

# Lecture 6: Cutting Plane Method and Strong Valid Inequalities

(3 units)

## Outline

- ▶ Valid inequality
- ▶ Gomory's cutting plane method
- ▶ Polyhedra, faces and facets
- ▶ Cover inequalities for 0-1 knapsack set
- ▶ Lifting and separation of cover inequalities
- ▶ Branch-and-cut algorithm

# Valid Inequality

- ▶ Let  $X = \{x \mid Ax \leq b, x \in \mathbb{Z}_+^n\}$ . Consider the following linear program:

$$\begin{array}{ll} (CIP) & \min \quad c^T x \\ & \text{s.t. } x \in \text{conv}(X). \end{array}$$

Then  $(IP)$  is equivalent to  $(CIP)$ .

- ▶ **Question:** How to describe or approximate  $\text{conv}(X) = \{x \mid \bar{A}x \leq \bar{b}, x \geq 0\}$ ?
- ▶ An inequality  $\pi^T x \leq \pi_0$  is said to be a **valid inequality** for  $X$  if  $\pi^T x \leq \pi_0$  for all  $x \in X$ .

► **Question:**

- (1) How to find good or useful valid inequality?
- (2) How to use valid inequality to solve an integer program?

► Chvátal-Gomory procedure for VI.  $X = \{x \in \mathbb{Z}_+^n \mid Ax \leq b\}$ ,  
 $A = (a_1, \dots, a_n)$ .

(i) (surrogate)  $\sum_{j=1}^n \mu^T a_j x_j \leq \mu^T b, \quad \mu \geq 0.$

(ii) (round off)  $\sum_{j=1}^n \lfloor \mu^T a_j \rfloor x_j \leq \mu^T b.$

(iii)  $\sum_{j=1}^n \lfloor \mu^T a_j \rfloor x_j \leq \lfloor \mu^T b \rfloor.$

► Theorem: every valid inequality of  $X$  can be obtained by applying Chvátal-Gomory procedure for a finite number of times.

# Gomory's Cutting Plane Procedure for IP

- ▶ Integer programming:

$$\min\{c^T x \mid x \in \mathbb{Z}_+^n, Ax = b\}.$$

- ▶ Create the valid inequalities (**cutting planes**) directly from the simplex tableau
- ▶ Given an (optimal) LP basis  $B$ , write the (pure) IP as

$$\begin{aligned} \min \quad & c_B^T B^{-1} b + \sum_{j \in NB} \bar{c}_j x_j \\ \text{s.t.} \quad & (x_B)_i + \sum_{j \in NB} \bar{a}_{ij} x_j = \bar{b}_i, \quad i = 1, 2, \dots, m, \\ & x_j \in \mathbb{Z}_+^1, \quad j = 1, 2, \dots, n, \end{aligned}$$

NB is the set of nonbasic variables.

- ▶  $\bar{c}_j \geq 0, j \in NB, \bar{b}_i \geq 0, i = 1, \dots, m.$

- ▶ If the LP solution is **not** integral, then there exists some row  $i$  with  $\bar{b}_i \notin \mathbb{Z}$ .
- ▶ The C-G cut for row  $i$  is

$$(x_B)_i + \sum_{j \in NB} \lfloor \bar{a}_{ij} \rfloor x_j \leq \lfloor \bar{b}_i \rfloor.$$

- ▶ Substitute for  $(x_B)_i$  to get

$$\sum_{j \in NB} (\bar{a}_{ij} - \lfloor \bar{a}_{ij} \rfloor) x_j \geq \bar{b}_i - \lfloor \bar{b}_i \rfloor.$$

- ▶ Let  $f_{ij} = \bar{a}_{ij} - \lfloor \bar{a}_{ij} \rfloor$ ,  $f_i = \bar{b}_i - \lfloor \bar{b}_i \rfloor$ .

$$\sum_{j \in NB} f_{ij} x_j \geq f_i.$$

- ▶ Since the LP optimal solution  $x_j^* = 0$  for  $j \in NB$ , and  $0 \leq f_{ij} < 1$ ,  $0 < f_i < 1$ , this inequality cut off  $x^*$ !

