Lecture 14: Shrinkage

Reading: Section 6.2

STATS 202: Data mining and analysis

Jonathan Taylor, 10/29
Slide credits: Sergio Bacallado

Shrinkage methods

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Why would shrunk coefficients be better?

- This introduces *bias*, but may significantly decrease the *variance* of the estimates. If the latter effect is larger, this would decrease the test error.
- There are Bayesian motivations to do this: the prior tends to shrink the parameters.

Ridge regression

Ridge regression solves the following optimization:

$$\min_{\beta} \quad \sum_{i=1}^{n} \left(y_i - \beta_0 - \sum_{j=1}^{p} \beta_j x_{i,j} \right)^2 + \lambda \sum_{j=1}^{p} \beta_j^2$$

In blue, we have the RSS of the model.

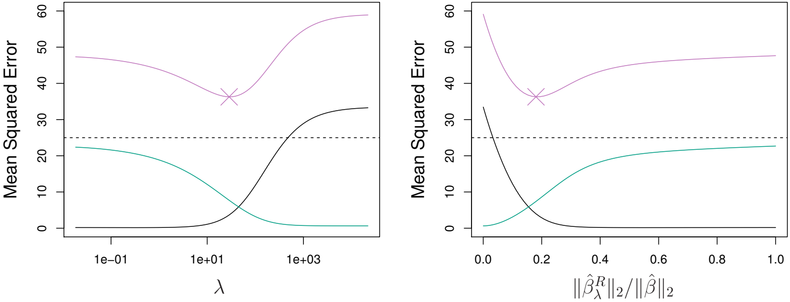
In red, we have the squared ℓ_2 norm of β, or $\|\beta\|_2^2$.

The parameter λ is a tuning parameter. It modulates the importance of fit vs. shrinkage.

We find an estimate $\hat{\beta}^R_\lambda$ for many values of λ and then choose it by cross-validation.

Bias-variance tradeoff

In a simulation study, we compute bias, variance, and test error as a function of λ.



Ridge regression

In least-squares linear regression, scaling the variables has no effect on the fit of the model:

$$Y = \beta_0 + \beta_1 X_1 + \beta_2 X_2 + \cdots + \beta_p X_p.$$

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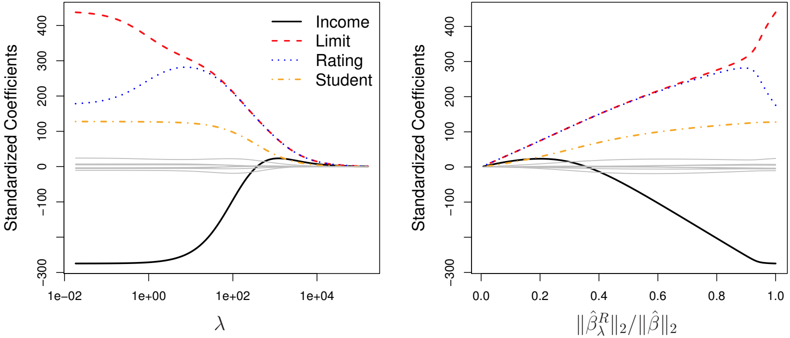
In practice, what do we do?

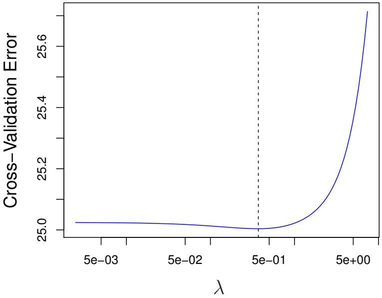
- Scale each variable such that it has sample variance 1 before running the regression.
- This prevents penalizing some coefficients more than others.

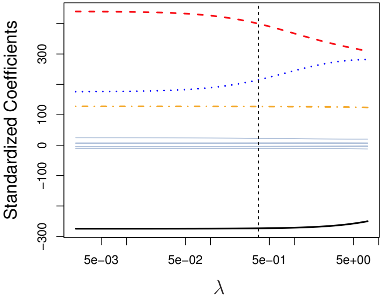
Example. Ridge regression

Ridge regression of `default` in the Credit dataset.



Selecting λ by cross-validation





The Lasso

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Why would we use the Lasso instead of Ridge regression?

- Ridge regression shrinks all the coefficients to a non-zero value.
- The Lasso shrinks some of the coefficients all the way to zero. Alternative **convex** to best subset selection or stepwise selection!

