Integer Programming
ISE 418

Lecture 7

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Reading for This Lecture

- Nemhauser and Wolsey Sections II.3.1, II.3.6, II.4.1, II.4.2, II.5.4
- Wolsey Chapter 7
- CCZ Chapter 1
- "Constraint Integer Programming," Achterberg, Chapter II

Computational Integer Optimization

- Before delving any deeper into the theory of integer optimization, we now turn to the basics of how integer optimization problems are solved in practice.
- Computationally, the most important aspects of solving integer optimization problems are
  - A method for obtaining good *bounds* on the value of the optimal solution (usually by solving a *relaxation* or *dual*; and
  - A method for generating *valid disjunctions* violated by a given (infeasible) solution.
- In this lecture, we will motivate this fact by introducing the *branch and bound* algorithm.
- We will then look at various methods of obtaining bounds.
- Later, we will examine branch and bound in more detail.

Integer Optimization and Disjunction

- As we know, the difficulty in solving an integer optimization problem arises from the requirement that certain variables take on integer values.
- Such requirements can be described in terms of logical *disjunctions*, constraints of the form
$$x \in \bigcup_{1 \le i \le k} X_i$$
for $X_i \subseteq \mathbb{R}^n, i \in 1, \ldots, k$.
- The integer variables in a given formulation may represent logical conditions that were originally expressed in terms of disjunction.
- In fact, the MILP Representability Theorem tells us that any MILP can be re-formulated as an optimization problem whose feasible region
$$\mathcal{F} = \bigcup_{i=1}^{k} \mathcal{P}_i + \mathrm{intcone}\{r^1, \ldots, r^t\}$$
is the *disjunctive set* $\mathcal{F}$ defined above, for some appropriately chosen polytopes $\mathcal{P}_1, \ldots, \mathcal{P}_k$ and vectors $r^1, \ldots, r^t \in \mathbb{Z}^n$.

Two Conceptual Reformulations

- From what we have seen so far, we have to two conceptual reformulations of a given integer optimization problem.
- The first is in terms of *disjunction*:

$$\max\left\{c^\top x \mid x \in \left(\bigcup_{i=1}^{k} \mathcal{P}_i + \mathrm{intcone}\{r^1, \ldots, r^t\}\right)\right\} \qquad \text{(DIS)}$$

- The second is in terms of *valid inequalities*:

$$\max\left\{c^\top x \mid x \in \mathrm{conv}(\mathcal{S})\right\} \qquad \text{(CP)}$$

  where $\mathcal{S}$ is the feasible region.

- In principle, if we had a method for generating either of these reformulations, this would lead to a practical method of solution.
- Unfortunately, these reformulations are necessarily of exponential size in general, so there can be no way of generating them efficiently.

Valid Disjunctions

- In practice, we dynamically generate parts of the reformulations (CP) and (DIS) in order to obtain a proof of optimality for a particular instance.
- We can think of the concept of a *valid inequality* as arising from our desire to approximate $\mathrm{conv}(\mathcal{S})$ (the feasible region of (CP)).
- Similarly, we also have the concept of *valid disjunction*, arising from a desire to approximate the feasible region of (DIS).

Definition 1. *Let $\{X_i\}_{i=1}^k$ be a collection of subsets of $\mathbb{R}^n$. Then if $\bigcup_{1 \le i \le k} X_i \supseteq \mathcal{S}$, the disjunction associated with $\{X_i\}_{i=1}^k$ is said to be* valid *for an MILP with feasible set $\mathcal{S}$.*

Definition 2. *If $\{X_i\}_{i=1}^k$ is a disjunction valid for $\mathcal{S}$ and X_i is polyhedral for all $i \in \{1, \ldots, k\}$, then we say the disjunction is* linear*.*

Definition 3. *If $\{X_i\}_{i=1}^k$ is a disjunction valid for $\mathcal{S}$ and $X_i \cap X_j = \emptyset$ for all $i, j \in \{1, \ldots, k\}$, we say the disjunction is* partitive*.*

Definition 4. *If $\{X_i\}_{i=1}^k$ is a disjunction valid for $\mathcal{S}$ that is both linear and partitive, we call it* admissible*.*

A Generic Algorithm

- Many algorithms in optimization consist of the iterative solution of a certain "dual" problem (or relaxation) that is improved dynamically.
- A simple algorithm for solving MILPs is to start by solving the LP relaxation to obtain

$$\hat{x} \in \operatorname{argmax}_{x \in \mathcal{P}} c^\top x$$

  and the upper bound $U = c^\top \hat{x} \geq z_{\text{IP}}$

- Then determine either a valid disjunction or a valid inequality that is *violated* by $\hat{x}$ and "add" it to the relaxation.
- Resolve the strengthened relaxation and continue this process until $U = z_{\text{IP}}$ (or the solution to the relaxation is in $\mathcal{S}$).
- This vague algorithm is, at a high level, how we solve MILPs.
- The condition that $U = z_{\text{IP}}$ is the basic optimality condition used in a wide range of optimization algorithms.

Optimality Conditions

- Let us now consider an MILP (A, b, c, p) with feasible set $\mathcal{S} = \mathcal{P} \cap (\mathbb{Z}^p_+ \times \mathbb{R}^{n-p}_+)$.
- Further, let $\{X_i\}_{i=1}^k$ be a linear disjunction valid for this MILP so that $X_i \cap \mathcal{P} \subseteq \mathbb{R}^n$ is polyhedral.
- Then $\max_{X_i \cap \mathcal{S}} c^\top x$ is an MILP for all $i \in 1, \ldots, k$.
- For each $i = 1, \ldots, k$, let $\mathcal{P}_i$ be a polyhedron such that $X_i \cap \mathcal{S} \subseteq \mathcal{P}_i \subseteq \mathcal{P} \cap X_i\}$.
- In other words, $\mathcal{P}_i$ is a valid formulation for subproblem i, possibly strengthened by additional valid inequalities.
- Note that $\{\mathcal{P}_i\}_{i=1}^k$ is itself a valid linear disjunction.
- We will see why there is a distinction between X_i and $\mathcal{P}_i$ later on.
- Conceptually, we are combining and relaxing the formulations (CP) and (DIS).

