HW5 677

1. (a) $X \sim \mathcal{M}(n, k, p)$, $L(p) = \binom{n-1}{k-1} p^k (1-p)^{n-k}$

$l(p) = \log[\binom{n-1}{k-1}] + k\log(p) + \log(1-p)(n-k)$

$\therefore \frac{\Delta l}{\Delta p} = \frac{k}{p} - \frac{n-k}{1-p} = 0 \quad \therefore \hat{p} = \frac{k}{n} = \boxed{\frac{5}{43}}$

(b) $L(p, x_1 \cdots x_2) = p^{\sum_{i=1}^{n}} (1-p)^{n-\sum_{i=1}^{n} x_i}$

$\therefore l(p) = \sum_{i=1}^{n} x_i \log(p) + (n - \sum_{i=1}^{n} x_i) \log(1-p)$

$\therefore \frac{\Delta l}{\Delta p} = \frac{\sum_{i=1}^{n} x_i}{p} - \frac{n - \sum_{i=1}^{n} x_i}{1-p} = 0 \quad \therefore \hat{p} = \frac{\sum_{i=1}^{n} x_i}{n} = \boxed{\frac{3}{58}}$

2. $x_i \sim$ Uniform $(0, \theta)$.

$L(\theta, x_1 \text{---} x_n) = \frac{1}{\theta^n} \Rightarrow l(\theta, x_1 \text{--} x_n) = -n\ln(\theta)$

$\therefore \frac{\Delta l}{\Delta \theta} = -\frac{n}{\theta} = 0 \quad \because 0 \le x_1 \le \cdots\cdots \le n \le \theta$

$l(\theta, x_1 \text{--} x_n) = \frac{1}{\theta^n} \le \frac{1}{x_1 \text{--} x_n} \quad \therefore \hat{\theta} = X_{(n)}$.

3. $f(x) = \frac{1}{\sqrt{2\pi}\sigma} \cdot e^{-\frac{1}{2\sigma^2}(x-\mu)^2}$

$L(\mu, \sigma^2, x_1 \text{--} x_n) = (\frac{1}{\sqrt{2\pi}\sigma})^n \cdot e^{-\frac{1}{2\sigma^2}[(x_1-\mu)^2 + \text{--} (x_n-\mu)^2]}$

$l(\mu, \sigma^2, x_1 \text{--} x_n) = n\log(\frac{1}{\sqrt{2\pi}}) - \frac{n}{2}\log(\sigma^2) - \frac{1}{2\sigma^2}\sum^{n}(x_i-\mu)^2$

$\frac{\Delta l}{\Delta \mu} = \frac{1}{\sigma^2}\sum_{i=1}^{n}(x_i-\mu) = 0, \frac{\Delta l}{\Delta \sigma^2} = \frac{1}{2\sigma^4}\sum_{i=1}^{n}(x_i-\mu)^2 = 0.$

$\therefore \hat{\mu} = \sum_{i=1}^{n} \frac{x_i}{n} \quad \hat{\sigma}^2 = \frac{\sum_{i=1}^{n}(x_i-\mu)^2}{n}$

$\therefore$ MLE of 95% quantile. $\hat{\mu} \pm 1.96\hat{\sigma} = \sum_{i=1}^{n} \frac{x_i}{n} + 1.96\sqrt{\frac{\sum_{i=1}^{n}(x_i-\mu)^2}{n}}$

4. $v = P(X>2) = P(Z > \frac{2-\mu}{\delta}) = 1 - P(Z \le \frac{2-\mu}{\delta}) = 1 - \Phi(\frac{2-\mu}{\delta})$

$\hat{\delta}^2 = \sum_{i=1}^{n} (X_i - \mu)^2/n \Rightarrow \hat{v} = 1 - \Phi[(2 - \sum_{i=1}^{n} X_i)/\sqrt{\sum_{i=1}^{n}(X_i-\mu)^2/n}$

5. $L(\theta, x) = \frac{1}{\pi^n} \cdot \frac{1}{[1+(X_1-\theta)^2] + \cdots [1+(X_n-\theta)^2]}$, $l(\theta, x) = -n\log\pi - \sum_{i=1}^{n} \log(1+(X_i-\theta)^2)$

$\frac{\Delta l}{\Delta\theta} = \sum_{i=1}^{n} \frac{2(X_i-\theta)}{1+(X_i-\theta)^2} = 0.$