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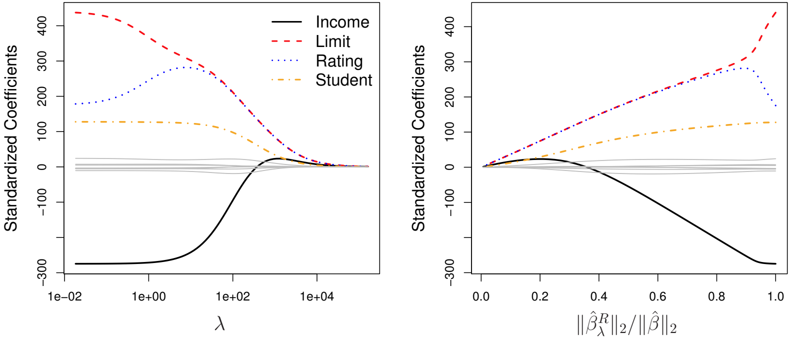
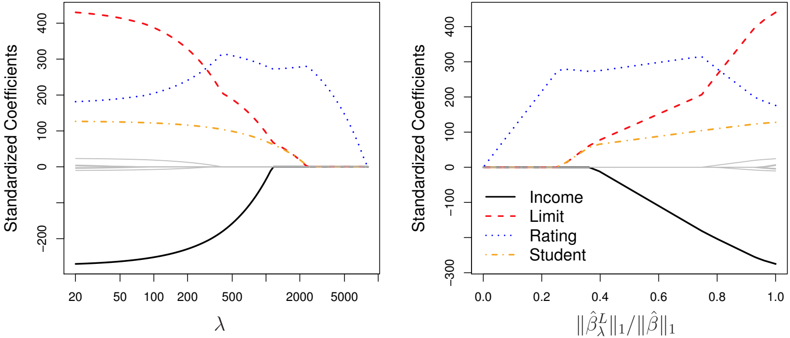


A lot of pesky small coefficients throughout the regularization path.

Example. The Lasso

Lasso regression of `default` in the Credit dataset.



Those coefficients are shrunk to zero.

An alternative formulation for regularization

- **Ridge:** for every λ, there is an s such that $\hat{\beta}^R_\lambda$ solves:

$$\underset{\beta}{\text{minimize}} \left\{ \sum_{i=1}^{n} \left(y_i - \beta_0 - \sum_{j=1}^{p} \beta_j x_{i,j} \right)^2 \right\} \quad \text{subject to} \quad \sum_{j=1}^{p} \beta_j^2 < s.$$

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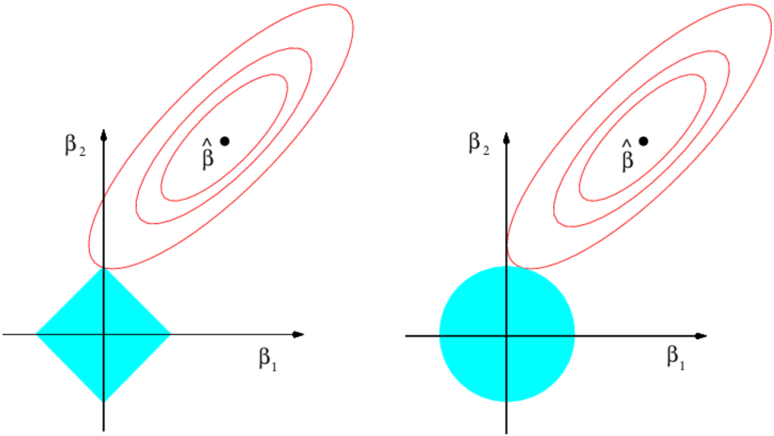
- **Best subset:**

$$\underset{\beta}{\text{minimize}} \left\{ \sum_{i=1}^{n} \left(y_i - \beta_0 - \sum_{j=1}^{p} \beta_j x_{i,j} \right)^2 \right\} \text{ s.t. } \sum_{j=1}^{p} \mathbf{1}(\beta_j \neq 0) < s.$$

Visualizing Ridge and the Lasso with 2 predictors



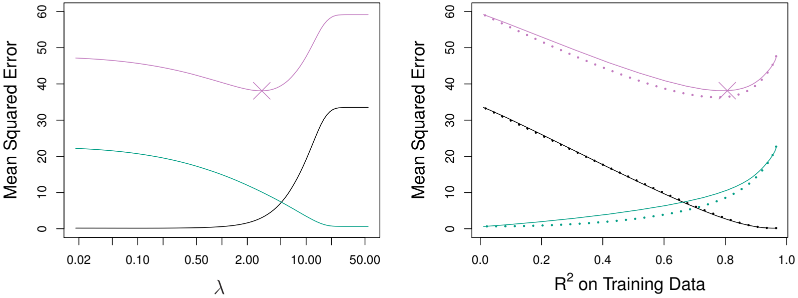
The Lasso ◆ : $\sum_{j=1}^{p} |\beta_j| < s$

Ridge Regression ● : $\sum_{j=1}^{p} \beta_j^2 < s$

Best subset with $s = 1$ is union of the axes...

