Week 10

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Basis

Definition

Let V be a vector space of and S be a finite set of vectors in V. If

- S spans V
- S is linearly independent

Then we say that S is a **basis** for V

Properties of Basis

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- Unique Expression
- All bases for a finite-dimensional vector space have the same number of vectors
- Basis is the **minimum** vectors set that spans the space

Coordinates

Definition. Let $S = \{\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_n\}$ be a basis for V and

$$\mathbf{v} = c_1\mathbf{u}_1 + c_2\mathbf{u}_2 + \cdots + c_n\mathbf{u}_n.$$

is the expression of $\mathbf{v} \in V$. Then the coordinate vector of $\mathbf{v}$ relative to S, and the coordinate matrix of $\mathbf{v}$ relative to S, are defined and denoted by

$$(\mathbf{v})_S = (c_1, c_2, \ldots, c_n), \qquad [\mathbf{v}]_S = \begin{bmatrix} c_1 \\ \vdots \\ c_n \end{bmatrix},$$

respectively.

Theorem

Theorem.
(1) Elementary row operations do not change the null space of a matrix.
(2) Elementary row operations do not change the row space of a matrix.
(3) Elementary row operations do not change the dependence relationships among the column vectors.

Theorem. Suppose that a matrix A has row echelon form R.
◇ The row vectors with the leading $1'$s in R form a basis for $Row(A) = Row(R)$.
◇ The column vectors with the leading $1'$s in R form a basis for $Col(R)$; the corresponding column vectors of A form a basis for $Col(A)$.

Dimension

Definition

The **dimension** of a finite-dimensional vector space V, denoted by **dim(V)**, is defined to be the number of basis for V.

Properties

Refer to Your Slides!

- Let V be an n-dimensional vector space, and let S be a set in V with exactly n vectors. Then S is a basis for V if and only if S spans V or S is linearly independent.
- Let W be a finite-dimensional vector space with U, V two subspaces. Then

$$dim(U + V) = dim(U) + dim(V) - dim(V \cap V)$$

- $\cdots$

Recall

Definition. Suppose that an echelon form of a matrix A has r non-zero rows. Then we say that A has rank r, and denote $\text{rank}(A) = r$.

Theorem. For any matrix $A \in M_{m\times n}$, there is an integer $r \leq \min\{m, n\}$, an invertible matrix $P \in M_m$ and an invertible matrix $Q \in M_n$ such that

$$PAQ = \begin{bmatrix} I_r & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{bmatrix}$$

Rank of a Matrix

Nullity

The dimension of the null space of A is called the nullity of A denoted by nullity(A).

Theorem: Matrix A $\in M_{m\times n}$

- rank(A) $=$ dim($Row(A)$) $=$ dim($Col(A)$)
- rank(A) $\leq min\{m, n\}$, rank(A) $=$ rank(A^T)
- Elementary row or column operations do not change the rank of a matrix
- rank(A) $+$ nullity(A) $=$ n

We will see this again later

Theorem

If A is an $n \times n$ matrix, then the following statements are equivalent:

- A is invertible
- $Ax = 0$ has only the trivial solution
- The reduced row echelon form of A is I_n
- A is expressible as a product of elementary matrices
- $Ax = b$ is consistent for every b, and x is unique
- $det(A) \neq 0$
- rank(A) = n, nullity(A) = 0

Exercise

Let $A \in M_{m \times n}$, prove that the null space of A is equivalent to the null space of $A^T A$