Integer Programming
ISE 418

Lecture 12

Dr. Ted Ralphs

Reading for This Lecture

- Nemhauser and Wolsey Sections II.1.1-II.1.3, II.1.6
- Wolsey Chapter 8
- CCZ Chapters 5 and 6
- "Valid Inequalities for Mixed Integer Linear Programs," G. Cornuejols.
- "Generating Disjunctive Cuts for Mixed Integer Programs," M. Perregaard.

Generating Cutting Planes: Two Basic Viewpoints

- There are a number of different points of view from which one can derive the standard methods used to generate cutting planes for general MILPs.
- As we have seen before, there is an *algebraic* point of view and a *geometric* point of view.
- Algebraic:
  - Take combinations of the known valid inequalities.
  - Use rounding to produce stronger ones.
- Geometric:
  - Use a disjunction (as in branching) to generate several disjoint polyhedra whose union contains $\mathcal{S}$.
  - Generate inequalities valid for the convex hull of this union.
- Although these seem like very different approaches, they turn out to be very closely related.

Generating Valid Inequalities: Algebraic Viewpoint

- Consider the polyhedron $\mathcal{Q} = \{x \in \mathbb{R}^n \mid Ax \leq b\}$.
- Valid inequalities for $\mathcal{Q}$ can be obtained by taking non-negative linear combinations of the rows of (A, b).
- Except for one pathological case[1], all valid inequalities for $\mathcal{Q}$ are either equivalent to or dominated by an inequality of the form

$$uAx \leq ub, u \in \mathbb{R}^m_+.$$

- We are taking combinations of inequalities existing in the description, so any such inequalities will be redundant for $\mathcal{Q}$ itself.
- Nevertheless, such redundant inequalities can be strengthened by a simple procedure when $\mathcal{Q}$ is the LP relaxation of an MILP.

[1] The pathological case occurs when one or more variables have no explicit upper bound *and* both the primal and dual problems are infeasible.

Generating Valid Inequalities for $\operatorname{conv}(\mathcal{S})$

As usual, we consider the MILP

$$z_{IP} = \max\{c^\top x \mid x \in \mathcal{S}\}, \tag{MILP}$$

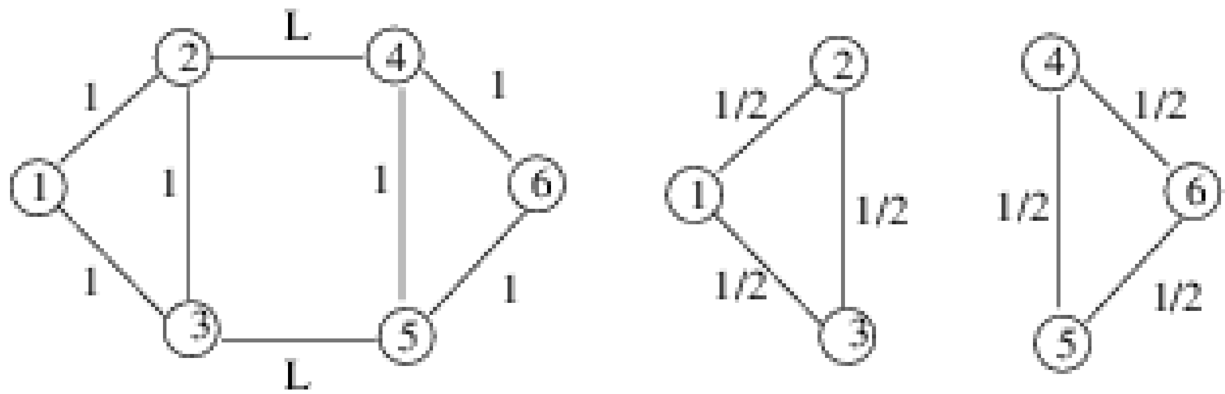
where

$$\mathcal{P} = \{x \in \mathbb{R}^n_+ \mid Ax \leq b\} \tag{FEAS-LP}$$

$$\mathcal{S} = \mathcal{P} \cap (\mathbb{Z}^p_+ \times \mathbb{R}^{n-p}_+) \tag{FEAS-MIP}$$

- All inequalities valid for $\mathcal{P}$ are also valid for $\operatorname{conv}(\mathcal{S})$, but they are not cutting planes.
- We can do better.
- We need the following simple principle: if $a \leq b$ and a is an integer, then $a \leq \lfloor b \rfloor$.
- Believe it or not, this simple fact is all we need to generate all valid inequalities for $\operatorname{conv}(\mathcal{S})$!