When is the Lasso better than Ridge?

Example 1. Most of the coefficients are non-zero.



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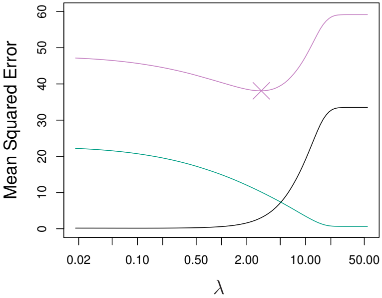

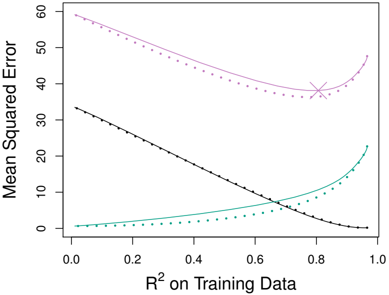

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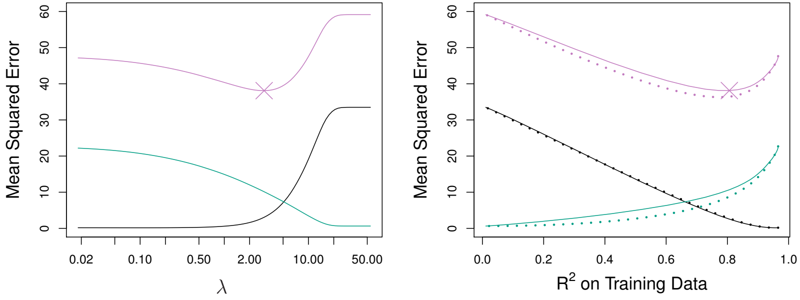

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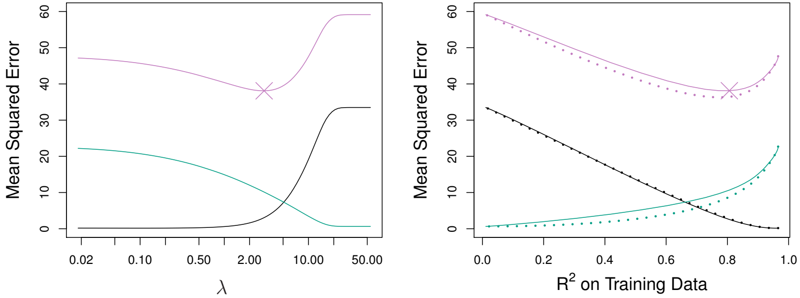

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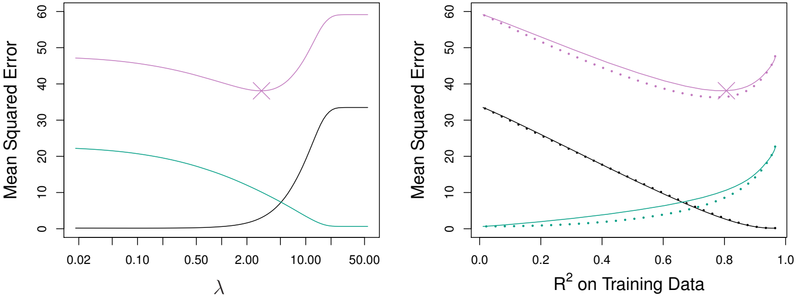

- Bias, Variance, MSE. The Lasso (—), Ridge ($\cdots$).
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- The variance of Ridge regression is smaller, so is the MSE.

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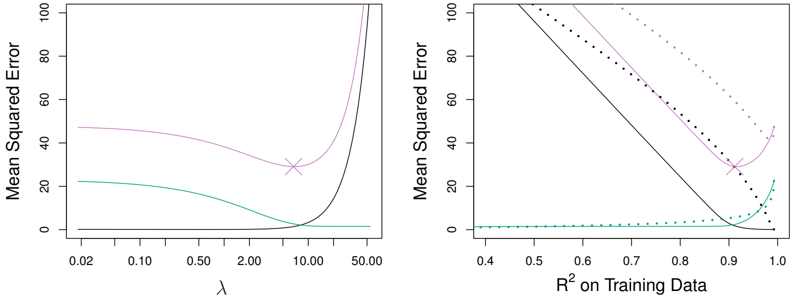

