Integer Programming
ISE 418

Lecture 23

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Reading for This Lecture

- Wolsey Section 2.5
- Nemhauser and Wolsey II.3.1-II.3.3
- "Duality for Mixed-Integer Linear Programs," Güzelsoy and Ralphs.

Duality

- An alternative to relaxation for obtaining bounds is to formulate a *dual problem*.
- Let a pure integer program be defined by

$$z_{IP} = \max\{cx \mid x \in \mathcal{S}\}, S = \{x \in \mathbb{Z}^n_+ \mid Ax = b\}$$

  where $c \in \mathbb{R}^n, A \in \mathbb{Q}^{m\times n}, b \in \mathbb{R}^m$.

- We refer to this instance as the *primal problem*.
- A *weak dual problem* is an optimization problem of the form

$$z_D = \min_{v \in V} f(v),$$

  with $f : V \to \mathbb{R}$, $V \subseteq \mathbb{R}^k$ for $k \in \mathbb{N}$ such that $z_D \geq z_{IP}$.

- A *strong dual* is a weak dual if z_{IP} is finite and also $z_D = z_{IP}$.

Importance of Duality

- Note again that if we have a dual to IP, then we can easily obtain bounds on the value of an optimal solution.
- The advantage of a dual is that we need not solve it to optimality.
- Any feasible solution to the dual yields a valid bound.
- The three main categories of duals used most frequently are
  - LP duals
  - Combinatorial duals
  - Lagrangian duals

The Duality Gap

- In the case of weak duals, there is a gap between the optimal solution to the dual problem and the optimal solution to IP.
- This gap is known as the *duality gap* or just *the gap*.
- It is typically measured as a percentage of the value of an optimal solution.
- The size of the gap is a measure of the difficulty of a problem.
- It can help us estimate how long it will take to solve a given problem by branch and bound.
- As a rule of thumb, problems with a gap of more than 5-10% are too difficult to solve in practice.
- Note that in most cases, we don't know the exact gap because we don't know the exact value of an optimal solution.
- Usually, the gap is estimated based on the best known solution.

Generalized Dual

- The previously introduced definition of a dual problem is not at useful, since the dual problem is not selected by any measure of goodness.
- Conceptually, we can improve the situation by choosing the "best" from a family of dual problem to obtain

$$z_D = \min_{f,V} \min_{v \in V} f(v),$$

where each pair (f, V) is required to comprise a dual problem.

- It is not clear how to solve this generalized dual.
- Even if this were possible, the dual would not be useful for analyzing perturbed instances, as the LP dual is.
- We would like to generalize the concept of duality we had in LP in order to be able to perform sensitivity analyses and warm start.
- What we would like is a *dual function* that can produce a valid bound across a range of perturbed instances.
- This is essentially what we have in the LP case.

Value function

- What do we mean by the *neighborhood* of a given instance?
- Here, we only consider varying the right-hand-side.
- The *value function* is defined as:

$$z(d) = \max_{x \in \mathcal{S}(d)} cx,$$

where $\mathcal{S}(d) = \{x \in \mathbb{Z}^n_+ \mid Ax = d\}$, $d \in \mathbb{R}^m$.

- We let $z(d) = -\infty$ if $d \notin \Omega$, where $\Omega = \{d \in \mathbb{R}^m \mid S(d) \neq \emptyset\}$

