Week 4

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October 17, 2023

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Multiplication of Partitioned Matrices

Example. Let $A \in M_{m\times r}$ and $B \in M_{r\times n}$. Suppose that

$$A = \begin{bmatrix} \mathbf{a}_1 \\ \hdashline \mathbf{a}_2 \\ \hdashline \vdots \\ \hdashline \mathbf{a}_m \end{bmatrix}, \quad B = \left[\begin{array}{c:c:c:c} \mathbf{b}_1 & \mathbf{b}_2 & \dots & \mathbf{b}_n \end{array}\right].$$

Then

$$AB = \begin{bmatrix} \mathbf{a}_1\mathbf{b}_1 & \mathbf{a}_1\mathbf{b}_2 & \dots & \mathbf{a}_1\mathbf{b}_n \\ \mathbf{a}_2\mathbf{b}_1 & \mathbf{a}_2\mathbf{b}_2 & \dots & \mathbf{a}_2\mathbf{b}_n \\ \dots & \dots & \dots & \dots \\ \mathbf{a}_m\mathbf{b}_1 & \mathbf{a}_m\mathbf{b}_2 & \dots & \mathbf{a}_m\mathbf{b}_n \end{bmatrix}$$

$$= \begin{bmatrix} \mathbf{a}_1 B \\ \hdashline \mathbf{a}_2 B \\ \hdashline \vdots \\ \hdashline \mathbf{a}_m B \end{bmatrix} = \left[\begin{array}{c:c:c:c} A\mathbf{b}_1 & A\mathbf{b}_2 & \dots & A\mathbf{b}_n \end{array}\right].$$

Linear Combination of Matrices

$$\begin{cases} a_{11}x_1 + & \cdots & + & a_{1n}x_n & = & b_n \\ & \vdots & & & & \\ a_{m1}x_1 + & \cdots & + & a_{mn}x_n & = & b_m \end{cases} \Rightarrow \begin{bmatrix} a_{11} & \cdots & a_{1n} \\ & \vdots & \\ a_{m1} & \cdots & a_{mn} \end{bmatrix} \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} b_1 \\ \vdots \\ b_n \end{bmatrix}$$

$$\Rightarrow x_1 \begin{bmatrix} a_{11} \\ \vdots \\ a_{m1} \end{bmatrix} + \cdots + x_n \begin{bmatrix} a_{1n} \\ \vdots \\ a_{mn} \end{bmatrix} = \begin{bmatrix} b_1 \\ \vdots \\ b_n \end{bmatrix}$$

Properties of Matrix Arithmetic

Properties

- $A + B = B + A$
- $A(BC) = (AB)C$
- $A(B + C) = AB + AC$
- $(B + C)A = BA + CA$
- AB is not always equivalent to BA

$A = \mathbf{0}$?

If $\mathbf{x}$ is not zero vector, and $A\mathbf{x} = \mathbf{0}$, can we infer $A = \mathbf{0}_{m\times n}$?

Concepts

Please Remember These

- Zero Matrix
- Identity Matrix
- Inverse
- Powers of Matrix
- *Polynomials of Matrices*
- Elementary Matrices

Elementary Matrices

Basic Understanding

- **Def:** An n $\times$ n matrix is called an elementary matrix if it can be obtained from In by performing a single elementary row operation.
- If we left-multiple one elementary matrix, we are simply doing the corresponding elementary row operation on the right matrix!

Inverse

Properties

- For a fixed matrix, its inverse is unique
- $(AB)^{-1} = B^{-1}A^{-1}$

The inverse algorithm

- Known Fact: (1) A is invertible $\Longrightarrow E_k \dots E_2 E_1 A = I$;
  (2) $A^{-1} = E_k \dots E_2 E_1$;
  (2) $E_k \dots E_2 E_1 = (E_k \dots E_2 E_1) I$.
- ALGORITHM: By row operations,

$$\left[\begin{array}{c:c:c} A & I & B \end{array}\right] \xrightarrow{\text{row}} \left[\begin{array}{c:c:c} I & A^{-1} & A^{-1}B \end{array}\right]$$

Transpose and Trace

Properties

- $(A^T)^T = A \qquad (A+B)^T = A^T + B^T$
- $(AB)^T = B^T A^T$
- $(A^T)^{-1} = (A^{-1})^T$
- $tr(A+B) = tr(A) + tr(B)$
- $tr(AB) = tr(BA)$

Equivalence Theorem

We will see this throughout this course

Theorem. If A is an $n \times n$ matrix, then the following statements are equivalent.
(1) A is invertible.
(2) $A\mathbf{x} = \mathbf{0}$ has only the trivial solution.
(3) The reduced row echelon form of A is I_n.
(4) A is expressible as a product of elementary matrices.

Need to Remember

Need To Know

- No division in matrix operations!
- Do not arbitrarily change the order of matrix multiplication
- Look at the PPT carefully!s

Problem

Problem 2.2: (2.3 #29. *Introduction to Linear Algebra:* Strang) Find the triangular matrix E that reduces "*Pascal's matrix*" to a smaller Pascal:

$$E\begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 2 & 1 & 0 \\ 1 & 3 & 3 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 2 & 1 \end{bmatrix}.$$

Which matrix M (multiplying several E's) reduces Pascal all the way to I?

Solution

Solution:

$$\text{The matrix is } E = \begin{bmatrix} 1 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & -1 & 1 \end{bmatrix}$$

One can eliminate the second column with the matrix

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & -1 & 1 \end{bmatrix}$$

and the third column with the matrix

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & -1 & 1 \end{bmatrix}$$

Multiplying these together, we get

$$M = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & -1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & -1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & -1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ 1 & -2 & 1 & 0 \\ -1 & 3 & -3 & 1 \end{bmatrix}.$$

Since M reduces the Pascal matrix to I, M must be the inverse matrix!