3. Conversations with $\mathbb{R}$

In previous two chapters, we learned about the alphabet and words used in mathematical language, which is $\mathbb{R}$. Now we are ready to have conversations with these words. In this chapter, we will mainly be talking about 1 input 1 response conversations, meaning that for every spoken input, there will be only one response. This is really not something that happens in daily language use, it is rather a huge simplification. However, with this simplification, we can still analyze lots of interesting conversations, their emotions and information presented in these conversations.

3.1 Functions

3.1.1 The Basics

Let us start by defining general conversation between two person using two alphabets (which are sets). Let us keep in mind that when we are dealing with specific mathematical structures that are closed under some operation, alphabets have the same meaning as words.

> **Definition 3.1.1** A **function** f between sets X and Y (conversation between two person using two alphabets) is a communication log that corresponds each $x \in X$ with an element $y \in Y$. We say that "x maps to y by f", and we often denote $f : X \to Y, x \mapsto y$ and $y = f(x)$. One calls y the **image** of x and x the **preimage** of y. We call X the **domain** of f and Y the **codomain** of f. We call $f(X) = \{f(x) : x \in X\}$ to be the **image** of f.

■ Example 3.1 — Functions. ■

- In the set $X = \{1,2,3\} = Y$, we can have a function $f : X \to Y$ such that $f(1) = 2$, $f(2) = 3$ and $f(3) = 1$. We can also have a function $g : X \to Y$ such that $g(1) = g(2) = g(3) = 1$. However, if we associate $1 \in X$ to both $2 \in Y$ and $3 \in Y$, then we will not get a function. An important thing to note here is that for every $x \in X$ and any function $f : X \to Y$, $f(x)$ is **unique**.
- When $X = Y = \mathbb{R}$, we can get a lot of familiar functions, such as $f : \mathbb{R} \to \mathbb{R}$ with $f(x) = x^2$ for all $x \in \mathbb{R}$. We can see that $1 \mapsto 1$, $2 \mapsto 4$ and $-3 \mapsto 9$. We can see that the *image* of f is $\mathbb{R}^+ \cup \{0\}$. Another example can be $g : \mathbb{R} \to \mathbb{R}$ where $g(x) = \sin(x)$ for all $x \in \mathbb{R}$. In this case,

$0 \mapsto 0$, $\frac{\pi}{2} \mapsto 1$ and $\frac{3\pi}{2} \mapsto -1$. In this case, the image of g is $[-1,1]$.
- When $X = \mathbb{N}$, $Y = \mathbb{R}$, and if we have any function $f : X \to Y$, then we call the set of images of f, namely $\{f(1), f(2), \dots\}$ a (real) **sequence**. As an example, if we have $f : \mathbb{N} \to \mathbb{R}$ such that $f(x) = x+2$, then $f(1) = 3, f(2) = 4, f(3) = 5, \dots$ is a sequence.
- In previous sections, we said we haven't defined the notion of "norm" rigorously. In fact, a norm N on $\mathbb{R}$ is a function $N : \mathbb{R} to \mathbb{R}$ such that $x \mapsto |x|$ and safistying the norm defining properties.

R A very important thing to notice is that in the definition of functions, the person with alphabet X uses *all* letters in the alphabet, while the person with alphabet Y does not necessarily use up all letters in Y, i.e. every $x \in X$ is mapped by f to some $y \in Y$, but not every $y \in Y$ is mapped from some $x \in X$.

From above remark and the definition of functions, we understand that not necessarily all letters in Y are mapped from some letter in X ($f(X)$ may not be equal to Y) and not necessarily each letter in X is mapped differently to some letter in Y. Functions with these type of properties are more desirable and very useful for analysis, so we would like to give them specific terminologies.

Definition 3.1.2 We say a function $f : X \to Y$ is **injective** if for all $x, x' \in X$, $f(x) \neq f(x')$. We say a function $f : X \to Y$ is **surjective** if for all $y \in Y$, there exists $x \in X$ such that $x \mapsto y$, i.e. $f(X) = Y$. We say f is **bijective** if it is both injective and surjective.

