Lecture 9: Classification, LDA

Reading: Chapter 4

STATS 202: Data mining and analysis

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Review: Main strategy in Chapter 4

Find an estimate $\hat{P}(Y \mid X)$. Then, given an input x_0, we predict the response as in a Bayes classifier:

$$\hat{y}_0 = \text{argmax}_{\,y}\ \hat{P}(Y = y \mid X = x_0).$$

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$$\hat{P}(Y = k \mid X = x) = \frac{\hat{P}(X = x \mid Y = k)\hat{P}(Y = k)}{\hat{P}(X = x)}$$

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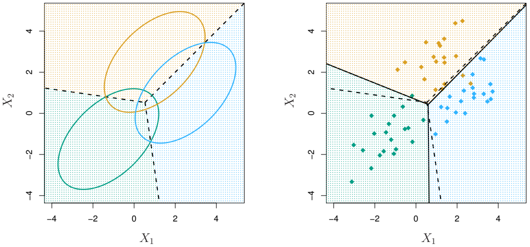
Linear Discriminant Analysis (LDA)

Instead of estimating $P(Y \mid X)$, we will estimate:

1. We model $\hat{P}(X = x \mid Y = k) = \hat{f}_k(x)$ as *a Multivariate Normal Distribution*:



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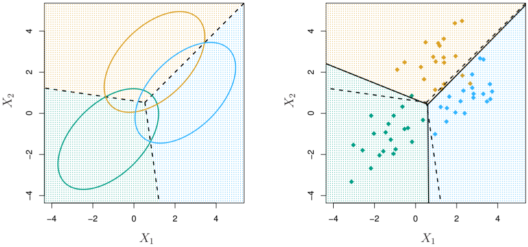

2. $\hat{P}(Y = k) = \hat{\pi}_k$ is estimated by the fraction of training samples of class k.

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Then, what is the Bayes classifier?

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Now, expanding $f_k(x)$:

$$P(Y = k \mid X = x) = \boxed{\frac{C\pi_k}{(2\pi)^{p/2}|\mathbf{\Sigma}|^{1/2}} e^{-\frac{1}{2}(x-\mu_k)^T \mathbf{\Sigma}^{-1}(x-\mu_k)}}$$

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So we want to find the maximum of this over k.

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Goal, maximize the following over k:

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We define the objective:

$$
\delta_k(x) = \log \pi_k - \frac{1}{2}\mu_k^T \mathbf{\Sigma}^{-1}\mu_k + x^T \mathbf{\Sigma}^{-1}\mu_k
$$

At an input x, we predict the response with the highest $\delta_k(x)$.

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What is the decision boundary? It is the set of points in which 2 classes do just as well:

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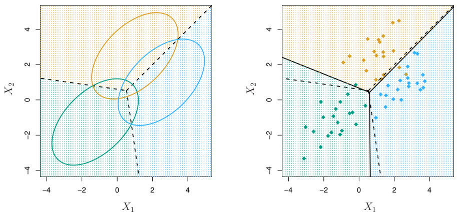
This is a linear equation in x.



Estimating π_k

$$\hat{\pi}_k = \frac{\#\{i \; ; \; y_i = k\}}{n}$$

In English, the fraction of training samples of class k.

Estimating the parameters of $f_k(x)$

Estimate the center of each class μ_k:

$$\hat{\mu}_k = \frac{1}{\#\{i\ ;\ y_i = k\}} \sum_{i\ ;\ y_i=k} x_i$$

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- One predictor ($p = 1$):

$$\boxed{\hat{\sigma}^2 = \frac{1}{n - K} \sum_{k=1}^{K} \sum_{i \; ; \; y_i = k} (x_i - \hat{\mu}_k)^2.}$$

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- Many predictors ($p > 1$): Compute the vectors of deviations $(x_1 - \hat{\mu}_{y_1}), (x_2 - \hat{\mu}_{y_2}), \ldots, (x_n - \hat{\mu}_{y_n})$ and use an unbiased estimate of its covariance matrix, $\mathbf{\Sigma}$.

LDA prediction

For an input x, predict the class with the largest:

$$\hat{\delta}_k(x) = \log \hat{\pi}_k - \frac{1}{2}\hat{\mu}_k^T \hat{\mathbf{\Sigma}}^{-1} \hat{\mu}_k + x^T \hat{\mathbf{\Sigma}}^{-1} \hat{\mu}_k$$

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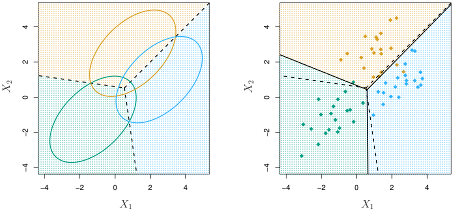
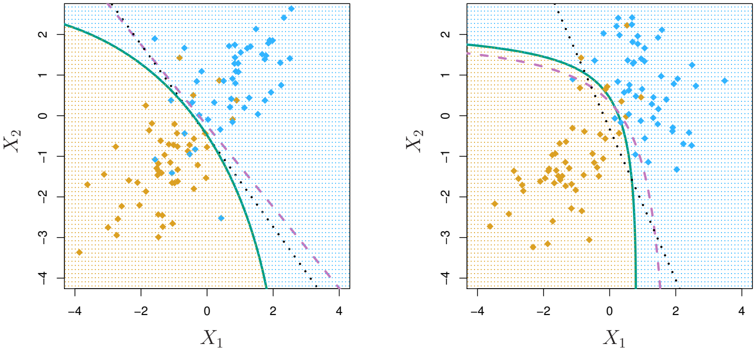
Solid lines in:



