2. The Words $\mathbb{R}$

In this chapter, we will be talking and learning about our words for (Real) analysis, which is the real numbers $\mathbb{R}$. The words for complex analysis is the complex numbers $\mathbb{C}$, which we will not talk about it here. Introduction to complex analysis course MAT354 takes care of this topic. There are many types of words, and most of the time to learn a language is spent on learning nouns, and here in real analysis, majority of the time will also be about *nouns*. As we will see, the nouns for $\mathbb{R}$ are the rationals, $\mathbb{Q}$. There are also many types of nouns such as $\mathbb{Z}$ and $\mathbb{N}$, which we will introduce in this chapter. In mathematical language, we associate emotions with words. We will also talk about emotions in this chapter, so that we can begin our dialogue(conversation) in the next one.
In this chapter, we will start talking about building blocks of analysis, in fact of all mathematics, which are logics and proofs, and we will end off the chapter with a specific proof technique of mathematical induction which gives properties of our specific nouns $\mathbb{N}$ and an official definition of our words $\mathbb{R}$.

2.1 Logics and Proofs

This section might be a little boring with symbols and abstract logic, however it serves as the foundation for all analysis. In fact, both math language and our natural language can be thought as a formal system, where the "rules" of this system is the *grammar of the language*. A *proof* in math language intuitively is a paragraph written using math words, but it must follow the correct grammar of mathematical language, which is *logic*.
In this section, we aim to be able to understand the logic behind the proofs, and throughout the notes, I am describing the natural language analogy to build intuition behind almost all proofs. With good intuition and clear logic flows, successful proofs come out of good understanding of mathematics and the beauty of math arises from these proofs.

2.1.1 Implications and Negations

In this section, we will first explain the mathematical equivalent of the "if...then..." sentence, which is implication and the symbol $\Rightarrow$, $\Leftarrow$ and $\iff$. Before this, we need to understand that a **boolean variable** is a variable that can only have value "True" or "False", and a **logic statement** is a statement consisting of math symbols, numbers and boolean variables that can only be evaluated "True" or "False". We will often write "statement" instead of "logic statement".

Definition 2.1.1 We denote $A \Rightarrow B$ to represent "if A then B". Here, A, B are boolean variables or logic statements. The statement $A \Rightarrow B$ is true *precisely when* A is False or B is True. Similarly, the statement $A \Leftarrow B$ is equivalent to $B \Rightarrow A$. The statement $A \iff B$ is True only when both $A \Rightarrow B$ and $B \Rightarrow A$ are True. We sometimes call A and B to be **equivalent** if $A \iff B$ is True.

Example 2.1 — Implications.

- The statement "If U of T is easy, then $\sqrt{2}$ is irrational." is True since in this case $A =$"U of T is easy", which is *False*. However, this is *not a valid proof* of $\sqrt{2}$ being irrational, as we will see later on.
- The statement "If U of T is hard, then $\sqrt{2}$ is rational" is False since in this case although $A =$"U of T is hard" is True, $\sqrt{2}$ is not rational.

R We can have a *chain* of *same direction implications*, for example $A_1 \Rightarrow A_2, A_2 \Rightarrow A_3, \ldots$. In this case, we can simply notation and write $A_1 \Rightarrow A_2 \Rightarrow A_3 \Rightarrow \ldots$.

Next, we will talk about the mathematical equivalence of the word "not", which is negations.

Definition 2.1.2 A negation is a logical statement of the form "not A", denoted $\neg A$, is true *only when* A is False.

Exercise: Verify the following statements are True:

$$(A \Rightarrow B) \iff (\neg B \Rightarrow \neg A)$$

$$\neg(\neg A) \iff A$$

R A good method of verifying two logic statements to be equivalent is to compare the cases when they are each True/False, and verify that they are both True or they are both False.

2.1.2 Conjunctions and Disjunctions

In this section, we will describe two important logical connections – conjunctions and disjunctions. This is the mathematical equivalence of "and" and "or", respectively.

Definition 2.1.3 We use A AND B to denote the logical conjunction, which is True *only when* A and B are both True. We use A OR B to denote the logical disjunction, which is True if one of A or B is True, or if both of them are True.

Exercise: Verify that the following logic statements are True (**De Morgan's Laws**):

$$\neg(A \text{ OR } B) \iff \neg A \text{ AND } \neg B.$$
$$\neg(A \text{ AND } B) \iff \neg A \text{ OR } \neg B.$$

■

Exercise: Verify that the following logic statement is True:

$$(A \Rightarrow B) \iff \neg A \text{ OR } B$$

■

2.1.3 Quantifiers

In this section, we are going to talk about two important quantifiers, **existential quantifier** (which corresponds to "there exists") and **universal quantifier** (which corresponds to "for all"). Using quantifiers and previously described logic symbols, we can form comprehensive logical statements used in mathematical proofs.