Back to the Matching Problem



Recall again the matching problem.

$$
\begin{aligned}
\min \quad & \sum_{e=\{i,j\}\in E} c_e x_e \\
s.t. \quad & \sum_{\{j \mid \{i,j\}\in E\}} x_{ij} = 1, \ \forall i \in N \\
& x_e \in \{0,1\}, \qquad \forall e = \{i,j\} \in E.
\end{aligned}
$$

Generating the Odd Cut Inequalities

- Recall that each odd cutset induces a possible valid inequality.

$$\sum_{e \in \delta(S)} x_e \geq 1, S \subset N, |S| \text{ odd}.$$

- Let's derive these another way.
  - Consider an odd set of nodes U.
  - Sum the (relaxed) constraints $\sum_{\{j|\{i,j\} \in E\}} x_{ij} \leq 1$ for $i \in U$.
  - This results in the inequality $2\sum_{e \in E(U)} x_e + \sum_{e \in \delta(U)} x_e \leq |U|$.
  - Dividing through by 2, we obtain $\sum_{e \in E(U)} x_e + \frac{1}{2}\sum_{e \in \delta(u)} x_e \leq \frac{1}{2}|U|$.
  - We can drop the second term of the sum to obtain

$$\sum_{e \in E(U)} x_e \leq \frac{1}{2}|U|.$$

  - What's the last step?

Chvátal Inequalities

- Suppose we can find a $u \in \mathbb{R}^m_+$ such that $\pi = uA$ is integer and $\pi_0 = ub \notin \mathbb{Z}$.
- In this case, we have $\pi^\top x \in \mathbb{Z}$ for all $x \in \mathcal{S}$, and so $\pi^\top x \leq \lfloor \pi_0 \rfloor$ for all $x \in \mathcal{S}$.
- In other words, $(\pi, \lfloor \pi_0 \rfloor)$ is both a valid inequality *and* a split disjunction.
- In other words, it is a split disjunction for which

$$\{x \in \mathcal{P} \mid \pi^\top x \geq \lfloor \pi_0 \rfloor + 1\} = \emptyset \tag{1}$$

- Such an inequality is called a *Chvátal inequality*.
- Note that we have not used the non-negativity constraints in deriving this inequality.

Chvátal-Gomory Inequalities

- If we allow the non-negativity constraints to be combined with the other constraints of $\mathcal{P}$ (with weight vector $v \in \mathbb{R}^n_+$), then integrality of π requires

$$\begin{aligned} \pi_i = uA_i - v_i &\in \mathbb{Z} \text{ for } 1 \leq i \leq p \\ \pi_i = uA_i - v_i &= 0 \text{ for } p+1 \leq i \leq n. \end{aligned}$$

- Since v_i must be non-negative, we then have that

$$\begin{aligned} v_i &\geq uA_i - \lfloor uA_i \rfloor && \text{for } 1 \leq i \leq p \\ v_i &= uA_i \geq 0 && \text{for } p+1 \leq i \leq n \end{aligned}$$

- Taking $v_i = uA_i - \lfloor uA_i \rfloor$ for $1 \leq i \leq p$, we then obtain that

$$\sum_{0 \leq i \leq p} \pi_i x_i = \sum_{0 \leq i \leq p} \lfloor uA_i \rfloor x_i \leq \lfloor ub \rfloor = \pi_0 \tag{C-G}$$

  is valid for all $u \in \mathbb{R}^m_+$ such that $uA_C \geq 0$.

- Note that validity of this cut could also be derived from first principles.
- This is the *Chvátal-Gomory Inequality*.

The Chvátal-Gomory Procedure

1. Choose a weight vector $u \in \mathbb{R}^m_+$ such that $uA_C \geq 0$.
2. Obtain the valid inequality $\sum_{0 \leq i \leq p} (uA_i) x_i \leq ub$.
3. Round the coefficients down to obtain $\sum_{0 \leq i \leq p} (\lfloor uA_i \rfloor) x_i \leq ub$.
4. Finally, round the right-hand side down to obtain the valid inequality

$$\sum_{0 \leq i \leq p} (\lfloor uA_i \rfloor) x_i \leq \lfloor ub \rfloor$$

- This procedure is called the *Chvátal-Gomory* rounding procedure, or simply the *C-G procedure*.
- Surprisingly, for pure ILPs ($p = n$), any inequality valid for $\mathrm{conv}(\mathcal{S})$ can be produced by a finite number of applications of this procedure!
- Note that this procedure is recursive and requires exploiting inequalities derived in previous rounds to get new inequalities.
- This is not true for the general mixed case.