Optimality Conditions (cont'd)

- From the disjunction on the previous slide, we obtain a relaxation of a general MILP.
- This relaxation yields a practical set of optimality conditions.
- In particular,

$$\max_{i \in 1,\ldots,k} \max_{x \in \mathcal{P}_i \cap \mathbb{R}^n_+} c^\top x \geq z_{\mathsf{IP}}, \tag{1}$$

which implies that if we have $x^* \in \mathcal{S}$ such that

$$\max_{i \in 1,\ldots,k} \max_{x \in \mathcal{P}_i \cap \mathbb{R}^n_+} c^\top x = c^\top x^*, \tag{OPT}$$

then x^* must be optimal.

More on Optimality Conditions

- Although it is not obvious, these optimality conditions can be seen as a generalization of those from LP.
- They are also the optimality conditions implicitly underlying many advanced algorithms.
- There is an associated duality theory that we will see later.
- By parameterizing (1), we obtain a "dual function" that is the solution to a dual that generalizes the LP dual.

Branch and Bound

- *Branch and bound* is the most commonly-used algorithm for solving MILPs.
- It is a recursive, divide-and-conquer approach.
- Suppose $\mathcal{S}$ is the feasible set for an MILP and we wish to compute $\max_{x \in \mathcal{S}} c^\top x$.
- Consider a partition of $\mathcal{S}$ into subsets $\mathcal{S}_1, \ldots \mathcal{S}_k$. Then

$$\max_{x \in \mathcal{S}} c^\top x = \max_{\{1 \le i \le k\}} \{\max_{x \in \mathcal{S}_i} c^\top x\}$$

.

- In other words, we can optimize over each subset separately.
- Idea: If we can't solve the original problem directly, we might be able to solve the smaller *subproblems* recursively.
- Dividing the original problem into subproblems is called *branching*.
- Taken to the extreme, this scheme is equivalent to complete enumeration.

A Generic Branch-and-Bound Algorithm

Add root optimization problem $\mathcal{S}_0 := \mathcal{S}$ to a priority queue Q. Set global upper bound $U \leftarrow \infty$ and global lower bound $L \leftarrow -\infty$

while $U > L$ **do**

  Remove the highest priority subproblem $\mathcal{S}_i$ from Q.

  Bound $\mathcal{S}_i$ to obtain (updated) final upper bound $U(i)$ and (updated) final lower bound $L(i)$.

  Set $L \leftarrow \max\{L(i), U\}$.

  if $U(i) > L$ **then**

    Branch to create child subproblems $\mathcal{S}_{i_1}, \ldots, \mathcal{S}_{i_k}$ of subproblem $\mathcal{S}_i$ with

    - lower bounds $L(i_1), \ldots L(i_k)$ (initialized to $-\infty$ by default); and
    - initial upper bounds $U(i_1), \ldots, U(i_k)$ (initialized to $U(i)$ by default).

    by partitioning $\mathcal{S}_i$ (imposing a violated valid disjunction)

    Add $\mathcal{S}_{i_1}, \ldots, \mathcal{S}_{i_k}$ to Q.

    Set $U \leftarrow \max_{i \in Q} U(i)$.

  end if

end while

Branching in Branch and Bound

- Branching is achieved by selecting an admissible disjunction $\{X_i\}_{i=1}^k$ and using it to partition $\mathcal{S}$, e.g., $\mathcal{S}_i = \mathcal{S} \cap X_i$.
- We only consider linear disjunctions so that the subproblem remain MILPs after branching.
- The reason for choosing partitive disjunctions is self-evident.
- The way this disjunction is selected is called the *branching method* and is a topic we will examine in some depth.
- Generally speaking, we want $x^* \notin \cup_{1 \leq i \leq k} X_i$, where x^* is the (infeasible) solution produced by solving the *bounding problem*.
- In this case, we say the disjunction is *violated* by x^*.
- A typical disjunction is

$$X_1 = \{x \in \mathbb{R}^n \mid x_j \leq \lfloor x_j^* \rfloor\}, \quad (2)$$

$$X_2 = \{x \in \mathbb{R}^n \mid x_j \geq \lceil x_j^* \rceil\}, \quad (3)$$

where $x^* \in \operatorname{argmax}_{x \in \mathcal{P}} c^\top x$.

Bounding in Branch and Bound

- The *bounding problem* is a problem solved to obtain a bound on the optimal solution value of a subproblem $\max_{\mathcal{S}_i} c^\top x$.
- Typically, the bounding problem is either a relaxation or a dual of the subproblem (these concepts will be defined formally in Lecture 7).
- Solving the bounding problem serves two purposes.
  - In some cases, the solution x^* to the relaxation may actually be a feasible solution ($x^* \in \mathcal{S}$), in which case $c^\top x^*$ is a global lower bound.
  - *Bounding* enables us to inexpensively a bound $U(i)$ on the optimal solution value of subproblem i.
- If $U(i) \leq L$, then $\mathcal{S}_i$ can't contain a solution strictly better than the best one found so far.
- Thus, we may discard or *prune* subproblem i.

