$i=2,\ p=2$ Orthogonal metric tensor

$k=2:\Rightarrow$

$$-\left[2\sigma_1^2 M_{22}\frac{\partial^2 X_2}{\partial S_1^2}+2\sigma_2^2 M_{22}\frac{\partial^2 X_2}{\partial S_2^2}\right]$$

$$-\left[\sigma_1^2\left(2\frac{\partial M_{22}}{\partial X_2}\frac{\partial X_2}{\partial S_1}\frac{\partial X_2}{\partial S_1}-\frac{\partial M_{22}}{\partial X_2}\frac{\partial X_2}{\partial S_1}\frac{\partial X_2}{\partial S_1}\right)\right.$$

$$\left.+\sigma_2^2\left(2\frac{\partial M_{22}}{\partial X_2}\frac{\partial X_2}{\partial S_2}\frac{\partial X_2}{\partial S_2}-\frac{\partial M_{22}}{\partial X_2}\frac{\partial X_2}{\partial S_2}\frac{\partial X_2}{\partial S_2}\right)\right]$$

$$=-2M_{22}\left[\sigma_1^2\frac{\partial^2 X_2}{\partial S_1^2}+\sigma_2^2\frac{\partial^2 X_2}{\partial S_2^2}\right]$$

$$-\frac{\partial M_{22}}{\partial X_2}\left[\sigma_1^2\frac{\partial X_2}{\partial S_1}\frac{\partial X_2}{\partial S_1}+\sigma_2^2\frac{\partial X_2}{\partial S_2}\frac{\partial X_2}{\partial S_2}\right]$$

Central difference

$$X''(s)=X_{i-1}\frac{2}{ds_{i-\frac{1}{2}}(2ds_i)}+X_i\frac{2}{ds_{i-\frac{1}{2}}ds_{i+\frac{1}{2}}}+X_{i+1}\frac{2}{2ds_i\,ds_{i+\frac{1}{2}}}$$

$$=\frac{X_{i-1}}{ds_{i-\frac{1}{2}}ds_i}+\frac{2X_i}{ds_{i-\frac{1}{2}}ds_{i+\frac{1}{2}}}+\frac{X_{i+1}}{ds_i\,ds_{i+\frac{1}{2}}}$$

M_{ij} depends on both x_1 and x_2

$$-\left[2\sigma_1^2 M_{kj}\frac{\partial^2 X_j}{\partial s_1^2}+2\sigma_2^2 M_{kj}\frac{\partial^2 X_j}{\partial s_2^2}\right]$$

$$-\left[\sigma_1^2\left(2\frac{\partial M_{kj}}{\partial X_p}\frac{\partial X_p}{\partial s_1}\frac{\partial X_j}{\partial s_1}-\frac{\partial M_{ij}}{\partial X_k}\frac{\partial X_i}{\partial s_1}\frac{\partial X_j}{\partial s_1}\right)+\sigma_2^2\left(2\frac{\partial M_{kj}}{\partial X_p}\frac{\partial X_p}{\partial s_2}\frac{\partial X_j}{\partial s_2}-\frac{\partial M_{ij}}{\partial X_k}\frac{\partial X_i}{\partial s_2}\frac{\partial X_j}{\partial s_2}\right)\right]$$

$j=1,\ p=1$ Orthogonal metric tensor

$k=1$:

$$-\left[2\sigma_1^2 M_{11}\frac{\partial^2 X_1}{\partial s_1^2}+2\sigma_2^2 M_{11}\frac{\partial^2 X_1}{\partial s_2^2}\right]$$

$$-\left[\sigma_1^2\left(2\frac{\partial M_{11}}{\partial X_1}\frac{\partial X_1}{\partial s_1}\frac{\partial X_1}{\partial s_1}+2\frac{\partial M_{11}}{\partial X_2}\frac{\partial X_2}{\partial s_1}\frac{\partial X_1}{\partial s_1}-\frac{\partial M_{11}}{\partial X_1}\frac{\partial X_1}{\partial s_1}\frac{\partial X_1}{\partial s_1}-\frac{\partial M_{22}}{\partial X_1}\frac{\partial X_2}{\partial s_1}\frac{\partial X_2}{\partial s_1}\right)\right.$$
$$\left.\sigma_2^2\left(2\frac{\partial M_{11}}{\partial X_1}\frac{\partial X_1}{\partial s_2}\frac{\partial X_1}{\partial s_2}+2\frac{\partial M_{11}}{\partial X_2}\frac{\partial X_2}{\partial s_2}\frac{\partial X_1}{\partial s_2}-\frac{\partial M_{11}}{\partial X_1}\frac{\partial X_1}{\partial s_2}\frac{\partial X_1}{\partial s_2}-\frac{\partial M_{22}}{\partial X_1}\frac{\partial X_2}{\partial s_2}\frac{\partial X_2}{\partial s_2}\right)\right]$$