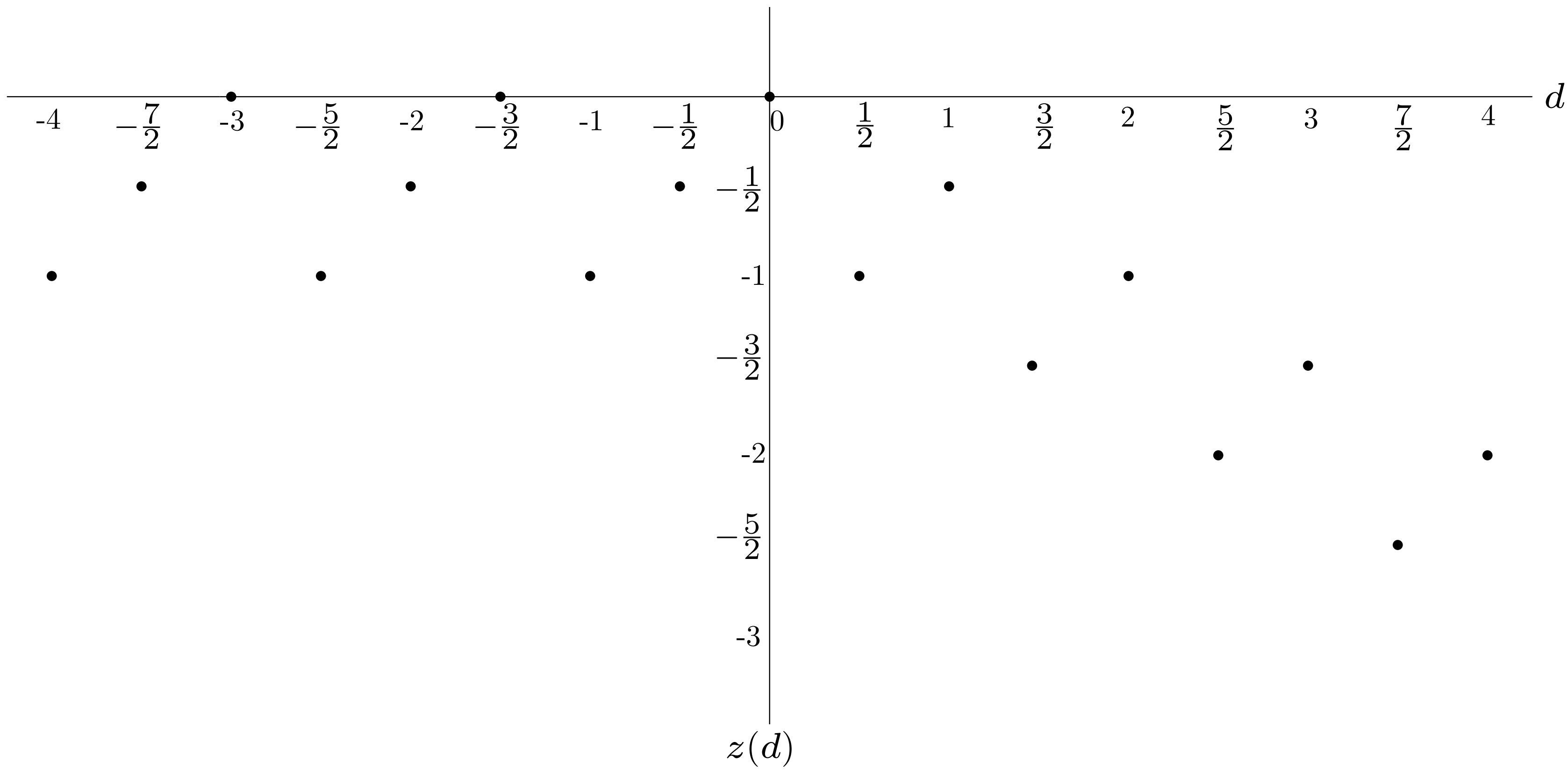
Example

Consider the following instance

$$\begin{aligned} z_{IP} = \max \quad & -\tfrac{1}{2}x_1 - 2x_3 - x_4 \\ \text{s.t} \quad & x_1 - \tfrac{3}{2}x_2 + x_3 - x_4 = b \quad \text{and} \\ & x_i \in \mathbb{Z}_+, i = 1, ..., 4 \end{aligned}$$

In closed form, we have $z(d) = -\frac{3}{2}\max\{\lceil\frac{2d}{3}\rceil,\ \lceil d\rceil\} + d, d \in \Omega$.

See Figure.


d
z(d)
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-3

Dual Function

- It is difficult to construct the value function itself.
- It is easier to obtain an approximate function that bounds the value function from above.
  - A *dual function* $F : \mathbb{R}^m \to \mathbb{R}$ is one that satisfies $F(d) \geq z(d)$ for all $d \in \mathbb{R}^m$.
  - What is a "good" dual function?
  - We cab choose one that provides the best bound for the current right-hand side b.
  - This results in the dual

$$z_D = \min \{F(b) : F(d) \geq z(d),\ d \in \mathbb{R}^m, F \in \Upsilon^m\},$$

    where $\Upsilon^m \subseteq \{f \mid f : \mathbb{R}^m \to \mathcal{R}\}$.
  - We call F^* *strong* if F^* is a *feasible* dual function and $F^*(b) = z_{IP}$.
  - This dual problem always has a solution F^* that is strong if the primal problem is finite and $\Upsilon^m \equiv \{f \mid f : \mathbb{R}^m \to \mathcal{R}\}$. Why?

