Integer Programming
ISE 418

Lecture 5

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Reading for This Lecture

- N&W Sections I.4.1-I.4.3
- Wolsey, Chapters 8 and 9
- CCZ Chapter 3

Dimension of Polyhedra

- As usual, let $\mathcal{P}$ be a rational polyhedron

$$\mathcal{P} = \{x \in \mathbb{R}^n \mid Ax \leq b\}$$

- $\mathcal{P}$ is of *dimension* k, denoted $dim(\mathcal{P}) = k$, if the maximum number of affinely independent points in $\mathcal{P}$ is $k+1$.
- Alternatively, the dimension of $\mathcal{P}$ is exactly the dimension of $\mathrm{aff}(\mathcal{P})$.
- A polyhedron $\mathcal{P} \subseteq \mathbb{R}^n$ is *full-dimensional* if $dim(\mathcal{P}) = n$.
- Let
  - $M = \{1, \ldots, m\}$,
  - $M^= = \{i \in M \mid a_i^\top x = b_i \ \forall x \in \mathcal{P}\}$ (the *equality set*),
  - $M^\leq = M \setminus M^=$ (the *inequality set*).
- Let $(A^=, b^=), (A^\leq, b^\leq)$ be the corresponding rows of (A, b).

Proposition 1. *If* $\mathcal{P} \subseteq \mathbb{R}^n$*, then* $dim(\mathcal{P}) + rank(A^=, b^=) = n$

Dimension and Rank

- $x \in \mathcal{P}$ is called an *inner point* of $\mathcal{P}$ if $a_i^\top x < b_i \; \forall i \in M^{\leq}$.
- $x \in \mathcal{P}$ is called an *interior point* of $\mathcal{P}$ if $a_i^\top x < b_i \; \forall i \in M$.
- Every nonempty polyhedron has an inner point.
- The previous proposition showed that a polyhedron has an interior point if and only if it is full-dimensional.

Computing the Dimension of a Polyhedron

- To compute the dimension of a polyhedron, we generally use these two equations

$$\begin{aligned} dim(\mathcal{P}) &= n - rank(A^=, b^=), and \\ dim(\mathcal{P}) &= \max\{|D| : D \subseteq \mathcal{P} \text{ and the points in } D \text{ are aff. indep.}\} - 1. \end{aligned}$$

- In general, it is difficult to determine $dim(\mathcal{P})$ using either one of these formulas alone, so we use them together.
  1. Determine a conjectured form for $(A^=, b^=)$ to obtain an upper bound d on $dim(\mathcal{P})$.
  2. Display a set of $d + 1$ affinely independent points in $\mathcal{P}$.
- In some cases, it is possible to avoid step 2 by proving the exact form of $(A^=, b^=)$.
- Usually, this consists of showing that any other equality satisfied by all members of the polytope is a linear combination of the known ones.

Dimension of the Feasible Set of an MILP

- We have so far defined what we mean by the dimension of a polyhedron.
- What do we mean by the "dimension of the feasible set of a mixed integer optimization problem"?
- Suppose we are given an integer optimization problem described by (A, b, c, p), with feasible set

$$\mathcal{S} = \{x \in \mathbb{Z}^p_+ \times \mathbb{R}^{n-p}_+ \mid Ax \le b\}$$

- We will see later that $\mathrm{conv}(\mathcal{S})$ is a polyhedron.
- It is the dimension of this polyhedron, which could be different from that described by the linear constraints, that we are interested in.
- Knowing its dimension can help us determine which inequalities in the formulation are necessary and which are not.

Determining the Dimension of $\text{conv}(\mathcal{S})$

- The procedure for determining the dimension of $\text{conv}(\mathcal{S})$ is more difficult because we do not have an explicit description of $\text{conv}(\mathcal{S})$.
- We therefore have to use only points in $\mathcal{S}$ itself to determine the dimension.
- Note that the equality set may not consist only of constraints from the original formulation.
- In general, we need to determine $(D^=, d^=)$ such that $D^= x = d^=$ for all $x \in \mathcal{S}$.
- In many cases, however, the equality set will be a subset of the inequalities from the original formulation.
- The procedure is then as follows.
  - Determine a conjectured form for $(D^=, d^=)$ to obtain an upper bound d on $dim(\text{conv}(\mathcal{S}))$.
  - Display a set of $d+1$ affinely independent points in $\mathcal{S}$.

