Integer Programming
ISE 418

Lecture 3

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Reading for This Lecture

- N&W Sections I.1.1-I.1.6
- Wolsey Chapter 1
- CCZ Chapter 2

Alternative Formulations

- Recall our definition of a valid formulation from the last lecture.
- A key concept in the rest of the course will be that every mathematical model has many alternative formulations.
- Many of the key methodologies in integer programming are essentially automatic methods of reformulating a given model.
- The goal of the reformulation is to make the model easier to solve.

Simple Example: Knapsack Problem

- We are given a set $N = \{1, \ldots n\}$ of items and a capacity W.
- There is a profit p_i and a size w_i associated with each item $i \in N$.
- We want to choose the set of items that maximizes profit subject to the constraint that their total size does not exceed the capacity.
- The most straightforward formulation is to introduce a binary variable x_i associated with each item.
- x_i takes value 1 if item i is chosen and 0 otherwise.
- Then the formulation is

$$\begin{aligned} \min \sum_{j=1}^{n} p_j x_j & \\ \text{s.t. } \sum_{j=1}^{n} w_j x_j \le W & \\ x_i \in \{0, 1\} \qquad & \forall i \end{aligned}$$

- Is this formulation correct?

An Alternative Formulation

- Let us call a set $C \subseteq N$ a *cover* is $\sum_{i \in C} w_i > W$.
- Further, a cover C is *minimal* if $\sum_{i \in C \setminus \{j\}} w_i > W$ for all $j \in C$.
- Then we claim that the following is also a valid formulation of the original problem.

$$
\begin{aligned}
\min \sum_{j=1}^{n} p_j x_j & \\
\text{s.t.} \sum_{j \in C} x_j \leq |C| - 1 & \quad \text{for all minimal covers } C \\
x_i \in \{0, 1\} & \quad i \in N
\end{aligned}
$$

- Which formulation is "better"?

Back to the Facility Location Problem

- Recall our earlier formulation of this problem.
- Here is another formulation for the same problem:

$$
\begin{aligned}
\min \sum_{j=1}^{n} c_j y_j + \sum_{i=1}^{m}\sum_{j=1}^{n} d_{ij} x_{ij} & \\
\text{s.t. } \sum_{j=1}^{n} x_{ij} = 1 & \qquad \forall i \\
x_{ij} \le y_j & \qquad \forall i, j \\
x_{ij}, y_j \in \{0,1\} & \qquad \forall i, j
\end{aligned}
$$

- Notice that the set of integer solutions contained in each of the polyhedra is the same (why?).
- However, the second polyhedron is strictly included in the first one (how do we prove this?).
- Therefore, the second polyhedron will yield a better lower bound.
- The second polyhedron is a better approximation to the convex hull of integer solutions.

