Week 7

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Basic Computation

Definition:

- Basic idea is to expand along a row or a column.
- Cofactor Expansion: $det(A) = \sum_{i}^{n} a_{ij} C_{ij}$
- Determinant is about things like area, volumn and so on...

Properties

Read this: https://zhuanlan.zhihu.com/p/358591975

- $det(F_i(c)) = c, \quad det(F_{i,j}(c)) = -1, \quad det(F_{i,j}) = 1$
- $\begin{vmatrix} a & b \\ c & d \end{vmatrix} + \begin{vmatrix} a' & b' \\ c & d \end{vmatrix} = \begin{vmatrix} a + a' & b + b' \\ c & d \end{vmatrix}$
- Let $A, B \in M_n$, $det(AB) = det(A)det(B)$
- Let $A \in M_n$, $det(cA) = c^n det(A)$
- Let $A \in M_n$, $det(A) = det(A^T)$
- $det(A^{-1}) = 1/det(A)$
- The det of some special matrices

Adjunct of A

Definition. If A is any $n \times n$ matrix and C_{ij} is the cofactor of a_{ij}, then the matrix

$$\begin{bmatrix} C_{11} & C_{21} & \cdots & C_{n1} \\ C_{12} & C_{22} & \cdots & C_{n2} \\ \vdots & \vdots & \ddots & \vdots \\ C_{1n} & C_{2n} & \cdots & C_{nn} \end{bmatrix}$$

is called the adjunct of A, and is denoted by $\text{adj}(A)$.

Remark: Pay attention to the subscripts!

$$A^{-1} = \frac{1}{det(A)} adj(A)$$

We will see this again later

Theorem

If A is an $n \times n$ matrix, then the following statements are equivalent:

- A is invertible
- $Ax = 0$ has only the trivial solution
- The reduced row echelon form of A is I_n
- A is expressible as a product of elementary matrices
- $Ax = b$ is consistent for every b, and x is unique
- $det(A) \neq 0$

Theorem. (Cramer's Rule) If $A\mathbf{x} = \mathbf{b}$ is a linear system with A an invertible square matrix of order n, then the system has a unique solution given by

$$x_i = \frac{\det(A_i)}{\det(A)},$$

for $i = 1, \ldots, n$. Here A_i is the matrix obtained by replacing the ith column of A by $\mathbf{b}$.

Concept

MY IDEA

- To make the matrix sparse
- To expand it and seek for clues

Example

Example. Evaluate the following determinant of order $2n$:

$$D_n = \begin{vmatrix} a_n & & & & & & & b_n \\ & a_{n-1} & & & & & b_{n-1} & \\ & & \ddots & & & \iddots & & \\ & & & a_1 & b_1 & & & \\ & & & c_1 & d_1 & & & \\ & & \iddots & & & \ddots & & \\ & c_{n-1} & & & & & d_{n-1} & \\ c_n & & & & & & & d_n \end{vmatrix}.$$

Solution: