Approximation Algorithms Part II: week 2 assignment

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Primal-dual algorithm for k-median problem

For quicker reference, we write down LP2 here explicitly:

$$\begin{array}{lll} \text{minimize} & \sum_{i,j\in F\times C} x_{ij}c_{ij} + \lambda \sum_{i\in F} y_i - \lambda k & \\ \text{s.t.} & \sum_{i\in F} x_{ij} \geq 1 & \forall j \in C \\ & y_i - x_{ij} \geq 0 & \forall i,h \in F\times C \\ & y_i \geq 0 & \forall i \in F \\ & x_{ij} \geq 0 & \forall i,j \in F\times C \end{array}$$

Q1.

$$\begin{array}{lll} \text{maximize} & \sum_{j\in C} \alpha_j & \\ \text{s.t.} & \alpha_j - \beta_{ij} \leq c_{ij} & \forall i,j \in F\times C \\ & \sum_{j\in C} \beta_{ij} \leq \lambda & \forall i \in F \\ & \alpha_j \geq 0 & \forall j \in C \\ & \beta_{ij} \geq 0 & \forall i,j \in F\times C \end{array}$$

Q2. Using the algorithm for facility location problem discussed in the lectures to solve LP2, let's denote the corresponding primal solution by $\{C_0\}$ and dual solutions by $\{\alpha_j\}$ and $\{\beta_{ij}\}$. We have:

$$\sum_{\text{cluster} C_0} 3\lambda + \sum_{j\in C_0} c(i,j) \leq 3\sum_{j\in C} \alpha_j$$

Q3. If we're lucky to have exactly k opened facilities, the algorithm gives 3-approximation to the k-median problem.

Q4. When $\lambda = \lambda_1$, all facilities are opened to minimize the objective. Therefore at least k facilities are opened (assuming $|F| \geq k$)

For the case of $\lambda = \lambda_2$, the idea is to show we can achieve a much better solution by opening fewer than k facilities compared to opening more than k facilities. Therefore, the 3-approximation ratio should fail when opening more than k facilities.

Suppose $k+k^{'}$ facilities are opened for some positive integer $k^{'}$, the objective value becomes $\sum_{i,j\in F\times C} x_{ij}c_{ij} + k^{'}\lambda_2$. It's easy to see that $\sum_{i,j\in F\times C} x_{ij}c_{ij} <= k^{'}\lambda_2$. Therefore, we can achieve a better objective by opening k facilities, where the objective value becomes $\sum_{i,j\in F\times C} x_{ij}c_{ij}$.

Q5.

$$3|S_1|\lambda_1 + \text{cost}(S_1) \leq 3\sum_{j\in C}\alpha_j^1$$

$$3|S_2|\lambda_2 + \text{cost}(S_2) \leq 3\sum_{j\in C}\alpha_j^2$$

Q6.

$$\begin{aligned}
\text{cost}(S_1) &\leq 3\sum_{j\in C}\alpha_j^1 - 3\lambda_1|S_1| \\
&\leq 3\sum_{j\in C}\alpha_j^1 - 3|S_1|(\lambda_2 - \epsilon c_{min}/(3|F|)) \\
&\leq 3\sum_{j\in C}\alpha_j^1 - 3\lambda_2|S_1| + \epsilon c_{min}|S_1|/|F| \\
&\leq 3\sum_{j\in C}\alpha_j^1 - 3\lambda_2|S_1| + \epsilon \text{OPT}|S_1|/|F| \\
&\leq 3\sum_{j\in C}\alpha_j^1 - 3\lambda_2|S_1| + \epsilon \text{OPT}|F|/|F| \\
&\leq 3(\sum_{j\in C}\alpha_j^1 - \lambda_2|S_1|) + \epsilon \text{OPT}
\end{aligned}$$

Q7. Using the results from Q5 and Q6:

$$\begin{aligned}
&\delta_1\text{cost}(S_1) + \delta_2\text{cost}(S_2) \\
\leq &3(\delta_1\sum_j\alpha_j^1 + \delta_2\sum_j\alpha_j^2) - 3\lambda_2(\delta_1|S_1| + \delta_2|S_2|) + \delta_1\epsilon\text{OPT} \\
\leq &3\sum_j\tilde{\alpha}_j - 3\lambda_2 k + \delta_1\epsilon\text{OPT} \\
\leq &3\sum_j\tilde{\alpha}_j + 3\delta_1\epsilon\text{OPT} \\
\leq &3\text{OPT} + 3\delta_1\epsilon\text{OPT}
\end{aligned}$$

Q8.

$$\begin{aligned}\mathrm{cost}(S_2) &\le 2(\delta_1 \mathrm{cost}(S_1) + \delta_2 \mathrm{cost}(S_2)) \\ &\le 2(3 + \delta_1 \epsilon)\mathrm{OPT} \\ &\le 2(3 + \epsilon)\mathrm{OPT}\end{aligned}$$

Q9. The probability should be δ_1. I assume f_1 is not the closest among S_1 to any of the opened facilities in S_2.

Q10. $c(i, f_2) \le c(f_1, f_2)$ because i is the closest facility in S_1 to f_2.

Q11. Using triangle inequality on $c(f_1, f_2)$:
$c(i, j) \le c(j, f_2) + c(f_1, f_2) \le c(j, f_2) + c(f_1, j) + c(j, f_2) = c_j^1 + 2c_j^2$

Q12.

$$\begin{aligned}&\text{Expected cost for client } j \\ \le&\mathrm{Prob}[f_1 \text{ is opened}]c_j^1 + \mathrm{Prob}[f_1 \text{ is not opened}](c_j^1 + 2c_j^2) \\ =&\delta_1 c_j^1 + (1 - \delta_1)(c_j^1 + 2c_j^2) \\ =&\delta_1 c_j^1 + \delta_2(c_j^1 + 2c_j^2)\end{aligned}$$

Q13.

$$\begin{aligned}&\text{Total cost} \\ \le&\sum_{j \in C} \delta_1 c_j^1 + \delta_2(c_j^1 + 2c_j^2) \\ \le&\sum_{j \in C} \delta_1 \alpha_j^1 + \delta_2(\alpha_j^1 + 2\alpha_j^2)\end{aligned}$$