Week 13

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Maps

Similarity

Eigen

Equivalence Theorem

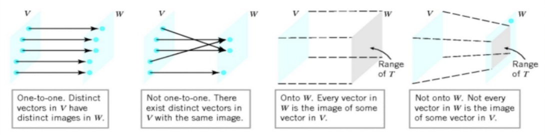
Definitions

Definition. Let $f : V \to W$ be a map from a set V to a set W.

◇ We say that f is injective (one-to-one),
  if it maps distinct elements in V into distinct elements in W.

◇ We say that f is surjective (onto),
  if every element in V is in the range of f.

◇ We say that f is bijective, if it is both injective and surjective.



Definition. A bijective linear transformation from V to W is called an isomorphism. In such case, we say that the two vector spaces V and W are isomorphic.

Remark: They have "same" linear properties!

Equivalence Theorem

Theorem. Suppose that V and W are vector spaces with $\dim(V) = n$, $\dim(W) = m$. Let $T : V \to W$ be a linear transformation.

The following three statements are equivalent:
(1) T is injective;
(2) $\text{Ker}(T) = \{\mathbf{0}\}$; or $\text{nullity}(T) = 0$;
(3) $\text{rank}(T) = n$.

The following three statements are equivalent:
(1) T is surjective;
(2) $\text{Ran}(T) = W$; or $\text{rank}(T) = m$;
(3) $\text{nullity}(T) = n - m$.

When $m = n$, the following three statements are equivalent:
(1) T is bijective (isomorphic);
(2) T is surjective (onto);
(3) T is injective (one-to one).

Remark: If $n > m$, then T cannot be injective.
If $n < m$, then T cannot be surjective.

Transformations

- Composition of Transformations
- Inverse of transformations
- Properties of an Inverse

Maps

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Similarity

Definition

Let $A, B \in M_n$. We say that A and B are similar, if there's an invertible matrix $P \in M_n$ such that

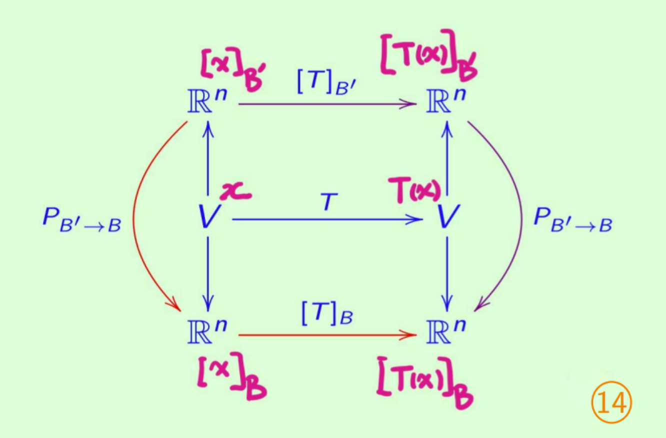
$$P^{-1}AP = B$$

Remark: Similarity means the same linear transformation under distinct bases.

Similar Invariants

- $tr(A) = tr(B)$
- $det(A) = det(B)$
- $rank(A) = rank(B)$

The transition graph



Maps

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Eigen

Definition

For $A \in M_n(\mathbb{F})$, if

$$Ax = \lambda x$$

for some non-zero vector $x \in \mathbb{F}^n$ and some scalar $\lambda \in \mathbb{F}$, then λ is called an eigenvalue of A, and x is called an eigenvector of A corresponding to λ

Compute eigenvalues

Characteristic polynomial

Let $A \in M_n(\mathbb{F})$. Then $\lambda \in \mathbb{F}$ is an eigenvalue of A, if and only if

$$det(\lambda I_n - A) = 0$$

Do not forget about **Eigenspace**!

Maps

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Part 1

Theorem

If A is an $n \times n$ matrix, then the following statements are equivalent:

- A is invertible
- $Ax = 0$ has only the trivial solution
- The reduced row echelon form of A is I_n
- A is expressible as a product of elementary matrices
- $Ax = b$ is consistent for every b, and x is unique
- $det(A) \neq 0$
- rank(A) $=$ n, nullity(A) $=$ 0

Part 2

Continue ...

- Column(row) vectors of A are linearly independent
- Column(row) vectors of A span $\mathbb{R}^n$
- Column(row) vectors of A form a basis of $\mathbb{R}^n$
- $Null(A)^\perp = \mathbb{R}^n$, $Row(A)^\perp = \mathbf{0}$
- $\lambda = 0$ is not an eigenvalue of A