Definition 2.1.4 We use symbol $\exists$ to denotes that "there exists" and the symbol $\forall$ to denote "for all". Both quantifiers need to combine with sets (called the **domains** of the quantifiers) to make sense. A statement of $\exists$ is True if there is an element in the domain that makes the statement True. A statement of $\forall$ is True if every element in the domain makes the statement True.

■ **Example 2.2 — Quantifiers.** ■

- Consider the statement $\forall x \in \mathbb{R}.\exists a \in \mathbb{Z}.a < x$. This statement is True only when for x being any real number, we can find a specific integer a smaller than this x. The choice of *a can depend* on the value of x.
- Consider the statement $\exists x \in \mathbb{Z}.\forall a \in \mathbb{R}^+.|x| < |a|$. This statement is True only when we can find an integer x such that any positive real number a satisfies $|x| < |a|$. The value of x does not depend on any value of a.

Exercise: Verify that the following statements are True (A is a domain and B is a statement):

$$\neg(\forall x \in A.B.) \iff \exists x \in A.\neg B.$$
$$\neg(\exists x \in A.B.) \iff \forall x \in A.\neg B.$$

■

After the above exercises, you can get a complete understanding of how to negate a specific logic statement. There are several steps to this process, including:

1. Change any implications into conjunctions and disjunctions by definition and the second exercise in 2.1.2.
2. Distribute $\neg$ inward, changing conjunctions and disjunctions accordingly by De Morgan's Laws, and swap $\exists$ and $\forall$ by the above exercises.
3. When there is double $\neg$, remove them, according to previous exercises in 2.1.1.

■ **Example 2.3** We would like to change the statement

$$A = \text{"}\forall \varepsilon > 0.\forall x < 1.\exists N \in \mathbb{N}.\forall n \in \mathbb{N}.n > N \Rightarrow x^n < \varepsilon.\text{"}$$

to $\neg A$.

1. Firstly, we rewrite A to be:
$$A = \text{"}\forall \varepsilon > 0.\forall x < 1.\exists N \in \mathbb{N}.\forall n \in \mathbb{N}.[\neg(n > N) \text{ OR } (x^n < \varepsilon)].\text{"}$$
2. Next, we distribute $\neg$ inward, to get:
$$\neg A = \text{"}\exists \varepsilon > 0.\exists x < 1.\forall N \in \mathbb{N}.\exists n \in \mathbb{N}.\neg(\neg(n > N)) \text{ AND } \neg(x^n < \varepsilon).\text{"}$$
3. Finally, we negate the $<$ and $>$, as well as removing double $\neg$ and get:
$$\neg A = \text{"}\exists \varepsilon > 0.\exists x < 1.\forall N \in \mathbb{N}.\exists n \in \mathbb{N}.n > N \text{ AND } x^n \geq \varepsilon.\text{"}$$

■

2.1.4 Proofs, finally

A proof is describing a *logical flow* to show that a statement is True via a *fixed set* of True statements. This is the most important technique we use to establish new math results (to add to the set of True statements) and to practise mathematical language. There are various types of proof techniques, and here in this chapter, we will only describe the basic ones. We will start by the most standard proof technique – *proof by logical deduction*.

Definition 2.1.5 A **logical deduction** is the method of showing a statement Y is True by showing that X is *True* and the statement $X \Rightarrow Y$ is *True*.

Let us denote the statement we would like to prove G. Proof by logical deduction is a proof technique that starts with known-True statement A in the fixed set of True statements (this can be axioms, definitions or proved theorems, propositions and statements) and use a series of logical deductions, to finally show that G is True.

■ **Example 2.4** Prove that if $\mathbb{F}$ is a field, then we have **cancellation law:** if $a \cdot b = c \cdot b$ where $a, b, c \in \mathbb{F}$, $b \neq 0 \in \mathbb{F}$, then $a = c$.
Proof. We will start with the known-True statement in the question, which is $a \cdot b = c \cdot b$.
Then, by properties of equality, we know that if $x = y$, then $x \cdot z = y \cdot z$ for any z. This allows us to make the logical deduction to show that:

$$a \cdot b \cdot b^{-1} = c \cdot b \cdot b^{-1}.$$

(Of course this also uses the fact that $b^{-1} \in \mathbb{F}$ exists since $\mathbb{F}$ is a field, which is a more subtle logical deduction.)
Then, we can use the associative law of field $\mathbb{F}$ and properties of equality to show that:

$$a \cdot b \cdot b^{-1} = a \cdot (b \cdot b^{-1}) = a \cdot e = a = c \cdot b \cdot b^{-1} = c \cdot (b \cdot b^{-1}) = c \cdot e = c.$$

(Above is another logical deduction.)
Finally, we have shown that $a = c$, which is the statement we would like to prove (G), hence above completes this proof. □

■

Next, we will talk about another very important technique of proof - *proof by contradiction*.

Definition 2.1.6 A **proof by contradiction** is an argument of showing that a statement G is True through assuming $\neg G$ is True, and prove $\neg Y$ is True via *logical deduction* for some known True statement Y in the fixed set of True statements. The fact that $\neg Y$ is proved and Y is True is called a **contradiction**.

