

# On Clique and Cluster Problems and their Applications

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## **Abstract**

To be written ...

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# Chapter 1

## Introduction

Nowadays, computer science is playing a more and more important role in many fields, such as natural sciences, engineering and social science. With the advances of technology, there are many ways how computer science impacts a vast variety of areas.

While a number of computational problems are solvable efficiently, in the general case<sup>1</sup>Why don't we put the footnote right after "efficiently", many such problems have the property to be intractable, that is they are not solvable efficiently in the general case. An important complexity class of intractable computational decision problems are the ones that are NP-hard. For any NP-hard problem there is no algorithm known that solves the problem for any given instanceI think it should be for all given instances. in polynomial time. Finding a polynomial time algorithm that solved an NP-hard problem would answer the famous "P=NP?"-question affirmative. Typical running times of algorithms for NP-hard problems are exponential in the size of the input. Much progress has been achieved to tackle NP-hard problems despite its intractability property. Techniques to deal with these problems include fixed-parameter algorithms [8, 9, 23], approximation algorithms [27], and heuristics [21]. Approximation algorithms<sup>2</sup> and heuristics<sup>3</sup> can be fast but do not necessarily deliver optimal solutions.

Parameterized complexity deals with NP-hardness (or classical intractability in general) and includes the design of fixed-parameter algorithms [19, 8, 9, 23]. In parameterized complexity, the problem complexity is studied using a second dimension, the *problem parameter*. Intuitively, problems are distinguished to be either tractable for the specific parameterization considered (such prob-

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<sup>1</sup>A computational problem is understood to be solvable efficiently if it can be solved by an algorithm in polynomial time, for any input.

<sup>2</sup>An approximation algorithm is an algorithm that returns a solution with a guaranteed quality: the solution obtained is within a multiplicative factor of the optimal one [27].

<sup>3</sup>A heuristic algorithm is an algorithm that reaches a possible solution to the given problem with no optimization guarantee [21]

lems are said to be *fixed-parameter tractable* or members of FPT, the class of fixed-parameter-tractable parameterized decision problems) or intractable (that is they are shown to be *hard* for a parameterized class such as complexity class  $W[1]$ , a superclass of FPT).

To understand membership in FPT, we consider a decision problem that is associated with one (or more) parameter(s). For simplicity, assume the problem parameter is called  $k$ , and an instance to our parameterized problem is described as  $I = \langle X, k \rangle$ . Then our parameterized problem is in FPT if it can be solved in time  $O(n^{O(1)} + f(k))$ , where  $n = |X|$  is the size of problem instance and  $f(k)$  is a function that depends only on the parameter  $k$ , that is it is independent of  $n$ .

Note that, while function  $f(k)$  can be exponential, when the value of  $k$  is small and  $f$  does not grow too fast, such a running time can indeed describe a practical algorithm for a large number of instances of substantial size. Problems that are hard or complete for  $W[1]$  typically have running times of  $\binom{n}{k}$ . A parameterized problem that is hard for  $W[1]$  is assumed to be not in FPT unless  $FPT = W[1]$ , which is assumed to be false for most computer scientists (just like  $P = NP$  is assumed to be false). For further reading on parameterized intractability, see e.g. [19, 8, 7].

A rich toolkit of algorithm-design techniques has been developed for fixed-parameter algorithms of problems that are in FPT. Can we say for solving FPT problems. This toolkit includes the use of polynomial time *reduction rules*, *kernelization* and the technique of *bounded search trees*.

In this report, we concentrate on computational problems discussed in the literature that benefit from determining large complete or dense subgraphs. We will investigate their solution approaches and their applications. Complete subgraphs are also called *cliques*.

The goal of the clique problem is to determine, for a given graph, whether there exists a clique of size at least  $k$ . While this classic problem when parameterized by  $k$ , is complete for class  $W[1]$ <sup>4</sup>, a number of its variants are proven to be in FPT [19].

Our main focus in this report is to use approaches to deal with clique-based graph problems as a tool to assist solving problems in other fields. We exhibit some existing types of clique-based or cluster-based graph problems, especially those in the class of FPT. We discuss the complexity of these problems as well as some of their algorithmic solutions. Then we will survey some existing applications of cliques on fields of hardware design, bioinformatics, social networks and data mining. We model some FPT problems from those fields as clique-based

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<sup>4</sup>A parameterized decision problem is complete for class  $W[1]$  if it is (1) hard for class  $W[1]$  and (2) a member of  $W[1]$ .

*FPT* problems, shrink the graph with *reduction* rules and search for the exact solutions with *bounded search tree* algorithm. For each modelled problem, we also analyze how much we can improve its running time by reducing them into clique-based problems. this last sentences need to be looked at once the report is pretty much done.

## Chapter 2

# Background and Terminology

In this chapter, we introduce terminology required in this document, including basics in graph theory. For further resources on terminology used in the report see [4, 12, 8].

**Graph:** All *graphs* in this report, typically denoted as  $G = (V, E)$  with vertex set  $V$  and edge set  $E$ , are *undirected simple graphs*, unless otherwise stated.

**Open Neighborhood:** Given a graph  $G = (V, E)$  and a vertex  $v \in V$ , the *open neighborhood* of  $v$  in  $G$  is defined as  $N(v) = \{w : v \neq w, vw \in E\}$ .

**Closed Neighborhood:** Given a graph  $G = (V, E)$  and a vertex  $v \in V$ , the *closed neighborhood* of  $v$  in  $G$  is defined as  $N[v] = N(v) \cup \{v\}$ .

**Induced Subgraph:** For a graph  $G = (V, E)$  with  $V' = \{x_1, x_2, \dots, x_m\} \subseteq V$ ,  $G(x_1, x_2, \dots, x_m) = (V', E')$  denotes the graph that is the by  $V'$  *induced subgraph* of  $G$ . That is, for  $G(x_1, x_2, \dots, x_m)$ , for every pair of vertices  $x, y \in V'$ :  $xy \in E'$  if and only if  $xy \in E$ . In this project report,  $E(V')$  denotes the set of edges in the by  $V'$  *induced subgraph*.

**Complete Graph/Clique:** A graph  $G = (V, E)$  is *complete* if and only if for every pair of vertices  $u, v \in V$  there is an edge  $uv \in E$ . The vertex set of a complete (sub)graph is also called *clique*. A singleton is also considered to be a *clique*.

Note that graphs containing no vertices or graphs that contain exactly one vertex are also complete graphs.

**Maximal Clique:** A clique  $C$  is maximal if and only if it cannot be extended

by including any vertex  $v$  such that  $v \in V$  but  $v \notin C$ . In other words, a maximal clique  $C$  is not a [proper](#) subset of any other clique in  $G$ . [2].

**$K_n$  Graph:** The complete graph with  $n$  vertices is denoted  $K_n$ .

**$K_n$ -free Graph:** A  $K_n$ -free graph is a graph that does not contain a complete subgraph consisting of  $n$  vertices.

**Bipartite Clique:** a *bipartite clique*  $B$  is a subset of vertices from graph  $G = (V, E)$  such that the vertices in  $B$  can be divided into two disjoint sets  $V_1, V_2$ , with:

1.  $V_1 \cup V_2 = B$ .
2.  $V_1 \cap V_2 = \emptyset$ .
3. Every vertex in  $V_1$  is connected to every vertex in  $V_2$  in graph  $G$ .
4. For every pair of vertices  $u, v \in B$ , if  $u, v \in V_1$  or  $u, v \in V_2$ , then  $uv \notin E$ .

**Edge Clique Partition:** Given a graph  $G = (V, E)$ , an *edge clique partition* of  $G$  is a set  $CP = \{C_1, C_2, \dots, C_l\}$ , where  $l$  is a positive integer, such that:

1. Each  $C_i \in CP$  is a clique in  $G$ .
2. For every edge  $uv \in E$ :  $u, v \in C_i$  for exactly one  $C_i \in P$ .

**Non-trivial Clique Partition:** A *non-trivial clique partition* of a complete graph  $G$  is a clique partition of size at least two, that is a clique partition that contains two or more cliques.

**Bipartite Clique Partition:** Given graph  $G = (V, E)$ , a *bipartite clique partition* of graph  $G$  is a set  $BP = \{B_1, B_2, \dots, B_k\}$  such that:

1.  $B_1 \cup B_2 \cup \dots \cup B_k = V$ .
2. Each  $B_1, B_2, \dots, B_k$  is a bipartite clique of  $G$ .
3. For every edge  $uv \in E$ , there is a unique  $B_i \in BP$  which contains both of its endpoints  $u$  and  $v$ .

**Dominating Set:** Given a graph  $G = (V, E)$ , a *dominating set* is a subset  $V' \subseteq V$  such that for every  $v \in V$ , there exists a vertex  $v' \in N[v]$  with  $v' \in V'$ .

**Polynomial-time Reduction:** Given decision problems  $X$  and  $Y$ , a *polynomial-time reduction* from  $X$  to  $Y$  is a polynomial-time algorithm  $A$  that converts any instance  $I$  of  $X$  into an instance  $I' = A(I)$  of  $Y$ , such that  $I'$  is a yes-instance for  $Y$  if and only if  $I$  is a yes-instance for  $X$ .



Note that  $I = \langle X, k \rangle$  is a yes-instance if and only if there exists a solution  $S$  to the given problem  $X$ , where the size of  $S$  is at most  $k$ . Sometimes larger than  $k$ , e.g.  $k$ -clique

**Class NP (class of *non-deterministic polynomial time problems*):** NP is the complexity class of all decision problems whose *yes-answers* can be *verified* in polynomial time.

**NP-hardness:** *NP-hard* problems are the decision problems that are at least as hard as the hardest problems in NP. That is, every problem in NP can be polynomial-time reduced to every NP-hard problem.

**NP-Completeness:** NP-complete problems are the hardest problems in NP. That is a decision problem is called *NP-complete* if it is NP-hard and a member of NP.

**Parameterized decision problem:** A *parameterized* decision problem is denoted as  $\langle X, k \rangle$ , where  $X$  denotes the decision problem and  $k$  denotes its parameter.

**Parameterized Complexity Class FPT (Fixed-parameter tractability):** A parameterized decision problem  $\langle X, k \rangle$  is *fixed-parameter tractable*, or a member of FPT, if there exists an algorithm  $A$  that solves every instance  $I$  of  $\langle X, k \rangle$  in running time  $O(|X|^{O(1)} + f(k))$  or  $O(|X|^{O(1)} \cdot f(k))$ .  $A$  is called a *fixed-parameter algorithm*.

**Reduction Rules:** A *reduction rule* is a polynomial-time mapping from a given parameterized instance  $I$  to a new parameterized instance  $I'$ , where  $I'$  is a yes-instance if and only if  $I$  is a yes-instance.

**Kernelization** [22]: A *kernelization* is a polynomial time reduction that maps a given instance  $I = \langle G, k \rangle$  to a new instance  $I' = \langle G', k' \rangle$ , such that:

1.  $k' \leq k$ .
2.  $|G'| \leq g(k)$  for some computable function  $g(k)$ .
3.  $I'$  is a yes-instance if and only if  $I$  is a yes-instance.

$I'$  is called *(problem) kernel*. Note that if  $g(k)$  is polynomial in  $k$  then we also call  $I'$  a *polynomial (problem) kernel*.

**Bounded Search Trees** [6]: The *technique of bounded search trees* is an algorithm-design technique that solves a parameterized decision problem  $\langle X, k \rangle$  by using a systematic enumeration of candidate solutions in fixed-parameter-tractable time. The resulting search tree is bounded in size in a function  $g(k)$  that depends on parameter  $k$  only. The root of the (bounded)

search tree corresponds to the original problem  $\langle X, k \rangle$ .

Typically, when applying the technique of bounded search trees, the problem is solved recursively by branching according to a constant number [constant number of cases?](#) of different cases. When using a recursive bounded-search-tree algorithm, the size of the search tree  $g(k)$  is a function that depends on  $k$  and its largest *branching number*, defined below.

**Branching Vector:** When applying the technique of bounded search trees, whenever branching is applied to an instance  $I = \langle X, k \rangle$ , the instances resulting from the branching are represented as the children of  $I$  in the search tree. If a branching rule applied to  $I$  creates  $h$  children that reduce their respective parameter to  $k - d_1, k - d_2, \dots$  and  $k - d_h$ , respectively, then this branching rule has a *branching vector* of  $(d_1, d_2, \dots, d_h)$ .

Note that branching vector  $(d_1, d_2, \dots, d_h)$  corresponds to the recursion  $T_k = T_{k-d_1} + T_{k-d_2} + \dots + T_{k-d_h}$  and its corresponding characteristic polynomial is  $z^d = z^{k-d_1} + z^{k-d_2} + \dots + z^{k-d_h}$ , where  $d = \max\{d_1, d_2, \dots, d_h\}$ . If  $\alpha \in \mathcal{R}$  is a root of maximum absolute value solving this characteristic polynomial, we call  $\alpha$  the *branching number* of branching vector  $(d_1, d_2, \dots, d_h)$ . Furthermore,  $t_k = O(\alpha^k)$  is the size of this search tree.

Below, we use the  $k$ -VERTEX COVER problem as an example to demonstrate the technique of bounded search trees. We also highlight the corresponding branching vector, branching number, and search-tree size.

#### $k$ -VERTEX COVER

*Instance:* A graph  $G = (V, E)$ , a positive integer  $k$

*Parameter:*  $k$

*Question:* Does there exist a set  $V' \subseteq V$  with  $|V'| \leq k$  such that every edge is covered by  $V'$  (i.e. for each edge  $uv$ , at least one of  $u$  and  $v$  is in  $V'$ )?

Consider the following branching rule from the bounded-search-tree algorithm for  $k$ -Vertex described in [24]:

**Branching Rule 1.** : *Let  $G$  be a graph where all vertices have degree at least two and no vertex has degree larger than six. Assume  $x \in V$  is a vertex of degree 2 in  $G$  with  $N(x) = \{a, b\}$  [No need to state degree 2](#), where  $\deg(a) \geq 3$ ,  $ab \notin E$ , and there is no vertex  $c \in V$ ,  $\deg(c) = 2$  that builds a cycle  $x, a, c, b$  in  $G$  [x is the only common neighbour](#). If  $|N(a) \cup N(b)| \geq 4$ , then branch according to  $\{a, b\}$  and  $N(a) \cup N(b)$ .*

**Observation** [24]: The branching vector resulting from Branching Rule 1 is  $(2, 4)$ .

Note that branching vector  $(2, 4)$  describes how much the value of parameter  $k$  is reduced in each of the two branches. To be more precise, when given instance of vertex cover problem  $I = \langle G, k \rangle$ , if Branching Rule 1 is applied, we obtain two branches of instances as follows:

1.  $I_1 = \langle G_1, k - 2 \rangle$  with  $G_1 = G - \{a, b\}$
2.  $I_2 = \langle G_2, k - |N(a) \cup N(b)| \rangle$  with  $G_2 = G - (N(a) \cup N(b))$

Note that the value of  $k_1$  in Branch One is  $k_1 = k - 2$  since we include  $a, b$  into vertex cover  $V'$ , and the value of  $k_2$  in Branch 2 is at most  $k_2 = k - 4$  since we include  $N(a) \cup N(b)$  into  $V'$ , where  $|N(a) \cup N(b)| \geq 4$ . Therefore, the branching vector of this branching rule is  $(2, 4)$ .

Now we prove the soundness of Branching Rule 1 for  $k$ -VERTEX COVER.

*Proof.* To prove soundness for this branching rule we must show that  $I$  is a yes-instance for  $k$ -VERTEX COVER if and only if  $I_1$  or  $I_2$  is a yes-instance for  $k$ -VERTEX COVER.

We first prove that if  $I$  is a yes-instance then at least one of  $I_1$  and  $I_2$  is a yes-instance by contradiction.

Assume  $I$  is a yes-instance and  $V'$  is a vertex cover of size at most  $k$  for  $I$ . Further assume both  $I_1$  and  $I_2$  are no-instances. Let  $E' \subseteq E$  be the set of edges incident to vertices  $a$  and  $b$ .

For an edge  $e \in E'$ , one of its endpoints is in  $\{a, b\}$  and the other is in  $N(a) \cup N(b)$ . If neither  $I_1$  or  $I_2$  is a yes-instance, then including any one  $e$ 's endpoint in  $V'$  leads to

[Below is old.](#) Then each edge  $e \in E'$  is covered in one of the following cases:

1. Include both
2.  $a, b \notin V'$  and  $(N(a) \cup N(b)) \subseteq V'$ .
3.  $a \in V'$ ,  $b \notin V'$  and  $N(b) \subset V'$ .
4.  $a, b \in V'$  and  $N(a) \subset V'$ .

Let us consider Case 1. Then  $V'_1 = V' - \{a, b\}$  is a vertex cover of size  $k - 2$  for graph  $G_1$ . Therefore,  $I_1$  is a yes-instance since

In Case 2,  $(N(a) \cup N(b)) \subseteq V'$  ( $a, b \notin V'$ ). Then  $V'_2 = V' - (N(a) \cup N(b))$  is a vertex cover of  $G_2 = G - (N(a) \cup N(b))$  with  $V'_2 = k - |N(a) \cup N(b)|$ . Therefore,  $I_2$  is a yes-instance.

In Case 3,  $a \in V'$  and  $N(b) \subset V'$  ( $b \notin V'$ ). In this case,  $a$  and  $x \in N(b)$  both cover edge  $ax$ ,  $x$  covers edge  $bx$ . The remaining edges in  $E'$  are covered by  $a$  and the vertices in  $N(b) - b$ . If we substitute in the vertex cover  $x$  by  $b$ , all edges in  $G$  are still covered with the same number of vertices. Thus, we can assume that whenever an instance satisfies Case 3, then it also satisfies Case 1 with  $V'' = (V' - \{x\}) \cup \{b\}$  and  $|V''| = |V'|$ . Therefore,  $I_1$  is a yes-instance.

