Fast Simulation of Mass-Spring Systems

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https://github.com/xiaojxkevin/cloth-simulation

Motivation

In hw5 we have used *(semi-implicit) symplectic Euler's method* as the solver for state iterations in a mass-spring system. However, this method is not accurate enough.

Newton's method as the solver provides accurate results, but it costs time.

To come up with a method that is **fast** and **accurate** in results.

Notations and Background

- A mechanical system with m points in 3D and s springs, evolving through a discrete set of time samples $t_1, t_2, \ldots$ with constant time step h.
- The positions of 3D points in time t_n is $\mathbf{q}_n \in \mathbb{R}^{3m}$.
- The force function is $\mathbf{f} : \mathbb{R}^{3m} \to \mathbb{R}^{3m}$. And forces are conservative, i.e., $\mathbf{f} = -\nabla E$, where $E : \mathbb{R}^{3m} \to \mathbb{R}$ is the energy function.
- Velocity of points in time t_n is $\mathbf{v}_n \in \mathbb{R}^{3m}$.
- The diagonal mass-matrix is $M \in \mathbb{R}^{3m \times 3m}$.

Notations and Background

- The basic implicit Euler's:

$$\begin{aligned} \mathbf{q}_{n+1} &= \mathbf{q}_n + h\mathbf{v}_{n+1} \\ \mathbf{v}_{n+1} &= \mathbf{v}_n + h\mathbf{M}^{-1}\mathbf{f}(\mathbf{q}_{n+1}) \end{aligned}$$

- Reformulation:

$$\begin{aligned} &\mathbf{q}_{n+1} - 2\mathbf{q}_n + \mathbf{q}_{n-1} = h^2\mathbf{M}^{-1}\mathbf{f}(\mathbf{q}_{n+1}) \\ &\Rightarrow M(\mathbf{x} - \mathbf{y}) = h^2\mathbf{f}(\mathbf{x}) \end{aligned}$$

  where $\mathbf{x} \circeq \mathbf{q}_{n+1}$ and $\mathbf{y} \circeq 2\mathbf{q}_n - \mathbf{q}_{n-1}$.

- Convert to an optimization problem:

$$\min_{\mathbf{x}} g(\mathbf{x}) = \frac{1}{2}(\mathbf{x} - \mathbf{y})^T M(\mathbf{x} - \mathbf{y}) + h^2 E(\mathbf{x})$$

Method

The main problem is to define the potential energy of springs. Authors come up with the idea of introducing auxiliary unknown variables:

$$E_{\text{spring}} \propto (||\mathbf{p}_1 - \mathbf{p}_2|| - r)^2 = \min_{||\mathbf{d}||=r} ||(\mathbf{p}_1 - \mathbf{p}_2) - \mathbf{d}||^2$$

where $\mathbf{p}_{\{1,2\}}$ are the endpoints of the spring, $\mathbf{d}$ is the unit direction of the spring and r is the rest length.

Method

The idea is to define the energy as an optimization problem. Thus with a few reformulations, we get

$$E(\mathbf{x}) = \min_{\mathbf{d}\in U} \frac{1}{2}\mathbf{x}^T L\mathbf{x} - \mathbf{x}^T J\mathbf{d} - \mathbf{x}^T \mathbf{f}_{\text{ext}}$$

where $L \in \mathbb{R}^{3m\times 3m}$ is a stiffness-weighted Laplacian of the mass-spring system graph, $J \in \mathbb{R}^{3m\times 3s}$ is a matrix connects mass points with springs, U is the set of rest-length spring directions and $\mathbf{f}_{\text{ext}} \in \mathbb{R}^{3m}$ denotes external forces.

