b d a fg h z p ok (divide)
Step5: b d a f g h z p o k (divide)
Step6: bd a fg hz p ko (conquer and merge)
Step7: bd afg hz kop (conquer and merge)
Step8: abdfg hkopz (conquer and merge)
Step9: abdfghkopz (conquer and merge)

Question 6

Use quick sort to sort 3, 5, 7, 8, 1, 9, 2, 4, 6 into increasing order. Show all steps used by the algorithm.

Step1: 3 5 7 8 1 9 2 4 6(pivot) (the initial array)
Step2: 3 5 1 2 4(pivot) 6 8 7 9(pivot) (6 is put at the correct position)
Step3: 3 1 2(pivot) 4 5(pivot) 6 8 7(pivot) 9 (4, 9 are put at the correct position)
Step4: 1(pivot) 2 3(pivot) 4 5 6 7 8(pivot) 9 (2, 5, 7 are put at the correct position)
Step5: 1 2 3 4 5 6 7 8 9 (1,3, 8 are put at the corerect position, the algorithm terminates)

Question 7

(a) Give a recursive definition of the length of a string.

(b) Use the recursive definition from part (a) to prove that, given two strings x and y, it holds that $l(xy) = l(x) + l(y)$, where $l(\cdot)$ denotes the length of a string.

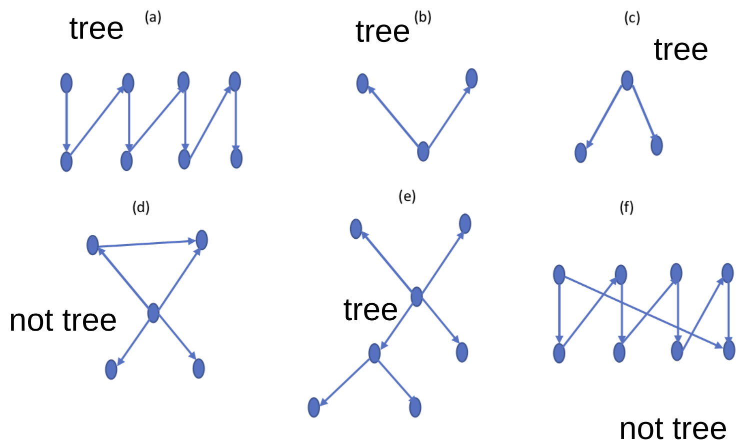

Figure 1: Graphical representations of trees, used in Question 8.

(a) 1. if a string is empty, then its length is 0; if a string has one character, then its length is 1
2. if a string has more than 1 character, its length is 1 + (the length of the tail), where the tail is the all characters in the string except the first character.
(b) suppose the length of string x and y are a and b, respectively.
left = l(xy) = l(x(y-1)) + 1 = l(x(y-2)) + 2 = ... = l(x(y-(y-1))) + (b-1) = l(x) + b
= l(x-1) + 1 + b = l(x-2) + 2 + b = ... = l(0) + a + b = a + b
right = l(x) + l(y) = a + b
left = right, proved!

Question 8

Consider the following graphs, represented in Figure 1. Which of the presented graphs are trees? If you think that some of the graphs (a)–(f) are not trees, please explain why.

(d) and (f) are not trees, because in both (d) and (f) there exists a node which has more than one parent, which does not meet the definition of a tree

Question 9

Draw graphs that have the following adjacency matrices:

(a) $\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 2 \\ 1 & 2 & 0 \end{bmatrix}$

(b) $\begin{bmatrix} 0 & 2 & 0 \\ 2 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix}$

(c) for graph a:

vertex	adjacency vertices
a	a, c
b	b, c
c	a, b

for graph b:

initial vertex	terminal vertices
a	b
b	a, b, c
c	c

(c) Find adjacency lists that correspond to the adjacency matrices from parts (a) and (b).

(a) since this matrix is symmetric, we can represent it in an undirected graph. Use a, b, c to represent the vertices, respectively, the connections are as follows:
a: one edge to a and one edge to c
b: one edge to b and two edges to c
c: one edge to a and two edges to b

(b) since the matrix is not symmetric, we can only represent it in a directed graph.Use a, b and c to represent the vertices, respectively. The connections are as follows:
a: two edges pointing to b
b: two edges pointing to a, two pointing to c, one pointing to b
c: one edge pointing to c

Question 10

(a) Represent the following expressions using binary trees, such that operands are leaves and operations are internal nodes:
- $(x + xy) + (x/y)$
- $x + ((xy + x)/y)$

Now, traverse the constructed trees using the following traversals:

(b) Pre-order traversal.

(c) In-order traversal.

(d) Post-order traversal.