$$=-2M_{11}\left[\sigma_1^2\frac{\partial^2 X_1}{\partial s_1^2}+\sigma_2^2\frac{\partial^2 X_1}{\partial s_2^2}\right]-\frac{\partial M_{11}}{\partial X_1}\left[\sigma_1^2\frac{\partial X_1}{\partial s_1}\frac{\partial X_1}{\partial s_1}+\sigma_2^2\frac{\partial X_1}{\partial s_2}\frac{\partial X_1}{\partial s_2}\right]$$

$$A=-\left[2\sigma_1^2 M_{11}\frac{\partial^2 X_1}{\partial s_1^2}+2\sigma_2^2 M_{11}\frac{\partial^2 X_1}{\partial s_2^2}\right]$$

$$b=\frac{\partial M_{11}}{\partial X_1}\left[\sigma_1^2\frac{\partial X_1}{\partial s_1}\frac{\partial X_1}{\partial s_1}+\sigma_2^2\frac{\partial X_1}{\partial s_2}\frac{\partial X_1}{\partial s_2}\right]+2\frac{\partial M_{11}}{\partial X_2}\left[\sigma_1^2\frac{\partial X_2}{\partial s_1}\frac{\partial X_1}{\partial s_1}+\sigma_2^2\frac{\partial X_2}{\partial s_2}\frac{\partial X_1}{\partial s_2}\right]-\frac{\partial M_{22}}{\partial X_1}\left[\sigma_1^2\frac{\partial X_2}{\partial s_1}\frac{\partial X_2}{\partial s_1}+\sigma_2^2\frac{\partial X_2}{\partial s_2}\frac{\partial X_2}{\partial s_2}\right]$$

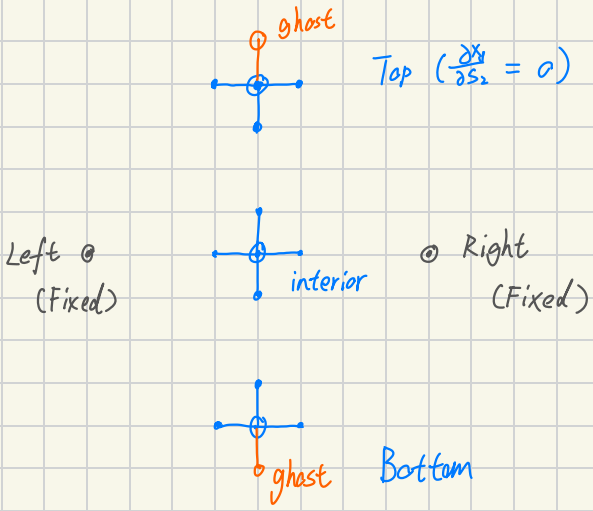
$k=2$:

$$-\left[2\sigma_1^2 M_{22}\frac{\partial^2 X_2}{\partial s_1^2}+2\sigma_2^2 M_{22}\frac{\partial^2 X_2}{\partial s_2^2}\right]$$