R Intuitively, this is just assuming the statement we are proving to be False, then find a contradiction with a known True statement. Then, this implies that the statement we are proving is not False, hence True.

■ Example 2.5 ■

Prove that $\sqrt{2}$ is irrational (i.e. $\sqrt{2}$ cannot be written in the form of $\frac{p}{q}$ for any integers $p,q \in \mathbb{Z}$.)

Proof. Suppose that $\sqrt{2}$ is not irrational, i.e. $\sqrt{2} = \frac{p}{q}$ for some $p,q \in \mathbb{Z}$. (First step in proof by contradiction). We can assume that they are reduced into lowest terms, i.e. p,q have no common factors (since if they have common factors, we can further reduce them until they don't).

Then, by properties of equality, we know that $\sqrt{2}\cdot q = \frac{p}{q}\cdot q = p$. Next, by properties of equality again, we can square both sides of the equality and get:

$$2q^2 = p^2.$$

Next, we would like to form contradiction with the help of even/odd properties statements that are known to be True. Firstly, since $2q^2$ is even, $2q^2 = p^2$, p^2 is even. Since the square of an odd integer is odd, the only way for p^2 to be even is that p is even. By definition of "even", we can write $p = 2k$ for some $k \in \mathbb{Z}$, then we know that $2q^2 = p^2 = (2k)^2 = 4k^2$, hence by properties of equality, we can divide 2 on both sides and get:

$$q^2 = 2k^2.$$

With similar arguments as above, we know that q^2 is even, thus q is even. However, p,q are both even suggests that they have a common factor 2.

However, at the start of the proof, we assumed that p,q have no common factors, this gives us a contradiction. Therefore, we have proved by contradiction that $\sqrt{2}$ is irrational. $\square$

The third proof technique that we will talk about in this section is proof by cases, which is very useful if adding in additional property assumptions (i.e., cases), the statement of interest is easier to prove.

Definition 2.1.7 A **proof by cases** is a proof for a statement of interest involving elements from a specific domain D, given by first choose finitely many subsets $D_1, D_2, \ldots, D_n \subset D$ such that $D_1 \cup D_2 \cdots D_n = D$. Then, for each specific subset, prove the statement using proof by logical deduction or contradiction. Each subset $D_1, D_2, \ldots$ are called **cases**.

This proof technique becomes very intuitive once we understand an example.

■ Example 2.6 ■

Prove that for all $a, b \in \mathbb{R}$, the absolute value $|ab| = |a||b|$. The definition of the absolute value is that for all $x \in \mathbb{R}$, $|x| = x$ if $x \geq 0$, and $|x| = -x$ otherwise.
Proof. We will consider four cases and do a proof by cases in the following way:
Case 1. $a, b \geq 0$.
Then, we know that $ab \geq 0$, thus $|ab| = ab = |a||b|$ by definition of absolute value. Hence, the statement is proved for Case 1.
Case 2. $a \geq 0, b < 0$.
Then, we know that $ab \leq 0$, thus $|ab| = -ab = a(-b) = |a||b|$ by definition of absolute value. Hence, the statement is proved for Case 2.
Case 3. $a < 0, b \geq 0$.
Similar to Case 2, we know that $ab \leq 0$, $|ab| = -ab = |a||b|$ by definition of absolute value. Hence, the statement is proved for Case 3.
Case 4. $a < 0, b < 0$.
In this case, we know that $ab > 0$, thus $|ab| = ab = (-a)(-b) = |a||b|$ by definition of absolute value. Hence, the statement is proved for Case 4.
Then, since the four cases above cover every possible case for $a, b \in \mathbb{R}$, we know that $|ab| = |a||b|$ is proved via a proof by cases. $\square$

The proof by logical deduction and proof by contradiction are two classic proof techniques that are used frequently. Often times, when the domain of proof is too "general", we can break it down into cases and use logical deduction and contradiction to prove each case separately. This constitutes a proof by cases. There is another well-known and important proof technique - *proof by mathematical induction*. However, that depends on a bit more knowledge about the math language, especially the words that we are speaking.

2.2 Nouns

As I explained before, learning nouns are a core part of studying and understanding a language. In real analysis, we will treat $\mathbb{R}$ as the whole universe of words, and $\mathbb{Q}$, the rationals are the nouns here. Let us start by understanding various type of nouns and then understand what $\mathbb{Q}$ is.

2.2.1 The Proper Nouns

In natural language, proper nouns are nouns that represent specific *people, places and things.* In our daily language, they are often the items with lots of functionalities, memories or knowledge associated. In mathematics language, the proper nouns are very "nice" numbers in the universe of nouns $\mathbb{Q}$.

