LECTURE 2: PRELIMINARIES

1. Standard form LP
2. Embedded assumptions
3. Converting to standard form

Standard form LP

- Key elements:
  - n variables:
    $x_1, x_2, \ldots, x_n$
  - 1 objective function:
    $\mathbf{z} = c_1x_1 + c_2x_2 + \cdots + c_nx_n$
  - m constraints:

$$
\begin{aligned}
a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= b_1 \\
a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= b_2 \\
&\vdots \\
a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n &= b_m
\end{aligned}
$$

  - non-negativity requirements:
    $x_1 \geq 0, \; x_2 \geq 0, \cdots, \; x_n \geq 0$

Explicit form

Minimize $\mathbf{z} = c_1x_1 + c_2x_2 + \cdots + c_nx_n$

subject to

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= b_2 \\ &\vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n &= b_m \\ x_1 \geq 0,\ x_2 \geq 0, \cdots,\ x_n &\geq 0 \end{aligned}$$

- Minimizing one objective function
- Equality constraints
- Non-negative variables

Matrix form

- Cost vector solution vector right-hand-side vector

$$\mathbf{c} = \begin{pmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{pmatrix}, \quad \mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}, \quad \mathbf{b} = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{pmatrix}$$

- Constraint matrix

$$\mathbf{A} = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix}$$

$$\boxed{\begin{aligned} \text{Min } \mathbf{c}^T\mathbf{x} \\ \text{s. t. } \mathbf{A}\mathbf{x} = \mathbf{b} \\ \mathbf{x} \geq 0 \end{aligned}}$$

Example – transportation problem

Supply	tons
NC	4,000
OK	3,000
IO	2,500
VA	1,500

Demand	tons
NY	5,000
LA	2,000
DC	4,000

Cost	NY	LA	DC
NC	30	10	40
OK	15	20	60
IO	60	35	25
VA	5	45	75

A management question

- How to meet customer demands in a most cost effect manner?

- LP model
  - What are that variables to be involved?
  - What's the objective function
  - How are the variables constrained?
  - Which one comes to picture first?

Formulation

- Variables

	NY	LA	DC
NC	x_{11}	x_{12}	x_{13}
OK	x_{21}	x_{22}	x_{23}
IO	x_{31}	x_{32}	x_{33}
VA	x_{41}	x_{42}	x_{43}

Objective function

$$\begin{aligned} \mathbf{z} = \; & 30x_{11} + 10x_{12} + 40x_{13} \\ + \; & 15x_{21} + 20x_{22} + 60x_{23} \\ + \; & 60x_{31} + 35x_{32} + 25x_{33} \\ + \; & 5x_{41} + 45x_{42} + 75x_{43} \end{aligned}$$

- Constraints

(supply side)

$$\begin{aligned} x_{11} + x_{12} + x_{13} &= 4,000 \\ x_{21} + x_{22} + x_{23} &= 3,000 \\ x_{31} + x_{32} + x_{33} &= 2,500 \\ x_{41} + x_{42} + x_{43} &= 1,500 \end{aligned}$$

(demand side)

$$\begin{aligned} x_{11} + x_{21} + x_{31} + x_{41} &= 5,000 \\ x_{12} + x_{22} + x_{32} + x_{42} &= 2,000 \\ x_{13} + x_{23} + x_{33} + x_{43} &= 4,000 \end{aligned}$$

LP model in standard form

$$
\begin{aligned}
\text{Minimize } \mathbf{z} = \quad & 30x_{11} + 10x_{12} + 40x_{13} \\
+ \quad & 15x_{21} + 20x_{22} + 60x_{23} \\
+ \quad & 60x_{31} + 35x_{32} + 25x_{33} \\
+ \quad & 5x_{41} + 45x_{42} + 75x_{43}
\end{aligned}
$$

$$
\begin{aligned}
\text{subject to} \quad & x_{11} + x_{12} + x_{13} &= 4,000 \\
& x_{21} + x_{22} + x_{23} &= 3,000 \\
& x_{31} + x_{32} + x_{33} &= 2,500 \\
& x_{41} + x_{42} + x_{43} &= 1,500 \\
& x_{11} + x_{21} + x_{31} + x_{41} &= 5,000 \\
& x_{12} + x_{22} + x_{32} + x_{42} &= 2,000 \\
& x_{13} + x_{23} + x_{33} + x_{43} &= 4,000 \\
& x_{11}, x_{12}, \ldots, x_{33}, x_{43} \geq 0
\end{aligned}
$$

Embedded assumptions in LP

1. Proportionality Assumption
   - No discount.
   - No economy of return to scale.
2. Additivity Assumption
   - Total contribution = Sum of contributions of individual variables
3. Divisibility Assumption
   - Any fractional value is allowed.
4. Certainty Assumption
   - Each parameter is known for sure.

