Integer Programming
ISE 418

Lecture 4

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Reading for This Lecture

- N&W Sections I.4.1-I.4.3

Some Conventions

If not otherwise stated, the following conventions will be followed for lecture slides during the course:

- A will denote a matrix of dimension m by n (rational).
- b will denote a vector of dimension m (rational).
- x will denote a vector of dimension n.
- c will denote a vector of dimension n (rational).
- p will be the number of integer variables.
- $\mathcal{P}$ will denote a polyhedron contained in $\mathbb{R}^n$, usually given in the form

$$\mathcal{P} = \{x \in \mathbb{R}^n \mid Ax \leq b\}$$

- $\mathcal{S}$ will be $\mathcal{P} \cap (\mathbb{Z}_+^p \times \mathbb{R}_+^{n-p})$.
- An integer program is then described fully by the quadruplet (A, b, c, p).
- Vectors will be column vectors unless otherwise noted.
- When taking the product of vectors, we will sometimes leave off the transpose.

Additional Notation

- The notation A_N will denote a submatrix formed by taking the columns indexed by set $N \subseteq \{1, \ldots, n\}$.
- The i^{th} column of A will be denoted A_i.
- The i^{th} row of A will be denoted a_i.

Linear Algebra Review: Linear Independence

Definition 1. *A finite collection of vectors* $x^1, \ldots, x^k \in \mathbb{R}^n$ *is* linearly independent *if the unique solution to* $\sum_{i=1}^{k} \lambda_i x^i = 0$ *is* $\lambda_i = 0, i \in [1..k]$. *Otherwise, the vectors are* linearly dependent*.*

Let A be a square matrix. Then, the following statements are equivalent:

- The matrix A is invertible.
- The matrix $A^\top$ is invertible.
- The determinant of A is nonzero.
- The rows of A are linearly independent.
- The columns of A are linearly independent.
- For every vector b, the system $Ax = b$ has a unique solution.
- There exists some vector b for which the system $Ax = b$ has a unique solution.

Linear Algebra Review: Affine Independence

Definition 2. *A finite collection of vectors* $x^1, \ldots, x^k \in \mathbb{R}^n$ *is* affinely independent *if the vectors* $x^2 - x^1, \ldots, x^k - x^1 \in \mathbb{R}^n$ *are linearly independent.*

- Linear independence implies affine independence, but not vice versa.
- The property of linear independence is with respect to a given origin.
- Affine independence is essentially a "coordinate-free" version of linear independence.

Proposition 1. *The following statements are equivalent:*

1. $x_1, \ldots, x_k \in \mathbb{R}^n$ *are affinely independent.*
2. $x_2 - x_1, \ldots, x_k - x_1$ *are linearly independent.*
3. $(x_1, 1), \ldots, (x_k, 1) \in \mathbb{R}^{n+1}$ *are linearly independent.*

Linear Algebra Review: Subspaces

Definition 3. *A nonempty subset $H \subseteq \mathbb{R}^n$ is called a* subspace *if $\alpha x + \gamma y \in H$ $\forall x, y \in H$ and $\forall \alpha, \gamma \in \mathbb{R}$.*

Definition 4. *A* linear combination *of a collection of vectors $x^1, \ldots x^k \in \mathbb{R}^n$ is any vector $y \in \mathbb{R}^n$ such that $y = \sum_{i=1}^k \lambda_i x^i$ for some $\lambda \in \mathbb{R}^k$.*

Definition 5. *The* span *of a collection of vectors $x^1, \ldots x^k \in \mathbb{R}^n$ is the set of all linear combinations of those vectors.*

Definition 6. *Given a subspace $H \subseteq \mathbb{R}^n$, a collection of linearly independent vectors whose span is H is called a* basis *of H. The number of vectors in the basis is the* dimension *of the subspace.*

Linear Algebra Review: Subspaces and Bases

- A given subspace has an infinite number of bases.
- Each basis has the same number of vectors in it.
- If S and T are subspaces such that $S \subseteq T \subseteq \mathbb{R}^n$, then a basis of S can be extended to a basis of T.
- The span of the columns of a matrix A is a subspace called the *column space* or the *range*, denoted $range(A)$.
- The span of the rows of a matrix A is a subspace called the *row space*.
- The dimensions of the column space and row space are always equal. We call this number $rank(A)$.
- Clearly, $rank(A) \leq \min\{m, n\}$. If $rank(A) = \min\{m, n\}$, then A is said to have *full rank*.
- The set $\{x \in \mathbb{R}^n \mid Ax = 0\}$ is called the *nullspace* of A (denoted $null(A)$) and has dimension $n - rank(A)$.