Definition 2.2.1 The **integers** $\mathbb{Z}$, or the *proper nouns* in the math language is a set of numbers containing the multiplicative identity $1 \in \mathbb{R}$, the additive identity $0 \in \mathbb{R}$, all additive multiples of 1 and their additive inverses. The notation is to denote $2 = 1 + 1$, $3 = 1 + 1 + 1$, $\ldots$ as we normally use. We use $-x$ to denote the additive inverses of x for all $x \in \mathbb{R}$. Thus, $\mathbb{Z} = \{\ldots, -3, -2, -1, 0, 1, 2, 3, \ldots\}$.

R $\mathbb{Z}$ has the same addition and multiplication defined for $\mathbb{R}$, and in section 1, we have already showed that $\mathbb{Z}$ is a *ring* but not a **field** since it does not have multiplicative inverses.

With the definition of the "proper nouns" $\mathbb{Z}$, we can define another very important type of nouns - the natural numbers $\mathbb{N}$.

Definition 2.2.2 The set of **natural numbers**, denoted $\mathbb{N}$ is the set of all additive multiples of $1 \in \mathbb{R}$, i.e. $\mathbb{N} = \{1, 2, \ldots\}$.

Based on this definition, we can see that the set of natural numbers is a subset of the integers $\mathbb{Z}$ but is not even a group since we do not include 0 and additive inverses. However, with this definition, we are able to find very important properties of the natural numbers, and propose another proof technique later on. After our discussion of emotions and ordered fields, we can also form intuition on $\mathbb{N}$.

2.2.2 All Nouns from Proper Nouns

With the definition, or rather classification of proper nouns $\mathbb{Z}$ from all words, we can finally classify the universe of nouns, namely $\mathbb{Q}$.

Definition 2.2.3 A number $x \in \mathbb{R}$ is a *mathematical noun* (or we usually call it **rational number** if $x = \frac{p}{q}$ for some $p, q \in \mathbb{Z}$. We denote the set of all rational numbers $\mathbb{Q}$, and they are all nouns of the math language. q in $x = \frac{p}{q}$ is called the **denominator** of x, and p is called the **numerator** of x.

R We have briefly discussed that $\mathbb{Q}$ with the same addition and multiplication defined as in $\mathbb{R}$ is also a field. This can be shown since for every $x = \frac{p}{q} \in \mathbb{Q}$, $x^{-1} = \frac{q}{p}$. We say $\mathbb{Q}$ is a **subfield** of $\mathbb{R}$.

With the definition, we notice that there exists multiple ways to write a rational number x as the quotient of two integers. For example, $\frac{3}{4} = \frac{9}{12} = \frac{12}{16} = \ldots$. In the proof that $\sqrt{2}$ is irrational before, we briefly talked about the concept of "*rational in lowest terms*". We will now make it formal.

Definition 2.2.4 A rational $x \in \mathbb{Q}$ is said to be written **in lowest terms** $\frac{p}{q}$ $(p, q \in \mathbb{Z})$ if $x = \frac{p}{q}$ and for any $p', q' \in \mathbb{Z}$ such that $x = \frac{p'}{q'}$, we know that $q' \geq q$.

R In the above definition, we are saying that $x = \frac{p}{q}$ is written in lowest terms if $\frac{p}{q}$ is the way to write x into quotients of integers for q to be the *smallest*.

2.3 Emotions

In fact, the most important intuition to have about the mathematical language used in analysis is the *"emotions"* of words. In the current real analysis domain, our universe of words is $\mathbb{R}$, our nouns are $\mathbb{Q}$ which are generated from proper nouns $\mathbb{Z}$. In English, we often have subtle emotions associated with each noun, "table", "professor", "mathematics", "U of T", "The Cows", or "Tim Hortons".

We put them into conversations where emotions move in. In fact, every math word has emotions associated with it and in this section, we will make sense of the idea of "emotion" in math words. Before we try to understand the strength of emotions and compare emotions of words, we first need to gain the sense of **order**. More generally, we will talk about the notion of **order** on *general fields*.

Definition 2.3.1 An **ordered field** is a field $\mathbb{F}$ with the notion of the set of positive elements P and the set of negative elements N, such that it satisfies three properties:
- $P \cap N = \emptyset$; $0 \notin P, N$; $P \cup N \cup \{0\} = \mathbb{F}$, where 0 is the additive identity of $\mathbb{F}$.
- For all $a, b \in P$, $a + b \in P$.
- For all $a, b \in P$, $ab \in P$.

With the notion of "order" defined as above, we can finally make *concrete* of the notion of "less than" and "greater than".

Definition 2.3.2 For all $a, b \in \mathbb{F}$ which is an ordered field, we say $a > b$ (a is greater than b) if $a - b \in P$, $a < b$ (a is less than b) if $a - b \in N$ and $a = b$ if $a - b = 0$.

R We often use the notation of $a \geq b$ to represent "$a > b$ OR $a = b$" is True, and $a \leq b$ likewise.

There are various properties of order and some of them are listed below. Proofs are left as exercises since almost all the properties directly follow from definition of order, or previously proved properties.