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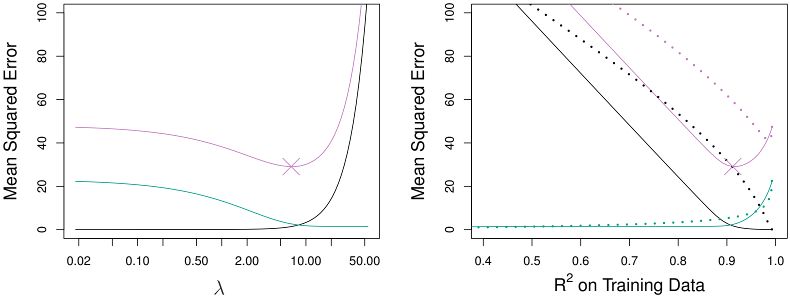

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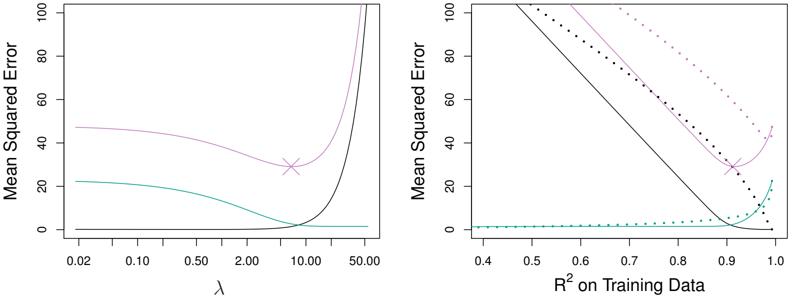

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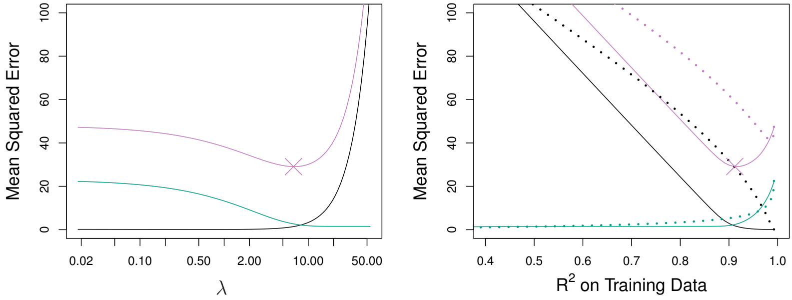
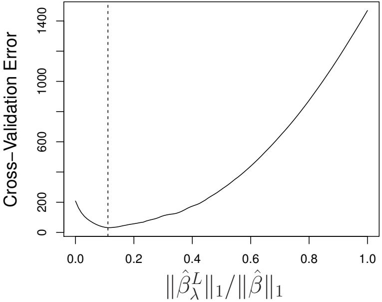
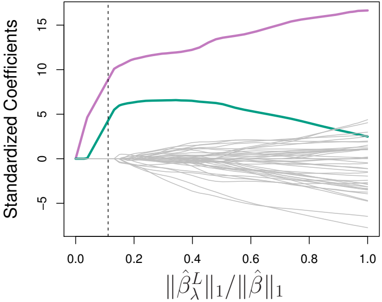


- Bias, Variance, MSE. The Lasso (—), Ridge ($\cdots$).
- The bias, variance, and MSE are lower for the Lasso.

Choosing λ by cross-validation





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It is easy to show that

$$\hat{\beta}_j^R = \frac{y_j}{1 + \lambda}.$$

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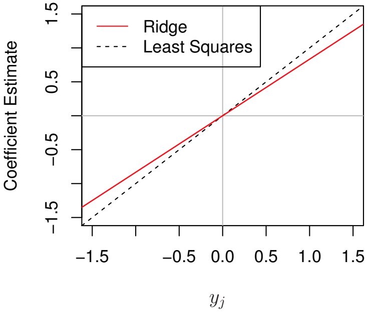
It is easy to show that

$$\hat{\beta}_j^L = \begin{cases} y_j - \lambda/2 & \text{if } y_j > \lambda/2; \\ y_j + \lambda/2 & \text{if } y_j < -\lambda/2; \\ 0 & \text{if } |y_j| < \lambda/2. \end{cases}$$

Lasso and Ridge coefficients as a function of λ



Bayesian interpretations

Ridge: $\hat{\beta}^R$ is the posterior mean, with a Normal prior on β.

Lasso: $\hat{\beta}^L$ is the posterior mode, with a Laplace prior on β.