Integer Programming
ISE 418

Lecture 9

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Reading for This Lecture

- Wolsey Sections 7.4-7.5
- Nemhauser and Wolsey Section II.4.2
- Linderoth and Savelsburgh, (1999)
- Martin (2001)
- Achterberg, Koch, Martin (2005)
- Karamanov and Cornuejols, Branching on General Disjunctions (2007)
- Achterberg, Conflict Analysis in Mixed Integer Programming (2007)

A Generic Branch-and-Bound Algorithm

1: Add root optimization problem $\mathcal{S}_0 := \mathcal{S}$ to a priority queue Q. Set global upper bound $U \leftarrow \infty$ and global lower bound $L \leftarrow -\infty$

2: **while** $U > L$ **do**

3: Remove the highest priority subproblem $\mathcal{S}_i$ from Q.

4: **Bound** $\mathcal{S}_i$ to obtain (updated) final upper bound $U(i)$ and (updated) final lower bound $L(i)$.

5: Set $L \leftarrow \max\{L(i), U\}$.

6: **if** $U(i) > L$ **then**

7: **Branch** to create child subproblems $\mathcal{S}_{i_1}, \ldots, \mathcal{S}_{i_k}$ of subproblem $\mathcal{S}_i$ with

- lower bounds $L(i_1), \ldots L(i_k)$ (initialized to $-\infty$ by default); and
- initial upper bounds $U(i_1), \ldots, U(i_k)$ (initialized to $U(i)$ by default).

by partitioning $\mathcal{S}_i$ (imposing a violated valid disjunction)

8: Add $\mathcal{S}_{i_1}, \ldots, \mathcal{S}_{i_k}$ to Q.

9: Set $U \leftarrow \max_{i \in Q} U(i)$.

10: **end if**

11: **end while**

Branching

- In the last lecture, we discussed basic methods for bounding.
- Obtaining tight bounds is the most important aspect of the branch-and-bound algorithm.
- Branching effectively is a very close second.
- Choosing an effective method of branching can make orders of magnitude difference in the size of the search tree and the solution time.

Disjunctions and Branching

- Recall that branching is generally achieved by selecting an admissible disjunction $\{X_i\}_{i=1}^k$ and using it to partition $\mathcal{S}$, e.g., $\mathcal{S}_i = \mathcal{S} \cap X_i$.
- The way this disjunction is selected is called the *branching method* and is the topic we now examine.
- Generally speaking, we want $x^* \notin \cup_{1 \le i \le k} X_i$, where x^* is the (infeasible) solution produced by solving the *bounding problem* associated with a given subproblem.

Split Disjunctions

- The most easily handled disjunctions are those based on dividing the feasible region using a given hyperplane.
- In such cases, each term of the disjunction can be imposed by addition of a single inequality.
- A hyperplane defined by a vector $\alpha \in \mathbb{R}^n$ is said to be *integer* if $\alpha_i \in \mathbb{Z}$ for $0 \leq i \leq p$ and $\alpha_i = 0$ for $p + 1 \leq i \leq n$.
- Note that if α is integer, then we have $\alpha^\top x \in \mathbb{Z}$ whenever $x \in \mathbb{Z}^p \times \mathbb{R}^{n-p}$.
- Then the disjunction defined by

$$X_1 = \{x \in \mathbb{R}^n \mid \alpha x \leq \beta\}, X_2 = \{x \in \mathbb{R}^n \mid \alpha x \geq \beta + 1\}, \qquad (1)$$

  is valid when $\beta \in \mathbb{Z}$.
- This is known as a *split disjunction*.

Variable Disjunctions

- The simplest split disjunction is to take $\alpha = e_i$ for $0 \leq i \leq p$, where e_i is the i^{th} unit vector.
- If we branch using such a disjunction, we simply say we are *branching on* x_j.
- For such a branching disjunction to be admissible, we should have $\beta < x_i^* < \beta + 1$.
- In the special case of a 0-1 IP, this dichotomy reduces to

$$x_j = 0 \text{ OR } x_j = 1$$

- In general IP, branching on a variable involves imposing new bound constraints in each one of the subproblems.
- This is easily handled implicitly in most cases.
- This is the most common method of branching.
- What are the benefits of such a scheme?