Proposition 2.3.1 For any $a, b, c, d \in \mathbb{F}$, which is an ordered field, we have:
- Exactly one of $a > b$, $a < b$, $a = b$ is True.
- $a > b \iff b < a$
- $a > b \iff a + c > b + c$
- $a > b, c > 0 \Rightarrow ac > bc$.
- $a > b, c < 0 \Rightarrow ac < bc$.
- $a > 0 \iff a^{-1} > 0$
- $a > 0 \iff -a < 0$
- $a \neq 0 \iff a^2 > 0$
- $a > b > 0 \iff b^{-1} > a^{-1} > 0$
- $a > b, c > d \Rightarrow a + c > b + d$
- $a > b > 0, c > d > 0 \Rightarrow ac > bd$.
- If $x \geq 0, y \geq 0$, $x \geq y \iff x^2 \geq y^2$.
- $1 > 0$. (*Hint:* Consider $1 = 1^2$)
- $\mathbb{F}$ cannot have finitely many elements. *Hint: consider 1, 1 + 1, 1 + 1 + 1, ...*

Theorem 2.3.2 $\mathbb{R}$, with the definition of **positive real numbers** $\mathbb{R}^+ = \{x \in \mathbb{R} : x > 0\}$ and **negative real numbers** $\mathbb{R}^- = \{x \in \mathbb{R} : x < 0\}$ is an **ordered field**.

This theorem is very easy to verify by the definition of ordered fields, and I need to remark here that the of "order" for $\mathbb{R}$ is purely a **definition**, as we always know, such as $1.6 > 1$, $3 < 4.2$, ...

In the mathematics language, the presence of emotion is often made explicit, in fact, the ways to measure the emotions of mathematical words can be varied and are often defined explicitly (depending on the order). This gives rise to the definition of **norm**, which is the measurement of emotions for each word. Below is rather an informal definition of a *norm* on $\mathbb{R}$, a formal one will be given once one understands "conversation".

Definition 2.3.3 A **norm** on math words $\mathbb{R}$ is an assignment of emotion value in $\mathbb{R}$ for each word $x \in \mathbb{R}$, denoted $|x|$ such that it satisfies the following three properties:

- The emotion value of the word 0 is 0, i.e., $|0| = 0$. Also, the only word with emotion value of 0 is 0, i.e. $|x| = 0 \Rightarrow x = 0$ is True.
- The emotion value of a word produced by multiplication is the multiplication of the emotional values, i.e. $|ab| = |a||b|$ for all $a, b \in \mathbb{R}$.
- The emotion value of a word produced by addition is less than or equal to the addition of the emotional values, i.e. $|a+b| \leq |a| + |b|$ for all $a, b \in \mathbb{R}$.

R The second and third property of a norm can be understood in the following way: when multiplying two words, the emotions do not have a "cancel-out" effect but when adding two words together, the result word might have less emotion value than the emotional value of two words added together, which is a *cancel-out effect*.

Example 2.7 — Norms.

- The first, and the simplest norm is the following one, where:

$$|x| = \begin{cases} 0 & \text{if } x = 0, \\ 1 & \text{otherwise.} \end{cases}$$

This is indeed a norm, which is not hard to verify.

With the definition of order, The most important norm for $\mathbb{R}$ is the **Euclidean norm**, which can be defined in the following way with the use of $>$ and $<$. For all $x \in \mathbb{R}$,

$$|x| = \begin{cases} x & \text{if } x \geq 0, \\ -x & \text{otherwise.} \end{cases}$$

It is simple to verify that the Euclidean norm satisfies the first property of norm, and in fact the proof of the second and the third property is not as straightforward. It is rather straight forward to see that the definition of Euclidean norm is in fact the definition of **absolute value**. From example 2.6, we can then verify the second property. We will prove the third property in the following Theorem.

Theorem 2.3.3 — Euclidean Norm. The Euclidean Norm satisfies three properties of norm.

Proof. The first property is very direct from the definition of Euclidean Norm, and will be omitted here as an exercise.

The proof of the second property is provided in example 2.6 via proof by cases.

While the third property can also be proved using proof by cases, we will rather prove it in a simpler way. By the property of Euclidean norm, we know that $|x|^2 = x^2$ for all $x \in \mathbb{R}$. Thus, for any $a, b \in \mathbb{R}$, $|a+b|^2 = (a+b)^2 = a^2 + 2ab + b^2$. Also, we have:

$$(|a| + |b|)^2 = |a|^2 + 2|a||b| + |b|^2 = a^2 + 2|a||b| + b^2.$$

Next, it is easy to verify based on the definition of Euclidean norm that $|a||b| \geq ab$ (can use proof by cases, left as an exercise). We then know that, by properties of inequalities:

$$(|a|+|b|)^2 = a^2 + 2|a||b| + b^2 \geq a^2 + 2ab + b^2 = (|a+b|)^2.$$

Next, since $|a|+|b| \geq 0$, $|a+b| \geq 0$, we can conclude that $|a|+|b| \geq |a+b|$, which is the statement we would like to prove. $\square$

R In fact, the notion of "Euclidean norm" (i.e. "absolute value") can be defined for all ordered fields. This is the emotional measurement that we use for our words $\mathbb{R}$, representing the "emotions" of $\mathbb{R}$.