Case 4 can be argued similar to Case 3: Also here,  $I_1$  is a yes-instance since.

Therefore, if  $I$  is a yes-instance, at least one of  $I_1$  and  $I_2$  is a yes-instance.

We now prove that if  $I_1$  or  $I_2$  is a yes-instance, then  $I$  is a yes-instance.

First, assume  $I_1$  is a yes-instance. We show that then also  $I$  is a yes-instance. Note that, since  $I_1$  is a yes-instance, it has a vertex cover of size at most  $k - 2$ . Considering  $G$ , we now argue that all edges incident to set  $\{a, b\}$  can be covered by no more than 2 additional vertices than the vertices in vertex cover  $V'_1$  for graph  $G_1$ , since  $G_1 = G - \{a, b\}$ .  $V' = V'_1 \cup \{a, b\}$  satisfies this condition. Therefore,  $I$  is a yes-instance.

Finally, we prove that if  $I_2$  is a yes-instance, then also  $I$  is a yes-instance.

Note that, since  $I_2$  is a yes-instance, it has a vertex cover of size at most  $k - |N(a) \cup N(b)|$ . Considering  $G$ , we argue that all edges incident to set  $\{a, b\} \cup (N(a) \cup N(b))$  can be covered by including  $N(a) \cup N(b)$  into the vertex cover. Thus,  $I$  is a yes-instance.

Therefore, if  $I_1$  or  $I_2$  is a yes-instance, there exists a vertex cover of size at most  $k$  for instance  $G$ , this,  $I$  is a yes-instance. □

The corresponding branching number of Branching Rule 1 is 1.272020, which is decided by its branching vector  $(2, 4)$  and computed using its corresponding characteristic polynomial (described above).

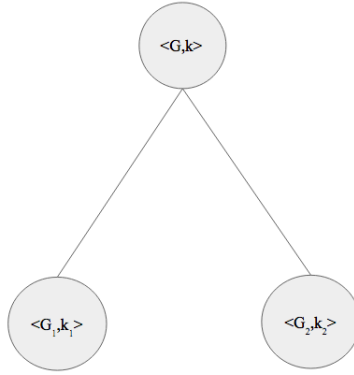


Figure 2.1: Branches on Rule 1, where  $G_1 = G - \{a, b\}$ ,  $k_1 = k - 2$  and  $G_2 = G - (N(a) \cup N(b))$ ,  $k_2 = k - |N(a) \cup N(b)|$ .

## Chapter 3

# Problem Definitions

In this chapter, we list the definitions of graph-based parameterized decision problems that we discuss in this report. Many of their classic versions can be found in [12].

*k*-CLIQUE [8]

**Input:** A graph  $G = (V, E)$ , a positive integer  $k$

**Parameter:**  $k$

**Question:** Does there exist a clique  $C \subseteq V$  of size at least  $k$ , that is  $|C| \geq k$ ?

*k*-EDGE CLIQUE PARTITION [8]

**Input:** A graph  $G = (V, E)$ , a positive integer  $k$

**Parameter:**  $k$

**Question:** Does there exist an *edge clique partition*  $CP = \{C_1, C_2, \dots, C_l\}$  of size  $l \leq k$  where each  $C_i \in CP$  is a clique in  $G$ , such that for each edge  $uv \in E$ , there is exactly one  $C_i \in CP$  with  $u, v \in C_i$ ?

*k*-VERTEX CLIQUE PARTITION

**Input:** A graph  $G = (V, E)$ , a positive integer  $k$

**Parameter:**  $k$

**Question:** Does there exist a *vertex clique partition*  $VP = \{C_1, C_2, \dots, C_l\}$  of size  $l \leq k$  where each  $C_i \in VP$  is a clique in  $G$ , such that for each vertex  $v \in V$ ,  $v$  is in exactly one  $C_i \in VP$ ?

*k*-EDGE CLIQUE COVER [8]

**Input:** A graph  $G = (V, E)$ , a positive integer  $k$

**Parameter:**  $k$

**Question:** Does there exist an *edge clique cover*  $CC = \{C_1, C_2, \dots, C_l\}$  of size  $l \leq k$  where each  $C_i \in CC$  is a clique in  $G$ , such that for each edge  $uv \in E$ ,  $u, v \in C_i$  for some  $C_i \in CC$ ?

*k*-ANNOTATED EDGE CLIQUE COVER [15]

**Input:** A graph  $G = (V, E)$ , a set  $A \subseteq E$ , a positive integer  $k$

**Parameter:**  $k$

**Question:** Does there exist an *annotated edge clique cover*  $ACC = \{C_1, C_2, \dots, C_l\}$  of size  $l \leq k$  where each  $C_i \in ACC$  is a clique in  $G$ , such that for each edge  $uv \in E - A$ ,  $u, v \in C_i$  for some  $C_i \in ACC$ ?

*k*-VERTEX CLIQUE COVER [8]

**Input:** A graph  $G = (V, E)$ , a positive integer  $k$

**Parameter:**  $k$

**Question:** Does there exist a *vertex clique cover*  $VCC = \{C_1, C_2, \dots, C_l\}$  of size  $l \leq k$  where each  $C_i \in VCC$  is a clique in  $G$  such that for each vertex  $v \in V$ ,  $v \in C_i$  for some  $C_i \in VCC$ ?

*k*-COMPRESSION CLIQUE COVER [8]

**Input:** A graph  $G = (V, E)$ , an edge clique cover  $CC$  for  $G$  of size  $k + 1$  for  $G$  and a positive integer  $k$

**Parameter:**  $k$

**Question:** Does there exist an edge clique cover  $CC'$  for  $G$  of size at most  $k$ ?

*k*-BIPARTITE CLIQUE PARTITION [25]

**Input:** A graph  $G = (V, E)$ , a positive integer  $k$

**Parameter:**  $k$

**Question:** Does there exist a *bipartite clique partition*  $BP = \{B_1, B_2, \dots, B_l\}$  of size  $l \leq k$  such that for each edge  $uv \in E$ , there is exactly one bipartite clique  $B_i \in BP$  with  $u, v \in B_i$ ?

*k*-BIPARTITE CLIQUE COVER [25]

**Input:** A graph  $G = (V, E)$ , a positive integer  $k$

**Parameter:**  $k$

**Question:** Does there exist a *bipartite clique cover*  $BC = \{B_1, B_2, \dots, B_l\}$  of size  $l \leq k$  such that for each edge  $uv \in E$ ,  $u, v \in B_i$  for some bipartite clique  $B_i \in BC$ ?

$k$ -CLIQUE EDITING [8]

**Input:** A graph  $G = (V, E)$ , a positive integer  $k$

**Parameter:**  $k$

**Question:** Is it possible to use at most  $k$  operations, consisting of edge additions or edge deletions, to edit  $G$  into a union of vertex-disjoint cliques?

$k$ -CLIQUE DELETION [8]

**Input:** A graph  $G = (V, E)$ , a positive integer  $k$

**Parameter:**  $k$

**Question:** Is it possible to use at most  $k$  edge deletions to edit  $G$  into a union of vertex-disjoint cliques?

$k$ -DOMINATING CLIQUE [8]

**Input:** A graph  $G = (V, E)$ , a positive integer  $k$

**Parameter:**  $k$

**Question:** Does there exist a *dominating clique*  $DC \subseteq V$  of size at least  $k$  such that  $DC$  is both a dominating set and a clique in  $G$ ?

$k$ -MULTICOLORED CLIQUE [8]

**Input:** A graph  $G = (V, E)$  with a vertex coloring  $\chi : V(G) \rightarrow [k]$ , a positive integer  $k$

**Parameter:**  $k$

**Question:** Does there exist a *multicolored clique*  $MC = \{v_1, v_2, \dots, v_l\}$  such that  $MC$  is a clique and for each  $v_i, v_j \in MC$ ,  $\chi(v_i) \neq \chi(v_j)$  if and only if  $i \neq j$ ?

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## Chapter 4

# Complexity Class of Clique-based problems

In our project, we found that many clique-base problems are in the class of NP-hard, which means that there are no known polynomial-time algorithms to solve them. However, many of clique-base problems are in the class of  $W[1]$  and some of them are also in the class of  $FPT$ .

In this chapter we will prove that  $k$ -VERTEX CLIQUE COVER and  $k$ -CLIQUE problems are in the complexity class of  $W[1]$ .

### 4.1 $k$ -VERTEX CLIQUE COVER to $k$ -COLORABLE

In this section, we prove  $k$ -VERTEX CLIQUE COVER by reducing it to an existing  $W[1]$  problem:  $k$ -COLORABLE. We also prove the correctness of our reduction.

$k$ -VERTEX CLIQUE COVER [8]

**Input:** A graph  $G = (V, E)$ , a positive integer  $k$

**Parameter:**  $k$

**Question:** Does there exist a *vertex clique cover*  $VCC = \{C_1, C_2, \dots, C_l\}$  of size  $l \leq k$  where each  $C_i \in VCC$  is a clique in  $G$  such that for each vertex  $v \in V$ ,  $v \in C_i$  for some  $C_i \in VCC$ ?

$k$ -COLORABLE [8]

**Input:** A graph  $G = (V, E)$ , a positive integer  $k$

**Parameter:**  $k$

**Question:** Can we color all the vertices in  $V$  with at most  $k$  colors such that for every  $v_i v_j \in E$ ,  $v_i$  and  $v_j$  are of different colors?

#### Problem Reduction

Given an instance of  $k$ -VERTEX CLIQUE COVER  $I = \langle G, k \rangle$  where  $G = (V, E)$ ,

we reduce  $I$  to an instance of  $k$ -COLORABLE  $I' = \langle G', k' \rangle$  where  $G' = (V', E')$  as follow:

1.  $V' = V$
2.  $v'_i v'_j \in E'$  iff  $v_i v_j \notin E$ .
3.  $k' = k$

Now we prove the correctness of our reduction.

*Proof.* If we can prove  $I$  is a yes-instance if and only if  $I'$  is a yes-instance, then our reduction is correct.

First we prove if  $I$  is a yes-instance then  $I'$  is a yes-instance. Assume  $VCC = \{C_1, C_2, \dots, C_k\}$  is a vertex clique cover of size  $k$  for  $G$ . In instance  $I' = (G', k')$ ,  $G'$  is the complement graph of  $G$ . We generate a color assignment  $CA$  as follow: color vertex  $v'$  in  $G'$  with color  $c_i$ , if its corresponding vertex  $v \in V$  belongs to clique  $C_i \in VCC$ . There are  $k$  colors required in  $CA$  since there are  $k$  cliques in  $VCC$ . Since  $G'$  is the complement graph of  $G$ , for any  $v'_i v'_j \in E'$ ,  $v_i v_j \notin E$ . Therefore, for any edge  $v'_i v'_j \in E'$ ,  $v_i$  and  $v_j$  do not appear together in any of  $C_1, C_2, \dots, C_k$  and  $v'_i, v'_j$  are assigned with different colors. Thus,  $CA$  is a correct color assignment of size  $k$  for  $I'$ .

Now we prove if  $I'$  is a yes-instance then  $I$  is a yes-instance. Since  $I'$  is a yes-instance, there exists a color assignment  $CA$  with  $k$  colors on vertices on  $G'$  such that for each  $v'_i v'_j \in E'$ ,  $v_i$  and  $v_j$  are in different colors. Assume the colors in  $CA$  are  $c_1, c_2, \dots, c_k$ . We can build a vertex clique cover  $VCC$  of size  $k$  as follows:  $VCC = \{C_1, C_2, \dots, C_k\}$ . For a vertex  $v \in V$ , if its corresponding vertex  $v'$  is of color  $c_i$ , then include  $v$  in clique  $C_i$ . For any pair of vertices  $v_i, v_j$  in some  $C \in CP$ , their corresponding vertices  $v'_i, v'_j$  are of the same color so  $v'_i v'_j \notin E'$ . Since  $G'$  is a complement graph of  $G$ , then  $v_i v_j \in E$  and the property of cliques are preserved in every clique of  $VCC$ . Therefore,  $VCC$  is a vertex clique cover of size  $k$  for  $I$ .  $\square$

Since  $k$ -COLORABLE problem is in complexity class of  $W[1]$  and  $k$ -VERTEX CLIQUE COVER problem can be reduced to it in polynomial time, then  $k$ -VERTEX CLIQUE COVER problem is in  $W[1]$  as well.

## 4.2 $k$ -CLIQUE to $(n - k)$ -VERTEX COVER

In this section, we prove  $k$ -CLIQUE be reducing it to an existing  $W[1]$  problem:  $(n - k)$ -VERTEX COVER. We also prove the correctness of our reduction.

$k$ -CLIQUE [8]

**Input:** A graph  $G = (V, E)$ , a positive integer  $k$

**Parameter:**  $k$

**Question:** Does there exist a clique  $C \subseteq V$  of size at least  $k$ , that is  $|C| \geq k$ ?

$n - k$ -VERTEX COVER

**Input:** A graph  $G = (V, E)$ , a positive integer  $k$

**Parameter:**  $k$

**Question:** Is there a set of vertices  $V' \subseteq V$  of size at most  $n - k$  in  $G$  such that for every edge  $v_i v_j \in E$ , either  $v_i \in V'$  or  $v_j \in V'$ ?

### Problem Reduction

Given an instance of  $k$ -CLIQUE  $I = \langle G, k \rangle$  where  $G = (V, E)$ , we reduce  $I$  to an instance of  $n - k$ -VERTEX COVER  $I' = \langle G', k' \rangle$  where  $G' = (V', E')$  as follow:

1.  $V' = V$ .
2.  $v'_i v'_j \in E'$  iff  $v_i v_j \notin E$ .
3.  $k' = n - k$

Now we prove the correctness of our reduction.

*Proof.* If  $I$  is a yes-instance if and only if  $I'$  is a yes-instance, then our reduction is correct.

First we prove if  $I$  is a yes-instance then  $I'$  is a yes-instance. Since  $I$  is a yes-instance, there is a clique  $C$  of size  $k$  in  $G$ . We label the vertices in this clique as  $C = \{v_1, v_2, \dots, v_k\}$ . Since  $C$  is a clique and  $G'$  is a complement graph of  $G$ , the corresponding vertices of  $\{v_1, v_2, \dots, v_k\}$ , denoted as  $V'_r = \{v'_1, v'_2, \dots, v'_k\}$  in  $G'$  are pairwise disconnected, which means there are no edges in  $E(V'_r)$ . For any edge  $u'v' \in E'$ , at least one of  $u'$  and  $v'$  is in  $V' - V'_r$ . Thus,  $V' - V'_r$  is a vertex cover of size  $n - k$  for  $G'$ .

Now we prove if  $I'$  is a yes-instance, then  $I$  is a yes-instance. Since  $I'$  is a yes-instance, there exists a vertex cover  $V' = \{v'_1, v'_2, \dots, v'_{n-k}\}$  of size  $n - k$  for  $G'$ . If we remove  $V'$  and their incident edges from  $G'$ , there remains  $k$  vertices, denoted as  $V'_r$ , and zero edges. Assume  $V_r$  is the set of corresponding vertices for  $V'_r$ . Since  $G$  is the complement of  $G'$ , in  $G$  every pair of vertices  $v_i, v_j$  are connected by an edge. Therefore,  $V_r$  is a clique of size  $k$ . □

Since  $n - k$ -VERTEX COVER problem is in complexity class of  $W[1]$  and  $k$ -CLIQUE problem can be reduced to it in polynomial time, then  $k$ -CLIQUE problem is in  $W[1]$  as well.

## Chapter 5

# Algorithms for Clique-based Problems

As it is discussed in Chapter 4, many clique-based problems are NP-complete, and there are no known polynomial-time algorithms to solve them. While  $k$ -CLIQUE, the classic CLIQUE problem parameterized by the clique size  $k$  is W[1]-complete, some interesting parameterizations of these NP-hard clique-based problems are members of the parameterized complexity class FPT. Techniques from the FPT toolkit such as kernelization or bounded-search-tree algorithms can be used to solve such problems to achieve a running time in the form of  $O(f(k) * n^c)$  or  $O(f(k) + n^c)$  for parameter  $k$ , input size  $n$ , and constant  $c$ . Examples of three clique-based problems that are in FPT, as well as their running times, are listed below. Here, for input graph  $G$  the number of vertices and edges are denoted by  $n$  and  $m$ , respectively.

1.  $k$ -EDGE CLIQUE PARTITION:  $O(kn^3 + 2^{k^2-2})^k 2^{k^2-2} k^4$  for general graphs [22].
2.  $k$ -EDGE CLIQUE COVER:  $O(n^4 + 2^{k^{k+2}})$  [15].
3.  $k$ -CLIQUE EDITING:  $O(1.62^k + m + n)$  [5].

In below sections we describe fixed-parameter algorithms for

$k$ -EDGE CLIQUE PARTITION and  $k$ -EDGE CLIQUE COVER:  $O()$  [15] problems: reduction rules followed by search-tree algorithms.

When introducing clique-based problems and algorithms in this chapter, we assume the vertices and the edges are stored in adjacency lists. Therefore, it takes time  $O(1)$  to search for a vertex and time  $O(n)$  to search for an edge.

Note that we assume the indices of the array correspond to the names of the vertices in  $G$  (see Figure 5.1.). Furthermore, in a list with array index  $u$ , there

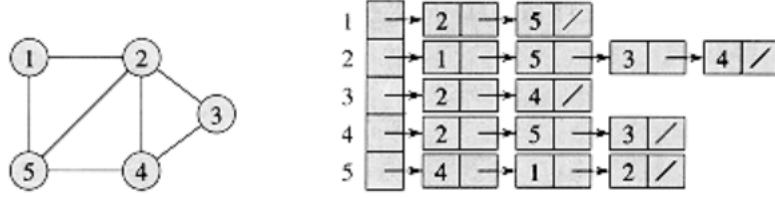


Figure 5.1: A graph  $G = (V, E)$  and its corresponding adjacency list.

exists a node with value  $v$  if and only if  $uv \in E$ .