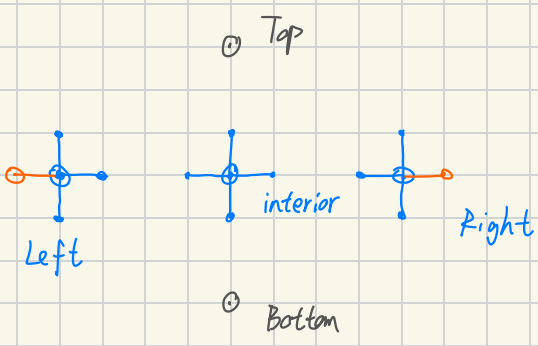
$$-\left[\sigma_1^2\left(2\frac{\partial M_{22}}{\partial X_1}\frac{\partial X_1}{\partial s_1}\frac{\partial X_2}{\partial s_1}+2\frac{\partial M_{22}}{\partial X_2}\frac{\partial X_2}{\partial s_1}\frac{\partial X_2}{\partial s_1}-\frac{\partial M_{11}}{\partial X_2}\frac{\partial X_1}{\partial s_1}\frac{\partial X_1}{\partial s_1}-\frac{\partial M_{22}}{\partial X_2}\frac{\partial X_2}{\partial s_1}\frac{\partial X_2}{\partial s_1}\right)\right.$$
$$\left.\sigma_2^2\left(2\frac{\partial M_{22}}{\partial X_1}\frac{\partial X_1}{\partial s_2}\frac{\partial X_2}{\partial s_2}+2\frac{\partial M_{22}}{\partial X_2}\frac{\partial X_2}{\partial s_2}\frac{\partial X_2}{\partial s_2}-\frac{\partial M_{11}}{\partial X_2}\frac{\partial X_1}{\partial s_2}\frac{\partial X_1}{\partial s_2}-\frac{\partial M_{22}}{\partial X_2}\frac{\partial X_2}{\partial s_2}\frac{\partial X_2}{\partial s_2}\right)\right]$$

$$b=\frac{\partial M_{22}}{\partial X_2}\left[\sigma_1^2\frac{\partial X_2}{\partial s_1}\frac{\partial X_2}{\partial s_1}+\sigma_2^2\frac{\partial X_2}{\partial s_2}\frac{\partial X_2}{\partial s_2}\right]+2\frac{\partial M_{22}}{\partial X_1}\left[\sigma_1^2\frac{\partial X_1}{\partial s_1}\frac{\partial X_2}{\partial s_1}+\sigma_2^2\frac{\partial X_1}{\partial s_2}\frac{\partial X_2}{\partial s_2}\right]-\frac{\partial M_{11}}{\partial X_2}\left[\sigma_1^2\frac{\partial X_1}{\partial s_1}\frac{\partial X_1}{\partial s_1}+\sigma_2^2\frac{\partial X_1}{\partial s_2}\frac{\partial X_1}{\partial s_2}\right]$$

x_1 component stencil



x_2 component stencil



Approximate PDE:

$$-\left[8\sigma_1^4 M_{kl}\frac{\partial X_l}{\partial s_1}M_{ij}\frac{\partial X_i}{\partial s_1}\frac{\partial^2 X_j}{\partial s_1^2}+8\sigma_2^4 M_{kl}\frac{\partial X_l}{\partial s_2}M_{ij}\frac{\partial X_i}{\partial s_2}\frac{\partial^2 X_j}{\partial s_2^2}\right]$$

$$=\left[4\sigma_1^4 M_{kl}\frac{\partial X_l}{\partial s_1}\frac{\partial M_{ij}}{\partial X_p}\frac{\partial X_p}{\partial s_1}\frac{\partial X_i}{\partial s_1}\frac{\partial X_j}{\partial s_1}+4\sigma_2^4 M_{kl}\frac{\partial X_l}{\partial s_2}\frac{\partial M_{ij}}{\partial X_p}\frac{\partial X_p}{\partial s_2}\frac{\partial X_i}{\partial s_2}\frac{\partial X_j}{\partial s_2}\right]$$

$k=1$: $\Rightarrow l=1$

$$LHS=-2\left[\sigma_1^4 M_{11}\frac{\partial X_1}{\partial s_1}\left(M_{11}\frac{\partial X_1}{\partial s_1}\frac{\partial^2 X_1}{\partial s_1^2}+M_{22}\frac{\partial X_2}{\partial s_1}\frac{\partial^2 X_2}{\partial s_1^2}\right)+\sigma_2^4 M_{11}\frac{\partial X_1}{\partial s_2}\left(M_{11}\frac{\partial X_1}{\partial s_2}\frac{\partial^2 X_1}{\partial s_2^2}+M_{22}\frac{\partial X_2}{\partial s_2}\frac{\partial^2 X_2}{\partial s_2^2}\right)\right]$$