With the definition of Euclidean norm, we have actually another more intuitive way to define types of emotions, such as *positive* and *negative* emotions for $\mathbb{R}$.

Definition 2.3.4 We call **positive real numbers**, denoted $\mathbb{R}^+$ to be the set of nonzero real numbers (words) with emotion value equal to their own value under Euclidean norm, i.e. $\{x \in \mathbb{R} \setminus \{0\} : |x| = x\}$, and **negative real numbers** denoted $\mathbb{R}^-$ to be the set of nonzero real numbers (words) with own value being the additive inverse of emotion value under Euclidean norm, i.e. $\{x \in \mathbb{R} \setminus \{0\} : |x| = -x\}$.

R We can easily see that based on the definition of Euclidean norm, $\mathbb{R}^+ = \{x \in \mathbb{R} : x > 0\}$ and $\mathbb{R}^- = \{x \in \mathbb{R} : x < 0\}$. With this definition, we can see that the set $\mathbb{N}$ is the set of *positive integers*, which intuitively can be seen as the "*proper nouns with good (positive) emotions*".

There are some other properties of Euclidean norm which can be easily verified with the definition of ordered fields, or norms. We will end this subsection by listing some of them below (proofs omitted as exercises):

Proposition 2.3.4 In $\mathbb{R}$ with Euclidean norm $|\cdot|$, we have:
m$a \neq 0 \iff |a| > 0$. $a^2 = |a|^2$. $|a| = |-a|$. $|a^{-1}| = |a|^{-1}$.

2.4 Well-ordering and Mathematical Induction

In previous subsections, we have described various categories of nouns, and with the help of emotions and the notion of "norm" as the measurement of emotions, we gain intuitive understanding of some more categories of nouns. In the end of this chapter, we will focus on $\mathbb{N}$, since there are very important properties of $\mathbb{N}$ and also since it gives rise to another important proof technique called *mathematical induction*.

The other reason for studying $\mathbb{N}$ is that natural numbers appear naturally in our daily lives as "counting", and appears naturally in mathematical language as "sequences" and "series", which we will study later on.

Let us first begin by talking about some elementary properties of the natural numbers:

Theorem 2.4.1 Properties of $\mathbb{N}$:
- For all $m,n \in \mathbb{N}$, $m < n \iff n-m \in \mathbb{N}$.
- For all $n \in \mathbb{N}$, there are no natural numbers m such that $n < m < n+1$.

Proof. We will prove these properties one by one.
- Consider any $m,n \in \mathbb{N}$. We know by definition that $m,n \in \mathbb{Z}$. Hence, since $\mathbb{Z}$ is a group under addition, $n-m \in \mathbb{Z}$. Since $m < n$, $n-m > 0$, thus by definition, $n-m \in \mathbb{N}$.
- Suppose by contradiction that there exists $m \in \mathbb{N}$ such that for some $n \in \mathbb{N}$, $n < m < n+1$. Then, we know that $0 < m < 1$ by properties of inequalities, however there are no integers between 0 and 1 by definition, hence no natural numbers, which gives us a contradiction. □

We will prove other stronger properties of $\mathbb{N}$ by first develop a very important proof technique specifically used for $\mathbb{N}$ called **proof by mathematical induction**. Before we go into details about this proof technique, we first need to understand what *"induction"* means. We will begin by defining inductive subsets of $\mathbb{R}$.

Definition 2.4.1 An **inductive subset** $S \subset \mathbb{R}$ is a set satisfying the property that:
- $1 \in S$.
- $x \in S \Rightarrow x+1 \in S$ is True for all $x \in \mathbb{R}$.

Theorem 2.4.2 $\mathbb{N}$ is the smallet inductive subset of $\mathbb{R}$, i.e. for all inductive subsets $S \subset \mathbb{R}$, $\mathbb{N} \subset S$.

Proof. Let S be any inductive subset of $\mathbb{R}$. Then, we know that $1 \in S$, and $x \in S \Rightarrow x+1 \in S$ is True for all $x \in \mathbb{R}$. Since $1 \in S$, we know that $2 \in S$, then $3 \in S$, and thus all additive multiples of 1 is in S. Since all additive multiples of 1 are positive integers, by definition of $\mathbb{N}$, we know that $\mathbb{N} \subset S$. □

From above, we not only know that $\mathbb{N}$ is inductive, but also $\mathbb{N}$ is the smallest subset that is inductive. With this property of $\mathbb{N}$, we can actually mimick the definition of inductive subsets to develop a way to *prove* properties of $\mathbb{N}$.

