Lecture 12: The Bootstrap

Reading: Chapter 5

STATS 202: Data mining and analysis

Jonathan Taylor, 10/19
Slide credits: Sergio Bacallado

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  - No calculators necessary.
  - SCPD students: if you haven't chosen your proctor already, you must do it ASAP. For guidelines see:

    `http://scpd.stanford.edu/programs/courses/graduate-courses/exam-monitor-information`

Cross-validation vs. the Bootstrap

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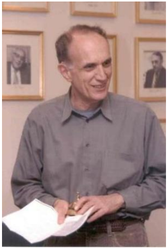

- One of the most important techniques in all of Statistics.
- Computer intensive method.
- Popularized by Brad Efron, from Stanford.

Standard errors in linear regression

Standard error: SD of an estimate from a sample of size n.

```
Residuals:
    Min      1Q  Median      3Q     Max
-15.594  -2.730  -0.518   1.777  26.199

Coefficients:
              Estimate Std. Error t value Pr(>|t|)
(Intercept)  3.646e+01  5.103e+00   7.144 3.28e-12 ***
crim        -1.080e-01  3.286e-02  -3.287 0.001087 **
zn           4.642e-02  1.373e-02   3.382 0.000778 ***
indus        2.056e-02  6.150e-02   0.334 0.738288
chas         2.687e+00  8.616e-01   3.118 0.001925 **
nox         -1.777e+01  3.820e+00  -4.651 4.25e-06 ***
rm           3.810e+00  4.179e-01   9.116  < 2e-16 ***
age          6.922e-04  1.321e-02   0.052 0.958229
dis         -1.476e+00  1.995e-01  -7.398 6.01e-13 ***
rad          3.060e-01  6.635e-02   4.613 5.07e-06 ***
tax         -1.233e-02  3.761e-03  -3.280 0.001112 **
ptratio     -9.527e-01  1.308e-01  -7.283 1.31e-12 ***
black        9.312e-03  2.686e-03   3.467 0.000573 ***
lstat       -5.248e-01  5.072e-02 -10.347  < 2e-16 ***
---
Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1

Residual standard error: 4.745 on 492 degrees of freedom
Multiple R-Squared: 0.7406,     Adjusted R-squared: 0.7338
F-statistic: 108.1 on 13 and 492 DF,  p-value: < 2.2e-16
```

Classical way to compute Standard Errors

Example: Estimate the variance of a sample $x_1, x_2, \ldots, x_n$:

$$\hat{\sigma}^2 = \frac{1}{n-1} \sum_{i=1}^{n} (x_i - \overline{x})^2.$$

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3. For large n, $\hat{\sigma}^2$ is normally distributed around σ^2.
4. The SD of this *sampling distribution* is the Standard Error.

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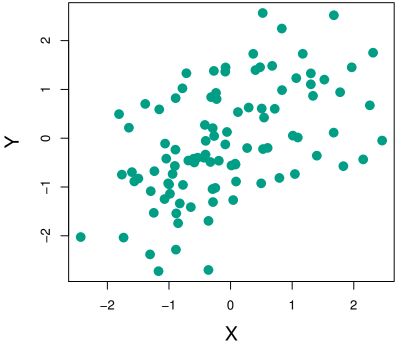
- The distributional assumption — for example, $x_1, \ldots, x_n$ being normal — breaks down?
- The estimator does not have a simple form and its sampling distribution cannot be derived analytically?

Example. Investing in two assets

Suppose that X and Y are the returns of two assets.

These returns are observed every day: $(x_1, y_1), \ldots, (x_n, y_n)$.



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Proposal: Use an estimate:

$$\hat{\alpha} = \frac{\hat{\sigma}_Y^2 - \hat{\mathsf{Cov}}(X,Y)}{\hat{\sigma}_X^2 + \hat{\sigma}_Y^2 - 2\hat{\mathsf{Cov}}(X,Y)}.$$

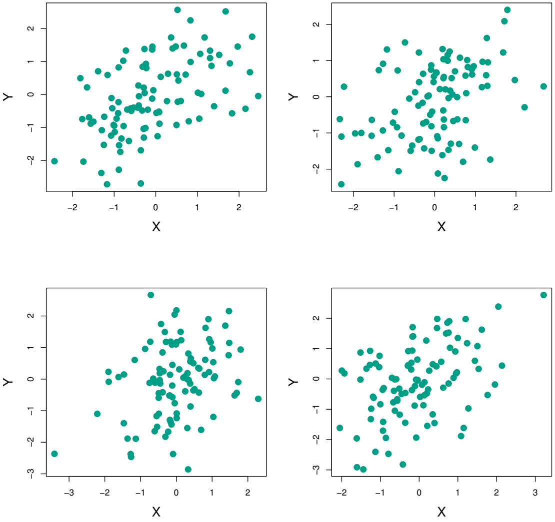
Example. Investing in two assets

Suppose we compute the estimate $\hat{\alpha} = 0.6$ using the samples $(x_1, y_1), \ldots, (x_n, y_n)$.