A Dual Function from LP Relaxation

- Consider the value function of the LP relaxation of the primal problem:

$$F_{LP}(d) = \min \{vd : vA \geq c,\ v \in \mathbb{R}^m\}.$$

- By linear programming duality theory, we have $F_{LP}(d) \geq z(d)$ for all $d \in \mathbb{R}^m$.
- F_{LP} is not necessarily strong.

Example

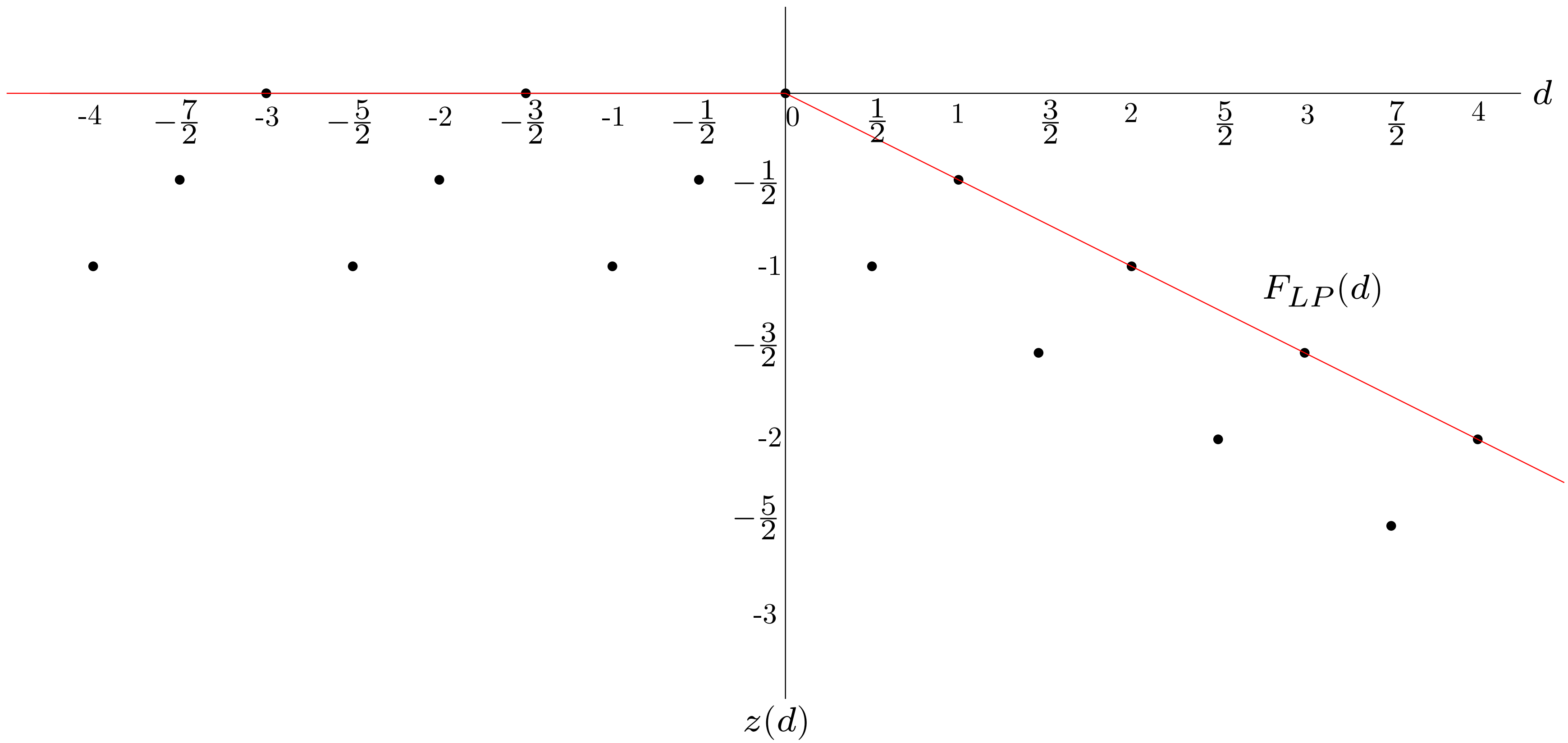
Consider the value function of the LP relaxation of our instance

