

Lecture 9: Classification, LDA

Reading: Chapter 4

STATS 202: Data mining and analysis

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Review: Main strategy in Chapter 4

Find an estimate $\hat{P}(Y | X)$. Then, given an input x_0 , we predict the response as in a Bayes classifier:

$$\hat{y}_0 = \operatorname{argmax}_y \hat{P}(Y = y | X = x_0).$$

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$$\hat{P}(Y = k | X = x) = \frac{\hat{P}(X = x | Y = k)\hat{P}(Y = k)}{\hat{P}(X = x)}$$

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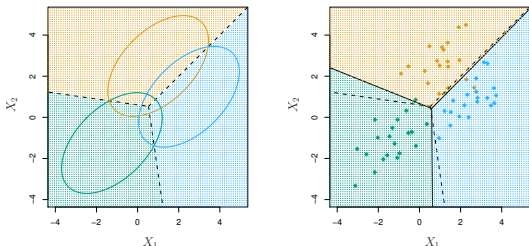
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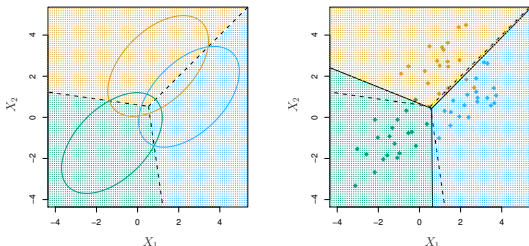
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2. $\hat{P}(Y = k) = \hat{\pi}_k$ is estimated by the fraction of training samples of class k .

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Then, what is the Bayes classifier?

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Now, expanding $f_k(x)$:

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So we want to find the maximum of this over k .

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Goal, maximize the following over k :

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We define the objective:

$$\delta_k(x) = \log \pi_k - \frac{1}{2} \mu_k^T \Sigma^{-1} \mu_k + x^T \Sigma^{-1} \mu_k$$

At an input x , we predict the response with the highest $\delta_k(x)$.

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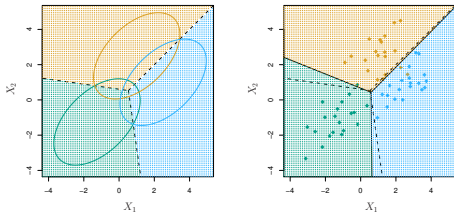
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This is a linear equation in \mathbf{x} .



Estimating π_k

$$\hat{\pi}_k = \frac{\#\{i ; y_i = k\}}{n}$$

In English, the fraction of training samples of class k .

Estimating the parameters of $f_k(x)$

Estimate the center of each class μ_k :

$$\hat{\mu}_k = \frac{1}{\#\{i ; y_i = k\}} \sum_{i ; y_i = k} x_i$$

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- Many predictors ($p > 1$): Compute the vectors of deviations $(x_1 - \hat{\mu}_{y_1}), (x_2 - \hat{\mu}_{y_2}), \dots, (x_n - \hat{\mu}_{y_n})$ and use an unbiased estimate of its covariance matrix, Σ .

LDA prediction

For an input x , predict the class with the largest:

$$\hat{\delta}_k(x) = \log \hat{\pi}_k - \frac{1}{2} \hat{\mu}_k^T \hat{\Sigma}^{-1} \hat{\mu}_k + x^T \hat{\Sigma}^{-1} \hat{\mu}_k$$

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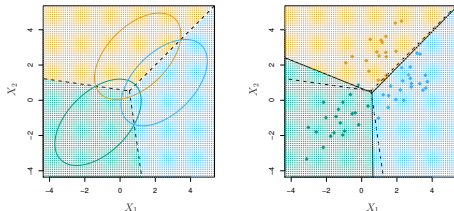
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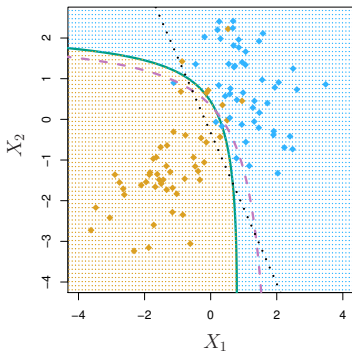
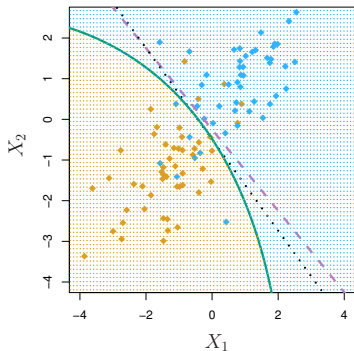
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Solid lines in:



Quadratic discriminant analysis (QDA)

