Week 5

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Number of Solutions of a Linear System

Theorem

- A system of linear equations has zero, one, or infinitely many solutions. There are no other possibilities
- If A is an invertible $n \times n$ matrix, then for each $n \times 1$ matrix b, the system of equations $Ax = b$ has exactly one solution

We will see this again later

Theorem

If A is an $n \times n$ matrix, then the following statements are equivalent:

- A is invertible
- $Ax = 0$ has only the trivial solution
- The reduced row echelon form of A is I_n
- A is expressible as a product of elementary matrices
- $Ax = b$ is consistent for every b, and x is unique

Three important kinds of the matrices

- Diagonal Matrices
- Triangular Matrices
- Symmetric Matrices

LU? Every matrix A can be decomposed to LU

Properties of Triangular Matrices

Theorem. The following properties of triangular matrices hold.
(1) The transpose of a lower/upper triangular matrix is upper/lower triangular.
(2) The product of upper/lower triangular matrices is still upper/lower triangular.
(3) A triangular matrix is invertible if and only if its diagonal entries are all nonzero.
(4) The inverse of an upper/lower triangular matrix is still upper/lower triangular.

Properties of Symmetric Matrices

Theorem. Let $A, B, C \in M_n$ be symmetric. Let $D \in M_{m\times n}$ and c be a scalar. Also suppose that C is invertible.
(1) A^T is symmetric.
(2) $A + B$ is symmetric.
(3) cA is symmetric.
(4) C^{-1} is symmetric.
(5) DD^T and D^TD are symmetric.
(6) AB is symmetric if and only if $AB = BA$.

Need to Remember

Need To Know

- Practice your calculation skills
- Revise the PPT
- Do not take things as granted

Problem

This question is about an m by n matrix A for which

$$Ax = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \text{ has } \textbf{no solutions} \quad \text{and} \quad Ax = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \text{ has } \textbf{exactly one solution}.$$

(a) Give all possible information about m and n and the rank r of A.

(b) Find all solutions to $Ax = 0$ and **explain your answer**.

(c) Write down an example of a matrix A that fits the description in part (a).

Solution

Solution.

(a) $Ax = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$ has *one* solution $\Longrightarrow N(A) = \{0\}$ so $r = n$. (Also, $m = 3$ since $Ax \in \mathbb{R}^3$.)

$Ax = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ has no solution $\Longrightarrow C(A) \neq \mathbb{R}^3$, so $r < m$.

There are two possibilities: $\begin{matrix} m = 3 \\ r = n = 1 \end{matrix}$ and $\begin{matrix} m = 3 \\ r = n = 2 \end{matrix}$.

(b) Since $N(A) = \{0\}$ (because $Ax = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$ has **1** solution), there is a unique solution to $Ax = 0$, which is clearly $x = 0$. (Can be either $x = \begin{bmatrix} 0 \end{bmatrix}$ or $x = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ depending on if $n = 1$ or $n = 2$.)

(c) A could be $\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$ or $\begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$ (many more possibilities).