COMPACT KÄHLER 3-FOLDS WITH NEF ANTI-CANONICAL BUNDLE

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Abstract. In this paper, we prove that a non-projective compact Kähler 3-fold with nef anti-canonical bundle is, up to a finite étale cover, one of the following: a manifold with vanishing first Chern class; the product of a K3 surface and the projective line; the projective space bundle of a numerically flat vector bundle over a torus. This result extends Cao-Höring's structure theorem for projective manifolds to compact Kähler manifolds in dimension three. For the proof, we study the Minimal Model Program for compact Kähler 3-folds with nef anti-canonical bundle by using the positivity of direct image sheaves, $\mathbb{Q}$-conic bundles, and orbifold vector bundles.

Contents

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1. Introduction

1.1. **Background and the Main Results.** In this paper, we study a structure theorem for compact Kähler manifolds with nef anti-canonical bundle, motivated by the conjecture below. This conjecture is regarded as a natural generalization of the pioneering studies for nef tangent bundles [DPS94] and non-negative holomorphic bisectional curvatures [HSW81, Mok88]. The study of nef anti-canonical bundles has been developed in interaction with other studies of "non-negatively curved" varieties (e.g., see [Mat20, Mat22a, Mat22b, HIM22] for recent developments).

Conjecture 1.1. *Let X be a compact Kähler manifold with the nef anti-canonical bundle $-K_X$. Then, there exists a fibration $\varphi : X \to Y$ with the following*:

- *$\varphi : X \to Y$ is a locally constant fibration*;
- *Y is a compact Kähler manifold with $c_1(Y) = 0$*;
- *F, which is the fiber of $\varphi : X \to Y$, is rationally connected.*

The notion of locally constant fibrations (e.g., see [MW21, Definition 2.3]) is stronger than that of locally trivial fibrations, but the readers unfamiliar with locally constant fibrations may regard them as locally trivial fibrations except for Proposition 4.6.

Conjecture 1.1 had been settled by the theory of holonomy groups [CDP15, DPS96] under the stronger assumption that $-K_X$ is semi-positive (i.e., it has a smooth Hermitian metric with semi-positive curvature). Nevertheless, the nefness is much more difficult to handle than the semi-positivity, just as it was highly non-trivial to generalize the structure theorem from non-negative holomorphic bisectional curvatures to nef tangent bundles. A recent breakthrough [Cao19, CH19] solved the conjecture when X is a projective manifold (see [CCM21, MW21, Wan21] for projective klt pairs), but the proof needs ample line bundles on X; therefore we cannot at least directly apply this method to compact Kähler manifolds.

The purpose of this paper is to solve Conjecture 1.1 in the case of $\dim X = 3$ (see Theorem 1.2). It is sufficient for this purpose to consider non-projective Kähler 3-folds; hence, Theorem 1.2 is a direct consequence of Theorem 1.3 by Proposition 4.6. Therefore, in this paper, we focus on proving Theorem 1.3.

Theorem 1.2. *Conjecture 1.1 is true in the case of* $\dim X = 3$.

Theorem 1.3. *Let X be a non-projective compact Kähler 3-fold with nef anti-canonical bundle. Then X admits a finite étale cover that is one of the following*:

- *a compact Kähler manifold with vanishing first Chern class*;
- *the product of a K3 surface and the projective line* $\mathbb{P}^1$;
- *the projective space bundle* $\mathbb{P}(E)$ *of a numerical flat vector bundle* E *of rank* 2 *over a* 2*-dimensional* (*compact complex*) *torus.*

1.2. **Strategy of the proof of Theorem 1.3.** In this subsection, we outline the proof of Theorem 1.3. Let X be a non-projective compact Kähler 3-fold and let $\varphi : X \dashrightarrow R(X)$ be an MRC (maximally rationally connected) fibration of X (see [KoMM92, Cam92] for MRC fibrations). We will show that MRC fibrations of X determine the candidates of X in Theorem 1.3.

First, we prove that it is sufficient to consider the case of $\dim R(X) = 2$. Indeed, in the case of $\dim R(X) = 0$, the manifold X is rational connected, and hence projective. In the case of $\dim R(X) = 1$, a general fiber F of $\varphi : X \dashrightarrow R(X)$ is rationally connected, and hence has no (non-zero) holomorphic differential forms; therefore we have $h^2(X, \mathcal{O}_X) = h^0(X, \Omega_X^2) = 0$ by $\dim R(X) = 1$, which implies that X is projective. In the case of $\dim R(X) = 3$, the manifold X is non-uniruled; hence K_X is pseudo-effective, which follows from [BDPP13] for projective manifolds of any dimension and from [Bru06] for compact Kähler manifolds of dimension ≤ 3. This implies that $c_1(X) = c_1(K_X) = 0$ since $-K_X$ is nef.

Next, we recall Cao-Höring's argument [CH19] showing that X admits a (holomorphic and) locally constant fibration when X is projective. For simplicity, we suppose that $\varphi : X \dashrightarrow R(X)$ is a holomorphic map (everywhere defined on X) onto a smooth projective variety $R(X)$. The core of the proof is to construct a φ-ample line bundle B on X such that the direct image sheaf $\varphi_*(pB) := \varphi_*\mathcal{O}_X(pB)$ is weakly positively curved and satisfies that $c_1(\varphi_*(pB)) = 0$ for $1 \ll p \in \mathbb{Z}$ (see Subsections 2.2 and 2.3 for details). Hence, Simpson's result [Sim92] shows that $\varphi_*(pB)$ admits a flat connection, which implies that $X \to R(X)$ is a locally constant fibration.

Let us go back to the case where X is non-projective. In this case, even if $\varphi : X \dashrightarrow R(X)$ is holomorphic, the manifold X may not have even φ-ample line bundles, which causes an obvious problem. Our idea to solve this problem is to apply the MMP (Minimal Model Program) for compact Kähler 3-folds developed in [HP15a, HP15b, HP16]. By running the MMP, we can find $X \dashrightarrow X' \to Z$ as an outcome of the MMP, where $X \dashrightarrow X'$ is a bimeromorphic map consisting of divisorial contractions and flips and $\varphi : X' \to S$ is a MF (Mori fiber) space (see Theorem 3.6 for details). Note that S is a surface by $\dim R(X) = 2$. An advantage of running the MMP is that $-K_{X'}$ is φ-ample by construction; thus, we can expect that Cao-Höring's argument works for $\varphi : X' \to S$.

In considering $\varphi : X' \to S$ instead of $\varphi : X \dashrightarrow R(X)$, we face the new difficulties compared to Cao-Höring's argument: The first difficulty is that $-K_{X'}$ is not necessarily nef although $-K_X$ is nef. To solve this difficulty, based on the observation in [EIM], we focus on the fact that the non-nef locus of $-K_{X'}$ is not dominant over S (see Subsection 3.3), which enables us to treat our situation as in the case where $-K_{X'}$ is nef. Thus, we can construct a φ-ample line bundle B on X' such that $\varphi_*(pB)$ is weakly positively curved and satisfies that $c_1(\varphi_*(pB)) = 0$. The second difficulty is that S may have singularities, which prevents us to use Simpson's result to obtain a flat connection on $\varphi_*(pB)$. To solve this difficulty, we observe that $\varphi : X' \to S$ is a toroidal $\mathbb{Q}$-conic bundle and $\varphi_*(pB)$ is an orbifold vector bundle on S, which enables us to obtain flat connections. For this observation, we study $\mathbb{Q}$-conic bundles and conic bundles in the non-projective setting (see Subsections 3.1 and 3.2).

We consider the case where X is simply connected in Subsection 4.3. In this case, using the theory of orbifold vector bundles, we show that $\varphi_*(pB)$ is a trivial vector bundle, which implies that $\varphi : X' \to S$ gives the product structure of X' and $X \dashrightarrow X'$ is isomorphic.

We finally consider the remaining case where X is not simply connected in Subsection 4.2. In this case, we focus on the Albanese map $\alpha : X \to A(X)$ after taking a finite étale cover of X. Each step in the MMP contracts rational curves and $A(X)$ has no rational curve; hence we can find a morphism $\beta : S \to A(X)$. Then, comparing S to $A(X)$ by $\beta : S \to A(X)$, we show that the MF space $\varphi : X' \to S$ is actually a conic bundle. Then, by studying conic bundles in the non-projective setting, we deduce that $\varphi : X' \to S$ is a projective space bundle and $X \dashrightarrow X'$ is isomorphic.

Notation and Conventions. We interchangeably use the terms "Cartier divisors," "invertible sheaves," and "line bundles," and use the additive notation for tensor products (e.g., $L+M := L\otimes M$ for line bundles L and M). Furthermore, we interchangeably use the terms "locally free sheaves" and "vector bundles," and often simply abbreviate singular Hermitian metrics to "metrics." The term of "fibrations" denotes a proper surjective morphism with connected fibers, the term of "analytic varieties" denotes an irreducible and reduced complex analytic space, the term of "Kähler spaces" denotes a normal analytic variety admitting a Kähler form (i.e., a smooth positive $(1, 1)$-form on X with local potential).

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2. Preliminary Results

2.1. **Bott-Chern cohomology groups on normal analytic varieties.** In this subsection, following [BEG13], we review Bott-Chern cohomology groups and positive currents on normal analytic varieties.

Let X be a normal analytic variety. A pluriharmonic function on X can be locally written as the real part of a holomorphic function, that is, the kernel of the $\partial\overline{\partial}$-operator on the sheaf of distributions of bidegree $(0,0)$ coincides with the sheaf $\mathbb{R}\mathcal{O}_X$ of real parts of holomorphic functions (e.g., see [BEG13, Lemma 4.6.1]). Then, the Bott-Chern cohomology group of X is defined by

$$H^{1,1}_{BC}(X, \mathbb{C}) := H^1(X, \mathbb{R}\mathcal{O}_X).$$

The first Chern class $c_1(L) \in H^{1,1}_{BC}(X, \mathbb{C})$ of a line bundle L on X is defined by the Bott-Chern cohomology class of $(\sqrt{-1}/2\pi)\Theta_h(L)$, where $\Theta_h(L)$ denotes the Chern curvature of a smooth metric h on L (e.g., which is constructed by a partition of unity). Note that $c_1(L)$ does not depend on the choice of smooth metrics and the first Chern class of $\mathbb{Q}$-Cartier divisors can be also defined by linearity.

The proposition below, which is often used, is an extension theorem for positive currents representing Bott-Chern cohomology classes (see [Dem85] for currents on analytic varieties). The lemma below is a generalization of the support theorem to analytic varieties.

Proposition 2.1 ([BEG13, Proposition 4.6.3])**.** *Let $\alpha \in H^{1,1}_{BC}(X, \mathbb{C})$ be a Bott-Chern cohomology class on a normal analytic variety X, and let T be a positive current on X_{reg} representing the restriction $\alpha|_{X_{\mathrm{reg}}} \in H^{1,1}_{BC}(X_{\mathrm{reg}}, \mathbb{C})$. Then, the current T is uniquely extended to the positive current with local potential on X representing $\alpha \in H^{1,1}_{BC}(X, \mathbb{C})$.*

Lemma 2.2. *Let X be an analytic variety, and let T_1, T_2 be d-closed positive currents of bidimension (p,p) (without assuming that they admit local potentials). If the support of the difference $T := T_1 - T_2$ is contained in a Zariski closed subset $A \subset X$ of dimension $< p$, then we have $T = 0$.*

Proof. The statement is local in X; therefore we may assume that there exists an embedding $i : X \to B \subset \mathbb{C}^N$ of X into an open set $B \subset \mathbb{C}^N$. Since the pushforward i_*T is a normal current, the support theorem for smooth varieties shows that $i_*T = 0$, which implies that $T = 0$. □

2.2. **Positivity of sheaves on normal analytic varieties.** In this subsection, following [HPS18, PT18, Mat22a], we shortly review singular Hermitian metrics on torsion-free sheaves on normal analytic varieties.

Let $\mathcal{E}$ be a torsion-free coherent sheaf on a normal analytic variety X. *A singular Hermitian metric* h on $\mathcal{E}$ is a possibly singular Hermitian metric on the vector bundle

$\mathcal{E}|_{X_0}$ (see [HPS18, PT18] for metrics on vector bundles). Here $\mathcal{E}|_{X_0}$ is the restriction of $\mathcal{E}$ to $X_0 := X_{\mathrm{reg}} \cap X_{\mathcal{E}}$, where X_{reg} is the non-singular locus of X and $X_{\mathcal{E}}$ is the maximally locally free locus of $\mathcal{E}$. Note that $X_0 \subset X$ is a Zariski open set with $\operatorname{codim}(X \setminus X_0) \geq 2$. For a smooth $(1,1)$-form θ on X with local potential, we write as

$$\sqrt{-1}\Theta_h \geq \theta \otimes \mathrm{id} \text{ on } X$$

if the function $\log |e|_{h^*} - f$ is psh for any local section e of $\mathcal{E}^*$, where f is a local potential of θ (i.e., $\theta = \sqrt{-1}\partial\overline{\partial} f$) and h^* is the induced metric on the dual sheaf $\mathcal{E}^* := \mathcal{H}om(\mathcal{E}, \mathcal{O}_X)$. The plurisubharmonicity can be extended through a Zariski closed set of codimension ≥ 2; therefore it is sufficient to check that $\log |e|_{h^*} - f$ is a psh function on an open set of X_0 by $\operatorname{codim}(X \setminus X_0) \geq 2$.