■ Example 3.2 — Injectivity, Surjectivity and Bijectivity. ■

- Same as in example 3.1.1, if $X = \{1,2,3\} = Y$, f defined in 3.1.1 is both *injective* and *surjective*, hence *bijective*. However, the g defined in 3.1.1 is not *injective* nor *surjective*. For any set $X = Y = \{1,2,3,\dots,n\} \subset \mathbb{N}$, the set of bijective functions $f : X \to Y$ is called the set of **permutations** on X, since each function is essentially "permuting" the elements $1,2,\dots,n$.
- We call a function $f : \mathbb{R} \to \mathbb{R}$ such that $f(x) = a_0 + a_1 x + \dots + a_n x^n$ for all $x \in \mathbb{R}$ where $a_i \in \mathbb{R}$ for all i and $n \in \mathbb{N}$ a (real) **polynomial** of degree n. An example would be $f(x) = x^2$, this is a real polynomial of degree 2, we can see that in this case f is not surjective or injective, since $f(1) = f(-1)$ and $f(x) \geq 0$ for all $x \in \mathbb{R}$. However, one can show that $f(x) = x^3$ is *bijective*. (*Hint:* Showing it is surjective is simple, since we can solve $x^3 = a$ for all $a \in \mathbb{R}$. To show it is injective, consider $x^3 - y^3 = 0$ and factor.)

It is a common scenario for us to consider only the conversation involving 1 person saying specific "words" and the other person's responses. This natural language intuition gives rise to the "restriction" of a function.

Definition 3.1.3 Given $f : X \to Y$, and $A \subset X$, we define the **restriction of f to A** to be the function $f|_A : A \to Y$ such that $f|_A(a) = f(a)$ for all $a \in A$. It is easy to see that since f is a function, $f|_A$ is also a function.

R $f|_A$ can be seen as the conversation log for usage of words in A and their responses.

■ Example 3.3 — Function Restriction. ■

- Sometimes non-injective functions can restrict to injective ones. As an example, $f(x) = x^2$ on $\mathbb{R}$ restricts to an injective function $f|_{\mathbb{R}^+} : \mathbb{R}^+ \to \mathbb{R}$.

- By changing the codomain to be the function's image, we get a surjective function. With same example as above, we know that $f(x) = x^2$ is surjective if we consider $f : \mathbb{R} \to \mathbb{R}^+ \cup \{0\}$.

3.1.2 Functions with Group Structure

Let us consider two functions $f : X \to Y$ and $g : X \to Y$ between set X and group Y closed under $\star$. Then, we can define $f \star g$ such that $f \star g : X \to Y$ where $x \mapsto f(x) \star g(x)$. It is easy to see that $f \star g$ is indeed a function.
Intuitively, this means that if we have two separate conversations between two people using the same two alphabets, we can consider the response emotion together for each input. This is what we do all the time, consider everybody's opinion on the same input we provide. Everybody needs to use the same alphabet otherwise the combination won't make any sense. In fact, this gives the group operation for functions from X to Y, since multiplying two such functions still gives us a function from X to Y, provided that Y has a group structure.
Similarly, if Y is a (ring) field with another operation $\cdot$, we can also define $f \cdot g : x \mapsto f(x) \cdot g(x)$, and $-f : x \mapsto -f(x)$ and possibly also $f^{-1} : x \mapsto f(x)^{-1}$. We can see that the set of functions from given set X to given set Y has the exact same structure that Y has, with the operations defined as above.
What is truly powerful is that the set of functions $f : X \to X$ for any set X with or without any structure is already a semigroup, equipped with the following multiplication method:

Definition 3.1.4 Given any sets X,Y,Z, and functions $f : X \to Y$, $g : Y \to Z$, we define the **composition of f and g**, read as "g compose f" or "f pull back g" to be $f^*g : X \to Z$ such that $x \mapsto (f^*g)(x) = g(f(x))$.

R The above definition is a bit general, but we can see that for two functions $f,g : X \to X$, we can compose them together using composition defined above. It is important that we keep the codomain of f and domain of g to be the same for composition to make sense. Many sources use $g \circ f$ to represent g compose f, however it is applying f first, then g. The backward order in the terminology here is why I do not like it. I will use f^*g in all later occurrences.