The assumption that the inputs of every class have the same covariance Σ can be quite restrictive:



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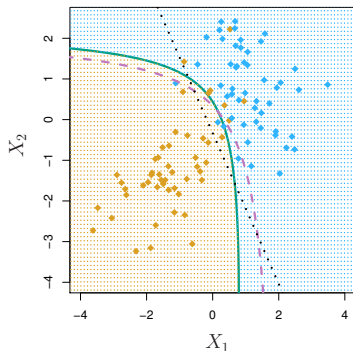
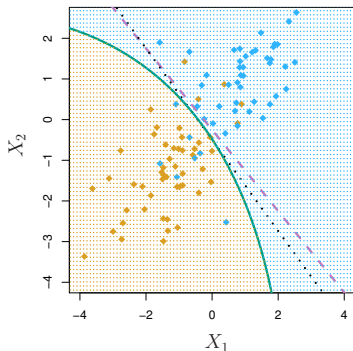
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This objective is now quadratic in x and so are the decision boundaries.

Quadratic discriminant analysis (QDA)

- ▶ Bayes boundary (---)
- ▶ LDA (.....)
- ▶ QDA (—).



Evaluating a classification method

We have talked about the 0-1 loss:

$$\frac{1}{m} \sum_{i=1}^m \mathbf{1}(y_i \neq \hat{y}_i).$$

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A much more informative summary of the error is a **confusion matrix**:

		<i>Predicted class</i>		
		– or Null	+ or Non-null	Total
<i>True class</i>	– or Null	True Neg. (TN)	False Pos. (FP)	N
	+ or Non-null	False Neg. (FN)	True Pos. (TP)	P
	Total	N*	P*	

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Used LDA to predict credit card default in a dataset of 10K people.

Predicted “yes” if $P(\text{default} = \text{yes}|X) > 0.5$.

		<i>True default status</i>		
		No	Yes	Total
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	Yes	23	81	104
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- ▶ However, the rate of false negatives is 76%.
- ▶ It is possible that false negatives are a bigger source of concern!
- ▶ One possible solution: Change the **threshold**.

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Changing the threshold to 0.2 makes it easier to classify to “yes”.

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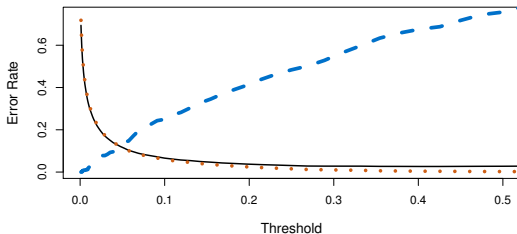
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Note that the rate of false positives became higher! That is the price to pay for fewer false negatives.

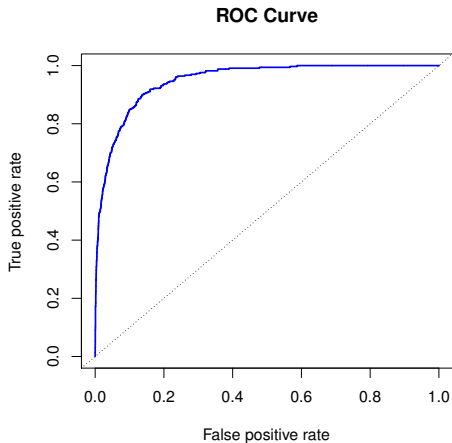
Example. Predicting default

Let's visualize the dependence of the error on the threshold:



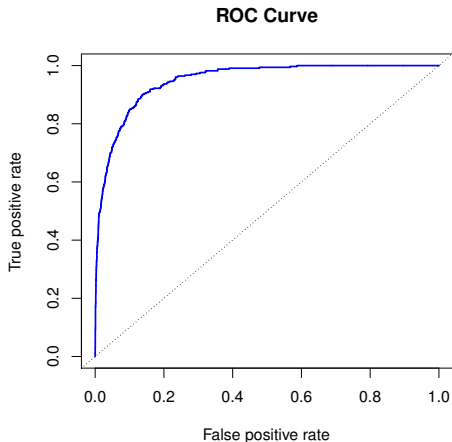
- ▶ — — — False negative rate (error for defaulting customers)
- ▶ False positive rate (error for non-defaulting customers)
- ▶ — 0-1 loss or total error rate.

Example. The ROC curve



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Example. The ROC curve



- ▶ Displays the performance of the method for any choice of threshold.
- ▶ The area under the curve (AUC) measures the quality of the classifier:
 - ▶ 0.5 is the AUC for a random classifier
 - ▶ The closer AUC is to 1, the better.

Next time

- ▶ Comparison of logistic regression, LDA, QDA, and KNN classification.
- ▶ Start Chapter 5: Resampling.