► Example 1

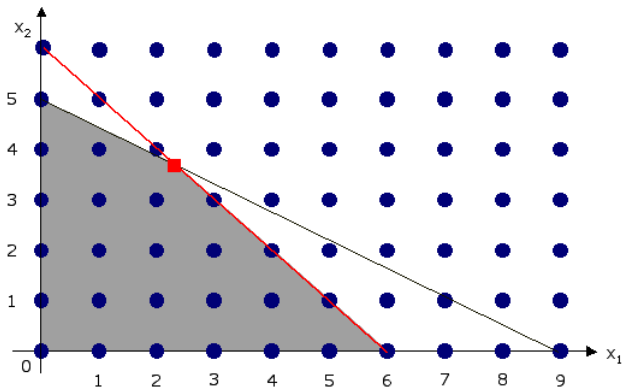
$$\min -5x_1 - 8x_2$$

$$\text{s.t. } x_1 + x_2 \leq 6$$

$$5x_1 + 9x_2 \leq 45$$

$$x_1, x_2 \in \mathbb{Z}_+$$

► The feasible region is



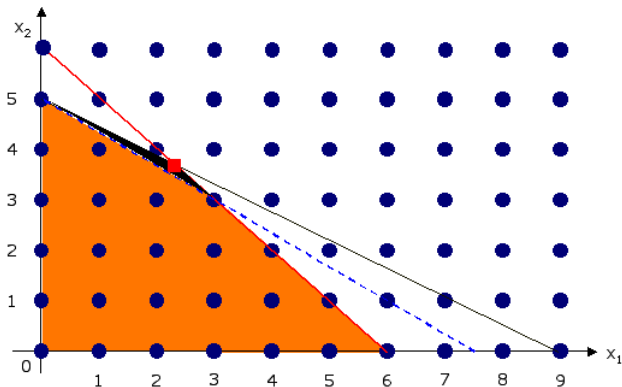
- Optimal Simplex Tableau:

$x_1$	$x_2$	$x_3$	$x_4$	$b$
1	0	2.25	-0.25	2.25
0	1	-1.25	0.25	3.75
0	0	1.25	0.75	41.25

- The Gomory cut from the second row of the tableau is:

$$0.75x_3 + 0.25x_4 \geq 0.75$$

- The modified feasible set is



- The fractional optimal solution  $x = (2.35, 3.75)$  violates the cutting plane and is removed from the new feasible set.



► Example 2

$$\min -4x_1 + x_2$$

$$\text{s.t. } 7x_1 - 2x_2 \leq 14$$

$$x_2 \leq 3$$

$$2x_1 - 2x_2 \leq 3$$

$$x_1, x_2 \in \mathbb{Z}_+$$

► Optimal Simplex Tableau:

$x_1$	$x_2$	$x_3$	$x_4$	$x_5$	$b$
1	0	$\frac{1}{7}$	$\frac{2}{7}$	0	$\frac{20}{7}$
0	1	0	1	0	3
0	0	$-\frac{2}{7}$	$\frac{10}{7}$	1	$\frac{23}{7}$
0	0	$\frac{4}{7}$	$\frac{1}{7}$	0	$-\frac{59}{7}$

Cut from the first row of the tableau is:

$$\frac{1}{7}x_3 + \frac{2}{7}x_4 \geq \frac{6}{7}.$$

► Reoptimization:

$x_1$	$x_2$	$x_3$	$x_4$	$x_5$	$x_6$	$b$
1	0	0	0	0	1	2
0	1	0	0	$-\frac{1}{2}$	1	$\frac{1}{2}$
0	0	1	0	-1	-5	1
0	0	0	1	$\frac{1}{2}$	6	$\frac{5}{2}$
0	0	0	0	$\frac{1}{2}$	3	$-\frac{15}{2}$

Cut from the second row of the tableau is:

$$\frac{1}{2}x_5 \geq \frac{1}{2}.$$

► Reoptimization:

$x_1$	$x_2$	$x_3$	$x_4$	$x_5$	$x_6$	$x_7$	$b$
1	0	0	0	0	1	0	2
0	1	0	0	0	1	-1	1
0	0	1	0	0	-5	-2	2
0	0	0	1	0	6	1	2
0	0	0	0	1	0	-1	1
0	0	0	0	0	3	1	-7

Done! Optimal solution  $x^* = (2, 1)^T$ .

# Linear algebra and convex analysis review

- **Affine independence:** A finite collection of vectors  $x^1, \dots, x^k \in \mathbb{R}^n$  is **affinely independent** if the unique solution to

$$\sum_{i=1}^k \alpha_i x^i = 0, \quad \sum_{i=1}^k \alpha_i = 0,$$

is  $\alpha_i = 0, i = 1, 2, \dots, k$ .

