# ON THE GENERALIZATION ERROR OF NORM PENALTY LINEAR REGRESSION MODELS

JOSÉ LUIS MONTIEL OLEA, CYNTHIA RUSH, AMILCAR VELEZ, AND JOHANNES WIESEL

ABSTRACT. We study linear regression problems  $\inf_{m{\beta} \in \mathbb{R}^d} \left( \mathbb{E}_{\mathbb{P}_n}[|Y - \mathbf{X}^\top \boldsymbol{\beta}|^r] \right)^{1/r} + \delta \rho(\boldsymbol{\beta})$ , with  $r \geq 1$ , convex penalty  $\rho$ , and empirical measure of the data  $\mathbb{P}_n$ . Well known examples include the square-root lasso, square-root sorted  $\ell_1$  penalization, and penalized least absolute deviations regression. We show that, under benign regularity assumptions on  $\rho$ , such procedures naturally provide robust generalization, as the problem can be reformulated as a distributionally robust optimization (DRO) problem for a type of max-sliced Wasserstein ball  $B_{\delta}^{\rho}(\mathbb{P}_n)$ , i.e.  $\hat{\boldsymbol{\beta}}$  solves the linear regression problem iff it solves  $\inf_{m{\beta} \in \mathbb{R}^d} \sup_{\mathbb{Q} \in B_{\delta}^{\rho}(\mathbb{P}_n)} \mathbb{E}_{\mathbb{Q}}[|Y - \mathbf{X}^\top \boldsymbol{\beta}|^r]$ . Our proof of this result is constructive: it identifies the worst-case measure in the DRO problem, which is given by an additive perturbation of  $\mathbb{P}_n$ . We argue that  $B_{\delta}^{\rho}(\mathbb{P}_n)$  are the natural balls to consider in this framework, as they provide a computationally efficient procedure comparable to non-robust methods and optimal robustness guarantees. In fact, our generalization bounds are of order d/n, up to logarithmic factors, and thus do not suffer from the curse of dimensionality as is the case for known generalization bounds when using the Wasserstein metric on  $\mathbb{R}^d$ . Moreover, the bounds provide theoretical support for recommending a regularization parameter  $\delta$  of the same order for the linear regression problem.

#### 1. Introduction

We study the classical problem of predicting an outcome variable, Y, using a linear combination of a d-dimensional covariate vector,  $\mathbf{X}$ . A sample  $((\mathbf{x}_1, y_1), \dots, (\mathbf{x}_n, y_n))$  is available and we are interested in linear predictors whose coefficients solve:

(1) 
$$\inf_{\boldsymbol{\beta} \in \mathbb{R}^d} \sqrt[r]{\frac{1}{n} \sum_{i=1}^n |y_i - \mathbf{x}_i^\top \boldsymbol{\beta}|^r} + \delta \rho(\boldsymbol{\beta}),$$

where  $\delta > 0$  is the regularization parameter,  $\rho : \mathbb{R}^d \to \mathbb{R}_+$  is a convex penalty function, and  $r \geq 1$ . The square-root Lasso [2], the square-root group Lasso [13], the square-root sorted  $\ell_1$  penalized estimator [40], and the  $\ell_1$ -penalized least absolute deviation estimator [43] provide examples of estimators obtained by solving (1).

This paper makes two contributions. We first show that linear predictors that solve (1) optimize the worst-case, out-of-sample prediction error over balls defined by the max-sliced Wasserstein metric (MSW)

(2) 
$$\widehat{\mathcal{W}}_{r,\rho,\sigma}(\mathbb{P},\widetilde{\mathbb{P}}) := \sup_{\boldsymbol{\gamma} \in \mathbb{R}^d} \inf_{\substack{\boldsymbol{\pi} \in \Pi(\mathbb{P},\widetilde{\mathbb{P}}):\\ ((\mathbf{X},Y),(\widetilde{\mathbf{X}},\widetilde{Y})) \sim \boldsymbol{\pi}}} \frac{1}{\sigma + \rho(\boldsymbol{\gamma})} \left( \mathbb{E}_{\boldsymbol{\pi}} \left[ \left| (\widetilde{Y} - Y) + (\mathbf{X} - \widetilde{\mathbf{X}})^{\top} \boldsymbol{\gamma} \right|^r \right] \right)^{1/r}.$$

Here, the set  $\Pi(\mathbb{P}, \widetilde{\mathbb{P}})$  denotes the collection of probability measures with marginals  $(\mathbb{P}, \widetilde{\mathbb{P}})$ . We assume  $r, \sigma \geq 1$ , where r denotes the Wasserstein exponent, while we think of  $\sigma$  as a hyperparameter controlling the moments of Y under a baseline measure. One can check that  $\widehat{W}_{r,\rho,\sigma}$  is a metric for

any convex  $\rho$ . In comparison to the d-dimensional Wasserstein metric, defined by

(3) 
$$\mathcal{W}_r(\mathbb{P}, \widetilde{\mathbb{P}}) := \inf_{\substack{\pi \in \Pi(\mathbb{P}, \widetilde{\mathbb{P}}): \\ ((\mathbf{X}, Y), (\widetilde{\mathbf{X}}, \widetilde{Y})) \sim \pi}} \left( \mathbb{E}_{\pi} \left[ \left| (\widetilde{\mathbf{X}}, \widetilde{Y}) - (\mathbf{X}, Y) \right|^r \right] \right)^{1/r},$$

the MSW metric,  $\widehat{W}_r(\mathbb{P}, \mathbb{P})$ , is less rigid, as it takes only the one-dimensional projections of the random vector  $(\mathbf{X}, Y)$  into account. If  $\rho(\cdot) = \|\cdot\|_2$  is equal to the Euclidean norm, it is indeed easy to see that  $\widehat{W}_r(\mathbb{P}, \widetilde{\mathbb{P}}) \leq W_r(\mathbb{P}, \widetilde{\mathbb{P}})$ , see Section 3. The MSW derives from the sliced Wasserstein metric, which was introduced in [34, 11] due to its computational benefits: computing the Wasserstein distance in  $\mathbb{R}^d$  is very costly in high-dimensional problems, while the computational cost for d=1 is relatively small. The MSW was first considered in [32, 31, 16].

More precisely, in our first contribution we show that  $\widehat{\beta}$  solves (1) if and only if it solves the distributionally robust optimization problem

(4) 
$$\inf_{\beta \in \mathbb{R}^d} \sup_{\widetilde{\mathbb{P}} \in B^{r,\rho,\sigma}_{\delta}(\mathbb{P}_n)} \mathbb{E}_{\widetilde{\mathbb{P}}}[|Y - \mathbf{X}^{\top} \boldsymbol{\beta}|^r].$$

The ball  $B^{r,\rho,\sigma}_{\delta}(\mathbb{P}_n)$  is centered around the empirical distribution of the data,  $\mathbb{P}_n$ , and collects all the distributions  $\widetilde{\mathbb{P}}$  for which  $\widehat{\mathcal{W}}_{r,\rho,\sigma}(\mathbb{P}_n,\widetilde{\mathbb{P}})$  is smaller than  $\delta$ .

A version of the above result (4) for the case r=2, penalty  $\rho(\cdot) = \|\cdot\|_p$ ,  $p \ge 1$ , and  $\widehat{\mathcal{W}}_r$  replaced by  $\mathcal{W}_r$  was first established in [5, Theorem 1], using the optimal transport (OT) duality of [22, 8]. Our proof is independent and does not rely on this duality. Instead, it explicitly identifies a worst-case measure  $\mathbb{P}^* \in B^{r,\rho,\sigma}_{\delta}(\mathbb{P}_n)$  for (4). The measure  $\mathbb{P}^*$  is given by an additive perturbation of  $\mathbb{P}_n$ , see Figure 1 for an illustration in the  $\sqrt{\text{LASSO}}$  case. In this sense, our proof can be seen as a natural extension of [4, Theorem 1] to the Wasserstein space.

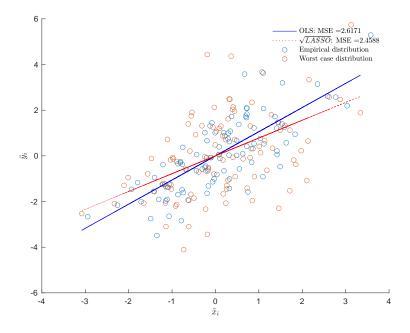


FIGURE 1. OLS regression vs.  $\sqrt{LASSO}$  regression trained with the empirical distribution (blue) and evaluated on a sample form the worst case distribution (red).

Defining  $B_{\delta}^{r,\rho,\sigma}(\mathbb{P}_n)$  in (4) with respect to  $\widehat{W}_{r,\rho,\sigma}$  is new to the best of our knowledge. In fact, we believe that  $\widehat{W}_{r,\rho,\sigma}$  is the natural Wasserstein metric to consider for (4). Next to its lower computational complexity, the main statistical reason for this is that the d-dimensional Wasserstein metric  $W_r$  leads to undesirable artifacts for statistical inference: it captures a high-dimensional structure of  $\mathbb{P}_n$  that does not actually appear in (4) due to the linear projections of the covariates, see e.g. [26]. On the other hand,  $\widehat{W}_{r,\rho,\sigma}$  is in itself a max-min problem over one-dimensional projections, mirroring the min-max problem over linear measurements  $\mathbf{X}^{\top}\boldsymbol{\beta}$  in (4). In conclusion, our procedure is computationally efficient, while simultaneously providing rigorous robustness guarantees.

Our second contribution is a detailed analysis of the statistical properties of (2). We are particularly interested in understanding how to choose  $\delta$  to guarantee that predictors based on (4) have good generalization at the true, unknown distribution of the data  $\mathbb{P}$ . We focus on the case in which  $\rho(\cdot) = \|\cdot\|_2$  but, as we detail in Remark 3, Section 3, the result then generalizes to all norms on  $\mathbb{R}^d$ , as any norm  $\|\cdot\|$  is equivalent to  $\|\cdot\|_2$ . We show that if

$$\Gamma := \mathbb{E}_{\mathbb{P}} \left[ \sup_{\boldsymbol{\gamma} : \ \|\boldsymbol{\gamma}\|_2 = 1} \left| (\mathbf{X}, Y)^{\top} \boldsymbol{\gamma} \right|^s \right] < \infty, \quad \text{for some } s > 2r,$$

then with probability greater than  $1 - \alpha$  we have

(5) 
$$\widehat{\mathcal{W}}_{r,\rho,\sigma}(\mathbb{P}_n,\widetilde{\mathbb{P}})^r \le \frac{C \log(2n+1)^{r/s}}{\sqrt{n}},$$

where

$$C := 2^r r \left(180\sqrt{d+2} + \sqrt{2\log\left(\frac{3}{\alpha}\right)} + \sqrt{\frac{3\Gamma}{\alpha}} \frac{8}{s/2 - r} \sqrt{\log\left(\frac{24}{\alpha}\right) + (d+2)}\right).$$

Thus, our statistical analysis provides a concrete recommendation to select the regularization parameter  $\delta$  in (1) to be equal to the right-hand side of (5). These rates are optimal, up to logarithmic factors (cf. e.g. [20, Remark (a), p.2]).

The proof of this result is stated in Section 3, and is based on a novel connection between an inequality for the Wasserstein distance for d=1, and classical bounds from empirical process theory for self-normalized processes. Its relative simplicity enables us to give the explicit constants above.

1.1. Related Literature. The distributionally robust optimization problem in (4) has been shown to be equivalent to various forms of penalized regression, variance-penalized estimation, and dropout training [29, 23, 25, 7, 8, 18, 30], depending on the choice of uncertainty set. The equivalence between (1) and (4) has been established in [5, Theorem 1] using a ball defined in terms of the Wasserstein metric, which is a common choice for the uncertainty set in the distributionally robust optimization literature [24, 6, 39, 26, 23, 29, 21, 37]; however, there are many other metrics one can consider: e.g., total variation, Hellinger, Gelbrich distance or Kullback-Leibler (KL) divergence [35]. It is known that for certain cost and metric, the optimization in (4) is equivalent to dropout training, various forms of penalized regression, or variance-penalized estimation [29, 23, 25, 7, 8, 18, 30].