Assessing the Procedure

- Although it is *theoretically* possible to generate any valid inequality using the C-G procedure, this is not true in practice.
- The two biggest challenges are numerical errors and slow convergence.
- The inequalities produced may be very weak—we may not even obtain a supporting hyperplane.
- This is is because the rounding only "pushes" the inequality until it meets some point in $\mathbb{Z}^n$, which may or may not even be in $\mathcal{S}$.
- The coefficients of the generated inequality must be relatively prime to ensure the generated hyperplane even includes an integer point!

Proposition 1. *Let* $\mathcal{S} = \{x \in \mathbb{Z}^n \mid \sum_{j \in N} a_j x_j \leq b\}$*, where* $a_j \in \mathbb{Z}$ *for* $j \in N$*, and let* $k = gcd\{a_1, \ldots, a_n\}$*. Then* $\mathrm{conv}(\mathcal{S}) = \{x \in \mathbb{R}^n \mid \sum_{j \in N}(a_j/k)x_j \leq \lfloor b/k \rfloor\}$*.*

Gomory Inequalities

- Let's consider T, the set of solutions to a pure ILP with one equation:

$$T = \left\{ x \in \mathbb{Z}^n_+ \;\middle|\; \sum_{j=1}^n a_j x_j = a_0 \right\}$$

- For each j, let $f_j = a_j - \lfloor a_j \rfloor$. Then equivalently

$$T = \left\{ x \in \mathbb{Z}^n_+ \;\middle|\; \sum_{j=1}^n f_j x_j = f_0 + k \text{ for some integer k} \right\}$$

- Since $\sum_{j=1}^n f_j x_j \geq 0$ and $f_0 < 1$, then we must have $k \geq 0$ and so

$$\sum_{j=1}^n f_j x_j \geq f_0$$

is a valid inequality for $\mathcal{S}$ called a *Gomory inequality*.

Gomory Cuts from the Tableau

- Gomory cutting planes can also be derived directly from the tableau while solving an LP relaxation.
- We assume for now that A has integral coefficients so that the slack variables also have integer values implicitly.
- Consider the set

$$\left\{(x, s) \in \mathbb{Z}_+^{n+m} \mid Ax + Is = b\right\}$$

in which the LP relaxation of an ILP is put in standard form.

- The tableau corresponding to basis matrix B is

$$B^{-1}Ax + B^{-1}s = B^{-1}b$$

- Each row of this tableau corresponds to a weighted combination of the original constraints.
- The weight vectors are the rows of B^{-1}.

Gomory Cuts from the Tableau (cont.)

- A row of the tableau is obtained by combining the equations in the standard representation with weight vector $\lambda = B_j^{-1}$ to obtain

$$\sum_{j=1}^{n}(\lambda A_j)x_j + \sum_{i=1}^{m}\lambda_i s_i = \lambda b,$$

where A_j is the j^{th} column of A and λ is a row of B^{-1}.

- Applying the previous procedure, we can obtain the valid inequality

$$\sum_{j=1}^{n}(\lambda A_j - \lfloor \lambda A_j \rfloor)x_j + \sum_{i=1}^{m}(\lambda_i - \lfloor \lambda_i \rfloor)s_i \geq \lambda b - \lfloor \lambda b \rfloor.$$

- We will show that this Gomory cut is equivalent to the C-G inequality with weights $u_i = \lambda_i - \lfloor \lambda_i \rfloor$.

Gomory Versus C-G

- To show the Gomory cut is a C-G cut, we first apply the C-G procedure directly to the tableau row, resulting in the inequality

$$\sum_{j=1}^{n} \lfloor \lambda A_j \rfloor x_j + \sum_{i=1}^{m} \lfloor \lambda_i \rfloor s_i \leq \lfloor \lambda b \rfloor.$$

- Note that we can relax the equality to an inequality because of non-negativity.
- We could have also obtained this inequality by summing the associated Gomory cut and the original tableau row.
- Now, we substitute out the slack variables using the equation

$$s = b - Ax.$$

to obtain

$$\sum_{j=1}^{n} \left(\lfloor \lambda A_j \rfloor - \sum_{i=1}^{m} \lfloor \lambda_i \rfloor a_{ij} \right) x_j \leq \lfloor \lambda b \rfloor - \sum_{i=1}^{m} \lfloor \lambda_i \rfloor b_i,$$

Gomory Versus C-G (cont.)

- The final inequality from the previous slide can be re-written as

$$\sum_{j=1}^{n}\left(\sum_{i=1}^{m}(\lambda_i - \lfloor \lambda_i \rfloor) a_{ij}\right) x_j \leq \sum_{i=1}^{m}(\lambda_i - \lfloor \lambda_i \rfloor) b_i,$$

  which is a C-G inequality.

- The substitution of slack variables is more than just a textbook procedure to show the Gomory cut is a C-G cut.

- In practice, the slack variables are substituted out in this fashion in order to derive a cut in terms of the original variables.

Strength of Gomory Cuts from the Tableau

- Consider a row of the tableau in which the value of the basic variable is not an integer.
- Applying the procedure from the last slide, the resulting inequality will only involve nonbasic variables and will be of the form

$$\sum_{j \in NB} f_j x_j \geq f_0$$

where $0 \leq f_j < 1$ and $0 < f_0 < 1$.

- The left-hand side of this cut has value zero with respect to the solution to the current LP relaxation.
- We can conclude that the generated inequality will be violated by the current solution to the LP relaxation.

Example: Gomory Cuts

Consider the polyhedron $\mathcal{P}$ described by the constraints

$$4x_1 + x_2 \leq 28 \tag{2}$$

$$x_1 + 4x_2 \leq 27 \tag{3}$$

$$x_1 - x_2 \leq 1 \tag{4}$$

$$x_1, x_2 \geq 0 \tag{5}$$

Graphically, it can be easily determined that the facet-inducing valid inequalities describing $\mathrm{conv}(\mathcal{S} = \mathcal{P} \cap \mathbb{Z}^2)$ are

$$x_1 + 2x_2 \leq 15 \tag{6}$$

$$x_1 - x_2 \leq 1 \tag{7}$$

$$x_1 \leq 5 \tag{8}$$

$$x_2 \leq 6 \tag{9}$$

$$x_1 \geq 0 \tag{10}$$

$$x_2 \geq 0 \tag{11}$$

Example: Gomory Cuts (cont.)

