Week 14

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Theorem

Let $A \in M_n(\mathbb{F})$. Let $q(x)$ be a polynomial with coefficients in $\mathbb{C}$. Then

$$q(A)x = q(\lambda)x$$

Theorem

Let $A \in M_n(\mathbb{F})$, the following are equivalent:

- A has n linearly independent eigenvectors.
- $P^{-1}AP = D$ for some invertible P and diagonal D.

Linearly Independent Eigenvectors

Linearly Independent Eigenvectors

Theorem. If $\lambda_1, \lambda_2, \ldots, \lambda_k$ are distinct eigenvalues of A, and $E_{\lambda_i}(A)$ $(1 \le i \le k)$ has basis

$$\mathbf{v}_{i,1}, \mathbf{v}_{i,2}, \ldots, \mathbf{v}_{i,m_k}.$$

Then the vectors

$$\mathbf{v}_{1,1}, \ldots, \mathbf{v}_{1,m_1}, \quad \mathbf{v}_{2,1}, \ldots, \mathbf{v}_{2,m_2}, \quad , \ldots, \quad \mathbf{v}_{k,1}, \ldots, \mathbf{v}_{k,m_k}$$

are linearly independent.

Proof:

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Remark: Let $A \in M_n$. Suppose that all the distinct eigenvalues of A are $\lambda_1, \lambda_2, \ldots, \lambda_k$. Then A is diagonalizable if and only if

$$\dim E_{\lambda_1}(A) + \dim E_{\lambda_2}(A) + \ldots + \dim E_{\lambda_k}(A) = n.$$

Corollary. Let $A \in M_n$. If A has n distinct eigenvalues then A is diagonalizable.

Definition

Definition. Let V be an $\mathbb{F}$-vector space. An inner product on V is a function $\langle \cdot, \cdot \rangle : V \times V \to \mathbb{F}$ such that the following axioms are satisfied.

(i) $\langle \mathbf{u}, \mathbf{v} \rangle = \overline{\langle \mathbf{v}, \mathbf{u} \rangle}$ $(\forall \mathbf{u}, \mathbf{v} \in V)$.
(ii) $\langle k\mathbf{u} + \ell\mathbf{v}, \mathbf{w} \rangle = k\langle \mathbf{u}, \mathbf{w} \rangle + \ell\langle \mathbf{v}, \mathbf{w} \rangle$ $(\forall \mathbf{u}, \mathbf{v}, \mathbf{w} \in V, \forall k, l \in \mathbb{R})$.
(iii) $\langle \mathbf{v}, \mathbf{v} \rangle \geq 0$ $(\forall \mathbf{v} \in V)$. And $\langle \mathbf{v}, \mathbf{v} \rangle = 0$ if and only if $\mathbf{v} = 0$.

In this case, we call $(V, \langle \cdot, \cdot \rangle)$ a inner product space.

Remark: When $\mathbb{F} = \mathbb{R}$, we may ignore the "conjugate".
Remark: When $\mathbb{F} = \mathbb{R}$, we also call it real inner product space;
When $\mathbb{F} = \mathbb{C}$, we also call it unitary space.

Remark: One always has $\langle \mathbf{0}, \mathbf{v} \rangle = \langle \mathbf{u}, \mathbf{0} \rangle = 0$.
Remark: By (i) and (ii), we have

$$\langle \mathbf{w}, k\mathbf{u} + l\mathbf{v} \rangle = \overline{k}\langle \mathbf{w}, \mathbf{u} \rangle + \overline{l}\langle \mathbf{w}, \mathbf{v} \rangle.$$

Norm and Distance

If $(V, \langle \cdot, \cdot \rangle)$ is an inner product space, then the norm of a vector v is:

$$||v|| = \sqrt{\langle v, v \rangle}$$

The distance between two vectors is

$$d(u, v) = ||u - v||$$

Theorem

Define u, v be two vectors and k a scalar

- $||v|| \geq 0$, and $d(u, v) \geq 0$
- $d(u, v) = d(v, u)$
- $||ku|| = |k| \cdot ||u||$
- $||u + v||^2 + ||u - v||^2 = 2(||u||^2 + ||v||^2)$
- $\langle u, v \rangle = \frac{1}{4}(||u + v||^2 - ||u - v||^2)$

Problems

- Define A, B be two 3 by 3 square matrices, where B is of rank 2 and rank(AB) = 1. Given that

$$A = \begin{bmatrix} 1 & 0 & -1 \\ 2 & x & 1 \\ 1 & 2 & 1 \end{bmatrix}$$

  Find the value of x

- Let A be a square matrix of order n with distince r ($r \leq n$) eigenvalues $\lambda_1, ..., \lambda_r$.
  If A is diagonalizable, then prove that

$$(\lambda_1 I - A)(\lambda_2 I - A) \cdots (\lambda_r I - A) = 0$$