Constructing a Bounding Problem

- There are many ways to construct a bounding problem and this will be the topic of later lectures.
- The easiest of the these is to form the *LP relaxation* $\max_{\mathcal{P} \cap \mathbb{R}^n_+ \cap X_i}$, obtained by dropping the integrality constraints.
- For the rest of the lecture, assume all variables have finite upper and lower bounds.

LP-based Branch and Bound: Initial Subproblem

- In LP-based branch and bound, we first solve the LP relaxation of the original problem. The result is one of the following:

  1. The LP is infeasible $\Rightarrow$ MILP is infeasible.
  2. We obtain a feasible solution for the MILP $\Rightarrow$ optimal solution.
  3. We obtain an optimal solution to the LP that is not feasible for the MILP $\Rightarrow$ upper bound.

- In the first two cases, we are finished.

- In the third case, we must branch and recursively solve the resulting subproblems.

Branching in LP-based Branch and Bound

- In LP-based branch and bound, the most commonly used disjunctions are the *variable disjunctions*, imposed as follows:
  - Select a variable i whose value $\hat{x}_i$ is fractional in the LP solution.
  - Create two subproblems.
    * In one subproblem, impose the constraint $x_i \leq \lfloor \hat{x}_i \rfloor$.
    * In the other subproblem, impose the constraint $x_i \geq \lceil \hat{x}_i \rceil$.
- What does it mean in a 0-1 problem?

The Geometry of Branching

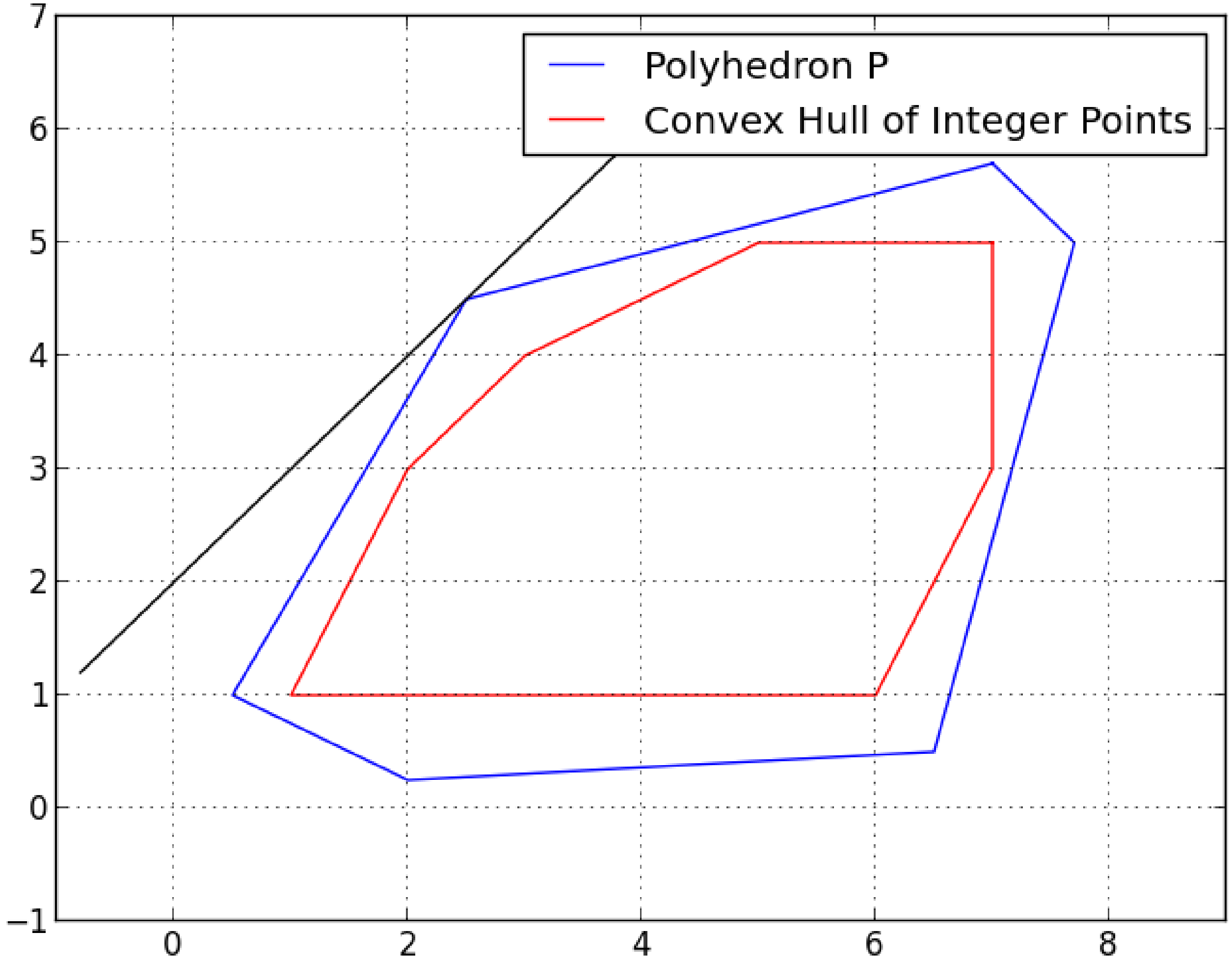


Figure 1: The original feasible region

The Geometry of Branching (cont'd)