- Linear independence implies affine independence, but not vice versa.
- The following statements are equivalent:
  1.  $x^1, \dots, x^k \in \mathbb{R}^n$  are affinely independent.
  2.  $x^2 - x^1, \dots, x^k - x^1$  are linearly independent.
  3.  $(x^1, 1), \dots, (x^k, 1) \in \mathbb{R}^{n+1}$  are linearly independent.

- ▶ A **polyhedron** is a set of the form  

$$\{x \in \mathbb{R}^n \mid Ax \leq b\} = \{x \in \mathbb{R}^n \mid a_i x \leq b_i, i \in M\},$$
where  $A \in \mathbb{R}^{m \times n}$  and  $b \in \mathbb{R}^m$ .
- ▶ **Dimension of Polyhedra:** A polyhedron  $P$  is of **dimension  $k$** , denoted  $\dim(P) = k$ , if the maximum number of affinely independent points in  $P$  is  $k + 1$ . A polyhedron  $P \in \mathbb{R}^n$  is **full-dimensional** if  $\dim(P) = n$ . Let  
 $M = \{1, \dots, m\}$ ,  
 $M^= = \{i \in M \mid a_i x = b_i, \forall x \in P\}$ , (the equality set),  
 $M^{\leq} = M \setminus M^=$  (the inequality set).  
Let  $(A^=, b^=)$ ,  $(A^{\leq}, b^{\leq})$  be the corresponding rows of  $(A, b)$ .  
If  $P \in \mathbb{R}^n$ , then  $\dim(P) + \text{rank}(A^=, b^=) = n$ .

# Valid Inequalities and Faces

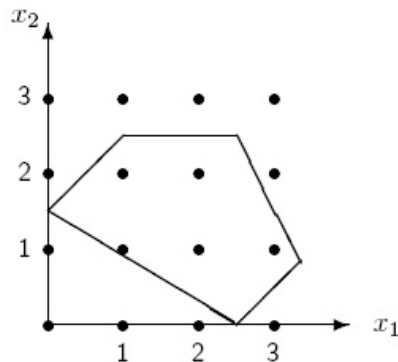
- ▶ The inequality denoted by  $(\pi, \pi_0)$  is called a **valid inequality** for  $P$  if  $\pi x \leq \pi_0, \forall x \in P$ .
- ▶ Note that  $(\pi, \pi_0)$  is a valid inequality if and only if  $P$  lies in the half-space  $\{x \in \mathbb{R}^n \mid \pi x \leq \pi_0\}$ .
- ▶ If  $(\pi, \pi_0)$  is a valid inequality for  $P$  and  $F = \{x \in P \mid \pi x = \pi_0\}$ ,  $F$  is called a **face** of  $P$  and we say that  $(\pi, \pi_0)$  represents or defines  $F$ .
- ▶ A face is said to be proper if  $F \neq \emptyset$  and  $F \neq P$ . Note that a face has multiple representations.
- ▶ The face represented by  $(\pi, \pi_0)$  is nonempty if and only if  $\max\{\pi x \mid x \in P\} = \pi_0$ .
- ▶ If the face  $F$  is nonempty, we say it supports  $P$ . Note that the set of optimal solutions to an LP is always a face of the feasible region.

# Facets

- ▶ Let  $P$  be a polyhedron with equality set  $M^=$ . If  $F = \{x \in P \mid \pi x = \pi_0\}$  is nonempty, then  $F$  is a polyhedron. We can get the polyhedron  $F$  by taking some of the inequalities of  $P$  and making them equalities.
- ▶ The number of distinct faces of  $P$  is finite.
- ▶ A face  $F$  is said to be a **facet** of  $P$  if  $\dim(F) = \dim(P) - 1$ . The inequality corresponding to a facet is called a **strong valid inequality**.
- ▶ If  $F$  is a facet of  $P$ , then in any **description** of  $P$ , there exists some inequality representing  $F$ . (By setting the inequality to equality, we get  $F$ ).
- ▶ Every inequality that represents a face that is not a facet is unnecessary in the description of  $P$ .