- How sure can we be of this value?
- If we resampled the observations, would we get a wildly different $\hat{\alpha}$?

In this thought experiment, we know the actual joint distribution $P(X, Y)$, so we can resample the n observations to our hearts' content.

Resampling the data from the true distribution



Computing the standard error of $\hat{\alpha}$

For each resampling of the data,

$$(x_1^{(1)}, \dots, x_n^{(1)})$$

$$(x_1^{(2)}, \dots, x_n^{(2)})$$

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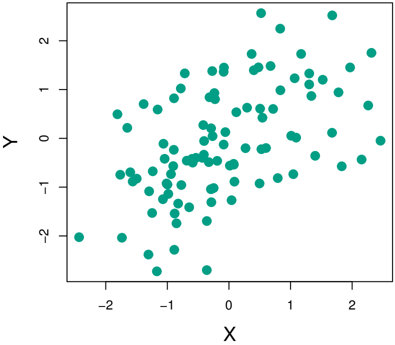
The Standard Error of $\hat{\alpha}$ is approximated by the standard deviation of these values.

In reality, we only have n samples



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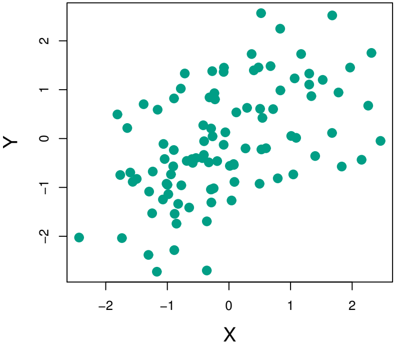

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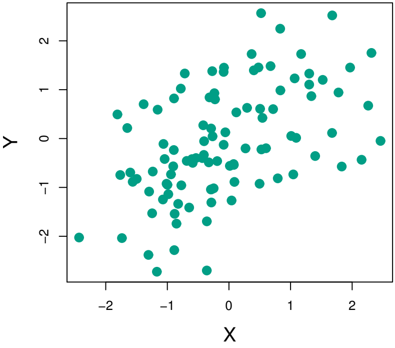

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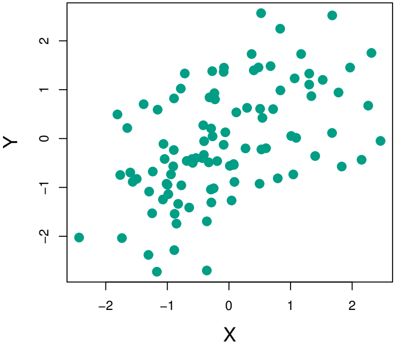

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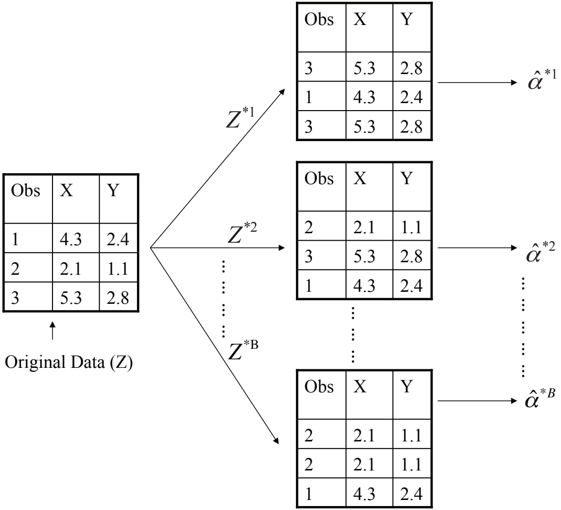
- Equivalently, resample the data by drawing n samples *with replacement* from the actual observations.

A schematic of the Bootstrap



Comparing Bootstrap resamplings to resamplings from the true distribution