Starting from [15, 20], the question of establishing finite sample bounds on the Wasserstein metric and its variants has seen a spike in research activity over the last years: an incomplete list is [10, 38, 44, 45, 27, 14], see also the references therein. Tight rates for  $W_r(\mathbb{P}_n, \mathbb{P})$  are of the order  $n^{-1/(rd)}$ , i.e. they suffer from the curse of dimensionality. As our results shows, this is not the case for the MSW distance. The faster rates of convergence for the max-sliced Wasserstein metric were first observed in [31] for subgaussian probability measures and in [28] under a projective

Poincaré/Bernstein inequality. More recently, [1] have obtained sharp rates for r=2 and isotropic distributions. Our rates are of the same order, up to logarithmic factors, and simultaneously hold for all  $r \geq 1$  and all distributions with finite higher-order moments. Lastly, let us mention that most of the papers cited above only give explicit rates, while the constants are often non-explicit and large, cf. [19].

- 1.2. **Outline.** The rest of the paper is organized as follows. We give a detailed discussion of our first contribution, the equivalence of (1) and (4) in Section 2. Section 3 discusses our second contribution: it states rates for the MSW distance  $\widehat{W}_{r,\rho,\sigma}$  between the true and empirical measure, both for  $\mathbb{P}$  with compact support and for  $\mathbb{P}$  satisfying  $\Gamma < \infty$ . Section 4 collects remaining proofs, while Section 1.3 states the notation used throughout the paper.
- 1.3. **Notation.** We use capital, bold letters—such as  $\mathbf{Z}$  and  $\widetilde{\mathbf{Z}}$ —to denote Borel measurable random vectors in  $\mathbb{R}^d$ , and use  $Z_j$  to denote the j-th coordinate of  $\mathbf{Z}$ . We denote the set of all Borel probability measures in  $\mathbb{R}^d$  by  $\mathcal{P}(\mathbb{R}^d)$  and let  $\mathcal{P}_r(\mathbb{R}^d) \subset \mathcal{P}(\mathbb{R}^d)$  denote all Borel probability measures with finite rth moments. If the random vector  $\mathbf{Z}$  has distribution or law  $\mathbb{P} \in \mathcal{P}(\mathbb{R}^d)$ , we write  $\mathbf{Z} \sim \mathbb{P}$ . The expectation of  $\mathbf{Z}$  is denoted as  $\mathbb{E}_{\mathbb{P}}[\mathbf{Z}]$ .

For two probability measures  $\mathbb{Q}$  and  $\mathbb{P}$ , we define a *coupling* of  $\mathbb{Q}$  and  $\mathbb{P}$  as any element of  $\mathcal{P}(\mathbb{R}^d \times \mathbb{R}^d)$  that preserves the marginals over  $\mathbb{R}^d$ . We denote the collection of all such couplings as  $\Pi(\mathbb{Q}, \mathbb{P})$ . By definition, if  $(\widetilde{\mathbf{Z}}, \mathbf{Z})$  is an  $\mathbb{R}^d \times \mathbb{R}^d$ -valued random vector with distribution  $\pi \in \Pi(\mathbb{Q}, \mathbb{P})$ , then  $\widetilde{\mathbf{Z}} \sim \mathbb{Q}$  and  $\mathbf{Z} \sim \mathbb{P}$ .

A function  $\rho: \mathbb{R}^d \to [-\infty, \infty]$  is said to be *proper* if  $\rho(\beta) < \infty$  for at least one  $\beta \in \mathbb{R}^d$  and  $\rho(\beta) > -\infty$  for every  $\beta \in \mathbb{R}^d$  [36, p. 24]. The *effective domain* of  $\rho$ —denoted dom $(\rho)$ —is the set of all elements  $\beta \in \mathbb{R}^d$  for which  $\rho(\beta) < \infty$ . The conjugate of  $\rho(\cdot)$  (see [36]; p. 104) is defined as the function

$$\rho^*(\boldsymbol{\beta}) := \sup_{\mathbf{x} \in \mathbb{R}^d} \left\{ \boldsymbol{\beta}^\top \mathbf{x} - \rho(\mathbf{x}) \right\}.$$

If  $\rho$  is convex, a vector  $\boldsymbol{\beta}^*$  is said to be a subgradient of  $\rho$  at a point  $\boldsymbol{\beta}$  if:

$$\rho(\mathbf{x}) \ge \rho(\boldsymbol{\beta}) + {\boldsymbol{\beta}^*}^{\top}(\mathbf{x} - \boldsymbol{\beta}), \qquad \forall \mathbf{x} \in \mathbb{R}^d.$$

The set of all subgradients of  $\rho$  at  $\beta$  is called the subdifferential of  $\rho$  at  $\beta$  and is denoted  $\partial \rho(\beta)$ , ([36]; pp. 214-215).

Lastly, let us mention two important facts that will be relevant in Section 2.1. If  $\rho$  is differentiable, then its subdifferential  $\partial \rho(\beta)$  is a singleton that contains the gradient of  $\rho$  at  $\beta$ ; see, for example, Theorem 25.1 in ([36]). If  $\rho$  is a norm in  $\mathbb{R}^d$ , then  $\rho^*$  is only equal to zero or infinity; see ([12]; p. 93).

## 2. Equivalence of penalized estimation and DRO

Let  $\rho \colon \mathbb{R}^d \to [0, +\infty]$  be a non-negative, proper, convex function with effective domain,  $\operatorname{dom}(\rho)$ , and conjugate  $\rho^*$ . For any fixed  $1 \le r < \infty$  and  $1 \le \sigma < \infty$ , define the ball of distributions

$$\begin{split} B^{r,\rho,\sigma}_{\delta}(\mathbb{P}) &:= \left\{ \mathbb{Q} \in \mathcal{P}_r(\mathbb{R}^d) : \widehat{\mathcal{W}}_r(\mathbb{Q},\mathbb{P}) \leq \delta \right\} \\ &= \left\{ \mathbb{Q} \in \mathcal{P}_r(\mathbb{R}^d) : \ \forall \boldsymbol{\gamma} \in \operatorname{dom}(\rho) \ \exists \ \operatorname{a \ coupling} \ \pi(\boldsymbol{\gamma}) \in \Pi(\mathbb{P},\mathbb{Q}) \ \operatorname{for \ which} \right. \\ & \mathbb{E}_{\pi} \big[ \big| (\widetilde{Y} - Y) + (\mathbf{X} - \widetilde{\mathbf{X}})^\top \boldsymbol{\gamma}) \big|^r \big] \leq \delta^r \big( \sigma + \rho(\boldsymbol{\gamma}) \big)^r, \ \operatorname{where} \ \big( (\mathbf{X}, Y), \ (\widetilde{\mathbf{X}}, \widetilde{Y}) \big) \sim \pi \right\}. \end{split}$$

In the above, notice that the penalty function  $\rho$  controls the possible deviations from the baseline distribution of the covariates. For notational simplicity we will suppress the dependence of the ball  $B_{\delta}^{r,\rho,\sigma}(\mathbb{P})$  on  $r,\rho,\sigma$  in the following and write  $B_{\delta}(\mathbb{P})$  instead.

We remark that the infimum in (6) is attained for fixed  $\gamma$ : indeed, note that the function  $(\mathbf{x}, \widetilde{\mathbf{x}}) \mapsto |(\widetilde{\mathbf{x}} - \mathbf{x})^{\top} \gamma|^r$  is continuous and non-negative. The result then follows from [41, Theorem 4.1]. Furthermore, for any norm  $\rho$ , the supremum over  $\gamma \in \text{dom}(\rho)$  is also attained.

**Remark 1.** Take any norm  $\|\cdot\|$  on  $\mathbb{R}^{d+1}$  satisfying  $\|(1,0,\ldots,0)\|=1$  and recall that its dual norm is given by

(7) 
$$\|\mathbf{x}\|_* := \sup_{\mathbf{y}: \|\mathbf{y}\| = 1} \mathbf{x}^\top \mathbf{y}.$$

Assume that  $\mathbb{E}_{\mathbb{P}}[\|(\mathbf{X},Y)\|_{*}^{r}] < \infty$  and consider a Wasserstein ball  $\mathcal{B}_{\delta}(\mathbb{P})$  with cost  $\|\cdot\|_{*}$ , defined as

(8) 
$$\mathcal{B}_{\delta}(\mathbb{P}) = \left\{ \widetilde{\mathbb{P}} \in \mathcal{P}_r(\mathbb{R}^d) : \ \mathcal{W}_r(\mathbb{P}, \widetilde{\mathbb{P}}) \leq \delta \right\},$$

where

$$\mathcal{W}_r(\mathbb{P}, \widetilde{\mathbb{P}}) = \inf_{\substack{\pi \in \Pi(\mathbb{P}, \widetilde{\mathbb{P}}): \\ ((\mathbf{X}, Y), (\widetilde{\mathbf{X}}, \widetilde{Y})) \sim \pi}} \sqrt[r]{\mathbb{E}_{\pi} \left[ \| (\mathbf{X}, Y) - (\widetilde{\mathbf{X}}, \widetilde{Y}) \|_*^r \right]}.$$

We show that for  $\rho(\gamma) = ||\gamma||$ , the ball defined in (6) contains the ball in (8), i.e.  $\mathcal{B}_{\delta}(\mathbb{P}) \subseteq B_{\delta}(\mathbb{P})$ . For this, we note that by (7) we have

$$\mathbb{E}_{\pi} \left[ \left| (\widetilde{Y} - Y) + (\mathbf{X} - \widetilde{\mathbf{X}})^{\top} \boldsymbol{\gamma} \right|^{r} \right] \leq \|(\boldsymbol{\gamma}, -1)\|^{r} \mathbb{E}_{\pi} \left[ \|(\mathbf{X}, Y) - (\widetilde{\mathbf{X}}, \widetilde{Y})\|_{*}^{r} \right] \\ \leq (1 + \|\boldsymbol{\gamma}\|)^{r} \mathbb{E}_{\pi} \left[ \|(\mathbf{X}, Y) - (\widetilde{\mathbf{X}}, \widetilde{Y})\|_{*}^{r} \right]$$

so that

$$\sup_{\boldsymbol{\gamma} \in \text{dom}(\rho)} \inf_{\boldsymbol{\pi} \in \Pi(\mathbb{P}, \widetilde{\mathbb{P}})} \frac{1}{\sigma + \|\boldsymbol{\gamma}\|} \sqrt[r]{\mathbb{E}_{\boldsymbol{\pi}} \left[ \left| (\widetilde{Y} - Y) + (\mathbf{X} - \widetilde{\mathbf{X}})^{\top} \boldsymbol{\gamma} \right|^{r} \right]} \\ \leq \inf_{\boldsymbol{\pi} \in \Pi(\mathbb{P}, \widetilde{\mathbb{P}})} \sqrt[r]{\mathbb{E}_{\boldsymbol{\pi}} \left[ \left\| (\mathbf{X}, Y) - (\widetilde{\mathbf{X}}, \widetilde{Y}) \right\|_{*}^{r} \right]}.$$

The above can be applied in particular to  $\|\cdot\| = \|\cdot\|_p$  and  $\|\cdot\|_* = \|\cdot\|_q$ , where 1/p + 1/q = 1.

Next, we establish a connection between the solutions to the penalized regression problem in (1) and the minimax formulation in (4) using the MSW ball  $B_{\delta}(\mathbb{P})$  defined in (6).