Quadratic discriminant analysis (QDA)

The assumption that the inputs of every class have the same covariance $\boldsymbol{\Sigma}$ can be quite restrictive:



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Given an input, it is easy to derive an objective function:

$$\delta_k(x) = \log \pi_k - \frac{1}{2}\mu_k^T \mathbf{\Sigma}_k^{-1}\mu_k + x^T \mathbf{\Sigma}_k^{-1}\mu_k - \frac{1}{2}x^T \mathbf{\Sigma}_k^{-1} x - \frac{1}{2}\log|\mathbf{\Sigma}_k|$$

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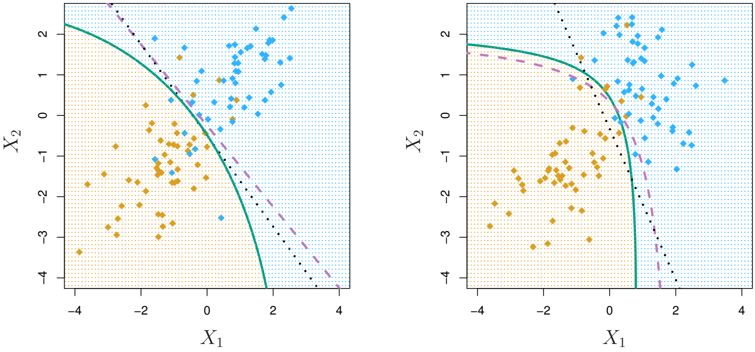
This objective is now quadratic in x and so are the decision boundaries.

Quadratic discriminant analysis (QDA)

- Bayes boundary (– – –)
- LDA (· · · · · ·)
- QDA (——).



Evaluating a classification method

We have talked about the 0-1 loss:

$$\frac{1}{m}\sum_{i=1}^{m}\mathbf{1}(y_i \neq \hat{y}_i).$$

It is possible to make the wrong prediction for some classes more often than others. The 0-1 loss doesn't tell you anything about this.

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A much more informative summary of the error is a **confusion matrix**:

		Predicted class		
		− or Null	+ or Non-null	Total
True class	− or Null	True Neg. (TN)	False Pos. (FP)	N
	+ or Non-null	False Neg. (FN)	True Pos. (TP)	P
	Total	N*	P*	

Example. Predicting `default`

Used LDA to predict credit card default in a dataset of 10K people.

Predicted "yes" if $P(\texttt{default} = \text{yes}|X) > 0.5$.

		True default status		
		No	Yes	Total
Predicted default status	No	$9,644$	252	$9,896$
	Yes	23	81	104
	Total	$9,667$	333	$10,000$

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- However, the rate of false negatives is 76%.
- It is possible that false negatives are a bigger source of concern!
- One possible solution: Change the threshold.

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Changing the threshold to 0.2 makes it easier to classify to "yes".

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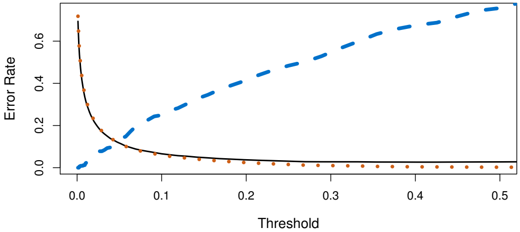
Note that the rate of false positives became higher! That is the price to pay for fewer false negatives.

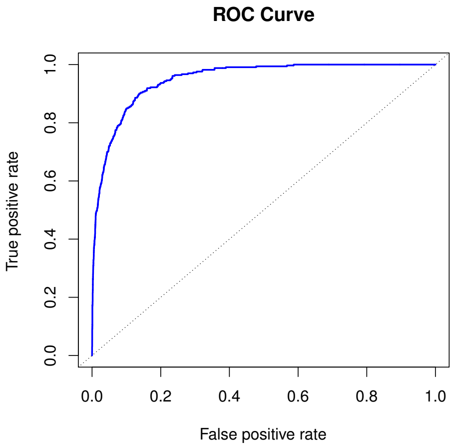
Example. Predicting `default`

Let's visualize the dependence of the error on the threshold:



- – – – False negative rate (error for defaulting customers)
- · · · · · False positive rate (error for non-defaulting customers)
- —— 0-1 loss or total error rate.

Example. The ROC curve



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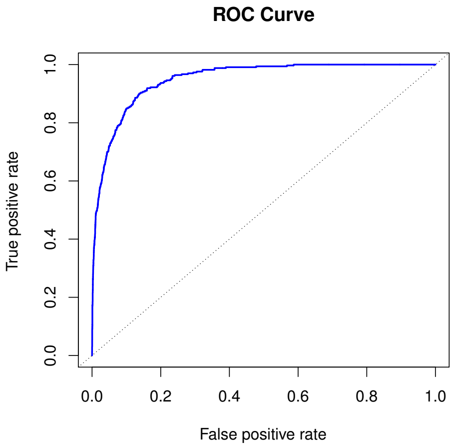

- Displays the performance of the method for any choice of threshold.
- The area under the curve (AUC) measures the quality of the classifier:
  - 0.5 is the AUC for a random classifier
  - The closer AUC is to 1, the better.

Next time

- Comparison of logistic regression, LDA, QDA, and KNN classification.
- Start Chapter 5: Resampling.