Definition 2.4.2 A **proof by mathematical induction** is a proof technique specifically used to prove properties of $\mathbb{N}$. The proof first defines a statement $P(n)$ that we would like to prove for all $n \in \mathbb{N}$, then show $P(1)$ is true, and the statement "$P(n) \Rightarrow P(n+1)$" is True for all $n \in \mathbb{N}$.

The above definition being a *valid proof technique* can be shown by the inductive property of $\mathbb{N}$. Let $T = \{n \in \mathbb{N} : P(n) \text{ is True.}\}$. Then by definition of T, we know that $T \subset \mathbb{N}$.
Then, if we have successfully done a proof by mathematical induction, we know that $1 \in T$, $x \in T \Rightarrow x+1 \in T$ is True for all $x \in \mathbb{N}$. Since for all $x \in \mathbb{R} \setminus \mathbb{N}, x \notin T$, thus the statement $x \in T \Rightarrow x+1 \in T$ is True. Hence, by definition, T is inductive, and thus $\mathbb{N} \subset T$, hence we know that $T = \mathbb{N}$, i.e. $P(n)$ is True for all $n \in \mathbb{N}$.

R The proof by induction structure sometimes can be used to prove properties for some other sets as well, for example to prove something for all even numbers, we can start by proving $P(0)$, and then show $P(n) \Rightarrow P(n+2)$ is True for all n even. To prove something for all odd numbers, we can start by proving $P(1)$ and then show $P(n) \Rightarrow P(n+2)$ is True for all n odd. We can also prove statements for $\mathbb{Z}$ by proving $P(0)$, then show $P(n) \Rightarrow P(n+1)$ is True for all $n \in \mathbb{Z}, n \geq 0$ and show $P(n) \Rightarrow P(-n)$ is True for all $n \in \mathbb{Z}, n \geq 0$.

Example 2.8 — Mathematical Induction.

Prove that for all $n \in \mathbb{N}$, $\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$.

Proof. It is very straight-forward for us to notice that the statement we are trying to prove here is about a property for $\mathbb{N}$. In this case, $P(n)$ corresponds to the statement "$\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$".

Our first step is to show $P(1)$. For $n = 1$, we know that $\sum_{i=1}^{1} i = 1 = \frac{1(1+1)}{2}$, thus $P(1)$ is True by simple calculation.

Our next step is to show that $P(n) \Rightarrow P(n+1)$ is True for all $n \in \mathbb{N}$, which requires us to assume $P(n)$ being True and prove $P(n+1)$. Let n be any natural number, and assume $P(n)$ is True. Then, we know that $\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$.

We also know that $\sum_{i=1}^{n+1} i = \sum_{i=1}^{n} i + n + 1 = \frac{n(n+1)}{2} + n + 1 = \frac{(n+2)(n+1)}{2} = \frac{(n+1)(n+1+1)}{2}$.

Thus, we have verified that "$P(n) Rightarrow P(n+1)$" is True. Therefore, by proof of mathematical induction, $P(n)$ is True for all $n \in \mathbb{N}$. ☐

R In the proof above, and in any proof by mathematical induction, we call the assumption that "$P(n)$ is correct" **the inductive hypothesis**.

There are a lot of amazing properties about natural numbers, and later on about polynomial rings (MAT347Y1) that can be proved with mathematical induction. I will try to provide an exercise sheet regarding properties of a very interesting polynomial called binomial, and their coefficients.

With the tool of mathematical induction, we can go on to prove an extremely useful and important result for $\mathbb{N}$ called the **well-ordering principle**.

Theorem 2.4.3 The **well ordering principle** states that for every nonempty subset $S \subset \mathbb{N}$, S contains a minimum element, i.e. there exists $s \in S$ such that for all $x \in S$, $s \leq x$.

Actually, the well-ordering principle is very intuitive, since the natural number represents proper nouns with good/positive emotions, for any nonempty subset of proper nouns with good/positive emotions, we might not have a noun with the *best* positive emotion, but we definitely have one with the *worst* positive emotion, that is the "minimum" in this subset.

Proof. This elegant proof combines both proof by contradiction and proof by mathematical induction. We first suppose we can find an nonempty $S \subset \mathbb{N}$ such that S does not have a smallest element.

The beautiful part is where induction comes in. We ask the question "What element is not in S?" Let $P(n)$ be the statement that for all $k \in \mathbb{N}, k \leq n$, $k \notin S$. Then, we know that $P(1)$ is True since if $1 \in S$, and $S \subset \mathbb{N}$, S has a smallest element since 1 is indeed the smallest element of $\mathbb{N}$ by definition.