**Theorem 1.** For fixed  $1 \le r < \infty$ , suppose that, for any  $\beta \in \text{dom}(\rho)$ , there exists  $\beta^* \in \partial \rho(\beta)$  such that

(9) 
$$|\gamma^{\top} (\beta^* - \frac{\beta}{\beta^{\top} \beta} \rho^* (\beta^*))| \le \rho(\gamma), \qquad \forall \gamma \in \text{dom}(\rho).$$

Then, for any  $\delta \geq 0$  and any  $\beta \in dom(\rho)$  we have

(10) 
$$\sup_{\widetilde{\mathbb{P}}\in B_{\delta}(\mathbb{P})} \mathbb{E}_{\widetilde{\mathbb{P}}} \left[ \left| Y - \mathbf{X}^{\top} \boldsymbol{\beta} \right|^{r} \right] = \left( \sqrt[r]{\mathbb{E}_{\mathbb{P}} \left[ \left| Y - \mathbf{X}^{\top} \boldsymbol{\beta} \right|^{r} \right]} + \delta \left( \sigma + \rho(\boldsymbol{\beta}) \right) \right)^{r}.$$

Clearly, the result in (10) implies

(11) 
$$\operatorname{arg\ min}_{\boldsymbol{\beta} \in \operatorname{dom}(\rho)} \sup_{\widetilde{\mathbb{P}} \in B_{\delta}(\mathbb{P})} \mathbb{E}_{\widetilde{\mathbb{P}}} \left[ \left| Y - \mathbf{X}^{\top} \boldsymbol{\beta} \right|^{r} \right] = \operatorname{arg\ min}_{\boldsymbol{\beta} \in \operatorname{dom}(\rho)} \sqrt[r]{\mathbb{E}_{\mathbb{P}} \left[ \left| Y - \mathbf{X}^{\top} \boldsymbol{\beta} \right|^{r} \right]} + \delta \rho(\boldsymbol{\beta})$$

We sketch the proof of Theorem 1 here, and refer to Section 4.1 for details. The proof precedes in two steps:

Step 1. We show that

(12) 
$$\mathbb{E}_{\widetilde{\mathbb{P}}}\left[\left|Y - \mathbf{X}^{\top} \boldsymbol{\beta}\right|^{r}\right] \leq \sqrt[r]{\mathbb{E}_{\mathbb{P}}\left[\left|Y - \mathbf{X}^{\top} \boldsymbol{\beta}\right|^{r}\right]} + \delta\left(\sigma + \rho(\boldsymbol{\beta})\right),$$

holds for any  $\beta \in \text{dom}(\rho)$  and any  $\widetilde{\mathbb{P}} \in B_{\delta}(\mathbb{P})$ .

Step 2. We show that for any  $\beta \in \text{dom}(\rho)$ , the upper bound given in Step 1 is tight; i.e. we construct  $\mathbb{P}^* \in B_{\delta}(\mathbb{P})$ , for which the bound holds exactly.

Step 2 implies the following corollary, characterising a worst-case measure in the Wasserstein ball:

Corollary 1. The supremum in (10) is attained for the distribution  $\mathbb{P}^*$  corresponding to the random vector  $(\widetilde{\mathbf{X}}, \widetilde{Y})$  defined as

(13) 
$$\widetilde{\mathbf{X}} = \mathbf{X} - e \left( \boldsymbol{\beta}^* - \frac{\boldsymbol{\beta}}{\boldsymbol{\beta}^{\top} \boldsymbol{\beta}} \rho^* (\boldsymbol{\beta}^*) \right), \qquad \widetilde{Y} = Y + \sigma e,$$

where

$$e \equiv \frac{\delta(Y - \mathbf{X}^{\top} \boldsymbol{\beta})}{\sqrt[r]{\mathbb{E}_{\mathbb{P}}\left[|Y - \mathbf{X}^{\top} \boldsymbol{\beta}|^{r}\right]}}, \qquad (\mathbf{X}, Y) \sim \mathbb{P}.$$

**Remark 2** (On condition (9)). If  $\rho$  is a norm, then the condition in (9) is automatically satisfied; meaning, there exists a  $\beta^* \in \partial \rho(\beta)$  such that (9) is true. Thus, the conclusion of Theorem 1 holds for all  $\rho(\cdot) = \|\cdot\|$  that are norms. Indeed, in that case [12, Example 3.26] states that  $\rho^*(\mathbf{x}) = \infty \mathbb{1}_{\{\|\mathbf{x}\|_* > 1\}}$ . Recall further that  $\beta^* \in \partial \rho(\beta)$  if and only if

$$(\mathbf{\beta}^*)^{\top} \boldsymbol{\beta} - \rho^* (\boldsymbol{\beta}^*) = \rho(\boldsymbol{\beta}).$$

Both facts together imply that  $\rho^*(\boldsymbol{\beta}^*) = 0$ ; thus,  $\|\boldsymbol{\beta}^*\|_* \leq 1$  for all  $\boldsymbol{\beta}^* \in \partial \rho(\boldsymbol{\beta})$ . Hence in (9), as claimed, we have

(15) 
$$\left| \boldsymbol{\gamma}^{\top} \left( \boldsymbol{\beta}^* - \frac{\boldsymbol{\beta}}{\boldsymbol{\beta}^{\top} \boldsymbol{\beta}} \, \rho^*(\boldsymbol{\beta}^*) \right) \right| = \left| \boldsymbol{\gamma}^{\top} \boldsymbol{\beta}^* \right| \leq \|\boldsymbol{\gamma}\| \|\boldsymbol{\beta}^*\|_* \leq \|\boldsymbol{\gamma}\|, \quad \forall \, \boldsymbol{\gamma} \in \text{dom}(\rho).$$

On the other hand, the following example shows that condition (9) is strictly more general:

**Example 1** (Condition (9) for a function  $\rho$  that is not a norm). Fix any compact set  $K \subseteq \mathbb{R}^d$  such that -K = K and consider

$$\rho(\boldsymbol{\beta}) = \sup_{\mathbf{y} \in K} \boldsymbol{\beta}^{\top} \mathbf{y}.$$

Then  $\rho$  is convex (as a supremum of linear functions), finite (as K is compact), non-negative (as K = -K), symmetric  $\rho(\beta) = \rho(-\beta)$  (as K = -K), and homogeneous  $\rho(\lambda \beta) = \lambda \rho(\beta)$ . Thus,

$$\rho^*(\boldsymbol{\beta}^*) = \sup_{\boldsymbol{\gamma} \in \mathbb{R}^d} \left( (\boldsymbol{\beta}^*)^\top \boldsymbol{\gamma} - \rho(\boldsymbol{\gamma}) \right) = \begin{cases} \infty & \text{if } \exists \boldsymbol{\gamma} \in \mathbb{R}^d \text{ s.t. } (\boldsymbol{\beta}^*)^\top \boldsymbol{\gamma} - \rho(\boldsymbol{\gamma}) > 0, \\ 0 & \text{if } (\boldsymbol{\beta}^*)^\top \boldsymbol{\gamma} - \rho(\boldsymbol{\gamma}) \leq 0 \text{ for all } \boldsymbol{\gamma} \in \mathbb{R}^d. \end{cases}$$

By (14) we conclude that  $\rho^*(\boldsymbol{\beta}^*) = 0$  for all  $\boldsymbol{\beta} \in \mathbb{R}^d$  and  $(\boldsymbol{\beta}^*)^\top \boldsymbol{\gamma} \leq \rho(\boldsymbol{\gamma})$ , and by symmetry of  $\rho$ , then we also have that  $|(\boldsymbol{\beta}^*)^\top \boldsymbol{\gamma}| \leq \rho(\boldsymbol{\gamma})$ . It follows that

$$\left| \boldsymbol{\gamma}^\top \left( \boldsymbol{\beta}^* - \frac{\boldsymbol{\beta}}{\boldsymbol{\beta}^\top \boldsymbol{\beta}} \, \rho^*(\boldsymbol{\beta}^*) \right) \right| \leq \rho(\boldsymbol{\gamma}), \quad \forall \, \boldsymbol{\gamma} \in \mathrm{dom}(\rho).$$

#### 2.1. Examples for Theorem 1.

2.1.1. Square-root Lasso. Let us take r=2 and

$$\rho(\boldsymbol{\beta}) = \|\boldsymbol{\beta}\|_1 = \sum_{j=1}^d |\beta_j|.$$

Under this choice of penalty function, the regression problem (11) is the objective function of the  $\sqrt{\text{LASSO}}$  of [2], also studied in [3]. These papers have shown that the  $\sqrt{\text{LASSO}}$ -estimator achieves the near-oracle rates of convergence in sparse, high-dimensional regression models over data distributions that extend significantly beyond normality.

Clearly  $\rho$  is a norm; in particular, it is nonnegative, convex, and its effective domain is  $\mathbb{R}^d$ . Moreover, Condition (9) of Theorem 1 is satisfied, cf. Remark 2.

One set of distributions that render the square-root LASSO a worst-case best linear predictor is

$$B_{\delta}^{\sqrt{\text{LASSO}}}(\mathbb{P}) := \{ \mathbb{Q} \in \mathcal{P}_{2}(\mathbb{R}^{d}) \mid \exists \text{ a coupling } \pi \in \Pi(\widetilde{\mathbb{P}}, \mathbb{P}) \text{ for which:}$$

$$\mathbb{E}_{\pi} [|\widetilde{X}_{j} - X_{j}|^{2}] \leq \delta^{2}, \ \forall \ j = 1, \dots, d, \text{ and } \mathbb{E}_{\pi}[|\widetilde{Y} - Y|^{2}] \leq (\delta\sigma)^{2},$$

$$\text{where } ((\mathbf{X}, Y), (\widetilde{\mathbf{X}}, \widetilde{Y})) \sim \pi \}.$$

We verify that  $B_{\delta}^{\sqrt{\mathrm{LASSO}}}(\mathbb{P}) \subseteq B_{\delta}(\mathbb{P})$  used in Theorem 1 and defined in (6). To see this, notice that  $\mathbb{E}_{\pi}\left[|\widetilde{X}_{j}-X_{j}|^{2}\right] \leq \delta^{2}$  for all  $j=1,\ldots,d$  implies condition (6), i.e.

$$\mathbb{E}_{\pi} [|(\widetilde{Y} - Y) + (\mathbf{X} - \widetilde{\mathbf{X}})^{\top} \gamma|^{2}] \leq (\delta(\sigma + \rho(\gamma)))^{2}.$$

Indeed, the triangle inequality implies that for any  $\gamma \in \text{dom}(\rho)$  and any coupling  $\pi \in \Pi(\mathbb{P}, \mathbb{Q})$ :

$$\sqrt{\mathbb{E}_{\pi}\left[\left|(\widetilde{Y} - Y) + (\mathbf{X} - \widetilde{\mathbf{X}})^{\top} \boldsymbol{\gamma}\right|^{2}\right]} \leq \sqrt{\mathbb{E}_{\pi}\left[\left|\widetilde{Y} - Y\right|^{2}\right]} + \sum_{j=1}^{d} |\gamma_{j}| \sqrt{\mathbb{E}_{\pi}\left[(\widetilde{X}_{j} - X_{j})^{2}\right]}$$

$$\leq \delta \sigma + \delta \sum_{j=1}^{d} |\gamma_{j}| = \delta(\sigma + \rho(\boldsymbol{\gamma})).$$

Consequently,  $B_{\delta}^{\sqrt{\text{LASSO}}}(\mathbb{P}) \subseteq B_{\delta}(\mathbb{P})$ . We note that the other direction, namely,  $B_{\delta}(\mathbb{P}) \subseteq B_{\delta}^{\sqrt{\text{LASSO}}}(\mathbb{P})$  does not hold in general.

It is worth mentioning that the set  $B_{\delta}^{\sqrt{\text{LASSO}}}(\mathbb{P})$  contains different versions of  $(\mathbf{X}, Y)$  measured with error. For example, any additive measurement error model of the form

$$\widetilde{X}_j = X_j + u_j, \quad \widetilde{Y} = Y,$$

where  $\mathbb{E}[u_j^2] \leq \delta^2$  is a possible distribution that can be used to compute mean-squared error. Furthermore,  $B_{\delta}^{\sqrt{\mathrm{LASSO}}}(\mathbb{P})$  contains multiplicative errors-in-variables models where

$$\widetilde{X}_j = X_j u_j, \quad \widetilde{Y} = Y,$$

with u's independent of  $(\mathbf{X}, Y)$ , mean one and  $\mathbb{E}_{\mathbb{P}}[X_j^2]\mathbb{E}[(u_j - 1)^2] \leq \delta$ .

It is well known that the conjugate of  $\rho$  is

$$\rho^*(\boldsymbol{\beta}) = \begin{cases} 0 & \max\{|\beta_1|, \dots, |\beta_d|\} \le 1, \\ \infty & \text{otherwise.} \end{cases}$$

The argument is analogous to Remark 2. Moreover, algebra shows that

$$\boldsymbol{\beta}^* = (\operatorname{sign}(\beta_1), \dots, \operatorname{sign}(\beta_d))^{\top}$$

is a subgradient of  $\rho$  at  $\beta$ . Using these facts, we report the worst-case distribution for each particular  $\beta$ . Corollary 1 states that:

$$\widetilde{\mathbf{X}} = \mathbf{X} - e(\operatorname{sign}(\beta_1), \dots, \operatorname{sign}(\beta_d))^{\top}, \qquad \widetilde{Y} = Y + \sigma e,$$

where

$$e := \frac{\delta(Y - \mathbf{X}^{\top} \boldsymbol{\beta})}{\sqrt{\mathbb{E}_{\mathbb{P}}[(Y - \mathbf{X}^{\top} \boldsymbol{\beta})^2]}}, \qquad (\mathbf{X}, Y) \sim \mathbb{P}.$$

Notably the worst-case mean-squared error of  $\sqrt{LASSO}$  is attained at distributions where there is a (possibly correlated) measurement error that has a factor structure.

