Computers & Operations Research 37 (2010) 341 – 350

Contents lists available at ScienceDirect

Computers & Operations Research

journal homepage: www.elsevier.com/locate/cor

On determining optimal fleet size and vehicle transfer policy for a car rental company

Zhaolin Li[a,*], Feng Tao[b]

[a]Faculty of Economics and Business, The University of Sydney, Australia
[b]School of Management, University of Science and Technology of China, China

ARTICLE INFO

Available online 6 June 2009

Keywords:
Automobile rental
Markov decision process
Fleet size management
Vehicle transfer

ABSTRACT

This paper considers how to determine the optimal fleet size and vehicle transfer policy for a rental-car company that serves two cities. In each city, there are single-trip and round-trip customers, where the former is given a higher priority. Because of the single-trip traffic, the number of cars at these two cities may become unbalanced. Hence, the central planner in each day needs to decide whether to transfer any cars from one city to the other. We develop a two-stage dynamic programming model, in which we determine the vehicle transfer policy in the second stage and the optimal fleet size in the first stage. Although the objective function could be neither concave nor quasi-concave due to lost sales, we can find the optimal fleet size and vehicle transfer policy by solving a series of linear programming problems. We propose a heuristic solution, which is based on a special case analysis, for the fleet size problem. A numerical study reveals that our heuristic solution for the fleet size performs well. However, if the corresponding vehicle transfer policy is not appropriate, the overall performance can drastically deteriorate even with the optimal fleet size. Several extensions of our basic model are also analyzed.

1. Introduction

Although some restrictions are applied to the car rental market in China (e.g., the international driver license is not recognized and national credit-checking is almost non-existent), the rising economy and increasing application of GPS are boosting this industry. The car rental trade in China is undergoing a large-scale development. In 2002, Hertz Corp., of the US, and Avis Europe PLC, of the United Kingdom, struck deals with Chinese rental-car agencies to enter a market that opens as part of China's entry to the World Trade Organization (Hutzler [7]). China's car rental market is now crowded with more than 400 rental-car agencies. While most of them are small and confined to single cities, a few built solid reputations and nascent national networks.

The analysis presented in this paper is motivated by a rental-car company that mainly operates in Suzhou and Shanghai in China. The distance between these two cities is about 80 km. Everyday the company faces two types of customers: (i) the single-trip (also called the inter-city) customers who pick up the vehicle in one city and drop it off in the other and (ii) the round-trip (also called the local) customers who pick up and return the vehicle in the same city. Because the inter-city rental rate is much higher than the local rental rate, it is the corporate policy that a higher priority is given to the inter-city demand. However, the number of cars in these two cities may become unbalanced when the inter-city demand surges in one city and remains flat in the other. As such, the company often needs to decide whether to transfer any vehicles from one city to the other, which is normally done overnight. In addition, the company predicts that the demand will rise significantly in the next few years since Shanghai will host the World Exhibition in 2010. The general manager planned to expand and upgrade the existing fleet. The stake related to these decisions is high, particularly for a small company.

The planning process is composed of two phases: fleet size planning and day-to-day vehicle transfer planning. Capacity planning is one of the well-studied topics in the operations management literature. However, to the best of our knowledge, the research on the car rental industry with such operational details as (i) lost sales, (ii) random inter-city and local demands, and (iii) joint optimization of fleet size and vehicle transfer is limited.

We formulate a two-stage dynamic programming model that determines the vehicle transfer policy in the second stage and the optimal fleet size in the first stage. Although the objective function could be neither concave nor quasi-concave due to lost sales, we can find the optimal fleet size and vehicle transfer policy by solving a series of linear programming (LP) problems. A sensitivity analysis suggests that the optimal fleet size is not sensitive to the transfer cost. Inspired by this sensitivity analysis, we propose a heuristic solution, which is based on a special case analysis, for the fleet size

* Corresponding author. Tel.: +61 2 9114 0751; fax: +61 2 9351 6409.
E-mail addresses: E.Li2@econ.usyd.edu.au (Z. Li), tfeng@mail.ustc.edu.cn (F. Tao).

0305-0548/$ - see front matter © 2009 Elsevier Ltd. All rights reserved.
doi:10.1016/j.cor.2009.05.010

problem. A numerical study reveals that our heuristic solution for fleet size indeed hits the optimum in 76% of the random samples and the maximum error is plus/minus one vehicle. However, it is very important to use a proper vehicle transfer policy. Our numerical study finds that if the vehicle transfer policy is not properly determined (e.g., a myopic policy is used), the overall performance can drastically deteriorate even when the fleet size is optimal.

Several extensions of the basic model are presented. First, we consider relaxing the assumption of lost sales (e.g., the company can use leased cars to satisfy customer demands when there are not enough owned cars). We prove that the optimal vehicle transfer policy indeed is a two-limit policy and the total cost is also convex in the fleet size. This extension can be applied in other business contexts such as empty containers and railcars. Second, we quantify the value of the advanced demand information by assuming that the company can observe customer demands before transferring vehicles. Third, we discuss the formulation when the transfer time is longer than one day.

The rest of this paper is organized as follows. Section 2 reviews the related literature. Section 3 introduces the notation and describes the model. Section 4 analyzes the single-period problem and a special case. Section 5 provides numerical examples. Section 6 discusses several extensions and Section 7 concludes this paper.

2. Literature review

Fleet size planning models can be treated as a subset of capacity planning models, which are concerned with determining the size, timing, and location of buying additional capacity or disposing of existing capacity. For literature review on general capacity planning models, the readers are directed to Luss [9] and Van Mieghem [21].

Our research is related to inventory transfer models, which have received growing attention. Robinson [13] formulates a multi-location inventory problem where the central planner has the option to transfer surplus inventory to some places with shortage after observing the demand at the end of each planning period. A simulation-based algorithm is proposed to estimate the gradient and solve the inventory replenishment problem. Archibald et al. [1] considered a two-depot inventory system where inventory transfer can take place any time between two replenishment orders. The decision of stock transfer depends on the real-time inventory level at both sites and the remaining time to receive the next replenishment. Rudi et al. [14] studied the coordination issue in a one-period two-location newsvendor problem with inventory transfer. They determine the optimal transfer price (e.g., the price to be charged by the surplus location to the shortage location) that achieves the joint-profit-maximization when inventory decisions are made locally. Dong and Rudi [4] investigated how to determine the wholesale price between a manufacturer and n retailers with inventory transfer to achieve supply chain coordination.