## An example

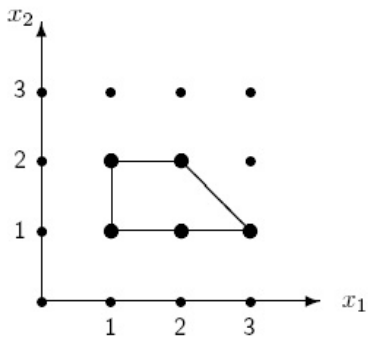


$P$  is defined by five inequalities:

$$2x_2 \leq 5, 6x_1 + 10x_2 \geq 15$$

$$2x_1 - 2x_2 \leq 5, 2x_1 - 2x_2 \geq -3$$

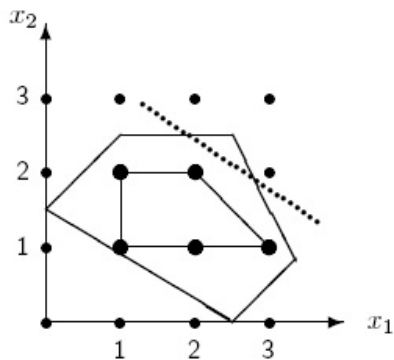
$$4x_1 + 2x_2 \leq 15$$



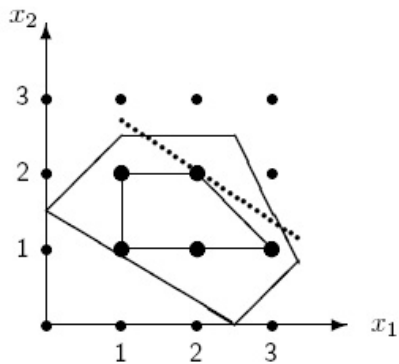
$P_I$  has four facets:

$$x_1 \geq 1, x_2 \geq 1$$

$$x_2 \leq 2, x_1 + x_2 \leq 4$$



A weak cutting plane:  
doesn't even touch  $P_I$



A stronger cutting  
plane: touches  $P_I$

## 0-1 Knapsack Inequalities

- ▶ Valid Inequalities for the Knapsack Problem. We are interested in valid inequalities for the **knapsack** set

$$S = \{x \in \{0, 1\}^n \mid \sum_{j=1}^N a_j x_j \leq b\}.$$

$N = \{1, 2, \dots, n\}$ . Assume that  $a_j > 0$ ,  $j \in N$ ,  $a_j < b$ ,  $j \in N$ . We are interested in finding **facets** of  $\text{conv}(S)$ .

- ▶ Simple facets. What is  $\dim(\text{conv}(S))$ ?  $0, e_j$ ,  $j \in N$  are  $n + 1$  affinely independent points in  $\text{conv}(S)$ , so  $\dim(\text{conv}(S)) = n$ .
- ▶  $x^k \geq 0$  is a **facet** of  $\text{conv}(S)$ .
- ▶ Proof.  $0, e_j$ ,  $j \in N \setminus \{k\}$  are  $n$  affinely independent points that satisfy  $x_k = 0$ .

- ▶  $x_k \leq 1$  is a **facet** of  $\text{conv}(S)$  if  $a_j + a_k \leq b, \forall j \in N \setminus \{k\}$ .
- ▶ Proof.  $e_k, e_j + e_k, j \in N \setminus \{k\}$  are  $n$  affinely independent points that satisfy  $x_k = 1$ .
- ▶ A set  $C \subseteq N$  is a **cover** if  $\sum_{j \in C} a_j > b$ . A cover  $C$  is a **minimal cover** if  $C \setminus \{j\}$  is not a cover  $\forall j \in C$ .
- ▶ If  $C \subseteq N$  is a cover, then the cover inequality

$$\sum_{j \in C} x_j \leq |C| - 1$$

is a **valid inequality** for  $S$ .

► Example:

$$S = \{x \in B^7 \mid 11x_1 + 6x_2 + 6x_3 + 5x_4 + 5x_5 + 4x_6 + x_7 \leq 19\}.$$

► Minimal covers:

$$\begin{aligned} C &= \{1, 2, 3\}, C = \{1, 2, 6\}, \\ C &= \{1, 5, 6\}, C = \{3, 4, 5, 6\}. \end{aligned}$$

## Can we do better?

- ▶ Are these inequalities the strongest ones we can come up with? What does strongest mean? We all know that facets are the “strongest”, but can we say anything else?
- ▶ If  $\pi x \leq \pi_0$  and  $\mu x \leq \mu_0$  are two valid inequalities for  $P \in \mathbb{R}_+^n$ , we say that  $\pi x \leq \pi_0$  **dominates**  $\mu x \leq \mu_0$  if  $\exists u \geq 0$  such that  $\pi \geq u\mu$ ,  $\pi_0 \leq u\mu_0$  and  $(\pi, \pi_0) \neq u(\mu, \mu_0)$ .
- ▶ If  $\pi x \leq \pi_0$  **dominates**  $\mu x \leq \mu_0$ , then

$$\{x \in \mathbb{R}_+^n \mid \pi x \leq \pi_0\} \subseteq \{x \in \mathbb{R}_+^n \mid \mu x \leq \mu_0\}.$$



- Strengthening cover inequalities. If  $C \subseteq N$  is a minimal cover, the **extended cover**  $E(C)$  is defined as  $E(C) = C \cup \{j \in N \mid a_j \geq a_i, \forall i \in C\}$ . If  $E(C)$  is an extended cover for  $S$ , then the extended cover inequality

$$\sum_{j \in E(C)} x_j \leq |C| - 1$$

is a valid inequality for  $S$ .