Definition 2.3. Let X be a Kähler space, ω_X be a Kähler form on X, and θ be a $(1,1)$-form on X with local potential. A torsion-free sheaf $\mathcal{E}$ on X is said to be *θ-weakly positively curved* if there exist singular Hermitian metrics $\{h_\varepsilon\}_{\varepsilon>0}$ on $\mathcal{E}$ such that $\sqrt{-1}\Theta_{h_\varepsilon} \geq (\theta - \varepsilon\omega_X) \otimes \mathrm{id}$ on X. We simply say that $\mathcal{E}$ is weakly positively curved in the case of $\theta = 0$.

When X is compact, the notion of θ-weakly positively curved sheaves does not depend on the choice of ω_X and is stronger than pseudo-effective sheaves in the sense of [Mat, Definition 2.1].

2.3. **On direct image sheaves for projective morphisms.** This subsection aims to prove Theorem 2.6. For this purpose, we prepare the following proposition:

Proposition 2.4 (cf. [CH19, 2.8 Proposition], [CCM21, Theorem 2.2 (1)])**.** *Let $\varphi : X \to Y$ be a fibration between (not necessarily compact) Kähler manifolds X and Y with the Kähler forms ω_X and ω_Y. Let L be a line bundle on X and θ be a d-closed $(1,1)$-form on Y. Assume the following conditions:*

(a) *The non-nef locus of $-K_{X/Y}$ is not dominant over Y in the following sense: $-K_{X/Y}$ has singular metrics $\{g_\delta\}_{\delta>0}$ such that $\sqrt{-1}\Theta_{g_\delta} \geq -\delta\omega_X$ holds on X and the upper-level set $\{x \in X \mid \nu(g_\delta, x) > 0\}$ of Lelong numbers is not dominant over Y, where $\nu(g_\delta, x)$ is the Lelong number of a local potential of g_δ at x;*

(b) *L is a φ-big line bundle in the following sense: L has a singular Hermitian metric g such that $\sqrt{-1}\Theta_g + \varphi^*\omega_Y \geq \omega_X$ holds on X;*

(c) *L is $\varphi^*\theta$-weakly positive in the following sense: L has singular metrics $\{h_{\delta'}\}_{\delta'>0}$ such that $\sqrt{-1}\Theta_{h_{\delta'}} \geq \varphi^*\theta - \delta'\omega_X$ on X.*

Then, we have:

(1) *The direct image sheaf $\varphi_*(-mK_{X/Y} + L)$ is $((1-\varepsilon)\theta - \varepsilon\omega_Y)$-positively curved for any $m \in \mathbb{Z}_+$ and $\varepsilon > 0$.*

(2) *If we further assume that $\omega_Y \geq \theta$ holds, then $\varphi_*(-mK_{X/Y} + L)$ is θ-weakly positively curved. In particular, if L is a pseudo-effective line bundle, then $\varphi_*(-mK_{X/Y} + L)$ is weakly positively curved.*

Remark 2.5. For a line bundle M on X, we use the notation $\varphi_*(M)$ to denote the direct image sheaf $\varphi_*(\mathcal{O}_X(M))$ of the invertible sheaf $\mathcal{O}_X(M)$ for simplicity of the notation.

Proof. We construct the desired singular Hermitian metrics on $\mathcal{W}_m := \varphi_*(-mK_{X/Y}+L)$ by applying the theory of positivity of direct image sheaves [PT18, HPS18]. Note that the results in [PT18, HPS18] are stated for projective fibrations, but in fact they are valid for Kähler fibrations [Wan21]. The sheaf $\mathcal{W}_m$ can be regarded as the direct image sheaf of the pluri-adjoint bundle

$$-mK_{X/Y} + L = kK_{X/Y} \overbrace{-(m+k)K_{X/Y}}^{\text{with } g_\delta^{m+k}} + \overbrace{L}^{\text{with } g^\varepsilon \cdot h_{\delta'}^{1-\varepsilon}}.$$

Let us consider the curvature current and multiplier ideal sheaf associated to the metric

$$G := g_\delta^{m+k} \cdot g^\varepsilon \cdot h_{\delta'}^{1-\varepsilon} \text{ on } -(m+k)K_{X/Y} + L.$$

We can easily confirm that

$$\mathcal{I}(G^{1/k})|_{X_y} = \mathcal{I}\big(g_\delta^{m/k+1} \cdot g^{\varepsilon/k} \cdot h_{\delta'}^{(1-\varepsilon)/k}\big)|_{X_y} = \mathcal{O}_{X_y} \tag{2.1}$$

holds for a very general fiber X_y and a sufficiently large $k \gg 1$ (which depends on δ', but not depend on ε and δ). Indeed, by Condition (a), we have that $\nu(g_\delta, x) = 0$ for any $x \in X_y$ since X_y is a very general fiber. Hence, we obtain

$$\nu(g_\delta^{m/k+1} \cdot g^{\varepsilon/k} \cdot h^{(1-\varepsilon)/k}, x) \leq \nu(g^{1/k} \cdot h_{\delta'}^{1/k}, x) < 1$$

for $k \gg 1$. Then, Skoda's lemma shows that $\mathcal{I}(G^{1/k})|_{X_y} = \mathcal{O}_{X_y}$; hence the natural inclusion

$$\varphi_*((-mK_{X/Y} + L) \otimes \mathcal{I}(G^{1/k})) \to \varphi_*(-mK_{X/Y} + L) \tag{2.2}$$

is generically surjective. By the construction of metrics, we can easily see that

$$\begin{aligned}\sqrt{-1}\Theta_G &\geq -\delta(m+k)\omega_X + \varepsilon\omega_X - \varepsilon\varphi^*\omega_Y + (1-\varepsilon)\varphi^*\theta - (1-\varepsilon)\delta'\omega_X \\ &\geq \big(\varepsilon - \delta(m+k) - \delta'\big)\omega_X - \varepsilon\varphi^*\omega_Y + (1-\varepsilon)\varphi^*\theta.\end{aligned}$$

For a given $\varepsilon > 0$, after taking $\delta' > 0$ with $\delta' < (1/2)\varepsilon$, we fix a sufficiently large k satisfying (2.1). Furthermore, we take $1 \gg \delta > 0$ so that $\delta(m+k) < (1/2)\varepsilon$. Then, the right-hand side is bounded below by $\varphi^*(-\varepsilon\omega_Y + (1-\varepsilon)\theta)$. By (2.2) and the above curvature estimate, [PT18, HPS18] asserts that $\varphi_*(-mK_{X/Y}+L)$ has the desired singular metrics in Conclusion (1). Conclusion (2) directly follows from $-\varepsilon\omega_Y + (1-\varepsilon)\theta \geq -2\varepsilon\omega_Y + \theta$. □

Theorem 2.6. *Let $\varphi : X \to Y$ be an equi-dimensional fibration between compact Kähler spaces X and Y with Kähler forms ω_X and ω_Y. Let $Y_0 \subset Y$ be a Zariski open set with* $\operatorname{codim}(Y \setminus Y_0) \geq 2$ *such that $X_0 := \varphi^{-1}(Y_0)$ and Y_0 are smooth and that $\varphi_0 := \varphi|_{X_0} : X_0 \to Y_0$ is a smooth fibration. Let L be a line bundle on X. Assume the following conditions*:

(a) *$-K_X$ is $\mathbb{Q}$-Cartier and the non-nef locus of $-K_X$ is not dominant over Y in the sense of Proposition 2.4 (a)*;

(b) *$-K_Y$ is $\mathbb{Q}$-Cartier and numerically trivial*;

(c) *L is a pseudo-effective and φ-ample line bundle on X*;

(d) *For any $p \in \mathbb{Z}_+$ with $\varphi_*(pL) \neq 0$, the line bundle* $\det(\varphi_*(pL))|_{Y_0}$ *has a smooth metric g_p such that $\eta_p := \sqrt{-1}\Theta_{g_p} \geq -\omega_Y$ holds on Y_0.*

Let r be the rank of $\varphi_*(L)$ and p be a sufficiently large integer with $p/r \in \mathbb{Z}_+$. Define the sheaf $\mathcal{V}_p$ on Y by

$$\mathcal{V}_p := \varphi_*(pL) \otimes \Big(\frac{p}{r} \det \varphi_*(L)\Big)^*.$$

Then, both $\mathcal{V}_p$ and $(\det \mathcal{V}_p)^*$ are weakly positively curved.

Remark 2.7. The determinant sheaf $\det \varphi_*(L) := (\Lambda^r \varphi_*(L))^{**}$ is a reflexive sheaf of rank 1, but not necessarily invertible when Y has singularities; therefore, the notation $(p/r) \det \varphi_*(L)$ should be replaced by $((\det \varphi_*(L))^{\otimes (p/r)})^{**}$. Nevertheless, we mainly handle only the restriction of $\det \varphi_*(L)$ to Y_0, which is a line bundle on Y_0; hence this notation does not cause confusion.

In applying this theorem, the sheaf $\varphi_*(pL)$ is an orbifold vector bundle; hence Condition (d), which looks a technical assumption, is automatically satisfied.

Proof. We will attempt to use the argument in [Cao19, CH19] in terms of the φ-ample line bundle L instead of ample line bundles, which is the basic strategy of the proof. We assume that Y_0 is smooth and $\varphi_*(pL)$ is locally free on Y_0 by removing the singular locus $Y_{\rm sing}$ and the non locally free locus of $\varphi_*(pL)$ from Y_0.

Claim 2.8. *$\varphi_*(pL)$ is weakly positively curved on Y for any $p \in \mathbb{Z}_+$.*

Proof. We can construct singular metrics $\{H_\varepsilon\}_{\varepsilon>0}$ on $\varphi_*(pL)|_{Y_0}$ such that $\sqrt{-1}\Theta_{H_\varepsilon} \geq -\varepsilon\omega_Y \otimes \mathrm{id}$ holds on Y_0 by applying Proposition 2.4 to $\varphi_0 = \varphi|_{X_0} : X_0 \to Y_0$. Indeed, the assumptions of Proposition 2.4 for $\theta = 0$ are satisfied from Conditions (a), (b), (c) of Theorem 2.6. Hence, by Conclusion (2) of Proposition 2.4, we obtain the desired singular metrics $\{H_\varepsilon\}_{\varepsilon>0}$ on $\varphi_*(pL)|_{Y_0}$. Then, by $\operatorname{codim}(Y \setminus Y_0) \geq 2$, the metrics H_ε can be automatically extended to Y, where we implicitly used that ω_Y is a Kähler form defined on Y (not only on Y_0). This shows that $\varphi_*(pL)$ is weakly positively curved on Y. □

Claim 2.9. *Let r_p be the rank of $\varphi_*(pL)$. Then, the sheaf*

$$r_p pL \otimes \big(\varphi^* \det \varphi_*(pL)\big)^*$$

is weakly positively curved on X.

Proof. The basic strategy is the same as in [Cao19, Proposition 3.15], but we should make the discussion without using ample line bundles unlike [Cao19, Proposition 3.15]. For simplicity of the notation, we assume that $p = 1$ and $\eta_1 := \sqrt{-1}\Theta_{g_1} \geq -(1/r)\omega_Y$ by replacing L with pL and ω_Y with $(r+1)\omega_Y$, where $r := r_1$. Furthermore, we assume that $\omega_X \geq \varphi^*\omega_Y$. Let Z be the r-times fiber product $X \times_Y X \times_Y \cdots \times_Y X$ with the i-th projection $\mathrm{pr}_i : Z \to X$ and the natural morphism $\psi : Z \to Y$:

$$\begin{tikzcd} Z \arrow[r, "\mathrm{pr}_j"] \arrow[d, "\mathrm{pr}_i"'] \arrow[dr, "\psi"] & X \arrow[d, "\varphi"] \\ X \arrow[r, "\varphi"'] & Y. \end{tikzcd}$$

Set

$$L_r := \sum_{i=1}^{r} \mathrm{pr}_i^* L \text{ and } L' := L_r \otimes (\psi^* \det \varphi_*(L))^*.$$

To apply Proposition 2.4 to

$$\psi_0 = \psi|_{Z_0} : Z_0 := \psi^{-1}(Y_0) \to Y_0 \text{ equipped with } L'|_{Z_0} \text{ and } \theta := \eta_1,$$

we will examine the non-nef locus of $-K_{Z/Y}$ and metrics on L'.

By Conditions (a), (b), we obtain singular metrics $\{g_\delta\}_{\delta>0}$ on $-K_X|_{X_0} = -K_{X_0}$ such that $\sqrt{-1}\Theta_{g_\delta} \geq -\delta\omega_X$ holds on X_0 and the upper-level set $\{x \in X_0 \,|\, \nu(g_\delta, x) > 0\}$ is not dominant over Y_0. Note that

$$K_{Z_0} = \sum_{i=1}^{r} \mathrm{pr}_i^* K_{X_0} \text{ on } Z_0$$

since $\psi : Z \to Y$ be a smooth fibration over Y_0. Let us consider the metric $G_\delta := \sum_{i=1}^{r} \mathrm{pr}_i^* g_\delta$ on $-K_{Z_0/Y_0}$ defined by the pull-back. By construction, the upper-level set $\{x \in Z_0 \,|\, \nu(G_\delta, x) > 0\}$ is not dominant over Y_0 and the curvature current satisfies that

$$\sqrt{-1}\Theta_{g_\delta} \geq -\delta \sum_{i=1}^{r} \mathrm{pr}_i^* \omega_X \text{ on } Z_0.$$

Hence $-K_{Z_0/Y_0}$ satisfies Condition (a) of Proposition 2.4 for the Kähler form $\sum_{i=1}^{r} \mathrm{pr}_i^* \omega_X$ on Z_0. By Condition (c) of Theorem 2.6, we obtain a smooth metric g on L such that

$\sqrt{-1}\Theta_g + \varphi^*\omega_Y \geq \omega_X$ holds on X. Let us consider the smooth metric

$$G := (\sum_{i=1}^{r} \mathrm{pr}_i^* g) \cdot (\psi^* g_1)^{-1} \text{ on } L' = (\sum_{i=1}^{r} \mathrm{pr}_i^* L) \otimes (\psi^* \det \varphi_*(L))^*,$$

where g_1 is a smooth metric on $\det \varphi_* L|_{Y_0}$ in Condition (d). Then, we obtain that

$$\sqrt{-1}\Theta_G(L') + \sum_{i=1}^{r} \mathrm{pr}_i^* \varphi^* \big(\omega_Y + \frac{1}{r}\eta_1\big) \geq \sum_{i=1}^{r} \mathrm{pr}_i^* \omega_X \text{ on } Z_0.$$

Here we used $\psi^* = \mathrm{pr}_i^* \varphi^*$ for any $1 \leq i \leq r$. Since $\omega_Y + (1/r)\eta_1$ is a Kähler form on Y_0, the line bundle $L'|_{Z_0}$ satisfies Condition (b) of Proposition 2.4. On the other hand, there exists the (non-zero) natural morpshim

$$\det \varphi_*(L) \to (\varphi_*(L))^{\otimes r} \cong \psi_*(L_r) \text{ on } Y_0,$$

which shows that $h^0(Z_0, L') \neq 0$ by the definition of L'. In particular, the line bundle $L'|_{Z_0}$ satisfies Condition (c) of Proposition 2.4 for $\theta = 0$ (and $\delta' = 0$). The above arguments enable us to apply Proposition 2.4, and then we obtain singular metrics $\{H_\varepsilon\}_{\varepsilon>0}$ on $\psi_*(L')|_{Y_0}$ such that $\sqrt{-1}\Theta_{H_\varepsilon} \geq -\varepsilon\omega_Y \otimes \mathrm{id}$ on Y_0.

We finally prove the desired conclusion using the metrics on L' induced by H_ε. Let us consider the natural morphism

$$\psi^*\psi_*(L') \to L' \text{ on } Z_0,$$

which is generically surjective by $h^0(Z_0, L') \neq 0$. Let G_ε be the metric on $L'|_{Z_0}$ induced by $\varphi^* H_\varepsilon$ and the above morphism. We identify the diagonal subset Δ of the fiber product Z_0 with X_0. Note that $L'|_\Delta \cong rL \otimes \big(\varphi^* \det \varphi_* L\big)^*$ holds under this identification. By construction, the metric $G_\varepsilon|_\Delta$ on $L'|_\Delta \cong rL \otimes (\varphi^* \det \varphi_* L)^*$ is well-defined (i.e., $G_\varepsilon|_\Delta \not\equiv \infty$) and

$$\sqrt{-1}\Theta_{G_\varepsilon}|_\Delta \geq -\varepsilon\psi^*\omega_Y|_\Delta \geq -\varepsilon\omega_X \text{ holds on } \Delta \cong X_0.$$

Note that the well-definedness follows since G_ε is constructed by the pull-back $\psi^* H_\varepsilon$. This curvature condition can be extended to X by $\mathrm{codim}(X \setminus X_0) \geq 2$. Here we used that $\varphi : X \to Y$ has equi-dimensional fibers. □

We finally finish the proof of Theorem 2.6. Let p be an integer with $p/r \in \mathbb{Z}_+$. By Claim 2.9 and Condition (d), there exist singular metrics $\{g_{\delta'}\}_{\delta'>0}$ on $L|_{X_0}$ such that

$$\sqrt{-1}\Theta_{g_{\delta'}}(L) \geq \varphi^*\Big(\frac{1}{pr_p}\eta_p\Big) - \delta'\omega_X \text{ on } X_0.$$

Let us apply Proposition 2.4 for $\theta := (1/pr_p)\eta_p$. Then, since η_p is the curvature of $\det \varphi_*(pL)$, we see that

$$\varphi_* L \otimes \Big(\frac{1}{pr_p} \det \varphi_*(pL)\Big)^* \text{ is weakly positively curved} \tag{2.3}$$

on Y_0 (with respect to ω_Y); hence it is weakly positively curved on Y since ω_Y is defined on Y. The determinant sheaf

$$\det \varphi_* L \otimes \Big(\frac{r}{pr_p} \det \varphi_*(pL)\Big)^* = \det \varphi_* L - \frac{r}{pr_p} \det \varphi_*(pL)$$

is also weakly positively curved on Y. Here we use the additive notation on the left-hand side, which is justified on Y_0 (see Remark 2.7). This implies that

$$\begin{aligned}(\det \mathcal{V}_p)^* &= -\det \varphi_*(pL) + \frac{r_p p}{r} \det \varphi_* L \\ &\geq -\det \varphi_*(pL) + \frac{r_p p}{r} \cdot \frac{r}{pr_p} \det \varphi_*(pL) \text{ on } Y_0\end{aligned}$$

is weakly positively curved, where the notation $\geq$ denotes the difference is weakly positively curved. On the other hand, since L is φ-ample, the natural morphism

$$\mathcal{W}_p := \mathrm{Sym}^p(\varphi_* L) \otimes \Big(\frac{p}{r} \det \varphi_* L\Big)^* \to \varphi_*(pL) \otimes \Big(\frac{p}{r} \det \varphi_* L\Big)^* = \mathcal{V}_p$$

is generically surjective for $p \gg 1$. The sheaf $\mathcal{W}_p$ can be written as the p-th symmetric tensor of (2.3) of $p = 1$; therefore $\mathcal{W}_p$ is weakly positively curved. By the above morphism, we see that $\mathcal{V}_p$ is also weakly positively curved. □

2.4. **Hermitian metrics on orbifold vector bundles.** In this subsection, following [MM07, Wu], we review some basic facts related to orbifold structures. We first define the categories $\mathcal{M}_s$ and $\mathcal{M}_k$ as follows:

Definition 2.10. (1) *The objects* of $\mathcal{M}_s$ (resp. $\mathcal{M}_k$) consists of the class of pairs (G, M), where M is a smooth manifold (resp. Kähler manifold) and G is a finite group effectively acting on M.

(2) *The morphism* $\Phi : (G, M) \to (G', M')$ for objects (G, M), (G', M') consists of holomorphic open embeddings $\varphi : M \to M'$ (resp. preserving the Kähler form) satisfying the following conditions:

- For any $\varphi \in \Phi$, there exists an injective group homomorphism $\lambda_\varphi : G \to G'$ such that $\varphi : M \to M'$ is λ_φ -equivariant;
- For any $g \in G'$ and $\varphi \in \Phi$, we define $g\varphi : M \to M'$ by $(g\varphi)(x) = g \cdot \varphi(x)$ for $x \in M$. If $(g\varphi)(M) \cap \varphi(M) \neq \emptyset$, then $g \in \lambda_\varphi(G)$;
- For any $\varphi \in \Phi$, we have $\Phi = \{g\varphi \,|\, g \in G'\}$.

Definition 2.11. Let X be a paracompact Hausdorff space and let $\mathcal{U}$ be an open cover of X consisting of connected open subsets in the category $\mathcal{M}_s$ satisfying that for any $x \in U \cap U'$ $(U, U' \in \mathcal{U})$, there is $U'' \in \mathcal{U}$ such that $x \in U'' \subset U \cap U'$. *An orbifold structure $\mathcal{V}$ of* $(X, \mathcal{U})$ consists in the following data:

- A ramified covering $\mathcal{V}(U)$ for any $U \in \mathcal{U}$ defined by
$$\mathcal{V}(U) = \big\{(G_U, \tilde{U} \xrightarrow{\tau} U) \text{ giving an identification } U \simeq \tilde{U}/G_U\big\};$$
- A morphism $\varphi_{VU} : (G_U, \tilde{U}) \to (G_V, \tilde{V})$ for any $U, V \in \mathcal{U}$ with $U \subset V$ such that φ_{VU} is compatible with the inclusion $U \subset V$ and satisfies $\varphi_{WU} = \varphi_{WV} \circ \varphi_{VU}$ for any $U, V, W \in \mathcal{U}$ with $U \subset V \subset W$.

A space with an orbifold structure is called a *complex orbifold* and is said to have *an orbifold Kähler structure* when it is covered by objects in the category $\mathcal{M}_k$. Note that a complex orbifold is equivalent to an analytic variety with only quotient singularities.

Let $\mathcal{F}$ be a reflexive sheaf on a compact complex orbifold X. Without loss of generality, we may assume that X is covered by finite open sets $\{U_\alpha\}_\alpha$ such that $U_\alpha \cong \tilde{U}_\alpha/G_\alpha$ holds, where $\tilde{U}_\alpha$ is an Euclidean open set and G_α is a finite subgroup in the general linear group with free-in-codimension-1 action on $\tilde{U}_\alpha$. Let $\pi_\alpha : \tilde{U}_\alpha \to \tilde{U}_\alpha/G_\alpha \cong U_\alpha$ be the corresponding quotient map. To simplify the notation, we often call $\{\tilde{U}_\alpha\}_\alpha$ local smooth ramified covers.

Definition 2.12. A reflexive sheaf $\mathcal{F}$ is called *an orbifold vector bundle* if $\{(\pi_\alpha^*\mathcal{F})^{**}\}_\alpha$ is locally free for any α. The quotients of total spaces $\{(\pi_\alpha^*\mathcal{F})^{**}/G_\alpha\}_\alpha$ glue to the analytic variety X, which we will denote by E. The *determinant orbifold line bundle* of $\mathcal{F}$ is defined to be the determinant line bundle of $\{(\pi_\alpha^*\mathcal{F})^{**}\}_\alpha$ on the local smooth ramified covers, which itself has the natural orbifold structure.

Note that for an orbifold vector bundle E, the projective space bundle$\mathbb{P}(E)$ of E and its dual E^* can be naturally defined as a complex orbifold with a tautological orbifold line bundle $\mathcal{O}_{\mathbb{P}(E)}(1)$ and natural projection $p : \mathbb{P}(E) \to X$. We can define the positivity of an orbifold vector bundle by using G_α-invariant metrics on $\{(\pi_\alpha^*\mathcal{F})^{**}\}_\alpha$ that is compatible with the orbifold structure, which are called *orbifold metrics*.

Definition 2.13. Let (X, ω) be a compact Kähler orbifold and let E be an orbifold vector bundle. The orbifold vector bundle E is said to be *nef* (resp. *strongly pseudo-effective*) if $\mathcal{O}_{\mathbb{P}(E)}(1)$ has orbifold smooth (resp. singular) metrics $\{h_\varepsilon\}_{\varepsilon>0}$ such that $\sqrt{-1}\Theta_{h_\varepsilon}(\mathcal{O}_{\mathbb{P}(E)}(1)) + \varepsilon p^*\omega$ is positive (resp. positive in the sense of currents such that the projection of the polar set $\{h_\varepsilon = +\infty\}$ under p is not dominant) on each local smooth ramified cover using the usual formula of Chern curvature.

The orbifold vector bundle E is said to be *numerically flat* if E, E^* are nef orbifold vector bundles. The orbifold vector bundle E is said to be *Hermitian flat* if it admits a Hermitian flat metric over each local smooth ramified cover.

3. $\mathbb{Q}$-Conic Bundles and the Minimal Model Program

$\mathbb{Q}$-conic bundles naturally appear as an outcome of the MMP in our situation (see Subsection 3.3 for details). For this reason, we respectively study $\mathbb{Q}$-conic bundles and conic bundles in Subsections 3.1, 3.2, and clarify what the nefness of anti-canonical bundles brings to the geometry of $\mathbb{Q}$-conic bundles.

3.1. $\mathbb{Q}$-conic bundles.

In this subsection, following [MP08a, MP08b, Pro07], we summarize basic properties of $\mathbb{Q}$-conic bundles. We first review the definition of $\mathbb{Q}$-conic bundles.

Definition 3.1. (1) Let X and S be normal analytic varieties. A fibration $\varphi : X \to S$ is called *a* $\mathbb{Q}$*-conic bundle* if it satisfies following conditions:

- X has terminal singularities;
- $\varphi : X \to S$ is equi-dimensional and of relative dimension 1;
- $-K_X$ is φ-ample.

Throughout this paper, except for Subsection 3.2, we promise that a $\mathbb{Q}$-conic bundle $\varphi : X \to S$ satisfies $\dim X = 3$ (and hence $\dim S = 2$). Note that the results in Subsection 3.2 are valid even for conic bundles with $\dim X \neq 3$.

(2) *The discriminant divisor* Δ is defined by the union of divisorial components of the non-smooth locus $\{s \in S \,|\, \varphi \text{ is not a smooth fibration at s}\}$.

(3) A $\mathbb{Q}$-conic bundle $\varphi : X \to S$ is said to be *toroidal* at $s \in S$ with respect to $\mu_m := \mathbb{Z}/m\mathbb{Z}$ if X is isomorphic to the quotient of $\mathbb{P}^1 \times \mathbb{C}^2$ over a neighborhood of s by the μ_m-action defined by

$$(t; z_1, z_2) \to (\varepsilon^b t; \varepsilon z_1, \varepsilon^{-1} z_2),$$

where b is an integer with $\gcd(m, b) = 1$ and ε is a primitive m-th root of unity. Note that the singularities of X consists of cyclic quotient singularities of types $(1/m)(b, 1, -1)$ and $(1/m)(-b, 1, -1)$; furthermore, the singularities of the base $S \cong \mathbb{C}^2/\mu_m$ are the cyclic quotient of type A_{m-1}.

A $\mathbb{Q}$-conic bundle $\varphi : X \to S$ with $\dim X = 3$ can be explicitly described locally over S, which is the reason why we require $\dim X = 3$. The following corollary is a direct consequence of the classification [Pro18, Corollary 10.85].