2.1.2. Square-Root SLOPE. Now suppose again that r=2, but let

$$\rho(\boldsymbol{\beta}) = \sum_{j=1}^{d} \lambda_j |\beta|_{(j)},$$

where  $\lambda_1 \geq \cdots \geq \lambda_d \geq 0$  and  $|\beta|_{(j)}$  are the decreasing order statistics of the absolute values of the coordinates of  $\beta$ . Under this penalty function—which is nonnegative and has  $\mathbb{R}^d$  as effective domain—the penalized regression problem in (11) is the objective function of the square-root SLOPE of [40].

An equivalent definition for this penalty function is

(17) 
$$\rho(\boldsymbol{\beta}) = \max_{pm} \sum_{j=1}^{d} \lambda_{pm(j)} |\beta_j|,$$

where we maximize over all permutations, pm, of the coordinates  $\{1, \ldots, d\}$ . It follows that  $\rho$  is a norm, so Condition (9) of Theorem 1 is satisfied (see Remark 2).

For a given  $\boldsymbol{\beta} \in \mathbb{R}^d$ , let  $pm^*$  be a permutation that solves (17). Define  $\boldsymbol{\beta}^*$  by  $\beta_j^* = \lambda_{pm^*(j)} \operatorname{sign}(\beta_j)$ . Algebra shows that  $\rho(\boldsymbol{\beta}) = (\boldsymbol{\beta}^*)^\top \boldsymbol{\beta}$  and  $(\boldsymbol{\beta}^*)^\top \boldsymbol{\gamma} \leq \rho(\boldsymbol{\gamma})$ , for any  $\boldsymbol{\gamma} \in \mathbb{R}^d$ . It follows that  $\rho(\boldsymbol{\gamma}) \geq \rho(\boldsymbol{\beta}) + (\boldsymbol{\beta}^*)^\top \boldsymbol{\gamma} - (\boldsymbol{\beta}^*)^\top \boldsymbol{\beta}$ , which implies that  $\boldsymbol{\beta}^*$  is a subgradient of  $\rho$  at  $\boldsymbol{\beta}$ . Recall that  $\rho^*(\boldsymbol{\beta}^*) = 0$  and thus (9) holds.

In this case, a set of distributions rendering the square-root SLOPE a worst-case best linear predictor is given by

 $B_{\delta}^{\mathrm{SLOPE}}(\mathbb{P}) := \{ \mathbb{Q} \in \mathcal{P}_2(\mathbb{R}^d) : \exists \text{ a coupling } \pi \in \Pi(\mathbb{Q}, \mathbb{P}) \text{ for which: }$ 

$$\mathbb{E}_{\pi}\left[|\widetilde{X}_{(j)} - X_{(j)}|^2\right] \leq (\delta \lambda_j)^2, \ \forall \ j = 1, \dots, d,$$
  
and 
$$\mathbb{E}_{\pi}[|\widetilde{Y} - Y|^2| \leq (\delta \sigma)^2, \ \text{where} \ ((\mathbf{X}, Y), (\widetilde{\mathbf{X}}, \widetilde{Y})) \sim \pi\}.$$

where the decreasing order statistic is induced by the vector  $(\mathbb{E}_{\pi}[|\widetilde{X}_{j} - X_{j}|^{2}])_{j=1,\dots,d}$ . As for the  $\sqrt{\text{LASSO}}$ , we check that  $B_{\delta}^{\text{SLOPE}}(\mathbb{P}) \subseteq B_{\delta}(\mathbb{P})$ . The triangle inequality implies that for any coupling  $\pi \in \Pi(\mathbb{P}, \mathbb{Q})$ :

$$\sqrt{\mathbb{E}_{\pi}[|(\widetilde{Y} - Y) + (\mathbf{X} - \widetilde{\mathbf{X}})^{\top} \gamma|^{2}]} \leq \sqrt{\mathbb{E}_{\pi}[|\widetilde{Y} - Y|^{2}]} + \sum_{j=1}^{d} |\gamma_{j}| \sqrt{\mathbb{E}_{\pi}[|\widetilde{X}_{j} - X_{j}|^{2}]},$$

$$= \sqrt{\mathbb{E}_{\pi}[|\widetilde{Y} - Y|^{2}]} + \sum_{j=1}^{d} |\gamma_{(j)}| \sqrt{\mathbb{E}_{\pi}[|\widetilde{X}_{(j)} - X_{(j)}|^{2}]},$$

$$\leq (\sigma + \rho(\gamma))\delta,$$

where the last equality follows by the definition of  $B_{\delta}^{\text{SLOPE}}(\mathbb{P})$  and (17).

Finally, we report the worst-case distribution for each particular  $\beta$ . Corollary 1 shows that

$$\widetilde{\mathbf{X}} = \mathbf{X} - e\beta^*, \quad \widetilde{Y} = Y + \sigma e,$$

where the j-coordinate of  $\beta^*$  is  $\lambda_{pm^*(j)} \operatorname{sign}(\beta_j)$  and

$$e := \frac{\delta(Y - \mathbf{X}^{\top} \beta)}{\sqrt{\mathbb{E}_{\mathbb{P}}[|Y - \mathbf{X}^{\top} \beta|^2]}}, \quad (\mathbf{X}, Y) \sim \mathbb{P}.$$

As it was mentioned for the case of the  $\sqrt{\rm LASSO}$ , it is interesting that the worst-case mean-squared error of SLOPE is also attained at distributions where there is a (possibly correlated) measurement error that has a factor structure.

#### 3. Rates for the MSW-distance

In this section, we assume that the data  $\{(\mathbf{X},Y)_i\}_{i=1}^n$  is drawn i.i.d. from  $\mathbb{P}$  and we let  $\mathbb{P}_n$  be the associated empirical measure. In what follows, we give explicit upper bounds on the radius  $\delta$  of the ball  $B_{\delta}(\mathbb{P}_n)$  defined in (6), such that  $\mathbb{P} \in B_{\delta}(\mathbb{P}_n)$  for all  $d, n \in \mathbb{N}$  with constant probability. Recall from (2) that

(18) 
$$\widehat{\mathcal{W}}_{r,\rho,\sigma}(\mathbb{P},\widetilde{\mathbb{P}}) = \sup_{\boldsymbol{\gamma} \in \text{dom}(\rho)} \inf_{\substack{\boldsymbol{\pi} \in \Pi(\mathbb{P},\widetilde{\mathbb{P}}):\\ ((\mathbf{X},Y), (\widetilde{\mathbf{X}},\widetilde{Y})) \sim \boldsymbol{\pi}}} \frac{1}{\sigma + \rho(\boldsymbol{\gamma})} \sqrt[r]{\mathbb{E}_{\boldsymbol{\pi}}[|(\widetilde{Y} - Y) + (\mathbf{X} - \widetilde{\mathbf{X}})^{\top} \boldsymbol{\gamma}|^{r}]}.$$

Notice that we can rewrite this as

(19) 
$$\widehat{\mathcal{W}}_{r,\rho,\sigma}(\mathbb{P},\widetilde{\mathbb{P}}) = \sup_{\boldsymbol{\gamma} \in \text{dom}(\rho)} \frac{1}{\sigma + \rho(\boldsymbol{\gamma})} \mathcal{W}_r \left( \left[ (\mathbf{X}, Y)^\top (\boldsymbol{\gamma}, -1) \right]_* \mathbb{P}, \left[ (\widetilde{\mathbf{X}}, \widetilde{Y})^\top (\boldsymbol{\gamma}, -1) \right]_* \widetilde{\mathbb{P}} \right),$$

where  $f_*\mathbb{P}$  denotes the pushforward of  $\mathbb{P}$  with respect to a map  $f:\mathbb{R}^d\to\mathbb{R}$  and  $\mathcal{W}_r$  is the one-dimensional Wasserstein metric defined as

(20) 
$$\mathcal{W}_r(\mathbb{Q}, \widetilde{\mathbb{Q}}) = \inf_{\substack{\pi \in \Pi(\mathbb{Q}, \widetilde{\mathbb{Q}}): \\ (X, \widetilde{X}) \sim \pi}} \sqrt[r]{\mathbb{E}_{\pi}[|\mathbf{X} - \widetilde{\mathbf{X}}|^r]}.$$

Focusing on the case  $\rho(\gamma) = ||\gamma||_2$  (the Euclidean metric), from (19) we find the upper bound (21)

$$\widehat{\mathcal{W}}_{r,\rho,\sigma}(\mathbb{P},\widetilde{\mathbb{P}}) = \sup_{\boldsymbol{\gamma} \in \text{dom}(\rho)} \frac{\|(\boldsymbol{\gamma},-1)\|_2}{\sigma + \|\boldsymbol{\gamma}\|_2} \frac{1}{\|(\boldsymbol{\gamma},-1)\|_2} \mathcal{W}_r \left( \left[ (\mathbf{X},Y)^\top (\boldsymbol{\gamma},-1) \right]_* \mathbb{P}, \left[ (\widetilde{\mathbf{X}},\widetilde{Y})^\top (\boldsymbol{\gamma},-1) \right]_* \widetilde{\mathbb{P}} \right)$$

$$\leq \sup_{\widetilde{\boldsymbol{\gamma}}: \|\widetilde{\boldsymbol{\gamma}}\|_2 = 1} \mathcal{W}_r \left( \left[ (\mathbf{X},Y)^\top \widetilde{\boldsymbol{\gamma}} \right]_* \mathbb{P}, \left[ (\widetilde{\mathbf{X}},\widetilde{Y})^\top \widetilde{\boldsymbol{\gamma}} \right]_* \widetilde{\mathbb{P}} \right)$$

$$=: \overline{\mathcal{W}}_r (\mathbb{P},\widetilde{\mathbb{P}}),$$

where we used the triangle inequality for the inequality.

The quantity  $\overline{\mathcal{W}}_r$  defined in (21) is the max-sliced Wasserstein distance on  $(\mathbb{R}^{d+1}, \|\cdot\|_2)$ . It is also a special case of the Projection Robust Wasserstein (PRW) distance, also called the Wasserstein Projection Pursuit (WPP), see [33, Definition 1]. The work in [33, Proposition 1] shows that  $\overline{\mathcal{W}}_r(\mathbb{P}, \mathbb{P})$  is a metric (the proof is stated for the case r=2, but carries over line by line to arbitrary  $r \geq 1$ ). As stated in the Introduction, it is well known in the literature that, in the worst case,  $\mathcal{W}_r(\mathbb{P}_n, \mathbb{P})^r \sim n^{-1/(d+1)}$ . In what follows, we show that the MSW distance  $\overline{\mathcal{W}}_r$  does not have this limitation.