Next, let us suppose $P(n)$ is true for any $n \in \mathbb{N}$. Then we know that for all $k \in \mathbb{N}, k \leq n$, $k \notin S$. Now, can $n+1 \in S$ be True? No, because if so, since $n+1$ is the next natural number after n, and we know that all natural numbers from 1 up to n are not in S, $n+1$ would then be the smallest natural number

in S, which is impossible. Therefore, $n+1 \notin S$, hence for all $k \in \mathbb{N}, k \leq n+1$, $k \notin S$, $P(n+1)$ is True.
Hence, by mathematical induction, we know that $P(n)$ is True for all $n \in \mathbb{N}$, i.e., for all $n \in \mathbb{N}, n \notin S$. However, since $S \subset \mathbb{N}$, this imples $S = \emptyset$, which forms the contradiction. Therefore, S has a smallest element. □

R We can see that the above induction hypothesis is carefully cooked up in the way that instead of assuming for a single $n \in \mathbb{N}$, we in fact assumed that $P(k)$ is True for all $k \in \mathbb{N}, k \leq n$. This technique is a sub-technique of proof by mathematical induction, often called **proof by strong mathematical induction**, where we first prove $P(1)$, then prove that $P(1)$, AND$\ldots$,AND $P(n) \Rightarrow P(n+1)$ is True. This is a valid proof technique to use since it is in fact using mathematical induction, with a little cleverness in the inductive hypothesis. This proof technique is sometimes useful and easier to prove properties of $\mathbb{N}$.

2.5 The Final Axiom

In this chapter, we aim to fully study the basics of the words $\mathbb{R}$. From Chapter 1, we know that $\mathbb{R}$ is **defined to be a field** with addition and multiplication. In this chapter, we also know that $\mathbb{R}$ is **defined to be an ordered field**. In fact, to make it easier to talk about and understand conversations, we need to define another intuitive property of emotions called **completeness**.
To make it simple, *completeness* simply means that for every set of words from $\mathbb{R}$ that is "emotionally bounded", their emotions approaches both upwards and downwards to some emotion values that are unique. This is just very intuitive, since we can just "order" the words and "figure out" the approaching emotional value. However, this cannot be formulated as a Theorem, because all previous definitions and properties of $\mathbb{R}$ do not deal with the idea of "approaching", hence it is impossible to prove this idea (Feel free to try it out, but you will get into trouble talking about "approaching").

First of all, let us make clear of the concept of *"approaching"* by defining "*emotional bounds*" for any ordered field $\mathbb{F}$.

Definition 2.5.1 For any ordered field $\mathbb{F}$, A set $S \subset \mathbb{F}$ is called **bounded** (above) if there exists an $a \in \mathbb{F}$ such that for all $s \in S$, $s \leq a$. **Bounded below** is defined similarly. In this case, a is called an **upper bound** of S. **Lower bound** can be defined similarly as well.

Example 2.9 — Bounds.

Let the ordered field be $\mathbb{R}$.
- Let $S = \{1,2,3\} \subset \mathbb{R}$. It is very easy to see that $3, 3.1, 4, 50$ are all upper bounds of S, $0, 0.2, -5, 1$ are all lower bounds of S. This tells us that upper and lower bounds are far from being *unique*.
- Let $S = \mathbb{R} \subset \mathbb{R}$. We can actually see that S is not bounded above or below. This can be proved using proof by contradiction. If S is bounded above by some upper bound x, then since $x \in \mathbb{R}$, $x+1 \in \mathbb{R}$, which means that x is not an upper bound of S, which gives us the desired contradiction. Lower bounds can be proved similarly. Hence, not all subsets of $\mathbb{R}$ have upper bounds or lower bounds.

Definition 2.5.2 An ordered field $\mathbb{F}$ is called **complete** if every non-empty bounded above $S \subset \mathbb{F}$ has a **least upper bound** $x \in \mathbb{F}$, which means for all upper bounds $x' \in \mathbb{F}$ of S, $x' \geq x$. x is denoted as $\sup(S)$.

Theorem 2.5.1 $\mathbb{R}$ is complete.

This is purely another **definition**, and we assume without proof that this is True.

R It is actually equivalent to denote the completeness definition for $\mathbb{R}$ (or for any ordered $\mathbb{F}$) to be every non-empty bounded below $S \subset \mathbb{R}$ has a **greatest lower bound** $x \in \mathbb{R}$, which means for all lower bounds $x' \in \mathbb{R}$, $x' \leq x$. x is denoted as $\inf(S)$. The equivalence can be seen by forming $S' = \{x : -x \in S\}$ for any bounded above nonempty $S \subset \mathbb{R}$.

Now, we have defined and everything we need to understand our words $\mathbb{R}$. However, I do want to remind you of a major problem: we took the definition of the set $\mathbb{R}$ for granted!!! We simply never defined what $\mathbb{R}$ is, just made a couple of property definitions (axioms) without proof, and defined $\mathbb{Z}$, $\mathbb{Q}$, and $\mathbb{N}$ using $\mathbb{R}$. In fact, there is the beautiful definition for $\mathbb{R}$ which summarizes our Math Language 101: Learning the words. After this, we can go on the journey to explore and have interesting conversations.

Definition 2.5.3 We define $\mathbb{R}$, the **real numbers** to be the **unique ordered and complete field**, i.e. the **unique** set satisfying Theorem 1.2.1, Theorem 2.3.2, and Theorem 2.5.1.