Lemma 3.2 ([Pro18, Corollary 10.85])**.** *Let $\varphi : X \to S$ be a $\mathbb{Q}$-conic bundle and $\Delta \subset S$ be the discriminant divisor. Then $s \notin \Delta$ if and only if $\varphi : X \to S$ is toroidal at s.*

3.2. **Conic bundles.** In this subsection, after reviewing conic bundles following [Sar82], we generalize some properties to the non-projective case (see Propositions 3.4 and 3.5).

We first deduce Proposition 3.5 from Proposition 3.4. Proposition 3.4 is proved by the same argument as in [Sar82] even in the non-projective case.

Definition 3.3. Let $\varphi : X \to S$ be a $\mathbb{Q}$-conic bundle (see Definition 3.1). The fibration $\varphi : X \to S$ is said be a *conic bundle* if X and S are smooth.

Proposition 3.4. *Let $\varphi : X \to S$ be a conic bundle. Then we have:*

(1) *$-K_X$ is φ-very ample, $E := \varphi_*(-K_X)$ is a locally free sheaf of rank 3, and $E = \varphi_*(-K_X)$ defines an embedding of $\varphi : X \to S$ into $p : \mathbb{P}(E) \to S$:*

$$\begin{tikzcd} X \arrow[rr, hook] \arrow[dr, "\varphi"'] & & \mathbb{P}(E) \arrow[dl, "p"] \\ & S. & \end{tikzcd}$$

Furthermore, the scheme-theoretic fiber X_s at a point $s \in S$ is a (possibly reducible or non-reduced) conic on the projective plane $\mathbb{P}(E_s)(\cong \mathbb{P}^2)$.

(2) *$X \subset \mathbb{P}(E)$ can be written as the zero locus of a section*
$$\sigma \in H^0(\mathbb{P}(E), \mathcal{O}_{\mathbb{P}(E)}(2) \otimes p^*(-\det E - K_S)).$$

(3) *Take a sufficiently small open set $U \subset S$ with a coordinate z and identify $\mathbb{P}(E)$ with $\mathbb{P}^2 \times U$ over U. Then, the embedding $X \subset \mathbb{P}(E) = \mathbb{P}^2 \times U$ over U can be written as*
$$X = \{([x_0 : x_1 : x_2], z) \in \mathbb{P}^2 \times U \mid \sum_{0 \le i,j \le 2} a_{i,j}(z) x_i x_j = 0\},$$
where $a_{i,j} \in \mathcal{O}_Y(U)$. Furthermore, the discriminant divisor Δ coincides with the non-smooth locus $\{s \in S \mid \varphi \text{ is not smooth at s}\}$ and is described as
$$\Delta = \{z \in S \mid \det[a_{i,j}(z)]_{i,j=0}^2 = 0\}.$$

(4) *By changing coordinates for a given point $s \in U$, we can assume that $a_{i,j} = 0$ for $i \neq j$. Furthermore, we can assume that*
- *$a_{i,i} \in \mathcal{O}_Y^*(U)$ for any $0 \le i \le 2$ when X_s is smooth;*
- *$a_{i,i} \in \mathcal{O}_Y^*(U)$ for any $1 \le i \le 2$ and $\mathrm{mult}_s(a_{0,0}) = 1$ when X_s is reduced and reducible.*

(5) *The discriminant divisor Δ is normal crossing in codimension 2 in S. Furthermore, the fiber X_s of a general point $s \in \Delta$ is reduced and reducible.*

(6) *$c_1(\Delta) = -c_1(\det E) - 3c_1(K_S)$ holds.*

Proposition 3.5. *Let $\varphi : X \to S$ be a conic bundle. Then, we have*
$$\varphi_*(c_1(K_X)^2) = -4c_1(K_S) - c_1(\Delta).$$

Proof. The proposition has been proved in [Mi83, 4.11] when X is projective, but the non-projective case is needed for our purpose. By Proposition 3.4 (6), the conclusion is equivalent to

$$\varphi_*(c_1(K_X)^2) = \frac{4}{3}c_1(\det E) + \frac{1}{3}c_1(\Delta).$$

We fix a smooth Hermitian metric h on $E = \varphi_*(-K_X)$ and use the same notation h to denote the induced metric on $\mathcal{O}_{\mathbb{P}(E)}(1)$. Then, by the adjunction formula and Proposition 3.4 (2), the conclusion is equivalent to the following formula:

$$p_*(c_1(\mathcal{O}_{\mathbb{P}(E)}(1), h)^2 \wedge [X]) = \frac{4}{3}c_1(\det E, \det h) + \frac{1}{3}[\Delta], \tag{3.1}$$

where $[X], [\Delta]$ are the integration currents and $c_1(\mathcal{O}_{\mathbb{P}(E)}(1), h), c_1(\det E, \det h)$ are the Chern curvatures divided by 2π.

We first prove the desired formula on $S \setminus \Delta$. For this purpose, we summarize some formulas for the curvatures of vector bundles (e.g., see [Dem, Section 15.C, Chap. V]). Set $n = \dim S$ and $r := \operatorname{rank} E (= 3)$. For a given $s \in S \setminus \Delta$, we take a local frame $(e_\lambda)_{\lambda=1}^r$ of E giving an orthonormal basis of E_s at $s \in S$, and then write the Chern curvature of E as

$$\Theta_h(E) = \sum_{1\le j,k\le n, 1\le\lambda,\mu\le r} c_{jk\lambda\mu} dz^j \wedge d\overline{z}^k \otimes e_\lambda^* \otimes e_\mu, \tag{3.2}$$

where $(z_j)_{j=1}^n$ is a local coordinate of S. Let $[x] \in \mathbb{P}(E_s)$ be the point represented by a vector $\sum_{\lambda=1}^r x_\lambda e_\lambda^* \in E_s^*$ with $\sum_{\lambda=1}^r |x_\lambda|^2 = 1$. Then, the curvature of $\mathcal{O}_{\mathbb{P}(E)}(1)$ at $[x]$ can be written as

$$\Theta_h(\mathcal{O}_{\mathbb{P}(E)}(1))_{[x]} = \sum_{1\le j,k\le n, 1\le\lambda,\mu\le r} c_{jk\lambda\mu} x_\lambda \overline{x}_\mu dz^j \wedge d\overline{z}^k + \sum_{1\le\lambda\le r-1} d\xi_\lambda \wedge d\overline{\xi}_\lambda, \tag{3.3}$$

where $(\xi_\lambda)_{\lambda=1}^{r-1}$ is the coordinate of $\mathbb{P}(E)$ induced by unitary coordinates on the hyperplane $(\mathbb{C}x)^\perp \subset E_s^*$. We identify $\mathbb{P}(E)$ with $\mathbb{P}^{r-1} \times U$ over an open neighborhood $U \subset S$ of s, and regard the Fubini-study form Ω on $\mathbb{P}^{r-1}$ as the $(1,1)$-form on $\mathbb{P}(E)$. Then, we can easily check that

$$\frac{\sqrt{-1}}{2\pi}\Theta_h(\mathcal{O}_{\mathbb{P}(E)}(1))_{[x]} = \Omega - \frac{\sqrt{-1}}{2\pi}\frac{\langle p^*\Theta_{h^*}(E^*)x, x\rangle}{|x|^2} \quad \text{for any } [x] \in p^{-1}(s).$$

The push-forward of smooth forms can be described as a fiber integration near $s \in S \setminus \Delta$ since $\varphi : X \to S$ is a smooth morphism at $s \in S \setminus \Delta$ (e.g., see [GPR94, Theorem 1.14, Proposition 2.15, Chap. II]). Hence, we obtain

$$p_*(c_1(\mathcal{O}_{\mathbb{P}(E)}(1), h)^2 \wedge [X])_s = \int_{X_s} c_1(\mathcal{O}_{\mathbb{P}(E)}(1), h)^2 = -2 \cdot \frac{\sqrt{-1}}{2\pi} \int_{X_s} \Omega \wedge \frac{\langle p^*\Theta_{h^*}(E^*)x, x\rangle}{|x|^2},$$

where X_s is the fiber of $\varphi : X \to S$ at $s \in S$.

We consider the special case of $X_s = \{x_0^2 + x_1^2 + x_2^2 = 0\}$. We can easily see that

$$\int_{X_s} \frac{x_\lambda \overline{x}_\mu}{|x|^2}\, \Omega = \frac{2}{3} \cdot \delta_{\lambda\mu},$$

where $\delta_{\lambda\mu}$ is the Kronecker-delta. Hence, by the formula (3.2), we obtain

$$\int_{X_s} c_1(\mathcal{O}_{\mathbb{P}(E)}(1), h)^2 = \frac{4}{3} \frac{\sqrt{-1}}{2\pi} \sum_{\lambda} c_{jk\lambda\lambda} dz^j \wedge d\overline{z}^k = \frac{4}{3} c_1(\det E, \det h).$$

The general case can be reduced to the special case. Indeed, by Proposition 3.4 (4), we may assume that $X_s = \{C_0 x_0^2 + C_1 x_1^2 + C_2 x_2^2 = 0\}$, where C_i is a non-zero constant. Hence, we see that X_s is cohomologeous to $\{x_0^2 + x_1^2 + x_2^2 = 0\}$ in $\mathbb{P}(E_s)$. By the Stokes formula, we can conclude that

$$\int_{X_s} c_1(\mathcal{O}_{\mathbb{P}(E)}(1), h)^2 = \int_{\{x_0^2+x_1^2+x_2^2=0\}} c_1(\mathcal{O}_{\mathbb{P}(E)}(1), h)^2 = \frac{4}{3} c_1(\det E, \det h).$$

The Fubini theorem shows that

$$\int_{S} \varphi_*(c_1(\mathcal{O}_{\mathbb{P}(E)}(1), h)^2) \wedge \alpha = \int_{S \setminus \Delta} \alpha \wedge \int_{X_s} c_1(\mathcal{O}_{\mathbb{P}(E)}(1), h)^2$$

for any smooth form α with compact support in $S \setminus \Delta$, which implies that

$$p_*(c_1(\mathcal{O}_{\mathbb{P}(E)}(1), h)^2 \wedge [X]) = \frac{4}{3} c_1(\det E, \det h) \text{ on } S \setminus \Delta.$$

We finally prove the desired formula (3.1) on S. The push-forward $p_*(c_1(\mathcal{O}_{\mathbb{P}(E)}(1), h)^2 \wedge [X])$ is a normal current; hence, by applying the support theorem (see Lemma 2.2), we can find $c_i \in \mathbb{R}$ such that

$$\mathbb{1}_\Delta\, p_*(c_1(\mathcal{O}_{\mathbb{P}(E)}(1), h)^2 \wedge [X]) = \sum_i c_i [\Delta_i],$$

where $\{\Delta_i\}_{i \in I}$ are the irreducible components of Δ. We show that $c_i = 1/3$ for any i by regarding c_i as the generic Lelong number along Δ_i. Note that c_i is independent of the choice of metric h on E. Indeed, let $h' = he^{-\psi}$ be another smooth metric on $\mathcal{O}_{\mathbb{P}(E)}(1)$ for some $\psi \in C^\infty(\mathbb{P}(E))$. Then, we have

$$p_*(c_1(\mathcal{O}_{\mathbb{P}(E)}(1), h')^2 \wedge [X]) - p_*(c_1(\mathcal{O}_{\mathbb{P}(E)}(1), h)^2 \wedge [X]) = \frac{\sqrt{-1}}{2\pi} \partial\overline{\partial} F,$$

where F is a function defined by

$$F(s) = \int_{X_s} \psi \cdot (c_1(\mathcal{O}_{\mathbb{P}(E)}(1), h') + c_1(\mathcal{O}_{\mathbb{P}(E)}(1), h)) \text{ for } s \in S.$$

To check that the Lelong number is independent of the choice of h, it is sufficient to show that F is continuous over any small 1-dimensional disc passing through a general point of Δ, but this follows from the theory of cycle spaces (e.g. by [Bar78, Corollaire 1]).

To finish the proof, for a general point $s \in \Delta$, we construct a smooth metric h on E (e.g., by a partition of unity) such that h is flat on a neighborhood of s. Then, by $c_1(\mathcal{O}_{\mathbb{P}(E)}(1), h) = \Omega$, it is sufficient to show that

$$p_*(\Omega^2 \wedge [X]) = \frac{1}{3}[\Delta_i]$$

on a neighborhood U of s. We reduce this problem to the projective case as follows: Let $z_0x_0^2 + f_1x_1^2 + f_2x_2^2$ with $f_1, f_2 \in \mathcal{O}^*(U)$ be a local defining function of $X \subset \mathbb{P}(E)$. Let us regard $U \subset S$ as an open subset in $\mathbb{P}^n$. Then, we can find $g_N \in H^0(\mathbb{P}^n \times \mathbb{P}^2, \mathcal{O}(N) \boxtimes \mathcal{O}(2))$ such that

$$[X \cap \mathbb{P}(E)] = \lim_{N \to \infty} [X_N \cap \mathbb{P}(E)] \text{ over } U$$

by using the polynomial approximation of f_1, f_2 (e.g., we can use the Taylor expansion). Here $X_N := \{g_N = 0\}$. By the Bertini theorem, a general member in $H^0(\mathbb{P}^n \times \mathbb{P}^2, \mathcal{O}(N) \boxtimes \mathcal{O}(2))$ determines a conic bundle over $\mathbb{P}^n$. Thus, by replacing g_N with the general member, we may assume that $X_N = \{g_N = 0\} \to \mathbb{P}^n$ is a conic bundle with discriminant divisor Δ_N. Since $X_N \to \mathbb{P}^n$ is a projective conic bundle, we have

$$p_*(\Omega^2 \wedge [X_N \cap \mathbb{P}(E)]) = \frac{1}{3}[\Delta_N] \text{ over } U$$

by [Mi83, 4.11]. Then, as $N \to \infty$, we obtain the desired conclusion. □

3.3. **Minimal Model Program.** In this subsection, we review the MMP for Kähler 3-folds developed in [HP15a, HP15b, HP16], and observe what the MMP brings to Conjecture 1.1, particularly, a geometric condition guaranteeing that $\mathbb{Q}$-conic bundles are toroidal or $\mathbb{P}^1$-bundles (see Corollary 3.10).