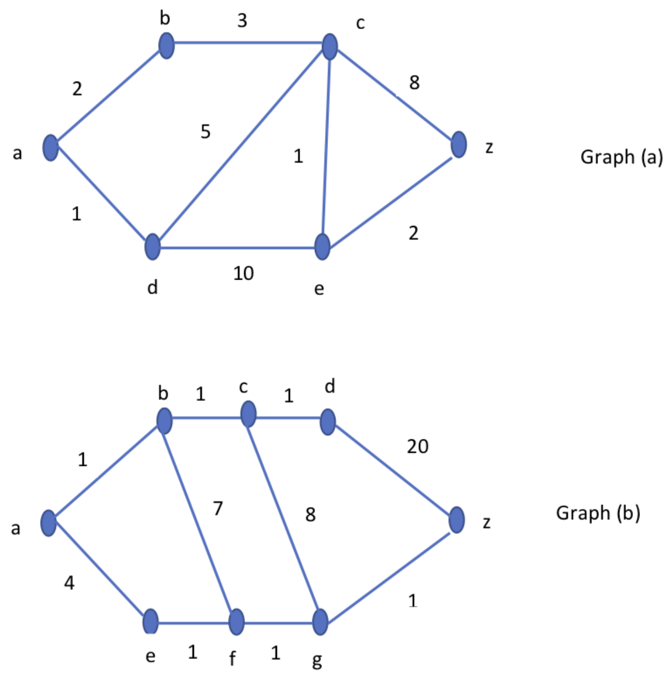

Figure 2: Graphical representations of graphs, used in Question 11.

Question 11

Use Dijkstra's algorithm to find the shortest path from a to z for graphs represented in Figure 2. For both graphs, list all steps of the algorithm.

Points: ______________ out of 6

Shortest path from a to z in graph a: a->b->c->e->z, the distance is 8
Shortest path from a to z in graph b: a->e->f->g->z, the distance is 7

Use S to record the vertices that has been visted, Q as a priority queue to record the vertices that has not been recorded. The union of S and Q are the whole set of vertices.

Graph(a):
1.S: empty
  Q: a(0), b(∞), c(∞), d(∞), e(∞), z(∞)
2.S: a(0)
  Q: d(a, 1), b(a, 2), c(∞), e(∞), z(∞)
3.S: a(0), d(a, 1)
  Q: b(a, 2), c(d, 6), e(d, 11), z(∞)
4.S: a(0), d(a, 1), b(a, 2)
  Q: c(b, 5), e(d, 11), z(∞)
5.S: a(0), d(a, 1), b(a, 2), c(b, 5)
  Q: e(c, 6), z(c, 13)
6.S: a(0), d(a, 1), b(a, 2), c(b, 5), e(c, 6)
  Q: z(e, 8)
7.S: a(0), d(a, 1), b(a, 2), c(b, 5), e(c, 6), z(e, 8)
  Q: empty

Graph(:b):
1.S: empty
  Q: a(0), b(∞), c(∞), d(∞), e(∞), f(∞), g(∞), z(∞)
2.S: a(0)
  Q:b(a, 1), e(a, 4), c(∞), d(∞), f(∞), g(∞), z(∞)
3.S:a(0), b(a, 1)
  Q:c(b, 2), e(a, 4), f(b, 8), d(∞), g(∞), z(∞)
4.S:a(0), b(a, 1), c(b, 2)
  Q:d(c, 3), e(a, 4), f(b, 8), g(c, 10), z(∞)
5.S:a(0), b(a, 1), c(b, 2), d(c, 3)
  Q:e(a, 4), f(b, 8), g(c, 10), z(d, 23)
6.S:a(0), b(a, 1), c(b, 2), d(c, 3), e(a, 4)
  Q:f(e, 5), g(c, 10), z(d, 23)
6.S:a(0), b(a, 1), c(b, 2), d(c, 3), e(a, 4), f(e, 5)
  Q:g(f, 6), z(d, 23)
7.S:a(0), b(a, 1), c(b, 2), d(c, 3), e(a, 4), f(e, 5), g(f,6)
  Q:z(g, 7)
6.S:a(0), b(a, 1), c(b, 2), d(c, 3), e(a, 4), f(e, 5), g(f, 6)
  z(g, 7)
  Q: empty

PROBLEMS

Question 12

Using mathematical induction, prove that 2 divides $n^2 + n$ whenever n is a positive integer.

Basic step: when n = 1, 1^2+1 = 2, 2mod2 = 0, so base case is true
Inductive hypothesis: suppose for any arbitrary positive integer k, 2 divides k^2+k, which means (k^2+k)mod2 = 0
Inductive steps: prove for k + 1, 2 divides (k+1)^2 + (k+1)
Since (k + 1)^2 + (k+1) = k^2 + 2k + 1 + k + 1= k^2 + 3k + 2 = (k^2+k) + 2(k+1).
Based on the inductive hypothesis, we have (k^2+k)mod2=0,
besides, 2(k+1) must be an even number -> 2(k+1)mod 2= 0
hence ((k^2+k) + 2(k+1))mod2 = 0. The conclusion is also true for k + 1.
Hence 2 divides n^2+n whenever n is a positive integer

Question 13

Prove that $f_1 + f_3 + f_3 + \cdots + f_{2n-1} = f_{2n}$. condition: fibonacci sequence

Basic step: according to the definition of fibonacci sequence, we have f1 = f2 = 1, hence the basic step (when n = 1, f1 = f2) is true
Inductive hypothesis: suppose for any arbitrary positive integer k, the conclusion that
f1 + f3 + ... + f(2k-1) = f2k is true
Inductive step: prove for k+1, the conclusion is also true.
left = f1 + f3 +... + f(2k-1) + f(2k+1), based on the inductive hypothesis,left = f(2k) + f(2k+1)
Since fn stands for fibonacci sequence, according to its property, we have
left = f(2k) + f(2k+1) = f(2k+2) , for k+1, right = f(2k+2), which means left = right for k+1.
Hence the conclusion is proved!