Converting to standard form

- Example

$$\begin{array}{lrcrcrl}
\text{Maximize} & 3x_1 & - & 2x_2 & - & 4|x_3| & \\
\text{s. t.} & -x_1 & + & 2x_2 & & & \leq -5 \\
& & & 3x_2 & - & x_3 & \geq 6 \\
& 2x_1 & & & + & x_3 & = 12 \\
& x_1, & & x_2 & & \geq 0 &
\end{array}$$

Converting to standard form

- What went wrong?
- How to fix them? In which order?

$$
\begin{array}{llllllll}
\text{Maximize} & 3x_1 & - & 2x_2 & - & 4|x_3| & & \\
\text{s. t.} & -x_1 & + & 2x_2 & & & \leq & -5 \\
 & & & 3x_2 & - & x_3 & \geq & 6 \\
 & 2x_1 & & & + & x_3 & = & 12 \\
 & x_1, & & x_2 & & \geq 0 & &
\end{array}
$$

Rule 1

- Rule 1: Unrestricted (free) variables

$$x_i \in R$$

$$x_i^+ = \begin{cases} x_i, & \text{if } x_i \geq 0 \\ 0, & \text{otherwise} \end{cases}$$

$$\Updownarrow$$

$$x_i^- = \begin{cases} 0, & \text{if } x_i \geq 0 \\ -x_i, & \text{otherwise} \end{cases}$$

$$x_i = x_i^+ - x_i^-$$

$$x_i^+,\ x_i^- \geq 0$$

- By-product: $|x_i| = x_i^+ + x_i^-$
- Potential problem: the requirement of $x_i^+ \times x_i^- = 0$

Example

$$
\begin{array}{lcccccr}
\text{Maximize} & 3x_1 & - & 2x_2 & - & 4(x_3^+ + x_3^-) & \\
\text{s. t.} & -x_1 & + & 2x_2 & & & \leq -5 \\
& & & 3x_2 & - & (x_3^+ - x_3^-) & \geq 6 \\
& 2x_1 & & & + & (x_3^+ - x_3^-) & = 12 \\
& x_1, & & x_2, & & x_3^+,\ x_3^-, & \geq 0
\end{array}
$$

Rule 2

- Rule 2: Inequality constraints

- slack variable

$$\begin{aligned} a_{i1}x_1 + a_{i2}x_2 + \cdots + a_{in}x_n &\le b_i \\ \Updownarrow \text{ add a } \underline{\textit{slack}}\ \underline{\textit{variable}}\ s_i &\ge 0 \\ a_{i1}x_1 + a_{i2}x_2 + \cdots + a_{in}x_n + s_i &= b_i \\ s_i &\ge 0 \end{aligned}$$

- excess variable

$$\begin{aligned} a_{i1}x_1 + a_{i2}x_2 + \cdots + a_{in}x_n &\ge b_i \\ \Updownarrow \text{ subtract an } \underline{\textit{excess}}\ \underline{\textit{variable}}\ e_i &\ge 0 \\ a_{i1}x_1 + a_{i2}x_2 + \cdots + a_{in}x_n - e_i &= b_i \\ e_i &\ge 0 \end{aligned}$$

Example

Maximize

$$3x_1 - 2x_2 - 4x_3^+ - 4x_3^-$$

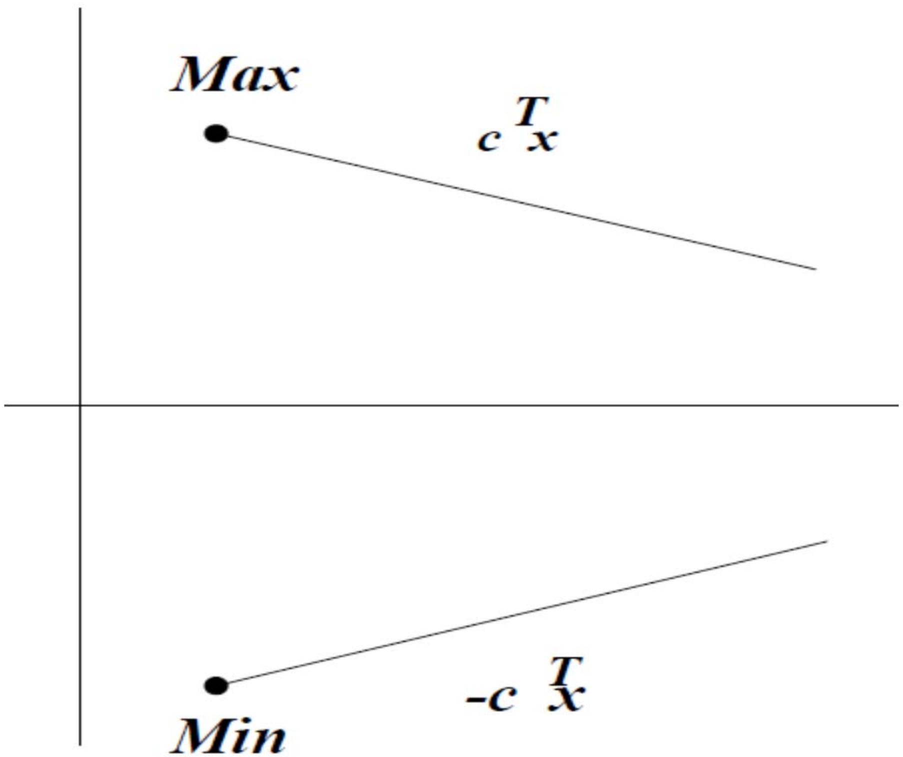
subject to

$$\begin{aligned}
-x_1 + 2x_2 \quad\quad\quad\quad + x_4 \quad\quad &= -5 \\
3x_2 - x_3^+ + x_3^- \quad\quad - x_5 &= 6 \\
2x_1 \quad\quad + x_3^+ - x_3^- \quad\quad\quad\quad &= 12 \\
x_1, \quad x_2, \quad x_3^+, \; x_3^- \quad x_4, \quad x_5 &\geq 0
\end{aligned}$$