Theorem 3.6 ([HP15b])**.** *Let X be a $\mathbb{Q}$-factorial compact Kähler space of dimension 3 with terminal singularities. Assume that* $\dim R(X) = 2$*, where $R(X)$ is the base of an MRC fibration $X \dashrightarrow R(X)$ of X. Then, we have*:

(1) *X is bimeromorphic to a MF (Mori fiber) space; more precisely, there exist*

a bimeromorphic map $\pi : X \dashrightarrow X'$ and a MF space $\varphi : X' \to S$ such that

- (a) *$X \dashrightarrow X'$ is obtained from the composition of divisorial contractions and flips;*
- (b) *X' is a $\mathbb{Q}$-factorial compact Kähler space with terminal singularities;*
- (c) *S is a $\mathbb{Q}$-factorial compact Kähler space of dimension 2 with klt singularities;*
- (d) *S is non-uniruled and K_S is pseudo-effective;*
- (e) *$-K_{X'}$ is φ-ample and the relative Picard number $\rho(X'/S)$ is 1;*
- (f) *$\varphi : X' \to S$ is equi-dimensional and of relative dimension 1.*

(2) *The outcome $X \dashrightarrow X' \to S$ of the MMP factors through the Albanese map $\alpha : X \to A(X)$, that is, there exists the morphism $\beta : S \to A(X)$ with the diagram:*

$$\begin{tikzcd} X \arrow[r, dotted, "\pi"] \arrow[d, "\alpha"'] & X' \arrow[d, "\varphi"] \\ A(X) & S. \arrow[l, "\beta"'] \end{tikzcd}$$

Proof. By running the MMP in [HP15b] for the initial variety X, we can find a bimeromorphic map $\pi : X \dashrightarrow X'$ such that $K_{X'}$ is nef or there exists a MF space $\varphi : X' \to S$. The other properties of (1) follow from [HP15b]; hence we check only (1). Note that X' has terminal singularities. By[Bru06], a compact Kähler manifold Y of dimension ≤ 3 is non-uniruled if and only if K_Y is pseudo-effective. The same statement holds for compact Kähler spaces with terminal singularities. Hence, if $K_{X'}$ is nef, the variety X' is non-uniruled, which contradicts to $\dim R(X) = 2$. The outcome $X \dashrightarrow X' \to S$ gives an MRC fibration of X, which implies that S is a non-uniruled surface by $\dim R(X) = 2$. The non-uniruledness shows that K_S is pseudo-effective. Indeed, for a minimal resolution $\pi : \bar{S} \to S$ of S, we have $\pi^* K_S = K_{\bar{S}} + E$ for some effective exceptional $\mathbb{Q}$-divisor E. Since $\bar{S}$ is non-uniruled, we see that $K_{\bar{S}}$ is pseudo-effective; hence so is K_S.

As in the case of the projective case, all the steps of the MMP (i.e., divisorial contractions, flips, MF spaces) are obtained from contractions of rational curves. This implies that $X \dashrightarrow X' \to S$ factors through $X \to A(X)$ since the torus $A(X)$ has no rational curve. □

Let us shortly observe the positivity of $-K_{X'}$. Suppose that we start from X with the nef anti-canonical bundle $-K_X$, and obtain a bimeromorphic map $X \dashrightarrow X'$ in Theorem 3.6. We may expect that $-K_{X'}$ is still nef "outside the exceptional locus" of $X \dashrightarrow X'$, but this cannot be shown immediately. In fact, after taking a smooth form $T_\varepsilon \in c_1(-K_X)$ such that $T_\varepsilon \geq -\varepsilon\omega$, we can obtain $\pi_* T_\varepsilon \geq -\varepsilon \pi_* \omega$ on X', where ω is a Kähler form on X. Then, it is not clear how the current $\pi_*\omega$ relates to a Kähler form on X'. To overcome this difficulty, we prepare Lemma 3.8, which compares Kähler forms between X and X'.

Setting 3.7. Before stating the lemma, we fix the notation. Assume that X in Theorem 3.6 is smooth and $-K_X$ is nef. The bimeromorphic map $X \dashrightarrow X'$ is decomposed as follows:

$$X =: X_0 \dashrightarrow X_1 \dashrightarrow \cdots \dashrightarrow X_N := X', \tag{3.4}$$

where each bimeromorphic map $\pi_i : X_i \dashrightarrow X_{i+1}$ is a divisorial contraction or flip. Let $\bar{X}$ be a compact Kähler manifold with a bimeromorphic morphism $p_i : \bar{X} \to X_i$ that resolves the indeterminacy locus of π_i (when π_i is a flip). Depending on whether π_i is a divisorial contraction or flip, we obtain the following diagrams:

$$\begin{tikzcd} \bar{X} \arrow[d, "p_i"'] \arrow[dr, bend left, "p_{i+1}"] & \\ X_i \arrow[r, "\pi_i"] & X_{i+1}, \end{tikzcd} \qquad \begin{tikzcd} \bar{X} \arrow[dr, "p_i"] \arrow[drrr, bend left, "p_{i+1}"] \arrow[ddrr, bend right, "r_i"'] & & & \\ & X_i \arrow[rr, dotted, "\pi_i"'] \arrow[dr, "q_i"] & & X_{i+1} \arrow[dl, "q_{i+}"] \\ & & Z_i. & \end{tikzcd}$$

Note that Z_i and X_i are Kähler spaces by [HP15b, Theorem 3.15].

Lemma 3.8. *We consider Setting 3.7. Then, for any $0 \leq i \leq N$, there exists a Kähler form ω_i on X_i such that the Bott-Chern class*

$$\{p_{0*}(p_{i+1}^*\omega_{i+1} - p_i^*\omega_i)\} + O(E, K_X)$$

is represented by a positive current that is smooth outside the exceptional locus of $X \dashrightarrow X'$, where $O(E, K_X)$ is a linear combination of the first Chern classes of K_X and the exceptional divisors. In particular, the Bott-Chern cohomology class $\{p_{0*}p_i^*\omega_i - \omega_0\} + O(E, K_X)$ is represented by a positive current that is smooth outside the exceptional locus of $X \dashrightarrow X'$.

Proof. We first consider the case where $\pi_i : X_i \to X_{i+1}$ is a divisorial contraction with the exceptional divisor E_i. Since $-E_i$ is π_i-ample, we can take a smooth form $\theta_i \in c_1(E_i)$ such that $\omega_i := \pi_i^*\omega_{i+1} - \varepsilon\theta_i$ is a Kähler from on X_i for $1 \gg \varepsilon > 0$. Then, we see that

$$p_{0*}(p_{i+1}^*\omega_{i+1} - p_i^*\omega_i) = p_{0*}p_i^*(\pi_i^*\omega_{i+1} - \omega_i) = \varepsilon p_{0*}p_i^*\theta_i.$$

The current $p_{0*}p_i^*\theta_i$ represents $c_1(p_{0*}p_i^*E_i) = O(E)$, which finishes the proof.

We now consider the case where $\pi_i : X_i \dashrightarrow X_{i+1}$ is a flip. For a fixed Kähler form ω_{Z_i} on Z_i, we take a Kähler form ω_{i+1} on X_{i+1} with $\omega_{i+1} \geq q_{i+}^*\omega_{Z_i}$. Furthermore, since $-K_{X_i}$ is q_i-ample, we can take a smooth form $\eta_i \in c_1(K_{X_i})$ such that $\omega_i := q_i^*\omega_{Z_i} - \varepsilon\eta_i$ is a Kähler from on X_i for $1 \gg \varepsilon > 0$. Then, we can easily see that

$$\begin{aligned} p_{0*}(p_{i+1}^*\omega_{i+1} - p_i^*\omega_i) &= p_{0*}(p_{i+1}^*\omega_{i+1} - p_i^*(q_i^*\omega_{Z_i} - \varepsilon\eta_i)) \\ &= p_{0*}(p_{i+1}^*(\omega_{i+1} - q_{i+}^*\omega_{Z_i}) + \varepsilon p_i^*\eta_i)). \end{aligned}$$

The current $p_{0*}p_i^*\eta_i$ represents $c_1(p_{0*}p_i^*K_i) = O(K_X, E)$ and $p_{0*}(p_{i+1}^*(\omega_{i+1} - q_{i+}^*\omega_{Z_i}))$ is smooth outside the exceptional locus, which finishes the proof. □

Later we will show that $X \dashrightarrow X'$ in Theorem 3.6 is actually an isomorphism when $-K_X$ is nef. For this purpose, we need the following corollary on the intersection number, which is similar to [Wu22a, Lemma 3].

Proposition 3.9. *In Setting 3.7, let ω_i be a Kähler form on X_i. Then, we have*:

(1) *There exists a positive current $T_\varepsilon \in c_1(-K_{X_i}) + \varepsilon\{\omega_i\}$ such that T_ε is smooth outside the exceptional locus of $X \dashrightarrow X'$.*

(2) *For a surface $V \subset X_i$, we have*

$$(c_1(K_{X_i})^2 \cdot \{\omega_i\}) \geq 0 \text{ and } (c_1(-K_{X_i}) \cdot c_1(V) \cdot \{\omega_i\}) \geq 0.$$

Proof. For every $\varepsilon > 0$, we take a smooth (semi-)positive form $T_\varepsilon \in c_1(-K_X)+\varepsilon\{\omega_0\}$ by the nefness of $-K_X$. By Lemma 3.8, we can find a positive current $P \in \{p_{0*}p_i^*\omega_i - \omega_0\} + O(E, K_X)$ such that P is smooth outside the exceptional locus. Then, we first prove that the pushforward current $p_{i*}p_0^*(T_\varepsilon + \varepsilon P)$ represents $c_1(-K_{X_i}) + \varepsilon\{\omega_i\} + \varepsilon O(K_{X_i})$. By construction, the current $p_{i*}p_0^*(T_\varepsilon + \varepsilon P)$ defined on X_i is positive and represents

$$p_{i*}p_0^*(c_1(-K_X) + \varepsilon p_{0*}p_i^*\{\omega_i\} + \varepsilon O(E, K_X))$$

outside the exceptional locus. The above class coincides with $c_1(-K_{X_i}) + \varepsilon\{\omega_i\} + \varepsilon O(K_{X_i})$ outside the exceptional locus, which is a Bott-Chern cohomology class on X_i. Hence, by Proposition 2.1 the current $p_{i*}p_0^*(T_\varepsilon + \varepsilon P)$ actually represents $c_1(-K_{X_i}) + \varepsilon\{\omega_i\} + \varepsilon O(K_{X_i})$.

Conclusion (1) easily follows since the $O(K_{X_i})$-part can be absorbed into the Kähler class and $p_{i*}p_0^*(T_\varepsilon + \varepsilon P)$ is smooth outside the exceptional locus by construction.

To prove Conclusion (2), we first remark that $c_1(V)$ is well-defined since X_i is $\mathbb{Q}$-factorial. Let Q be a $(2,2)$-form representing either $c_1(V)\cdot\{\omega_i\}$ or $c_1(-K_{X_i})\cdot\{\omega_i\}$ (in the de Rham cohomology). In any case, the pull-back p_i^*Q defines a class represented by a positive $(2,2)$-current since $p_i^*(-K_{X_i})$ is pseudo-effective by Conclusion (1). Note that $\bar{X}$ is smooth on which all cohomology theories coincide in the natural way. By a simple computation, we have

$$\begin{aligned}
(c_1(-K_{X_i}) \cdot Q) &= \lim_{\varepsilon\to 0}((c_1(-K_{X_i}) + \varepsilon\{\omega_i\} + \varepsilon O(K_{X_i})) \cdot Q) \\
&= \lim_{\varepsilon\to 0}(\{p_{i*}p_0^*(T_\varepsilon + \varepsilon P)\} \cdot Q) \\
&= \lim_{\varepsilon\to 0}(\{p_0^*(T_\varepsilon + \varepsilon P)\} \cdot p_i^*Q) \\
&= \lim_{\varepsilon\to 0}(\{p_0^*T_\varepsilon\} \cdot p_i^*Q).
\end{aligned}$$

The right-hand side is non-negative since p_i^*Q is pseudo-effective and T_ε is a smooth (semi-)positive $(1,1)$-form. □

Corollary 3.10. *We consider Setting 3.7 and the MF space $\varphi : X' = X_N \to S$ in Theorem 3.6. Then, we have*:

(1) *The Bott-Chern cohomology class $-4c_1(K_S) - c_1(\Delta)$ is pseudo-effective, where Δ is the discriminant divisor of the MF space $\varphi : X \to S$ (which is a $\mathbb{Q}$-conic bundle).*

(2) *The relation $\Delta = 0$ and $c_1(K_S) = 0$ holds; in particular, $\varphi : X' \to S$ is toroidal over S. Furthermore, when S are smooth, the variety X is automatically smooth and $\varphi : X' \to S$ is a (locally trivial) $\mathbb{P}^1$-bundle.*