- The cover inequality  $x_3 + x_4 + x_5 + x_6 \leq 3$  is dominated by the extended cover inequality  $x_1 + x_2 + x_3 + x_4 + x_5 + x_6 \leq 3$ .

- Let  $C$  be a **minimal cover**. If  $C = N$ , then

$$\sum_{j \in C} x_j \leq |C| - 1$$

is a facet of  $\text{conv}(S)$ .

- Proof  $R_k = C \setminus \{k\}$ ,  $\forall k \in C$ .  $x^{R_k}$  satisfies

$$\sum_{j \in C} x_j^{R_k} = |C| - 1.$$

Also,  $x^{R_1}, \dots, x^{R_{|C|}}$  are affinely independent. Since  $C = N$ , there are  **$n$  affinely independent vectors** satisfy  $\sum_{j \in C} x_j^{R_k} = |C| - 1$  at equality.

- ▶ Let  $C = \{j_1, \dots, j_r\}$  be a **minimal cover**. Let  $p = \min\{j \mid j \in N \setminus E(C)\}$ . If  $C = E(C)$ , and  $\sum_{j \in C \setminus \{j_1\}} a_j + a_p \leq b$ , then  $\sum_{j \in C} x_j \leq |C| - 1$  is a facet of  $\text{conv}(S)$ .
- ▶ Proof.  $T_k = C \setminus \{j_1\} \cup \{k\}$ ,  $\forall k \in N \setminus E(C)$ .  
 $|T_k \cap E(C)| = |C| - 1$  and

$$\sum_{j \in T_k \cap E(C)} x_j^{T_k} = |C| - 1.$$

$|R_k| + |T_k| = N$ .  $x^{R_k}$ ,  $k \in C$ ,  $x^{T_k}$ ,  $k \in N \setminus C$ , are  **$n$**  affinely independent vectors.

- ▶ Example:

$$S = \sum \{x \in \{0, 1\}^5 \mid 79x_1 + 53x_2 + 53x_3 + 45x_4 + 45x_5 \leq 178\}.$$

- ▶ Consider minimal cover  $C = \{1, 2, 3\}$ . The valid inequality is:

$$x_1 + x_2 + x_3 \leq 2.$$

$$C = E(C). \quad p = 4, \quad C \setminus \{1\} \cup \{4\} = \{2, 3, 4\}.$$

$53 + 53 + 45 = 151 < 178$ . So  $x_1 + x_2 + x_3 = 2$  gives a facet of  $\text{conv}(S)$ .

## Lifting Cover Inequalities

- ▶ **Question:** Can we find the the a valid inequality as strong as possible?
- ▶ Example:  $C = \{3, 4, 5, 6\}$ , the valid inequality for  $C$  is:

$$x_3 + x_4 + x_5 + x_6 \leq 3.$$

- ▶ Setting  $x_1 = x_2 = x_7 = 0$ , the cover inequalities  $x_3 + x_4 + x_5 + x_6 \leq 3$  is valid for  $\{x \in \{0, 1\}^4 \mid 6x_3 + 5x_4 + 5x_5 + 4x_6 \leq 19\}$ .
- ▶ If  $x_1$  is not fixed at 0, can we strengthen the inequality? For what values of  $\alpha_1$  is the inequality

$$\alpha_1 x_1 + x_3 + x_4 + x_5 + x_6 \leq 3$$

valid for

$$P_{2,7} = \{x \in \{0, 1\}^5 \mid 11x_1 + 6x_3 + 5x_4 + 5x_5 + 4x_6 \leq 19\}.$$

- ▶  $\Leftrightarrow \alpha_1 + x_3 + x_4 + x_5 + x_6 \leq 3$  is valid for all  $x \in \{0, 1\}^4$  satisfying  $6x_3 + 5x_4 + 5x_5 + 4x_6 \leq 19 - 11$ ;  
 $\Leftrightarrow \alpha_1 + \zeta \leq 3$ , where

$$\zeta = \max\{x_3 + x_4 + x_5 + x_6 \mid 6x_3 + 5x_4 + 5x_5 + 4x_6 \leq 8\}.$$

- ▶  $\zeta = 1 \Rightarrow \alpha_1 \leq 2$ . Thus  $\alpha_1 = 2$  gives the strongest inequality.
- ▶ How to find the best value  $\alpha_j, j \in N \setminus C$  such that

$$\sum_{j \in C} x_j + \sum_{j \in N \setminus C} \alpha_j x_j \leq |C| - 1$$

is valid for  $S$ ?