$$RHS=\sigma_1^4 M_{11}\frac{\partial X_1}{\partial s_1}\left[\frac{\partial M_{11}}{\partial X_p}\frac{\partial X_p}{\partial s_1}\frac{\partial X_1}{\partial s_1}\frac{\partial X_1}{\partial s_1}+\frac{\partial M_{22}}{\partial X_p}\frac{\partial X_p}{\partial s_1}\frac{\partial X_2}{\partial s_1}\frac{\partial X_1}{\partial s_1}\right]+\sigma_2^4 M_{11}\frac{\partial X_1}{\partial s_2}\left[\frac{\partial M_{11}}{\partial X_p}\frac{\partial X_p}{\partial s_2}\frac{\partial X_1}{\partial s_2}\frac{\partial X_1}{\partial s_2}+\frac{\partial M_{22}}{\partial X_p}\frac{\partial X_p}{\partial s_2}\frac{\partial X_2}{\partial s_2}\frac{\partial X_2}{\partial s_2}\right]$$

$$=\sigma_1^4 M_{11}\frac{\partial X_1}{\partial s_1}\left[\left(\frac{\partial M_{11}}{\partial X_1}\frac{\partial X_1}{\partial s_1}+\frac{\partial M_{11}}{\partial X_2}\frac{\partial X_2}{\partial s_1}\right)\frac{\partial X_1}{\partial s_1}\frac{\partial X_1}{\partial s_1}+\left(\frac{\partial M_{22}}{\partial X_1}\frac{\partial X_1}{\partial s_1}+\frac{\partial M_{22}}{\partial X_2}\frac{\partial X_2}{\partial s_1}\right)\frac{\partial X_2}{\partial s_1}\frac{\partial X_1}{\partial s_1}\right]+\sigma_2^4 M_{11}\frac{\partial X_1}{\partial s_2}\left[\left(\frac{\partial M_{11}}{\partial X_1}\frac{\partial X_1}{\partial s_2}+\frac{\partial M_{11}}{\partial X_2}\frac{\partial X_2}{\partial s_2}\right)\frac{\partial X_1}{\partial s_2}\frac{\partial X_1}{\partial s_2}+\left(\frac{\partial M_{22}}{\partial X_1}\frac{\partial X_1}{\partial s_2}+\frac{\partial M_{22}}{\partial X_2}\frac{\partial X_2}{\partial s_2}\right)\frac{\partial X_2}{\partial s_2}\frac{\partial X_2}{\partial s_2}\right]$$

$$-\left[8\sigma_1^4 M_{kl}\frac{\partial X_l}{\partial s_1}M_{ij}\frac{\partial X_i}{\partial s_1}\frac{\partial^2 X_j}{\partial s_1^2}+8\sigma_2^4 M_{kl}\frac{\partial X_l}{\partial s_2}M_{ij}\frac{\partial X_i}{\partial s_2}\frac{\partial^2 X_j}{\partial s_2^2}\right]$$

$$=\left[4\sigma_1^4 M_{kl}\frac{\partial X_l}{\partial s_1}\frac{\partial M_{ij}}{\partial X_p}\frac{\partial X_p}{\partial s_1}\frac{\partial X_i}{\partial s_1}\frac{\partial X_j}{\partial s_1}+4\sigma_2^4 M_{kl}\frac{\partial X_l}{\partial s_2}\frac{\partial M_{ij}}{\partial X_p}\frac{\partial X_p}{\partial s_2}\frac{\partial X_i}{\partial s_2}\frac{\partial X_j}{\partial s_2}\right]$$

$k=2$: $\Rightarrow l=2$

$$LHS=-2\left[\sigma_1^4 M_{22}\frac{\partial X_2}{\partial s_1}\left(M_{11}\frac{\partial X_1}{\partial s_1}\frac{\partial^2 X_1}{\partial s_1^2}+M_{22}\frac{\partial X_2}{\partial s_1}\frac{\partial^2 X_2}{\partial s_1^2}\right)+\sigma_2^4 M_{22}\frac{\partial X_2}{\partial s_2}\left(M_{11}\frac{\partial X_1}{\partial s_2}\frac{\partial^2 X_1}{\partial s_2^2}+M_{22}\frac{\partial X_2}{\partial s_2}\frac{\partial^2 X_2}{\partial s_2^2}\right)\right]$$