Rule 3

- Rule 3: Minimization of the objective function

$$\text{Max } c_1x_1 + c_2x_2 + \cdots + c_nx_n$$



$$Max\ c^T x = -\,Min\ (-c^T x)$$

Example

$$
\begin{array}{llllll}
(-)\text{Minimize} & & & & & \\
& -3x_1 & +2x_2 & +4x_3^+ + 4x_3^- & & \\
\text{subject to} & & & & & \\
& -x_1 & +2x_2 & & +x_4 & & = -5 \\
& & 3x_2 & -x_3^+ + x_3^- & & -x_5 & = 6 \\
& 2x_1 & & +x_3^+ - x_3^- & & & = 12 \\
& x_1, & x_2, & x_3^+,\ x_3^- & x_4, & x_5 & \geq 0
\end{array}
$$

More on free variable and absolute value

- Potential problems:

$$x_i = x_i^+ - x_i^- \qquad |x_i| = x_i^+ + x_i^-$$

$$x_i^+,\ x_i^- \geq 0$$

1. one quadratic constraint is missing
2. increasing dimensionality
3. one original solution corresponds to many new solutions
4. |x| is a convex function while -|x| is a concave function
5. Maximize c|x| could be problematic with c being positive

Reference

'Linear' Programming with Absolute-Value Functionals

David F. Shanno; Roman L. Weil

Operations Research, Vol. 19, No. 1. (Jan. - Feb., 1971), pp. 120-124.

Consider the problem $Ax=b$; max $z=\sum c_j|x_j|$. This problem cannot, in general, be solved with the simplex method. The problem has a simplex-method solution (with unrestricted basis entry) only if c_j are nonpositive (nonnegative for minimizing problems).

Example: where multiple solutions occur

- Consider

$$\begin{aligned} \min \quad & z = x \\ s.t. \quad & x \geq -1 \\ & x \in \mathcal{R} \end{aligned}$$

$$x^* = -1$$

$$z^* = -1$$

- Unique optimum

Standard Form

$$\begin{aligned} \min \quad & x^+ - x^- \\ s.t. \quad & x^+ - x^- - s = -1 \\ & x^+, x^-, s \geq 0 \end{aligned}$$

$$\begin{cases} x^+ = 0 \\ x^- = 1 \end{cases} \qquad z^* = -1$$

$$\begin{cases} x^+ = 1 \\ x^- = 2 \end{cases} \qquad z^* = -1$$

$$\begin{cases} x^+ = t \\ x^- = 1 + t \end{cases} (t \geq 0) \qquad z^* = -1$$

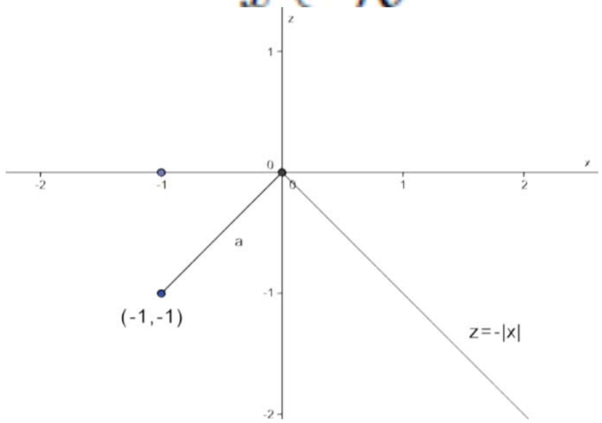
-

Multiple optimal solutions

Example: where simplex method fails

- Consider

$$\begin{aligned} \min \quad & z = -|x| \\ s.t. \quad & x \geq -1 \\ & x \in \mathcal{R} \end{aligned}$$



Standard Form

$$\begin{aligned} \min \quad & z = -x^+ - x^- \\ s.t. \quad & x^+ - x^- - s = -1 \\ & x^+, x^-, s \geq 0 \end{aligned}$$

$$\mathbf{x}^1 = \begin{pmatrix} x^+ \\ x^- \\ s \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \qquad z(\mathbf{x}^1) = -1$$

$$\mathbf{x}^2 = \begin{pmatrix} x^+ \\ x^- \\ s \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, \qquad z(\mathbf{x}^2) = 0$$

$$\mathbf{x} = \begin{pmatrix} t \\ 0 \\ t+1 \end{pmatrix} \text{ is feasible as } t \geq 0 \text{ with } z(\mathbf{x}) = -t.$$

Hence $z^* = -\infty$

- Start from x1 won't go to x2 that leads to x*