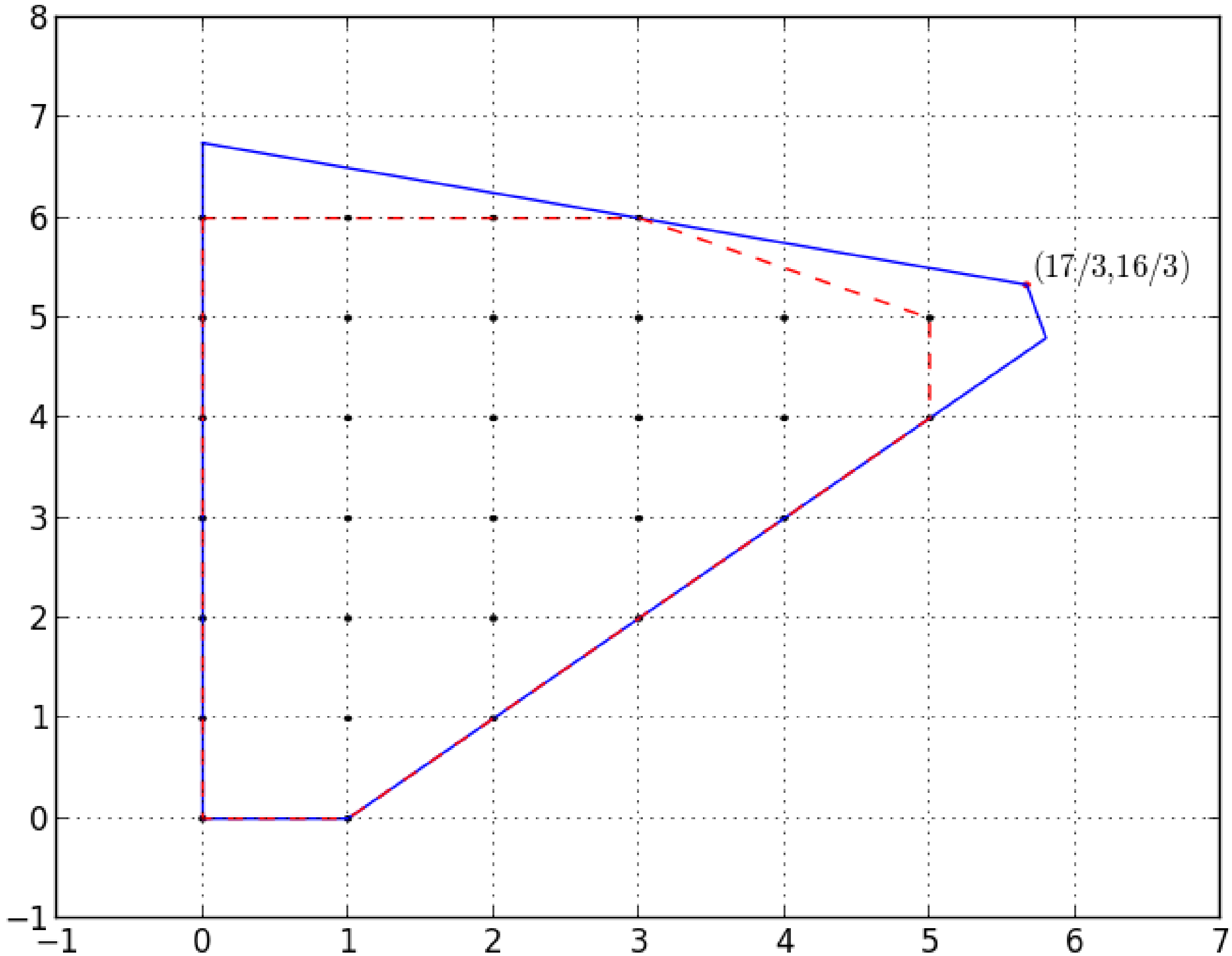


Figure 1: Convex hull of $\mathcal{S}$

Example: Gomory Cuts (cont.)

Consider the optimal tableau of the LP relaxation of the ILP

$$\max\{2x_1 + 5x_2 \mid x \in \mathcal{S}\},$$

shown in Table 1.

Basic var.	x_1	x_2	s_1	s_2	s_3	RHS
x_2	0	1	-2/30	8/30	0	16/3
s_3	0	0	-1/3	1/3	1	2/3
x_1	1	0	8/30	-2/30	0	17/3

Table 1: Optimal tableau of the LP relaxation

The associated optimal solution to the LP relaxation is also shown in Figure 1.

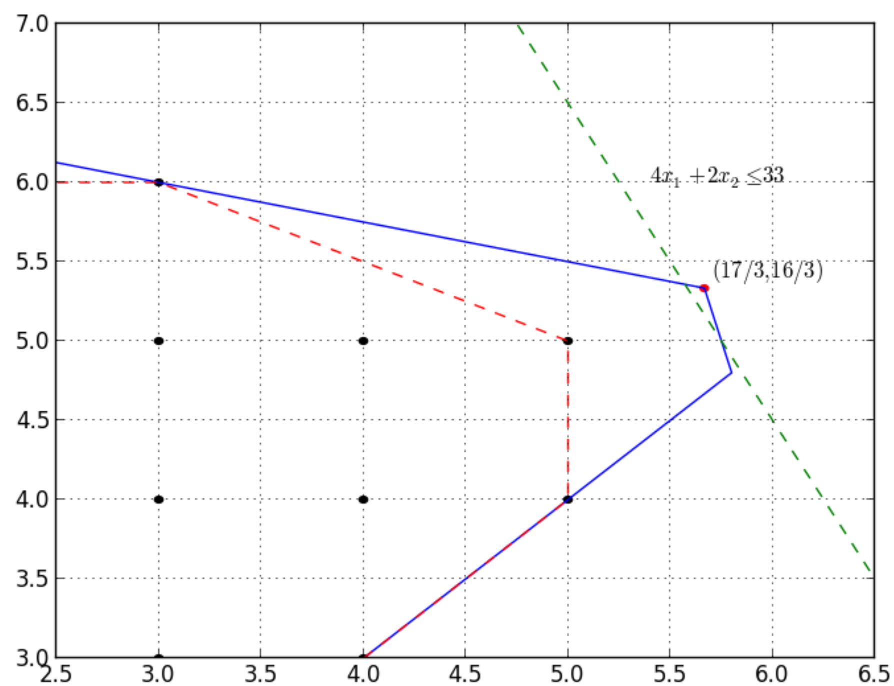
Example: Gomory Cuts (cont.)