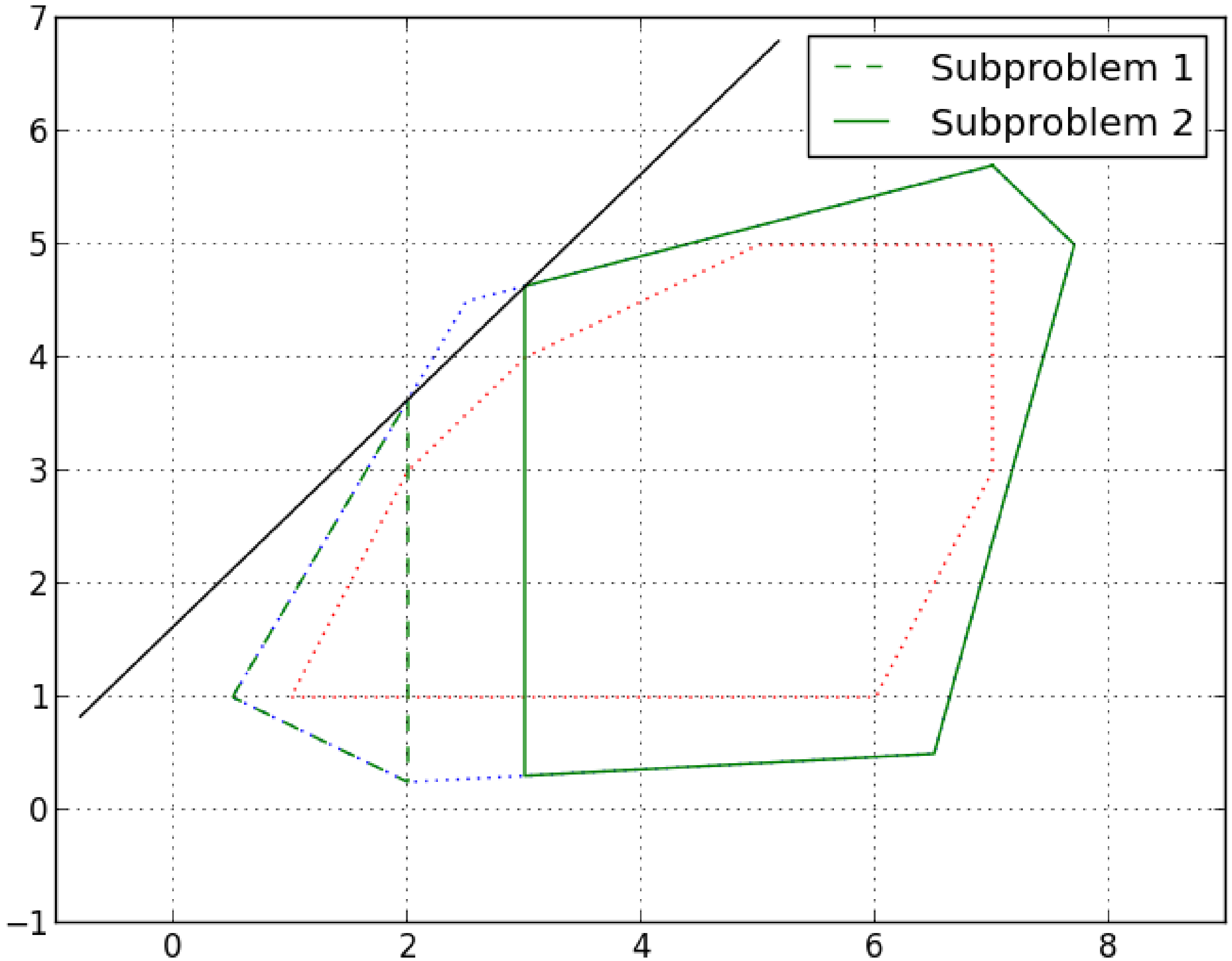


Figure 2: Branching on disjunction $x_1 \leq 2$ OR $x_1 \geq 3$

Continuing the Algorithm After Branching

- After branching, we solve each of the subproblems recursively.
- Now we have an additional factor to consider.
- As mentioned earlier, if the optimal solution value to the LP relaxation is smaller than the current lower bound, we need not consider the subproblem further.
- This is the key to the efficiency of the algorithm.
- Terminology
  - If we picture the subproblems graphically, they form a *search tree*.
  - Each subproblem is linked to its *parent* and eventually to its *children*.
  - Eliminating a problem from further consideration is called *pruning*.
  - The act of bounding and then branching is called *processing*.
  - A subproblem that has not yet been considered is called a *candidate* for processing.
  - The set of candidates for processing is called the *candidate list*.

