Angle and Orthogonality

Gram-Schmidt Process

QR-Decomposition

Advise for review

Week 15

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Norm and Distance

Theorm

- $$|\langle u, v\rangle| \leq ||u|| \cdot ||v||$$
- $$||u+v|| \leq ||u|| + ||v||$$

Definition

$$\theta = \arccos \frac{\langle u, v\rangle}{||u|| \cdot ||v||}$$

Orthogonality

Def

We say that u and v are orthogonal if $\langle u, v\rangle = 0$

Orthogonal Complements

Definition. If W is a subspace of an inner product space V, then the orthogonal complement of W is defined by

$$W^{\perp} = \{\mathbf{v} : \langle \mathbf{v}, \mathbf{w}\rangle = 0\,(\forall \mathbf{w} \in W)\}.$$

Theorem. Let W be a subspace of an inner product space V.
(1) $W^{\perp}$ is a subspace of V.
(2) $W^{\perp} \cap W = \{\mathbf{0}\}$.
(3) If V is finite-dimensional, then $(W^{\perp})^{\perp} = W$.

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Orthonormal Sets

Recall

Notice that standard bases that we have learnt are all orthonormal under the basic inner product.

Philosophy: Basis is designed for a vector space. Now there is a inner product, the basis vectors can have "angles".

Definition.

◇ A set of two or more vectors in an inner product space is called orthogonal if all pairs of distinct vectors in the set are orthogonal.

◇ An orthogonal set in which each vectors has norm 1 is called orthonormal.

◇ If a basis is orthogonal, then we say it is an orthogonal basis.

◇ If a basis is orthonormal, then we say it is an orthonormal basis.

Properties of Orthononal

Properties of Orthogonal Basis

Theorem. If $S = \{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n\}$ is an orthogonal set of non-zero vectors in an inner product space $(V, \langle\cdot,\cdot\rangle)$, then S is linearly independent.

Proof:

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Theorem. If $S = \{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n\}$ is an orthogonal basis for an inner product space V, and if $\mathbf{u}$ is any vector in V, then

$$\mathbf{u} = \frac{\langle \mathbf{u}, \mathbf{v}_1\rangle}{\|\mathbf{v}_1\|^2}\mathbf{v}_1 + \frac{\langle \mathbf{u}, \mathbf{v}_2\rangle}{\|\mathbf{v}_2\|^2}\mathbf{v}_2 + \cdots + \frac{\langle \mathbf{u}, \mathbf{v}_n\rangle}{\|\mathbf{v}_n\|^2}\mathbf{v}_n.$$

Proof:

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Remark: When S is an orthonormal basis,

$$\mathbf{u} = \langle \mathbf{u}, \mathbf{v}_1\rangle\mathbf{v}_1 + \langle \mathbf{u}, \mathbf{v}_2\rangle\mathbf{v}_2 + \cdots \langle \mathbf{u}, \mathbf{v}_n\rangle\mathbf{v}_n.$$

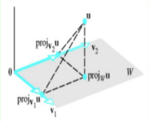
Orthogonal Projections

Orthogonal Projections

Theorem. (Projection Theorem) If W is a subspace of a finite-dimensional inner product space V, then every vector $\mathbf{u}$ in V can be expressed in exactly one way as

$$\mathbf{u} = \mathbf{w}_1 + \mathbf{w}_2,$$

where $\mathbf{w}_1 \in W$ and $\mathbf{w}_2 \in W^{\perp}$.



Remark: We denote $\mathbf{w}_1 = \text{proj}_W(\mathbf{u})$ and $\mathbf{w}_2 = \text{proj}_{W^{\perp}}(\mathbf{u})$.

Theorem. Let $(V, \langle \cdot, \cdot \rangle)$ be an inner product space and W be a subspace of V with an orthogonal basis $S = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_r\}$. Then for any $\mathbf{u} \in V$,

$$\text{proj}_W\mathbf{u} = \frac{\langle \mathbf{u}, \mathbf{v}_1 \rangle}{\|\mathbf{v}_1\|^2}\mathbf{v}_1 + \frac{\langle \mathbf{u}, \mathbf{v}_2 \rangle}{\|\mathbf{v}_2\|^2}\mathbf{v}_1 + \cdots + \frac{\langle \mathbf{u}, \mathbf{v}_r \rangle}{\|\mathbf{v}_r\|^2}\mathbf{v}_r$$

Proof:

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Remark: When S is an orthonormal basis for W,

$$\text{proj}_W\mathbf{u} = \langle \mathbf{u}, \mathbf{v}_1 \rangle \mathbf{v}_1 + \langle \mathbf{u}, \mathbf{v}_2 \rangle \mathbf{v}_2 + \cdots , + \langle \mathbf{u}, \mathbf{v}_r \rangle \mathbf{v}_r .$$

Gram-Schmidt Process

Gram-Schmidt Process

Theorem. Every nonzero finite-dimensional inner product space has an orthonormal basis.

Let $S = \{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n\}$ be a basis for V.