The Geometry of Branching

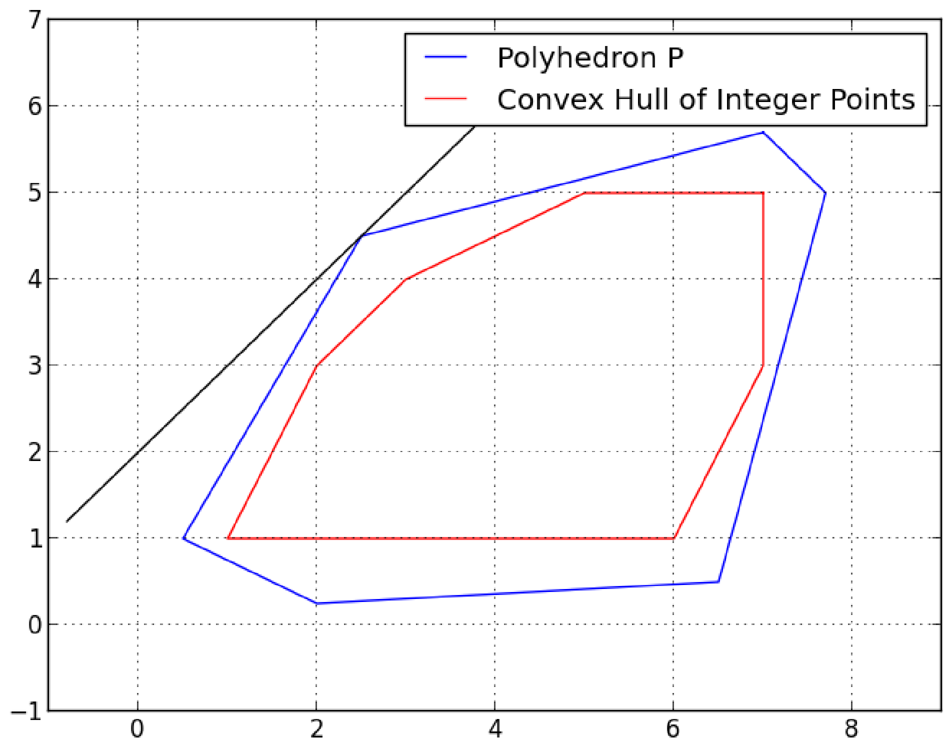


Figure 1: Feasible region of an MILP

The Geometry of Branching (Variable Disjunction)

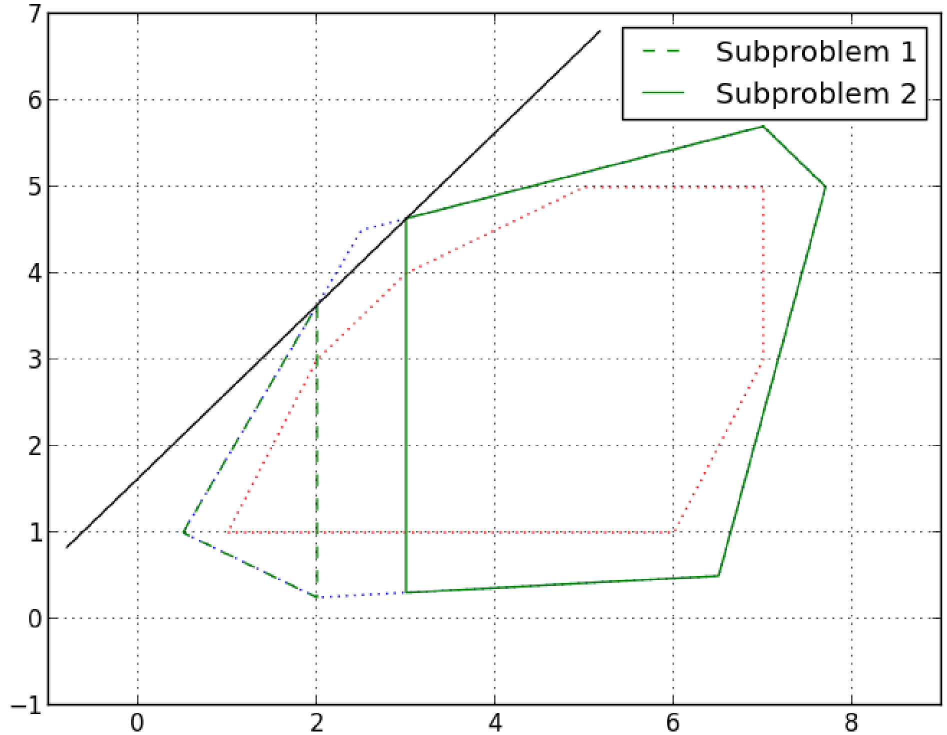


Figure 2: Branching on disjunction $x \leq 2$ OR $x \geq 3$

The Geometry of Branching (Variable Disjunction)