Proof. Since $-K_{X'}$ is φ-ample by Theorem 3.6 (d), we can take a Kähler form $\omega' \in c_1(-K_{X'}) + \{\varphi^*\omega_S\}$, where ω_S is a fixed Kähler form on S. By Proposition 3.9 (1), there exists a positive current

$$T_\varepsilon \in -c_1(K_{X'}) + \varepsilon\{\omega'\} = -(1+\varepsilon)c_1(K_{X'}) + \varepsilon\{\varphi^*\omega_S\}$$

such that T_ε is smooth outside a Zariski closed subset of codimension ≥ 2. Then, we can defined the Bedford-Taylor product T_ε^2 on X'_{reg}. Since X' is smooth in codimension 2, there exists a Zariski open set S_0 with $\operatorname{codim}(S \setminus S_0) \geq 2$ such that $\varphi|_{X_0} : X_0 := \varphi^{-1}(S_0) \to S_0$ is a conic bundle. The pushforward $\varphi_*(T_\varepsilon^2)$ defined on S_0 is a positive current representing the following class on S_0:

$$\begin{aligned}
&\varphi_*\big((-(1+\varepsilon)c_1(K_{X'}) + \varepsilon\{\varphi^*\omega_S\})^2\big)\\
=&-(1+\varepsilon)^2(4c_1(K_S) + c_1(\Delta)) - 2\varepsilon(1+\varepsilon)\varphi_* c_1(K_{X'}) \cdot \{\omega_S\} + \varepsilon^2\varphi_*\{\varphi^*\omega_S^2\}\\
=&-(1+\varepsilon)^2(4c_1(K_S) + c_1(\Delta)) - 4\varepsilon(1+\varepsilon) \cdot \{\omega_S\}
\end{aligned}$$

by Proposition 3.5. Here we used that $\varphi_* c_1(K_{X'}) = 2$ and $\varphi_*\{\varphi^*\omega_S^2\} = 0$ holds on S_0. Proposition 2.1 shows that $\varphi_*(T_\varepsilon^2)$ is actually a positive current on S representing the Bott-Chern cohomology class of the right-hand side. The mass measure of $\varphi_*(T_\varepsilon^2)$ is uniformly bounded, we may assume that $\varphi_*(T_\varepsilon^2)$ has the weak limit after taking a subsequence by the weak compactness (see [Dem, (1.14), (1.23) Propositions, Chapter III]). Note that we can apply the weak compactness after taking a resolution of singularities of S although S may have singularities. Then, the weak limit of $\varphi_*(T_\varepsilon^2)$ is a positive current representing the class

$$\lim_{\varepsilon \to 0} -(1+\varepsilon)^2(4c_1(K_S) + c_1(\Delta)) - 4\varepsilon(1+\varepsilon) \cdot \{\omega_S\} = -4c_1(K_S) - c_1(\Delta).$$

This implies that $-4c_1(K_S) - c_1(\Delta)$ is pseudo-effective.

Conclusion (2) follows from Therem 3.6 (4) and Lemma 3.2. When S is smooth, the latter statement of Conclusion (2) is a special case of [AR14, Theorem 2]. However, for the convenience of the readers, we give a full proof in our special case by using an explicit construction of metrics and extending coherent sheaves.

It is sufficient to show that X' is smooth. Assume that there exists a singular point $x \in X$. For a small polydisc $\mathbb{D}$ of S centered at $\varphi(x)$, we have

$$H^2(\mathbb{D} \setminus \{\varphi(x)\}, \mathcal{O}^*_{\mathbb{D}\setminus\{\varphi(x)\}}) \cong H^3(\mathbb{D} \setminus \{\varphi(x)\}, \mathbb{Z}) \cong \mathbb{Z}$$

by

$$H^2(\mathbb{D} \setminus \{\varphi(x)\}, \mathcal{O}_{\mathbb{D}\setminus\{\varphi(x)\}}) \cong H^3(\mathbb{D} \setminus \{\varphi(x)\}, \mathcal{O}_{\mathbb{D}\setminus\{\varphi(x)\}}) = 0.$$

Since $H^3(\mathbb{D} \setminus \{\varphi(x)\}, \mathbb{Z})$ is torsion-free, there exists a holomorphic vector bundle E_0 on $\mathbb{D} \setminus \{\varphi(x)\}$ such that $X'_0 := \varphi^{-1}(\mathbb{D} \setminus \{\varphi(x)\}) \cong \mathbb{P}(E_0)$ holds over $\mathbb{D} \setminus \{\varphi(x)\}$ by [Ele82].

Since X' is terminal and $\mathbb{Q}$-factorial, some multiple of $-K_{X'}$ is a line bundle with a smooth metric. We have that $\varphi_*(c_1(-K_{X'_0})^2) = 8c_1(\det(E_0))$ on $\mathbb{D} \setminus \{\varphi(x)\}$ since $-K_{X'_0} = \mathcal{O}_{\mathbb{P}(E_0)}(2) + \varphi^* \det(E_0)$. In particular, there exists a smooth metric on $\det(E_0)$ whose curvature is locally integrable. By the removable singularity theorem [Ban91, Theorem 1], the line bundle $\det(E_0)$ is extended to a line bundle on $\mathbb{D}$.

The line bundle $\mathcal{O}_{\mathbb{P}(E_0)}(1)$ can also be endowed with a smooth metric whose curvature has finite L^2 norm by endowing some smooth metric on the extended line bundle of $\det(E_0)$ denoted by $\det(E)$. More precisely, the local potential of $\mathcal{O}_{\mathbb{P}(E_0)}(1)$ is the local potentials of $-K_{X'} - \varphi^*(\det(E))$ divided by 2. By the removable singularity theorem (see [BS94, Lemma 1] and [Siu69, Theorem 1]), the line bundle $\mathcal{O}_{\mathbb{P}(E_0)}(1)$ has a reflexive extension whose direct image is a reflexive extension of E_0. Since X' is smooth in codimension 2, the reflexive sheaf $\mathcal{O}_{\mathbb{P}(E_0)}(1)$ is extended through the possible singular set of X' by [Siu69, Theorem 1]. By $\dim S = 2$, the reflexive extension E is locally free. By [Kol91, Corollary 2.1.13], since $\varphi^{-1}(\mathbb{D})$ and $\mathbb{P}(E)$ is isomorphic outside finite curves, we obtain the isomorphism $\varphi^{-1}(\mathbb{D}) \cong \mathbb{P}(E)$. □

Remark 3.11. In the proof, if X' is smooth, the Bedford-Taylor product T_ε^2 is defined on X' as a positive current representing $(c_1(-K_i) + \varepsilon\{\omega_i\})^2$. This is expected to be true even when X' has singularities, which gives a more direct proof of Proposition 3.9.

4. Proof of the Main Results

This section is devoted to the proof of Theorem 1.3. Throughout this section, let X be a non-projective compact Kähler 3-fold with nef anti-canonical bundle. As explained in Subsection 1.2, it is sufficient for Theorem 1.3 to consider the case of $\dim R(X) = 2$. Furthermore, after replacing X with a finite étale cover, we may assume that $\pi_1(X) \cong \mathbb{Z}^{\oplus 2q}$ by [Pău97], where q is the irregularity of X. We treat the case of $q \neq 0$ in Subsection 4.2 and the case of $q = 0$ in Subsection 4.3.

4.1. On the base of MRC fibrations.

Before starting the proof of Theorem 1.3, we prove the following proposition in this subsection.

Proposition 4.1. *Let X be a non-projective compact Kähler 3-fold with nef anti-canonical bundle such that $\dim R(X) = 2$, where $R(X)$ denotes the smooth minimal base of MRC fibrations of X. Then, up to a finite étale cover of X, the base $R(X)$ is either a torus or K3 surface. In particular, the augmented irregularity is either 0 or 2.*

Proof. By replacing X with a finite étale cover, we assume that $\pi_1(X) \cong \mathbb{Z}^{\oplus 2q}$ by [Pău97]. The Albanese map $\alpha : X \to A(X)$ is a fibration that is smooth outside a Zariski closed subset of codimension ≥ 2 by [Cao13]. In particular, we have $q =$

$q(X) = \bar{q}(X) = \dim A(X)$, where $q(X)$ (resp. $\bar{q}(X)$) is the irregularity (resp. augmented irregularity) of X.

In the case of $\dim A(X) = 3$, the manifold X is non-uniruled, which contradicts to $\dim R(X) = 2$. In the case of $\dim A(X) = 2$, if a general fiber F is a curve with genus ≥ 1, which also contradicts to $\dim R(X) = 2$. If F is a rational curve, the Albanese map $\alpha : X \to A(X)$ gives an MRC fibration of X; hence, we have $R(X) = A(X)$.

We now consider the case of $\dim A(X) = 1$. We first show that $\pi_1(F)$ is finite. Note that $\alpha : X \to A(X)$ is a smooth fibration by $\dim A(X) = 1$. Furthermore, since $-K_F$ is nef, the fundamental group $\pi_1(F)$ is almost abelian by [Pău97] and the Albanese map $\bar{F} \to A(\bar{F})$ is also a smooth fibration for any finite étale cover $\bar{F} \to F$ by $\dim F = 2$. Then, by [DPS94, Proposition 3.12 (iii)], we obtain that

$$\bar{q}(F) = \bar{q}(X) - \bar{q}(A(X)) = q(X) - \dim A(X) = 0,$$

which implies that $\pi_1(F)$ is finite. Hence, the universal cover of F is either a K3 surface or rational connected since $-K_F$ is nef. Since X is uniruled by $\dim R(X) = 2$ and $A(X)$ has no rational curve, the fiber should be uniruled, which excludes the first case. When F is rational connected, the Albanese map $X \to A(X)$ gives an MRC fibration, which contradicts to $\dim R(X) = 2$.

We consider the remaining case of $\dim A(X) = 0$. Then, the manifold X is simply connected, which implies that the base $R(X)$ is also simply connected since the MRC-fibration induces the isomorphism $\pi_1(X) \cong \pi_1(R(X))$ by [Kol93, Theorem 5.2] and [BC15, Theorem 4.1]. Since $R(X)$ is simply connected, non-uniruled, and compact Kähler, but non-projective, we see that $R(X)$ is a K3 surface or a minimal elliptic surface $R(X) \to C$ by the classification of surfaces (e.g., see [BHPV04]). In the latter case, the Kodaira dimension $\kappa(R(X))$ is larger than or equal to 1. On the other hand, we see that $X \dashrightarrow X' \to S$ is an MRC fibration of X and satisfies $\kappa(S) = 0$. This is a contradiction since the base of MRC fibrations is uniquely determined up to bimeromorphic models. We finally show $\kappa(S) = 0$. The $\mathbb{Q}$-conic bundle $\varphi : X' \to S$ satisfies $c_1(K_S) = 0$ and $\Delta = 0$ by Corollary 3.10; hence $\varphi : X' \to S$ is toroidal by Lemma 3.2 and the singularities of S are rational double points. Therefore, for the minimal resolution $h : \bar{S} \to S$, we have $K_{\bar{S}} = h^*K_S$, which implies that $\kappa(S) = \kappa(\bar{S}) = 0$. □

4.2. **The case of X being non-simply connected.** In this subsection, we prove Theorem 1.3 by assuming that $\pi_1(X) \cong \mathbb{Z}^{\oplus 2q}$, $q \neq 0$, and $\dim R(X) = 2$.

Theorem 4.2. *Under the same situation as above, there exists a numerically flat vector bundle E on $A(X)$ such that X is isomorphic to the projective space bundle $\mathbb{P}(E)$ over $A(X)$.*

Proof. Recall that $\varphi : X' \to S$ is the MF space and $\beta : S \to A(X)$ is the morphism in Theorem 3.6. We first show that $\beta : S \to A(X)$ is actually isomorphic. Note that

$\dim A(X) = 2$ by the latter conclusion of Proposition 4.1. The morphism $\beta : S \to A(X)$ is a bimeromorphic map since $\alpha : X \to A(X)$ is a fibration by [Cao13]. For the minimal resolution $h : \bar{S} \to S$, since the singularities of S are rational double points by Corollary 3.10, we have $K_{\bar{S}} = h^* K_S$, which implies that $c_1(K_{\bar{S}}) = 0$. Hence, the composition $\bar{S} \to S \to A(X)$ should be isomorphic by $c_1(K_{A(X)}) = 0$ and $c_1(K_{\bar{S}}) = 0$; thus, we see that $\beta : S \to A(X)$ is isomorphic.

By Corollary 3.10, the variety X' is smooth and $\varphi : X' \to S$ is a $\mathbb{P}^1$-bundle; hence [Ele82] shows that X' is isomorphic to the projective bundle $\mathbb{P}(E)$. We show that E is numerically flat. Since $-K_{X'} = \mathcal{O}_{\mathbb{P}(E)}(2) + \varphi^* \det(E)$ and $\varphi_*(c_1(-K_{X'})^2) = 0$ by Corollary 3.10, we have $c_1(E) = 0$. Note that the fact that X' and S are smooth works intrinsically in the discussion below. By applying the regularization theorem [Dem92] to Proposition 3.9 (1), we obtain positive currents $\{T_\varepsilon\}_{\varepsilon>0}$ with analytic singularities in $c_1(-K_{X'}) + \varepsilon\{\omega_{X'}\}$ such that the singular locus of T_ε is not dominant over S. (Recall that the singular locus is contained in the exceptional locus $X \dashrightarrow X'$; hence it is not dominated over S). By the proof of Corollary 3.10, we have $\varphi_*(c_1(-K_{X'})^2) = 0$ and $\lim_{\varepsilon\to 0} \varphi_*(T_\varepsilon^2) = 0$; hence the Lelong number of T_ε uniformly converges to 0 on S by [Wu22b, Lemma 15]. By Demailly's regularization by smooth forms [Dem82], we see that $-K_{X'}$ is nef; hence E is numerically flat.