Example: Knapsack Problem

- We are given n items and a capacity W.
- There is a profit p_i and a size w_i associated with each of the items.
- We want to choose the set of items that maximizes profit subject to the constraint that their total size does not exceed the capacity.
- We thus have a binary variable x_i associated with each items that is 1 if item i is included and 0 otherwise.

$$\begin{aligned} \min \sum_{j=1}^{n} p_j x_j & \\ \text{s.t. } \sum_{j=1}^{n} w_j x_j \leq W & \\ x_i \in \{0,1\} & \qquad \forall i \end{aligned}$$

- What is the dimension of $\mathrm{conv}(\mathcal{S})$?

Valid Inequalities

- The inequality denoted by (π, π_0) is called a *valid inequality* for $\mathcal{P}$ if $\pi^\top x \leq \pi_0 \; \forall x \in \mathcal{P}$.
- Note (π, π_0) is a valid inequality if and only if $\mathcal{P} \subseteq \{x \in \mathbb{R}^n \mid \pi^\top x \leq \pi_0\}$.
- Consider the polyhedron $\mathcal{Q} = \{x \in \mathbb{R}^m \mid Ax \leq b, Cx = d\}$.
- An inequality (π, π_0) is valid for $\mathcal{Q}$ if and only if the system

$$\begin{aligned} uA + vC &= \pi \\ ub + vd &\leq \pi_0 \\ u &\geq 0 \end{aligned}$$

  has a solution.
- When the above system has a solution, we say that the inequality (π, π_0) is *implied by* the system of inequalities and equations that describe $\mathcal{Q}$.

Checking Containment

- The procedure on the last slide gives us straightforward way of determining whether one polyhedron is contained in another.
- We simply check whether all the inequalities describing one of the polyhedra are implied by the inequalities describing the other.
- In principle, this could be used to compare the strength of two formulations for a given MILP.
- This procedure is computationally prohibitive in general, though.

Minimal Descriptions

- If $\mathcal{P} = \{x \in \mathbb{R}^n_+ \mid Ax \leq b\}$, then the inequalities corresponding to the rows of $[A \mid b]$ are called a *description* of $\mathcal{P}$.
- Given that there are an infinite number of descriptions, we would like to determine a minimal one.

Definition 1. *If (π, π_0) and (μ, μ_0) are two inequalities valid for a polyhedron $\mathcal{P} \subseteq \mathbb{R}^n_+$, we say (π, π_0)* dominates *(μ, μ_0) if there exists $u > 0$ such that $\pi \geq u\mu$ and $\pi_0 \leq u\mu_0$.*

- Although this concept is defined in terms of $\mathcal{P}$, the relationship is independent of $\mathcal{P}$ itself.
- The assumption that the polyhedron is contained in the non-negative orthant is crucial.

Redundant Inequalities

- It is easy to show that all inequalities valid for a polyhedron are either combinations of those in the description or dominated by some such combination.

Definition 2. *An inequality (π, π_0) that is part of a description of $\mathcal{P}$ is* redundant *in that description if there exists a non-negative combination of the inequalities in the description that dominates (π, π_0).*

- Note again that this definition depends strongly on our assumption that $\mathcal{P}$ is contained in the non-negative orthant.
- We could also define a redundant inequality as one that is implied by the system of all inequalities in the description of $\mathcal{P}$ *except* for (π, π_0) itself.
- This latter definition would be independent on the non-negativity assumption.
- It seems clear that any minimal description will have to be free of redundant inequalities, but can we say more than this?

Faces

- If (π, π_0) is a valid inequality for $\mathcal{P}$ and $F = \{x \in \mathcal{P} \mid \pi^\top x = \pi_0\}$, F is called a *face* of $\mathcal{P}$ and we say that (π, π_0) *represents* or *defines* F.
- The face F represented by (π, π_0) is itself a polyhedron and is said to be *proper* if $F \neq \emptyset$ and $F \neq \mathcal{P}$.
  - F is nonempty (and we say it *supports* $\mathcal{P}$) if and only if $\max\{\pi^\top x \mid x \in \mathcal{P}\} = \pi_0$.
  - $F \neq \mathcal{P}$ if and only (π, π_0) is not in the equality set.
- Note that a face has multiple representations in general.
- The set of optimal solutions to an LP is always a face of the feasible region.
- For polyhedron $\mathcal{P}$, we have
  1. Two faces F and F' are distinct if and only if $\text{aff}(F) \neq \text{aff}(F')$.
  2. If F and F' are faces of $\mathcal{P}$ and $F \subseteq F'$, then $\dim(F) \leq \dim(F')$.
  3. Given a face F of $\mathcal{P}$, the faces of F are exactly the faces of $\mathcal{P}$ contained in F.