## ► Lifting Procedure

- Let  $j_1, \dots, j_r$  be an ordering of  $N \setminus C$ . Set  $t = 1$ .
- The valid inequality

$$\sum_{i=1}^{t-1} \alpha_{j_i} x_{j_i} + \sum_{j \in C} x_j \leq |C| - 1$$

is given. Solve the following knapsack problem:

$$\begin{aligned} \zeta_t = \quad & \max \sum_{i=1}^{t-1} \alpha_{j_i} x_{j_i} + \sum_{j \in C} x_j \\ \text{s.t.} \quad & \sum_{i=1}^{t-1} a_{j_i} x_{j_i} + \sum_{j \in C} a_j x_j \leq b - a_{j_t} \\ & x \in \{0, 1\}^{|C|+t-1}. \end{aligned}$$

- Set  $\alpha_{j_t} = |C| - 1 - \zeta_t$ . Stop if  $t = r$ .

- Example:  $C = \{3, 4, 5, 6\}$ ,  $j_1 = 1$ ,  $j_2 = 2$ ,  $j_3 = 7$ .  $\alpha_1 = 2$ .  
Consider  $x_2$ , we have

$$\begin{aligned}\zeta_2 = & \max 2x_1 + x_3 + x_4 + x_5 + x_6 \\ \text{s.t. } & 11x_1 + 6x_3 + 5x_4 + 5x_5 + 4x_6 \leq 19 - 6 = 13, \\ & x \in \{0, 1\}^5.\end{aligned}$$

So  $\zeta_2 = 2$  and  $\alpha_{j_2} = \alpha_2 = 3 - 2 = 1$ .

Consider  $x_7$  now, we have

$$\begin{aligned}\zeta_3 = & \max 2x_1 + x_2 + x_3 + x_4 + x_5 + x_6 \\ \text{s.t. } & 11x_1 + 6x_2 + 6x_3 + 5x_4 + 5x_5 + 4x_6 \leq 19 - 1 = 18, \\ & x \in \{0, 1\}^6.\end{aligned}$$

So  $\zeta_3 = 3$  and  $\alpha_{j_3} = \alpha_7 = 3 - 3 = 0$ . We obtain a valid inequality:

$$2x_1 + x_2 + x_3 + x_4 + x_5 + x_6 + x_7 \leq 3.$$



# Separation of Cover Inequalities

- ▶ We often try to solve problems that have knapsack rows with lots more variables than that... Obviously I do not want to add all of those facets at once. **What to do?**
- ▶ Given some  $P$ , find an inequality of the form  $\sum_{j \in C} x_j \leq |C| - 1$  such that  $\sum_{j \in C} x_j^* > |C| - 1$ . This is called a **separation problem**.
- ▶ Note that  $\sum_{j \in C} x_j \leq |C| - 1$  can be rewritten as

$$\sum_{j \in C} (1 - x_j) \geq 1.$$

- **Separation Problem:** Given a fractional LP solution  $x^*$ , does  $\exists$  cover  $C \subseteq N$  such that  $\sum_{j \in C} (1 - x_j^*) < 1$ ? or is

$$\gamma = \min_{C \subseteq N} \left\{ \sum_{j \in C} (1 - x_j) \mid \sum_{j \in C} a_j > b \right\} < 1?$$

- If  $\gamma \geq 1$ , then  $x^*$  satisfies all the cover inequalities.
- If  $\gamma < 1$  with optimal solution  $z^R$ , then  $\sum_{j \in R} x_j \leq |R| - 1$  is a violated cover inequality.

# Branch-and-Cut Method

- ▶ Branch and cut is an LP-based branch and bound scheme in which the linear programming relaxations combined with by cutting plane method.
- ▶ The valid inequalities are generated dynamically using separation procedures.
- ▶ At each node of the search tree, cuts are generated and used to improve the LP relaxation.
- ▶ Branch-and-cut method is very efficient for some hard integer programming problems.