There are two major differences between the inventory transfer models and ours. First, the purpose of inventory transfer is to tranship the surplus inventory to some locations with shortage; whereas the purpose of vehicle transfer is to re-balance the number of vehicles in different locations after the inter-city traffic causes imbalances. Second, in the inventory transfer models the inventory is replenished in each period; while in our model the total number of cars remains unchanged.

In the transportation literature, a great deal of attention has been devoted to fleet management for the airline and rail–road industries. For a conceptual discussion about fleet size management in railway industry, the readers are referred to Sherali and Maguire [17]. The needs to re-balance the capacity such as planes and empty rail-cars are driven by the changes in flight schedules or train routes and the central planner has direct influence on the inter-site traffic. In our model, the needs to re-balance the number of vehicles are driven by the imbalances caused by the inter-city traffic and the central planner does not have direct influence on the inter-city traffic. Typically, the fleet management problem in transportation literature is formulated as a time-space LP model with a large number of destination and origin nodes (e.g., Beaujon and Turnquist [3], Sherali and Tuncbilek [18], Powell and Carvalho [12], Wu et al. [22], and Sayarshad and Ghoseiri [16]) or an integer programming problem (e.g., Ernst et al. [6]). The research in this area also focuses on developing efficient algorithms or effective approximations when some of the parameters are stochastic.

There are several articles dealing with continuous-time control models. Savin et al. [15], hereafter SCGK, have studied a single-location rental problem with two classes of customers. Whenever a customer arrives, the decision maker needs to decide whether to admit or reject a customer. If a customer is admitted, the customer pays a rental fee for his or her stay; if a customer is rejected, he or she exits the system and does not queue and the decision maker incurs a penalty cost. Clearly, the difference between SCGK and ours is that SCGK deals with the customer admission for one location while our model considers balancing the capacity in two locations by moving vehicles around. Song and Earl [20] developed a continuous-time model to determine the transfer policy and fleet size decision for empty vehicles in a two-depot service system. Lost sales are not permitted as customer demands are satisfied by either the owned empty vehicle or the leased empty vehicle. They demonstrate that the optimal vehicle transfer policy has the following properties: first, either do not transfer or if transfer then use the fastest mode only; second, the timing of transferring vehicles is triggered by a threshold policy. On the basis of this work, Song and Carter [19] extended the model to a hub-and-spoke transportation system and presented a decomposition procedure to solve the problem.

The work by Pachon et al. [11] is also related. They decompose the tactical fleet deployment and operational transportation problems and solve them separately. Their fleet deployment solution resembles a multi-location constrained newsvendor problem, namely there are n newsvendors in the network and the total number of inventories in the network cannot exceed the fleet size. Once the fleet deployment problem is solved, the transportation problem is to minimize the total transportation costs for adjusting the number of vehicles at each location to the level prescribed in the fleet deployment solution. In contrast to Pachon et al. [11], we optimize the fleet deployment and transportation problems jointly. Our vehicle transfer policy is a two-limit policy. Formally, for each city there are two control limits (which are city-specific). If the number of cars in a city is below the first limit, it is optimal to increase the number of cars in this city up to the first limit by moving some cars from the other; if the number of cars in a city is above the second limit, it is optimal to reduce the number of cars down to the second limit by moving some cars to the other city; if the number of cars in a city is between the first and second limits, no car movement is needed. It can be seen that the vehicle transfer policy of Pachon et al. [11] is a special case of the two-limit policy. In general, a single-limit policy involves more vehicle transfers than a two-limit policy. Additionally, Pachon et al. [11] considered a single-period problem whereas we consider a multi-period problem.

3. The basic model

We consider a rental-car company that opens two branches in cities 1 and 2. There are two types of customers at each branch: the single-trip and round-trip customers (we call them type-1 and type-2 customers, respectively). For simplicity, we assume that both types of customers use the car for one day only. In words, a type-1 customer in city 1 will pick up the car in the morning and return it to city 2 in the evening; whereas a type-2 customer in city 1 will pick up the car in the morning and return it to city 1 in the evening. Because the rental rate for type-1 demand is higher, the company gives a higher priority to type-1 demand.

3.1. Chronology and notation

For convenience, we count the time index backward, e.g., $t=0$ marks the end of the planning horizon and day T represents the beginning of the planning horizon. The sequence of events proceeds as the following:

1. At the beginning of the planning horizon, the firm decides the fleet size n (e.g., total number of cars) and allocates x_T cars to city 1 and $n-x_T$ cars to city 2. Demand arrives periodically starting from day $T-1$.
2. In the morning of a typical day t ($\le T-1$), transferred cars, if any, arrive.
3. Demand for day t arrives. Type-1 demand is given a higher priority and unmet demand (irrespective its type) is lost.
4. In the evening, cars are returned. Now, the company needs to decide whether to transfer some vehicles between two cities overnight.
5. Repeat events 2–4 until the end of the planning horizon.

Before formulating the dynamic programming problem for vehicle transfer, we define the following notation in alphabetical order.

D_{ij} = Demand of type-j at city i, where $i,j=1,2$. The distribution of D_{ij} varies from city to city and from type to type, i.e., the distribution of D_{ij} is a function of i and j. For any given (i,j), D_{ij} is assumed to be identically and independently distributed in each day.

h = Daily operating cost per car, e.g., vehicle insurance, interest expenses, depreciation of the vehicle, and amortized expenses. Such cost is incurred irrespective of whether the car is rented or not.

k = Cost for transferring one car between two cities.

r_i = Rental rate per car per day for type-i demand, $i=1,2$, and $r_1>r_2$, e.g., the inter-city rental rate is higher than the local rental rate.

$V_t^n(x_t)$= Maximum expected income attainable when there are x_t cars in city 1 and $n-x_t$ cars in city 2 before any vehicle transfer.

y_t = The number of cars available in city 1 after transfer is completed in the morning of day t. So y_t is the decision variable for the second stage problem. If $y_t-x_t>0$, cars were transferred from city 2 to city 1 in previous evening; if $y_t-x_t<0$, cars are transferred from city 1 to city 2 in previous evening; if $y_t=x_t$, no cars were transferred in previous evening.