MLE of θ can be derived by given inputs of X and $\frac{\Delta l}{\Delta\theta} = 0$

6. $\because f(x) = \lambda e^{-\lambda x}$ $L(\lambda, X_1 \cdots X_n) = \lambda^n e^{-\lambda \sum_{i=1}^{n} X_i}$ $l(\lambda, X_1, \cdots, X_n)$

$= n\log(\lambda) - \lambda \sum_{i=1}^{n} X_i$ $\frac{\Delta l}{\Delta\lambda} = \frac{n}{\lambda} - \sum_{i=1}^{n} X_i = 0$

$\therefore \hat{\lambda} = \frac{n}{\sum_{i=1}^{n} X_i} = \frac{21}{15+6\times20} = \frac{21}{135} = \frac{7}{45}$ $\hat{\mu} = \frac{1}{\lambda} = \frac{45}{7}.$

7. $\because f(x) = \frac{\lambda^{X_i}}{X_i!} e^{-\lambda}$, $v(x) = E(x) = \lambda$,

$\Rightarrow L(\lambda, X_1, \dots, X_n) = \lambda^{\sum_{i=1}^{n} X_i} \cdot [X_1! \cdots X_n!]^{-1} e^{-n\lambda}$

$\Rightarrow l(\lambda, X_1, \dots, X_n) = \sum_{i=1}^{n} X_i \log(\lambda) - \log(X_1! \cdots X_n!) - n\lambda.$

$\frac{\Delta l}{\Delta\lambda} = \frac{\sum_{i=1}^{n} X_i}{\lambda} - n = 0 \Rightarrow \hat{\lambda} = \frac{\sum_{i=1}^{n} X_i}{n}.$

8. Assume α is median $\because f(x) = \lambda e^{-\lambda x}$

$\therefore \int_0^{\alpha} \lambda e^{-\lambda x} dx = e^{-\frac{\alpha}{\lambda}} + 1 = 0.5 \Rightarrow e^{-\frac{\alpha}{\lambda}} = -0.5 \Rightarrow \alpha = \ln 2\lambda.$

$\because \hat{\lambda} = \frac{n}{\sum_{i=1}^{n} X_i}$ $\therefore \hat{\alpha} = \ln 2 \cdot \frac{n}{\sum_{i=1}^{n} X_i}.$

1. $T=\sum_{i=1}^{n} x_i$ $\because f(x,p)=p^x = p^x(1-p)^{1-x}$

$L(p,x)=p^{\sum x_i}(1-p)^{n-\sum x_i} = p^T(1-p)^{n-T}$

$\begin{cases} \mu(x)=1 \\ v(t,p)=p^{t(x)}(1-p)^{n-t(x)} \end{cases}$

2. $\because f(x,p)=p(1-p)^{x-1}$

$L(p,x)=p^n(1-p)^{\sum x_i - n} = p^n(1-p)^{\sum x_i - n}\cdot 1$

$\begin{cases} \mu(x)=1 \\ V(t,p)=p^n(1-p)^{-n+t(x)} \end{cases}$

3. $\because f(x,r,p)=\binom{x+r-1}{x-1}p^x(1-p)^r$

$L(r,p,x)=p^{\sum x_i}(1-p)^{nr}\left[\binom{x_i+r-1}{x_i-1}\cdots\binom{x_n+r-1}{x_n-1}\right]$

$\begin{cases} \mu(x)=\prod_{i=1}^{n}\binom{x+r-1}{x-1} \\ V(t,r,p)=p^{t(x)}(1-p)^{nr} \end{cases}$

4. $\because f(x,\alpha,\beta)=\frac{\beta^\alpha}{\Gamma(\alpha)}x^{\alpha-1}e^{-\beta x}$

$L(x,\alpha,\beta)=\left(\frac{\beta^\alpha}{\Gamma(\alpha)}\right)^n e^{-\beta\sum x_i}(x_1\cdots x_n)^{\alpha-1}$

$\begin{cases} \mu(x)=\Gamma(\alpha)^{-n}\left(\prod_{i=1}^{n}x_i\right)^{\alpha-1} \\ V(t,\beta)=\beta^{n\alpha}e^{-\beta t(x)} \end{cases}$

5. $\begin{cases} \mu(x)=e^{-\beta\sum x_i} \\ V(t,\alpha)=(t)^{\alpha-1}\left(\frac{\beta^\alpha}{\Gamma(\alpha)}\right)^n \end{cases}$