To show this, we first make a few notational simplifications: if there is no confusion, we write  $\sup_{\gamma}$  instead of  $\sup_{\gamma: \|\gamma\|_2=1}$ . We also write  $\mathbb{P}_{\gamma}$  and  $F_{\gamma}$ , respectively, for the distribution and cdf of  $(\mathbf{X}, Y)^{\top} \gamma$  under  $\mathbb{P}$ . Similarly, we write  $\mathbb{P}_{\gamma,n}$  and  $F_{\gamma,n}$ , respectively, for the probability measure and cdf of  $(\mathbf{X}, Y)^{\top} \gamma$  under  $\mathbb{P}_n$ . Note then, in particular, by (21) we have that

$$\overline{\mathcal{W}}_r(\mathbb{P},\widetilde{\mathbb{P}}) = \sup_{\gamma} \mathcal{W}_r(\mathbb{P}_{\gamma},\widetilde{\mathbb{P}}_{\gamma}).$$

Furthermore,

$$F_{\boldsymbol{\gamma},n}(t) = \frac{1}{n} \sum_{i=1}^{n} \mathbf{1}_{\left\{ (\boldsymbol{X},Y)_{i}^{\top} \boldsymbol{\gamma} \leq \right\}} = \mathbb{E}_{\mathbb{P}_{n}} \left[ \mathbf{1}_{\left\{ (\boldsymbol{X},Y)^{\top} \boldsymbol{\gamma} \leq t \right\}} \right],$$

is a (rescaled) sum of i.i.d. Bernoulli random variables  $\mathbf{1}_{\{(\mathbf{X},Y)_1^\top \boldsymbol{\gamma} \leq t\}}, \dots, \mathbf{1}_{\{(\mathbf{X},Y)_n^\top \boldsymbol{\gamma} \leq t\}}$  with success probability  $\mathbb{P}\left((\mathbf{X},Y)^\top \boldsymbol{\gamma} \leq t\right)$ . Consequently, we also have

$$F_{\gamma,n}(t) - F_{\gamma}(t) = \mathbb{E}_{\mathbb{P}_n} \left[ \mathbf{1}_{\left\{ \boldsymbol{X}^{\top} \boldsymbol{\gamma} \leq t \right\}} \right] - \mathbb{E}_{\mathbb{P}} \left[ \mathbf{1}_{\left\{ \boldsymbol{X}^{\top} \boldsymbol{\gamma} \leq t \right\}} \right].$$

We are now in a position to give explicit upper bounds for  $\overline{W}_r(\mathbb{P}, \mathbb{P}_n)$ . We first cover compactly supported measures  $\mathbb{P}$  in Section 3.1 and then the general case in Section 3.2.

3.1. The compactly supported case. We use the following well-known result, where we refer to Section 4 for a computation of the constants appearing:

**Lemma 1** (cf. [42, Theorem 4.10], [17, Chapter 4]). Let

(22) 
$$\mathcal{H} := \left\{ \mathbf{1}_{\left\{ \mathbf{x}^{\top} \boldsymbol{\gamma} \leq t \right\}} : \boldsymbol{\gamma} \in \mathbb{R}^{d+1}, \, t \in \mathbb{R} \right\},$$

be the set of indicator functions of half spaces. Then, with probably at least  $1-\alpha$ ,

$$\sup_{(\gamma,t)\in\mathbb{R}^{d+1}\times\mathbb{R}} |F_{\gamma,n}(t) - F_{\gamma}(t)| = \sup_{f\in\mathcal{H}} |\mathbb{E}_{\mathbb{P}_n}[f] - \mathbb{E}_{\mathbb{P}}[f]| \le 180\sqrt{\frac{d+2}{n}} + \sqrt{\frac{2}{n}\log\left(\frac{1}{\alpha}\right)}.$$

The proof of Lemma 1 can be found in Section 4.

Let us now assume that  $\mathbb{P}$  has compact support. Denote the diameter of a set  $\mathbf{X} \subseteq \mathbb{R}^{d+1}$  with respect to the Euclidean norm by diam( $\mathbf{X}$ ).

**Theorem 2.** With probability at least  $1 - \alpha$ , we have

$$\overline{\mathcal{W}}_r(\mathbb{P}_n, \mathbb{P})^r \le \frac{C}{\sqrt{n}}$$

where

$$C := \left(180\sqrt{d+2} + \sqrt{2\log\left(\frac{1}{\alpha}\right)}\right) \operatorname{diam}\left(\operatorname{supp}(\mathbb{P})\right)^r.$$

The above rates also capture the high-dimensional regime  $d/n \to c \in \mathbb{R}_+$  as  $n, d \to \infty$ .

*Proof.* We first recall the representations for the 1-dimensional Wasserstein distance

$$\mathcal{W}_r(\mathbb{P}_{\boldsymbol{\gamma},n},\mathbb{P}_{\boldsymbol{\gamma}})^r = \int_0^1 \left| F_{\boldsymbol{\gamma},n}^{-1}(p) - F_{\boldsymbol{\gamma}}^{-1}(p) \right|^r dp,$$

for  $r \geq 1$  and

(24) 
$$\mathcal{W}_1(\mathbb{P}_{\gamma,n}, \mathbb{P}_{\gamma}) = \int_{\mathbb{R}} |F_{\gamma,n}(t) - F_{\gamma}(t)| \ dt,$$

see e.g. [9, Theorem 2.9, Theorem 2.10]. We also note that  $W_r$  is translation invariant, which implies in particular that

$$\mathcal{W}_r\left(\mathbb{P}_{\boldsymbol{\gamma},n},\,\mathbb{P}_{\boldsymbol{\gamma}}\right) = \mathcal{W}_r\left(\left[\left((\mathbf{X},Y) - (\mathbf{x_0},y_0)\right)^{\top}\boldsymbol{\gamma}\right]_{\star}\,\mathbb{P}_{\boldsymbol{\gamma},n},\,\left[\left((\mathbf{X},Y) - (\mathbf{x_0},y_0)\right)^{\top}\boldsymbol{\gamma}\right]_{\star}\,\mathbb{P}_{\boldsymbol{\gamma}}\right),$$

for any  $\mathbf{x_0} \in \mathbb{R}^d$  and  $y_0 \in \mathbb{R}$ . There is thus no loss of generality if we assume that

(25) 
$$\operatorname{supp}(\mathbb{P}_{\gamma}) \subseteq [0, \operatorname{diam}(\operatorname{supp}(\mathbb{P}))].$$

Define  $c := \text{diam} \left( \text{supp}(\mathbb{P}) \right)$ . Noting that  $|F_{\gamma,n}(p)^{-1} - F_{\gamma}^{-1}(p)| \le c$  for all  $p \in (0,1)$ , we estimate

$$\overline{\mathcal{W}}_r(\mathbb{P}_{\gamma,n},\mathbb{P}_{\gamma})^r = \sup_{\gamma} \left( \int_0^1 \left| F_{\gamma,n}^{-1}(p) - F_{\gamma}^{-1}(p) \right|^r dp \right) 
\leq c^{r-1} \sup_{\gamma} \left( \int_0^1 \left| F_{\gamma,n}^{-1}(p) - F_{\gamma}^{-1}(p) \right| dp \right) 
= c^{r-1} \sup_{\gamma} \left( \int_{\mathbb{R}} \left| F_{\gamma,n}(t) - F_{\gamma}(t) \right| dt \right),$$

where the final inequality follows from (23) and (24). Next, recalling (25),

$$\begin{split} \sup_{\gamma} \int_{\mathbb{R}} |F_{\gamma,n}(t) - F_{\gamma}(t)| \ dt &\leq \sup_{\gamma} \int_{0}^{c} \sup_{t} |F_{\gamma,n}(t) - F_{\gamma}(t)| \ ds \\ &\leq c \sup_{f \in \mathcal{H}} |\mathbb{E}_{\mathbb{P}_{n}}[f] - \mathbb{E}_{\mathbb{P}}[f]| \,. \end{split}$$

The final inequality in the above follows from the definition of  $\mathcal{H}$  in (22). The claim now follows from Lemma 1.

**Remark 3.** Theorem 2 yields rates for all norms  $\|\cdot\|$  on  $\mathbb{R}^{d+1}$ , not just  $\|\cdot\|_2$ , as we have the following inequality: there exists a constant  $c_d > 0$  such that

$$c_d \|\mathbf{x}\|_2 \le \|\mathbf{x}\|, \quad \text{for all } \mathbf{x} \in \mathbb{R}^{d+1}.$$

Indeed, we can estimate

$$\frac{1}{\|\boldsymbol{\gamma}\|^{r}} \sup_{\boldsymbol{\gamma} \in \mathbb{R}^{d}} \inf_{\pi \in \Pi(\mathbb{P}, \mathbb{P}_{n})} \mathbb{E}_{\pi} \left[ \left| \left( (\widetilde{\mathbf{X}}, \widetilde{Y}) - (\mathbf{X}, Y) \right)^{\top} \boldsymbol{\gamma} \right|^{r} \right] \\
\leq \frac{1}{(c_{d})^{r} \|\boldsymbol{\gamma}\|_{2}^{r}} \sup_{\boldsymbol{\gamma} \in \mathbb{R}^{d}} \inf_{\pi \in \Pi(\mathbb{P}, \mathbb{P}_{n})} \mathbb{E}_{\pi} \left[ \left| \left( (\widetilde{\mathbf{X}}, \widetilde{Y}) - (\mathbf{X}, Y) \right)^{\top} \boldsymbol{\gamma} \right|^{r} \right] \\
\leq \frac{1}{(c_{d})^{r}} \overline{W}_{r} \left( \mathbb{P}_{n}, \mathbb{P} \right)^{r} \\
\leq \frac{C_{d}}{(c_{d})^{r}} \frac{1}{\sqrt{n}},$$

where the final inequality follows from the Theorem 2 result, now noting explicitly that the constant  $C_d := C$  depends on the dimension. If e.g.  $\|\cdot\| = \|\cdot\|_1$ , then we have  $c_d = 1$ , so that the rate is  $\sqrt{d/n}$ .

3.2. The general case. We now show that, under a mild moment condition, we obtain  $\sqrt{d/n}$ rates for  $\overline{\mathcal{W}}_r(\mathbb{P}_n, \mathbb{P})$ , up to logarithmic factors. These rates are optimal: indeed, in the case of
fixed  $\gamma$ , one can easily find  $\mathbb{P}$  for which  $\mathcal{W}_r(\mathbb{P}_{\gamma,n}, \mathbb{P}_{\gamma}) \sim 1/\sqrt{n}$ , see e.g. [20, Remark (a), p.2] and
[1, Lemma 2.7]. Our results hold under the moment assumption

$$\Gamma := \mathbb{E}_{\mathbb{P}} \left[ \sup_{\gamma} \left| (\mathbf{X}, Y)^{\top} \gamma \right|^{s} \right] < \infty, \quad \text{for some } s > 2r.$$

This assumption substantially generalises [31] and [28], two works that also give rates for  $\overline{W}_r(\mathbb{P}_n, \mathbb{P})$ , but that assume certain transport or Poincaré inequalities. What is more, our constants are fully explicit, with a linear dependence on the dimension on d.

Our main result is the following:

**Theorem 3.** Assume that  $\Gamma < \infty$ . Then with probability greater than  $1 - 3\alpha$  we have

(26) 
$$\overline{W}_r(\mathbb{P}_n, \mathbb{P})^r \le \frac{C \log(2n+1)^{r/s}}{\sqrt{n}},$$

where

$$C := 2^r r \left( 180\sqrt{d+2} + \sqrt{2\log\left(\frac{1}{\alpha}\right)} + \sqrt{\frac{\Gamma}{\alpha}} \frac{8}{s/2 - r} \sqrt{\log\left(\frac{8}{\alpha}\right) + 2(d+2)} \right).$$

The proof relies on two lemmas given below, both of which are proved in Section 4.