3.2. Formulation for finite planning horizon

Suppose that there are t days remaining and there are x_t cars in city 1 and $n-x_t$ cars in city 2. After the vehicle transfer, the number of cars available in city 1 in the morning of day t is y_t and that in city 2 is $n-y_t$. For convenience, we let $(x)^+=\max\{x,0\}$. The optimality equation is

$$V_t^n(x_t)=\max_{0\le y_t\le n}\left\{\begin{array}{l}-k|y_t-x_t|+r_1E[\min(y_t,D_{11})\\ \quad+\min(n-y_t,D_{21})]\\ \quad+r_2E[\min((y_t-D_{11})^+,D_{12})\\ \quad+\min((n-y_t-D_{21})^+,D_{22})]\\ \quad+EV_{t-1}^n(y_t-\min(y_t,D_{11})\\ \quad+\min(n-y_t,D_{21}))\end{array}\right\}. \tag{1}$$

Inside the optimization operator, the first term is the cost of vehicle transfer, the second term is the inter-city rental revenue from both cities, the third term is the local rental revenue from both cities, and the fourth term is the expected income from the next day. By the assumption of one-day rental, the number of cars in city 1 in the evening of day t equals y_t minus the number of cars rented by type-1 customers in city 1 (i.e., $\min(y_t,D_{11})$), plus the number of cars rented by type-1 customers in city 2 (i.e., $\min(n-y_t,D_{21})$). As we shall show in Section 4.1, the fourth term inside the optimization operator could be neither concave nor quasi-concave. Therefore, it is difficult to explicitly characterize the optimal vehicle transfer policy for the multi-period problem.

When the fleet size is n, the daily operating cost is hn. Hence, the optimal fleet size can be obtained by solving

$$\max_{n,x_T} Z_T(n,x_T)=\max\left\{-hn+\frac{1}{T}V_T^n(x_T)\right\}. \tag{2}$$

As evidenced in Eq. (2), the initial number of cars in each city could have some impacts on the time-average profit for the finite planning horizon, but such impact will be diluted when we extend the analysis to the infinite planning horizon.

4. Analysis

In this section, we first consider the final-period problem, second, present an LP approach to solve the problem with an infinite planning horizon, and third, analyze a special case with zero transfer cost.

4.1. Final-period analysis

For the final day, the optimality equation for vehicle transfer is

$$V_1^n(x_1)=\max_{0\le y_1\le n}\left\{\begin{array}{l}-k|y_1-x_1|+r_1E[\min(y_1,D_{11})\\ \quad+\min(n-y_1,D_{21})]\\ \quad+r_2E[\min((y_1-D_{11})^+,D_{12})\\ \quad+\min((n-y_1-D_{21})^+,D_{22})]\end{array}\right\}. \tag{3}$$

Define

$$\begin{aligned}C(y_1)=&\ r_1E\min(y_1,D_{11})+r_2E\min((y_1-D_{11})^+,D_{12})\\ &+r_1E\min(n-y_1,D_{21})\\ &+r_2E\min((n-y_1-D_{21})^+,D_{22}),\end{aligned} \tag{4}$$

which represents the one-day reward if there are y_1 cars in city 1, $n-y_1$ cars in city 2, and no cars are being transferred. At the first glance, not every term of $C(y_1)$ is concave in y_1. For instance, the second term, $\min((y_1-D_{11})^+,D_{12})$, is actually quasi-concave in y_1. However, with the condition $r_1>r_2$, we can show that $C(y_1)$ is indeed concave in y_1. Hence, the optimal vehicle transfer policy for the final-period problem is a two-limit control policy.

Theorem 1. *For the final-period problem with fleet size n, there exist two unique control limits x^l and x^h such that* (i) *if the number of cars in city 1 is $x_1<x^l$, it is optimal to transfer x^l-x_1 cars from city 2 to city 1*; (ii) *if the number of cars in city 1 is $x_1>x^h$, it is optimal to transfer x_1-x^h cars from city 1 to city 2; and* (iii) *if the number of cars in city 1 is between x^l and x^h* (*inclusive*), *it is optimal to do nothing.*

The proof of Theorem 1 is in the Appendix. Note that the control limits for city 2 will be $n-x^h$ and $n-x^l$. Substituting Eq. (16) into Eq. (3), one can see that

$$V_1^n(x_1)=\begin{cases}C(x^l)-k(x^l-x_1), & x_1<x^l,\\ C(x_1), & x^l\le x_1\le x^h,\\ C(x^h)-k(x_1-x^h), & x_1>x^h.\end{cases}$$

It can be verified that $V_1^n(x_1)$ is continuous and concave in x_1. However, when we extend the analysis to multi-period, the proof of induction breaks down because we cannot establish that the objective

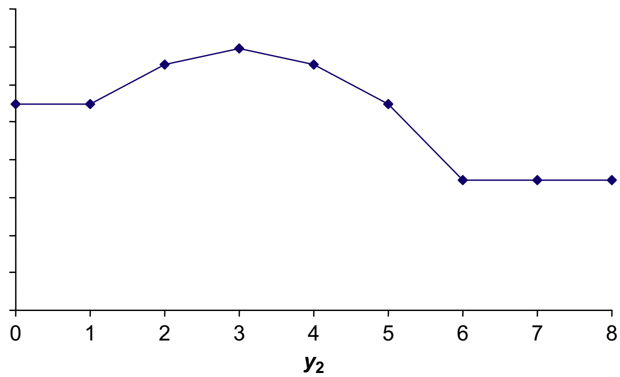

Fig. 1. The plausible shape of $V_1(y_2-\min(y_2,D_{11})+\min(n-y_2,D_{21}))$.

function is concave or quasi-concave. To illustrate, consider a two-period problem. The optimality equation is

$$V_2^n(x_2)=\max_{0\le y_2\le n}\{-k|y_2-x_2|+C(y_2)\\ +EV_1^n(y_2-\min(y_2,D_{11})+\min(n-y_2,D_{21}))\}.$$

As illustrated in Fig. 1, the term $V_1(y_2-\min(y_2,D_{11})+\min(n-y_2,D_{21}))$ is quasi-concave in y_2. Note that the one-day reward $-k|y_2-x_2|+C(y_2)$ is concave in y_2. It is well known that the sum of a quasi-concave function and a concave function can be neither concave nor quasi-concave (Avriel [2]). Hence, it is difficult to characterize the structure of the optimal vehicle transfer policy for the multi-period problem. The analytical difficulty is mainly due to lost sales, which makes the system transition to be non-linear, e.g., $y_2-\min(y_2,D_{11})+\min(n-y_2,D_{21})$ is piecewise linear.

In our numerical study, we generate hundreds of random samples and solve the optimal vehicle transfer policy for each one of them. We notice that the optimal vehicle transfer policy for each random sample indeed has a two-limit structure, although the values of control limits may not be the same as those in Eq. (15). Nevertheless, one of the future research directions is to pursue a proof that a two-limit policy is indeed optimal for Eq. (1).

4.2. Formulation for infinite planning horizon

We consider the time-average cost criterion when T approaches infinity. The optimal profit per day (excluding the daily operating cost) with fleet size n converges to

$$Z(n)=\lim_{T\to\infty}\frac{1}{T}EV_T^n(x),$$

where x follows the steady-state distribution with the corresponding optimal vehicle transfer policy being used. Although the concavity or quasi-concavity of $V_{t-1}^n(y_t-\min(y_t,D_{11})+\min(n-y_t,D_{21}))$ is difficult to establish, the optimal vehicle transfer policy can be obtained by solving a linear programming model. First, define the following notation.

$C_{ia}=$ The one-day reward if there are i cars in city 1 prior to vehicle transfer and the number of vehicles to be transferred is a. If a is positive (negative), cars are transferred from city 2 (1) to city 1 (2).

$p_{ij}(a)=$ The transition probability that there are j cars in city 1 at the end of the next day provided that vehicle transfer action a is taken and city 1 initially has i cars before transfer.

$z_{ia}=$ The long-run probability that vehicle transfer action a is taken when there are i cars in city 1.

The decision variables are $\{z_{ia}\}$ and the input parameters are $\{C_{ia}\}$ and $\{p_{ij}(a)\}$. Specifically, the one-day reward C_{ia} satisfies

$$C_{ia}=-k|a|+r_1E[\min(i+a,D_{11})+\min(n-i-a,D_{21})]\\ +r_2E[\min((i+a-D_{11})^+,D_{12})\\ +\min((n-i-a-D_{21})^+,D_{22})],$$

for $i=\{0,1,\ldots,n\}$ and $a\in\{-i,-i+1,\ldots,n-i\}$. The transition probability $p_{ij}(a)$ satisfies

$$p_{ij}(a)=\Pr((i+a-D_{11})^+\\ +\min(n-i-a,D_{21})=j),$$

for $i,j\in\{0,1,\ldots,n\}$ and $a\in\{-i,-i+1,\ldots,n-i\}$.

The standard form of LP model with fleet size n is

$$\max_{z_{ia}} W(n)=\sum_i\sum_a C_{ia}z_{ia}, \tag{5}$$

subject to

$$\begin{cases}\sum_i\sum_a z_{ia}=1,\\ \sum_a z_{ja}-\sum_i\sum_a z_{ia}p_{ij}(a)=0 \text{ for any } j,\\ z_{ia}\ge 0.\end{cases}$$

The first constraint means that the total probability equals 1. The second constraint means that, for each $j\in\{0,1,\ldots,n\}$, in the steady state, the probability that the system enters state j equals the probability that the system leaves state j. The third constraint means that the steady-state probability is non-negative. After obtaining the optimal solutions $\{z_{ia}^*\}$, the optimal vehicle transfer policy can be retrieved by

$$P(a|i)=\frac{z_{ia}^*}{\sum_a z_{ia}^*},$$

where $P(a|i)$ is the conditional probability that action a is taken when there are i cars in city 1. For example, if $P(a'|i)=1$ for a particular state i and a particular action a', then the optimal policy is to take action a' when the system is in state i.

The optimal fleet size is determined by

$$n^*=\arg\max_n\{W(n)-hn\}. \tag{6}$$

Let $W^*=W(n^*)-hn^*$ be the optimal time-average profit with optimal fleet size. The reasons for us to choose the LP approach to solve our dynamic programming problem are the following. The LP approach gives the exact solutions and is easy to write the computer program. But the LP matrix could consume a large amount of computer memory.

4.3. Special case

If the cost of transferring a car is zero, it is straightforward to show that the optimal vehicle transfer policy is myopic, i.e., it is optimal to keep y' cars in city 1 and $n-y'$ cars in city 2 in every morning, where

$$y'=\arg\max_y\{C(y)\}.$$

Note that y' can be understood as the optimal solution for the single-period problem with zero transfer cost. However, the value of y' depends on fleet size n.

Next, we move our attention to the optimal fleet size for this special case. Let

$$H_1(y)=r_1E\min(y,D_{11})+r_2E\min(y-D_{11})^+,D_{12})-hy$$

and

$$H_2(y) = r_1 E \min(y, D_{21}) + r_2 \min((y - D_{21})^+, D_{22}) - hy,$$

where $H_i(y)$ $(i = 1, 2)$ can be understood as the expected one-day profit in city i if there are y cars in city i in the morning.

Corollary 1. *The optimal fleet size with zero transfer cost is given by*

$$n' = \arg\max_y \{H_1(y)\} + \arg\max_y \{H_2(y)\}. \quad (7)$$

The proof of Corollary 1 is straightforward and is omitted. It is clear that n' in Eq. (7) is easy to compute. We shall use it as a heuristic solution for fleet size.

5. Numerical study

The purposes of our numerical study are (i) to conduct a sensitivity analysis on the optimal policy and (ii) to investigate the effectiveness of our heuristic solutions. All the MATLAB programmes for computational study are available upon request.

5.1. Sensitivity analysis

We first conduct a sensitivity analysis on cost parameters. Hereafter, we use $U(a, b)$ to represent the uniform distribution with lower bound a and upper bound b. The related parameters are specified below.

Demand distribution: Note that the local demand does not affect the transition probability $p_{ij}(a)$, only the inter-city demand does. Hence, we assume that the local demand in both cities follows the same discrete uniform distribution and the inter-city demand distribution is city-specific. Formally, D_{11} follows $U(0, 9)$, D_{21} follows $U(0, 4)$, and D_{i2} (for city $i = 1, 2$) follows $U(0, 15)$. It is clear that there are more inter-city demands in city 1 than city 2, hence, vehicle transfer decisions will be made frequently. If we assume that the inter-city demand D_{i1} also follows the same distribution for all city i, the performance of our heuristic described in the next section will improve because the vehicle transfer decisions will be made less frequently. Therefore, we use asymmetric inter-city demand distributions in our sensitivity analysis and numerical study.

Cost parameters: Without loss of generality, we let $h = 1$. The rental rate and transfer cost are scaled accordingly. For each set of (k, r_1, r_2), we solve the optimal fleet size and vehicle transfer policy and compute the time-average profit by following the procedures described in Section 4.2.

