Integer Programming
ISE 418

Lecture 12

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Reading for This Lecture

- Nemhauser and Wolsey Sections II.2.1
- Wolsey Chapter 9
- CCZ Chapter 7

Generating Stronger Valid Inequalities

- We have now seen some "generic" methods of generating valid inequalities.
- In general, these methods are not capable of generating strong inequalities (facets).
- To generate such inequalities, we must use our knowledge of the problem structure.

The Strength of a Valid Inequality

- Roughly speaking, for an inequality to be strong, the face it defines should have as high a dimension as possible.
- The facet-defining inequalities are those of maximal dimension, i.e., dimension one less than the dimension of the polyhedron.
- The facet-defining inequalities dominate all others and are the only ones necessary in a complete description of a polyhedron.
- To know which inequalities are facets, we use the following result based on methods for determining the dimension of polyhedra.

Proposition 1. *If (π, π_0) defines a face of dimension $k-1$ of $\mathrm{conv}(S)$, then there are k affinely independent points $x^1, \ldots, x^k \in S$ such that $\pi x^i = \pi_0$ for $i = 1, \ldots, k$.*

Facet Proofs

- How do we prove an inequality is facet-defining?
- Straightforward approach: use the definition.
- First, we need to find the dimension d of the polyhedron.
- Then, we need to exhibit a set of d affinely independent points in $\mathcal{S}$ satisfying the given inequality at equality.
- Example:
  - Set $\mathcal{S} = \{(x, y) \in \mathbb{R}^m \times \mathbb{B} \mid \sum_{i=1}^m x_i \leq my, 0 \leq x \leq 1\}$.
  - We want to show that the valid inequality $x_i \leq y$ is facet-defining for $\mathrm{conv}(S)$.
  - First, we show that $dim(\mathrm{conv}(S)) = m + 1$ (how?).
  - For a chosen i, we exhibit $m+1$ affinely independent points in X that satisfy $x_i = y$ (how?).

Example: Valid Inequalities for Node Packing

- Recall the node packing problem. The set of node packings of a graph $G = (V, E)$ is given by

$$S = \{x \in \mathbb{B}^n \mid x_i + x_j \leq 1 \text{ for all } \{i, j\} \in E\}.$$

- We are interested in the polytope $\mathcal{P} = \mathrm{conv}(S)$.
- This polytope is easily shown to be full-dimensional (how?).
- What are some valid inequalities?

The Clique Inequalities

- When C is a clique in G, the *clique constraint*

$$\sum_{j \in C} x_j \leq 1$$

  is valid for $\mathrm{conv}(S)$.

- In fact, when C is maximal, this constraint is facet-defining for $\mathrm{conv}(S)$.
- How do we prove this?

Back to Separation and Optimization

- We have just seen an example of a class of inequalities of which we have explicit knowledge of an inequality that is facet-defining.
- Yet, we know that this problem is a difficult one to solve.
- Question: Can we efficiently generate such inequalities?
- Answer: Yes and no.
  - It is easy to generate *some* maximal cliques in a graph.
  - It may be difficult to generate one that corresponds to an inequality violated by a given (fractional) solution to the LP relaxation.
  - In general, there are no efficient exact separation algorithms for the convex hull of feasible solutions to a "difficult" MILP.
- Why is it difficult to generate facets of $\mathrm{conv}(\mathcal{S})$ in general?

More Valid Inequalities for Node Packing

- The clique constraints are not enough to completely describe the convex hull for all instances.
- What other inequalities can we find?
- An *odd hole* is a set of nodes that lie on a *chordless cycle* of the graph G.
- If $H \subseteq V$ is an odd hole, then the inequality

$$\sum_{j \in H} x_j \leq \frac{|H| - 1}{2}$$

  is valid for $\mathrm{conv}(S)$.
- This new inequality is easily shown to be facet-defining for the subgraph induced by H.
- But it is not facet-defining in general.
- Can we strengthen it?

Strengthening Valid Inequalities

- The problem seems to be that we are not taking into account the interaction with other nodes in the graph.
- Let's try to generate a valid inequality of the form

$$\alpha x_i + \sum_{j \in H} x_j \leq \frac{|H| - 1}{2}$$

where $i \notin H$.

- We want to make α as big as possible. How big can it be?

The Lifting Principle

- Suppose we have an inequality $\sum_{i=2}^{n} \pi_i x_i \leq \pi_0$ that is facet-defining for $\mathcal{P}_0 = \{x \in \mathcal{P} \mid x_1 = 0\}$ where $\mathcal{P} = \mathrm{conv}(S)$ and $\mathcal{S} \subseteq \mathbb{B}^n$.
- We want to generate π_1 so that $\sum_{i=1}^{n} \pi_i x_i \leq \pi_0$ will be a facet of $\mathcal{P}$.
- This means making the new inequality as strong as possible.
- Hence, we set $\pi_1 := \pi_0 - \xi$, where $\xi = \max\{\sum_{i=2}^{n} \pi_i x_i \mid x \in \mathcal{P}, x_1 = 1\}$.
- If there are no feasible solutions with $x_1 = 1$, then we can simply fix x_1 to zero.
- For BIPs, this guarantees that the new inequality will be valid for $\mathcal{P}$ and will define a face of dimension one higher than the original inequality.
- Note that the new inequality will be valid as long $\pi_1 \leq \pi_0 - \xi$