Formulation Strength and Ideal Formulations

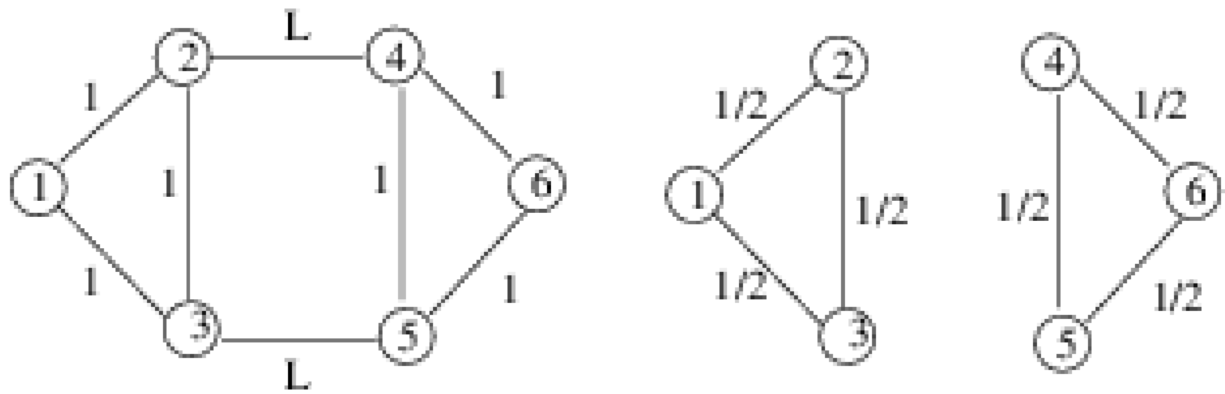
- Consider two formulations A and B for the same MILP.
- Denote the feasible regions corresponding to their LP relaxations as $\mathcal{P}_A$ and $\mathcal{P}_B$.
- Formulation A is said to be *at least as strong as* formulation B if $\mathcal{P}_A \subseteq \mathcal{P}_B$.
- If the inclusion is strict, then A is *stronger than* B.
- If $\mathcal{S}$ is the set of all feasible integer solutions for the MILP, then we must have $conv(\mathcal{S}) \subseteq \mathcal{P}_A$ (why?).
- A is *ideal* if $conv(F) = \mathcal{P}_A$.
- If we know an ideal formulation (of small enough size), we can solve the MILP (why?).
- How do our formulations of the knapsack problem compare by this measure?

Strengthening Formulations

- Often, a given formulation can be strengthened with additional inequalities satisfied by all feasible integer solutions.
- Example: The Perfect Matching Problem
  - We are given a set of n people that need to paired in teams of two.
  - Let c_{ij} represent the "cost" of the team formed by person i and person j.
  - We wish to maximize efficiency over all teams.
  - We can represent this problem on an undirected graph $G = (N, E)$.
  - The nodes represent the people and the edges represent pairings.
  - We have $x_e = 1$ if the endpoints of e are matched, $x_e = 0$ otherwise.

$$\begin{aligned}
\min \quad & \sum_{e=\{i,j\}\in E} c_e x_e \\
\text{s.t.} \quad & \sum_{\{j \mid \{i,j\}\in E\}} x_{ij} = 1, \ \forall i \in N \\
& x_e \in \{0,1\}, \qquad \forall e = \{i,j\} \in E.
\end{aligned}$$

Valid Inequalities for Matching



- Consider the graph on the left above.
- The optimal perfect matching has value $L + 2$.
- The optimal solution to the LP relaxation has value 3.
- This formulation can be extremely weak.
- Add the *valid inequality* $x_{24} + x_{35} \geq 1$.
- Every perfect matching satisfies this inequality.

The Odd Set Inequalities

- We can generalize the inequality from the last slide.
- Consider the cut S corresponding to any odd set of nodes.
- The *cutset* corresponding to S is

$$\delta(S) = \{\{i,j\} \in E | i \in s, j \notin S\}.$$

- An *odd cutset* is any $\delta(S)$ for which $|S|$ is odd.
- Note that every perfect matching contains at least one edge from every odd cutset.
- Hence, each odd cutset induces a possible valid inequality.

$$\sum_{e \in \delta(S)} x_e \geq 1, S \subset N, |S| \text{odd}.$$

Using the New Formulation

- If we add all of the odd set inequalities, the new formulation is ideal.
- Hence, we can solve this LP and get a solution to the IP.
- However, the number of inequalities is exponential in size, so this is not really practical.
- Recall that only a small number of these inequalities will be active at the optimal solution.
- Later, we will see how we can efficiently generate these inequalities on the fly to solve the IP.

Extended Formulations

- We have so far focused on strengthening formulations using additional constraints.

- However, changing the set of variables can also have a dramatic effect.