First, we examine the impacts of the inter-city rental rate. We fix the transfer cost to be $k = 3$ and the local rental rate to be $r_2 = 4$ and then change the inter-city rental rate r_1 incrementally from 6 to 12. In Fig. 2(a), we see that the optimal time-average profit is linearly increasing in r_1 and the optimal fleet size is not sensitive to the changes in r_1. The reasons for this observation are the following. We assume that the demand distributions are bounded and the inter-city demand is given a higher priority. As the optimal fleet size is actually more than the maximum total demand of inter-city traffic, all the inter-city demands will be met provided that the optimal vehicle transfer policy is adopted. Hence, the time-average inter-city revenue is equal to $r_1[E(D_{11}) + E(D_{21})]$, which is linear. In Fig. 2(b), we see that the optimal vehicle transfer policy (i.e., the upper and lower control limits) is not sensitive to the changes in r_1 either. If it is optimal to transfer a vehicle with a certain r_1, then it is also optimal to transfer when r_1 increases. Hence, the vehicle transfer policy is not sensitive to the inter-city rental rate.

Next, we examine the impacts of the local rental rate. We fix the transfer cost to be $k = 3$ and the inter-city rental rate to be $r_1 = 12$ and then change the local rental rate r_2 incrementally from 2 to 8. In Fig. 3(a), we see that the optimal fleet size is a roughly concave increasing function of r_2 and that the time-average profit is roughly linearly increasing in r_2. In Fig. 3(b), we see that the gap between the upper and the lower control limits is decreasing in r_2. The reason for the decreasing gap is that the opportunity cost for not transferring a vehicle increases when the local rental rate increases. Hence, it is optimal to do more transfers, which leads to a smaller gap between the upper and the lower control limits.

Third, we examine the impacts of the transfer cost. Fig. 4(a) shows how the optimal fleet size and time-average profit vary with the transfer cost. It turns out that the time-average profit is approximately linear and decreasing in transfer cost. But the optimal fleet size is not sensitive to the transfer cost. The observation in Fig. 4(a) increases our confidence in using Eq. (7) as an approximation for the optimal fleet size. On the other hand, in Fig. 4(b) we see that the gap between the upper and the lower control limits is widening when the transfer cost increases. When the transfer cost increases, it is less profitable to transfer. However, when the transfer cost is zero, it is always optimal to transfer, e.g., the upper limit is the same as the lower limit, such that the number of cars in each city remains unchanged.

5.2. Performance of heuristics

First of all, it is useful to reiterate the heuristic policies that we will use. For the first-stage problem, we shall use n' in Eq. (7) to determine the fleet size. For the second-stage problem, we use Eq. (15) to determine the vehicle transfer policy while the fleet size is set to be n'.

Demand distribution is the same as that specified in Section 5.1 and daily operating cost is $h = 1$. We consider two levels of transfer cost: low and high. The low and high transfer costs are generated by sampling from uniform distributions $U(0, 2)$ and $U(2, 4)$, respectively. Inter-city rental rate is generated by sampling from $U(6, 12)$. We generate the inter-to-local ratio by sampling from $U(0.4, 0.6)$ and $U(0.3, 0.4)$. Then the local rental rate equals the product of inter-to-local ratio and the inter-city rental rate. It is clear that there are four cases with two different levels of transfer cost and two different levels of inter-to-local ratio. The settings cover a wide range of plausible scenarios. We generate 100 random samples for each of the four cases, which means 400 random samples in total.

For each randomly generated sample, we do the following:

- We iterate all plausible fleet sizes.
- For each plausible fleet size n, we find the optimal time-average profit by solving the LP model described in Eq. (5).
- We identify the optimal fleet size n^* and the heuristic fleet size n' according to Eqs. (6) and (7), respectively.
- We evaluate the time-average profit of our heuristic solution, denoted by W'.

Let

$$\Delta E = \frac{W^* - W'}{W^*} \times 100\%$$

be the percentage difference between the true optimal time-average profit and the heuristic time-average profit. Define Hit% as the percentage of the samples that the heuristic fleet size is the same as the true optimum (i.e., $n^* = n'$). We summarize the data in Table 1

In terms of hit rate, the heuristic is able to find the optimal fleet size in 76% of the samples. In the 24% of the samples that the heuristic does not hit the optimum, the difference between n^* and n' is ± 1, implying that the error is just one car. Given such a good performance in finding the optimal fleet size, managers can use Eq. (7) to quickly

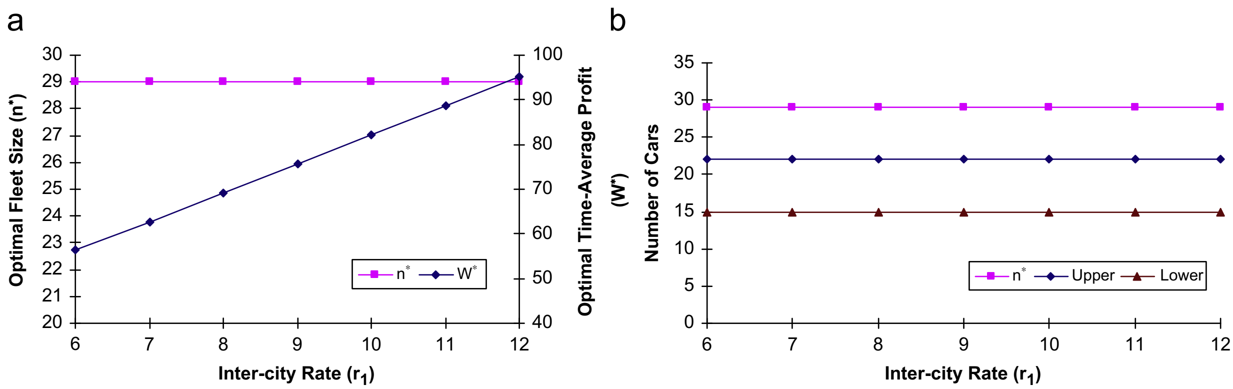

Fig. 2. Impacts of inter-city rental rate: (a) changes in optimal fleet size and profit and (b) changes in optimal transfer policy.