Some Properties of Subspaces

Proposition 2. *The following are equivalent:*

1. $H \subseteq \mathbb{R}^n$ *is a subspace.*
2. *There is an* $m \times n$ *matrix* A *such that* $H = \{x \in \mathbb{R}^n \mid Ax = 0\}$.
3. *There is a* $k \times n$ *matrix* B *such that* $H = \{x \in \mathbb{R}^n \mid x = uB, u \in \mathbb{R}^k\}$.

Proposition 3. *If* $\{x \in \mathbb{R}^n \mid Ax = b\} \neq \emptyset$*, the maximum number of affinely independent solutions of* $Ax = b$ *is* $n + 1 - rank(A)$.

Proposition 4. *If* $H \subseteq \mathbb{R}^n$ *is a subspace, the subspace* $\{x \in \mathbb{R}^n \mid x^\top y = 0 \; \forall \; y \in H\}$ *is a subspace called the* orthogonal subspace *and denoted* $H^\perp$.

Proposition 5. *If* $H = \{x \in \mathbb{R}^n \mid Ax = 0\}$*, with* A *being an* $m \times n$ *matrix, then* $H^\perp = \{x \in \mathbb{R}^n \mid x = A^\top u, u \in \mathbb{R}^m\}$.

Affine Spaces

Definition 7. *An* affine combination *of a collection of vectors* $x^1, \ldots x^k \in \mathbb{R}^n$ *is any vector* $y \in \mathbb{R}^n$ *such that* $y = \sum_{i=1}^k \lambda_i x^i$ *for some* $\lambda \in \mathbb{R}^k$ *with* $\sum_{j=1}^k \lambda_j = 1$*.*

Definition 8. *A nonempty subset* $\mathcal{A} \subseteq \mathbb{R}^n$ *is called an* affine space *if* $\mathcal{A}$ *is closed with respect to affine combination.*

Definition 9. *A* basis *of an affine space* $\mathcal{A} \subseteq \mathbb{R}^n$ *is maximal set of affinely independent points of* $\mathcal{A}$*.*

Definition 10. *The inclusionwise minimal affine space containing a set* $\mathcal{S}$ *is called the* affine hull *of* $\mathcal{S}$*, denoted* $\text{aff}(\mathcal{S})$*.*

Definition 11. *All bases of an affine space* $\mathcal{A}$ *have the same cardinality and this is the* dimension *of the affine space.*

Projections

Definition 12. *If $p \in \mathbb{R}^n$ and H is a subspace, the* projection of p onto H *is the vector $q \in H$ such that $p - q \in H^\perp$.*

- Note that this is a decomposition of a vector p into the sum of a vector in H and a vector in $H^\perp$.
- The projection of a set is the union of the projections of all its members.
- Projections play a very important role in discrete optimization, as we will see later in the course.

Polyhedra, Hyperplanes, and Half-spaces

Definition 13. *A polyhedron is a set of the form* $\{x \in \mathbb{R}^n \mid Ax \leq b\}$*, where* $A \in \mathbb{R}^{m \times n}$ *and* $b \in \mathbb{R}^m$*.*

Definition 14. *A polyhedron* $\mathcal{P} \subset \mathbb{R}^n$ *is bounded if there exists a constant* K *such that* $|x_i| < K \ \forall x \in S, \forall i \in [1, n]$*.*

Definition 15. *A bounded polyhedron is called a polytope.*

Definition 16. *Let* $a \in \mathbb{R}^n$ *and* $b \in \mathbb{R}$ *be given.*

- *The set* $\{x \in \mathbb{R}^n \mid a^\top x = b\}$ *is called a hyperplane.*
- *The set* $\{x \in \mathbb{R}^n \mid a^\top x \leq b\}$ *is called a half-space.*

Convex Sets

Definition 17. *A set* $S \subseteq \mathbb{R}^n$ *is* convex *if* $\forall x, y \in S, \lambda \in [0,1]$, *we have* $\lambda x + (1-\lambda)y \in S$.

Definition 18. *Let* $x^1, \ldots, x^k \in \mathbb{R}^n$ *and* $\lambda \in \mathbb{R}^k_+$ *be given such that* $\lambda^\top \mathbf{1} = 1$. *Then*

1. *The vector* $\sum_{i=1}^k \lambda_i x^i$ *is said to be a* convex combination *of* $x^1, \ldots, x^k$.
2. *The* convex hull *of* $x^1, \ldots, x^k$ *is the set of all convex combinations of these vectors.*

- The convex hull of two points is a line segment.
- A set is convex if and only if for any two points in the set, the line segment joining those two points lies entirely in the set.
- All polyhedra are convex.