Step 1-1. Take $\mathbf{u}_1 = \mathbf{v}_1$.

Step 1-2. Take $\mathbf{u}_2 = \mathbf{v}_2 - \frac{\langle \mathbf{v}_2, \mathbf{u}_1 \rangle}{\|\mathbf{u}_1\|^2}\mathbf{u}_1 = \mathrm{proj}_{\mathrm{span}\{\mathbf{u}_1\}^\perp}(\mathbf{v}_2)$.

Step 1-3. Take $\mathbf{u}_3 = \mathbf{v}_3 - \frac{\langle \mathbf{v}_3, \mathbf{u}_1 \rangle}{\|\mathbf{u}_1\|^2}\mathbf{u}_1 - \frac{\langle \mathbf{v}_3, \mathbf{u}_2 \rangle}{\|\mathbf{u}_2\|^2}\mathbf{u}_2 = \mathrm{proj}_{\mathrm{span}\{\mathbf{u}_1, \mathbf{u}_2\}^\perp}(\mathbf{v}_3)$.

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Step 1-n. Take

$$\mathbf{u}_n = \mathbf{v}_n - \frac{\langle \mathbf{v}_n, \mathbf{u}_1 \rangle}{\|\mathbf{u}_1\|^2}\mathbf{u}_1 - \dots - \frac{\langle \mathbf{v}_n, \mathbf{u}_{n-1} \rangle}{\|\mathbf{u}_{n-1}\|^2}\mathbf{u}_{n-1} = \mathrm{proj}_{\mathrm{span}\{\mathbf{u}_1, \dots, \mathbf{u}_{n-1}\}^\perp}(\mathbf{v}_n).$$

Conclusion of step 1:

We obtain an orthogonal basis $S' = \{\mathbf{u}_1, \mathbf{u}_2, \dots, \mathbf{u}_n\}$ for V.

Step 2. Let $\mathbf{w}_i = \frac{\mathbf{u}_i}{\|\mathbf{u}_i\|}$ $(1 \le i \le n)$.

Conclusion of step 2:

We obtain an orthonormal basis $S'' = \{\mathbf{w}_1, \mathbf{w}_2, \dots, \mathbf{w}_n\}$ for V.

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Definition

QR-Decomposition

Theorem. Suppose that $A \in M_{m\times n}$ and $\operatorname{rank}(A) = n$. Then A can be factored as $A = QR$, where $Q \in M_{m\times n}$ is an matrix with orthonormal column vectors, and R is an invertible upper triangular matrix.

Indeed, let $A = \left[\begin{array}{c:c:c:c} \mathbf{c}_1 & \mathbf{c}_2 & \cdots & \mathbf{c}_n \end{array}\right]$. Suppose that we obtain orthonormal vectors $\mathbf{q}_1, \ldots, \mathbf{q}_n$ by applying the Gram-Schmidt process to $\mathbf{c}_1, \ldots, \mathbf{c}_n$. Then

$$A = \left[\begin{array}{c:c:c:c} \mathbf{c}_1 & \mathbf{c}_2 & \cdots & \mathbf{c}_n \end{array}\right] = \left[\begin{array}{c:c:c:c} \mathbf{q}_1 & \mathbf{q}_2 & \cdots & \mathbf{q}_n \end{array}\right] \begin{bmatrix} \langle \mathbf{c}_1, \mathbf{q}_1\rangle & \langle \mathbf{c}_2, \mathbf{q}_1\rangle & \cdots & \langle \mathbf{c}_n, \mathbf{q}_1\rangle \\ \langle \mathbf{c}_1, \mathbf{q}_2\rangle & \langle \mathbf{c}_2, \mathbf{q}_2\rangle & \cdots & \langle \mathbf{c}_n, \mathbf{q}_2\rangle \\ \vdots & \vdots & & \vdots \\ \langle \mathbf{c}_1, \mathbf{q}_n\rangle & \langle \mathbf{c}_2, \mathbf{q}_n\rangle & \cdots & \langle \mathbf{c}_n, \mathbf{q}_n\rangle \end{bmatrix}$$

$$= \left[\begin{array}{c:c:c:c} \mathbf{q}_1 & \mathbf{q}_2 & \cdots & \mathbf{q}_n \end{array}\right] \begin{bmatrix} \langle \mathbf{c}_1, \mathbf{q}_1\rangle & \langle \mathbf{c}_2, \mathbf{q}_1\rangle & \cdots & \langle \mathbf{c}_n, \mathbf{q}_1\rangle \\ & \langle \mathbf{c}_2, \mathbf{q}_2\rangle & \cdots & \langle \mathbf{c}_n, \mathbf{q}_1\rangle \\ & & \ddots & \vdots \\ & & & \langle \mathbf{c}_n, \mathbf{q}_n\rangle \end{bmatrix} = QR.$$

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Notes

- Should attach importance to the final exam.
- The most important thing is to go over the slides, and understand the basic concepts
- Then practice some problems (hw, exercises in the discussion class ...)