Week 12

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Equivalence Theorem

Theorem. If $A \in M_n(\mathbb{R})$, then the following statements are equivalent.

(a) A is invertible.
(b) $A\mathbf{x} = 0$ has only the trivial solution.
(c) The reduced row echelon form of A is I_n.
(d) A can be expressed as a product of elementary matrices.
(e) $A\mathbf{x} = \mathbf{b}$ is consistent for every $n \times 1$ matrix $\mathbf{b}$.
(f) $A\mathbf{x} = \mathbf{b}$ has exactly one solution for every $n \times 1$ matrix $\mathbf{b}$.
(g) $\det(A) \neq 0$.
(h) The column vectors of A are linearly independent.
(i) The row vectors of A are linearly independent.
(j) The column vectors of A span $\mathbb{R}^n$.
(k) The row vectors of A span $\mathbb{R}^n$.
(l) The column vectors of A form a basis for $\mathbb{R}^n$.
(m) The row vectors of A form a basis for $\mathbb{R}^n$.
(n) $\text{rank}(A) = n$.
(o) $\text{nullity}(A) = 0$.
(p) $\text{Null}(A)^{\perp} = \mathbb{R}^n$.
(q) $Row(A)^{\perp} = \{\mathbf{0}\}$.

Transition Matrix

Remark

To know how to *prove* this (should be understood)

Definition. Let $B = \{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$ and $B' = \{\mathbf{v}'_1, \ldots, \mathbf{v}'_n\}$ be two bases of V. The transition matrix from B' to B is defined as

$$P_{B \leftarrow B'} = \left[\begin{array}{c:c:c:c} [\mathbf{v}'_1]_B & [\mathbf{v}'_2]_B & \cdots & [\mathbf{v}'_n]_B \end{array} \right]$$

Remark: Our textbook uses the notation $P_{B' \to B}$.

Proposition. Let $B = \{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$ and $B' = \{\mathbf{v}'_1, \ldots, \mathbf{v}'_n\}$ be two bases of V. Then for any vector $\mathbf{x} \in V$, it satisfies that

$$[\mathbf{x}]_B = P_{B \leftarrow B'} [\mathbf{x}]_{B'}.$$

Basic Ideas

Def

Let V, W be $\mathbb{F}$-vector spaces. A map $T : V \to W$ is called a linear transformation if the following two properties hold for all vectors $u, v \in V$ and for all scalars $k \in \mathbb{F}$:

- $T(u + v) = T(u) + T(v)$
- $T(ku) = kT(u)$

Examples

Please refer to your slides!

Matrices for General Linear Transformations

Definition. Let $T : V \to W$ be a linear transform. Let $B = \{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$ and $\widetilde{B} = \{\mathbf{w}_1, \ldots, \mathbf{w}_m\}$ be bases for V and W, respectively. Then the matrix for T relative to B and $\widetilde{B}$ is defined to be

$$[T]_{\widetilde{B},B} = \left[\ [T(\mathbf{v}_1)]_{\widetilde{B}} \ \vdots\ [T(\mathbf{v}_2)]_{\widetilde{B}} \ \vdots\ \cdots \ \vdots\ [T(\mathbf{v}_n)]_{\widetilde{B}}\ \right].$$

Notes

- To distinguish $T : V \to W$ from the matrix T_A
- Whenever we know the information about basis, we can make clear the information of whole space

Kernel and Range

Definition. For a linear transformation $T : V \to W$, the kernel and the range (or image) of T are defined by

$$\mathrm{Ker}(T) = \{\mathbf{v} \in V : T(\mathbf{v}) = \mathbf{0}\},$$

$$\mathrm{Ran}(T) = \{T(\mathbf{v}) : \mathbf{v} \in V\}.$$

Definition. Let $T : V \to W$ be a linear transform. The rank and nullity of T is defined by

$$\mathrm{rank}(T) = \dim \mathrm{Ran}(T), \quad \mathrm{nullity}(T) = \dim \mathrm{Ker}(T),$$

whenever the above subspaces are finite-dimensional.

Theorem. For any linear transformation $T : V \to W$ with V finite-dimensional, we have $\mathrm{rank}(T) + \mathrm{nullity}(T) = \dim(V)$.