We use hash tables to determine the intersection of the neighbourhoods of two vertices  $u$  and  $v$  as below, which can be done in time  $O(n)$ .

1. Initialize a hash table  $ht$ , an empty set  $R$  of vertices.
2. Push all vertices  $N[u]$  on  $ht$ .
3. For every  $w \in N[v]$ , if  $w$  in  $ht$ , add  $w$  in  $R$ .
4. Return  $R$ , which is the intersection of  $N[u]$  and  $N[v]$ .

## 5.1 Algorithms for $k$ -EDGE CLIQUE PARTITION

As defined in Chapter 3, when given an input to  $k$ -EDGE CLIQUE PARTITION, a solution  $CP$  of size  $k$  for  $G$  requires that for each edge in input graph  $G$  both of its endpoints are included in the same clique in  $CP$ . Furthermore, each edge is part of exactly one clique in clique partition  $CP$ .

In this section, we introduce a fixed-parameter algorithm by Mujuni and Rosamond [22] that solves  $k$ -EDGE CLIQUE PARTITION. This algorithm combines polynomial time reduction rules that reduce a given an instance  $I = \langle G, k \rangle$  with  $|V| = n$  to a kernelized instance that contains at most  $k^2$  vertices, with a bounded search-tree algorithm. We also give an upper bound of the algorithm's running time.

Following are four reduction rules selected from Section 2 in [22]. We also prove their correctness, which is not provided in [22].

**Reduction Rule 1.** *Let  $I = \langle G, k \rangle$  be an instance for  $k$ -EDGE CLIQUE PARTITION. If there exists a vertex  $v \in V$  with  $\deg(v) = 0$ , then remove  $v$  from  $G$ .*

*Proof.* Given  $I = \langle G, k \rangle$ , and let  $v$  be a vertex in  $G$  with  $\deg(v) = 0$ . Furthermore, let  $I' = \langle G - v, k \rangle$  be the instance obtained from instance  $I = \langle G, k \rangle$  and singleton  $v$  for  $k$ -EDGE CLIQUE PARTITION by applying Rule 1. We prove that  $I$  is a yes-instance if and only if  $I'$  is a yes-instance.

First, we show that if  $I$  is a yes-instance then  $I'$  is a yes-instance. Consider  $I = \langle G, k \rangle$ ,  $G = (V, E)$ , and vertex  $v \in V$  with  $\deg(v) = 0$ . Let  $CP$  be an edge clique partition for  $I$  of size  $k$ . Note that since  $v$  is a singleton in  $G$ ,  $v \notin C_i$  for any  $C_i \in CP$ . Therefore,  $CP$  is also an edge clique partition of size  $k$  for instance  $I'$ .

We next prove that if  $I' = \langle G', k \rangle$  with  $G' = (V - \{v\}, E')$  and  $E' = E$  is a yes-instance then  $I$  is a yes-instance. Assume  $I'$  is a yes-instance and  $CP'$  is a solution of size  $k$  for  $I'$ . Since  $E' = E$ ,  $CP'$  is also a solution for  $I$ .  $\square$

For the following reduction rule we assume that Rule 1 is not applicable to instance  $I$ .

**Reduction Rule 2.** *Let  $I = \langle G, k \rangle$  be an instance for  $k$ -EDGE CLIQUE PARTITION. If there is a vertex  $v \in V$  with  $\deg(v) = 1$ , then remove  $v$  from  $G$  and decrease  $k$  by one.*

*Proof.* Given  $I = \langle G, k \rangle$ , let  $v$  be a vertex in  $G$  with  $\deg(v) = 1$ . Furthermore, let  $I' = \langle G - v, k - 1 \rangle$  be the instance obtained from  $I = \langle G, k \rangle$  and pendant vertex  $v$  by applying Rule 2. We prove that  $I$  is a yes-instance if and only if  $I'$  is a yes-instance.

First we show that if  $I$  is a yes-instance then  $I'$  is a yes-instance. Given  $I = \langle G, k \rangle$ , let  $N(v) = \{u\}$ . Let  $CP$  be an edge clique partition for  $I$ , where  $CP$  is of size at most  $k$ . Note that there exists a  $C_i \in CP$  with  $C_i = \{u, v\}$ , since edge  $u$  and  $v$  cannot both be part of a larger clique in  $G$ 's edge clique partition since  $u$  and  $v$  have no common neighbours, but every edge must be included in some clique in  $CP$ . Therefore  $CP' = CP - \{C_i\}$  is an edge clique partition for  $I'$  of size at most  $k - 1$ , since  $G'$  does not contain edge  $uv$ .

We next prove that if  $I'$  is a yes-instance then  $I$  is a yes-instance. Let  $CP'$  be a solution of size at most  $k - 1$  for  $I'$ . Then  $CP = CP' \cup \{\{u, v\}\}$  is a solution for  $I$ , since  $uv$  is the only edge that is not included in partition  $CP'$ .  $\square$

For the following reduction rule we assume that neither Rule 1 nor Rule 2 can be applied to instance  $I$ .

**Reduction Rule 3.** *Let  $I = \langle G, k \rangle$  be an instance of  $k$ -EDGE CLIQUE PARTITION. If there exists an edge  $uv \in E$  with  $N(u) \cap N(v) = \emptyset$ , then remove edge  $uv$  from  $G$  and decrease  $k$  by one.*

*Proof.* Given  $I = \langle G, k \rangle$ ,  $G = (V, E)$ , with  $uv \in E$  with  $N(u) \cap N(v) = \emptyset$ , let  $I' = \langle G - uv, k - 1 \rangle$  be the instance obtained from  $I$  and  $uv$  by applying

Rule 3. We prove that  $I$  is a yes-instance if and only if  $I'$  is a yes-instance.

First we show that if  $I$  is a yes-instance then  $I'$  is a yes-instance. Assume  $CP$  is a solution for instance  $I$  of size at most  $k$ . Because  $N(u) \cap N(v) = \emptyset$ , there is a  $C_i \in CP$  with  $C_i = \{u, v\}$ . Then  $CP' = CP - \{C_i\}$  is a solution of size at most  $k - 1$  for  $I'$ .

We next prove if  $I'$  is a yes-instance then  $I$  is a yes-instance. Let  $CP'$  be a solution of size at most  $k - 1$  for  $I'$ . Then  $CP = CP' \cup \{\{u, v\}\}$  is a solution of size at most  $k$  for  $I$  since  $uv$  is the only edge in  $G$  that is not covered by  $CP'$ .  $\square$

In order to prove the correctness of the next reduction rule, Rule 4 described below, we consider the following lemma. For this, recall that a non-trivial edge clique partition of a complete graph is of size at least two (Section 2).

**Lemma 1.** *Let  $G = (V, E)$  be a complete graph with  $n > 2$  vertices. Then the smallest non-trivial edge clique partition of  $G$  is of size at least  $n$ .*

*Proof.* We prove this lemma by induction on  $n$ , the number of vertices in the graph.

*Hypothesis:* let  $G$  be a complete graph with  $n > 2$  vertices. Any non-trivial edge clique partition for  $G$  is of size at least  $n$ .

*Base Case:* Let  $n = 3$ , assume vertices in  $G$  are  $v_1, v_2, v_3$ . The smallest non-trivial edge clique partition for  $G$  is  $\{\{v_1, v_2\}, \{v_2, v_3\}, \{v_1, v_3\}\}$ .

*Induction Step:* We prove, by contradiction, that the smallest non-trivial edge clique partition of a complete graph  $G' = (V', E')$  with  $n + 1$  vertices is of size at least  $n + 1$ .

Let  $CP'$  be a smallest non-trivial edge clique partition of size  $h$  for  $G' = (V', E')$ , and let  $v \in V'$ . Note that each clique in  $CP'$  is of size at least 2.

Furthermore let  $G^* = G' - v$ . Then  $G^*$  is a complete graph of size  $n$  and therefore for  $G^*$  the hypothesis holds. That is  $G^*$  has a non-trivial edge clique partition of size at least  $n$ .

Now we prove that if there is an edge clique partition  $CP'$  of size  $h$  for  $G'$ , then there exists an edge clique partition  $CP^*$  of size at most  $h$  for  $G^*$  obtained by removing  $v$  from every clique in  $CP'$ .

For any clique  $C$ , removing any vertex  $v$  and  $v$ 's incident edges again a clique, namely  $C - v$ . Furthermore, removing  $v$  from  $G'$  and from all cliques in partition  $CP'$  does not create any edges that are not partitioned by  $CP^*$ . Note that if  $v \in C$ , with  $|C| = 2$  and  $C \in CP'$ , then when creating  $CP^*$  from  $CP'$ ,

we must remove  $C$  instead of including a size-one clique in  $CP^*$ . Therefore  $|CP^*| \leq |CP'|$ .

Let us assume that  $h < n + 1$ . We show a contradiction for each of the two cases:  $h < n$  and  $h = n$ .

*Case 1:  $h < n$ .* By removing  $v$  from every clique in  $CP'$  we obtain a clique partition  $CP^*$  for  $G$  of size smaller than  $n$ , which contradicts the hypothesis that any non-trivial edge clique partition for  $G^*$  is of size at least  $n$ .

*Case 2:  $h = n$ .* Assume there is a  $CP'$  of size  $n$  for  $G'$ , and let  $S' = \{C'_1, C'_2, \dots, C'_l\}$ ,  $l \leq n$ , be the subset of  $CP'$  which is composed of all cliques from  $CP'$  that contain vertex  $v$ .

Define  $S^* = \{C_1^*, C_2^*, \dots, C_l^*\}$  as the set by removing  $v$  from cliques in  $S'$ . To be more precise,  $C_1^* = C'_1 \setminus \{v\}$ ,  $C_2^* = C'_2 \setminus \{v\}$ , ...,  $C_l^* = C'_l \setminus \{v\}$ . We have proved above that there exists a solution  $CP^*$  for  $G^*$  such that  $|CP^*| \leq |CP'|$  by removing  $v$  from every clique in  $CP'$ . Then  $S^*$  is a subset of  $CP^*$ .  $S^*$  is of size at least two. Otherwise every vertex  $v \in V$  is in a clique  $C$  and  $V \setminus C = \emptyset$ , which leads to a trivial edge clique partition for  $G$ .

Note that if removing  $v$  from every  $C' \in CP'$  leads to a trivial clique partition  $CP^* = \{C\}$  for  $G$ , then  $C \in CP'$  and each edge  $u^*v \in E'$  such that  $u^*$  is also in  $V$  is partitioned in a different clique in  $CP'$ . There are  $n$  such edges. Thus  $CP'$  is of size at least  $1 + n$ .

Now we prove  $CP'$  is of size at least  $n + 1$  if removing  $v$  leads to a clique partition  $CP^*$  such that  $|CP^*| \geq 2$  for  $G^*$ . Based on our definition of edge clique partition, each clique in  $CP^*$  and  $CP'$  is of size at least two. Let  $C_i^*$  be such a clique that  $C_i^* \in S^*$  and  $|C_i^*| \geq 2$ . Since  $S^*$  is of size at least two, if we can prove that to partition out-coming edges from  $C_i^*$ , we need at least  $n - 1$  additional cliques, then the size of  $CP'$  is at least  $2 + n - 1 = n + 1$ .

Assume  $|C_i^*| = m$  that  $m \geq 2$ . There are  $n - m$  other vertices in  $G^*$  but not in  $C_i^*$ . Assume  $v_1^*, v_2^* \in C_i^*$ . There are  $n - m$  edges connecting  $v_1^*$  and other  $n - m$  vertices not in  $C_i^*$ . Denote them with a set  $E_1$ . And there are also  $n - m$  edges connecting  $v_2^*$  and other  $n - m$  vertices not in  $C_i^*$ . Denote them with a set  $E_2$ . Note that  $E_1 \cap E_2 = \emptyset$ .

For every pair of edges  $e_1 \in E_1$  and  $e_2 \in E_2$ ,  $e_1$  and  $e_2$  is not partitioned by the same clique, otherwise pair of vertices  $v_1^*$  and  $v_2^*$  are in two different cliques from  $CP'$ .

Consider following edges:



1.  $e_{1f}$  connects vertices  $v_1$  and  $f$  that  $f \notin C_i^*$ .
2.  $e_{1g}$  connects vertices  $v_1$  and  $g$  that  $g \notin C_i^*$ .
3.  $e_{2f}$  connects vertices  $v_2$  and  $f$  that  $f \notin C_i^*$ .
4.  $e_{2g}$  connects vertices  $v_2$  and  $g$  that  $g \notin C_i^*$ .

It takes one clique to partition edges  $e_{1f}$  and  $e_{1g}$ , since we can partition both edges with a clique  $C'_0 \in CP'$  that  $v_1, f, g \in C'_0$ . However, it takes two cliques to partition edges  $e_{2f}$  and  $e_{2g}$ , since  $f, g$  is in  $C'_0$  and cannot appear in any clique which is not  $C'_0$ .

For this reason, in an optimal solution  $CP'$  it takes exactly one clique to partition out-coming edges from vertex  $v_1$  and each of the rest out-coming edge takes one cliques to be put in the partition.

Therefore to partition all the out-coming edges from  $C_i^*$ , we need at least  $f(j) = 1 + (j-1)(n-j) = 1 + \frac{n^2-1}{4} - (j - \frac{(n+1)^2}{2})$  cliques.  $f(j)$  takes its minimum value  $n-1$  when  $j=2$ . Therefore  $CP'$  is of size at least  $2+n-1 = n+1$ , which is a contradiction.

Now we have shown that both Case 1 and Case 2 yield a contradiction. Therefore the smallest non-trivial edge clique partition of a complete graph  $G' = (V', E')$  with  $n+1$  vertices is of size at least  $n+1$ . □

For the following reduction rule we assume that none of Rules 1–3 can be applied.

**Reduction Rule 4.** *Let  $I = \langle G, k \rangle$  be an instance of  $k$ -EDGE CLIQUE PARTITION. If there exists a vertex  $v \in V$  with  $|N[v]| > k$  where  $N[v]$  induces a clique, then remove  $E(N[v])$  from  $G$ , and reduce  $k$  by one.*

*Proof.* Assume  $v$  is a vertex in  $G$  such that  $|N[v]| > k$  and vertices in  $N[v]$  induces a clique. Furthermore, let  $I' = \langle G - E(N[v]), k-1 \rangle$  be the instance obtained from applying Rule 4 to  $I$  and  $v$ . We prove that  $I$  is a yes-instance if and only if  $I'$  is a yes-instance.

First we show that if  $I$  is a yes-instance then  $I'$  is a yes-instance. Assume  $CP$  is a solution of size at most  $k$  for  $I$ . Further assume  $C$  is a clique in  $G$  where  $|C| > k$ . Then the subgraph induced by  $C$ , denoted as  $G(C)$ , is a complete graph. According to Lemma 1, if we want to do a non-trivial partition on  $G(C)$ , we need at least  $k+1$  cliques. Therefore the only way to put all the edges in  $E(C)$  in partition is to include  $C$  as a member into  $CP$ .

Therefore, if there is a vertex  $v \in V$  with  $|N[v]| > k$  where vertices in  $N[v]$  induces a clique, then  $C_i = N[v]$  is a member of  $CP$ . Then  $CP - \{C_i\}$  partitions all edges in  $E - E(N[v])$ , and  $CP' = CP - \{C_i\}$  is a solution of size at most

$k - 1$  for  $I'$ .

We next prove that if  $I'$  is a yes-instance then  $I$  is a yes-instance. Given instance  $I' = \langle G - E(N[v]), k - 1 \rangle$ , let  $CP'$  be a solution of size at most  $k - 1$  for  $I'$ . Then  $CP' \cup \{N[v]\}$  is a solution of size at most  $k$  for  $I = \langle G, k \rangle$  since clique  $N[v]$  partitions all the edges in  $E(N[v])$ .  $\square$

We now prove the running time of above reduction rules, and then prove the size of problem kernel when none of these rules is applicable.

**Lemma 2.** *In time  $O(kn^3)$ , one can use Rules 1–4 to generate a reduced instance where none of these rules applies.*

*Proof.* First, we only consider the application of Rule 1. Given a singleton in  $G$ , it takes constant time to remove it from  $G$ . The removal of singletons does not decrease the degree of any vertex in  $G$ . Therefore to determine all singletons in a given graph  $G$  (and their removal) can be done in time  $O(n)$ .

Then we consider the running time of applying Rule 1 and Rule 2 together using below algorithm:

1. Initialize an empty queue  $Q$ .
2. Inspect  $\deg(v)$  for all  $v \in V$ . If  $\deg(v) = 0$ , remove  $v$  from  $G$ . If  $\deg(v) = 1$ , add  $v$  to  $Q$ .
3. Keep dequeuing first vertex  $v'$  from  $Q$ , remove  $v'$  from  $G$  and reduce  $k$  by one. Assume  $u'$  is the only neighbor of  $v'$ . Update  $\deg(u')$  after the removal of  $v'$ . If  $\deg(u') = 0$ , remove  $u'$  from  $Q$  and  $G$ . If  $\deg(u') = 1$ , add  $u'$  to the rear of  $Q$ . Stop when  $Q$  is empty or  $k$  decreases to zero.

In above algorithm, we look up the degree of a vertex at most  $2n$  times in total. The first  $n$  times happen when we look up the degree of every vertex and enqueue the degree-one vertices into  $Q$ . Then when removing a vertex  $v'$  from a vertex in  $Q$ , we look up the degree of its neighbor  $u'$ , where  $u'$  is not in queue  $Q$ . Note that there are at most  $n$  vertices in  $Q$ . Therefore, we need to inspect at most  $n$  vertices in step 3. Overall, all look ups of degrees of vertices can be done in linear time.

After applying Rule 1 and Rule 2 in time  $O(n)$ , assuming that none of the Rules 1 and 2 can be applied any longer, there are neither singletons nor degree-one vertices in  $I$ .

We now consider the application of Rule 3.

To apply Rule 3, we need to determine an edge  $uv$  with  $N(u) \cap N(v) = \emptyset$ . For each edge  $uv \in E$ , we obtain  $N(u)$  and  $N(v)$  directly from the adjacency list

of the graph, which takes time  $O(n)$ . Determining the intersection of  $N(u)$  and  $N(v)$ , which can be done in linear time, e.g. by using a hash table. Thus, it takes time  $O(n)$  to determine whether a given edge  $uv$  can be removed using Rule 3. The removal of edge  $uv$  itself takes time  $O(1)$ . Inspecting all  $m$  edges in  $E$  for applicability of Rule 3, and applying the rule, takes time  $O(m) * O(n) = O(mn)$ .

Note that we only need to inspect all  $uv \in E$  once to determine and remove all  $uv$  with  $N(u) \cap N(v) = \emptyset$  from  $G$ , because the application of Rule 3 does neither remove any vertex from  $G$  nor decreases  $|N(u') \cap N(v')|$  for any edge  $u'v' \in E$  such that  $u'v' \neq uv$ . Therefore, each application of Rule 3 takes time  $O(mn)$ .

Note that nor singleton or degree-one vertex can be a common neighbour of vertices  $u$  and  $v$  when  $uv \in E$ . Therefore, after applying Rule 3, and when Rule 3 is not applicable, we require an extra  $O(n)$  time to re-apply Rule 1 and Rule 2. Only now we are guaranteed that none of Rules 1,2 and 3 applies to the reduced graph.

Therefore the overall running time to apply Rules 1–3 is  $O(n^3)$ .

Now let us assume that none of Rules 1–3 applies. We consider Rule 4.

It takes time  $O(n)$  to find a vertex  $v$  with  $|N[v]| > k$  from  $G$  by inspecting the degree of each  $v \in V$ . To determine if the vertices in  $N[v]$  induce a clique takes time  $O(n^2)$  (by checking whether there is an edge between every pair of vertices in  $N[v]$ ). Since there are  $n$  vertices in  $G$ , it takes time  $O(n) * O(n^2) = O(n^3)$  to determine a vertex that satisfies the preconditions of Rule 4 in  $I$ . There are at most  $n^2$  edges in  $E(N[v])$ . Therefore the removal of the edges in  $E(N[v])$  takes time  $O(n^2)$ .

Figure 5.2 illustrates that Rule 4 might be applicable many times: If we assume that we inspect vertex  $v_2$  before  $v_4$  when applying Rule 4, the first time we inspect vertex  $v_2$ , we note that  $|N[v_2]| = k$ , that is Rule 4 is not applicable at this time. Then we inspect  $v_4$ , which meets all the requirements to apply Rule 4. When applying Rule 4, we remove  $E(N[v_4])$  and reduce  $k$  by one. As it is shown in Figure 5.3, now  $E(N[v_4])$  is removed and  $k$  is reduced to two. Therefore at this time, Rule 4 can be applied on  $v_2$  since  $|N[v_2]| > k$  and vertices in  $N[v_2]$  induce a clique.

Thus, we need to inspect all  $v \in V$  again whenever Rule 4 is applied. Thus, the overall running time to detect and apply all Rule 4 cases is  $O(n^3k)$ .

Rule 4 has no influence on the applicability of Rule 3 because no vertex is removed in Rule 4. Rule 1 and 2 have no influence on the applicability of Rule 3 neither, which is proved before. Therefore when Rule 4 is not applicable, we keep applying Rule 1,2,4 until none of them is applicable.

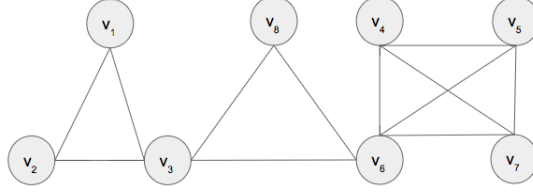


Figure 5.2: Given an instance  $I = \langle G, k \rangle$  with  $k = 3$ , Rule 4 is applicable on  $v_4$  but not  $v_2$ .

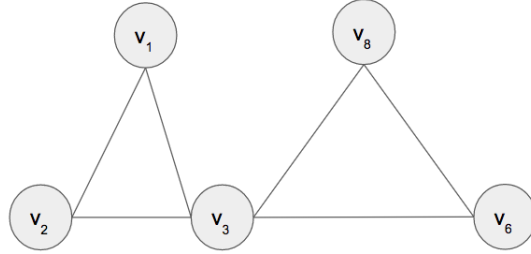


Figure 5.3: After the removal of  $v_4$  and  $E(N[v_4])$  in  $I$ ,  $k$  is reduced by one. Rule 4 now can be applied on  $v_2$ .

Every time we apply Rule 2 and 4,  $k$  is reduced by one. Therefore we can apply them at most  $k$  times in total. Among all four reduction rules, the time complexity of Rule 4 is the highest. In the worst case, we apply Rule 4  $k$  times, which takes time  $O(kn^3)$ .

□

We call an instance  $I = \langle G, k \rangle$  for  $k$ -EDGE CLIQUE PARTITION *reduced* if none of Rules 1–Rule 4 is applicable.

In general, we call an algorithm that transforms any parameterized instance  $I$  into a parameterized instance  $I'$  a *kernelization* if the size of instance  $I'$  is bounded by a function that only relies on parameter  $k$ . We also call  $I'$  a *kernelized* instance.

**Lemma 3.** *Let  $I = \langle G, k \rangle$  be a reduced instance of  $k$ -EDGE CLIQUE PARTITION. Then  $I$  is of size at most  $k^2$  [22].*

*Proof.* Assume there are more than  $k^2$  vertices in instance  $I = \langle G, k \rangle$  where Rule 1 to Rule 4 do not apply. Because Rule 1 is not applicable, there is no

singleton in  $G$ . That is, every vertex is incident to some edge  $e \in E$ . Because Rule 4 is not applicable, all the cliques in  $G$  are of size at most  $k$ . By including  $k$  cliques into  $CP$ , we can only put at most  $k^2$  vertices in the partition. Furthermore, if there are more than  $k^2$  vertices of degree  $\geq 2$ , there is at least one vertex  $v$  that is not in any clique  $C \in CP$ . Therefore  $v$ 's incident edge  $e_0$  is not induced by a clique in partition  $CP$ . Thus,  $I$  is a no-instance.

□

When none of the above four reduction rules can be applied, using a brute force algorithm to search for the solution of the reduced instance, could look as follows: Try all combinations for possible edge clique partitions of size  $k$  and test their validity. For each such partition, we arrange at most  $n^2$  pairs of vertices (which correspond to edges in the graph) into a set  $H = C_1^h, C_2^h, \dots, C_k^h$ . If each subset  $C_i^h \in H$  is a clique and for every edge  $xy$   $x, y \in C_i^h$  for some  $i$ ,  $1 \leq i \leq k$ , then  $H$  is an edge clique partition for the reduced instance  $I$ .

There are at most  $S_{(n^2, k)}$  ways to construct such set  $H$ , where  $S_{(n^2, k)}$  is the number of ways to put " $n^2$  balls into  $k$  identical bins". To determine whether each  $C_i$  is a clique can be done in time  $O(n^2)$ . To verify whether each  $H$  is an edge clique partition, we verify that each  $C_i \in H$  is a clique. This takes time  $O(kn^2)$ . Therefore it takes time  $O(S_{(n^2, k)}kn^2)$  to obtain the solution for the reduced instance. Since in the reduced instance  $n < k^2$ , the total running time is  $S_{(k^4, k)}O(k^5)$ .

To improve the running time, we provide a bounded search-tree algorithm below to determine our final answer for the obtained kernelized instance of size at most  $k^2$ .

**Lemma 4.** *Algorithm 1 solves the sized  $k^2$  kernelized instance of  $k$ -EDGE CLIQUE PARTITION in the running time of  $O((2^{k^2-2})^k 2^{k^2-2} k^4)$ .*

*Proof.* Assume  $I = \langle G, k \rangle$  is the kernelized instance of  $k$ -EDGE CLIQUE PARTITION. According to Lemma 3 there are at most  $k^2$  vertices in graph  $G$ . We apply Algorithm 1 to search for the solution to  $I = \langle G, k \rangle$ , where  $G = \langle V, E \rangle$  and  $|V|$  is bounded by  $k^2$  since none of Rules 1-4 applies.

We first calculate the maximum height of tree. In Line 13 of Algorithm 1, we create one branch for including each possible clique that partitions edge  $uv \in E$  where  $u, v$  has minimum number of common neighbours over other edges. The maximum height of the tree  $T$  is  $k$ , because whenever we branch on any subinstance of  $I$ ,  $k$  will be reduced by one for including a clique in solution  $CP$ .

Now we calculate the number of nodes in the tree. We first branch on the given instance  $I = \langle G, k \rangle$ , and then branch recursively on  $I$ 's subinstances until we find a solution or no more branches can be created. For each subinstance, we

---

**Algorithm 1** Bounded Search-Tree Algorithm for a kernelized instance of  $k$ -EDGE CLIQUE PARTITION

---

**Input:** A graph  $G = (V, E)$ , an initially empty set  $CP$  to store cliques, an integer  $k$ .

**Question:** Is there an edge clique partition of size at most  $k$  for  $G$ ?

```

1: function  $C_{Partition}(G, CP, k)$ 
2:   if  $E = \emptyset$  then return TRUE
3:   else if  $k = 0$  then return FALSE
4:   else
5:     choose  $uv \in E$  such that  $|N(u) \cap N(v)|$  is minimum
6:     find a set  $S$  of all cliques in  $N(u) \cap N(v)$ 
7:     for each  $K \in S$  do
8:        $K' = K \cup \{u, v\}$ 
9:        $CP' = CP \cup \{K'\}$ 
10:       $k' = k - 1$ 
11:       $G' := G - E(K')$ 
12:      if  $C_{Partition}(G', CP', k')$  then return TRUE
13:   return FALSE

```

---

search  $G$  and find the edge  $uv$  such that  $|N(u) \cap N(v)|$  is minimum. The size of  $|N(u) \cap N(v)|$  is at least one since none of Rule 1 to Rule 3 applies. There are at most  $k^2 - 2$  common neighbours for  $u$  and  $v$ . Therefore there are at most  $2^{k^2-2}$  different cliques in  $N(u) \cap N(v)$ . We branch on this subinstance by including each possible  $K \cup \{u, v\}$  in  $CP$  (see Lines 8 and 9), where  $K$  is a member of  $S$ . Therefore, there are at most  $2^{k^2-2}$  branches for each subinstance in  $T$ .

Now we calculate the running time of processing each node(subinstance) in the tree. When processing a subinstance of  $I$ , we first check all the edges in  $E$  to determine  $uv \in E$  such that  $|N(u) \cap N(v)|$  is minimum (see Line 5). The running time for this operation is  $O(k^6)$ . Then we generate set  $S$  which stores all the cliques in  $N(u) \cap N(v)$  (see Line 6). Since there are at most  $k^2 - 2$  vertices in  $N(u) \cap N(v)$ , we obtain  $2^{k^2-2}$  sets of vertices that have the potential to be cliques. To verify whether or not each of those sets is a clique, it takes time  $O(k^4)$ . Therefore, the time complexity of processing a subinstance to generate  $S$  is  $2^{k^2-2}k^4$ .

The height of  $T$  is  $k$  and each instance in  $T$  has at most  $2^{k^2-2}$  branches. There are at most  $(2^{k^2-2})^k$  nodes in the bounded search tree  $T$ . The time of processing a node (subinstance) in  $T$  is  $2^{k^2-2}k^4$ . Therefore, the upper bound for Algorithm 1 for a kernelized graph is  $O((2^{k^2-2})^k 2^{k^2-2}k^4)$ .

□

May keep it or delete later.

### 5.1.1 Bounded Search Tree Algorithm on $K_4$ -free Graph.

When given an instance  $I = \langle G, k \rangle$ , if  $G$  is a  $K_4$ -free graph, the algorithm can be further improved to achieve a better running time.

Note that the reduction rules and the size of kernelization on  $K_4$ -free graphs stay the same as general graph. The improvements are on the bounded search tree algorithm.

---

**Algorithm 2** Bounded Search Tree Algorithm for  $k$ -EDGE CLIQUE PARTITION on  $K_4$ -free Graph

---

**Input:** Graph  $G = (V, E)$ , an integer  $k$ .

**Output:** Is there an edge clique partition of size at most  $k$  for  $G$ ?

---

```

1: function  $C_{Partition}(G, CP, k)$ 
2:   if  $E(G) = \emptyset$  then return TRUE
3:   else if  $k = 0$  then return FALSE
4:   else
5:     choose an Edge  $uv \in E$ 
6:     if  $|N(u) \cap N(v)| > \frac{k+1}{2}$  then return FALSE
7:     else
8:        $CP' = CP \cup \{\{u, v\}\}$ 
9:        $k' = k - 1$ 
10:       $G' = G - \{uv\}$ 
11:      if  $C_{Partition}(G', CP', k')$  then return TRUE
12:      else
13:        for each  $w \in N(u) \cap N(v)$  do
14:           $CP' = CP \cup \{u, v, w\}$ 
15:           $k' = k - 1$ 
16:           $G' = G - \{uv\}$ 
17:          if  $C_{Partition}(G', CP', k')$  then return TRUE
18:      return FALSE

```

---

**Lemma 5.** : In a given instance  $I = \langle G, k \rangle$ , where  $G$  is a  $K_4$ -free graph, when none of Rule 1 to Rule 4 applies, if there exists an edge  $uv \in E$  with  $|N(u) \cap N(v)| > \frac{k+1}{2}$ ,  $I$  is a no-instance. [22]

*Proof.* Assume there is an edge  $uv \in E$  with  $|N(u) \cap N(v)| > \frac{k+1}{2}$ . Let  $V' = N(u) \cap N(v)$ . There are at least  $\frac{k+1}{2} + 1$  vertices in  $V'$  and  $k+3$  edges between  $V'$  and vertices  $u, v$  since for each vertex  $w \in V'$ , there are two edges  $uw, vw \in E$ .

For any pair of vertices  $v_0, v_1 \in V'$ , edge  $v_0v_1 \notin E$ , otherwise  $u, v, v_0, v_1$  induce a  $K_4$  subgraph. In an optimal solution,  $u, v$  is in the same clique  $C_i$  with a random vertex  $v_2 \in V'$ .  $C_i$  partitions 2 edges between  $V'$  and vertices  $u, v$ . Each of the

remaining  $k + 1$  edges is partitioned in one clique of size two. The minimum number of cliques to partition those  $k + 3$  edges is  $1 + (k + 1) = k + 2 > k$ . Therefore,  $I$  is a no-instance.  $\square$

**Lemma 6.** *When none of Rule 1 to Rule 4 applies on a given instance  $I = \langle G, k \rangle$  of  $k$ -EDGE CLIQUE PARTITION problem where  $G$  is a  $K_4$ -free graph, Algorithm 1 solves  $I$  in the running time of  $O(2^{((k+3)\log k)/2} n)$ .*

*Proof.* According to Lemma 3, when none of Rule 1 to Rule 4 applies, there are at most  $k^2$  vertices in graph  $G$ . In Algorithm 2, between two levels in the tree  $T$ , the difference of  $k$  is one. Therefore, the maximum height of the tree is  $k$ .

To construct the tree  $T$ , we start with the given instance  $I = \langle G, k \rangle$  where  $G$  is a  $K_4$ -free graph, and then branch recursively on  $I$  and  $I$ 's subinstances until we find a solution or no more branches can be created. For each subinstance, we randomly pick a edge  $uv \in E$ . According to Lemma 5, if  $|N(u) \cap N(v)| > \frac{k+1}{2}$ ,  $I$  is a no-instance. Otherwise, for each common neighbour of  $u$  and  $v$ , denoted as  $w$ , we create a branch for including  $C_i = \{u, v, w\}$  in solution  $CC$ . There are at most  $\frac{k+1}{2}$  such  $w$ 's, so at most  $\frac{k+1}{2}$  branches are created. Also, we create an additional branch for including  $C_i = \{u, v\}$  in  $CC$ . Therefore, the maximum number of branches for each subinstance in  $T$  is  $\frac{k+1}{2} + 1 = \frac{k+3}{2}$ .

The maximum height of the  $T$  is  $k$  and each node has at most  $\frac{k+3}{2}$  children. The number of nodes in  $T$  is  $k^{\frac{k+3}{2}}$ . For each edge  $uv$ , it takes  $O(n) = O(k^2)$  to compute  $|N(u) \cap N(v)|$  by traversing all  $v' \in V$ . Therefore, the upper bound for Algorithm 2 on  $K_4$ -free graph is  $O(k^{\frac{k+3}{2}} n) = O(2^{((k+3)\log k)/2} n)$ .  $\square$

## 5.2 Algorithms for $k$ -EDGE CLIQUE COVER

In a solution for  $k$ -EDGE CLIQUE COVER, when given an instance  $I = \langle G, k \rangle$ , for every edge  $uv \in E$ ,  $u$  and  $v$  appear together in some clique  $C_i \in CC$ , where  $|CC| \leq k$ . In general, we call an edge  $uv$  covered by a clique  $C$  in  $G$  if  $u, v \in C$ .

Compared to  $k$ -EDGE CLIQUE PARTITION, a solution for an instance to  $k$ -EDGE CLIQUE COVER allows an edge  $uv \in E$  to be covered by multiple cliques in the solution. Note that for a given graph  $G = (V, E)$ , a clique partition of size  $k$  for  $G$  is also a clique cover for  $G$ . However, a clique cover of size  $k$  for  $G$  is not guaranteed to be a clique partition for  $G$  (see Figure 5.4).