Question 14

A chain letter starts when a person sends a letter to five others. Each person who receives the letter either sends it to five other people who have never received it, or does not send it to anyone. Suppose that 10000 people send out the letter before the chain ends, and that no one receives more than one letter. How many people receive the letter, and how many do not send it out?

Since everyone who receives the letter either send it to 5 other people who have not received it yet, or does not send it to anyone. This problem is equivalent to a full 5-ary tree.
10000 people send out the letter, which means there are 10000 internal nodes in the tree.
Hence the total number of nodes in the tree is 5 * 10000 + 1 = 50001
The people who first sends the letter is the root in the tree, and he/she will not receive a letter.
So 50001 - 1 = 50000 poeple receive the letter.

The number of people who do not send it out equals the number of leaves in the tree. Since there are 10000 internal nodes, there are 50001-10000=40001 leaf nodes -> which means there are 40001 people who receive the letter but did not send it out.

Question 15

Prove that if some adjacency matrix A is an $m \times m$ symmetric matrix, then A^2 is also symmetric.

Prove:
If a matrix A is a symmetric matrix, we have A = A^T(the transpose matrix of A)
Now we only need to prove A^2 =(A^T)^2
Since A^2 = AA = (A^T)(A^T) = (A^T)^2, it can be conclued that A^2 is also symmetric.
Proved!

Question 16

Show that a simple graph is a tree if and only if it is connected, but the deletion of any of its edges produces a graph that is not connected.

1. A simple graph is a tree -> the graph is connected:
when a simple graph is a tree, by the properties of the tree we know that from root there is one and only one path to every node. Hence all vertices in the graph are connected
2. A simple graph is connected -> a simple graph is a tree:
If a simple graph is connected, by the definition of a simple graph, we know that between each pair of vertices, there is one and only one edge, and from any vertex, there is one and only one way to visit another vertex. We random pick any vertex and adopts dfs to visit the graph, and a tree will be formed.
Since in a connected simple graph, any two connected vertices have only one edge, and there is one way to go from one vertex to another vertex, deleting any edge will result in the vertices which were connected by the edge to lose their connection, and the graph will become not connected.

Question 17

Suppose there exists an integer k such that every man on a desert island is willing to marry exactly k of the women on the island, and every woman on the island is willing to marry exactly k of the men. Let's also assume that a man is willing to marry a woman if and only if she is willing to marry him. Show that it is possible to match men and women on the island so that everyone is matched with someone that they are willing to marry.

This problem can be constructed as a undirected bipartile Graph model.
Man and women are vertices in the graph, Use M and W to represent the set of men and women on the island, respectively. E stands for the set of edges, connecting a man vertex and a women vertex if they are williing to marry each other. Let G = (V, E), where V is the union of M and W.
We can use Hall's theorem to prove the conclusion. For any subset A of W, suppose there are m edges between A and N(A). Since any vertex in V has degree k, m = k|A|, since all the m edges are incident to N(A), m <= k|N(A)| (the number of degrees of all vertices in N(A) is at least the number of edges connecting A and N(A)).
Hence we have k|A| <= k|N(A)|, which is |A| < |N(A)|.
By Hall's theorem, it can be conclued that graph V has a complete matching from M to W,
In the same way we can also prove that graph V has a complete matching from W to M.
Hence it is possible to match men and women such that every is matched with someone that they are willing to marry.

Question	Points	Score
Simple Proof By Induction	6	
Lame Proof	4	
Another Proof by Induction	4	
Simple Recursive Algorithm	3	
Merge Sort	2	
Quick Sort	2	
Strings and Recursion	5	
Trees	3	
Adjacency Matrix	4	
Binary Trees	10	
Dijkstra's Algorithm	6	
Mathematical Induction	6	
Mathematical Induction	4	
Trees and Proofs	10	
Adjacency Matrix and Proofs	10	
Connected Graphs and Treees	10	
Marriage Problem	11	
Total:	100	

SUBMISSION DETAILS

Things to submit:

- Submit the following on Blackboard for Assignment 11:
  - The written parts of this assignment as a .pdf named “CS5002_[lastname]_A11.pdf”. For example, Ben Bitdiddle’s file would be named “CS5002_Bitdiddle_A11.pdf”. (There should be no brackets around your name).
  - Make sure your name is in the document as well (e.g., written on the top of the first page).