The Gomory cut from the first row is

$$\frac{28}{30}s_1 + \frac{8}{30}s_2 \geq \frac{1}{3},$$

In terms of x_1 and x_2, we have

$$4x_1 + 2x_2 \leq 33, \qquad \text{(G-C1)}$$



Note this inquality can be trivially strengthened by dividing by 2.

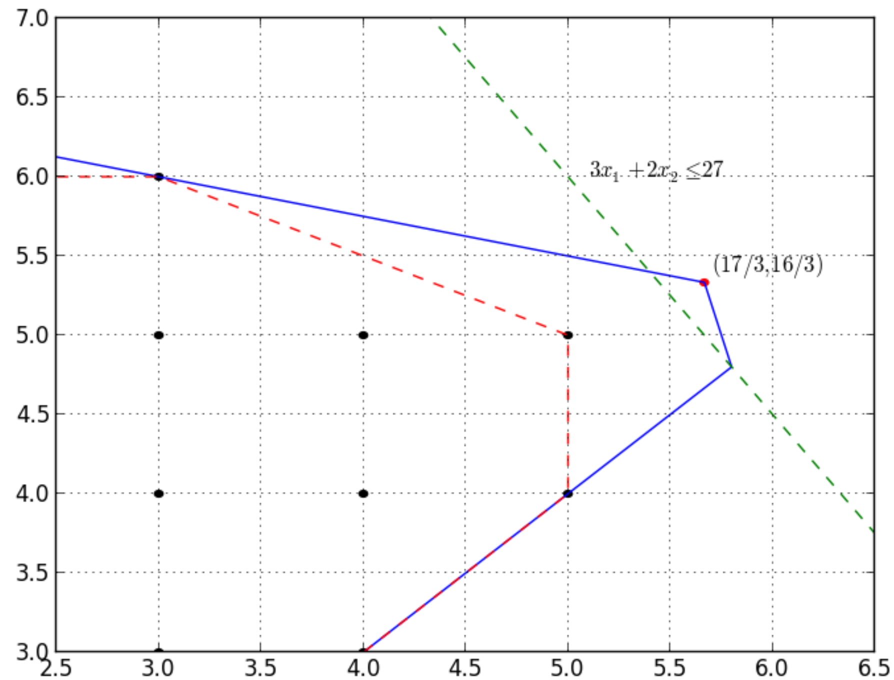
Example: Gomory Cuts (cont.)

The Gomory cut from the second row is

$$\frac{2}{3}s_1 + \frac{1}{3}s_2 \geq \frac{2}{3},$$

In terms of x_1 and x_2, we have

$$3x_1 + 2x_2 \leq 27, \tag{G-C2}$$



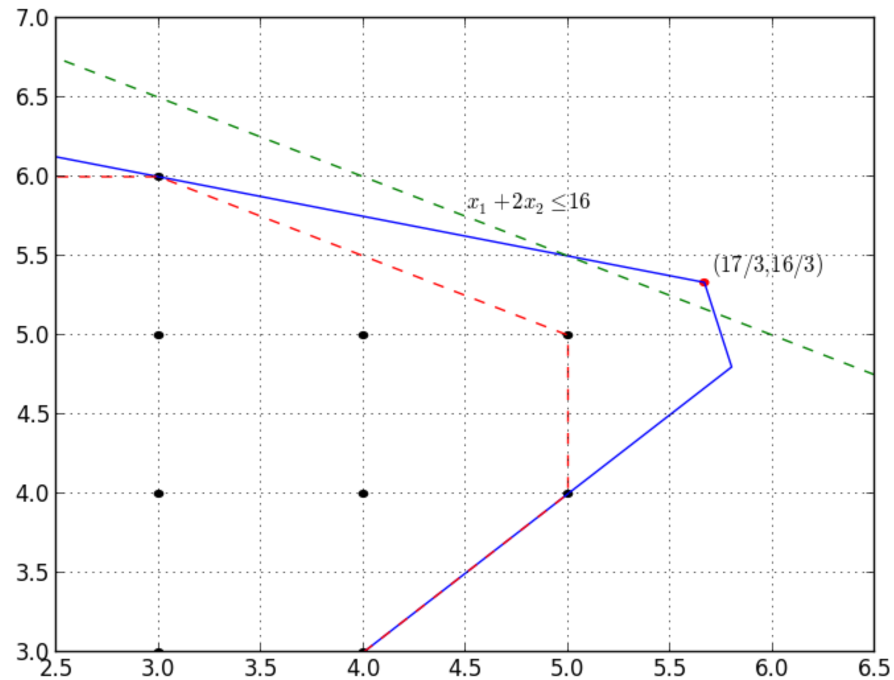
Example: Gomory Cuts (cont.)

