Lecture 7: Linear Regression (continued)

Reading: Chapter 3

STATS 202: Data mining and analysis

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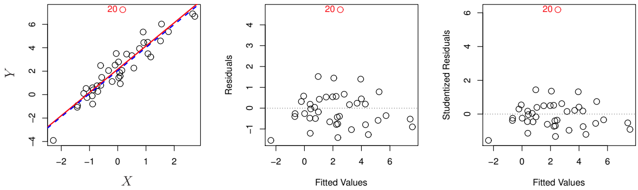
Potential issues in linear regression

1. Interactions between predictors
2. Non-linear relationships
3. Correlation of error terms
4. Non-constant variance of error (heteroskedasticity).
5. Outliers
6. High leverage points
7. Collinearity

Outliers

Outliers are points with very high errors.



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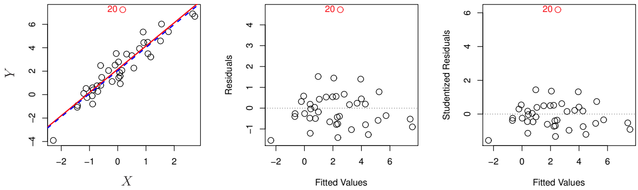

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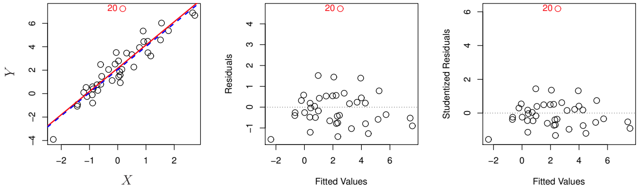

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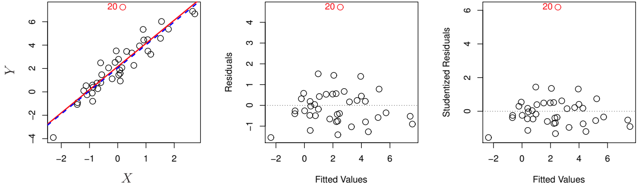

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Possible solutions:

- If we believe an outlier is due to an error in data collection, we can remove it.
- An outlier might be evidence of a missing predictor, or the need to specify a more complex model.

High leverage points

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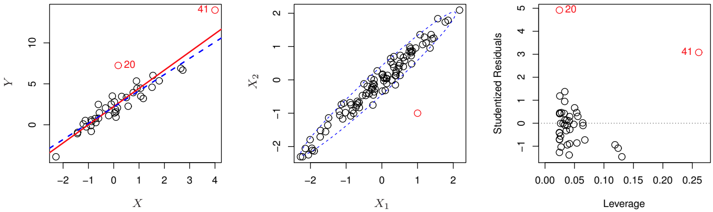

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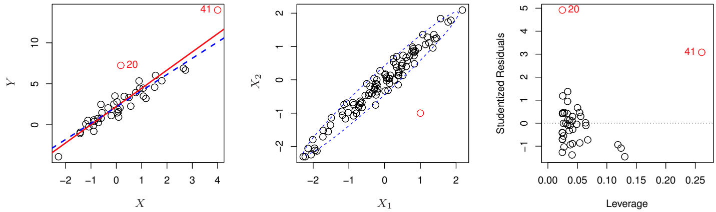

This can be measured with the **leverage statistic** or **self influence**:

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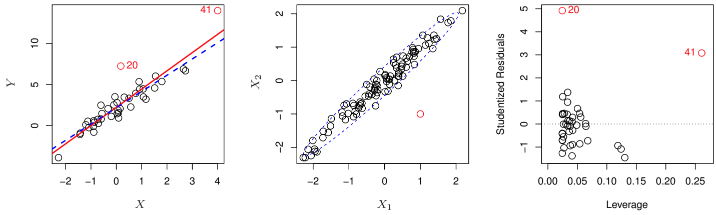