To be able to allow edges to be covered by several cliques  $k$ -EDGE CLIQUE COVER, we allow the possibility to mark an edge  $e$  as *covered*. To be able to solve  $k$ -EDGE CLIQUE COVER in the case where the graph contains edges covered edges in a current instance that are marked as covered, we add an auxiliary



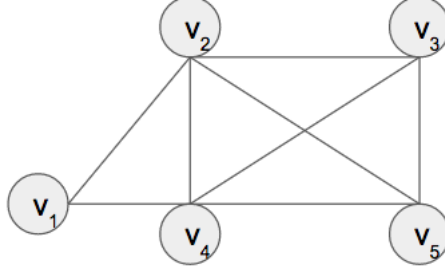


Figure 5.4: A graph  $G = (V, E)$  with a minimum clique partition of size 3 and a minimum clique cover of size 2: A minimum clique partition of size 3 for  $G$  is  $CP = \{\{v_1, v_2\}, \{v_1, v_4\}, \{v_2, v_3, v_4, v_5\}\}$ .  $CP$  is also a clique cover of size 3 for  $G$  since for each edge of  $G$  both endpoints are members of at least one clique in  $CP$ . However, a (minimum) clique cover for  $G$  of size 2 exists, i.e.  $CC = \{\{v_1, v_2, v_4\}, \{v_2, v_3, v_4, v_5\}\}$ .  $CC$  is not a clique partition for  $G$  since vertices  $v_2$  and  $v_4$  appear both in both cliques of  $CC$ .

set  $A$  to our instance  $I$  and consider solving  $k$ -ANNOTATED EDGE CLIQUE COVER, the annotated version of  $k$ -EDGE CLIQUE COVER (defined above). Note that  $k$ -EDGE CLIQUE COVER is a special case of  $k$ -ANNOTATED EDGE CLIQUE COVER where  $A = \emptyset$ .

We now describe an *FPT* algorithm for  $k$ -ANNOTATED EDGE CLIQUE COVER that was introduced in [15]. The preprocessing phase of this algorithm consists of four reduction rules that reduce a given instance  $I = \langle G, A, k \rangle$  of  $k$ -ANNOTATED EDGE CLIQUE COVER to a kernelized instance  $I'$  with a graph of size at most  $k^2$ . The preprocessing is followed by a bounded search-tree algorithm that solves  $I'$ . We prove that this algorithm solves  $k$ -ANNOTATED EDGE CLIQUE COVER in time  $O(2^{k^{k+2}})$ .

Before introducing reduction rules to be applied on instance  $I = \langle G, A, k \rangle$ , we run an initialization phase that computes some useful parameters of graph  $G$ .

**Initialization.** For every edge  $uv \in E$ , we compute a set  $N_{\{u,v\}}$  that stores all common neighbours of vertices  $u$  and  $v$ . We also record the size of  $N_{\{u,v\}}$  for every edge  $uv \in E$ . Furthermore, compute  $c_{\{u,v\}} = |E(N_{\{u,v\}})|$ .

After the initialization phase, we apply the following reduction rules to the instance.

**Reduction Rule 5.** Let  $I = \langle G, A, k \rangle$  be an instance of  $k$ -ANNOTATED EDGE CLIQUE COVER, and let  $v \in V$  be a vertex such that  $\deg(v) = 0$ . Then

remove  $v$  from  $G$ .

*Proof.* Assume  $I' = \langle G - v, A, k \rangle$  is an instance obtained by applying Rule 5 on  $I$ . We prove that  $I$  is a yes-instance if and only if  $I'$  is a yes-instance.

First, we show if  $I$  is a yes-instance then  $I'$  is a yes-instance. Given  $I = \langle G, A, k \rangle$ , consider vertex  $v$  in  $G$  with  $\deg(v) = 0$ . Assume  $CC$  is a solution for  $I$  with  $|CC| = k$ . Since  $v$  has no incident edges, there is no  $C_i \in CC$  with  $|C_i| > 1$  such that  $v \in C_i$ . Therefore,  $CC - C_j | C_j| = 1$  is a solution of size at most  $k$  for  $I'$ .

We now show that if  $I'$  is a yes-instance then  $I$  is a yes-instance. Given instance  $I' = \langle G - v, A, k \rangle$ , where  $CC'$  is a solution of size  $k$  for  $I'$ . Adding any new vertex  $v$  with  $\deg(v) = 0$  to  $G'$  does not create any additional edges. Therefore  $CC = CC'$  is a solution of size  $k$  for  $I$ . □

From now on we assume that Rule 5 is not applicable to instance  $I$ .

**Reduction Rule 6.** *If  $I = \langle G, A, k \rangle$  is an instance of  $k$ -ANNOTATED EDGE CLIQUE COVER with  $A \neq \emptyset$ , and where  $v \in V$  is a vertex in  $G$  with all incident edges of  $v$  are elements in  $A$ , then  $I'$  is obtained by creating  $G'$  from  $G$  via removing  $v$  and updating the set of covered edges to  $A' = A - \{e \in E | e \text{ is incident to } \text{vin}G\}$ .*

*Proof.* Let  $I' = \langle G - v, A', k \rangle$  with  $A' = A - \{e \in E | e \text{ is incident to } \text{vin}G\}$  is an instance obtained by applying Rule 6 on  $I$  for a vertex  $v \in V$  in  $G$  with all incident edges of  $v$  are elements in  $A$ .

We prove that  $I$  is a yes-instance if and only if  $I'$  is a yes-instance.

First, we show if  $I$  is a yes-instance then  $I'$  is a yes-instance. Let  $CC$  be a solution of size  $k$  for  $I = \langle G, A, k \rangle$ , and let  $v$  be a vertex in  $G$  such that all of  $v$ 's incident edges are members of the set of covered edges  $A$ . We observe that for every clique  $C_i \in CC$  such that  $v \in C_i$ ,  $C_i - \{v\}$  is also a clique in  $G$ . Furthermore, every edge  $u'v' \in E$  with  $u', v' \neq v$  is covered by at least one clique in  $CC' = \{C_i - \{v\} | C_i \in CC\}$ . Therefore  $CC'$  is a solution of size at most  $k$  for  $I'$ .

We next prove that if  $I'$  is a yes-instance then  $I$  is a yes-instance. Let  $I' = \langle G - v, A', k \rangle$  with  $A' = A - \{e \in E | e \text{ is incident to } \text{vin}G\}$  be a yes-instance. Let  $CC'$  be a solution of size  $k$  for  $I'$ . Since  $E' = E - \{e \in E | e \text{ is incident to } \text{vin}G\}$  and  $A' = A - \{e \in E | e \text{ is incident to } v\}$ ,  $E - A = E' - A'$ . Therefore  $CC'$  is also a solution for  $I$ . □

From now on we assume that neither Rule 5 nor Rule 6 are applicable to instance  $I$ . Before introducing and proving correctness of rules 7 and 8, we consider the following lemma.

**Lemma 7.** *Let  $I = \langle G, A, k \rangle$  be an instance of  $k$ -ANNOTATED EDGE CLIQUE COVER, and let  $CC$  be a solution of size  $k$  for  $I$ . Then there is a solution  $CC'$  for  $I$ ,  $|CC'| \leq |CC|$ , where every member of  $CC'$  is a maximal clique.*

*Proof.* Assume  $CC$  is a solution for  $I$ . Consider a clique  $C_i \in CC$  that is not a maximal clique—if such a member exists (if not, we are done since then  $CC' = CC$  is a solution).

We build a maximal clique  $C_j$  incrementally by including  $v$  in  $C_i$  such that  $v \in N(C_i)$  and  $vv' \in E$  for every  $v' \in C_i$ . When there is no such  $v$ ,  $C_j$  is maximal since it cannot be further enlarged. At each step we only add a vertex that is adjacent to every vertex of  $C_i$ , so  $C_j$  is indeed a clique.

Since  $C_i \subset C_j$ ,  $C_j$  covers all edges in  $E(C_i)$ . Therefore replacing every such  $C_i$  by a maximal clique  $C_j$  leads to a new solution  $CC'$  for instance  $I$ .  $\square$

**Reduction Rule 7.** *Let  $I = \langle G, A, k \rangle$  be an instance of  $k$ -ANNOTATED EDGE CLIQUE COVER. If there exists an edge  $uv$  in  $G$  such that  $N_{\{u,v\}}$  induces a clique, then include all edges induced by  $N_{\{u,v\}} \cup \{u,v\}$  into  $A$  and reduce  $k$  by one.*

*Proof.* Let  $I = \langle G, A, k \rangle$  be an instance of  $k$ -ANNOTATED EDGE CLIQUE COVER with  $uv \in E$  and  $N_{\{u,v\}}$  induce a clique. Furthermore, let  $I' = \langle G, A', k-1 \rangle$  with  $A' = A \cup E(N_{\{u,v\}} \cup \{u,v\})$  be the instance obtained by applying Rule 7 on  $I$  for  $uv \in E$ . We prove that  $I$  is a yes-instance if and only if  $I'$  is a yes-instance.

First, we show that if  $I$  is a yes-instance then  $I'$  is a yes-instance. Given instance  $I = \langle G, A, k \rangle$ , assume  $CC$  is a solution of size  $k$  for  $I$  with all its members are maximal cliques (Lemma 7).

Note that if  $N_{\{u,v\}}$  induces a clique and  $uv \in E$ , then  $N_{\{u,v\}} \cup \{u,v\}$  is a clique as well.

Furthermore note that  $C_i = N_{\{u,v\}} \cup \{u,v\}$  is the unique maximal clique that covers edge  $uv$ , since there is no other vertex in  $G$  that is adjacent to both  $u$  and  $v$ .

Since every clique in  $CC$  is maximal and  $C_i$  is the unique maximal clique that covers edge  $uv$ ,  $C_i \in CC$ . Since  $CC$  covers  $E - A$ ,  $CC - \{C_i\}$  covers  $E - A - E(N_{\{u,v\}} \cup \{u,v\}) = E - A'$ . Therefore  $CC' = CC - \{C_i\}$  is a solution of size at most  $k - 1$  for  $I'$ .

We are now prove that if  $I'$  is a yes-instance then  $I$  is a yes-instance. Let  $I' = \langle G, A', k-1 \rangle$  be an instance of  $k$ -ANNOTATED EDGE CLIQUE COVER, and let  $CC'$  be a solution of size  $k-1$  for  $I'$ . Since  $A' = A \cup E(N_{\{u,v\}} \cup \{u, v\})$  and clique  $C_i = N_{\{u,v\}} \cup \{u, v\}$  covers all edges in  $E(N_{\{u,v\}} \cup \{u, v\})$ ,  $CC = CC' \cup \{C_i\}$  is a solution of size at most  $k$  for  $I$ .  $\square$

From now on we assume that none of the Rules 5–7 applies to instance  $I$ .

**Reduction Rule 8.** *Let  $I = \langle G, A, k \rangle$  be an instance of  $k$ -ANNOTATED EDGE CLIQUE COVER, and let  $uv \in E$  with  $N[u] = N[v]$ . Then include all of  $u$ 's incident edges into  $A$ .*

*Proof.* Let  $I = \langle G, A, k \rangle$  be an instance of  $k$ -ANNOTATED EDGE CLIQUE COVER, and let  $uv \in E$  with  $N[u] = N[v]$ . Let  $I' = \langle G, A', k \rangle$  where  $A' = A \cup \{\text{all edges incident to } u\}$  be an instance obtained by applying Rule 8 to  $I$  on edge  $uv$ . We prove that  $I$  is a yes-instance if and only if  $I'$  is a yes-instance.

We first prove that if  $I$  is a yes-instance then  $I'$  is a yes-instance. Given an instance  $I = \langle G, A, k \rangle$ , let  $CC$  be a solution of size  $k$  for  $I$ . Since  $E - A'$  is a subset of  $E - A$ , if  $CC$  covers all edges in  $E - A$  then  $CC$  covers all edges in  $E - A'$ . Therefore  $CC' = CC$  is a solution of size  $k$  for  $I'$ .

Now we prove that if  $I'$  is a yes-instance then  $I$  is a yes-instance. Given an instance  $I' = \langle G, A', k \rangle$ , where  $A' = A \cup \{\text{all edges incident to } u\}$ , according to Lemma 7, there exists a solution  $CC'$  of size  $k$  for  $I'$  that is composed of maximal cliques only.

We show that for any member  $C_0 \in CC'$  where every clique in  $CC'$  is maximal, if  $v \in C_0$ , then  $u \in C_0$ . We prove this statement by contradiction. Assume that there is a clique  $C_0 \in CC'$  such that  $v \in C_0$  but  $u \notin C_0$ . Because  $v \in C_0$ , every vertex in clique  $C_0 - v$  is adjacent to  $v$ . Since  $N[u] = N[v]$ , every vertex in  $C_0$  is adjacent to  $u$  as well. Therefore adding  $u$  to  $C_0$  would result in a larger clique. This shows that  $C_0$  is not maximal, a contradiction to Lemma 7.

Let  $C_1, C_2, \dots, C_h \in CC'$  such that  $h < k$ ,  $v \in C_i$  for all  $1 \leq i \leq h$ . From above we know that  $u \in C_1$ ,  $u \in C_2, \dots, u \in C_h$ . Since  $C_1, C_2, \dots, C_h$  cover all edges incident to  $v$  and  $N[u] = N[v]$ , they cover all edges incident to  $u$  as well. Therefore,  $CC = CC'$  is a solution to  $I$  of size at most  $k$ .  $\square$

We call an instance of  $k$ -ANNOTATED EDGE CLIQUE COVER *reduced* if none of the Rules 5–8 can be applied.

**Lemma 8.** *Given instance  $I$  of  $k$ -ANNOTATED EDGE CLIQUE COVER, in time  $O(n^4)$  a reduced instance can be obtained.*

*Proof.* Before applying the reduction rules introduced above, we pre-process the given instance in the initialization phase and compute, for every edge in  $E$ ,  $N_{\{u,v\}}$  and  $c_{\{u,v\}}$ . We now show that this initialization can be done in time  $O(n^4)$ .

For each edge  $uv \in E$ , we use an adjacency list and a hash table (page 19) to calculate the intersection of  $N[u]$  and  $N[v]$ . This can be done in time  $O(n)$ . To obtain  $c_{\{u,v\}}$ , we inspect every edge  $e \in E$  and sum up the number of edges with both endpoints in  $N_{\{u,v\}}$ . This can be done in time  $O(n^2)$ . Therefore it takes time  $O(n^2)$  to compute  $N_{\{u,v\}}$  and  $c_{\{u,v\}}$  for one edge  $uv \in E$ . Since there are at most  $n^2$  edges in  $E$ , this step takes time  $O(n^4)$  in total for the initialization phase.

Next we consider applying Rule 5 on  $I$ . To determine and remove all singletons in  $G$ , in time  $O(n)$  we can inspect all  $v \in V$ , followed by time  $O(1)$  for each singleton found to apply the removal. Note that the removal of singletons does not affect the degree of other vertices. Therefore every vertex needs to be inspected only once. Thus, given graph  $G$  using Rule 5 to generate an instance such that Rule 5 is no longer applicable takes time  $O(n)$ .

Now we consider the application of Rule 6. Since Rule 5 is not applicable to  $I$ , we only determine and remove a vertex  $v$ , where  $v$  is incident to edges only in  $A$ . To achieve this goal, we use a hash table to store edges in  $A$ . For each vertex  $v \in V$ , we obtain all its incident edges and inspect for each of them whether or not it is in  $A$ . Each inspection takes time  $O(1)$ , and each  $v$  is incident to at most  $n - 1$  edges. Therefore it takes time  $O(n)$  to decide whether or not  $v$  is only incident to edges in  $A$ .

When applying Rule 6, we only remove vertex  $v$  and its incident edges. Assume a vertex  $v' \in V$ , where  $v' \neq v$ , is incident to an edge  $e \notin A$ . After the removal of  $v$ ,  $v'$  is still incident to edge  $e$ . Therefore we only inspect every vertex once and perform the removal. This procedure takes time  $O(n^2)$ . Afterwards, Rule 6 is no more applicable to  $G$ .

When applying Rule 6, since some vertex  $v \in V$  is removed, we need to update  $N_{\{u,v\}}$  and  $c_{\{u,v\}}$  for every  $v$ 's neighbour  $u'$ . This can be one in time  $O(n)$  since  $v$  has at most  $n - 1$  neighbours.

Note that the removal of  $v$  where  $v$  was only incident to edges in  $A$ , might again create singletons. It takes an additional  $O(n)$  time to execute Rule 5 once more to  $G$  after applying Rule 6.

Now assume neither Rule 5 nor Rule 6 applies.

To determine whether Rule 7 is applicable on instance  $I$ , we need to inspect every edge  $uv \in E$  to determine whether or not set  $N_{\{u,v\}}$  induces a clique.

For every edge  $uv$ , the size of  $N_{\{u,v\}}$  and the value of  $c_{\{u,v\}}$  are already computed during the initialization phase (and updated when necessary, as discussed above). Let  $n' = |N_{\{u,v\}}|$ . If  $c_{\{u,v\}} = \frac{n'(n'-1)}{2}$ , then  $N_{\{u,v\}}$  is a clique. Therefore it takes time  $O(1)$  to determine whether an edge satisfies the conditions of applying Rule 7.

Note that the application of Rule 7 does not change  $G$  (however, it does change  $A$ ).

We inspect every edge in the graph once for applicability of Rule 7. Since there are at most  $n^2$  edges in the graph, in time  $O(n^2)$  we can apply Rule 7 until the rule is no more applicable.

When Rule 7 is not applicable, we in an additional time of  $O(n^2)$  we can re-apply Rule 6 and then Rule 5, since the application of Rule 7 increases the number of edges in  $A$ .

Now assume that Rule 8 is the only applicable rule on  $I$ . Rule 8 considers whether or not  $N[u] = N[v]$ , for each  $uv \in E$ . This can be done in time  $O(n)$  by using a hash table to inspect edge  $uv$ . The application of Rule 8 does not change  $G$ . Therefore each edge needs to be inspected only once. There are at most  $n^2$  edges in the graph. Therefore in time  $O(n^3)$  we can determine all edges for which Rule 8 is applicable.