In fact, with above remark we can see that given any set X, the set of functions from X to itself is **closed** under composition. It is easy to see that this operation is associative by definition.

Theorem 3.1.1

$$(f^*g)^*h = f^*(g^*h)$$

Proof. On the left hand side, we get $xg(f(x))$ by f^*g, then $g(f(x)) \mapsto h(g(f(x)))$ by h pullback. On the right hand side, we get $x \mapsto h(g(x))$ by g^*h, then mapsto $h(g(f(x)))$ by f pullback. They are equal. □
Then the final question before we conclude that given any set X, the set of functions from X to itself is a semigroup under composition: What is the identity element? In fact, it is easy to see that $f : X \to X$ such that $x \mapsto x$ is the identity element under composition (Try it!). This function is called the **identity function**, denoted id.
In fact, we sometimes can also define the inverse operation of composition. However, in order for the definition of inverse of a function to make sense, we need the function to be **bijective**.

Definition 3.1.5 Given any bijective function $f : X \to Y$, we define the **inverse function** of f to be $f^{-1} : Y \to X$ such that $f^*(f^{-1}) = id = (f^{-1})^*(f)$, i.e. $f^{-1} : y = f(x) \mapsto x$.

R The reason for this definition requiring f being bijective is that we need every $y \in Y$ to be the image of some $x \in X$, suggesting that f must be *surjective*. Besides, in order for f^{-1} to be a function, we need each $y \in Y$ only has one $x \in X$ such that $f(x) = y$, meaning that f must also be *injective*.
Another thing is that we use f^{-1} both for multiplicative inverse at the beginning of this chapter and for composition inverse in the above definition. To ressolve ambiguity, we *always* use f^{-1} for composition inverse, and $\frac{1}{f}$ for multiplicative inverse, when Y is closed under multiplication.

Theorem 3.1.2 Given any set X, the set of functions from X to itself forms a semigroup, and the set of bijective functions from X to itself forms a group under operation composition.

When you have a story, you tell person A, person A retells it to person B. The emotion that person B gets will be the emotion from person A telling the emotion based on your version of the story. This is the composition operation! If the response of person A uses all letters in person A's alphabet and you have 1-1 correspondance between your input and person A's response, it is possible to based on person A's response to figure out what you said to person A, we usually use this technique when we are trying to remember what us have said. This is inversion.

Example 3.4 — Composition and Inversion of Functions. ■

- The set of polynomials on $\mathbb{R}$ of any finite degree (as defined before) is a **ring**. Why? First, it is closed under addition with additive identity being 0 (treated as a 0 degree polynomial). The additive inverse is also defined, being negative of each polynomial. Then, it is closed under function composition (since each polynomial is a function from $\mathbb{R}$ to $\mathbb{R}$), with $f(x) = x$ being the *identity polynomial* for composition. However, we can actually verify that the set of polynomials on $\mathbb{R}$ with any finite degree is *not a field*, since for $f(x) = x^2$, its inverse $g(x) = \sqrt{x}$ is not a polynomial.
- Given set $X = \{1, 2, 3, \ldots, n\}$, we know that the set of *permutations* is a group from Theorem 3.1.2, since all permutations are bijections. In fact, we call this group S_n for each $n \in \mathbb{N}$. It is a very important group in abstract algebra!

3.1.3 Graphs of Functions

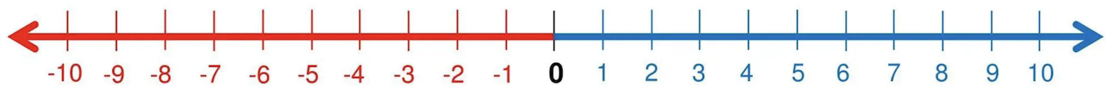
For functions from $\mathbb{R} \to \mathbb{R}$, we can actually plot them on a piece of paper. This involves the idea of defining the *graph* of functions, which also leads to one definition of the information of each conversation. Let us first define abstractly what it means for the graph of functions.
Before we define the graphs of functions, we will first define a new operation on sets, called the Cartesian Product.

Definition 3.1.6 Given two sets X, Y, we define the **Cartesian Product** of X and Y to be the set $X \times Y = \{(x, y) : x \in X, y \in Y\}$ which is the set of ordered pairs of elements in X and Y.