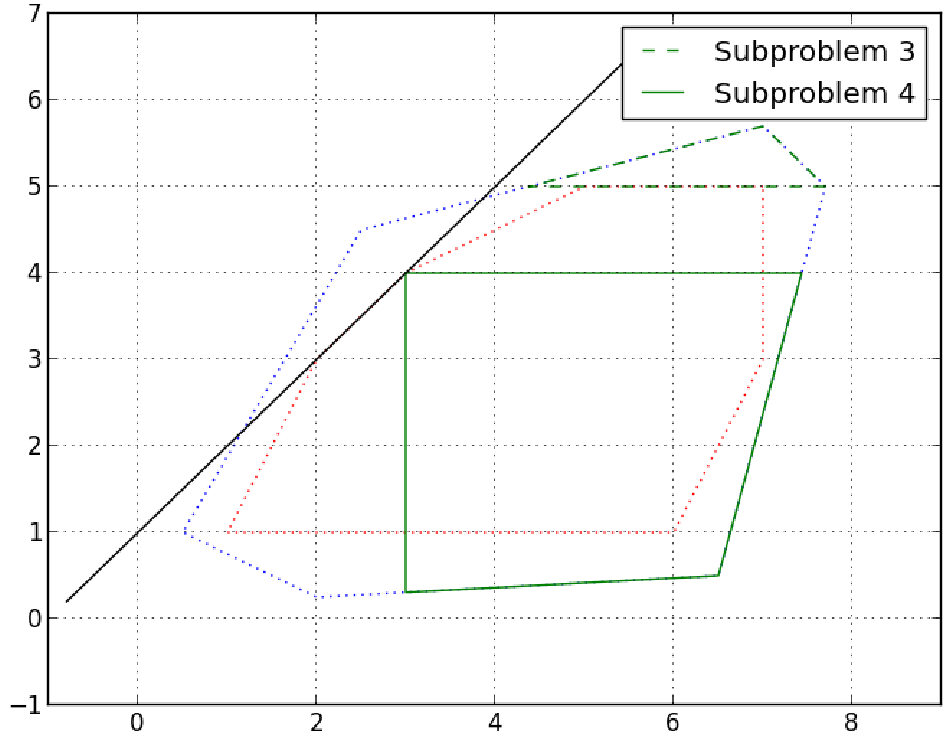


Figure 3: Branching on disjunction $y \leq 4$ OR $y \geq 5$ in Subproblem 2

The Geometry of Branching (General Split Disjunction)