The Geometry of Branching

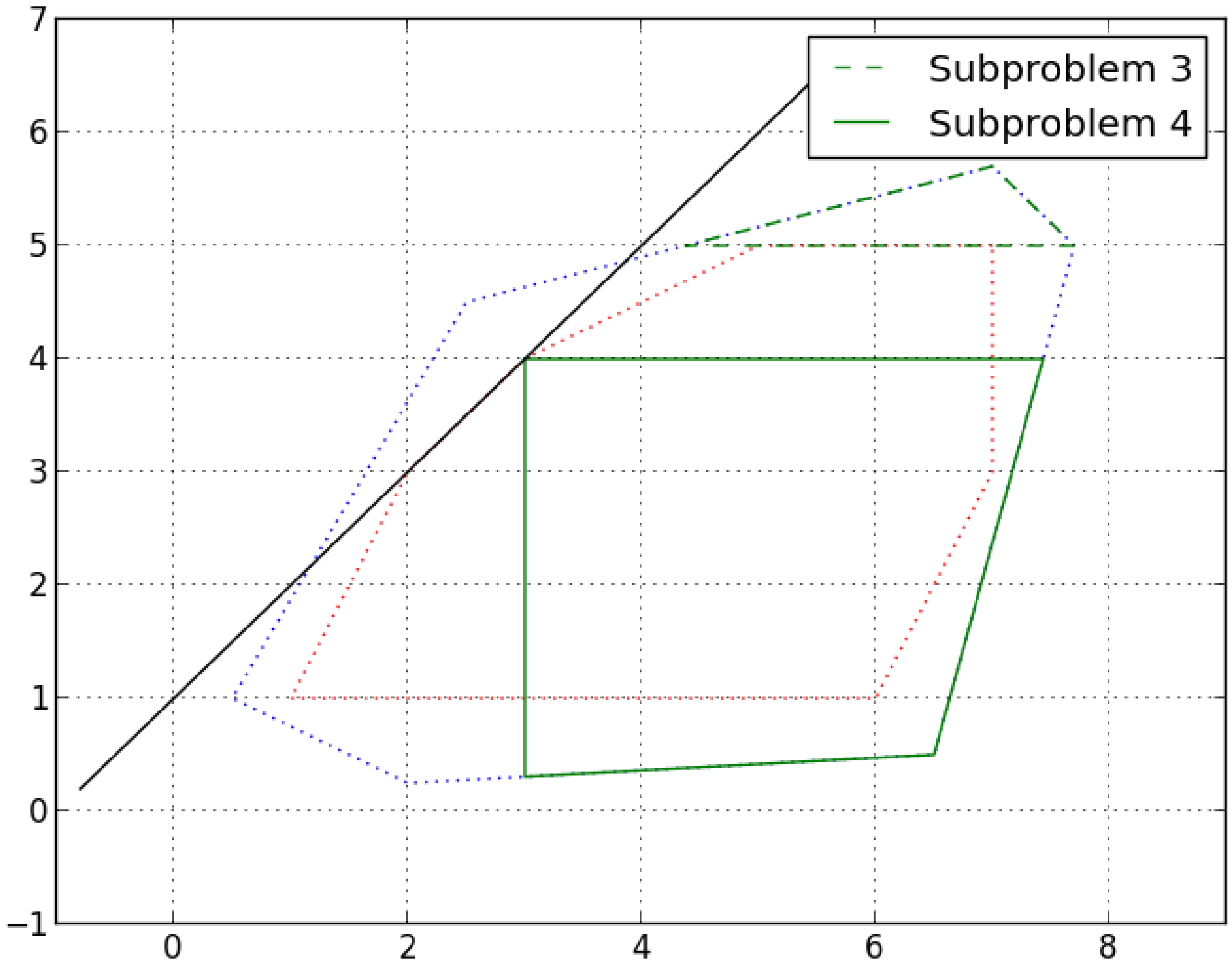
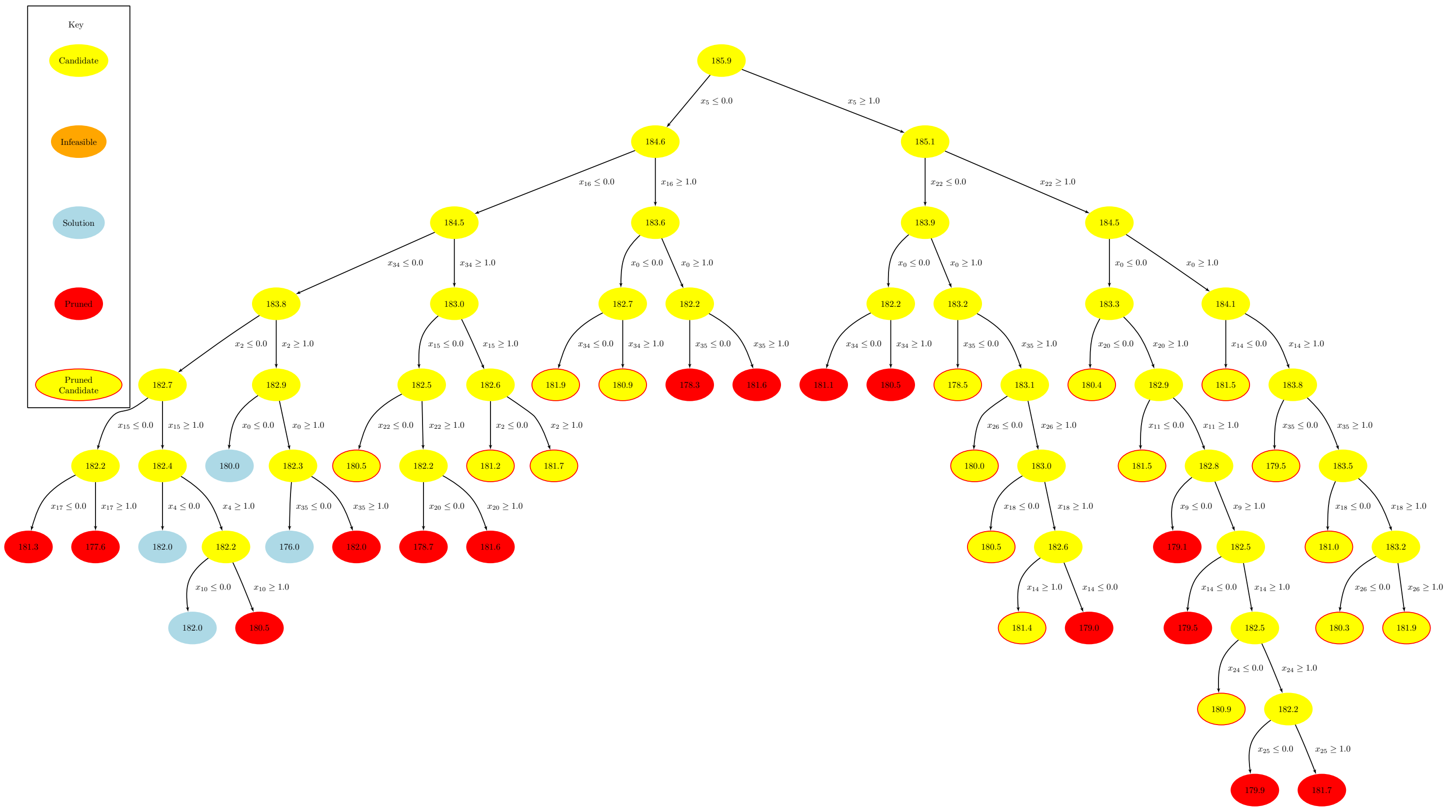