Note that E is an extension of Hermitian flat line bundles L_1, L_2:

$$0 \to L_1 \to E \to L_2 \to 0.$$

Indeed, by [DPS94, Theorem 1.18], the vector bundle E is constructed from a linear representation of $\pi_1(S)$. Furthermore, since $\pi_1(A(X))$ is abelian, this representation is the direct sum of 1-dimensional representation, which leads to the above exact sequence.

We finally show that $X \dashrightarrow X'$ is actually an isomorphism. Any rational curve $R \subset X'$ lies in a fiber of $\varphi : X' \cong \mathbb{P}(E) \to S \cong A(X)$; hence the intersection number $(R \cdot c_1(-K_{X'}))$ is negative. This implies that the last step $X_{N-1} \dashrightarrow X_N = X'$ of the MMP (see (3.4)) cannot be a flip. Hence, if $X \dashrightarrow X'$ is not isomorphic, the last step is a divisorial contraction.

We deduce a contradiction by computing intersection numbers. We first observe the case where the last step $X_{N-1} \to X_N = X'$ is a divisorial contraction contracting a surface to a point. Let E be the exceptional divisor and ω_{N-1} be a Kähler form on X_{N-1}. By Proposition 3.9 (2), we have $(c_1(K_{X_{N-1}})^2 \cdot \{\omega_{X_{N-1}}\}) \geq 0$. On the other hand, since X_{N-1} has terminal singularities, we have

$$-K_{X_{N-1}} = \pi^*(-K_{X_N}) - aE$$

for some $a > 0$. Since $X_N = X'$ is $\mathbb{P}^1$-bundle over $S \cong A(X)$, we have $c_1(K_{X_N})^2 = 0$; thus we obtain $c_1(-K_{X_N})^2 = a^2 c_1(E)^2$. Since $\mathcal{O}_E(-E)$ is ample, we obtain

$$(c_1(K_{X_{N-1}})^2 \cdot \{\omega_{X_{N-1}}\}) = a^2 (c_1(E)|_E \cdot \{\omega_{N-1}\}|_E) < 0.$$

This is a contradiction.

We observe the remaining case where the last step $X_{N-1} \to X_N = X'$ is a divisorial contraction contracting a surface to a curve. The Albanese map of X is flat in codimension 1 by [Cao13]; therefore all the divisorial contractions and flips of $\pi : X \dashrightarrow X'$ occur in some fibers of the Albanese map. In particular, the image $\pi(E)$ of the exceptional divisor E of $X_{N-1} \to X_N = X'$ is equal to a fiber F of $\varphi : X' \to S$. Let us consider a submanifold $V := \mathbb{P}(L_2) \subset \mathbb{P}(E) = X_N$ and the strict transform $\bar{V} \subset X_{N-1}$ of V. Since the intersection $\bar{V} \cap E$ is a rational curve, we have

$$(c_1(-E) \cdot c_1(\bar{V}) \cdot \omega_{N-1}) < 0.$$

This implies that $\mathcal{O}(-aE)|_{\bar{V}} = -K_{X_N}|_{\bar{V}}$ is not pseudo-effective. This is contradiction to the fact that $-K_{X_N} = -K_{X'}$ is nef. $\square$

4.3. **The case of X being simply connected.** Let X be a non-projective compact Kähler smooth 3-fold with nef anti-canonical bundle such that $\pi_1(X) = \{\mathrm{id}\}$ and $\dim R(X) = 2$. Compared to Subsection 4.2, the main difficulty is that S may have singularities, which prevents us to conclude that the MF space $\varphi : X' \to S$ is locally constant. The following example will help us to understand this difficulty.

Example 4.3. For a Kummer surface $S := A/\mu_2$ with a torus A of dimension 2, we consider

$$X' := (\mathbb{P}^1 \times A)/\mu_2 \to S = A/\mu_2,$$

where μ_2 acts on $\mathbb{P}^1 \times A$ by $-1 \cdot (t, z_1, z_2) = (-t, -z_1, -z_2)$. Both S and X' are simply connected and $\varphi : X' \to S$ is a $\mathbb{Q}$-conic bundle such that $-K_{X'}$ is nef. Nevertheless, the fibration $\varphi : X' \to S$ is not even locally trivial.

In fact, we show that the example above never appears as an outcome of the MMP of a compact Kähler 3-fold X with anti-canonical bundle thanks to the assumption that X is smooth. More precisely, we prove the following theorem:

Theorem 4.4. *Under the same situation as above, the manifold X is isomorphic to the product of a K3 surface and the projective line $\mathbb{P}^1$.*

Proof. Let us consider the same situation as in Theorem 3.6 by running the MMP. We first show that the sheaf $\mathcal{V}_m$ defined for $L := -K_{X'}$ satisfies the conclusion of Theorem 2.6. By Corollary 3.10 and Proposition 3.9, the assumptions of Theorem 2.6 are satisfied except for Condition (d). We check that Condition (d) is also satisfied by using orbifold structures. By Corollary 3.10, the $\mathbb{Q}$-conic bundle $\varphi : X' \to S$ has at most toroidal singularities; thus $\mathcal{V}_m$ is a reflexive sheaf over a compact complex orbifold S. By [Wu, Lemma 1], there exists a continuous function ψ on S whose pullback on each local smooth ramified cover is smooth such that $\omega_S + \sqrt{-1}\partial\bar{\partial}\psi$ defines an orbifold Kähler structure on S. Take any smooth orbifold metric $h_{\mathcal{V}_m}$ on the determinant orbifold line bundle of $\mathcal{V}_m$. Then $\sqrt{-1}\Theta_{h_{\mathcal{V}_m}} + C(\omega_S + \sqrt{-1}\partial\overline{\partial}\psi)$ is positive on each local smooth ramified cover for $C \gg 1$. Hence $h_{\mathcal{V}_m} e^{-C\psi}$ defines a metric satisfying Condition (d) of Theorem 2.6.

We prove that $\mathcal{V}_m$ is a numerical flat orbifold vector bundle. Note that there is a bijection between orbifold reflexive sheaves and reflexive sheaves over an orbifold. Since any reflexive sheaf over a smooth surface is locally free, the sheaf $\mathcal{V}_m$ is an orbifold vector bundle over the orbifold surface S. Let ω_S be a Kähler form on S. By Theorem 2.6, for any $\varepsilon > 0$, there exist a singular metric h_ε on $\mathcal{V}_m$ such that

$$\sqrt{-1}\Theta_{h_\varepsilon} \geq -\varepsilon\omega_S \otimes \mathrm{id}.$$

In particular, the orbifold vector bundle $\mathcal{V}_m$ is a pseudo-effective orbifold vector bundle in the strong sense. Then, [Wu, Theorem E] shows that $\mathcal{V}_m$ is numerically flat; hence there exists a successive extension of Hermitian flat orbifold vector bundles:

$$0 =: \mathcal{F}_0 \subset \mathcal{F}_1 \subset \cdots \subset \mathcal{F}_{p-1} \subset \mathcal{F}_p := \mathcal{V}_m, \tag{4.1}$$

such that each quotient $\mathcal{F}_k/\mathcal{F}_{k-1}$ is a Hermitian flat orbifold vector bundle. Note that a successive extension of vector bundles on each local smooth ramified cover which is compatible with the orbifold structure induces an exact sequence of reflexive sheaves on X by taking invariant part of the pushforward since higher direct image of finite morphism vanishes.

By [Cam04, Corollary 6.7], there exists a quasi-étale cover $\tau : S^\dagger \to S$ such that $\tau : S^\dagger \to S$ is an orbifold morphism and $S^\dagger$ is either a torus or normal K3 surface (i.e. a normal surface whose minimal resolution is a K3 surface). Note that the fundamental group of normal K3 surfaces is trivial. Let us consider the base change:

$$\begin{array}{ccc} X^\dagger & \xrightarrow{\nu} & X' \\ \Big\downarrow{\scriptstyle \varphi^\dagger} & & \Big\downarrow{\scriptstyle \varphi} \\ S^\dagger & \xrightarrow{\tau} & S. \end{array}$$

We will prove that $X^\dagger = S^\dagger \times \mathbb{P}^1$ by dividing our situation into the two cases.

Case 1 (The case where $S^\dagger$ is a torus.)**.** By applying Theorem 2.6 to $X^\dagger \to S^\dagger$, we see that the sheaf $\mathcal{V}_m^\dagger$ defined by $L := -K_{X^\dagger}$ is a numerical flat vector bundle over a torus $S^\dagger$ which satisfies the relation $\mathcal{V}_m^\dagger = (\tau^*\mathcal{V}_m)^{**}$. By Simpson's correspondence and [DPS94, Theorem 1.18], the vector bundle $\mathcal{V}_m^\dagger$ is flat; thus φ' is locally constant by [MW21, Proposition 2.5]. This implies that $X^\dagger = \mathbb{P}(E)$ for some numerical flat vector bundle E on $S^\dagger$ by the same argument as in the proof of Theorem 4.2.

We prove that E is a trivial vector bundle. For this purpose, we first show that any G-equivariant Hermitian flat line bundle over $S^\dagger$ is trivial, where G a finite group acting on $S^\dagger$ such that $S = S^\dagger/G$ which is free in codimension 1. Since $\pi_1(X) \cong$

$\pi_1(S)$ by [Kol93, Theorem 5.2] and [BC15, Theorem 4.1], the assumption shows that S is simply connected; thus we have $H^1(S^\dagger/G, \mathbb{C}) = 0$. Since the locally constant sheaf $\mathbb{C}_{S^\dagger/G}$ admits a soft resolution by G-equivariant smooth forms, we see that $0 = H^1(S^\dagger/G, \mathbb{C}) \cong H^1(S^\dagger, \mathbb{C})^G$. A Hermitian flat line bundle on $S^\dagger$ corresponds to an element in $\mathrm{Pic}^0(S^\dagger) := H^1(S^\dagger, \mathcal{O}_{S^\dagger})/\mathrm{Im}\,(H^1(S^\dagger, \mathbb{Z}))$; furthermore, if the line bundle L is G-equivariant, then L is represented by an element in $H^1(S^\dagger, \mathcal{O}_{S^\dagger})^G$; hence L is actually a trivial line bundle.

We finally show that $\mathcal{V}_m^\dagger = (\tau^*\mathcal{V}_m)^{**}$ is a trivial vector bundle on $S^\dagger$. Each graded piece of $\mathcal{V}_m^\dagger$ (i.e. $\tau^*(\mathcal{F}_k/\mathcal{F}_{k-1})$ in (4.1)) is G-equivariant by $\mathcal{V}_m^\dagger = (\tau^*\mathcal{V}_m)^{**}$ and a line bundle since $\pi_1(S^\dagger)$ is abelian; hence the graded pieces are trivial line bundles. Furthermore, the extension class of the pull-back of $0 \to \mathcal{F}_{k-1} \to \mathcal{F}_k \to \mathcal{F}_k/\mathcal{F}_{k-1} \to 0$ lies in $H^1(S^\dagger, \mathcal{O}_{S^\dagger})^G$; hence $\mathcal{V}_m^\dagger = (\tau^*\mathcal{V}_m)^{**}$ is a trivial vector bundle. We can conclude that $X^\dagger = S^\dagger \times \mathbb{P}^1$ by the proof of [MW21, Proposition 2.5].

Case 2 (The case where $S^\dagger$ is a normal K3 surface.)**.** We first show that $\pi_1(X^\dagger) = \{\mathrm{id}\}$. The variety $S^\dagger$ is the universal cover in the sense of orbifolds (see [Cam04, Définition 5.3]); hence we see that $\pi_1(S^\dagger_{\mathrm{reg}}) = \pi_1(S^\dagger) = \{\mathrm{id}\}$. Since φ is a smooth $\mathbb{P}^1$-bundle over S_{reg} (which is preserved under the base change), we see that $\pi_1((\varphi^\dagger)^{-1}(S^\dagger_{\mathrm{reg}})) = \{\mathrm{id}\}$. By Lemma 3.2, near a singular point of S, the MF space $\varphi : X' \to S$ can be described as $\varphi : (\mathbb{P}^1 \times \mathbb{C}^2)/\mu_m \to \mathbb{C}^2/\mu_m$. Then τ is étale or τ is given by $\mathbb{C}^2/\mu_{m^\dagger} \to \mathbb{C}^2/\mu_m$ induced by an inclusion $\mu_{m^\dagger} \to \mu_m$. In the latter case, the base change is locally given as follows:

$$\begin{tikzcd}
(\mathbb{P}^1 \times \mathbb{C}^2)/\mu_{m^\dagger} \arrow[r, "\nu"] \arrow[d, "\varphi^\dagger"] & (\mathbb{P}^1 \times \mathbb{C}^2)/\mu_m \arrow[d, "\varphi"] \\
\mathbb{C}^2/\mu_{m^\dagger} \arrow[r, "\tau"] & \mathbb{C}^2/\mu_m,
\end{tikzcd}$$

where the action is given by $\varepsilon \cdot (t, z_1, z_2) = (\varepsilon^b t, \varepsilon z_1, \varepsilon^{-1} z_2)$ for $\varepsilon \in \mu_{m^\dagger}$. The Van Kampan theorem shows that $\pi_1(X^\dagger_{\mathrm{reg}}) = \{\mathrm{id}\}$; In particular, we obtain $\pi_1(X^\dagger) = \{\mathrm{id}\}$.