$$\begin{aligned} F_{LP}(d) = \min \quad & vd, \\ \text{s.t} \quad & 0 \geq v \geq -\tfrac{1}{2}, \text{ and} \\ & v \in \mathcal{R}, \end{aligned}$$

which can be written explicitly as

$$F_{LP}(d) = \begin{cases} 0, & d \leq 0 \\ -\frac{1}{2}d, & d > 0 \end{cases} .$$

See Figure.



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$-\frac{3}{2}$
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$-\frac{5}{2}$
-3
$F_{LP}(d)$
$z(d)$

The Superadditive Dual

By considering that

$$\begin{aligned} F(d) \geq z(d),\ d \in \mathbb{R}^m \quad &\iff \quad F(d) \geq cx\ ,\ x \in \mathcal{S}(d),\ d \in \mathbb{R}^m \\ &\iff \quad F(Ax) \geq cx\ ,\ x \in \mathbb{Z}^n_+, \end{aligned}$$

the dual problem can be rewritten as

$$z_D = \min \{F(b) : F(Ax) \geq cx,\ x \in \mathbb{Z}^n_+,\ F \in \Upsilon^m\}.$$

Can we further restrict Υ^m and still guarantee a strong dual solution?

- The class of linear functions? NO!
- The class of concave functions? NO!
- The class of superadditive functions? YES!

The Superadditive Dual

- Let a function F be defined over a domain V. Then F is *superadditive* if $F(v_1) + F(v_2) \leq F(v_1 + v_2) \forall v_1, v_2, v_1 + v_2 \in V$.
- A strong motivation: value function z is superadditive over Ω. Why?

If $\Upsilon^m \equiv \Gamma^m \equiv \{F \text{ is superadditive} \mid F : \mathbb{R}^m \to \mathcal{R}, F(0) = 0\}$, we can rewrite the dual problem above as the *superadditive dual*

$$
\begin{aligned}
z_D = \min \quad & F(b) \\
& F(a^j) \geq c_j \quad j = 1, ..., n, \\
& F \in \Gamma^m
\end{aligned}
$$

where a^j is the j^{th} column of A.

Weak Duality

Theorem 1. *Let x be a feasible solution to the primal problem and let F be a feasible solution to the superadditive dual. Then, $F(b) \geq cx$.*

Proof.

Corollary 1. *For the primal problem and its superadditive dual:*

1. *If the primal problem (resp., the dual) is unbounded then the dual problem (resp., the primal) is infeasible.*
2. *If the primal problem (resp., the dual) is infeasible, then the dual problem (resp., the primal) is infeasible or unbounded.*

Strong Duality

Theorem 2. *If the primal problem (resp., the dual) has a finite optimum, then so does the superadditive dual problem (resp., the primal) and they are equal.*

Outline of the Proof. Show that the value function z or an extension to z is a feasible dual function.

- Note that z satisfies the dual constraints.
- $\Omega \equiv \mathbb{R}^m$: $z \in \Gamma^m$.
- $\Omega \subset \mathbb{R}^m$: $\exists\, z_e \in \Gamma^m$ with $z_e(d) = z(d)\ \forall\, d \in \Omega$ and $z_e(d) < \infty\ \forall d \in \mathbb{R}^m$.

