# Week 8

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# Matrix Transformation

## A Few Comments

- This topic will be discussed in the later chapters.
- In 2D and 3D space, this is of great importance. Since **geometry** is fascinating.

1. Matrix Transformation

2. Vector Space

3. Exercises
   - Matrix Operations
   - Dot Product
   - Determinant

# Eight Requirements

$$
\begin{array}{ll}
\textit{(A1)} & \mathbf{u}+\mathbf{v}=\mathbf{v}+\mathbf{u} \quad (\forall \mathbf{u},\mathbf{v}\in V). \\
\textit{(A2)} & \mathbf{u}+(\mathbf{v}+\mathbf{w})=(\mathbf{u}+\mathbf{v})+\mathbf{w} \quad (\forall \mathbf{u},\mathbf{v},\mathbf{w}\in V). \\
\textit{(A3)} & \exists \mathbf{0}\in V, \text{ s.t. } \mathbf{0}+\mathbf{u}=\mathbf{u}+\mathbf{0}=\mathbf{u} \quad (\forall \mathbf{u}\in V). \\
\textit{(A4)} & \forall \mathbf{u}\in V, \exists \mathbf{v}\in V, \text{ s.t. } \mathbf{u}+\mathbf{v}=\mathbf{v}+\mathbf{u}=\mathbf{0}. \\
\textit{(A5)} & 1\mathbf{u}=\mathbf{u} \quad (\forall \mathbf{u}\in V). \\
\textit{(A6)} & (kh)\mathbf{u}=k(h\mathbf{u}) \quad (\forall \mathbf{u}\in V \text{ and } k,h\in \mathbb{F}). \\
\textit{(A7)} & k(\mathbf{u}+\mathbf{v})=k\mathbf{u}+k\mathbf{v} \quad (\forall \mathbf{u},\mathbf{v}\in V \text{ and } k\in \mathbb{F}). \\
\textit{(A8)} & (k+h)\mathbf{u}=k\mathbf{u}+h\mathbf{u} \quad (\forall \mathbf{u}\in V \text{ and } k,h\in \mathbb{F}).
\end{array}
$$

# Examples

**Example.** The following are all examples of real vector spaces.

(1) $V = \{\mathbf{0}\}$; $\mathbf{0} + \mathbf{0} = \mathbf{0}; k\mathbf{0} = \mathbf{0}$.

(2) $\mathbb{R}^n$; vector addition and scalar multiplication.

(3) The set of all infinite sequence $\mathbb{R}^{\mathbb{N}} = \{(u_1, u_2, \ldots, u_n, \ldots)\}$;
componentwise addition and scalar multiplication.

(4) The set $\mathbb{R}^{(a,b)}$ of all real-valued functions on $(a, b)$;
function addition and scalar multiplication.

(5) The set $P_n$ of all polynomials with real coefficients and of degree $\leq n$.
polynomial addition and scalar multiplication.

(6) $M_{m\times n}$; matrix addition and scalar multiplication.

(7) The set of all upper triangular square matrices of order $n$;
matrix addition and scalar multiplication.

# Problem 1

Let A, B and C be square matrix of order $n$, where A and B are symmetric while C is skew-symmetric. Suppose that we have

$$A^2 + B^2 = C^2$$

Prove that

$$A = B = C = 0$$

# Problem 2

Recall the properties of dot product and norm. Suppose $x, y, z \in \mathbb{R}_+$, prove that

$$x + y + z \leq 2(\frac{x^2}{y+z} + \frac{y^2}{x+z} + \frac{z^2}{x+y})$$

# Problem 3

Suppose $q_1, q_2, q_3$ are orthonormal vectors in $\mathbb{R}^3$. Find **all possible values** for these 3 by 3 determinants and explain your thinking in 1 sentence each.

(a) $\det \begin{bmatrix} q_1 & q_2 & q_3 \end{bmatrix} =$

(b) $\det \begin{bmatrix} q_1 + q_2 & q_2 + q_3 & q_3 + q_1 \end{bmatrix} =$

(c) $\det \begin{bmatrix} q_1 & q_2 & q_3 \end{bmatrix}$ times $\det \begin{bmatrix} q_2 & q_3 & q_1 \end{bmatrix} =$

(*) For now, the answer for (a) is $\pm 1$

# Problem 4

Find the determinant of order n ($n \geq 2$):

$$\begin{vmatrix} 1 & 2 & 3 & \cdots & n-1 & n \\ 2 & 1 & 2 & \cdots & n-2 & n-1 \\ 3 & 2 & 1 & \cdots & n-3 & n-2 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ n-1 & n-2 & n-3 & \cdots & 1 & 2 \\ n & n-1 & n-2 & \cdots & 2 & 1 \end{vmatrix}$$