Lemma 2. Define

$$\Gamma_n := \sup_{\boldsymbol{\gamma}} \mathbb{E}_{\mathbb{P}_n} \left[ \left| (\mathbf{X}, Y)^{\top} \boldsymbol{\gamma} \right|^s \right] = \sup_{\boldsymbol{\gamma}} \frac{1}{n} \sum_{i=1}^n \left| (\mathbf{X}, Y)_i^{\top} \boldsymbol{\gamma} \right|^s.$$

For any  $k \in \mathbb{R}^+$ , we have

$$\begin{split} &(27)\\ \overline{\mathcal{W}}_r(\mathbb{P}_n,\mathbb{P})^r \leq r(2k)^r \sup_{(\gamma,t)\in\mathbb{R}^{d+1}\times\mathbb{R}} |F_{\gamma,n}(t)-F_{\gamma}(t))| \\ &+ \frac{2^r r \sqrt{\Gamma\vee\Gamma_n}}{s/2-r} k^{r-s/2} \Bigg( \sup_{(\gamma,t)\in\mathbb{R}^{d+1}\times\mathbb{R}} \frac{(F_{\gamma}(t)-F_{\gamma,n}(t))^+}{\sqrt{F_{\gamma}(t)(1-F_{\gamma,n}(t))}} + \sup_{(\gamma,t)\in\mathbb{R}^{d+1}\times\mathbb{R}} \frac{(F_{\gamma,n}(t)-F_{\gamma}(t))^+}{\sqrt{F_{\gamma,n}(t)(1-F_{\gamma}(t))}} \Bigg), \end{split}$$

with the convention that 0/0 = 0 and the notation  $x^+ := \max\{0, x\}$ 

**Lemma 3.** With probability greater than  $1 - \alpha$  we have

$$\sup_{(\gamma,t)\in\mathbb{R}^{d+1}\times\mathbb{R}} \frac{(F_{\gamma}(t)-F_{\gamma,n}(t))^{+}}{\sqrt{F_{\gamma}(t)(1-F_{\gamma,n}(t))}} \vee \sup_{(\gamma,t)\in\mathbb{R}^{d+1}\times\mathbb{R}} \frac{(F_{\gamma,n}(t)-F_{\gamma}(t))^{+}}{\sqrt{F_{\gamma,n}(t)(1-F_{\gamma}(t))}}$$
$$\leq 4\sqrt{\frac{\log(8/\alpha)+(d+2)\log(2n+1)}{n}}.$$

We are now ready for the proof of Theorem 3:

Proof of Theorem 3. Choosing  $k = \log(2n+1)^{1/s}$  in Lemma 2 we obtain

$$\overline{W}_{r}(\mathbb{P}_{n}, \mathbb{P})^{r} \leq 2^{r} r \log(2n+1)^{r/s} \left( \sup_{(\boldsymbol{\gamma}, t) \in \mathbb{R}^{d+1} \times \mathbb{R}} |F_{\boldsymbol{\gamma}}(t) - F_{\boldsymbol{\gamma}, n}(t))| + \frac{\sqrt{\Gamma \vee \Gamma_{n}}}{s/2 - r} \log(2n+1)^{-1/2} \left( \sup_{(\boldsymbol{\gamma}, t) \in \mathbb{R}^{d+1} \times \mathbb{R}} \frac{(F_{\boldsymbol{\gamma}}(t) - F_{\boldsymbol{\gamma}, n}(t))^{+}}{\sqrt{F_{\boldsymbol{\gamma}, n}(t)(1 - F_{\boldsymbol{\gamma}}(t))}} + \sup_{(\boldsymbol{\gamma}, t) \in \mathbb{R}^{d+1} \times \mathbb{R}} \frac{(F_{\boldsymbol{\gamma}, n}(t) - F_{\boldsymbol{\gamma}}(t))^{+}}{\sqrt{F_{\boldsymbol{\gamma}, n}(t)(1 - F_{\boldsymbol{\gamma}}(t))}} \right).$$

Next, by Markov's inequality and the triangle inequality

$$\mathbb{P}\left(\Gamma_n \geq C\right) \leq \frac{\mathbb{E}_{\mathbb{P}}[\Gamma_n]}{C} = \frac{1}{C} \mathbb{E}_{\mathbb{P}}\left[\sup_{\boldsymbol{\gamma}} \frac{1}{n} \sum_{i=1}^n \left| (\mathbf{X}, Y)_i^{\top} \boldsymbol{\gamma} \right|^s \right] \leq \frac{\Gamma}{C}.$$

Setting the last expression equal to  $\alpha$  yields  $\Gamma_n \leq \Gamma/\alpha$  on a set of probability at least  $1 - \alpha$ . Combining this with Lemma 1 yields, that with probability greater than  $1 - 3\alpha$  we have

$$\overline{W}_{r}(\mathbb{P}_{n}, \mathbb{P})^{r} \leq 2^{r} r \log(2n+1)^{r/s} \left( \frac{1}{\sqrt{n}} \left[ 180\sqrt{d+2} + \sqrt{2\log\left(\frac{1}{\alpha}\right)} \right] + \sqrt{\frac{\Gamma}{\alpha}} \frac{2}{s/2 - r} \frac{1}{\sqrt{\log(2n+1)}} \right]$$

$$\cdot \sup_{(\gamma,t) \in \mathbb{R}^{d+1} \times \mathbb{R}} \frac{(F_{\gamma}(t) - F_{\gamma,n}(t))^{+}}{\sqrt{F_{\gamma}(t)(1 - F_{\gamma,n}(t))}} \vee \sup_{(\gamma,t) \in \mathbb{R}^{d+1} \times \mathbb{R}} \frac{(F_{\gamma,n}(t) - F_{\gamma}(t))^{+}}{\sqrt{F_{\gamma,n}(t)(1 - F_{\gamma}(t))}} \right)$$

$$\leq \frac{2^{r} r \log(2n+1)^{r/s}}{\sqrt{n}} \left( 180\sqrt{d+2} + \sqrt{2\log\left(\frac{1}{\alpha}\right)} + \sqrt{\frac{\Gamma}{\alpha}} \frac{8}{s/2 - r} \sqrt{\log\left(\frac{8}{\alpha}\right) + (d+2)} \right),$$

which was the claim.

#### 4. Remaining proofs

#### 4.1. Proof of Theorem 1.

*Proof.* Following the statement of Theorem 1 in Section 2, we provide a proof sketch in two steps. Here we elaborate on the proof details in these two steps. The statements of these steps are repeated below for the reader's convenience.

#### Step 1. We show that

(28) 
$$\mathbb{E}_{\widetilde{\mathbb{P}}}\left[\left|Y - \mathbf{X}^{\top} \boldsymbol{\beta}\right|^{r}\right] \leq \sqrt[r]{\mathbb{E}_{\mathbb{P}}\left[\left|Y - \mathbf{X}^{\top} \boldsymbol{\beta}\right|^{r}\right]} + \delta \left(\sigma + \rho(\boldsymbol{\beta})\right),$$

holds for any  $\beta \in \text{dom}(\rho)$  and any  $\widetilde{\mathbb{P}} \in B_{\delta}(\mathbb{P})$ .

**Proving Step 1.** Take an arbitrary  $\widetilde{\mathbb{P}} \in B_{\delta}(\mathbb{P})$  and let  $\pi(\beta)$  be the coupling defined by  $B_{\delta}(\mathbb{P})$ . We emphasize with the chosen notation that the coupling will depend on  $\beta$ ; though, this matters little for the proof. Namely,  $((\mathbf{X},Y),(\widetilde{\mathbf{X}},\widetilde{Y})) \sim \pi(\beta)$  with  $(\mathbf{X},Y) \sim \mathbb{P}$  and  $(\widetilde{\mathbf{X}},\widetilde{Y}) \sim \widetilde{\mathbb{P}}$ . Consequently:

$$\mathbb{E}_{\widetilde{\mathbb{P}}}\left[\left|Y - \mathbf{X}^{\top} \boldsymbol{\beta}\right|^{r}\right] = \mathbb{E}_{\pi(\boldsymbol{\beta})}\left[\left|\widetilde{Y} - \widetilde{\mathbf{X}}^{\top} \boldsymbol{\beta}\right|^{r}\right].$$

By the triangle inequality:

$$\sqrt[r]{\mathbb{E}_{\pi(\boldsymbol{\beta})}\left[\left|\widetilde{Y} - \widetilde{\mathbf{X}}^{\top}\boldsymbol{\beta}\right|^{r}\right]} = \sqrt[r]{\mathbb{E}_{\pi(\boldsymbol{\beta})}\left[\left|(\widetilde{Y} - Y) + (Y - \mathbf{X}^{\top}\boldsymbol{\beta}) + (\mathbf{X} - \widetilde{\mathbf{X}})^{\top}\boldsymbol{\beta})\right|^{r}\right]} \\
\leq \sqrt[r]{\mathbb{E}_{\pi(\boldsymbol{\beta})}\left[\left|Y - \mathbf{X}^{\top}\boldsymbol{\beta}\right|^{r}\right]} + \sqrt[r]{\mathbb{E}_{\pi(\boldsymbol{\beta})}\left[\left|(\widetilde{Y} - Y) + (\mathbf{X} - \widetilde{\mathbf{X}})^{\top}\boldsymbol{\beta})\right|^{r}\right]}.$$

Next, for any  $\widetilde{\mathbb{P}} \in B_{\delta}(\mathbb{P})$  where  $B_{\delta}(\mathbb{P})$  is defined in (6), by the definition of  $\pi(\beta)$  we have that

(29) 
$$\sqrt[r]{\mathbb{E}_{\pi(\boldsymbol{\beta})}\left[\left|\widetilde{Y} - \widetilde{\mathbf{X}}^{\top}\boldsymbol{\beta}\right|^{r}\right]} \leq \sqrt[r]{\mathbb{E}_{\mathbb{P}}\left[\left|Y - \mathbf{X}^{\top}\boldsymbol{\beta}\right|^{r}\right]} + \delta\left(\sigma + \rho(\boldsymbol{\beta})\right).$$

**Step 2.** We show that for any  $\beta \in \text{dom}(\rho)$ , the upper bound given in **Step 1** is tight; i.e. we construct  $\mathbb{P}^* \in B_{\delta}(\mathbb{P})$ , for which the bound holds exactly.

**Proof Step 2.** Note first that because  $\rho$  is proper,  $dom(\rho)$  is nonempty. Also, because we have assumed that  $dom(\rho)$  is open,  $\partial \rho(\beta)$  is nonempty [36, Theorem 23.4 p. 217]. Let  $\beta^*$  be an element of  $\partial \rho(\beta)$  satisfying Equation (9) (there exists one such element by assumption).

Consider the distribution  $\mathbb{P}^*$  corresponding to the random vector  $(\widetilde{\mathbf{X}}, \widetilde{Y})$  defined by

(30) 
$$\widetilde{\mathbf{X}} = \mathbf{X} - e \left( \boldsymbol{\beta}^* - \frac{\boldsymbol{\beta}}{\boldsymbol{\beta}^{\top} \boldsymbol{\beta}} \rho^* (\boldsymbol{\beta}^*) \right), \qquad \widetilde{Y} = Y + \sigma e,$$

where

$$e \equiv \frac{\delta(Y - \mathbf{X}^{\top} \boldsymbol{\beta})}{\sqrt[r]{\mathbb{E}_{\mathbb{P}}\left[\left|Y - \mathbf{X}^{\top} \boldsymbol{\beta}\right|^{r}\right]}}, \qquad (Y, \mathbf{X}) \sim \mathbb{P}.$$

The distributions  $\mathbb{P}^*$  and  $\mathbb{P}$  are already coupled, since  $(\widetilde{\mathbf{X}}, \widetilde{Y})$  are measurable functions of  $(\mathbf{X}, Y) \sim \mathbb{P}$ . Let  $\pi^*(\beta)$  denote the coupling of  $(\widetilde{\mathbb{P}}^*, \mathbb{P})$ .

We show that the distribution  $\mathbb{P}^*$  of  $(\widetilde{\mathbf{X}}, \widetilde{Y})$  is an element of  $B_{\delta}(\mathbb{P})$ . By construction we have

$$\mathbb{E}_{\pi^*(\boldsymbol{\beta})} \left[ \left| (\widetilde{Y} - Y) + (\mathbf{X} - \widetilde{\mathbf{X}})^\top \boldsymbol{\gamma} \right|^r \right] = \mathbb{E}_{\pi^*(\boldsymbol{\beta})} \left[ |e|^r \left| \sigma + \left( \boldsymbol{\beta}^* - \frac{\boldsymbol{\beta}}{\boldsymbol{\beta}^\top \boldsymbol{\beta}} \rho^*(\boldsymbol{\beta}^*) \right)^\top \boldsymbol{\gamma} \right|^r \right]$$

$$= \left| \sigma + (\boldsymbol{\beta}^*)^\top \boldsymbol{\gamma} - \frac{\boldsymbol{\beta}^\top \boldsymbol{\gamma}}{\boldsymbol{\beta}^\top \boldsymbol{\beta}} \rho^*(\boldsymbol{\beta}^*) \right|^r \mathbb{E}_{\pi^*(\boldsymbol{\beta})} \left[ |e|^r \right]$$

$$\leq \left[ \delta \left( \sigma + \rho(\boldsymbol{\gamma}) \right) \right]^r.$$

The last inequality follows since  $\left|\left(\boldsymbol{\beta}^* - \frac{\boldsymbol{\beta}}{\boldsymbol{\beta}^\top \boldsymbol{\beta}} \rho^*(\boldsymbol{\beta}^*)\right)^\top \boldsymbol{\gamma}\right|$  is less than or equal  $\rho(\boldsymbol{\gamma})$  for any  $\boldsymbol{\gamma} \in \text{dom}(\rho)$  by the assumption in (9) and since  $\mathbb{E}_{\mathbb{P}}[|e|^r] = \delta^r$ .