Method

In general, we can obtain the target function as

$$\min_{\mathbf{x},\mathbf{d}\in U} g(\mathbf{x}) = \frac{1}{2}\mathbf{x}^T(M+h^2L)\mathbf{x} - h^2\mathbf{x}^T J\mathbf{d} - \mathbf{x}^T(h^2\mathbf{f}_{\text{ext}} + M\mathbf{y})$$

To optimize the problem above, we can first fix $\mathbf{x}$ and solve for $\mathbf{d}$, which is defined as a local step. Then we fix $\mathbf{d}$ and solve for $\mathbf{x}$ with

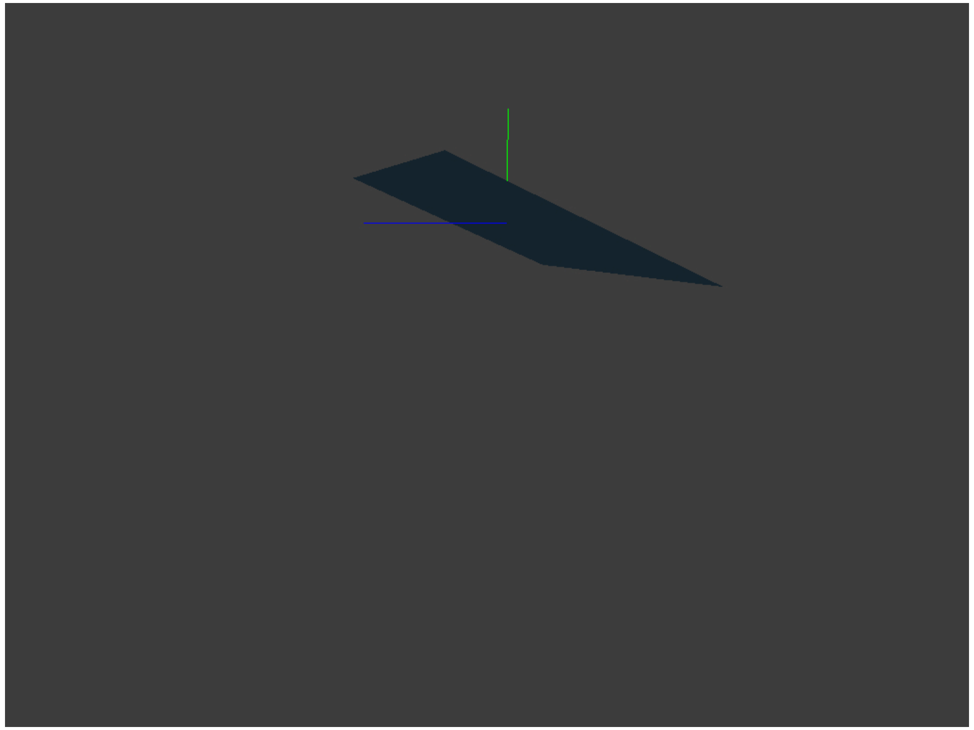
$$(M+h^2L)\mathbf{x} = h^2J\mathbf{d} + h^2\mathbf{f}_{\text{ext}} + M\mathbf{y}$$

and notice that $(M+h^2L)$ is PSD thus we can decompose it at the first place.

Implementation

Scratch Point: to make the system interactive.