$$RHS=$$

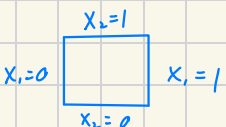
$$=\sigma_1^4 M_{22}\frac{\partial X_2}{\partial s_1}\left[\left(\frac{\partial M_{11}}{\partial X_1}\frac{\partial X_1}{\partial s_1}+\frac{\partial M_{11}}{\partial X_2}\frac{\partial X_2}{\partial s_1}\right)\frac{\partial X_1}{\partial s_1}\frac{\partial X_1}{\partial s_1}+\left(\frac{\partial M_{22}}{\partial X_1}\frac{\partial X_1}{\partial s_1}+\frac{\partial M_{22}}{\partial X_2}\frac{\partial X_2}{\partial s_1}\right)\frac{\partial X_2}{\partial s_1}\frac{\partial X_1}{\partial s_1}\right]+\sigma_2^4 M_{22}\frac{\partial X_2}{\partial s_2}\left[\left(\frac{\partial M_{11}}{\partial X_1}\frac{\partial X_1}{\partial s_2}+\frac{\partial M_{11}}{\partial X_2}\frac{\partial X_2}{\partial s_2}\right)\frac{\partial X_1}{\partial s_2}\frac{\partial X_1}{\partial s_2}+\left(\frac{\partial M_{22}}{\partial X_1}\frac{\partial X_1}{\partial s_2}+\frac{\partial M_{22}}{\partial X_2}\frac{\partial X_2}{\partial s_2}\right)\frac{\partial X_2}{\partial s_2}\frac{\partial X_2}{\partial s_2}\right]$$

$$LHS = -2\Big[\sigma_1^4 M_{kk}\frac{\partial X_k}{\partial S_1}\Big(M_{11}\frac{\partial X_1}{\partial S_1}\frac{\partial^2 X_1}{\partial S_1^2} + M_{22}\frac{\partial X_2}{\partial S_1}\frac{\partial^2 X_2}{\partial S_1^2}\Big) + \sigma_2^4 M_{kk}\frac{\partial X_k}{\partial S_2}\Big(M_{11}\frac{\partial X_1}{\partial S_2}\frac{\partial^2 X_1}{\partial S_2^2} + M_{12}\frac{\partial X_2}{\partial S_2}\frac{\partial^2 X_2}{\partial S_2^2}\Big)\Big]$$

RHS

$$= \sigma_1^4 M_{kk}\frac{\partial X_k}{\partial S_1}\Big[\Big(\frac{\partial M_{11}}{\partial X_1}\frac{\partial X_1}{\partial S_1} + \frac{\partial M_{11}}{\partial X_2}\frac{\partial X_2}{\partial S_1}\Big)\frac{\partial X_1}{\partial S_1}\frac{\partial X_1}{\partial S_1} + \Big(\frac{\partial M_{12}}{\partial X_1}\frac{\partial X_1}{\partial S_1} + \frac{\partial M_{22}}{\partial X_2}\frac{\partial X_2}{\partial S_1}\Big)\frac{\partial X_2}{\partial S_1}\frac{\partial X_1}{\partial S_1}\Big] +$$
$$\sigma_2^4 M_{kk}\frac{\partial X_k}{\partial S_2}\Big[\Big(\frac{\partial M_{11}}{\partial X_1}\frac{\partial X_1}{\partial S_2} + \frac{\partial M_{11}}{\partial X_2}\frac{\partial X_2}{\partial S_2}\Big)\frac{\partial X_1}{\partial S_2}\frac{\partial X_1}{\partial S_2} + \Big(\frac{\partial M_{12}}{\partial X_1}\frac{\partial X_1}{\partial S_2} + \frac{\partial M_{22}}{\partial X_2}\frac{\partial X_2}{\partial S_2}\Big)\frac{\partial X_2}{\partial S_2}\frac{\partial X_2}{\partial S_2}\Big]$$

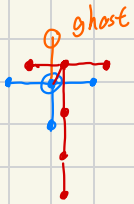
B.C.



$$\frac{\partial X_1}{\partial S_2}\Big|_{X_2=0,1} = 0$$

$$\frac{\partial X_2}{\partial S_1}\Big|_{X_1=0,1} = 0$$

x_1 component stencil



Top ($\frac{\partial x_1}{\partial s_2} = 0$)

Red: x_2 component involved

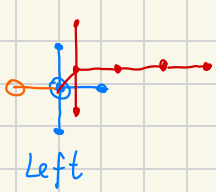
Left (Fixed)

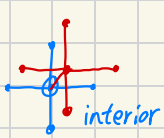
Right (Fixed)

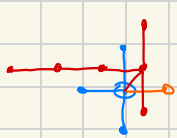
Bottom

x_2 component stencil

Top







Bottom