Thus, we only need to compute  $\mathbb{E}_{\mathbb{P}^*}[|Y - \mathbf{X}^{\top}\boldsymbol{\beta}|^r]$ , which equals  $\mathbb{E}_{\pi^*(\boldsymbol{\beta})}[|\widetilde{Y} - \widetilde{\mathbf{X}}^{\top}\boldsymbol{\beta}|^r]$ . Adding and subtracting  $\mathbf{X}^{\top}\boldsymbol{\beta}$  and Y to  $\widetilde{Y} - \widetilde{\mathbf{X}}^{\top}\boldsymbol{\beta}$  we have from (30)

(31) 
$$\widetilde{Y} - \widetilde{\mathbf{X}}^{\top} \boldsymbol{\beta} = \widetilde{Y} - Y + Y - \mathbf{X}^{\top} \boldsymbol{\beta} + (\mathbf{X} - \widetilde{\mathbf{X}})^{\top} \boldsymbol{\beta} = (Y - \mathbf{X}^{\top} \boldsymbol{\beta}) + e (\sigma + \rho(\boldsymbol{\beta})),$$

where the last term applies [36, Theorem 23.5, p. 218], which shows that for any proper, convex function  $\beta^* \in \partial \rho(\beta)$  if and only if

$$(\boldsymbol{\beta}^*)^{\top} \boldsymbol{\beta} - \rho^* (\boldsymbol{\beta}^*) = \rho(\boldsymbol{\beta}); \text{ hence, } \left( \mathbf{X} - \widetilde{\mathbf{X}} \right)^{\top} \boldsymbol{\beta} = e \left( \boldsymbol{\beta}^* - \frac{\boldsymbol{\beta}}{\boldsymbol{\beta}^{\top} \boldsymbol{\beta}} \rho^* (\boldsymbol{\beta}^*) \right)^{\top} \boldsymbol{\beta} = e \rho \left( \boldsymbol{\beta} \right).$$

Therefore, using (31) and writing  $(Y - \mathbf{X}^{\top} \boldsymbol{\beta})$  as  $e^{-r} \sqrt{\mathbb{E}_{\mathbb{P}}[|Y - \mathbf{X}^{\top} \boldsymbol{\beta}|^r]} / \delta$ , we have that

$$\begin{split} \mathbb{E}_{\mathbb{P}^*} \left[ \left| Y - \mathbf{X}^{\top} \boldsymbol{\beta} \right|^r \right] &= \mathbb{E}_{\widetilde{\mathbb{P}}^*} \left[ \left| \left( Y - \mathbf{X}^{\top} \boldsymbol{\beta} \right) + e(\sigma + \rho(\boldsymbol{\beta})) \right|^r \right] \\ &= \left| \frac{1}{\delta} \sqrt[r]{\mathbb{E}_{\mathbb{P}}[|Y - \mathbf{X}^{\top} \boldsymbol{\beta}|^r]} + (\sigma + \rho(\boldsymbol{\beta})) \right|^r \mathbb{E}_{\widetilde{\mathbb{P}}^*} \left[ |e|^r \right] \\ &= \left| \sqrt[r]{\mathbb{E}_{\mathbb{P}}[|Y - \mathbf{X}^{\top} \boldsymbol{\beta}|^r]} + \delta(\sigma + \rho(\boldsymbol{\beta})) \right|^r. \end{split}$$

In the final step above, we again use that  $\mathbb{E}_{\mathbb{P}}[|e|^r] = \delta^r$ .

### 4.2. Proofs for Section 3.

Proof of Lemma 1. By [42, Theorem 4.10], we have

$$\mathbb{P}\left(\sup_{f\in\mathcal{H}}\left|\mathbb{E}_{\mathbb{P}_n}\left[f\right] - \mathbb{E}_{\mathbb{P}}\left[f\right]\right| > 2\mathcal{R}_n(\mathcal{H}) + \epsilon\right) \le e^{-n\epsilon^2/2},$$

where

$$\mathcal{R}_n(\mathcal{H}) := \mathbb{E}_{\mathbb{P},\varepsilon} \left[ \sup_{f \in \mathcal{H}} \left| \frac{1}{n} \sum_{i=1}^n \varepsilon_i f\left(\mathbf{X}_i\right) \right| \right],$$

is the Rademacher complexity of  $\mathcal{H}$ , namely, we have that  $\varepsilon_i$  are i.i.d. Rademacher random variables. Next, following [17, statement and proof of Theorem 3.2], we obtain

(32) 
$$\mathcal{R}_n(\mathcal{H}) \leq \frac{12}{\sqrt{n}} \max_{\mathbf{x}_1, \dots, \mathbf{x}_n \in \mathbb{R}^{d+1}} \int_0^1 \sqrt{2 \log N\left(\mathbf{r}, \mathcal{H}(\mathbf{x}_1^n)\right)} d\mathbf{r},$$

where  $\mathbf{x}_1^n := \{\mathbf{x}_1, \dots, \mathbf{x}_n\}$  and

$$\mathcal{H}(\mathbf{x}_1^n) := \left\{ (f(\mathbf{x}_1), \dots, f(\mathbf{x}_n)) : f \in \mathcal{H} \right\},\,$$

and  $N(\mathsf{r},B)$  is defined as the cardinality of the smallest cover for any set  $B\subseteq\{0,1\}^n$  of radius  $\mathsf{r}$  with respect to the distance

$$\rho(\mathbf{b}, \mathbf{d}) := \sqrt{\frac{1}{n} \sum_{i=1}^{n} \mathbf{1}_{\{b_i \neq d_i\}}},$$

where in the above, vectors  $\mathbf{b}, \mathbf{d} \in B$ . [17, Theorem 4.3] states that

(33) 
$$N(r, \mathcal{H}(\mathbf{x}_1^n)) \le \left(\frac{4e}{r^2}\right)^{V/(1-1/e)} = \left(\frac{4e}{r^2}\right)^{Ve/(e-1)},$$

where V is the VC-dimension of  $\mathcal{H}$ . Furthermore, by [17, Corollary 4.2], the VC-dimension of  $\mathcal{H}$  is bounded by d+2, i.e.  $V \leq d+2$ . In conclusion, using (33),

(34) 
$$\log N\left(\mathbf{r}, \mathcal{H}(\mathbf{x}_1^n)\right) \le \frac{eV}{e-1} \log \left(\frac{4e}{\mathsf{r}^2}\right) \le \frac{e(d+2)}{e-1} \log \left(\frac{4e}{\mathsf{r}^2}\right).$$

Following [17, proof of Theorem 3.3] we estimate

(35) 
$$\int_0^1 \sqrt{\log\left(\frac{4e}{\mathsf{r}^2}\right)} \, d\mathsf{r} \le \sqrt{2\pi e},$$

so that from (34) and (35), we have

$$\int_0^1 \sqrt{2\log N(\mathsf{r}, \mathcal{H}(\mathbf{x}_1^n))} \, d\mathsf{r} \le 2e\sqrt{\frac{(d+2)\pi}{e-1}} \le 7.5\sqrt{d+2}.$$

Using (32), this yields

$$\mathcal{R}_n(\mathcal{H}) \le 90\sqrt{\frac{d+2}{n}}.$$

Proof of Lemma 2. We first note that [9, Proposition 7.14] yields,

$$\mathcal{W}_{r}(\mathbb{P}_{\gamma,n},\mathbb{P}_{\gamma})^{r} \leq r2^{r-1} \int |t|^{r-1} |F_{\gamma,n}(t) - F_{\gamma}(t)| dt \\
\leq r(2k)^{r} \sup_{t} |F_{\gamma,n}(t) - F_{\gamma}(t)| | \\
+ r2^{r-1} \int_{\mathbb{R}\setminus[-k,k]} |t|^{r-1} \sqrt{F_{\gamma}(t)(1 - F_{\gamma,n}(t))} \frac{(F_{\gamma}(t) - F_{\gamma,n}(t))^{+}}{\sqrt{F_{\gamma}(t)(1 - F_{\gamma,n}(t))}} dt \\
+ r2^{r-1} \int_{\mathbb{R}\setminus[-k,k]} |t|^{r-1} \sqrt{F_{\gamma,n}(t)(1 - F_{\gamma}(t))} \frac{(F_{\gamma,n}(t) - F_{\gamma}(t))^{+}}{\sqrt{F_{\gamma,n}(t)(1 - F_{\gamma}(t))}} dt.$$

By Markov's inequality, we have for any  $s \ge 1$  and any  $t \in \mathbb{R} \setminus \{0\}$ ,

$$\sqrt{F_{\gamma}(t)(1-F_{\gamma,n}(t))} \vee \sqrt{F_{\gamma,n}(t)(1-F_{\gamma}(t))} \leq \sqrt{\frac{\mathbb{E}_{\mathbb{P}}[|(\mathbf{X},Y)^{\top}\boldsymbol{\gamma}|^{s}] \vee \mathbb{E}_{\mathbb{P}_{n}}[|(\mathbf{X},Y)^{\top}\boldsymbol{\gamma}|^{s}]}{|t|^{s}}}.$$

Plugging these bounds into (36), we obtain

(37)

$$\widetilde{\mathcal{W}}_{r}(\mathbb{P}_{\boldsymbol{\gamma},n},\mathbb{P}_{\boldsymbol{\gamma}})^{r} \leq r(2k)^{r} \sup_{t} |F_{\boldsymbol{\gamma},n}(t) - F_{\boldsymbol{\gamma}}(t)| 
+ r2^{r-1} \int_{\mathbb{R}\setminus[-k,k]} |t|^{r-1-s/2} \sqrt{\mathbb{E}_{\mathbb{P}}[|(\mathbf{X},Y)^{\top}\boldsymbol{\gamma}|^{s}]} \vee \mathbb{E}_{\mathbb{P}_{n}}[|(\mathbf{X},Y)^{\top}\boldsymbol{\gamma}|^{s}]} \frac{(F_{\boldsymbol{\gamma}}(t) - F_{\boldsymbol{\gamma},n}(t))^{+}}{\sqrt{F_{\boldsymbol{\gamma}}(t)(1 - F_{\boldsymbol{\gamma},n}(t))}} dt 
+ r2^{r-1} \int_{\mathbb{R}\setminus[-k,k]} r|t|^{r-1-s/2} \sqrt{\mathbb{E}_{\mathbb{P}}[|(\mathbf{X},\mathbf{Y})^{\top}\boldsymbol{\gamma}|^{s}]} \vee \mathbb{E}_{\mathbb{P}_{n}}[|(\mathbf{X},Y)^{\top}\boldsymbol{\gamma}|^{s}]} \frac{(F_{\boldsymbol{\gamma},n}(t) - F_{\boldsymbol{\gamma}}(t))^{+}}{\sqrt{F_{\boldsymbol{\gamma},n}(t)(1 - F_{\boldsymbol{\gamma}}(t))}} dt,$$

where  $k \ge 0$  is some constant. Recall that we have assumed s/2 > r where  $r \ge 1$ . In particular, thi smeans that  $|t|^{r-1-s/2}$  is integrable on  $\mathbb{R} \setminus [-k, k]$  and

$$r2^{r-1} \int_{\mathbb{R}\backslash [-k,k]} |t|^{r-1-s/2} \, dt = \frac{2^r r}{s/2 - r} k^{r-s/2}.$$

Taking supremum over  $\gamma$  and t in (37) thus yields the claim.