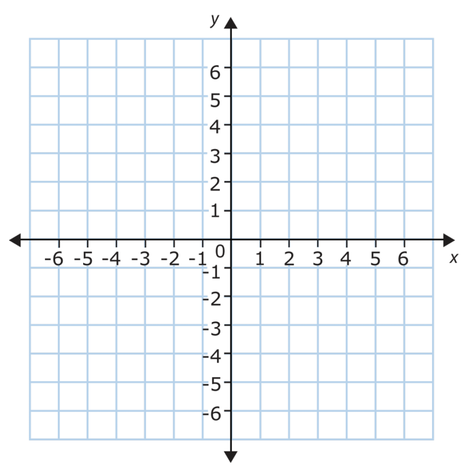
R By *ordered pairs*, we mean that even if $X = Y$, for $a \neq a' \in X = Y$, (a, a') and (a', a) are *two distinct elements* in $X \times Y$.

Definition 3.1.7 Given a function $f : X \to Y$, we define the graph of f, $\text{Graph}(f) : X \to X \times Y$, such that $x \mapsto (x, f(x))$. This is actually a function from X to the Cartesian Product of X and Y.

The intuition behind "graph" is that it is a conversation logging between two person using alphabets X and Y, but instead of the second person just responding to the first person, the second person first repeats what the first person said. In this way, it is a easier way for us to understand how we "log" a conversation on paper, we just write down what the second person says (including what he repeats). After defining abstractly what a *graph* of function means, we need to know how to draw the graph. As a starting point, we will define the notion of a "number line." A **number line** is a visual representation of $\mathbb{R}$, where numbers in $\mathbb{R}$ are ordered from left to right, as shown below:



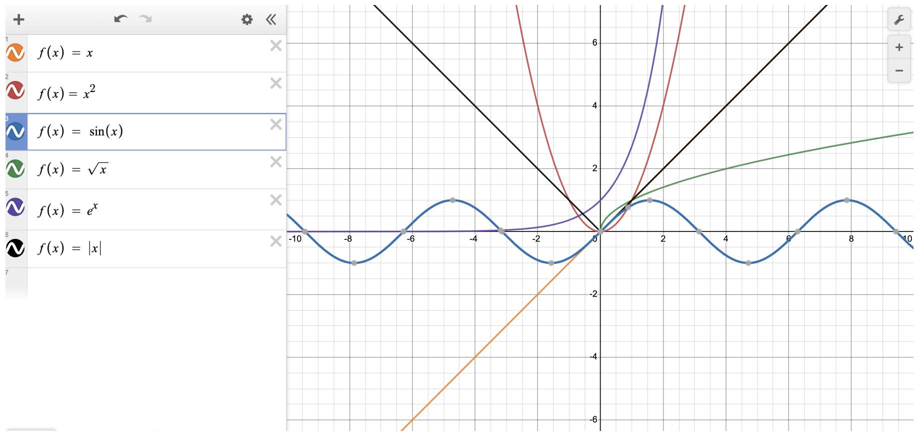
Moving from right to left, the number gets smaller. Moving from left to right the number gets bigger. The red half of the number line above represents *negative* emotions, while the blue half represents *positive* emotions. In order to graph a function $f : \mathbb{R} \to \mathbb{R}$, or even a function $f : X \to Y$ where $X, Y \subset \mathbb{R}$, we first combine two number lines together to make a number plane, like below:



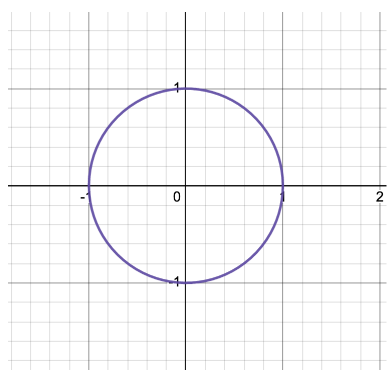
Then, we will label the horizontal axis the x-axis, and the vertical axis y-axis or $f(x)$-axis. Each point on the plane can be labelled by (x, y) where x represents its horizontal position and y represents its vertical position. We call the point $(0, 0)$ the **origin**.