Example

For our IP instance, the superadditive dual problem is

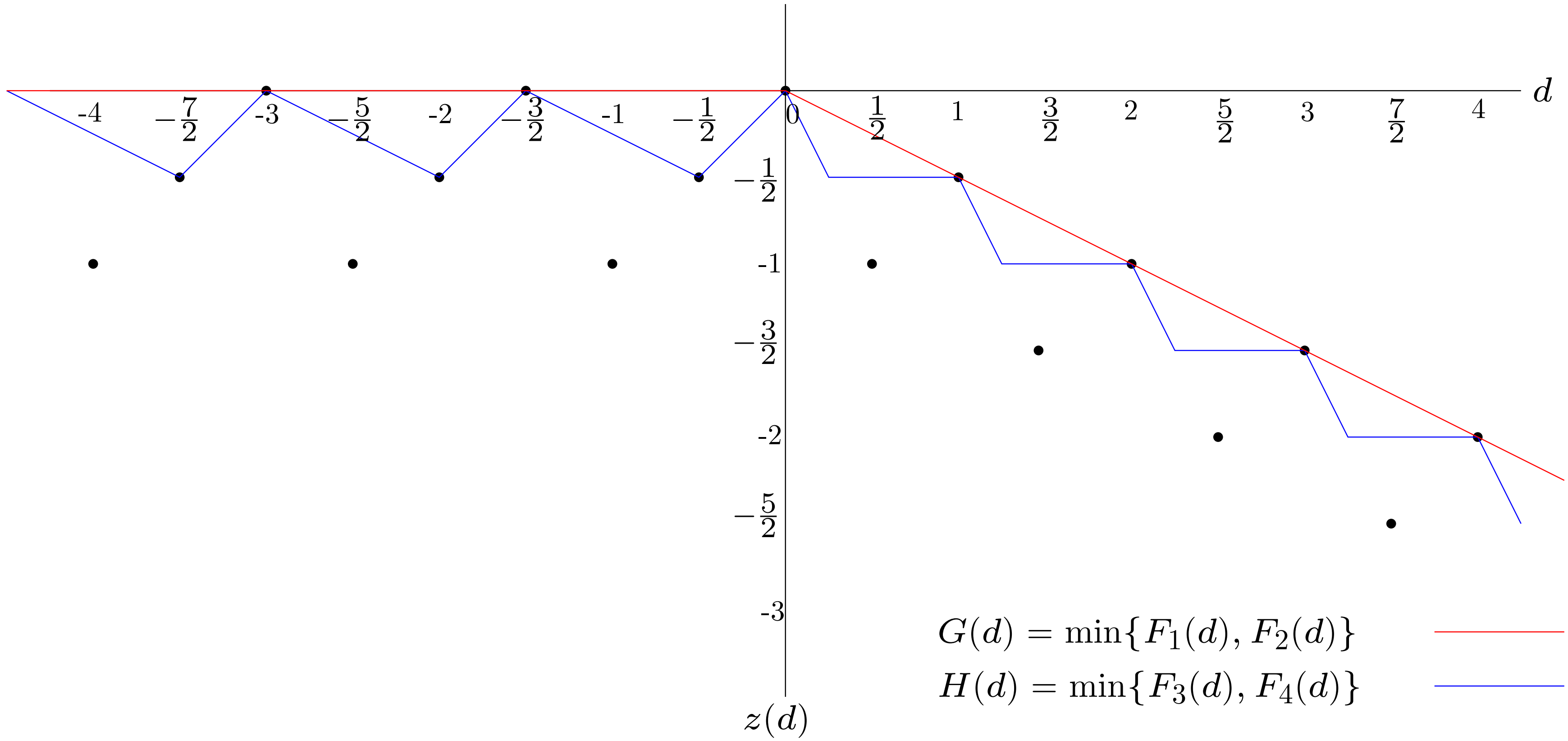
$$
\begin{array}{lrl}
\min & F(b) & \\
 & F(1) & \geq -\frac{1}{2} \\
 & F(-\frac{3}{2}) & \geq 0 \\
 & F(1) & \geq -2 \\
 & F(-1) & \geq -1 \\
 & F \in \Gamma^1. &
\end{array}
\qquad .
$$

Example

Notice how optimal solutions for different right-hand-sides give different bounds for other right-hand-sides.

1. $F_1(d) = -\frac{d}{2}$ is an optimal dual function for $b \in \{0, 1, 2, ...\}$.
2. $F_2(d) = 0$ is an optimal function for $b \in \{..., -3, -\frac{3}{2}, 0\}$.
3. $F_3(d) = -\max\{\frac{1}{2}\lceil d - \frac{\lceil \lceil d \rceil - d \rceil}{4} \rceil, 2d - \frac{3}{2}\lceil d - \frac{\lceil \lceil d \rceil - d \rceil}{4} \rceil\}$ is an optimal function for $b \in \{[0, \frac{1}{4}] \cup [1, \frac{5}{4}] \cup [2, \frac{9}{4}] \cup ...\}$.
4. $F_4(d) = -\max\{\frac{3}{2}\lceil \frac{2d}{3} - \frac{2\lceil \lceil \frac{2d}{3} \rceil - \frac{2d}{3} \rceil}{3} \rceil - d, -\frac{3}{4}\lceil \frac{2d}{3} - \frac{2\lceil \lceil \frac{2d}{3} \rceil - \frac{2d}{3} \rceil}{3} \rceil + \frac{d}{2}\}$ is an optimal function for $b \in \{... \cup [-\frac{7}{2}, -3] \cup [-2, -\frac{3}{2}] \cup [-\frac{1}{2}, 0]\}$