The Gomory cut from the third row is

$$\frac{8}{30}s_1 + \frac{28}{30}s_2 \geq \frac{2}{3},$$

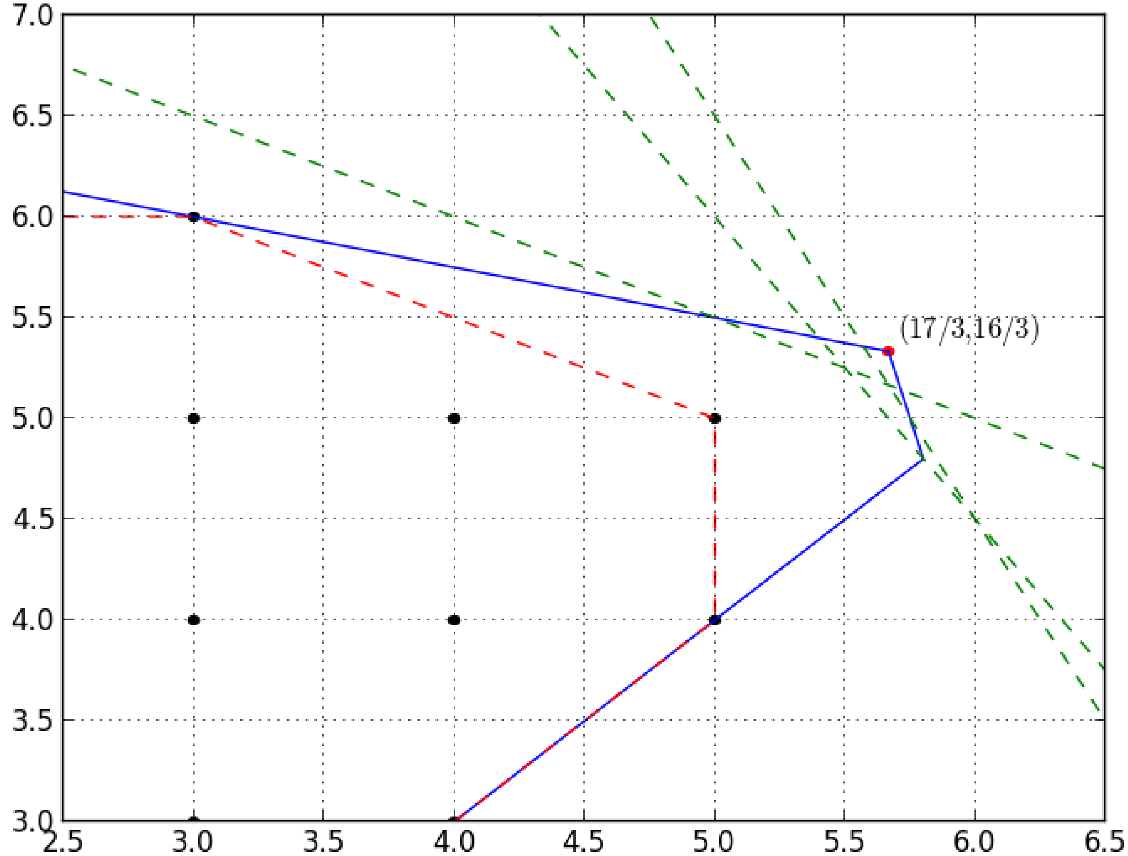
In terms of x_1 and x_2, we have

$$x_1 + 2x_2 \leq 16, \qquad \text{(G-C3)}$$



Example: Gomory Cuts (cont.)

This picture shows the effect of adding all Gomory cuts in the first round.



Applying the Procedure Recursively

- This procedure can be applied recursively by adding the generated inequalities to the formulation and performing the same steps again.
- Any inequality that can be obtained by recursive application of the C-G procedure (or is dominated by such an inequality) is a *C-G inequality*.
- For pure ILPs, all valid inequalities are C-G inequalities.

Theorem 1. *Let* $(\pi, \pi_0) \in \mathbb{Z}^{n+1}$ *be a valid inequality for* $\mathcal{S} = \{x \in \mathbb{Z}^n_+ \mid Ax \leq b\} \neq \emptyset$. *Then* (π, π_0) *is a C-G inequality for* $\mathcal{S}$.