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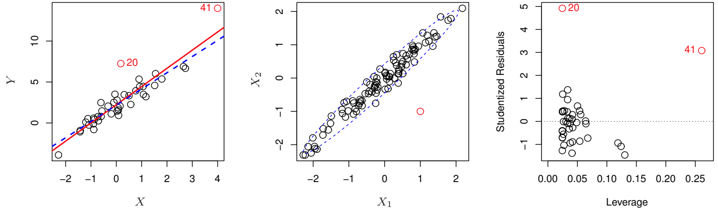

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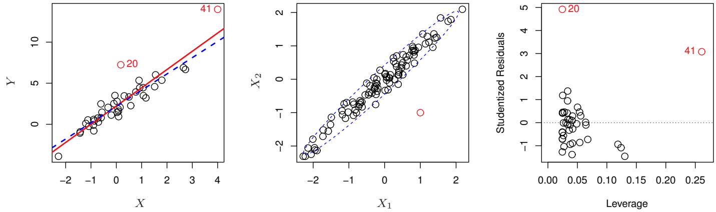
- When model is correct, it follows a Student-t distribution with $n-p-2$ degrees of freedom.

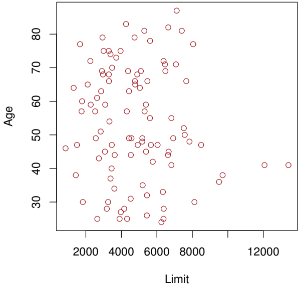


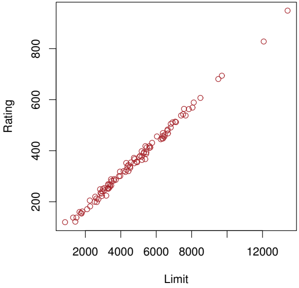
Collinearity

Two predictors are collinear if one explains the other well:

$$\texttt{limit} = a \times \texttt{rating} + b$$

i.e. they contain the same information





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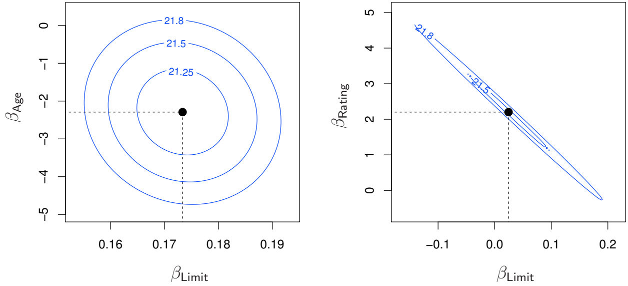

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The Variance Inflation Factor (VIF) measures how *necessary* a variable is, or how predictable it is given the other variables:

$$VIF(\hat{\beta}_j) = \frac{1}{1 - R^2_{X_j|X_{-j}}},$$

where $R^2_{X_j|X_{-j}}$ is the R^2 statistic for Multiple Linear regression of the predictor X_j onto the remaining predictors.

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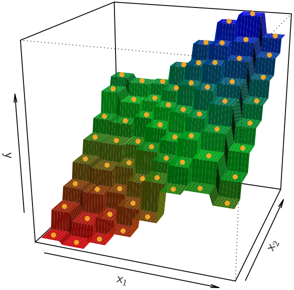
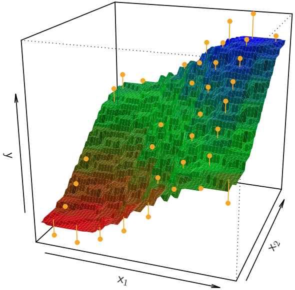


