Lecture 8: Classification

Reading: Chapter 4

STATS 202: Data mining and analysis

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Classification problems

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- *Online advertising:* Predict whether a user will click on an ad or not.

Review: Bayes classifier

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This minimum 0-1 loss (the best we can hope for) is the **Bayes error rate**.

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$$P(Y = 1|X) = \beta_0 + \beta_1 X_1 + \cdots + \beta_1 X_p$$

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Problems:

- This would allow probabilities <0 and >1.
- Difficult to extend to more than 2 categories.

Logistic regression

We model the joint probability as:

$$P(Y = 1 \mid X) = \frac{e^{\beta_0 + \beta_1 X_1 + \cdots + \beta_p X_p}}{1 + e^{\beta_0 + \beta_1 X_1 + \cdots + \beta_p X_p}},$$

$$P(Y = 0 \mid X) = \frac{1}{1 + e^{\beta_0 + \beta_1 X_1 + \cdots + \beta_p X_p}}.$$

This is the same as using a linear model for the log odds:

$$\log \left[\frac{P(Y = 1 \mid X)}{P(Y = 0 \mid X)} \right] = \beta_0 + \beta_1 X_1 + \cdots + \beta_p X_p.$$

Fitting logistic regression

The training data is a list of pairs $(y_1, x_1), (y_2, x_2), \ldots, (y_n, x_n)$. In the linear model

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We cannot use a least squares fit.

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Solution:

The likelihood is the probability of the training data, for a fixed set of coefficients $\beta_0, \ldots, \beta_p$:

$$\prod_{i=1}^{n} P(Y = y_i \mid X = x_i)$$

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- Choose estimates $\hat{\beta}_0, \ldots, \hat{\beta}_p$ which maximize the likelihood.
- Solved with numerical methods (e.g. Newton's algorithm).

Logistic regression in R

```
> glm.fit=glm(Direction~Lag1+Lag2+Lag3+Lag4+Lag5+Volume,
    data=Smarket,family=binomial)
> summary(glm.fit)

Call:
glm(formula = Direction ~ Lag1 + Lag2 + Lag3 + Lag4 + Lag5
    + Volume, family = binomial, data = Smarket)

Deviance Residuals:
   Min       1Q   Median       3Q      Max
 -1.45    -1.20     1.07     1.15     1.33

Coefficients:
            Estimate Std. Error z value Pr(>|z|)
(Intercept) -0.12600    0.24074   -0.52     0.60
Lag1        -0.07307    0.05017   -1.46     0.15
Lag2        -0.04230    0.05009   -0.84     0.40
Lag3         0.01109    0.04994    0.22     0.82
Lag4         0.00936    0.04997    0.19     0.85
Lag5         0.01031    0.04951    0.21     0.83
Volume       0.13544    0.15836    0.86     0.39
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- Other possible hypothesis tests: likelihood ratio test (chi-square distribution).

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- People with a high `balance` are more likely to default.
- Among people with a given `balance`, students are less likely to default.

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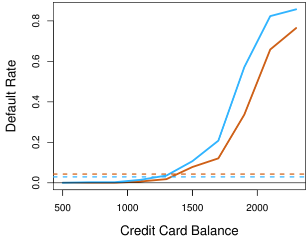

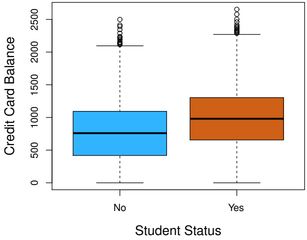

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Logistic regression using only `balance`:

	Coefficient	Std. error	Z-statistic	P-value
`Intercept`	−10.6513	0.3612	−29.5	<0.0001
`balance`	0.0055	0.0002	24.9	<0.0001

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Logistic regression using all 3 predictors:

	Coefficient	Std. error	Z-statistic	P-value
`Intercept`	−10.8690	0.4923	−22.08	<0.0001
`balance`	0.0057	0.0002	24.74	<0.0001
`income`	0.0030	0.0082	0.37	0.7115
`student[Yes]`	−0.6468	0.2362	−2.74	0.0062

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- When the classes are well separated, the coefficients become unstable. This is always the case when $p \geq n - 1$.

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Then, we use *Bayes rule* to obtain the estimate:

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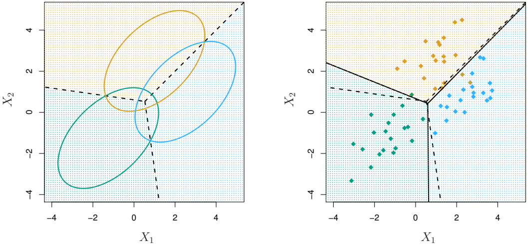

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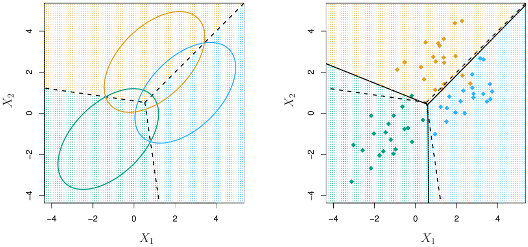

2. $\hat{P}(Y = k) = \hat{\pi}_k$ is estimated by the fraction of training samples of class k.

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- How to evaluate a classification method?
- Examples: comparing KNN, logistic regression and LDA.