- Roughly speaking, the *rank* of an inequality is the minimum number of recursive applications of the procedure required to produce the inequality.

C-G Inequalities of Rank 1

- The *elementary closure* of polyhedron $\mathcal{P}$ is the intersection of half-spaces defined by inequalities in the set

$$e(\mathcal{P}) = \{(\pi, \pi_0) \mid \pi_j = \lfloor ua_j \rfloor \text{ for } j \in N, \pi_0 = \lfloor ub \rfloor \text{ for some } u \in \mathbb{R}^m_+\}$$

- The elementary closure is described by all of the nondominated C-G inequalities obtained by combining inequalities in the original formulation.
- Although it is not obvious, one can show that the elementary closure *is a polyhedron*.
- Optimizing over this polyhedron is difficult ($\mathcal{NP}$-hard) in general.
- Inequalities valid for $e(\mathcal{P})$ but not for $\mathcal{P}$ have C-G rank 1 (inequalities valid for $\mathcal{P}$ have rank 0).

C-G Inequalities of Higher Rank

- The *rank k closure* $\mathcal{P}^k$ of $\mathcal{P}$ is defined recursively as follows.
  - The rank 1 closure of $\mathcal{P}$ is $\mathcal{P}^1 = e(\mathcal{P})$.
  - The rank k closure $\mathcal{P}^k = e(\mathcal{P}^{k-1})$ is the elementary closure of the $\mathcal{P}^{k-1}$.
  - An inequality is rank k if it is valid for the rank k closure $\mathcal{P}^k$ and not for $\mathcal{P}^{k-1}$.
- The *C-G rank* of $\mathcal{P}$ is the maximum rank of any facet-defining inequality of $\mathrm{conv}(\mathcal{S})$.

A Finite Cutting Plane Procedure

- Under mild assumptions on the algorithm used to solve the LP, this yields a general algorithm for solving (pure) ILPs.
- The details are contained in Section 5.2.5 of CCZ.

Determining the C-G Rank

- By solving an LP, it can be determined whether a given inequality has maximum rank 1.

  Proposition 2. *If* $(\pi, \pi_0) \in e(\mathcal{P})$*, then* $\pi_0 \geq \lfloor \pi_0^{LP} \rfloor$*, where* $\pi_0^{LP} = \max_{x \in \mathcal{P}} \pi^\top x$

- Alternatively, if $\pi \in \mathbb{Z}^n$, the inequality $(\pi, \lfloor \pi_0^{LP} \rfloor)$ is rank 1.
- Further, any valid inequality (π, π_0) for which $\pi_0 < \lfloor \pi_0^{LP} \rfloor$ has rank at least 2.
- This tells us that the effectiveness of the C-G procedure is strongly tied to the strength of our original formulation.
- In general it is difficult to determine the rank of any inequality that is not rank 1.

Example: C-G Rank

- Let's consider the C-G rank of the inequality

$$x_1 + 2x_2 \leq 15,$$

which is facet-defining for $\text{conv}(\mathcal{S})$ in our example.

- We have

$$\max_{x \in \mathcal{P}} x_1 + 2x_2 = 49/3. \tag{12}$$

- Since $\lfloor 49/3 \rfloor = 16$, we conclude that this is not a rank 1 cut.
- Note that the dual solution to the LP (12) gives us weights with which to combine the original inequalities to get a C-G cut.
- This is the strongest possible C-G cut of rank 1 with those coefficients.

Bounding The C-G Rank of a Polyhedron

- For most classes of MILPs, the rank of the associated polyhedron is an unbounded function of the dimension.
- Example:
  - $\mathcal{P} = \{x \in \mathbb{R}^n_+ \mid x_i + x_j \leq 1 \text{ for } i, j \in N, i \neq j\}$ and $S = \mathcal{P}^n \cap \mathbb{Z}^n$
  - $\text{conv}(\mathcal{S}) = \{x \in \mathbb{R}^n_+ \mid \sum_{j \in N} x_j \leq 1\}$.
  - $\text{rank}(\mathcal{P}) = O(\log n)$.
- For a family of polyhedra with bounded rank, there is a certificate for the validity of any given inequality.
- This leads to a certificate of optimality for the associated optimization problem.
- Hence, it is unlikely that the problem of optimizing over any family of MILPs formulated by polyhedra with bounded rank is in NP-hard[2].
- Conversely, for any family of MILPs that is in NP-hard, the associated family of polyhedra is likely to have unbounded rank.

[2] More on what this means later