$K = 1$



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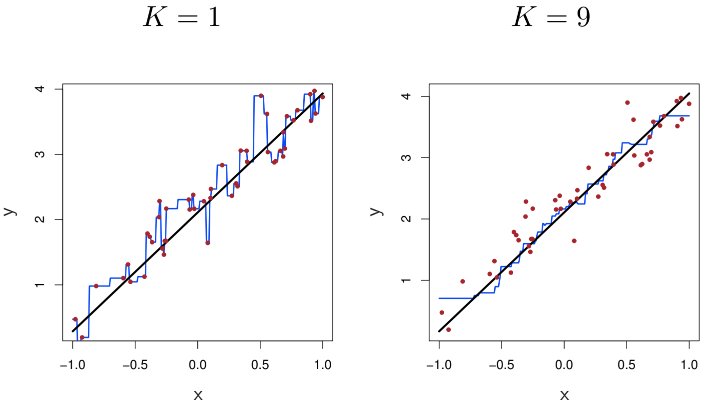
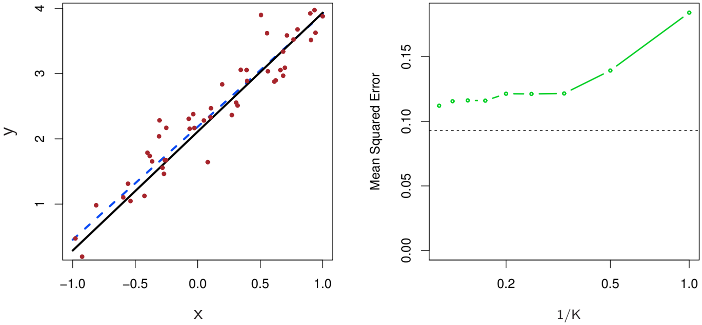
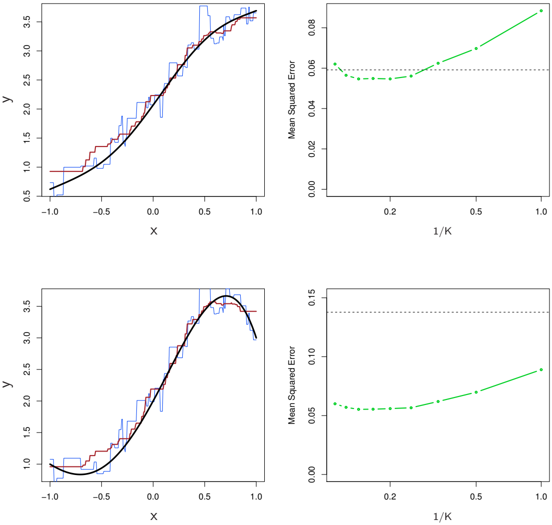
KNN estimates for a simulation from a linear model



Linear models dominate KNN



Increasing deviations from linearity



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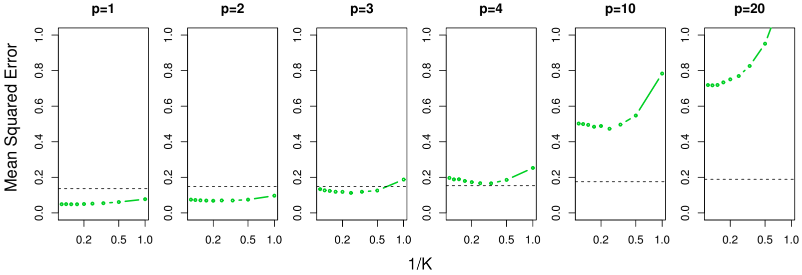

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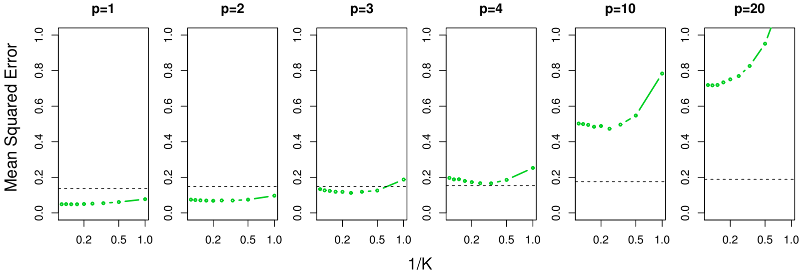

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