- Example: A Lot-sizing Problem

  - We want to minimize the costs of production, storage, and set-up.
  - Data for period $t = 1, \ldots, T$:
    * d_t: total demand,
    * c_t: production set-up cost,
    * p_t: unit production cost,
    * h_t: unit storage cost.
  - Variables for period $t = 1, \ldots, T$:
    *
    *
    *

Lot-sizing: The "natural" formulation

- Here is the formulation based on the "natural" set of variables:

$$
\begin{aligned}
\min \sum_{t=1}^{T} & (p_t y_t + h_t s_t + c_t x_t) \\
\text{s.t. } & y_1 = d_1 + s_1, \\
& s_{t-1} + y_t = d_t + s_t, \qquad \text{for } t = 2, \ldots, T, \\
& y_t \leq \omega x_t, \qquad \text{for } t = 1, \ldots, T, \\
& s_T = 0, \\
& s, y \in \mathbb{R}_+^T, \\
& x \in \{0,1\}^T.
\end{aligned}
$$

- Here, $\omega = \sum_{t=1}^{T} d_t$, an upper bound on y_t.

Lot-sizing: The "extended" formulation

- Suppose we split the production lot in period t into smaller pieces.
- Define the variables q_{it} to be the production in period i designated to satisfy demand in period $t \geq i$.
- Now, $y_i = \sum_{t=i}^{T} q_{it}$.
- With the new set of variables, we can impose the tighter constraint

$$q_{it} \leq d_t x_i \text{ for } i = 1, \ldots, T \text{ and } t = 1, \ldots, T.$$

- The additional variables strengthen the formulation.
- Again, this in contrary to conventional wisdom for formulating linear programs.

Strength of Formulation for Lot-sizing

- Although the formulation from the previous slide is much stronger than our original, it is still not ideal.
- Consider the following sample data.

```
# The demands for six periods
DEMAND = [6, 7, 4, 6, 3, 8]

# The production cost for six periods
PRODUCTION_COST = [3, 4, 3, 4, 4, 5]

# The storage cost for six periods
STORAGE_COST = [1, 1, 1, 1, 1, 1]

# The set up cost for six periods
SETUP_COST = [12, 15, 30, 23, 19, 45]

# Set of periods
PERIODS = range(len(DEMAND))
```

Strength of Formulation for Lot-sizing (cont'd)

```
Optimal Total Cost is:  171.42016761

Period  0 :  13 units produced,  7 units stored, 6 units sold
0.38235294 is the value of the fixed charge variable
Period  1 :  0 units produced,  0 units stored, 7 units sold
0.0 is the value of the fixed charge variable
Period  2 :  4 units produced,  0 units stored, 4 units sold
0.19047619 is the value of the fixed charge variable
Period  3 :  6 units produced,  0 units stored, 6 units sold
0.35294118 is the value of the fixed charge variable
Period  4 :  11 units produced, 8 units stored, 3 units sold
1.0 is the value of the fixed charge variable
Period  5 :  0 units produced,  0 units stored, 8 units sold
0.0 is the value of the fixed charge variable
```

What is happening here?

Strength of Formulation for Lot-sizing (cont'd)

Let's take a more detailed look:

```
production in period 0 for period 0 : 2.2941176
production in period 0 for period 1 : 2.6764706
production in period 0 for period 2 : 1.5294118
production in period 0 for period 3 : 2.2941176
production in period 0 for period 4 : 1.1470588
production in period 0 for period 5 : 3.0588235
```

What is the problem?

An Ideal Formulation for Lot-sizing

- We can further strengthen the formulation by adding the constraint

$$\sum_{i=1}^{t} q_{it} \geq d_t \text{ for } t = 1, \ldots, T$$

- In fact, adding these additional constraints makes the formulation ideal.
- If we *project* into the original space, we will get the convex hull of solutions to the first formulation.
- How would we prove this?

Contrast with Linear Programming

- In linear programming, the same problem can also have multiple formulations.
- In LP, however, conventional wisdom is that bigger formulations take longer to solve.
- In IP, this conventional wisdom does not hold.
- We have already seen two examples where it is not valid.
- Generally speaking, the size of the formulation does not determine how difficult the IP is.