Describing Polyhedra by Facets

Proposition 2. *Every face F of a polyhedron $\mathcal{P}$ can be obtained by setting a specified subset of the inequalities in the description of $\mathcal{P}$ to equality.*

- Note that this result is true for any description of $\mathcal{P}$.
- This result implies that the number of faces of a polyhedron is finite.
- A face F is said to be a *facet* of $\mathcal{P}$ if $dim(F) = dim(\mathcal{P}) - 1$.
- In fact, facets are all we need to describe polyhedra.

Proposition 3. *If F is a facet of $\mathcal{P}$, then in any description of $\mathcal{P}$, there exists some inequality representing F.*

Proposition 4. *Every inequality that represents a face that is not a facet is unnecessary in the description of $\mathcal{P}$.*

Putting It Together

Putting together what we have seen so far, we can say the following.

Theorem 1.

1. *Every full-dimensional polyhedron $\mathcal{P}$ has a unique (up to scalar multiplication) representation that consists of one inequality representing each facet of $\mathcal{P}$.*

2. *If $dim(\mathcal{P}) = n - k$ with $k > 0$, then $\mathcal{P}$ is described by any set of k linearly independent rows of $(A^=, b^=)$, as well as one inequality representing each facet of $\mathcal{P}$.*

Theorem 2. *If a facet F of $\mathcal{P}$ is represented by (π, π_0), then the set of all representations of F is obtained by taking scalar multiples of (π, π_0) plus linear combinations of the equality set of $\mathcal{P}$.*

Determining Whether an Inequality is Facet-defining

- One of the reasons we would like to know the dimension of a given polyhedron is to determine which inequalities are facet-defining.
- The face defined by any valid inequality is itself a polyhedron and its dimension can be determined in a similar fashion.
- Because the inequality defining F has been fixed to equality, F must have dimension at most $dim(\mathcal{P}) - 1$.
- The question of whether F is a facet is that of whether other (linearly independent) inequalities also hold at equality for F.
- These questions are relatively easy to answer in the case of an explicitly defined polyhedron.
- When we are asking the question of whether an inequality is facet-defining for $\mathrm{conv}(\mathcal{S})$, the question is more difficult.
- We must show that there are $dim(\mathrm{conv}(\mathcal{S}))$ affinely independent points in F.

Example: Facility Location Problem

- We are given n potential facility locations and m customers that must be serviced from those locations.
- There is a fixed cost c_j of opening facility j.
- There is a cost d_{ij} associated with serving customer i from facility j.
- We have two sets of binary variables.
  - y_j is 1 if facility j is opened, 0 otherwise.
  - x_{ij} is 1 if customer i is served by facility j, 0 otherwise.

$$
\begin{aligned}
\min \; & \sum_{j=1}^{n} c_j y_j + \sum_{i=1}^{m} \sum_{j=1}^{n} d_{ij} x_{ij} & \\
\text{s.t. } & \sum_{j=1}^{n} x_{ij} = 1 & \forall i \\
& x_{ij} \leq y_j & \forall i, j \\
& x_{ij}, y_j \in \{0, 1\} & \forall i, j
\end{aligned}
$$

Example: Facility Location Problem

- What is the dimension of the convex hull of feasible solutions?
- Which of the inequalities in the formulation are facet-defining?

Back to Formulation

- Aside: We will sometimes abuse terminology slightly and refer to any valid inequality representing a facet as a facet.
- The reason we are interested in facet-defining inequalities is because they are the "strongest" valid inequalities.
- We have shown that facet-defining inequalities can never be dominated.
- Although necessary for describing the convex hull of feasible solutions, they do not have to appear in the formulation.
- Adding a facet-defining inequality (that is not already represented) to a formulation necessarily increases its strength.
- In general, it is as difficult to generate facet-defining inequalities for $\mathrm{conv}(\mathcal{S})$ as it is to optimize over $\mathcal{S}$.
- We will see later in the course that we often settle for inequalities that are facet-defining for a given relaxation.