In fact, given any function $f : \mathbb{R} to \mathbb{R}$, each element in Graph$(f)$ is a point on the number plane. After we find all the points on the number plane, we connect them together to form the "graph" of function f.

■ Example 3.5 — Graphs of Functions. ■



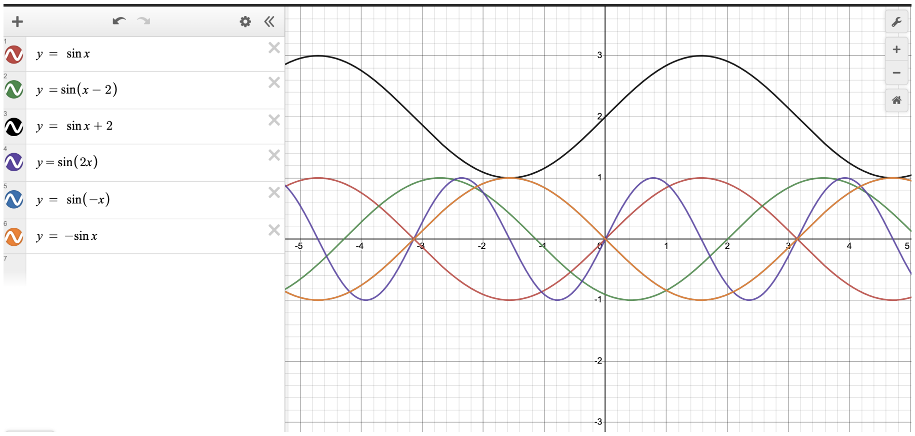
Above is the graphs of several important functions, some of which we haven't seen/defined, but we will in the future. A great tool to use is Desmos, where you can see the graph of all kinds of functions from $\mathbb{R}$ to $\mathbb{R}$. In fact, we can graph all kinds of shapes on the number plane, but is everything representing a function? Not necessarily, let us see an example.



If this is a function, the point 0.5 has two points being its image, which is impossible. It is not hard to realize that for every graph of functions, for every $x \in \mathbb{R}$, if we draw a vertical line through x on the x-axis, this line can only meet the graph at *one point*. In fact, this is a way to tell whether a graph represents a valid function, called **the Vertical Line Test.**

Let us consider the $f(x) = x^2$ graph included above, we know that this function is not *injective*, but can we tell from the graph? Yes, in fact, if we draw a horizontal line through 2 on the y-axis, we can see that it meets the graph at 2 points, meaning that two x values are mapped to the same y value, hence $f(x) = x^2$ is not injective. We can check every horizontal line from each value on the y-axis with the graph to see whether a function is injective or not, this is called the **Horizontal Line Test**.

There are several things you can do to a graph, including translation ($y = \sin(x) + 2$, $y = \sin(x - 2)$), stretching ($y = \sin(3x)$), and reflection $y = \sin(-x)$, $y = -sin(x)$. See below for the effects of these operations:



3.1.4 Countability

In this subsection, we will describe a very important idea called *countability*, meaning whether a set S can be counted with natural numbers. By definition of natural numbers, we know that S can be finite or infinite, but it should be "as many as" the natural numbers.

Definition 3.1.8 A set S is **countable** if we can find a surjective function $f : \mathbb{N} \to S$, i.e. we have a way to count *every* element of S using natural numbers.

R It can be seen from the above definition that we do not require f to be *injective*, meaning that we can count S's elements repetitively.

Example 3.6 — Countable Sets.

- Finite sets are countable. Why? Consider any finite set $S = \{s_1, \dots, s_n\}$ for some $n \in \mathbb{N}$. Simply take $f : \mathbb{N} \to S$ such that $f(x) = s_x$ if $1 \le x \le n$, and $f(x) = s_1$ otherwise. It is easy to see that f is surjective.
- $\mathbb{N}$ is countable. [Left as an exercise.]
- $\mathbb{Z}$ is countable. Why? Let us define $f : \mathbb{N} \to \mathbb{Z}$ such that $f(x) = -\frac{x-1}{2}$ if x is odd, and $f(x) = \frac{x}{2}$ if x is even. It can be shown that this function is surjective. This actually shows that on the set level, a set that is *bigger* than $\mathbb{N}$ may still be countable.
- Given a countable set S, every $A \subset S$ is countable. [By definition, left as an exercise.]