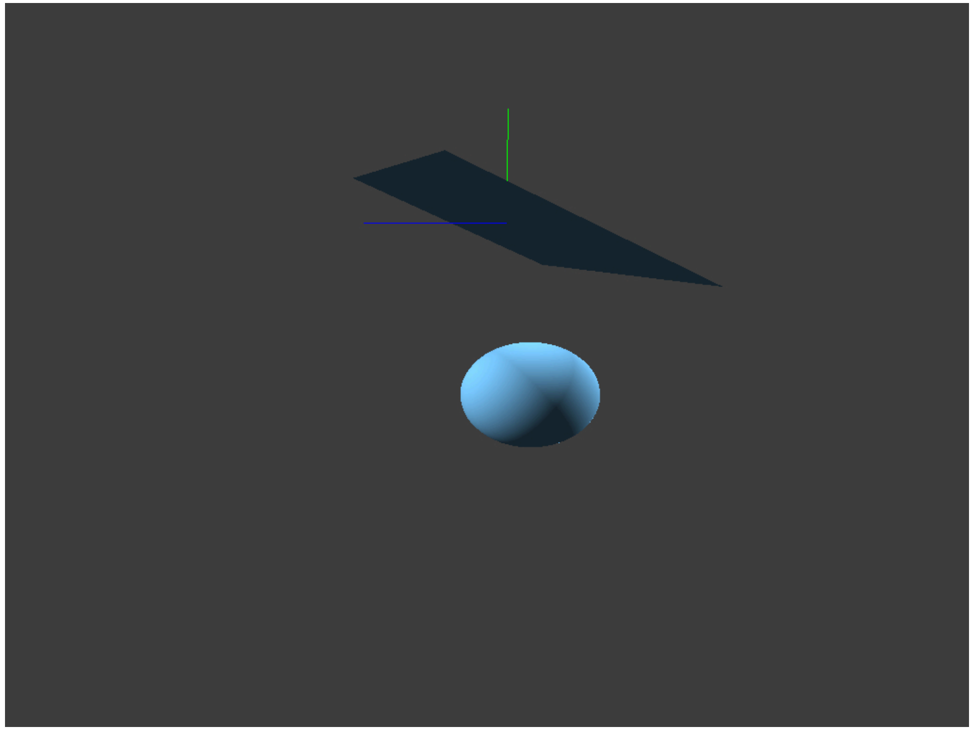
- Find the closest point k with the mouse.
- Fix its state, i.e., $\mathbf{q}_n^k$ and $\mathbf{q}_{n-1}^k$ are set to be the same wit the position of the mouse.



Implementation

Sphere Collision:

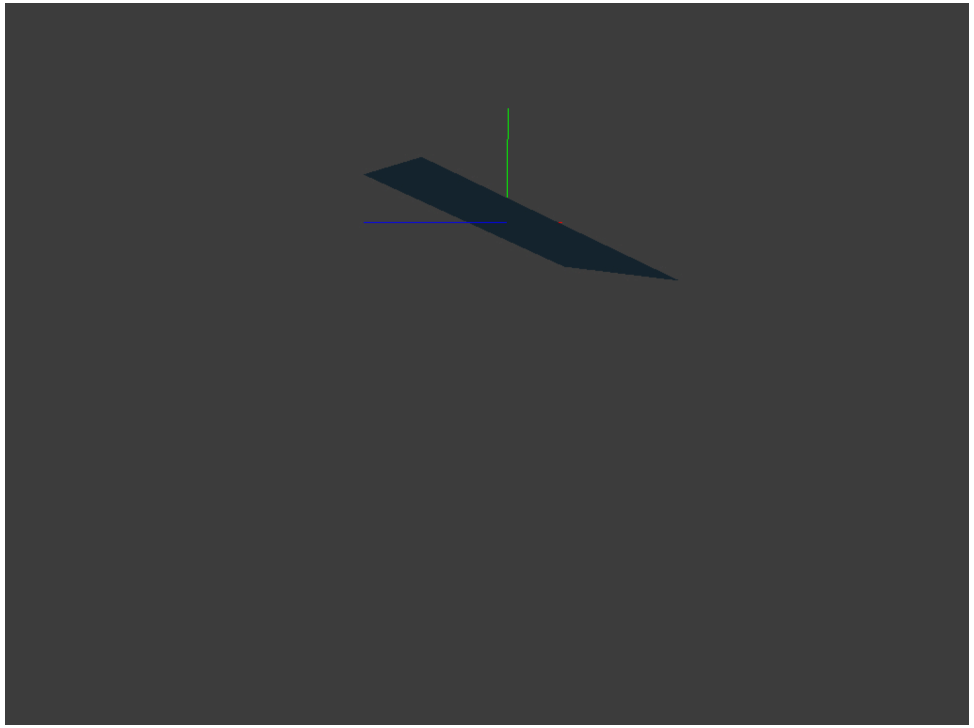
- Iterate all points (a set K) that may collide with the sphere.
- To force positions of points in set K, i.e., $\mathbf{q}_n^K$ to be on the sphere.



Implementation

Cut Cloth:

- To modify the shape of the cloth.
- Find the point with mouse, invalidate all springs connected to it and delete the triangles using it during rendering.
- Add triangles to make the cutting hole symmetric and dealing with special case.



Thanks