Week 6

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November 1, 2023

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Quiz and Exercise

Remark

If you want to learn something, you have to spend time on it.

This question is about an m by n matrix A for which

$$Ax = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \text{ has } \textbf{no solutions} \quad \text{and} \quad Ax = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \text{ has } \textbf{exactly one solution}.$$

(a) Give all possible information about m and n and the rank r of A.

(b) Find all solutions to $Ax = 0$ and **explain your answer**.

(c) Write down an example of a matrix A that fits the description in part (a).

Solution

Solution.

(a) $Ax = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$ has *one* solution $\implies N(A) = \{0\}$ so $r = n$. (Also, $m = 3$ since $Ax \in \mathbb{R}^3$.)

$Ax = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ has no solution $\implies C(A) \neq \mathbb{R}^3$, so $r < m$.

There are two possibilities: $\begin{matrix} m = 3 \\ r = n = 1 \end{matrix}$ and $\begin{matrix} m = 3 \\ r = n = 2 \end{matrix}$.

(b) Since $N(A) = \{0\}$ (because $Ax = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$ has **1** solution), there is a unique solution to $Ax = 0$, which is clearly $x = 0$. (Can be either $x = \begin{bmatrix} 0 \end{bmatrix}$ or $x = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ depending on if $n = 1$ or $n = 2$.)

(c) A could be $\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$ or $\begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$ (many more possibilities).

Linear Combination

Define a set of vectors in $\mathbb{R}^n$ as $\{v_1, v_2, \cdots, v_k\}$. The linear combination of this set of vectors is

$$w = c_1 v_1 + c_2 v_2 + \cdots + c_k v_k$$

L2 Norm

Let $v \in \mathbb{R}^n = (v_1, codts, v_n)$, we define its norm to be

$$\sqrt{v_1^2 + \cdots + v_n^2}$$

Concepts

- Normalization
- Standard Unit Vectors(bases)

Propositions

- $|u \cdot v| \leq ||u|| \cdot ||v||$
- $||u + v|| \leq ||u|| + ||v||$
- $||u + v||^2 + ||u - v||^2 = ||u||^2 + ||v||^2$
- $u \cdot v = \frac{1}{4}(||u + v||^2 + ||u - v||^2)$
- $u \cdot v = u^T v = v^T u$
- $\cdots$

Orthogonality and Projection

They are much more powerful than you think. In 3D world, it is more related to planes and lines.

$$Proj_a u = \frac{u \cdot a}{||a||^2} a$$

Definitions and Properties

Go and revise your PPT!

A few notes

- Can we change its sequence? $i \times j \times j$
- How to understand $(a \times b) \cdot c$?

More About Cross Product

$$a \times b = \begin{bmatrix} 0 & -a_3 & a_2 \\ a_3 & 0 & -a_1 \\ -a_2 & a_1 & 0 \end{bmatrix} b$$

已知：二维平面的两点X（x1，y1），Y（x2，y2），证明X，Y两点的齐次式叉乘为过XY的直线的系数.

证明： 叉乘的定义为已知向量a = (a1,a2,a3), b=(b1,b2,b3), a叉乘b=(a2b3-a3b2, a3b1-a1b3, a1b2-a2b1)

因为XY的齐次式为（x1,y1,1)和(x2,y2,1)，代入叉乘的定义得（y1-y2, x2-x1, x1y2-y1x2)

定义直线的表达式为y=kx + b,将XY代入得：

y1 = kx1 + b

y2 = kx2 + b

化简后得：

k = (y2-y1)/(x2-x1)

b = y1 - （(y2-y1)/(x2-x1)） * x1

将y = kx + b 转化为 ax + by + c = 0的形式得 （a b c） = （-k，1， -b） 化简后等于（y1-y2, x2-x1, x1y2-y1x2)

Need to Remember

Need To Know

- **Practice your calculation skills**
- Revise the PPT
- Do not take things as granted