Week 9

Jinxi Xiao

November 21, 2023

E-mail: xiaojx@shanghaitech.edu.cn

Definition

Theorem:

Let V be a $\mathbb{F}-$vector space and a non-empty set $W \subseteq V$. Then W is a subspace if and only if the following conditions holds:

- $\forall u, v \in W, u + v \in W$
- $\forall k \in \mathbb{F}$ and $u \in W, ku \in W$

Examples

Examples of Subspaces

Example. The following are some examples of subspaces.
(1) The set $C(-\infty, +\infty)$ of all continuous functions on $(-\infty, +\infty)$ is a subspace of the vector space $F(-\infty, +\infty)$ of all functions on $(-\infty, +\infty)$.
(2) The set $C^1(-\infty, \infty)$ of all functions on $(-\infty, +\infty)$ with continuous derivative is a subspace of $C(-\infty, +\infty)$.
(3) The set $C^\infty(-\infty, +\infty)$ of all functions on $(-\infty, +\infty)$ which have derivatives of all order is a subspace of $C^1(-\infty, +\infty)$.
(4) The set P_∞ of all polynomials if a subspace of $C(-\infty, +\infty)$.
(5) The set P_n of all polynomials of degree $\leq n$ is a subspace of P_∞.

Question: Is the set of all polynomials of degree n a subspace of P_∞?

Example. The following are some examples of subspaces.
(6) The set U of all symmetric matrix of order n is a subspace of M_n.
(7) The set V of all $n \times n$ upper triangular matrices is a subspace of M_n.
(8) The set of all diagonal matrix of order n is a subspace of either U or V.

Building Subspaces

Theorem

If U, W are subspaces of V, then

- $U \cap W$ and $U + W == span(U \cup W)$ are subspacs of V
- $U \cup W$ may not

Spanning a Subspace

$$W = span(s) = span\{w_1, w_2, \cdots, w_r\}$$

A lot more properties can be found here, please refer to the slides.

Column & Row Space

Definition. Let $A \in M_{m\times n}(\mathbb{F})$. Suppose that

$$A = [a_{ij}]_{m\times n} = \begin{bmatrix} \mathbf{r}_1 \\ \hdashline \mathbf{r}_2 \\ \hdashline \vdots \\ \hdashline \mathbf{r}_m \end{bmatrix} = \left[\begin{array}{c:c:c:c} \mathbf{c}_1 & \mathbf{c}_2 & \dots & \mathbf{c}_n \end{array} \right].$$

The subspace $\text{span}\{\mathbf{r}_1, \dots, \mathbf{r}_m\}$ of $\mathbb{F}^n$ is called the row space of A.
The subspace $\text{span}\{\mathbf{c}_1, \dots, \mathbf{c}_n\}$ of $\mathbb{F}^m$ is called the column space of A.

Notations (only in this class):

$$Row(A) := \text{span}\{\mathbf{r}_1, \dots, \mathbf{r}_m\}, \quad Col(A) := \text{span}\{\mathbf{c}_1, \dots, \mathbf{c}_n\}.$$

Concept

Definition. Let $S = \{\mathbf{v}_1, \cdots, \mathbf{v}_r\}$ be a non-empty set in a vector space V.

◇ If the equation

$$k_1\mathbf{v}_1 + k_2\mathbf{v}_2 + \cdots k_r\mathbf{v}_r = \mathbf{0}$$

has only the trivial solution $k_i = 0$ $(1 \leq i \leq r)$, then the set S (or these vectors) is said to be linearly independent.

◇ If the equation has non-trivial solutions, then S (or these vectors) is said to be linearly dependent.

Exercises

Problem 1

If A is a square matrix of order n and A is singular, then prove that adj(A) is also singular and Aadj(A) is zero matrix.

Problem 2

Prove that the solution set on a homogeneous linear system $Ax = 0$ with $x \in \mathbb{R}^n$ is a subspace of $\mathbb{R}^n$