Figure 3: Branching on disjunction $x_2 \leq 4$ OR $x_2 \geq 5$ in Subproblem 2

LP-based Branch and Bound Algorithm

1. To start, derive a lower bound L using a heuristic method.
2. Put the original problem on the candidate list.
3. Select a problem $\mathcal{S}_i$ from the candidate list and solve the LP relaxation to obtain the bound $U(i)$.
   - If the LP is infeasible $\Rightarrow$ node can be pruned.
   - Otherwise, if $U(i) \leq L \Rightarrow$ node can be pruned.
   - Otherwise, if $U(i) > L$ and the solution is feasible for the MILP $\Rightarrow$ set $L \leftarrow U(i)$.
   - Otherwise, branch and add the new subproblem to the candidate list.
4. If the candidate list in nonempty, go to Step 2. Otherwise, the algorithm is completed.

Branch and Bound Tree



Termination Conditions

- Note that although we use multiple disjunctions to branch during the algorithm, the tree can still be seen as encoding a single disjunction.
- To see this, consider the set $\mathcal{T}$ of subproblems associated with the leaf nodes in the tree.
  - Provided that we use admissible disjunctions for branching, the feasible regions of these subproblems are a partition of $\mathcal{S}$.
  - Furthermore, we will see that there exists a collection of polyhedra $\{\mathcal{P}_i\}_{i\in\mathcal{T}}$, where
    * $\mathcal{P}_i$ is a formulation for subproblem i; and
    * $\{\mathcal{P}_i\}_{i=1}^{k}$ is admissible with respect to $\mathcal{S}$.
- When this disjunction, along with the best solution found so far satisfies the optimality conditions (OPT), the algorithm terminates.
- We will revisit this more formally as we further develop the supporting theory.

Ensuring Finite Convergence

- For LP-based branch and bound, ensuring convergence requires a convergent branching method.
- Roughly speaking, a convergent branching method is one which will
  - produce a violated admissible disjunction whenever the solution to the bounding problem is infeasible; and
  - if applied recursively, guarantee that at some finite depth, any resulting bounding problem will either
    * produce a feasible solution (to the original MILP); or
    * be proven infeasible; or
    * be pruned by bound.
- Typically, we achieve this by ensuring that at some finite depth, the feasible region of the bounding problem contains at most one feasible solution.
- We will also revisit this result more formally as we develop the supporting theory.

Algorithmic Choices in Branch and Bound

- Although the basic algorithm is straightforward, the efficiency of it in practice depends strongly on making good algorithmic choices.
- These algorithmic choices are made largely by heuristics that guide the algorithm.
- Basic decisions to be made include
  - The bounding method(s).
  - The method of selecting the next candidate to process.
    - "Best-first" always chooses the candidate with the highest upper bound.
    - This rule minimizes the size of the tree (why?).
    - There may be practical reasons to deviate from this rule.
  - The method of branching.
    - Branching wisely is extremely important.
    - A "poor" branching can slow the algorithm significantly.
- We will cover the last two topics in more detail in later lectures.