See Figure.



d
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$-\frac{3}{2}$
-2
$-\frac{5}{2}$
-3
$G(d) = \min\{F_1(d), F_2(d)\}$
$H(d) = \min\{F_3(d), F_4(d)\}$

Farkas' Lemma

For the primal problem, exactly one of the following holds:

1. $\mathcal{S} \neq \emptyset$
2. There is an $F \in \Gamma^m$ with $F(a^j) \geq 0, j = 1, ..., n$, and $F(b) < 0$.

Proof. Let $c = 0$ and apply strong duality theorem to superadditive dual.

Complementary Slackness

For a given right-hand side b, let x^* and F^* be feasible solutions to the primal and the superadditive dual problems, respectively. x^* and F^* are optimal solutions if and only if

1. $x_j^*(c_j - F^*(a^j)) = 0, j = 1, ..., n$ and
2. $F^*(b) = \sum_{j=1}^n F^*(a^j)x_j^*$.

Proof. For an optimal pair we have

$$F^*(b) = F^*(Ax^*) = \sum_{j=1}^n F^*(a^j)x_j^* = cx^*.$$

Constructing Dual Functions

- Explicit construction
  - The Value Function
  - Generating Functions
- Relaxations
  - Lagrangian Relaxation
  - Quadratic Lagrangian Relaxation
  - Corrected Linear Dual Functions
- Primal Solution Algorithms
  - Cutting Plane Method
  - Branch-and-Bound Method
  - Branch-and-Cut Method

Branch-and-Bound Method

- Assume that the primal problem is solved to optimality. Let T be the set of leaf nodes.

- Note that we solve the LP relaxation of the following problem at node $t \in T$

$$z^{\top}(b) = \max \quad cx \\ \text{s.t} \quad x \in \mathcal{S}_t(b) \quad ,$$

  where $\mathcal{S}_t(b) = \{Ax = b, x \geq l^{\top}, -x \geq -u^{\top}, x \in \mathbb{Z}^n\}$ and $u^{\top},\ l^{\top} \in \mathbb{Z}^r$ are the branching bounds applied to the integer variables.

- Let $(v^{\top}, \underline{v}^{\top}, \overline{v}^{\top})$ be

  - the dual feasible solution used to prune node t, if t is feasibly pruned,
  - a dual feasible solution (that can be obtained from it parent) to node t, if t is infeasibly pruned.

Dual Function from Branch-and-Bound Tree

Then,

$$F_{BC}(d) = \max_{t \in T} \{v^\top d + \underline{v}^\top l^\top - \overline{v}^\top u^\top\}$$

is an optimal solution to the generalized dual problem.

Proof.

- We can also make use of the internal nodes.
- We can also get a dual function feasible to superadditive dual.

Superadditive Dual for MILPs

Let an MILP be defined by

$$z_{IP} = \max\{cx \mid x \in \mathcal{S}\}, S = \{x \in \mathbb{Z}^r_+ \times \mathbb{R}^{n-r}_+ \mid Ax = b\}.$$

Then, its superadditive dual problem is

$$\begin{aligned} z_D = \min \quad & F(b) \\ & F(a^j) \geq c_j \quad j = 1, ..., r, \\ & \bar{F}(a^j) \geq c_j \quad j = r+1, ..., n, \text{ and} \\ & F \in \Gamma^m, \end{aligned}$$

where the function $\bar{F}$ is defined by

$$\bar{F}(d) = \limsup_{\delta \to 0^+} \frac{F(\delta d)}{\delta} \quad \forall d \in \mathbb{R}^m.$$

Here, $\bar{F}$ is the *upper d-directional derivative* of F at zero.