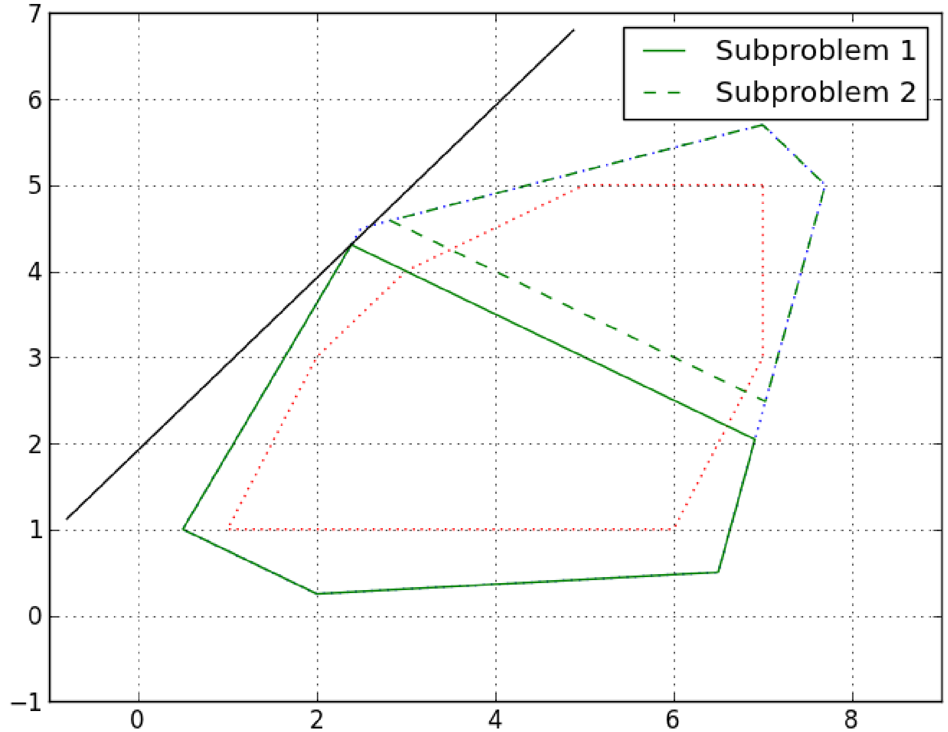


Figure 4: Branching on disjunction $x + 2y \leq 11$ OR $x + 2y \geq 12$

The Geometry of Branching (General Split Disjunction)

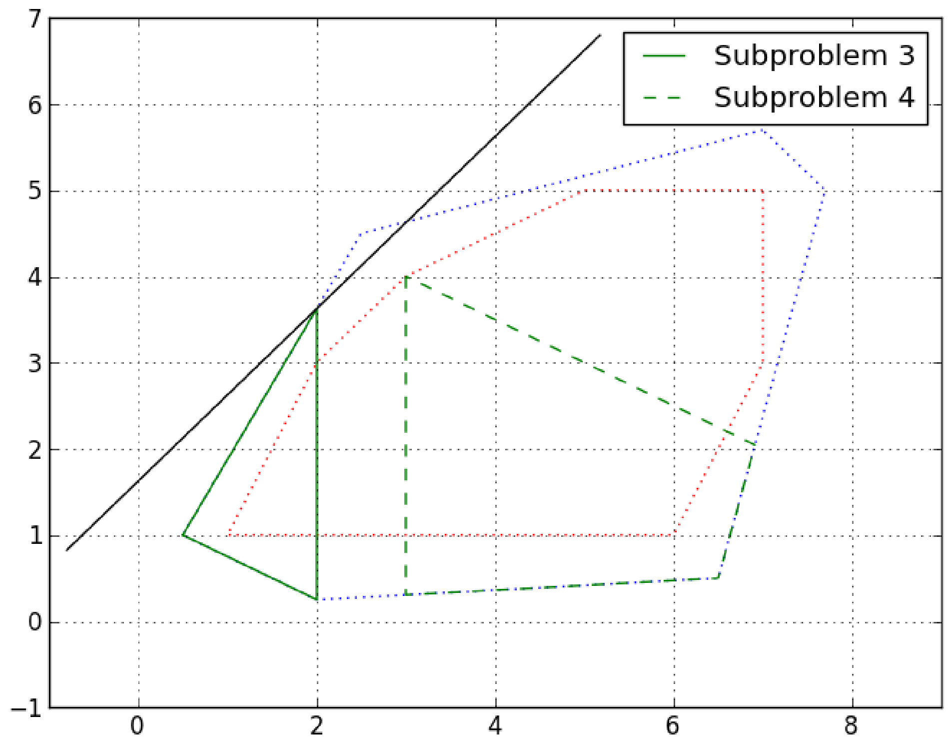


Figure 5: Branching on hyperplane $x \leq 2$ OR $x \geq 3$ in Subproblem 1

Other Disjunctions

- A *specially ordered set* (SOS) of Type I is a set Q of (binary) variables that must satisfy a constraint of the form:

$$\sum_{j \in Q} x_j = 1, \quad x \in \{0,1\}^Q$$

- Suppose $|Q| = 10$ and we branch on the disjunction $x_1 \leq 0 \text{ OR } x_1 \geq 1$.
- How many possible solutions to the above equation are there in each of the branches? Is this a "good" disjunction to branch on?
- Consider the disjunction $\sum_{j=1}^{5} x_j = 0 \text{ OR } \sum_{j=6}^{10} x_j = 0$.
- Is this better?
- There are also SOS Type II constraints in which two variables in a set may be nonzero.

Logical Disjunctions

- We can derive other types of branching based on logical considerations.
- Example:
  - y_i binary variable and $y_i = 0 \Rightarrow \pi x \leq \pi_0$.
  - Possible branching:

$$\begin{aligned} y_i &= 1, \\ y_i &= 0 \text{ and } \pi x \leq \pi_0. \end{aligned}$$

  - This avoids using the big M method.