When Rule 8 is not applicable, in an additional time of  $O(n^2)$  we can first apply Rule 6 and then Rule 5. We do this since Rule 8 increases the number of edges in  $A$  and therefore might invoke the applicability of Rule 6.

Based on above analysis, the  $O(n^4)$ -time initialization phase is the most expensive operation. After the initialization, in  $O(n^3)$  we apply Rules 5–8, which leads to a reduced instance. Thus, the overall running time for the preprocessing of given instance is  $O(n^4)$ . □

When none of the Rules 5–8 is further applicable,  $G$  is reduced to an instance with at most  $2^k$  vertices. To prove this kernelization result, we first consider the following lemma.

**Lemma 9.** *Assume  $u$  and  $v$  are two vertices in a given instance of  $k$ -ANNOTATED EDGE CLIQUE COVER, denoted as  $I = \langle G, k, A \rangle$ . Let  $CC$  be a solution for*

*I. If for every  $C_i \in CC$ ,  $C_i$  either contains both  $u$  and  $v$ , or neither  $u$  nor  $v$ , then  $N[u] = N[v]$ .*

*Proof.* We prove this lemma by contradiction. Assume we are given an instance of  $k$ -ANNOTATED EDGE CLIQUE COVER, denoted as  $I = \langle G, k, A \rangle$  with solution  $CC$  of size  $k$ . Assume that in  $CC$  for every  $C_i \in CC$ ,  $C_i$  either contains both  $u$  and  $v$ , or neither  $u$  nor  $v$ . For the sake of contradiction assume that  $N[u] \neq N[v]$ .

Since  $N[u] \neq N[v]$ , there exists a vertex  $v'$  such that  $uv' \in E$  but  $vv' \notin E$ . Let clique  $C_i$  be the clique in  $CC$  that covers edge  $uv'$ . Then, since  $u \in C_i$ , according to our assumption also  $v \in C_i$ . However,  $vv' \notin E$ , a contradiction to the definition of clique.  $\square$

Next we prove that our instance is kernelized when reduced.

**Lemma 10.** *Assume that none of the Rules 5–8 is applicable to a reduced instance  $I = \langle G, k, A \rangle$  of  $k$ -ANNOTATED EDGE CLIQUE COVER. Then there are at most  $2^k$  vertices in  $G$ , otherwise  $I$  is a no-instance.*

*Proof.* We prove Lemma 10 by contradiction. Assume  $I = \langle G, k, A \rangle$  is a reduced instance with more than  $2^k$  vertices. Since there is no singleton in  $G$  (Rule 5) and there are no vertices that are incident only to edges in  $A$  (Rule 6), every vertex  $v$  in  $G$  has to be a member of some  $C_i \in CC$ , since otherwise  $v$ 's incident edges are not covered.

Since Rule 8 does not apply, there is no pair of vertices  $u, v$  in  $G$  such that  $N[u] = N[v]$ . Now we show that if there are more than  $2^k$  vertices, there is a pair of vertices  $u, v$  with  $N[u] = N[v]$ :

Recall that we assume that there are more than  $2^k$  vertices in  $V$ . Let  $v_{2^k}$  be the  $(2^k + 1)^{th}$  vertex in  $V$ . Since there are at most  $k$  cliques in  $CC$  and there are at most  $2^k$  different ways to arrange a particular vertex into the cliques of  $CC$ , it is impossible for  $v_{2^k}$  to be part of cliques in  $CC$  while being different from all the configurations of vertices  $v_0, \dots, v_{2^k-1}$  in the arrangement.

According to Lemma 9, there is a  $v_i \in V$  with  $N[v_i] = N[v_{2^k}]$ , which contradicts our assumption that  $N[u] \neq N[v]$  (since Rule 8 is not applicable).

Therefore, a clique cover for  $I$  can contain at most  $2^k$  vertices in  $G$  if  $I$  is a reduced instance.  $\square$

Next we use a bounded search tree algorithm on the reduced instance  $I$  [15]:

Now we explain above algorithm in details and discuss its running time for a reduced instance with  $|V| \leq 2^k$  vertices.

As it is shown in Lemma 7, among all the minimum solutions for  $I$ , there is a solution  $CC$  where every member is a maximal clique. The above algorithm

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**Algorithm 3** Bounded Search Tree Algorithm for  $k$ -ANNOTATED CLIQUE COVER

---

**Input:** A reduced graph  $G = (V, E)$ , a set  $A$  of edges, an integer  $k$ .  
**Output:** A clique cover of size at most  $k$ .

---

```

1:  $A \leftarrow \emptyset, CC \leftarrow \emptyset$ .
2: return  $C_{cover}(G, k, A, CC)$ ;
3:
4: function  $C_{cover}(G, k, A, CC)$ 
5:   if  $CC$  covers  $G - A$  then: return  $CC$ 
6:   if  $k < 0$  then: return nil
7:   choose  $e_{ij}$  such that  $\binom{|N_{\{i,j\}}|}{2} - c_{\{i,j\}}$  is minimum
8:   for each maximal clique  $C$  in  $N[i] \cap N[j]$  do
9:      $CC' \leftarrow CC \cup \{C\}$ 
10:     $X' \leftarrow C_{cover}(G, k - 1, A, CC')$ 
11:    if  $CC' \neq nil$  then: return  $CC'$ 
12:   return nil

```

---

finds such a solution  $CC$  as below:

For an edge  $e \in E - A$ , we branch by including each possible maximal clique that contains  $e$  and then solve the obtained instance recursively (Line 10).

In each subinstance,  $k$  is reduced by one. When  $k = 0$ ,  $I$  is a no-instance if  $E' - A' \neq \emptyset$ . Otherwise, it is a yes-instance. Therefore the height of the tree is at most  $k$ .

The height of tree is bounded by  $k$ , so the size of the tree can be controlled by reducing its width.

At each node, after choosing  $e \in E$  (Line 7), we branch by including each possible maximal clique that contains  $e$ .

Note that in Line 7, for any edge  $v_i v_j \in E$ , if  $h = \binom{|N_{\{i,j\}}|}{2} - c_{\{i,j\}}$  is minimum,  $|E_m|$  is minimum where  $E_m$  is the set of missing edges that prevent  $N_{\{i,j\}}$  from being a clique. Then  $v_i v_j$  is contained in fewest maximal cliques: the removal of one endpoint from each  $e \in E_m$  leads a maximal clique that contains both vertices  $v_i$  and  $v_j$ .

Now we calculate the size of the search tree, that is its maximum number of nodes. In the worst case, the endpoints  $v_i, v_j$  of edge  $e \in E$  have  $n - 2$  common neighbours and no pair of their neighbours are adjacent to each other. Then every edge  $e$  is contained in  $|V|$  maximal cliques. Therefore each node has at most  $|V|$  branches. Since the height of the tree is at most  $k$ , there are at most



$|V|^k$  nodes in the tree. Recall that  $|V| \leq 2^k$ .

Now we calculate the time to process any node in the tree, and the total running time to solve the reduced instance. When processing each instance (node) in the tree, we need to determine an edge with a minimum number of maximal cliques by calculating  $\binom{|N_{\{i,j\}}|}{2} - c_{\{i,j\}}$  for each  $e_{ij} \in E$ . Since  $N_{\{i,j\}}$  and  $c_{\{i,j\}}$  are pre-calculated in initialization phase, it takes time  $O(|V|^2)$  to process each node. There are  $O(|V|^k)$  nodes in the tree. Therefore this bounded search algorithm takes time  $O(|V|^{k+2})$ . Since  $|V|$  is bounded by  $2^k$ , the overall running time for the bounded search tree algorithm is  $O(2^{k^2+k+2})$ .

Recall that preprocessing and kernelization are done in  $O(n^4)$ . Therefore  $O(n^4 + 2^{k^2+k+2})$  is the total running time to solve  $k$ -ANNOTATED EDGE CLIQUE COVER with bounded search tree Algorithm 3.

## Chapter 6

# Applications of Clique-based Problems

Considering the variants of cluster problems we mentioned in this project, `EDGE CLIQUE COVER` and `EDGE CLIQUE PARTITION` might be the ones that are most popular in application settings. While reviewing the literature, we found a number of papers discussing applications of these two problems in different fields. In the following sections of this chapter, we discuss three representative papers, which are in the fields of Hardware Design, Computational Geometry and Bioinformatics.

### 6.1 Applications of Clique Cover

As it is defined in Chapter 3, pairs of vertices in a solution of an instance for `EDGE CLIQUE COVER` are allowed to appear together in more than one clique. It means that if we model an application with this problem, we do not constrain the number of individual objects shared by any pair of clusters when constructing a solution. This property leads to a wide range of applications in the fields of compiler optimization, computational geometry and applied statistics. In this section, we present two application papers in details: `Handling Irregular ILP Within Conventional VLIW Schedulers Using Artificial Resource Constraints` by Subramanian [26] and `Can Visibility Graphs be Represented Compactly` by Panakaj [1].

#### 6.1.1 Handling Irregular ILP Within Conventional VLIW Schedulers Using Artificial Resource Constraints [26]

We discuss an application of `EDGE CLIQUE COVER` when implementing VLIW (Very Long Instruction Word) Processors. The goal of VLIW problems is to find

an assignment of resources to operation types while minimizing the number of resources and supporting all operation types (where some of the operation types might be required to be run in parallel) in a given instruction set architecture (ISA). The authors of [26] discuss models for and solutions of VLIW problems by first using EDGE CLIQUE COVER (see Chapter 5), and then converting the obtained solution of EDGE CLIQUE COVER to a solution of the given VLIW instance.

The size of VLIW problems is the number of operation types in the given ISA, which is in practice typically very small.

To understand this application, we first introduce the following definitions.

**Definition 1.** (*ISA*) *An instruction set architecture (ISA) or computer architecture is a low level abstract model of a computer. The description of the model uses native data types, instructions, registers, addressing modes, memory architecture, interrupt and exception handling, and external I/O.*

We can also view an ISA as a list of instructions that are all supported by a computer, and we need to consider how to program the instructions and assign resources to realize those instructions.

Famous examples of ISAs include: Hiperion DSP (digital signal processor), DEC VAX-11/780 and Intel 80x60.

**Definition 2.** (*Operation Types*) *are types of operations supported in a particular ISA.*

For example, a particular ISA may support the following four operations: loading, addition, subtraction and multiplication. The corresponding set of *operations types* is  $O = \{LOAD, ADD, SUB, MUL\}$ .

**Definition 3.** (*Operation Set*) *We define  $Q \subset I \times O$  to be the set of operations, where  $I$  is the set of identifiers.*

Note that a certain operation type might appear many times to execute a given set of instruction. All operations in an ISA (even those of the same type) are considered different from each other. To distinguish different operations of the same type, in the definition above, there is an identifier in each  $q = (i, t) \in Q$ .

We consider the following example. Let  $I = \{1, 2, 3, 4, 5\}$  and  $O = \{LOAD, ADD, SUB, MUL\}$ . Then  $Q = \{(1, LOAD), (2, ADD), (3, SUB), (4, LOAD)\}$  is a valid operation set.

**Definition 4.** (*Resource Set*) *Finite resource set  $R$  is used to denote the set of resources that operations may potentially occupy.*

For example,  $R = \{ADDER1, LOADER1, ADDER3, ISSUE2\}$ , where  $ADDER1, LOADER1, \dots$  are resources of the computer. Below are examples of the functionality of some types of resources.

**ADDER:** performs addition of binary numbers.

**LOADER:** assigns values to parameters.

**ISSUE:** obtains values from parameters.

**MULTIPLIER:** multiplies two binary numbers.

To execute an operation type in a given ISA, we might need to use several resources in a certain order, where each usage of resource is defined as a cycle.

**Definition 5.** (Resources Usage) *A resource usage for executing an operation of type  $o \in O$  is an element  $u$  of set  $U = N \times R$ , where  $N$  is the set of cycles, and  $R$  is the set of resources.*

For example, for  $n \in N$  and  $r \in R$   $u = (n, r)$  means that in the  $n^{th}$  cycle of our operation, resource  $r \in R$  is occupied. Note that a resource can only be occupied at most once in each cycle.

To perform a particular operation type, several cycles might be required, where each cycle occupies exactly one resource.

In such a case we can use a *resource usage table*  $T$ ,  $T \in 2^U$  where  $2^U$  is the power set of  $U$ . to show which resource is used in which cycle when executing an operation type. In other words a resource usage table shows which resources are occupied for which cycle of operations.



For example, assume in a given ISA, we require  $ISSUE1$  and  $ADDER1$  to perform operation ADD. Then  $T = \{(0, ISSUE1), (1, ADDER1)\}$  means that  $ISSUE1$  is occupied in Cycle 0 and  $ADDER1$  is occupied in Cycle 1. Figure 6.1 is the corresponding resource usage table (assuming that  $ISSUE1$ ,  $ADDER1$  and  $MULTIPLIER1$  are the only resources of the machine).

	ADDER1	MULTIPLIER1	ISSUE1
cycle 0			
cycle 1			

Figure 6.1: A usage table for operation ADD.

In a given ISA, there might be several alternative usage tables for an operation

type. For example, if there are resources *ADDER1*, *ADDER2*, *ISSUE1* and *MULTIPLIER1* in the machine, in Cycle 1 of our ADD operation, we can choose to use either *ADDER1* or *ADDER2*. Then we have two alternative usage tables as Figure 6.1.1.

	ADDER1	MULTIPLIER1	ISSUE1	ADDER2
cycle 0				
cycle 1				



	ADDER1	MULTIPLIER1	ISSUE1	ADDER2
cycle 0				
cycle 1				

Figure 6.2: Two alternative resource usage tables for operation ADD.

Note that two operation types  $o_i, o_j \in O$  can be executed in parallel if and only if they do not need to use the same resource in the same cycle.

Note that if we assign  $r_1 \in R$  to  $o_1 \in O$  and  $r_1, r_2 \in R$  to  $o_2 \in O$ , then  $o_1$  and  $o_2$  can run in parallel since we can let  $o_2$  use  $r_2$  and  $o_1$  use  $r_1$  whenever these two operation types run at the same time. But if we would instead assign  $r_1$  to  $o_2$ , then  $r_1$  and  $r_2$  cannot be executed in parallel.

In an ISA, if we do not require any pair of operation types to run in parallel, we need only one set of resources since we can run all operations one by one.

If we want to enable every pair of operation types to be executable in parallel, we need to assign a set of resources to each operation type, which is generally not practical, especially when resources are limited.

Therefore, computer engineers look for a trade-off between resource occupation and the possibility of parallel execution of operation types. When designing the ISA's, they specify which pairs of operation types should be executed in parallel and create a list accordingly. The list is notated as  $L$  in this section.

Based on this specification, our task is to assign resources to operation types to enable the parallel execution of every pair of operation types in  $L$ . The problem

to be solved is called VLIW and defined below.

For the sake of reserving some resources for other operations and saving energy, we do not want to assign more resources of the machine to the ISA than necessary. Thus, the goal of VLIW is to minimize the number of resources to support all parallel operation types in  $L$ .

We can formally describe VLIW as follows. Given are a set  $O$  of operation types and a list  $L$  of pairs of operations types from  $O$  that are required to be executed in parallel. The goal is to determine a smallest possible set  $R$  of resources such that there exists an assignment  $A$  of resources from  $R$  to operation types in  $O$  while  $A$  satisfies these two constraints:

1.  $A(o_i) \cap A(o_j) \neq \emptyset$  if  $(o_i, o_j) \notin L$ .
2.  $A(o_i) \cap A(o_j) = \emptyset$  if  $(o_i, o_j) \in L$ .

Here  $A(o)$  denotes the set of resources assigned to operation type  $o$ . For example, if resources  $r_1, r_3$  are assigned to operation type  $o_i$ , then  $A(o_i) = \{r_1, r_3\}$ .

**VLIW**

**Input:** A graph  $G = (O, L)$  with operation set  $O$  as vertices of  $G$  and pairs in list  $L$  as edges of  $G$ .

**Parameter:** *None*

**Question:** What is a resource set  $R$  and an assignment  $A$  for  $R$  onto  $O$  such that the size of  $R$  is minimized?

We describe next how to solve an instance of VLIW via EDGE CLIQUE COVER.

**ECC Algorithm for VLIW:**

1. Compute the complement graph  $G' = (O, E)$  for  $G$  where  $E = (O \times O) - L$ .
2. Compute a minimum edge clique cover  $CC$  for graph  $G'$ .
3. For each clique  $C_i \in CC$ , assign a unique resource  $r_i$  to every  $o \in C_i$ .

For example, given a set  $O = \{o_1, o_2, o_3, o_4\}$  and a list  $L = \{(o_1, o_4), (o_2, o_4)\}$  and we want to compute a assignment  $A$  of minimized  $R$  on  $O$ . First, we build a graph  $G = (O, L)$  as stated above. Then we compute the complement graph  $G' = (O, E)$  for  $G$ , where  $E = \{o_1 o_2, o_1 o_3, o_2 o_3, o_3 o_4\}$ . The minimum edge clique cover for  $G'$  is  $CC = \{\{o_1, o_2, o_3\}, \{o_3, o_4\}\}$ , the size of which is two. Therefore, the outputs are  $R = \{r_1, r_2\}$  and  $A(o_1) = r_1$ ,  $A(o_2) = r_1$ ,  $A(o_3) = r_1, r_2$  and  $A(o_4) = r_2$ . In our solution, operation types in pairs  $(o_1, o_4)$  share no resources, so they can be executed in parallel, so as pairs  $(o_2, o_4)$ .

Before proving the soundness of above algorithm to search for the minimum assignment, we first show the definition of EDGE CLIQUE COVER problems.

EDGE CLIQUE COVER

**Input:** A graph  $G = (V, E)$

**Parameter:** *None*

**Question:** What is the minimum *edge clique cover*  $CC$  for graph  $G$  such that for every edge  $uv \in E$ , vertices  $u$  and  $v$  appear together in at least one clique  $C \in CC$ ?

Since we compute the edge clique cover for  $G' = (O, E)$  in our algorithm, each edge in  $E$  represents a pair of operation types which is not required to execute in parallel. In fact, we are searching for clusters where each cluster stores operation types do not execute in parallel. Therefore, each cluster only needs to be assigned one resource from  $R$ .

Now we prove the correctness of our algorithm by proving that the assignment generated satisfies the two constraints of VLIW we discussed above.

*Proof.* We first prove the first constraint. An edge  $o_u o_v \in E$  exists if operation types  $o_u$  and  $o_v$  do not have to be executed in parallel. According to the definition of EDGE CLIQUE COVER, edge  $o_u o_v$  is covered by a clique  $C_i \in CC$  and both of  $o_u$  and  $o_v$  are assigned with resource  $r_i$ . As a consequence,  $r_i \in (A(o_u) \cap A(o_v))$ , which means  $A(o_u) \cap A(o_v)$  is not empty if  $o_u$  and  $o_v$  are not able to execute in parallel.

Then we prove the second constraint. If operation types  $o_u$  and  $o_v$  can be executed in parallel, then  $o_u o_v \notin E$  in our graph construction. Then  $u$  and  $v$  do not appear together in any  $C_i \in CC$ , otherwise definition of EDGE CLIQUE COVER is violated. Therefore, there is no  $r \in R$  assigned to both  $o_u$  and  $o_v$ , which means that  $A(o_u) \cap A(o_v) = \emptyset$  if  $o_u$  and  $o_v$  can be executed in parallel.

Thus, if we use the algorithm described above to assign resources to operation types, both constraints are satisfied. Assume  $A$  is the assignment generated from  $CC$ . We cannot find a smaller assignment than  $A$ , which can be proved by contradiction and we do not discuss the details here.  $\square$

Though EDGE CLIQUE COVER problem provide some sound strategies to solve the VLIW problem, as a well-known NP-Complete problem, it has no algorithms in polynomial running time. The paper [26] discusses a heuristic algorithms to solve EDGE CLIQUE COVER problem, but it has no guarantee about the quality of solution. As it is shown in Figure 6.3, in some data sets, the results from heuristic algorithm are of size twice larger than the smallest solutions from exact algorithm.

Stat	HIPERION				ELIXIR			
	<i>w ver.</i> <i>heur.</i>	<i>w ver.</i> <i>exact</i>	<i>w/o. ver.</i> <i>heur.</i>	<i>w/o ver.</i> <i>exact</i>	<i>w ver.</i> <i>heur.</i>	<i>w ver.</i> <i>exact</i>	<i>w/o. ver.</i> <i>heur.</i>	<i>w/o ver.</i> <i>exact</i>
Resources	22	9	7	7	17	9	7	7
$G(V, E)$	(38,46)	(38,46)	(20,29)	(20,29)	(33,43)	(33,43)	(17,27)	(17,27)
$G'(V, E)$	(38,657)	(38, 657)	(20,161)	(20,161)	(33,485)	(33,485)	(17,109)	(17,109)
$G'_1(V_1, E_1)$		(657,46824)		(161,3448)		(485,30520)		(109,1656)
CPU time (sec)	0.89	29.76	0.02	0.81	0.37	11.52	0.02	0.25

Figure 6.3: Implementation Results of VLIW.

However, sometimes engineers do not ask for the smallest assignment of resources to support parallel executions in  $L$ . Instead, they reserve a certain amount of resources and want to know whether the resources are sufficient for the ISA. In this case, we can model the problem with decision version of EDGE CLIQUE COVER problem as below, and solve the  $k$ -EDGE CLIQUE COVER with FPT algorithm (see Section 5.2).

1. Compute the complement graph  $G' = (O, E)$  for  $G$ , where  $E = O \times O - L$ .
2. Let  $k$  be the number of available resources.
3. Search solution of  $k$ -EDGE CLIQUE COVER for instance  $I = \langle G', k \rangle$ .  $k$  resources are sufficient to support parallel executions of operation types in  $L$  if and only if  $I$  is a yes-instance.

### 6.1.2 Algorithms for compact letter displays: Comparison and evaluation [14]

In fields like Biology, Social Science and Statistic, scientists design experiments and collect data to examine their hypothesis. These experiments are conducted in different environmental conditions. Each environmental condition is also called a treatment. By analyzing the results of experiments running in various treatments, scientists determine which factors have impact on the outputs of the experiment.

While some treatments are significantly different from each other, others are not. We say that two treatments are significantly different from each other if the difference of experimental results in treatments is not attributed chance. To store the significant difference between treatments and to provide convenience for further data analysis, there is a need to find a data structure that stores the information that which pairs of treatment have significant difference, within a small amount of space.

In this section, we introduce a binary matrix as the data structure to store the significant difference between treatments, as well as discuss how to apply EDGE



CLIQUE COVER to generate a binary matrix for a given set  $H$  of significantly different treatment pairs. We also introduce a heuristic algorithm to find binary matrix for  $H$  and compare it with the bounded search tree algorithm for that determines binary matrix for  $H$  via EDGE CLIQUE COVER and discuss the evaluation of their running time on real-world datasets from [14].

We first introduce some terminology.

**Definition 6.** (*Factor*) In an experiment, a factor is an independent variable manipulated by the experimenter. [28]

**Definition 7.** (*Levels*) Levels are the possible values of a factor. In an experiment, each factor contains at least two levels [28].

**Definition 8.** (*Treatments*) A treatment  $T$  is an element in the Cartesian product of set of factors  $F$  and set of levels  $L$ , denoted as  $T \in F \times L$ . [28]

We call experiments that contain multiple factors multifactor experiments. Since scientists want to determine how different levels of factors affect the experimental results, many experiments conducted are multifactor. Figure 6.4 is an example of a multifactor experiment.

		Irrigation regime				
		None	Light	Medium	Heavy	Xheavy
Polymer	No P4	No water No P4 (Treatment 1)	Light water No P4 (Treatment 2)	Medium water No P4 (Treatment 3)	Heavy water No P4 (Treatment 4)	Xheavy water No P4 (Treatment 5)
	With P4	No water With P4 (Treatment 6)	Light water With P4 (Treatment 7)	Medium water With P4 (Treatment 8)	Heavy water With P4 (Treatment 9)	Xheavy water With P4 (Treatment 10)

Figure 6.4: Experimental Design Table for Weight Gain of Golden Torch Cacti[28]

In Figure 6.4, there are two factors in set  $F = \{\text{'No P4'}, \text{'With P4'}\}$  and five levels in set  $L = \{\text{'None'}, \text{'Light'}, \text{'Medium'}, \text{'Heavy'}, \text{'Xheavy'}\}$ . The size of Cartesian product  $F \times L$  is 10. Therefore, there are 10 treatments in this experiment, one of which (Treatment 2) is 'Light water; No P4'.

In an experiment, it is important to determine whether two treatments are significantly different, since treatments have great impact on the outputs of the experiment.

**Definition 9.** (*Significant Difference*) For treatments  $T_1$  and  $T_2$ , if there are differences between experimental results under  $T_1$  and under  $T_2$ , and the differences are not attributed to chance, then we define treatment  $T_1$  and  $T_2$  to be

Treatment Number	Col 1	Col2	Col3
$T_1$		b	
$T_2$	a	b	
$T_3$	a	b	
$T_4$	a		
$T_5$	a		c

Table 6.1: Letter Display Table for  $H = \{(T_1, T_4), (T_1, T_5)\}$

significantly different.

After attaining the results under different treatments, experimenters want to analyze the effects of different treatments on their hypothesis. Therefore, they use a set  $H$  of treatment pairs to store the significant differences between treatments:  $(T_i, T_j) \in H$  if and only if  $T_i$  and  $T_j$  are significantly different from each other.

*Letter display* is a common method to display such a set  $H$ , in which a *letter display table*  $Ta$  is used to exhibit the significant differences between treatments:

1. The rows of  $Ta$  are the treatments in the experiment.
2. Each cell of  $Ta$  is empty or contains a letter.
3. Each letter can appear in at most one column.
4. If  $T_i$  and  $T_j$  are significant different then  $Ta(i, c) \neq Ta(j, c)$  for every column  $c$ .

Table 6.1 is an example of a letter display table for a *significant difference set*  $H = \{(T_1, T_4), (T_1, T_5)\}$ . In Table 6.1,  $T_1$  and  $T_2$  are not significant different. Therefore  $Ta(1, 2) = Ta(2, 2) = b$ .  $T_1$  and  $T_4$  are significantly different: Row 1 and Row 4 do not share the same letter in any column.

For further simplification, we use a binary matrix  $M$  to represent set  $H$  of significant difference between treatments:

1. The rows of matrix  $M$  are the treatment indices.
2.  $M[i][c] = M[j][c] = 1$  for some column  $c$  if  $(T_i, T_j) \in H$ . Otherwise, for every column  $c$ , at most one of  $M[i][c]$  and  $M[j][c]$  equals to 1.

In Figure 6.5,  $M$  is a binary matrix representation for set  $H = \{(T_1, T_4), (T_1, T_5)\}$ . Compared to the letter display table, binary matrix  $M$  takes less space to store the relation of significant difference since  $M$  has same number of rows and columns as  $Ta$  and the space to store a binary bit is only 1/8 of the space to store a character.

$$\begin{bmatrix} 0 & 1 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 0 \\ 1 & 0 & 0 \\ 1 & 0 & 1 \end{bmatrix}$$

Figure 6.5: Binary Matrix  $M$  for  $H = \{(T_1, T_4), (T_1, T_5)\}$

Given  $n$  treatments where  $n$  is a constant, the size of binary matrix  $M$  is defined by its number of columns. To minimize the size of  $M$  while preserving the relation of significant difference between treatments, we introduce CLD-C problem.

COMPACT LETTER DISPLAY (CLD-C) [14]

**Input:** A positive integer  $n$  and a set  $H$  (of size  $m$ ) of integer pairs where  $H = (i_1, j_1), (i_2, j_2), \dots, (i_m, j_m)$  with  $1 \leq i_r, j_r \leq n$  for  $r = 1, \dots, m$

**Parameter:** None

**Question:** Determine a smallest  $M$  to display  $H$ ?

We will show below that CLD-C and EDGE CLIQUE COVER are equivalent problems. Due to this equivalence, we can find a solution to CLD-C by solving EDGE CLIQUE COVER when given a set  $H$  of significant differences between treatments as below:

1. Build a graph  $G = (V, E)$ , where each vertex  $v_i \in V$  is a treatment  $T_i$  that appears in  $H$  and there is an edge  $e_{ij} \in E$  if and only if  $(T_i, T_j) \notin H$ .
2. Find a minimum edge clique cover for graph  $G$  using Algorithm 3.
3. Enumerate each clique in the obtained minimum edge clique cover.
4. Let the size of binary matrix  $M$  be defined by the size of a minimum edge clique cover for graph  $G$ .
5. We next build  $M$  as follows: if  $v_i$  is in clique  $j$  then  $M[i][j] = 1$ . Otherwise,  $M[i][j] = 0$ . Do this until every cell in  $M$  is filled.

Given  $M$  for  $H$  as instructed by above algorithm, assume that  $M$  has  $r$  rows and  $c$  columns.

We show that above algorithm is correct. For this we show that  $M$  satisfies the following two constraints:

1. If  $(T_i, T_j) \notin H$ ,  $M[i][l] = 1$  and  $M[j][l] = 1$  for some  $0 \leq l < c$ .
2. If  $(T_i, T_j) \in H$ ,  $M[i][l] \neq M[j][l]$  or  $M[i][l] = M[j][l] = 0$  for all  $0 \leq l < c$ .

*Proof.* Assume  $M$  is the binary matrix constructed from edge clique cover  $CC$  for graph  $G$ , where  $M$  has  $r$  rows and  $c$  columns.

We first prove the  $M$  satisfies the first constraint. If  $(T_i, T_j) \notin H$ , there is no edge  $e_{ij} \in E$ . When building a edge clique cover  $CC$  for  $G$ , vertex  $i$  and  $j$  appear together in some clique  $C_l \in CC$ , otherwise  $e_{ij}$  is not covered. Since  $i, j \in C_l$ ,  $M[i][l] = 1$  and  $M[j][l] = 1$ .

We now prove that the  $M$  satisfies the second constraint. If  $(T_i, T_j) \in H$ , edge  $e_{ij} \in E$ . Therefore, for every clique  $C_l \in CC$ , either neither  $i, j \notin C_l$ , or only one of  $i$  and  $j$  is in  $C_l$ . Then  $M[i][l] = M[j][l] = 0$  or  $M[i][l] \neq M[j][l]$  for every  $0 \leq l < c$ .

Since both constraints are satisfied,  $M$  is a correct binary matrix to represent  $H$ .  $\square$

To solve the corresponding Edge Clique Cover Problem when given an CLD-C, we can apply a search tree algorithm. In paper [14], the authors compare the search tree algorithm to two heuristic algorithms: Insert-Absorb Heuristic and Clique-Growing Heuristic, where Clique-Growing Heuristic applies the concepts of Edge Clique Cover while Insert-Absorb Heuristic does not.

Dataset	$n$	$ H $	Insert-Absorb			Clique-Growing			Search-Tree		
			Cols	1s	Time	Cols	1s	Time	Cols	1s	Time
Triticale 1	13	55	4	20	0.00	4	20	0.00	4	20	0.00
Triticale 2	17	86	5	32	0.00	5	33	0.00	5	32	0.00
Wheat 1	124	4847	56	986	1.93	50	711	0.20	49	663	4.00
Wheat 2	121	4706	50	902	1.66	48	605	0.17	48	656	3.69
Wheat 3	97	3559	39	691	1.03	32	484	0.15	31	487	2.08
Rapeseed 1	47	576	20	176	0.02	20	149	0.00	20	175	0.04
Rapeseed 2	57	1040	20	244	0.05	20	202	0.01	20	205	0.12
Rapeseed 3	64	1260	24	288	0.08	24	237	0.01	24	232	0.17
Rapeseed 4	62	1085	19	222	0.04	19	207	0.02	19	204	0.11
Rapeseed 5	64	1456	19	259	0.09	19	231	0.03	19	232	0.27
Rapeseed 6	70	1416	27	332	0.12	27	293	0.02	27	301	0.27
Rapeseed 7	74	1758	29	387	0.15	27	356	0.03	25	344	0.35
Rapeseed 8	59	1128	17	215	0.04	17	186	0.01	17	229	0.12
Rapeseed 9	76	1835	30	424	0.21	30	327	0.04	30	332	0.64

Figure 6.6: Comparison of Performances of Insert-Absorb Heuristic, Clique-Growing Heuristic and Search Tree Algorithm on datasets from real-world statistical analyses[14]

Figure 6.6 is the execution results of all three algorithms on different datasets. We can see that Insert-Absorb heuristic has highest running time and leads to

a results of poor quality (high number of columns).

Only the search tree algorithm provides the exact solution for CLD-C, but its running time is exponential, which is only slightly smaller than Insert-Absorb heuristic.

Though Clique-Growing heuristic does not guarantee the minimum number of columns in its result, the quality of its output is very similar to the one from search tree algorithm. Moreover, its running time is much smaller than the Insert-Absorb heuristic and the search tree algorithm, which is  $O(n^5)$  and can be improved to  $O(n^3)$  with an additional sweeping phase [13]. Therefore, Clique-Growing heuristic might be our best choice in this application.

However, if we only need to search for a solution to CLD-C problem of size at most  $k$ , we can reduce CLD-C to  $k$ -EDGE CLIQUE COVER, then use reduction rules in Section 5.2 on the corresponding  $k$ -Edge Clique Cover instance then solve the reduced instance with Algorithm 3.

## 6.2 $k$ -Edge Clique Partition on graph data compression.

When reviewing the literature, we found a number of application papers for EDGE CLIQUE COVER (e.g., [1, 14]) and VERTEX CLIQUE PARTITION (e.g. [3, 11]), but not any paper on EDGE CLIQUE PARTITION.

However, in [1], the authors discuss the application of a variant of EDGE CLIQUE COVER for determining a compact representation of visibility graphs, where a given visibility graph  $G$  is represented by a set of cliques and bipartite cliques to reduce its storing space. This paper inspires us to apply EDGE CLIQUE PARTITION to the problem of compact representation of general graphs: when given a graph  $G$ , we want to store only its edge clique partition  $CP$ , instead of  $V$  and  $E$ .

When we use an edge clique partition  $CP$  accompanied by a set  $S$ , which store singleton in  $G$ , to compactly represent a given graph  $G$ , we store the graph as sets of vertices where repetition of vertices between sets is limited. In our future work, we will investigate whether this representation have an average size of  $O(|V|)$  for general graphs  $G = (V, E)$ .

In this section, we prove the correctness of our approach for a compact representation of general graphs in EDGE CLIQUE PARTITION, and show its efficiency.

When given a graph  $G = (V, E)$ , we generate a compact representation  $CP$  with  $S$  for  $G$  as follows:

1. Include all singletons from  $G$  into a set  $S$ .
2. Compute an edge clique partition  $CP$  for  $G$ .
3.  $CP$  accompanied by  $S$  is the compact representation for  $G$ .

In above algorithm, we need a set  $S$  to store singletons before calculating an edge clique partition  $CP$  for  $G$  since we want to preserve those singletons in  $G$ , which will be discarded during the computation of  $CP$ .

To recover a compressed graph  $G$  from  $CP$  and  $S$ , we include every vertex that appears in  $CP$  and  $S$  as vertex of  $G$  and then connect every pair of vertices  $u, v \in C$  for every clique  $C \in CP$ . The running time for the recovery is linear to the number of edges in  $E$ .