We do have some important results with regards to countability of sets, one of which is that countability holds with $\mathbb{N}$-unions.

Theorem 3.1.3 If $\{S_i\}_{i \in \mathbb{N}}$ are a collection of countable sets, $S = \bigcup_{i=1}^{\infty} S_i$ is also countable.

Proof. This proof is a very clever trick to count the elements in the union. For each element in the union, we will give it one (multiple) natural numbers such that one natural number corresponds to a specific element. Each element $a \in S$ can be characterized uniquely by (i, x) where i represents which S_i that a is in, and x represents the natural number we give $a \in S_i$ since S_i is countable. Given this $a \in S$ and (i, x) representing a, we consider $m = i + x$, and we define the function $f : \mathbb{N} \to S$ such that

$$f\left(\frac{m(m-1)}{2} + i\right) = a$$

for all $a \in S$. Basically we are counting elements in S in the following order: $s_{11}, s_{12}, s_{21}, s_{13}, s_{22}, s_{31}, \ldots$, where s_{11} is the first element we count in S_1 and so on.
Since each $a \in S$ gets a unique (i, x) pair, and each pair corresponds to a unique $\frac{(m-1)(m-2)}{2} + i$ value, we know that f is surjective. Hence, S is countable. $\square$

> **R** In fact, we can try to do the above proof using mathematical induction, but even if we show that $S_n = \bigcup_{i=1}^{n} S_i$ is countable for all $n \in \mathbb{N}$, it is still different (requires more theory) to show that S is countable.

Above proof for the theorem also can be used to prove another important theorem, whose proof will left as an exercise:

Theorem 3.1.4 The rational numbers $\mathbb{Q}$ is countable.

Next, what about $\mathbb{R}$? In fact, $\mathbb{R}$ is *not* countable. Let us see why.

Theorem 3.1.5 $\mathbb{R}$ is not countable.

Proof. Suppose by contradiction that $\mathbb{R}$ is countable. Then, we can define a function $f : \mathbb{N} \to \mathbb{R}$ such that f is surjective. Since f is surjective, we can actually define an *injective* inverse function $f^{-1} : \mathbb{R} \to \mathbb{N}$, mapping each number in $\mathbb{R}$ to their counting order, and in fact $f^{-1}(\mathbb{R}) \subset \mathbb{N}$.
Let us consider the numbers $0.2111111\ldots, 0.22111111\ldots, 0.22211111\ldots, 0.222211111\ldots, \cdots$ Each of the numbers are *different* since they has different number of 2's. Let S be the set of these numbers. We know that they all have different counting orders. Also, we can define $g : S \to \mathbb{N}$ such that $g(s) =$"number of 2's in s.", and in fact g is a bijection onto $\mathbb{N}$.
Since we can find a bijective mapping from S to $\mathbb{N}$, it is not possible that $f^{-1}(S) \neq \mathbb{N}$ since we already know f^{-1} is injective on all of $\mathbb{R}$. Why is this? Consider any $h : \mathbb{N} \to \mathbb{N}$ such that $h(n) = f^{-1}(g^{-1}(n))$. Then, if h is not onto, we know that there are $n_1, n_2 \in \mathbb{N}$ such that $h(n_1) = h(n_2)$, however $g^{-1}(n_1) \neq g^{-1}(n_2)$, thus by definition of h, f^{-1} cannot be injective. Therefore, h must be onto $\mathbb{N}$. It is easy to see that $g^{*}h = f^{-1}|_S$ by definition of h, thus f^{-1} is bijective.
This means $f^{-1}(S) = \mathbb{N}$, then S must be equal to $\mathbb{R}$, by definition of f^{-1}. However, simply consider the number 0.3, which is different from all numbers in S but is still a real number in $\mathbb{R}$. Thus, $S \neq \mathbb{R}$, hence we reached a contradiction. $\mathbb{R}$ is uncountable. $\square$

3.2 Limits

3.3 Sequences

3.4 Continuity