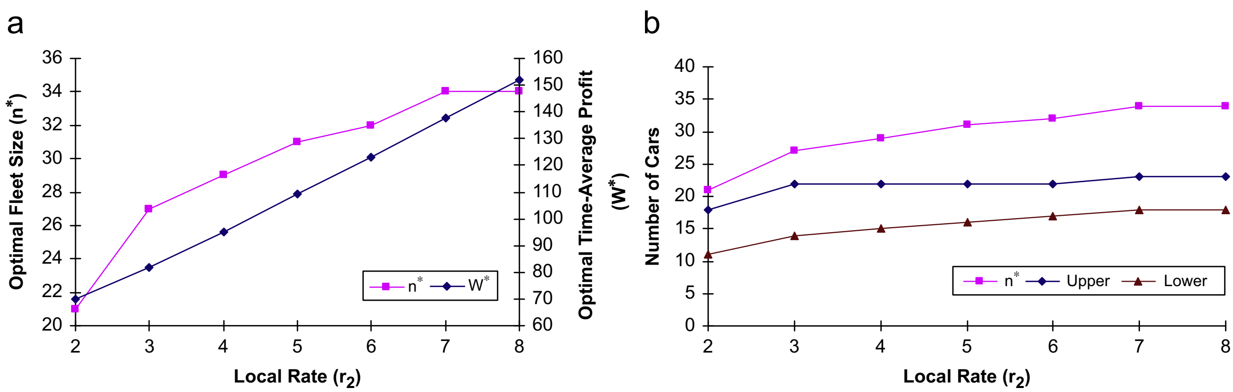

Fig. 3. Impacts of local rental rate: (a) changes in optimal fleet size and profit and (b) changes in optimal transfer policy.

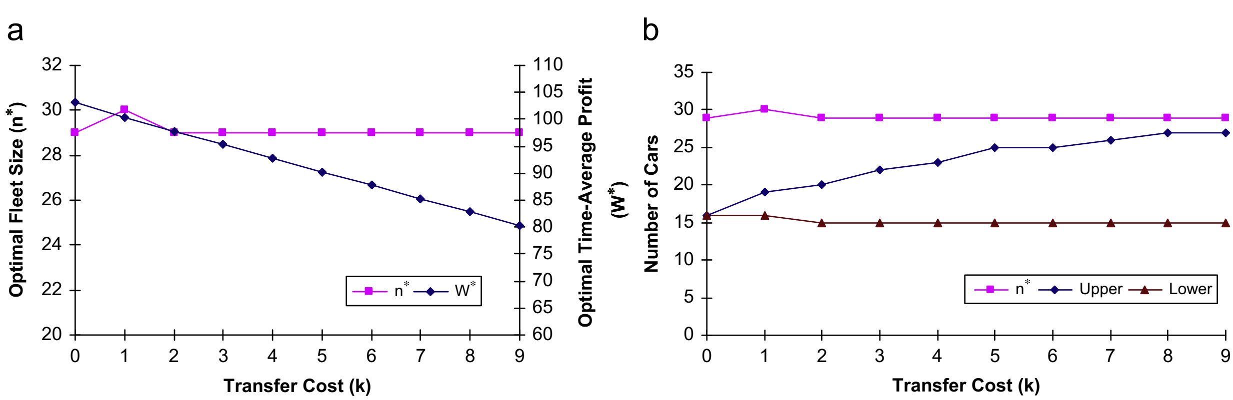

Fig. 4. Impacts of transfer cost: (a) changes in optimal fleet size and profit and (b) changes in optimal transfer policy.

Table 1
Performance of heuristics.

Case	Parameters		Hit (%)	Heuristic		Improved heuristic	
	r_2/r_1	k		Max ΔE (%)	Ave ΔE (%)	Max ΔE (%)	Ave ΔE (%)
1	U(0.3,0.4)	U(0,2)	81	7.8	1.7	0.04	0.00
2	U(0.3,0.4)	U(2,4)	74	23.7	10.4	0.06	0.01
3	U(0.4,0.6)	U(0,2)	71	8.5	1.3	0.08	0.01
4	U(0.4,0.6)	U(2,4)	78	26.2	10.9	0.08	0.01
Ave. hit			76				

determine if downsizing or expanding the fleet is necessary when the demand pattern changes due to such reasons as market competition and economic downturn.

In terms of time-average profit, the heuristic performs well when the transfer cost is lower than the local rental cost (e.g., cases 1 and 3). In the rental-car company we studied, the inter-city rental rate is about RMB450 and the local rental rate is about RMB300. The cost of transfer is about RMB200. Hence, our heuristic is likely to be a good solution for the company.

However, when the transfer cost is close to or higher than the local rental cost (e.g, cases 2 and 4), the heuristic performs poorly. We retrieved the true optimal policies by following the procedures described in Section 4.2 and compared them with our heuristic transfer policies. We found that the heuristic transfer policy is quite different from the true optimum. We noticed that a small difference in control limits could result in significant losses in time-average profit particularly when the cost of transfer is close to or higher than the local rental rate. Therefore, the poor performance of our heuristic in cases 2 and 4 is attributed to the transfer heuristic.

An improved heuristic is to use n' as the fleet size and then solve an LP to obtain the corresponding optimal transfer policy. In the last two columns of Table 1, we see that such an improved heuristic delivers better performance. In terms of the time-average profit, the improved heuristic is about 0.01% less than the optimum and the maximum difference is 0.08%. This observation underscores the importance of using appropriate vehicle transfer policy when the cost of transfer is close to or higher than the local rental rate.

6. Extensions

In this section, we present several extensions of the basic model.

6.1. Relaxing the assumption of lost sales

The difficulty in establishing the quasi-concavity or concavity of Eq. (1) is mainly due to lost sales. However, we can relax the assumption of lost sales if the company can access to a B2B leasing market. In the railway, empty container, or trucking business, the leasing market is so huge that the company can always satisfy customer demands by using either the owned or the leased capacities (see Song and Earl [20] for related industrial evidence). For our car rental application, we assume that if there are not enough owned cars available at a branch, cars are leased from a B2B leasing market immediately to satisfy the demand. After use, any leased vehicle must be returned to the B2B market before the customers arrive in the next day. Usually, if the leased cars are not returned on time, there could be a penalty cost that the company wants to avoid.

We consider route-specific leasing costs, i.e., the leasing cost for the inter-city traffic is b_1 per day and b_2 per day for the local traffic, where $b_1 \geq b_2 \geq h$ and $r_i \geq b_i$. Note that when customers are always satisfied, the rental revenues are constants, hence, can be excluded from the optimization operator. As such, we can re-formulate the problem as a cost-minimization problem. Let $V_t(x_t, n)$ be minimum expected cost attainable when there are x_t cars in city 1, the fleet size is n, and there are t days remaining.