Proof of Lemma 3. We first define

$$\mathcal{J} = \left\{ \mathbf{1}_{\left\{ \mathbf{x}^{\top} \boldsymbol{\gamma} \leq t \right\}}, \, \mathbf{1}_{\left\{ \mathbf{x}^{\top} \boldsymbol{\gamma} > t \right\}} : (\boldsymbol{\gamma}, t) \in \mathbb{R}^{d+1} \times \mathbb{R} \right\} \supset \mathcal{H}.$$

Considering the cases  $F_{\gamma,n}(t) < 1/2$  and  $F_{\gamma,n}(t) \ge 1/2$  separately—noting that e.g.  $\mathbb{E}_{\mathbb{P}_n}[\mathbf{1}_{\{\mathbf{x}^\top \gamma > t\}}] = 1 - \mathbb{E}_{\mathbb{P}_n}[\mathbf{1}_{\{\mathbf{x}^\top \gamma \leq t\}}] = 1 - F_{\gamma,n}(t)$ —one can check that

$$(38) \sup_{(\gamma,t)\in\mathbb{R}^{d+1}\times\mathbb{R}} \frac{(F_{\gamma}(t) - F_{\gamma,n}(t))^{+}}{\sqrt{F_{\gamma}(t)(1 - F_{\gamma,n}(t))}} \leq 2 \left( \sup_{f\in\mathcal{J}} \frac{(\mathbb{E}_{\mathbb{P}}[f] - \mathbb{E}_{\mathbb{P}_{n}}[f])^{+}}{\sqrt{\mathbb{E}_{\mathbb{P}}[f]}} \vee \sup_{f\in\mathcal{J}} \frac{(\mathbb{E}_{\mathbb{P}_{n}}[f] - \mathbb{E}_{\mathbb{P}}[f])^{+}}{\sqrt{\mathbb{E}_{\mathbb{P}_{n}}[f]}} \right).$$

By symmetry,

$$(39) \sup_{(\gamma,t)\in\mathbb{R}^{d+1}\times\mathbb{R}} \frac{(F_{\gamma,n}(t)-F_{\gamma}(t))^{+}}{\sqrt{F_{\gamma,n}(t)(1-F_{\gamma}(t))}} \leq 2\left(\sup_{f\in\mathcal{J}} \frac{(\mathbb{E}_{\mathbb{P}}[f]-\mathbb{E}_{\mathbb{P}_{n}}[f])^{+}}{\sqrt{\mathbb{E}_{\mathbb{P}}[f]}} \vee \sup_{f\in\mathcal{J}} \frac{(\mathbb{E}_{\mathbb{P}_{n}}[f]-\mathbb{E}_{\mathbb{P}}[f])^{+}}{\sqrt{\mathbb{E}_{\mathbb{P}_{n}}[f]}}\right).$$

Concentration for the terms on the right hand side of equations (38) and (39) are well studied: indeed, e.g. by [17, Examples 3.3 & 3.4] we have

$$\mathbb{P}\left(\sup_{f\in\mathcal{J}}\frac{\mathbb{E}_{\mathbb{P}}[f]-\mathbb{E}_{\mathbb{P}_{n}}[f]}{\sqrt{\mathbb{E}_{\mathbb{P}}[f]}}>\epsilon\right)\leq 4S_{\mathcal{J}}(2n)e^{-n\epsilon^{2}/4},$$

$$\mathbb{P}\left(\sup_{f\in\mathcal{J}}\frac{\mathbb{E}_{\mathbb{P}_{n}}[f]-\mathbb{E}_{\mathbb{P}}[f]}{\sqrt{\mathbb{E}_{\mathbb{P}_{n}}[f]}}>\epsilon\right)\leq 4S_{\mathcal{J}}(2n)e^{-n\epsilon^{2}/4}$$

for all  $\epsilon > 0$ , where  $S_{\mathcal{J}}(2n)$  is the shattering coefficient of  $\mathcal{J}$ . Note that by [17, Theorem 4.1] we have  $S_{\mathcal{J}}(2n) \leq 2S_{\mathcal{H}}(2n)$ . As the VC-dimension of  $\mathcal{H}$  is bounded by d+2, Sauer's lemma [17, Theorem Corollary 4.1] yields

$$\log(S_{\mathcal{I}}(2n)) < (d+2)\log(2n+1).$$

The claim follows by solving the above expression for  $\epsilon$ .

#### References

- [1] Bartl, D. and S. Mendelson (2022): "Structure preservation via the Wasserstein distance," arXiv preprint arXiv:2209.07058.
- [2] Belloni, A., V. Chernozhukov, and L. Wang (2011): "Square-root lasso: pivotal recovery of sparse signals via conic programming," *Biometrika*, 98, 791–806.

- [3] ——— (2014): "Pivotal estimation via square-root lasso in nonparametric regression," *The Annals of Statistics*, 42, 757–788.
- [4] Bertsimas, D. and M. S. Copenhaver (2018): "Characterization of the equivalence of robustification and regularization in linear and matrix regression," *European Journal of Operational Research*, 270, 931–942.
- [5] Blanchet, J., Y. Kang, and K. Murthy (2019): "Robust Wasserstein profile inference and applications to machine learning," *Journal of Applied Probability*, 56, 830–857.
- [6] BLANCHET, J., Y. KANG, K. MURTHY, AND F. ZHANG (2019): "Data-driven optimal transport cost selection for distributionally robust optimization," in 2019 winter simulation conference (WSC), IEEE, 3740–3751.
- [7] Blanchet, J., Y. Kang, J. L. M. Olea, V. A. Nguyen, and X. Zhang (2020): "Machine Learning's Dropout Training is Distributionally Robust Optimal," arXiv preprint arXiv:2009.06111.
- [8] Blanchet, J. and K. Murthy (2019): "Quantifying distributional model risk via optimal transport," *Mathematics of Operations Research*, 44, 565–600.
- [9] Bobkov, S. and M. Ledoux (2019): One-dimensional empirical measures, order statistics, and Kantorovich transport distances, vol. 261, American Mathematical Society.
- [10] BOISSARD, E. AND T. LE GOUIC (2014): "On the mean speed of convergence of empirical and occupation measures in Wasserstein distance," in *Annales de l'IHP Probabilités et statistiques*, vol. 50, 539–563.
- [11] Bonneel, N., J. Rabin, G. Peyré, and H. Pfister (2015): "Sliced and radon wasserstein barycenters of measures," *Journal of Mathematical Imaging and Vision*, 51, 22–45.
- [12] BOYD, S. AND L. VANDENBERGHE (2004): Convex optimization, Cambridge university press.
- [13] Bunea, F., J. Lederer, and Y. She (2013): "The group square-root lasso: Theoretical properties and fast algorithms," *IEEE Transactions on Information Theory*, 60, 1313–1325.
- [14] CHIZAT, L., P. ROUSSILLON, F. LÉGER, F.-X. VIALARD, AND G. PEYRÉ (2020): "Faster wasserstein distance estimation with the sinkhorn divergence," Advances in Neural Information Processing Systems, 33, 2257–2269.
- [15] DEREICH, S., M. SCHEUTZOW, AND R. SCHOTTSTEDT (2013): "Constructive quantization: Approximation by empirical measures," in *Annales de l'IHP Probabilités et statistiques*, vol. 49, 1183–1203.
- [16] DESHPANDE, I., Y.-T. HU, R. SUN, A. PYRROS, N. SIDDIQUI, S. KOYEJO, Z. ZHAO, D. FORSYTH, AND A. G. SCHWING (2019): "Max-sliced wasserstein distance and its use for gans," in *Proceedings of the IEEE/CVF Conference on Computer Vision and Pattern Recognition*, 10648–10656.
- [17] Devroye, L. and G. Lugosi (2001): Combinatorial methods in density estimation, Springer Science & Business Media.
- [18] Duchi, J., P. Glynn, and H. Namkoong (2016): "Statistics of robust optimization: A generalized empirical likelihood approach," arXiv preprint arXiv:1610.03425.
- [19] FOURNIER, N. (2022): "Convergence of the empirical measure in expected Wasserstein distance: non asymptotic explicit bounds in  $\mathbb{R}^d$ ," arXiv preprint arXiv:2209.00923.
- [20] FOURNIER, N. AND A. GUILLIN (2015): "On the rate of convergence in Wasserstein distance of the empirical measure," Probability Theory and Related Fields, 162, 707–738.
- [21] GAO, R., X. CHEN, AND A. J. KLEYWEGT (2017): "Distributional robustness and regularization in statistical learning," arXiv preprint arXiv:1712.06050.
- [22] GAO, R. AND A. KLEYWEGT (2022): "Distributionally robust stochastic optimization with Wasserstein distance," *Mathematics of Operations Research*.
- [23] Gao, R. and A. J. Kleywegt (2016): "Distributionally robust stochastic optimization with Wasserstein distance," arXiv preprint arXiv:1604.02199.

- [24] Kuhn, D., P. M. Esfahani, V. A. Nguyen, and S. Shafieezadeh-Abadeh (2019): "Wasserstein distributionally robust optimization: Theory and applications in machine learning," in *Operations research & management science in the age of analytics*, Informs, 130–166.
- [25] Lam, H. (2016): "Robust sensitivity analysis for stochastic systems," Mathematics of Operations Research, 41, 1248–1275.
- [26] LEE, J. AND M. RAGINSKY (2018): "Minimax statistical learning with wasserstein distances," Advances in Neural Information Processing Systems, 31.
- [27] Lei, J. (2020): "Convergence and concentration of empirical measures under Wasserstein distance in unbounded functional spaces," Bernoulli, 26, 767–798.
- [28] LIN, T., Z. ZHENG, E. CHEN, M. CUTURI, AND M. I. JORDAN (2021): "On projection robust optimal transport: Sample complexity and model misspecification," in *International Conference on Artificial Intelligence and Statistics*, PMLR, 262–270.
- [29] MOHAJERIN ESFAHANI, P. AND D. KUHN (2018): "Data-driven distributionally robust optimization using the Wasserstein metric: Performance guarantees and tractable reformulations," *Mathematical Programming*, 171, 115–166.
- [30] NGUYEN, V. A., S. S. ABADEH, D. FILIPOVIĆ, AND D. KUHN (2021): "Mean-covariance robust risk measurement," arXiv preprint arXiv:2112.09959.
- [31] NILES-WEED, J. AND P. RIGOLLET (2022): "Estimation of Wasserstein distances in the spiked transport model," *Bernoulli*, 28, 2663–2688.
- [32] Paty, F.-P. and M. Cuturi (2019): "Subspace robust Wasserstein distances," in *International conference on machine learning*, PMLR, 5072–5081.
- [33] ——— (2019): "Subspace Robust Wasserstein distances," ArXiv, abs/1901.08949.
- [34] Rabin, J., G. Peyré, J. Delon, and M. Bernot (2011): "Wasserstein barycenter and its application to texture mixing," in *International Conference on Scale Space and Variational Methods in Computer Vision*, Springer, 435–446.
- [35] RÉNYI, A. (1961): "On measures of entropy and information," in *Proceedings of the fourth Berkeley symposium on mathematical statistics and probability*, Berkeley, California, USA, vol. 1.
- [36] Rockafellar, R. T. (1997): Convex analysis, Princeton university press.
- [37] Shafieezadeh Abadeh, S., P. M. Mohajerin Esfahani, and D. Kuhn (2015): "Distributionally robust logistic regression," *Advances in Neural Information Processing Systems*, 28.
- [38] Singh, S. and B. Póczos (2018): "Minimax distribution estimation in Wasserstein distance," arXiv preprint arXiv:1802.08855.
- [39] Sinha, A., H. Namkoong, R. Volpi, and J. Duchi (2017): "Certifying some distributional robustness with principled adversarial training," arXiv preprint arXiv:1710.10571.
- [40] Stucky, B. and S. Van De Geer (2017): "Sharp oracle inequalities for square root regularization," *Journal of Machine Learning Research*, 18, 1–29.
- [41] VILLANI, C. (2008): Optimal Transport: Old and New, vol. 338, Springer.
- [42] Wainwright, M. J. (2019): High-dimensional statistics: A non-asymptotic viewpoint, vol. 48, Cambridge University Press.
- [43] WANG, H., G. LI, AND G. JIANG (2007): "Robust regression shrinkage and consistent variable selection through the LAD-Lasso," *Journal of Business & Economic Statistics*, 25, 347–355.
- [44] WEED, J. AND F. BACH (2019): "Sharp asymptotic and finite-sample rates of convergence of empirical measures in Wasserstein distance," *Bernoulli*, 25, 2620–2648.
- [45] WEED, J. AND Q. BERTHET (2019): "Estimation of smooth densities in Wasserstein distance," in *Conference on Learning Theory*, PMLR, 3118–3119.

José Luis Montiel Olea

DEPARTMENT OF ECONOMICS, CORNELL UNIVERSITY

 $Email\ address: {\tt montiel.olea@gmail.com}$ 

Cynthia Rush

DEPARTMENT OF STATISTICS, COLUMBIA UNIVERSITY

Email address: cynthia.rush@columbia.edu

Amilcar Velez

DEPARTMENT OF ECONOMICS, NORTHWESTERN UNIVERSITY

 $Email\ address: {\tt amilcareQu.northwestern.edu}$ 

Johannes Wiesel

DEPARTMENT OF STATISTICS, COLUMBIA UNIVERSITY  $Email\ address:$  johannes.wiesel@columbia.edu