Choosing a Branching Disjunction

- What is the real goal of branching?
- This may depend on the goal of the search
  - Find the best feasible solution possible in a limited time.
  - Find the provably optimal solution as quickly as possible.
- It is difficult to know how our branching decision will impact these goals, but we may want to choose a branching that
  - Decreases the upper bound,
  - Increases the lower bound, or
  - Result in one or more (nearly) infeasible subproblem.
- Most of the time, we focus on decreasing the upper bound.

Choosing a Branching Disjunction (cont'd)

- There are many possible disjunctions to choose from.
- We generally choose the branching disjunction based on the predicted amount of progress it will produce towards our goal.
- If the goal is to minimize time to optimality, bound improvement is often used as a proxy.
- How do we efficiently predict the bound improvement that will result from the imposition of a given disjunction?

Strong Branching

- *Strong branching* provides the most accurate estimate, but is computationally very expensive.
- The idea is to compute the *actual* change in bound by solving the bounding problems resulting from imposing the disjunction.
- This can be very costly. How can we moderate this?
  - Do only a limited number of dual-simplex pivots for each candidate for each child.
  - Use this as an estimate.
- Empirically, this reduces number of nodes, but this must be traded against the computational expense.

Pseudocost Branching

- An alternative to strong branching is *pseudocost branching*
- This is suitable primarily for branching on branching on variables.
- The pseudocost of a variable is an estimate derived by averaging observed changes resulting from branching on each of the variables.
- For each variable, we maintain an "up pseudocost" (P_j^+) and a "down pseudocost" (P_j^-).
- Then the change in bound for each child can be estimated as:

$$
\begin{aligned}
D_j^+ &= P_j^+(1 - f_j) \\
D_j^- &= P_j^- f_j,
\end{aligned}
$$

  where $f_j = x_j^* - \lfloor x_j^* \rfloor$.

- In other words, D_j^+ and D_j^- are estimates of the *change* in bound that will result from imposing $x_j \geq \lfloor x_j^* \rfloor$ and $x_j \geq \lceil x_j^* \rceil$, respectively.

Pseudocost Initialization

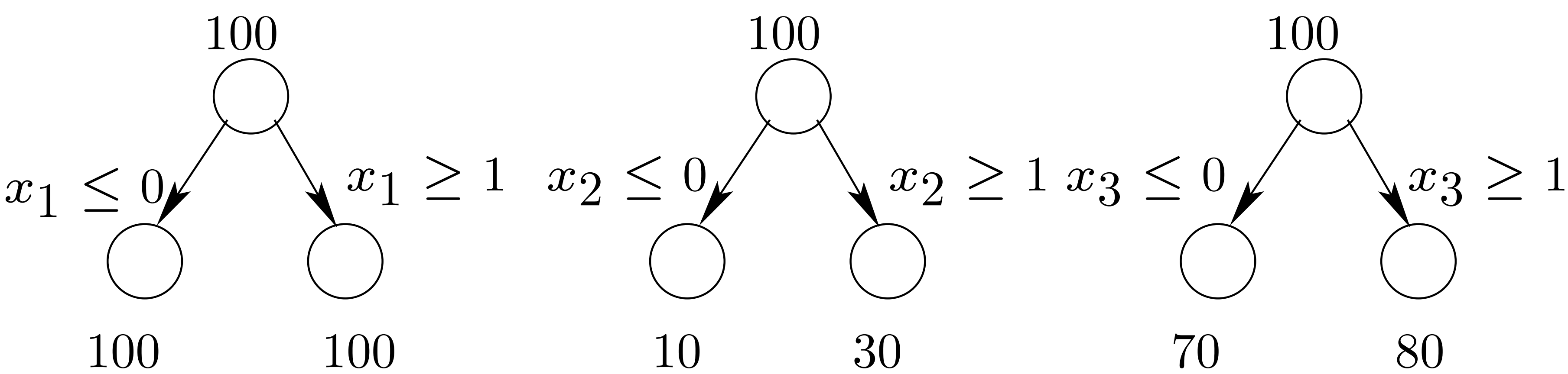
- Is it reasonable to assume that effect of branching on a particular variable is actually roughly the same in different parts of the tree?
- Empirical evidence shows that this is the case.
- Another important question is how to get initial estimates before any branching has occurred.
- This can be done initially using strong branching.
- After initialization, we switch to pseudocost branching, updating the pseudocost estimates after each bounding operation.
- A more systematic approach to doing this is to use what is called *reliability branching*.