A Thousand Words

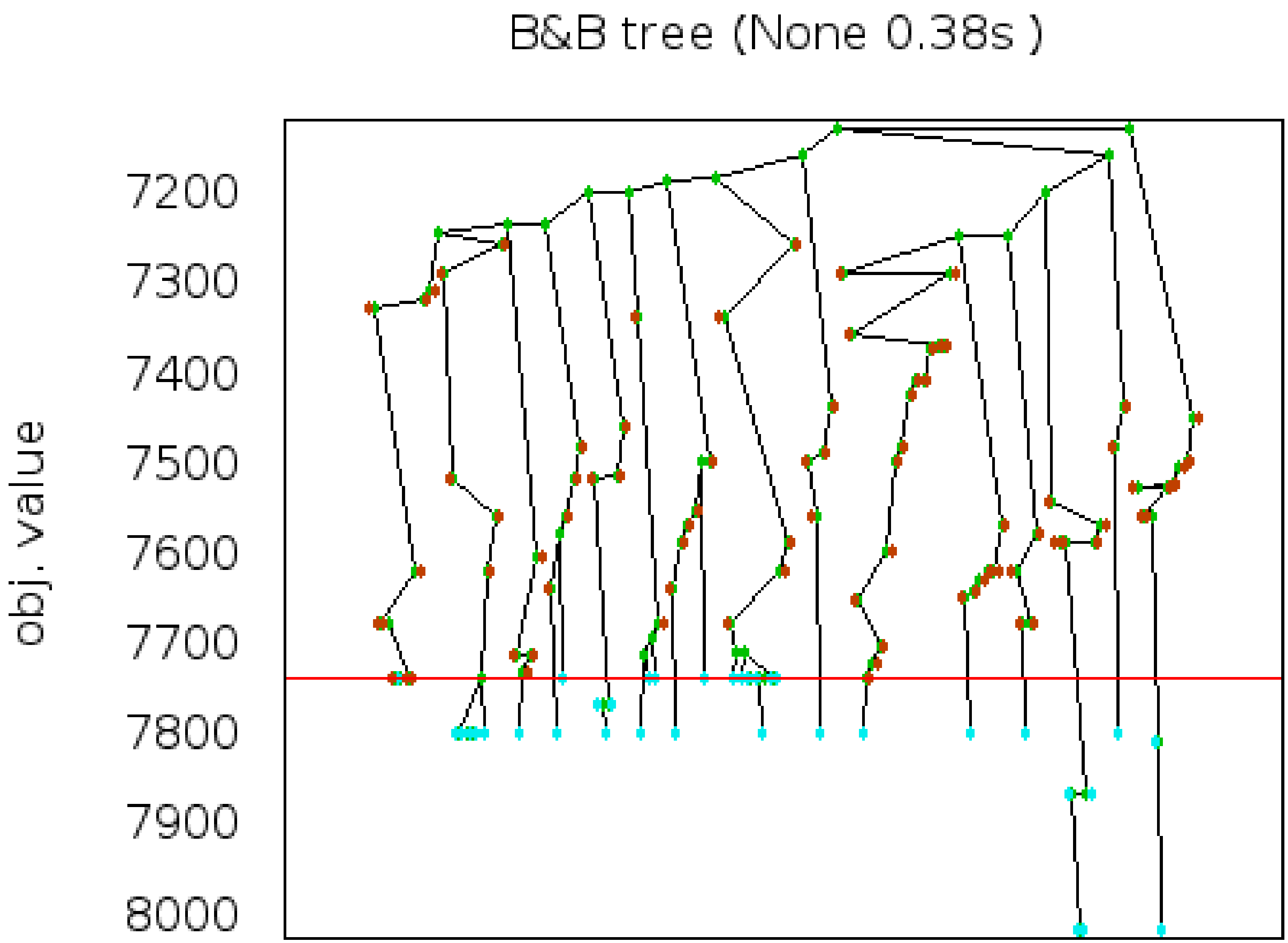


Figure 4: Tree after 400 nodes

Note that we are minimizing here!