The orbifold vector bundle $\varphi^{\dagger *}\mathcal{V}_m^\dagger$ is numerically flat and can be written as successive extension of Hermitian flat orbifold vector bundles $\varphi^{\dagger *}\tau^*(\mathcal{F}_k/\mathcal{F}_{k-1})$ as in (4.1). Thus, the graded pieces $\varphi^{\dagger *}\tau^*(\mathcal{F}_k/\mathcal{F}_{k-1})$ on $X^\dagger_{\mathrm{reg}}$ are given by a GL-representation of $\pi_1(X^\dagger_{\mathrm{reg}})$; hence they are trivial by $\pi_1(X^\dagger_{\mathrm{reg}}) = \{\mathrm{id}\}$. Since $X^\dagger$ is simply connected compact Kähler orbifold, we have $H^1(X^\dagger, \mathcal{O}_{X^\dagger}) = 0$. By Lemma 4.5 below (which is proved later), we see that $H^1(X^\dagger_{\mathrm{reg}}, \mathcal{O}_{X^\dagger_{\mathrm{reg}}}) = 0$. Thus, the extension class of the trivial vector bundles is trivial. This implies that $\varphi^{\dagger *}\mathcal{V}_m^\dagger$ is trivial over $X^\dagger_{\mathrm{reg}}$ (and thus on $X^\dagger$ by reflexivity).

The above argument shows that $\mathcal{V}_m^\dagger$ is trivial over $S^\dagger_{\mathrm{reg}}$. By [MW21, Proposition 2.5], we see that $\varphi^\dagger : X^\dagger \to S^\dagger$ is locally constant over $S^\dagger_{\mathrm{reg}}$; hence $X^\dagger$ is the product $S^\dagger_{\mathrm{reg}} \times \mathbb{P}^1$

over $S^\dagger_{\mathrm{reg}}$ by $\pi_1(S^\dagger_{\mathrm{reg}}) = \{\mathrm{id}\}$. Since the fiber over the singular point of $S^\dagger \times \mathbb{P}^1$ cannot be contracted, [Kol91, Corollary 2.1.13] implies that $X^\dagger = S^\dagger \times \mathbb{P}^1$.

We finally deduce that $\pi : X \dashrightarrow X'$ is actually isomorphic. The strategy is similar to Theorem 4.2, but we need to calculate intersection numbers on the quasi-étale cover $X^\dagger$. Then, since X' is smooth by $X \cong X'$, we see $\varphi : X \cong X' \to S$ is a locally constant $\mathbb{P}^1$-bundle by same argument as in Theorem 4.2, which finished the proof.

Assume that $\pi : X \dashrightarrow X'$ is not isomorphic. We show that the last step $\pi : X_{N-1} \dashrightarrow X_N = X'$ of the MMP cannot be a flip. Let d be the degree of finite morphism $\nu : X^\dagger \to X'$. Note that fibers of $\varphi : X' \to S$ over $S \setminus S_{\mathrm{reg}}$ are $K_{X'}$-negative rational curves by local structure of $\mathbb{Q}$-conic bundle (see Lemma 3.2). For other rational curves C in X', the restriction $\nu_{|\nu^{-1}(C)} : \nu^{-1}(C) \to C$ of ν is a ramified cover of degree d and we have the negative intersection number with $K_{X'}$ since

$$(c_1(K_{X'}) \cdot C) = \frac{1}{d}(c_1(K_{X^\dagger}) \cdot \nu^{-1}(C)) \leq 0.$$

Thus the last step in the MMP $X \dashrightarrow X'$ cannot be a flip.

If $\pi : X_{N-1} \dashrightarrow X_N = X'$ contracts a surface to a point, we have $K^2_{X'} = (1/d)\nu_* K^2_{X^\dagger} = 0$ since $K^2_{X^\dagger} = 0$ and $\nu : X^\dagger \to X'$ is an orbifold morphism. Then, the same argument as in Theorem 4.2 works; hence we deduce a contradiction, which implies that $X \cong X'$.

If $\pi : X_{N-1} \dashrightarrow X_N = X'$ contracts a surface to a curve, the variety X' is covered by disjoint surfaces $\nu(S^\dagger \times \{t\})$ for $t \in \mathbb{P}^1$. There exists a surface in X' whose strict transform intersects the exceptional divisor along a non-trivial effective curve. By the same argument as in Theorem 4.2, we deduce a contradiction, which implies that $X \cong X'$. ∎

The following lemma is an easy variant of [Wu22a, Lemma 4].

Lemma 4.5. *Let X be a (non necessarily compact) analytic variety of dimension 3 and let E be a vector bundle on X. Assume that X has isolated cyclic quotient singularities; more precisely, near any singular point, there is $m \in \mathbb{Z}+$ such that $X \cong \mathbb{D}/\mu_m$, where $\mathbb{D}$ is a polydisc centered at origin and the action of μ_m is given by for any $\varepsilon \in \mu_m$*

$$\varepsilon \cdot (z_0, z_1, z_2) = (\varepsilon^{p_0} z_0, \varepsilon^{p_1} z_1, \varepsilon^{p_2} z_2)$$

for some $p_i \in \mathbb{Z}$. Then the morphism $H^1(X, E) \to H^1(X_{\mathrm{reg}}, E)$ induced by the restriction morphism is surjective.

Proof. Set $\mathbf{0} := (0,0,0)$. We first prove that

$$H^1((\mathbb{C}^3 \setminus \{\mathbf{0}\})/\mu_m, \mathcal{O}_{(\mathbb{C}^3 \setminus \{\mathbf{0}\})/\mu_m}) = 0$$

by direct calculation. Cover $(\mathbb{C}^3 \setminus \{\mathbf{0}\})/\mu_m$ by three Stein open sets isomorphic to $(\mathbb{C}^* \times \mathbb{C}^2)/\mu_m$, say $U_i = \{z_i \neq 0\}$, with a coordinate (z_0, z_1, z_2). A 1-cochain can be identified with a triple of convergent power series (f_{01}, f_{02}, f_{12}) with f_{12} of type

$$\sum_{(\alpha,\beta,\gamma)\in\mathbb{Z}^2\times\mathbb{Z}_+} c_{\alpha\beta\gamma} z_0^\alpha z_1^\beta z_2^\gamma$$

over $\mathbb{C}^{*2}\times\mathbb{C}$ (the intersection of two Stein open sets) invariant under the group action. Similarly, we see that f_{02} is a sum over $(\alpha,\beta,\gamma)\in\mathbb{Z}\times\mathbb{Z}_+\times\mathbb{Z}$ and f_{01} is a sum over $(\alpha,\beta,\gamma)\in\mathbb{Z}_+\times\mathbb{Z}^2$.

The condition that (f_{01}, f_{02}, f_{12}) is closed means that $f_{01}-f_{02}+f_{12}=0$ on the intersection of the three Stein open sets $U_0\cap U_1\cap U_2$, isomorphic to $\mathbb{C}^{*3}/\mu_m$. We can write f_{01} as a sum of three convergent power series g_{01}^0, g_{01}^1, g_{01} such that g_{01} has only positive power terms, g_{01}^0 has only negative power terms in z_0 and g_{01}^1 has only negative power terms in z_1. Each term is invariant under the group action. Similarly, we decompose f_{02}, f_{12}. Now the closeness condition is equivalent to

$$g_{01}-g_{02}+g_{12}=0,\quad g_{01}^0=g_{02}^0,\quad g_{12}^2=g_{02}^2,\quad g_{01}^1+g_{12}^1=0.$$

We define a 0-cochain in such a way that its differential is (f_{01}, f_{02}, f_{12}). On U_0, resp. U_1, U_2, we take the convergent power series $g_{01}+g_{01}^0$, resp. g_{12}^1, $-g_{12}-g_{02}^2$. This implies that every 1-cocycle is exact, hence

$$H^1((\mathbb{C}^3\setminus\{\mathbf{0}\})/\mu_m, \mathcal{O}_{(\mathbb{C}^3\setminus\{\mathbf{0}\})/\mu_m})=0.$$

Now, on every polydisc $\mathbb{D}$ in $\mathbb{C}^3$, a holomorphic function is uniquely determined by its Taylor expansion at origin, and the same calculation shows that

$$H^1((\mathbb{D}\setminus\{\mathbf{0}\})/\mu_m, \mathcal{O}_{(\mathbb{D}\setminus\{\mathbf{0}\})/\mu_m})=0.$$

We now return to the general case. Cover X by the Stein open sets U_α and $B_\beta := \mathbb{D}_\beta/\mu_{m_\beta}$ such that X_{reg} is covered by U_α and $B_\beta\setminus\{\mathbf{0}\}$ where $\mathbb{D}_\beta$ are polydiscs with dimension 3 with some cyclic group action μ_{m_β} free in codimension 2. Without loss of generality, we may assume B_β's do not intersect each other. Assume that E is trivial on U_α and B_β. Cover $B_\beta\setminus\{\mathbf{0}\}$ by B_β^γ $(1\le\gamma\le 3)$ such that each B_β^γ is isomorphic to quotient of a polydisc minus a hyperplane defined as zero set of one coordinate. Since U_α, B_β^γ are Stein, the cohomology on X_0 can be calculated as the Čech cohomology with respect to this open covering of X_0, which we denote by $\mathcal{V}$. We also denote by $\mathcal{U}$ the open covering of X consisting of the sets U_α, B_β. Any element s of $H^1(X_0, E)$ can be represented by a family of sections

$$(s_{\alpha_1,\alpha_2}, s_{\alpha\beta}^\gamma, s_\beta^{\gamma_1,\gamma_2})\in\prod\Gamma(U_{\alpha_1}\cap U_{\alpha_2}, E)\times\prod\Gamma(U_\alpha\cap B_\beta^\gamma, E)\times\prod\Gamma(B_\beta^{\gamma_1}\cap B_\beta^{\gamma_2}).$$

Since $H^1(B_\beta, E)=0$ by the previous case, there exists

$$(s_\beta^\gamma)\in\prod\Gamma(B_\beta^\gamma, E)$$

such that for any β fixed

$$s_\beta^{\gamma_1,\gamma_2} = (-1)^{\gamma_1+1} s_\beta^{\gamma_1} + (-1)^{\gamma_2+1} s_\beta^{\gamma_2}.$$

Define a 0-cochain

$$(s_\beta^\gamma, 0) \in \prod \Gamma(B_\beta^\gamma, E) \times \prod \Gamma(U_\alpha, E).$$

Then we have $(s_{\alpha_1,\alpha_2}, s_{\alpha\beta}^\gamma, s_\beta^{\gamma_1,\gamma_2}) + \delta(-s_\beta^\gamma, 0)$ as another representative of the same cohomology class on X_0. The components in $\Gamma(B_\beta^{\gamma_1} \cap B_\beta^{\gamma_2}, E)$ are 0 by construction. Thus we can assume that the components in $\Gamma(B_\beta^{\gamma_1} \cap B_\beta^{\gamma_2}, E)$ are 0 from the beginning.

Since the representative is closed, the components in $\Gamma(B_\beta^\gamma \cap U_\alpha, E)$ glue to a section $s_{\alpha,\beta} \in \Gamma((B_\beta \setminus \{\mathbf{0}\}) \cap U_\alpha, E)$ when γ varies. By the Hartogs theorem, this section extends across the origin.

We claim that after performing this glueing, the sections

$$(s_{\alpha_1,\alpha_2}, s_{\alpha,\beta}) \in \prod \Gamma(U_{\alpha_1} \cap U_{\alpha_2}, E) \times \prod \Gamma(U_\alpha \cap B_\beta, E)$$

define a 1-cocycle of X with respect to the open covering U_α, B_β, and that its class in $H^1(X_0, E)$ is exactly s. Indeed, the image of $(s_{\alpha_1,\alpha_2}, s_{\alpha,\beta})$ from $H^1(\mathcal{U}, E)$ to $H^1(\mathcal{U} \cap X_0, E)$ is just the restriction of sections. The covering $\mathcal{V}$ is a refinement of $\mathcal{U} \cap X_0$ given by the inclusion of open sets: $U_\alpha \subset U_\alpha$, $B_\beta^\gamma \subset B_\beta$. The image under this refinement of open sets is precisely s. □

We finally check that Theorem 1.2 follows from Theorem 1.3 by using almost the same arguments as in [CH19, Theorem 1.4].

Proposition 4.6. *Theorem 1.2 follows from Theorem 1.3.*

Proof. Let X be a non-projective compact Kähler 3-fold with nef anti-canonical bundle. Then, the manifold X has a finite étale cover X' that is one of the list in Theorem 1.3. If $c_1(X') = 0$, then $c_1(X) = 0$; hence we may assume that X' admits a non-trivial locally constant MRC fibration. We first show that X admits a locally trivial MRC fibration $X \to Y$ onto a smooth surface with $c_1(Y) = 0$. Let $\mathcal{F} \subset T_X$ be the unique integrable saturated subsheaf such that for a very general point $x \in X$, the $\mathcal{F}$-leaf through x is a fiber of the MRC-fibration. We claim that $\mathcal{F}$ is a regular foliation which is invariant under passing to finite étale covers. The claim then follows from Theorem 1.3. By [Hor07, Corollary 2.11], there exists a smooth morphism $\varphi : X \to Y$ such that $T_{X/Y} = \mathcal{F}$ where Y is the leaf space. Since the general fiber is $\mathbb{P}^1$, by the Firscher-Grauert theorem, φ is locally trivial. By Proposition 3.5, the line bundle $-K_Y$ is pseudo-effective. Since Y is not uniruled, we have $c_1(Y) = 0$.

The fibration $\varphi : X \to Y$ is the MF space of X; hence $-K_X$ is φ-ample line bundle. By Theorem 2.6, since Y is smooth, we can find a φ-ample line bundle B such that $\varphi_*(pB)$ is numerically flat for $1 \ll p \in \mathbb{Z}_+$. Then, we see that $\varphi : X \to Y$ is locally constant by [MW21, Proposition 2.5]. (In this case, the base Y is a K3 surface, Enriques surface, or a étale quotient of a torus.) □

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