The system transition resembles the framework in Song and Earl [20]. To illustrate, assume that there are y_t cars in city 1 and $n - y_t$ cars in city 2 after vehicle transfers. At the evening of day t, the number of cars in city 1 will be $y_t - D_{11} + D_{21}$ as all the demands must be met. However, $y_t - D_{11} + D_{21}$ could be negative. This is due to the fact that the term $-D_{11} + D_{21}$ is random. For example, if $y_t = 5$, $n = 10$, $D_{11} = 6$, and $D_{21} = 0$, then there will be 11 cars in city 2 at the evening of day t and zero cars in city 1. Among the 11 cars in city 2, one of them is leased from the B2B market by the branch in city 1 and must be returned before the customers arrive in the next day. Hence, if x_t is negative, it means that there are $-x_t$ cars being leased from the B2B market by the branch in city 1 and are now in city 2.

The optimality equation is

$$
\begin{aligned}
&V_t(x_t, n) \\
&= \min_{0 \leq y_t \leq n} \left\{ \begin{array}{l} k|y_t - x_t| + b_1 E(D_{11} - y_t)^+ \\ \quad + b_1 E(D_{21} - n + y_t)^+ \\ \quad + b_2 E(D_{12} - (y_t - D_{11})^+)^+ \\ \quad + b_2 E(D_{22} - (n - y_t - D_{21})^+)^+ \\ \quad + EV_{t-1}(y_t - D_{11} + D_{21}, n) \end{array} \right\} \\
&= \min_{0 \leq y_t \leq n} \left\{ \begin{array}{l} k|y_t - x_t| + (b_1 - b_2) E[(D_{11} - y_t)^+ \\ \quad + (D_{21} - n + y_t)^+] \\ \quad + b_2 E[(D_{11} + D_{12} - y_t)^+ \\ \quad + (D_{21} + D_{22} - n + y_t)^+] \\ \quad + EV_{t-1}(y_t - D_{11} + D_{21}, n) \end{array} \right\}. \qquad (8)
\end{aligned}
$$

The constraint $y_t \geq 0$ guarantees that any leased vehicle must be returned after use.

Theorem 2. (a) *The optimal vehicle transfer policy for Eq.* (8) *is a two-limit policy*. (b) *The optimal value function $V_t(x_t, n)$ is convex in n for any t and x_t*.

The proof of Theorem 2 is in the Appendix. Theorem 2 resembles the results in Song and Earl [20] except that Song and Earl [20] considered a continuous time model with exponential vehicle transfer time and site-dependent backorder costs. The two-limit policy identified in Theorem 2 also resembles the ISD (invest-stay-decrease) policy in Eberly and Van Mieghem [5], hereafter EV. But there are two major differences between our model and EV. First, the capacity levels in EV can be freely adjusted; while in our model, the total number of owned cars remains the same and we re-balance the capacity in different sites by transferring vehicles. Second, in EV the initial capacity level in the next day is a result of the capacity adjustments in the current day, whereas in our model the number of vehicles in each city before any transfer is a random variable.

6.2. The value of demand information

Many car-rental companies in China are now accepting Internet and telephone reservations. In the ideal case, the central planner may be able to precisely observe the demand for the next day before making the vehicle transfer decisions. To handle this case, we reformulate the dynamic programming model as the following:

$$
\begin{aligned}
&V_t^n(x_t, d_{11}, d_{21}, d_{12}, d_{22}) \\
&= \max_{0 \leq y_t \leq n} \left\{ \begin{array}{l} -k|y_t - x_t| + r_1[\min(y_t, d_{11}) \\ \quad + \min(n - y_t, d_{21})] \\ \quad + r_2[\min((y_t - d_{11})^+, d_{12}) \\ \quad + \min((n - y_t - d_{21})^+, d_{22})] \\ \quad + EV_{t-1}^n(y_t - \min(y_t, d_{11}) \\ \quad + \min(n - y_t, d_{21}), D_{11}, D_{21}, D_{12}, D_{22}) \end{array} \right\}. \qquad (9)
\end{aligned}
$$

Here, $V_t^n(x_t, d_{11}, d_{21}, d_{12}, d_{22})$ represents the optimal profit attainable when there are x_t cars in city 1 and the observed demands are d_{11}, d_{21}, d_{12}, and d_{22}. However, the future reward depends on the realization of the demands in the next day. Hence, the expectation sign of the fourth term in Eq. (9) is taken with respect to $(D_{11}, D_{21}, D_{12}, D_{22})$. Note that we use d_{ij} to represent the observed demand for the current day and D_{ij} to represent the random demand in the next day.

Corollary 2. *When the demand information for the next day can be observed prior to the vehicle transfer, the central planner is better off.*

Corollary 2 is intuitive. After observing the demand information $\{d_{ij}\}$, the central planner can still use the optimal vehicle transfer policy determined by Eq. (1). Such a policy is equivalent to ignoring the observed demand information and performs as well as the case without demand information. Hence, when the central planner pays attention to the observed demand information and adjusts the vehicle transfer policy accordingly, he can do no worse with information than without.

The value of demand information can be quantified by taking the difference between Eqs. (1) and (9). But solving Eq. (9) requires more computational efforts because of the "curse of dimensions", e.g., the state space consists of $(n+1)D^4$ states, where n is the fleet size and the demand for each route takes D different values. The size of the LP matrix could be as large as $(n+1)^2 D^4 \times (n+1)^2 D^4$, which requires a huge memory to store. We attempted to quantify the profit increase due to advanced demand information with discrete uniform demand, e.g., $P(D_{ij} = k) = \frac{1}{3}$ for $k = 0, 1, 2$, all i and j. Even with this simple demand distribution, the size of the LP matrix is 4000×4000. We generated cost data randomly (the procedures to generate cost data are the same as those described in Section 5.2) and noted that the profit increases by 4.5% on average.

6.3. Long transfer time

In Section 3, we assume that vehicle transfer can be completed overnight. Such an assumption is consistent with Pachon et al. [11] and the practical situation in the company we studied since the distance between the two branches is 80 km. However, not every rental-car company opens branches in close proximity. For some companies, the time for vehicle transfer can take more than one day.

To illustrate how to modify the formulation to allow longer transfer time, we consider the following scenario where the transferred vehicles will show up in the evening rather than in the morning of the next day. Define

a_i=Number of cars to be transferred from city i to city j, where $i \neq j$ and $a_i \geq 0$. The decisions for vehicle transfer are made in the evening of every day after the transferred cars and returned cars arrive.

S_{ij} = The actual sales for demand type j in city i, where $i,j = 1,2$.