A Thousand Words

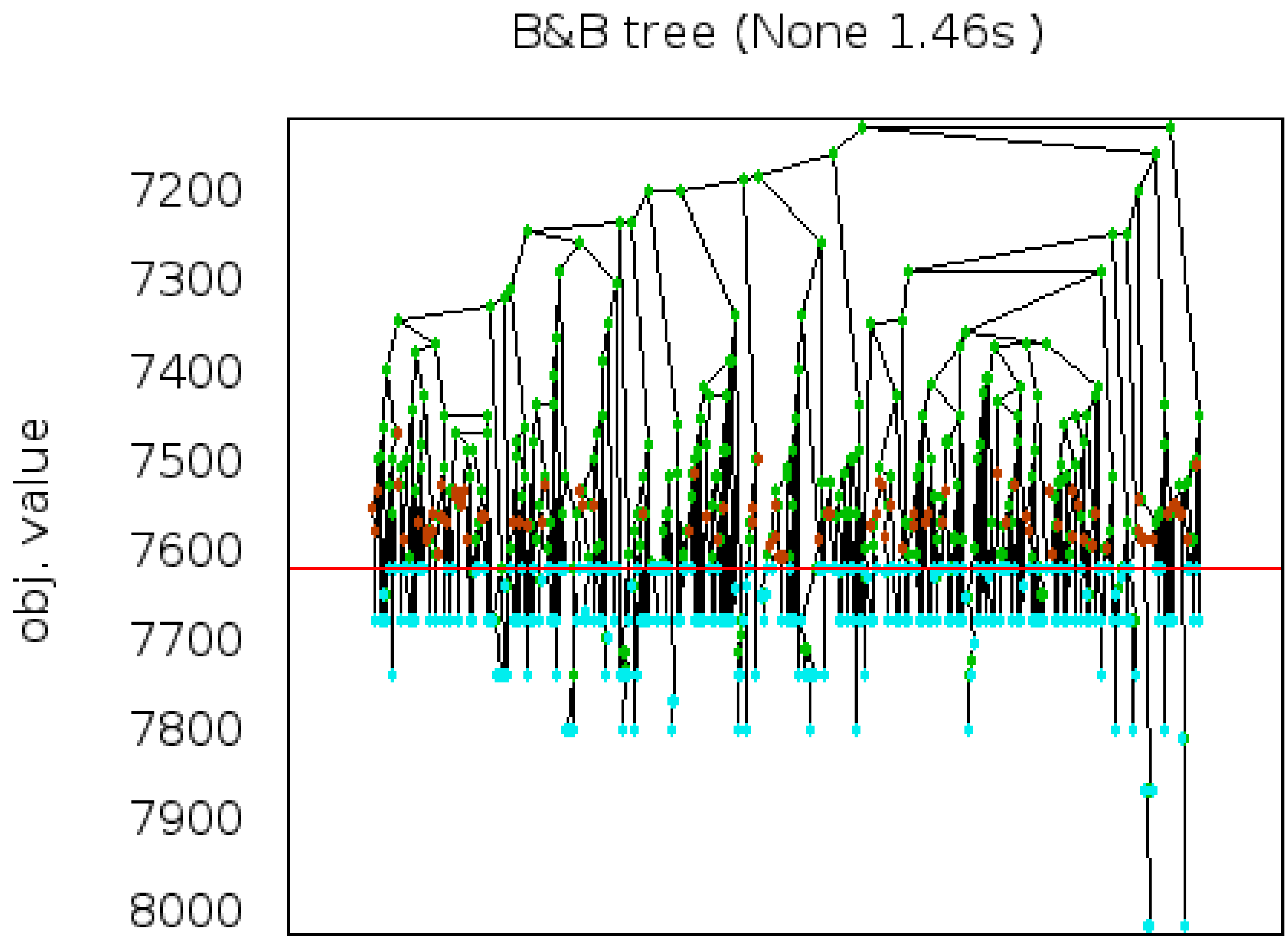


Figure 5: Tree after 1200 nodes

A Thousand Words

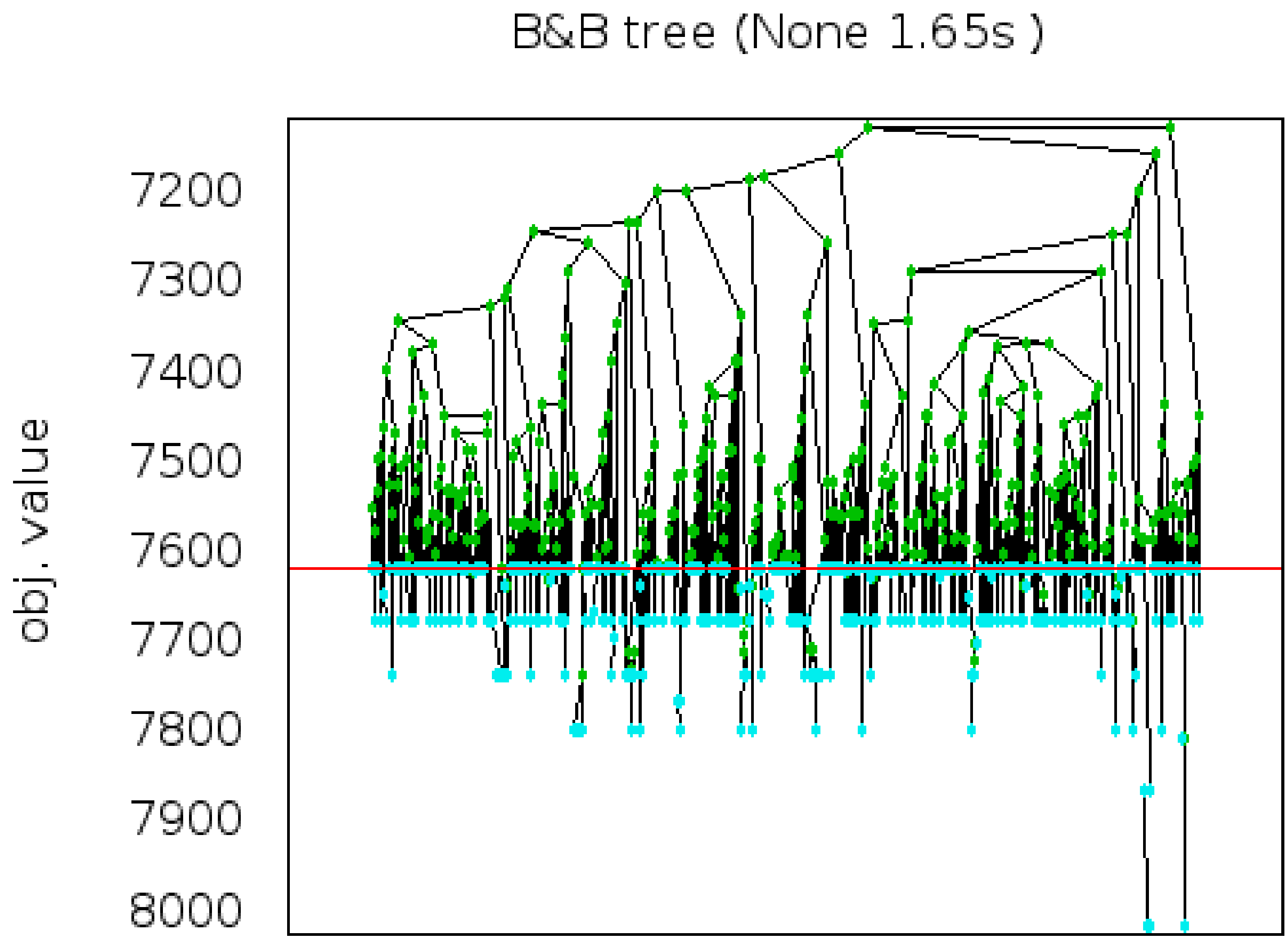


Figure 6: Final tree

Global Bounds

- The pictures show the evolution of the branch and bound process.
- Nodes are pictured at a height equal to that of their lower bound (we are minimizing in this case!!).
  - Red: candidates for processing/branching
  - Green: branched or infeasible
  - Turquoise: pruned by bound (possibly having produced a feasible solution) or infeasible.
- The red line is the level of the current best solution (global upper bound).
- The level of the highest red node is the global lower bound.
- As the procedure evolves, the two bounds grow together.
- The goal is for this to happen as quickly as possible.

Tradeoffs

- We will see that there are many tradeoffs to be managed in branch and bound.
- Note that in the final tree:
  - Nodes below the line were *pruned by bound* (and may or may not have generated a feasible solution) or were *infeasible*.
  - Nodes above the line were either *branched* or were *infeasible* or generated an *optimal solution*.
- There is a tradeoff between the goals of moving the upper and lower bounds
  - The nodes below the line serve to move the upper bound.
  - The nodes above the line serve to move the lower bound.
- It is clear that these two goals are somewhat antithetical.
- The search strategy has to achieve a balance between these two antithetical goals.

Tradeoffs in Practice

- In a practical implementation, there are many more choices and tradeoffs than those we have indicated so far.
- The complexity of the problem of optimizing the algorithm itself is immense.
- We have additional auxiliary methods, such as preprocessing and primal heuristics that we can choose to devote more or less effort to.
- We also have the choice of how much effort to devote to choosing a good candidate for branching.
- Finally, we have the choice of how much effort to devote to proving a good bound on the subproblem.
- It is the careful balance of the levels of effort devoted to each of these algorithmic processes the leads to a good algorithmic implementation.