Given a graph  $G = (V, E)$ . We apply EDGE CLIQUE PARTITION to compress  $G$  into  $CP$  and  $S$  then do the recovery as stated above. Assume  $G' = (V', E')$  is a graph recovered from  $CP$ . If  $V = V'$  and  $E = E'$ , then  $CP$  is a correct compact representation for  $G$ .

*Proof.* We first prove  $V = V'$ . All the singletons in  $G$  are included in  $S$  in our algorithm and will be recovered in  $V'$ . Every vertex  $v$  with  $\deg(v) > 1$  is incident to an edge  $e \in E$ , which is covered by some  $C \in CP$ , so  $v$  is in  $C$  and will be recovered in  $V'$  as well. Therefore, in the recovery, no vertex is missing and  $V = V'$ .

Then we prove  $E \subseteq E'$ . For every edge  $uv \in E$ ,  $u, v$  appear together in some  $C \in CP$ . In the recovery, every pair of vertices in  $C$  is connected by an edge, so  $uv \in E'$ . Therefore,  $E \subseteq E'$ .

We now prove  $E' \subseteq E$ . For every pair of vertices  $u, v$  in cliques of  $CP$ , there is an edge  $uv \in E$ , since otherwise the definition of EDGE CLIQUE PARTITION is violated. In the recovery, we only connect pairs of vertices appear in every  $C \in CP$  and include the corresponding edges in  $E'$ . Therefore,  $E' \subseteq E$ .

Finally,  $E \subseteq E'$  and  $E' \subseteq E$  together lead to  $E = E'$  and  $CP$  is a correct compact representation for graph  $G$ . □

Though there is no known polynomial time algorithm to generate a minimum edge clique partition for  $G$ , to store each  $G$  in graph database we only need to compute its minimum edge clique partition  $CP$  with a search tree algorithm (e.g. Algorithm 1) once and can use  $CP$  to recover  $G$  many times. Also, the time to recover  $G$  from  $CP$  is linear to the number of edges in  $G$ . Therefore, edge clique partition  $CP$  is a sound compact representation for  $G$ .

Moreover, there are many existing polynomial time heuristic algorithms to find an edge clique partition for  $G$ . Even a possibly non-minimum edge clique partition might save significant amount space for storing  $G$ .

We next prove that it takes no more space to store a graph  $G$  with edge clique partition  $CP$  than with  $V$  and  $E$  in an adjacency list, since adjacency list is a more space-efficient method to store  $G$  than the adjacency matrix.

*Proof.* Assume there are  $m$  edges in  $G$ . Further assume it takes 1 unit of space to store a vertex.

In an adjacency list, it takes 2 units of space to store an edge  $uv$ , since  $u$  appears in the linked-list rooted at vertex  $v$  and  $v$  appears in the linked-list rooted at vertex  $u$ . The total space to store  $G$  is  $2m$  units.

Now we consider the worst case for our compact representation. If there are no  $K_3$  subgraphs in  $G$ , then every clique in  $CP$  is of size two, since we cannot find a larger clique in  $G$ . It takes  $2m$  units to store  $G$  with  $CP$ . In this case, to store  $G$  with  $CP$  takes the same amount of space as with an adjacency list.

If there is a  $K_\ell$  subgraph in  $G$  such that  $\ell > 2$ , it takes  $\ell^2 - \ell$  units to store the  $K_\ell$  subgraph since there are  $\ell^2 - \ell/2$  edges in  $K_\ell$ . However, if we include  $K_\ell$  with a  $C \in CP$ , it takes only  $\ell$  units of space, which is smaller than storing all edges of  $K_\ell$  in  $E$ .

Therefore, if there is a clique of size at least three in  $CP$  (even  $CP$  is not minimum clique partition), it takes less space to store  $G$  with  $CP$  than with  $V$  and  $E$ .

□

Our compact representation in EDGE CLIQUE PARTITION not only uses less space than adjacency list, but also provides convenience for some graph-based problems such as DOMINATING SET and GRAPH CONNECTIVITY.

First we discuss the upper bound that our compact representation provides to DOMINATING SET. Assume  $CP$  is a size- $k$  edge clique partition for a graph  $G = (V, E)$ . For each  $C \in CP$ , ever pair of vertices  $u, v$  are connected by an edge  $uv$ , and  $CP$  includes all  $v \in V$ . Therefore, picking a  $v$  from each  $C \in CP$  leads to a DOMINATING SET for graph  $G$ . And  $k$  is an upper bound for the DOMINATING SET problem on  $G$ .

GRAPH CONNECTIVITY

**Input:** A graph  $G = (V, E)$

**Parameter:** None

**Question:** Whether  $G$  is a connected graph or not? That is, whether there is a path between every pair of vertices  $v_i, v_j \in G$ .

Then we discuss how to apply our compact representation on GRAPH CONNECTIVITY problem. Assume we determine whether  $G$  is a connected graph and given a size- $k$  edge clique partition  $CP$  for  $G$ . We can make the decision for  $G$  using following algorithm:

1. Build a clique graph  $G' = (V', E')$  for  $CP$ : each  $C_i \in CP$  is an vertex  $v_i \in V'$  and  $v_0 v_1 \in E'$  if and only if  $C_0 \cap C_1 \neq \emptyset$ .
2. Use Breadth-first-search (BFS) to decide whether  $G'$  is a connected graph.
3.  $G$  is a connected graph if and only if  $G'$  is a connected graph.

In this application, we do not need to recover  $G$  and the size of graph connectivity problem is reduced from  $|V|$  to  $k$ .



## Chapter 7

# Future Work

Due to the limitation of time for this project, there are two main tasks to be complete in the future: developing new lemmas and reduction rules for clique-based problem and researching new applications of EDGE CLIQUE PARTITION.

### 7.1 New Lemma and Reduction Rules

In our project, we research in the natural properties and existing FPT algorithms for clique-based problem, from which we have developed a number of new lemmas and reduction rules as well.

In the future, we will apply those new rules on graphs together with existing reduction rules and examine that whether they can further reduce the running time of current FPT algorithms in Chapter 5.

Below is the new lemmas and reduction rules we develop originally in this project, followed by the proof for their correctness.

**Lemma 11.** *Given a graph  $G = (V, E)$ , let  $CP$  be an EDGE CLIQUE PARTITION of  $G$  of size at most  $k$ . If  $G_0 = (V_0, E_0)$  is a complete subgraph of  $G$  with more than  $k$  vertices, then there is an element  $C \in CP$  such that  $V_0 \subseteq C$ . [22]*

*Proof.* Since  $CP$  is an EDGE CLIQUE PARTITION for graph  $G$  and  $E_0 \subseteq E$ , according to Lemma 1, if there is no  $C \in CP$  such that  $V_0 \subseteq C$ , then we need at least  $k + 1$  cliques to partition edges in  $E_0$  and even more cliques to partition edges in  $E$ , which contradicts to  $CP$  is of size at most  $k$ .  $\square$

**Reduction Rule 9.** *Let  $I = \langle G, k \rangle$  be an instance of  $k$ -EDGE CLIQUE PARTITION. Suppose there are 3 vertices  $u, v, w \in V$ , such that  $uv, uw, vw \in E$  and  $\deg(u) = 2, \deg(v) = 2, \deg(w) \geq 2$ . Then  $I = \langle G, k \rangle$  is yes if and only if  $I' = \langle G - u - v, k - 1 \rangle$  is yes.*

*Proof.* We prove that  $I$  is a yes-instance if and only if  $I'$  is a yes-instance.

We first prove if  $I$  is a yes-instance then  $I'$  is a yes-instance. Let  $CP$  be an optimal solution of size at most  $k$  for  $I$ . To partition edges  $uv, uw, vw$ , if we don't include clique  $C(u, v, w)$ , we need to include all  $C(u, v), C(u, w)$  and  $C(v, w)$  in our solution. Therefore  $C(u, v, w)$  is a member of  $CP$  otherwise  $CP$  is not optimal.  $CP' = CP - C(u, v, w)$  is a partition that covers all edges in  $G$  but not in  $E(u, v, w)$ , which means it is a partition of size at most  $k - 1$  for  $I' = \langle G - u - v, k - 1 \rangle$ .

Then we prove if  $I'$  is a yes-instance then  $I$  is a yes-instance. Assume  $CP'$  is a solution of size at most  $k - 1$  for  $I'$ . Clique  $C(u, v, w)$  partitions edges  $uv, uw, vw$ . Since  $CP'$  partitions all edges in graph  $G - u - v$ , then  $CP = CP' \cup \{C(u, v, w)\}$  is a partition of size at most  $k$  which partitions all edges in  $G$ . □

**Reduction Rule 10.** Let  $\langle G, k \rangle$  be an instance of  $k$ -EDGE CLIQUE PARTITION. Suppose  $G_1 = (V_1, E_1)$  and  $G_2 = (V_2, E_2)$  are two subgraphs of  $G$ , where  $G_1$  is complete. Further assume  $V_1 \cap V_2 = v$ ,  $V_1 \cup V_2 = V$ ,  $E_1 \cap E_2 = \emptyset$  and  $E_1 \cup E_2 = E$ .  $I = \langle G, k \rangle$  is a yes-instance if and only if  $I' = \langle G_2, k - 1 \rangle$  is a yes-instance.

*Proof.* We prove that  $I$  is a yes-instance if and only if  $I'$  is a yes-instance.

We first prove if  $I$  is a yes-instance then  $I'$  is a yes-instance. Let  $CP$  be an minimum solution of size  $k$  for  $I$ . If we don't include  $C_1 = V_1$  in  $CP$ , according to Lemma 1, we need at least  $|V_1|$  cliques to partition all edges in  $G_1$ . And none of those  $|V_1|$  cliques contains any vertex from  $V_2 - \{v\}$ . Therefore  $C_1 \in CP$  otherwise  $CP$  is not minimum. Then  $CP' = CP - \{C_1\}$  partitions all edges in graph  $G - G_1$ , which is the subgraph  $G_2$ . Then  $CP'$  is a solution of size at most  $k - 1$  for  $I' = \langle G_2, k - 1 \rangle$ .

Then we prove if  $I'$  is a yes-instance then  $I$  is a yes-instance. Clique  $C_1 = V_1$  partition all edges in  $G_1$  since  $G_1$  is complete. Then if we find a solution  $CP'$  of size at most  $k - 1$  for  $I'$  then  $CP = CP' \cup \{C_1\}$  is a solution of size at most  $k$  for  $I$ . □

**Reduction Rule 11.** Let  $I = \langle G, k \rangle$  be an instance of CLIQUE PARTITION. Suppose  $G_1 = (V_1, E_1)$  and  $G_2 = (V_2, E_2)$  are two subgraphs of  $G$ . Assume  $V_1 \cap V_2 = v$ ,  $V_1 \cup V_2 = V$ ,  $E_1 \cap E_2 = \emptyset$  and  $E_1 \cup E_2 = E$ . Further assume the size of minimum EDGE CLIQUE PARTITION for subgraph  $G_1$  is  $k_1$ .  $I = \langle G, k \rangle$  is a yes-instance if and only if  $I' = \langle G_2, k - k_1 \rangle$  is a yes-instance.

*Proof.* We prove that  $I$  is a yes-instance if and only if  $I'$  is a yes-instance.

We first prove if  $I$  is a yes-instance then  $I'$  is a yes-instance. Let  $CP$  be a minimum solution of size  $k$  for instance  $I$ . Let  $CP_1$  be a minimum EDGE CLIQUE PARTITION for  $G_1$  and  $CP_2$  be a minimum EDGE CLIQUE PARTITION for  $G_2$ . Since the only intersection of  $G_1$  and  $G_2$  is vertex  $v$ ,  $CP_1 \cap CP_2 = \emptyset$  and  $CP_1 \cup CP_2 = CP$ . We know  $CP_1$  is of size  $k_1$  and  $CP$  is of size at most  $k$ , then  $CP_2 = CP \setminus CP_1$  is of size at most  $k - k_1$ .

Then we prove that if  $I'$  is a yes-instance then  $I$  is a yes-instance. Since  $I'$  is a yes-instance, let  $CP_2$  be an EDGE CLIQUE PARTITION of size at most  $k - k_1$  for  $I'$ . Since the size of  $CP_1$ , the minimum EDGE CLIQUE PARTITION for subgraph  $G_1$ , is  $k_1$ ,  $CP = CP_1 \cup CP_2$  is a solution for  $I$ . The size of  $CP$  is  $|CP_1| + |CP_2|$ , which is at most  $k$ . □

**Lemma 12.** *Let  $I = \langle G, k \rangle$  be an instance of EDGE CLIQUE PARTITION. Suppose that  $G_1, G_2, \dots, G_m$  are all subgraphs of  $G$ . Assume  $V_1 \cup V_2 \cup \dots \cup V_m = V$ ,  $E_1 \cup E_2 \cup \dots \cup E_m = E$  and the pairwise intersections of  $G_1, G_2, \dots, G_m$  are vertex  $v$ . If  $m = k$ ,  $(G, k)$  is yes if and only if all  $V(G_1), V(G_2), \dots, V(G_m)$  are cliques.*

*Proof.* If all  $V(G_1), V(G_2), \dots, V(G_m)$  are cliques,  $CP = \{V(G_1), V(G_2), \dots, V(G_m)\}$  is a solution of size  $k$  for  $I$ .

Assume  $V(G_1)$  is not a clique. Since pairwise intersections of  $G_1, G_2, \dots, G_m$  are vertex  $v$ , we need at least three cliques to partition edges  $E(G_1)$  and need at least one clique to partition edges from each of  $E(G_2), E(G_3), \dots, E(G_m)$ , which makes solution  $CP$  is of size at least  $k + 2$ . Then  $I$  is a no-instance. □

**Lemma 13.** *Let  $I = \langle G, k \rangle$  be an instance of EDGE CLIQUE PARTITION. Suppose that  $G_1, G_2, \dots, G_m$  are all subgraphs of  $G$ . Assume  $V_1 \cup V_2 \cup \dots \cup V_m = V$ ,  $E_1 \cup E_2 \cup \dots \cup E_m = E$  and the pairwise intersection of  $G_1, G_2, \dots, G_m$  are vertex  $v$ . If  $m > k$ ,  $(G, k)$  is a no-instance.*

*Proof.* Since the pairwise intersection of  $G_1, G_2, \dots, G_m$  are vertex  $v$ , we need at least one clique to partition edges from each of  $E(G_2), E(G_3), \dots, E(G_m)$ . Therefore we need at least  $m > k$  cliques to partition edges in  $G$  and  $I$  is a no-instance. □

Note that if  $m < k$ , we can apply *Rule6* to shrink the instance.

## 7.2 New applications of EDGE CLIQUE PARTITION.

In this project, we have found a number of applications in EDGE CLIQUE COVER,  $k$ -CLIQUE, VETEX CLIQUE PARTITION. Though there are quite a

few papers describing the FPT algorithms and the running times complexity of EDGE CLIQUE PARTITION problem, we have not found any application papers on EDGE CLIQUE PARTITION.

However, we think the application of EDGE CLIQUE PARTITION is a field with great potential. It constrains the number of repetition between cliques in the solution while covering every edge in the graph. We think this property helps to model many real-world problems and provides them with sound solutions or strategies.

In our project, we have discovered some possible applications of EDGE CLIQUE PARTITION. In our future work, we will try to apply EDGE CLIQUE PARTITION on them and prove the correctness and efficiency of our applications.

### **7.2.1 Application of EDGE CLIQUE PARTITION on Administration of Clustered Communication on Distributed network.**

In distributed network, a cluster of computers work in parallel to execute a given task, which is often of large size. Occasionally, there is a need for computers in a cluster to communicate with each other and those communications need to be administrated and monitor by master servers.

If a communication between two computers is not administrated by a master server, there might appear consensus problems and inconsistency problems. If one communication is administrated by more than two different master serve, there exists a waste of resources and the communication might receive conflicting commands from different master servers.

Therefore, we can create a graph  $G = (V, E)$  where each computer is a vertex  $v \in V$  and there is an edge  $uv \in E$  if and only if communication between  $u, v$  are required to be administrated by a master server. Then we EDGE CLIQUE PARTITION on  $G$  to partition computers into different cliques and compute an minimum edge clique partition  $CP$  for  $G$ .

Finally, we assign a master server to a clique  $C$  to administrate all the communications between vertices of  $C$ , by doing which every required communication is administrated by exactly one master server while the number of total master servers are minimum.

In the future, we will use EDGE CLIQUE PARTITION on both real-world distributed network and distributed network simulated by Omnet++, in order to examine whether EDGE CLIQUE PARTITION creates a good assignment of master servers on communications on computer clusters. We mainly evaluate the occupation rate of master servers and the average waiting time of communica-

tions to be administrated: a high occupation rate with low average waiting time means a good resource assignment.

### 7.2.2 Application of EDGE CLIQUE PARTITION on compact representation of graphs.

As it is discussed in Section 6.2, we can represent graph  $G$  compactly with an edge clique partition  $CP$  instead of an adjacency list. We have proved the correctness of our compact representation and that the space to store  $CP$  is less than or equal to the adjacency list.

We have showed the best-case and worst-case space complexity of our compact representation. However, to determine whether our compact representations provides benefits on reducing the space of real-world datasets, we think the average space complexity is more important to us. Therefore, we will prove the average space complexity of our compact representation in the future, as well as running the compression on real-world datasets.

Also, we have stated a few applications that can be computed from our edge clique partition  $CP$  without recovering the original graph. To determine whether our compact representation can really benefits those applications is another work to be done in the future.

## Chapter 8

# Conclusion

This is an report on clique-based problems with their applications. We have introduced quite a few clique-based problems, but the main focus are on `EDGE CLIQUE PARTITION` and `EDGE CLIQUE COVER`. We not only survey on the existing FPT algorithms with running time analyzing on those two problems, but also discuss applications of them from papers.

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