Lecture 18: GAMs

Reading: Sections 7.7

STATS 202: Data mining and analysis

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Generalized Additive Models (GAMs)

Extension of non-linear models to multiple predictors:

$$\texttt{wage} = \beta_0 + \beta_1 \times \texttt{year} + \beta_2 \times \texttt{age} + \beta_3 \times \texttt{education} + \epsilon$$

$$\longrightarrow \quad \texttt{wage} = \beta_0 + f_1(\texttt{year}) + f_2(\texttt{age}) + f_3(\texttt{education}) + \epsilon$$

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The functions $f_1, \ldots, f_p$ can be polynomials, natural splines, smoothing splines, local regressions...

Fitting a GAM

- If the functions f_1 have a basis representation, we can simply use least squares:
  - Natural cubic splines
  - Polynomials
  - Step functions

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     as the response.

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  2. Keep $f_1, f_3, \ldots, f_p$ fixed, and fit f_2 using the partial residuals:

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- This works for smoothing splines and local regression. **For smoothing splines this is a descent method, descending on convex loss . . .**

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with $r_j^{(T-1)}$ the j-th partial residual at iteration T

$$r_j^{(T-1)} = Y - \hat{\beta}_0^{(T-1)} - \sum_{l:l\neq j} X_l \hat{\beta}_l^{(T-1)}.$$

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Properties of GAMs

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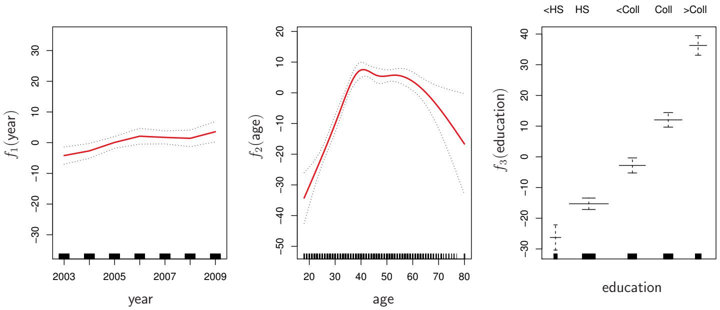
- We can report degrees of freedom for many non-linear functions.
- As in linear regression, we can examine the significance of each of the variables.

Example: Regression for `Wage`



`year`: natural spline with df=4.

`age`: natural spline with df=5.

`education`: factor.

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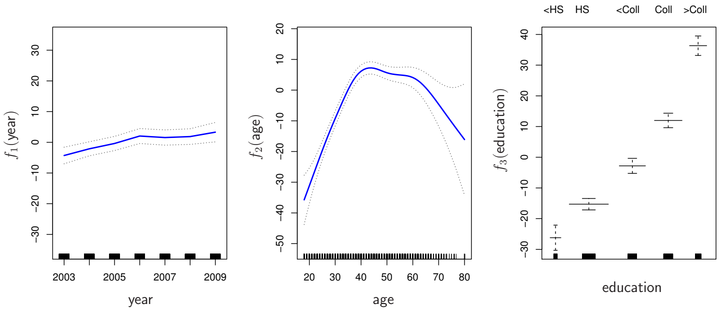

`year`: smoothing spline with df=4.
`age`: smoothing spline with df=5.
`education`: step function.

GAMs for classification

We can model the log-odds in a classification problem using a GAM:

$$\log \frac{P(Y=1 \mid X)}{P(Y=0 \mid X)} = \beta_0 + f_1(X_1) + \cdots + f_p(X_p).$$

Again fit by backfitting ...

Backfitting: GAM with logistic loss

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$$\hat{\alpha}_\lambda(T) = \text{argmin}_{\alpha \in \mathbb{R}^{n_{k(T)}}} \sum_{i=1}^{n} \ell\left(Y_i, \hat{\beta}_0^{(T-1)} + \sum_{j : j \neq k(T)} \sum_{l=1}^{n_j} f_{lj}(X_{ij}) \hat{\beta}_{lj}^{(T-1)} + \sum_{l=1}^{n_{k(T)}} \alpha_l f_{lk(T)}(X_{ik(T)}) \right)$$

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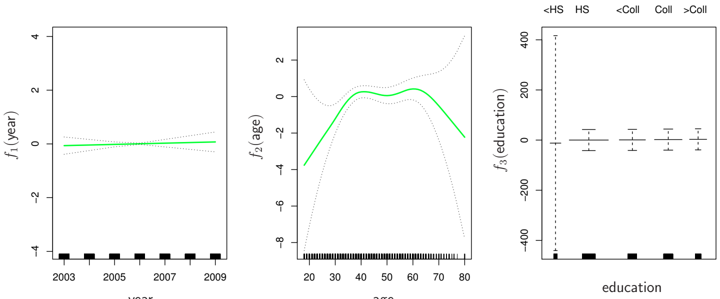
  3. Works for losses that have a *linear predictor*. For GAMs, the linear predictor is

$$\beta_0 + f_1(X_1) + \cdots + f_p(X_p)$$

Example: Classification for `Wage>250`

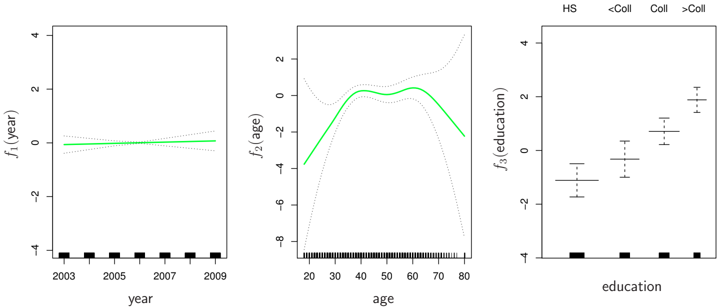


`year`: linear.

`age`: smoothing spline with df=5.

`education`: step function.

Example: Classification for `Wage>250`



`year`: linear.

`age`: smoothing spline with df=5.

`education`: step function.

Exclude samples with `education` < `HS`.