Projections and Restrictions

- We will define a *restriction* of $\mathcal{P}$ to be any polyhedron $\mathcal{Q}$ strictly contained in $\mathcal{P}$. $\mathcal{P}$ is also called a relaxation of $\mathcal{Q}$.
- If $\mathcal{P} = \{x \in \mathbb{R}^n \mid Ax \leq b\}$ and $\mathcal{Q}$ is a restriction of $\mathcal{P}$, then

$$\mathcal{Q} = \{x \in \mathcal{P} \mid Dx \leq d\}.$$

- It's important to understand the difference between a projection and a restriction.
  - $\mathcal{Q}_1 = \{(x, 0) \mid (x, y) \in \mathcal{P}\}$ is a projection of $\mathcal{P}$.
  - $\mathcal{Q}_2 = \{(x, y) \in \mathcal{P} \mid y = 0\}$ is a restriction of $\mathcal{P}$.
- $\mathcal{Q}_1$ and $\mathcal{Q}_2$ may or may not be the same polyhedron.

Lifting Inequalities Valid for a Restriction

- For our discussion here, we will only consider integer polytopes of the form $\mathcal{P}^I = \text{conv}(S)$, where $\mathcal{S} = \{x \in \mathbb{B}^n \mid Ax \leq b\}$.
- We will only consider restrictions of the form $\{x \in \mathcal{P}^I \mid x_j = 0 \text{ for } j \in N_0, x_j = 1 \text{ for } j \in N_1\}$ where $N_0, N_1 \subseteq \{1, \ldots, n\}$.
- The lifting principle allows us to do two things:
  - Transform inequalities that are valid for a restriction into inequalities that are valid for the original problem.
  - Transform inequalities that are strong for a restriction into inequalities that are strong for the original problem.
- In our example, the inequality was already valid for the original polyhedron and we wanted to strengthen it.
- It is not always the case that inequalities valid for a restriction are valid for the original polyhedron.

Determining Lifting Coefficients

- Suppose we have an inequality valid for a restriction defined by sets $N_0, N_1 \subseteq \{1, \ldots, n\}$.
- In the case where $N_0 = \{1\}$ and $N_1 = \emptyset$, we already have a procedure.
  - We choose π_1 such that $\pi_1 \leq \pi_0 - \xi$, where $\xi = \max\{\sum_{i=2}^n \pi_i x_i \mid x \in \mathcal{P}, x_1 = 1\}$.
  - If there are no feasible solutions with $x_1 = 1$, then we can simply fix x_1 to zero.
- How about the case where $N_0 = \emptyset$ and $N_1 = \{1\}$?
  - We chose π_1 such that $\pi_1 \geq \xi - \pi_0$, where $\xi = \max\{\sum_{i=2}^n \pi_i x_i \mid x \in \mathcal{P}, x_1 = 0\}$.
  - In this case, we must also add π_1 to the right hand side to obtain the inequality

$$\pi_1 x_1 + \sum_{i=2}^{n} \pi_i x_i \leq \pi_0 + \pi_1.$$

  - If there is no feasible solution with $x_1 = 0$, then we can fix x_1 to one.

Determining Multiple Lifting Coefficients (Sequentially)

- The same procedure can be used in cases where multiple variables are restricted.
- We simply determine one lifting coefficient at a time, as before.
- Note that the order matters.
- The earlier a variable is lifted in the sequence, the larger its coefficient will be.

Approximating Lifting Coefficients

- It is not always necessary or even possible to determine the best possible lifting coefficient.
- In general the problem of determining the best possible lifting coefficient is an optimization problem over a restricted polytope (usually NP-hard).
- In practice, lifting coefficients are often determined using heuristic algorithms that guarantee validity, but not strength.
- Note that generating approximate lifting coefficients destroys the property that the face defined by the inequality increase in dimension as it is lifted.

Determining Multiple Lifting Coefficients (Simultaneously)

- We can also determine multiple lifting coefficients simultaneously.
- Suppose the inequality $\sum_{j\in N\setminus(N_0\cup N_1)} \pi_j x_j$ is valid for the restriction $\{x \in \mathcal{P} \mid x_j = 0 \text{ for } j \in N_0, x_j = 1 \text{ for } j \in N_1\}$.
- We want to determining lifting coefficients π_i for $i \in N_0 \cup N_1 \subseteq \{1, \ldots, n\}$.
  - Choose M such that $M \leq \pi_0 - \xi$, where

$$\xi = \max \left\{ \sum_{\in N\setminus(N_0\cup N_1)} \pi_i x_i \;\middle|\; x \in \mathcal{P} \right\}.$$

  - Then the inequality

$$M \sum_{j\in N_0} x_j - M \sum_{j\in N_1} x_j + \sum_{j\in N\setminus(N_0\cup N_1)} \pi_j x_j \leq \pi_0 - M|N_1|$$

    is valid for $\mathcal{P}$.