Reliability Branching

- Strong branching is effective in reducing the number of nodes, but can be costly.
- Using pseudocosts is inexpensive, but requires good initialization.
- Reliability branching combines both.
  - Use strong branching in the early stages of the tree. Initialize/update pseudo-costs of variables using these bounds.
  - Once strong branching (or actual branching) has been carried out η number of times on a variable, only use pseudo-costs after that.
  - η is called reliability parameter.
  - What does $\eta = 0$ imply? What does $\eta = \infty$ imply?
  - Empirically $\eta = 4$ seems to be quite effective.

Comparing Branching Candidates

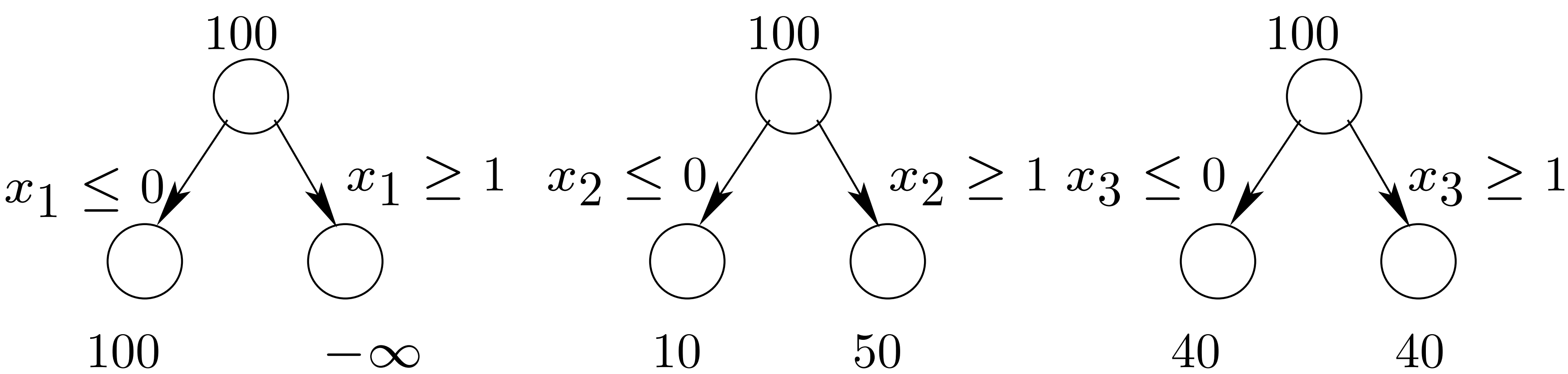
- So far we have seen, how to evaluate a candidate in several ways.
- Sometimes the choice of candidate is clear after this evaluation.



- Are we minimizing or maximizing?
- Which candidate would you choose?

Comparing Candidates

- However, choice of candidates is not always clear.
- Consider



- Possible metrics ($\tilde{z}_1, \tilde{z}_2, \ldots \tilde{z}_r$ are the estimates for r children of a candidate):
  - $\max \tilde{z}_i$
  - $\sum_i \tilde{z}_i / r$
  - $\max_i \tilde{z}_i - \min_i \tilde{z}_i$
  - $\alpha_1 \max_i \tilde{z}_i + \alpha_2 \min_i \tilde{z}_i$

Comparing Candidates

- The number of fractional variables (after full strong branching) is another possible criterion.
- For more criteria based on structure of constraints, see *Active-Constraint Variable Ordering for Faster Feasibility of MILPs*, by Patel and Chinneck, 2006.

Local Branching

- Local branching is a branching scheme that emphasizes finding feasible solutions over improving the upper bound.
- Consider the solution x^* to an LP relaxation at a certain node in the tree of a binary program.
- Let S be the set: $\{j | x_j^* = 0\}$.
- Consider the disjunction

$$\sum_{j \in S} x_j \leq k \text{ OR } \sum_{j \in S} x_j \geq k + 1$$

for small k.

- Is this a valid rule?
- Which child is easier to solve?
- For full details, see *Local Branching* by Fischetti and Lodi.
- We will discuss this and other methods when we talk about *primal heuristics*.

Valid Inequalities by Branching

- Note this one of the subproblems obtained by imposing a given binary disjunction is infeasible, then we obtain a valid inequality!
- This is in some sense what a valid inequality is.
- For the problem in Figure 1, branching on the valid disjunction $x_2 - x_1 \leq 1$ OR $x_2 - x_1 \geq 2$ immediately solves the problem.
- This may make it seem easy to find valid inequalities, but we will see later why this is not the case.