The system can be described by (x_1, x_2), where x_i represents the number of cars available in city i prior to vehicle transfer. The related dynamic programme for vehicle transfer problem is

$$V_t^n(x_1, x_2) = \max \left\{ \begin{array}{l} -k(a_1+a_2) + r_1 E(S_{11}+S_{21}) \\ \quad +r_2 E(S_{12}+S_{22}) \\ \quad +EV_{t-1}(x_1 - a_1 + a_2 - S_{11} \\ \quad +S_{21}, x_2 - a_2 + a_1 - S_{21} + S_{11}) \end{array} \right\}, \tag{10}$$

where the inter-city sales in city i satisfies

$$S_{i1} = \min\{x_i - a_i, D_{i1}\}; \tag{11}$$

the local sales in city i satisfies

$$S_{i2} = \min\{(x_i - a_i - D_{i1})^+, D_{i2}\}; \tag{12}$$

the vehicle transfer variables satisfy

$$x_i \geq a_i \geq 0, \quad i = 1,2; \tag{13}$$

and finally the fleet size constraint is

$$x_1 + x_2 = n. \tag{14}$$

The above model is similar to the one in Eq. (1) except that the vehicle in transit reduces the number of cars available for rent in the morning. The following Lemma identifies an important feature of the optimal vehicle transfer policy for Eq. (10).

Lemma 1. *The optimal vehicle transfer policy for Eq.* (10) *satisfies that* $a_1^* a_2^* = 0$, *meaning that simultaneous two-way transfer* (*e.g.*, $a_1^* > 0$ *and* $a_2^* > 0$) *is not optimal.*

For the same reason presented in Section 4.1, it is intractable to explicitly characterize the structure of the optimal vehicle transfer policy for Eq. (10). However, we can use the LP approach to find the optimal transfer policy with any given fleet size by modifying C_{ia} and $p_{ij}(a)$ properly. Nevertheless, it is clear that long transfer time reduces the profit.

6.4. Other extensions

In this section, we discuss several other extensions for the future research.

1. *Multi-city network*: We can extend the analysis to a network with more than two cities. However, such an extension is not straightforward as the state space increases rapidly and we need to consider route-specific transfer costs and priority rules (e.g., which route is given a higher priority). With lost sales, we can still derive the optimal fleet size by assuming that the transfer cost is zero and use such a solution as an approximation for the fleet size problem with positive transfer costs. When the customer demands can always be satisfied by either the leased or the owned cars, the optimal vehicle transfer policy is indeed tractable by adopting the methodology presented in Eberly and Van Mieghem [5].

2. *Multi-type vehicles*: A limitation of our model is that we only consider one type of vehicle for rent. In practice, rental-car company may provide four-door sedan, SUV, or mini-van for rent. If customers are *unwilling* to switch to a different type of vehicle when their initial choice is out-of-stock, then we can still use the current model to solve the problem for each type of vehicle separately. In fact, the company we studied conducted a survey and found that 90% of the customers are young professionals and that they would walk away if their first initial type of vehicle was out-of-stock. This phenomenon has to do with the intensive competition in Suzhou, a city crowded with a dozen rental-car companies and more to come. On the other hand, if customers accept a different type of vehicle as a substitute when their initial choice is out-of-stock (such a phenomenon is known as customer-based substitution, see Mahajan and van Ryzin [10] for a literature review), then it becomes necessary to modify the model to include multiple types of vehicles, customer-based substitution, and vehicle upgrade policy.

3. *Revenue management*: The car rental industry has embraced the practices of revenue management since early 1990s (Lev et al. [8]). Our model can be extended by considering price-sensitive demands. The optimal pricing and vehicle transfer policy may be difficult to characterize since the objective function may be neither concave nor quasi-concave. In China, the rental-car industry is slowly embracing the concept of dynamic pricing because of the government regulations and the difficulty in implementing the algorithms. Therefore, in this paper we concentrate on the scenario that price cannot be changed.

7. Concluding remarks

We conclude this paper by summarizing our contributions and results. First, we formulate a two-stage dynamic programming model to assist a rental-car company to determine the optimal fleet size and vehicle transfer policy. Our formulation considers lost sales, non-zero transfer costs, and single-trip and round-trip demands. Second, due to lost sales, we find that the objective function could be neither concave nor quasi-concave. We describe how to find the optimal fleet size and vehicle transfer policy by solving a series of linear programming models when the planning horizon is infinite and the time-average cost criterion is adopted. Third, we propose a heuristic to determine the fleet size. Numerical study shows that our heuristic performs well. We also show that using an appropriate vehicle transfer policy is important when the transfer cost is close to, or higher than, the local rental rate. If an improper vehicle transfer policy is used, the overall performance can drastically deteriorate even with the optimal fleet size. Fourth, we characterize the optimal policy when the company can satisfy all customer demands by leasing vehicles from a B2B leasing market. In such a case, the optimal vehicle transfer policy is indeed a two-limit policy. Overall, our research sheds light on optimal fleet size and vehicle transfer policy for a small-sized rental-car company.

Acknowledgements

We thank Professor Stefan Nickel (Editor-in-chief) and two anonymous reviewers (especially reviewer #2) for their valuable comments and suggestions that helped improve the model and analysis presented in this paper. The work described in this paper was partly based on the doctoral thesis of the second author when he was studying in the CityU and USTC joint program. The second author would also like to thank Professor Huaping Chen and Professor K.K. Lai for their guidance during his study.

Proof of Theorem 1

Let

$$C_1(y_1) = r_1 \min(y_1, D_{11}) + r_2 \min((y_1 - D_{11})^+, D_{12})$$

and

$$C_2(y_1) = r_1 \min(n - y_1, D_{21}) + r_2 \min((n - y_1 - D_{21})^+, D_{22}).$$

For any given D_{11} and D_{12}, we find

$$C_1(y_1) = \begin{cases} r_1 y_1, & 0 \le y_1 \le D_{11}, \\ r_1 D_{11} + r_2(y_1 - D_{11}), & D_{11} < y_1 \le D_{11} + D_{12}, \\ r_1 D_{11} + r_2 D_{12}, & D_{11} + D_{12} < y_1, \end{cases}$$

which is an increasing concave function of y. Similarly, for any given D_{21} and D_{22}, we find

$$C_2(y_1) = \begin{cases} r_1 D_{21} + r_2 D_{22}, & 0 \le y_1 \le n - D_{21} - D_{22}, \\ r_1 D_{21} + r_2(n - y_1 - D_{21}), & n - D_{21} - D_{22} < y_1 \le n - D_{21}, \\ r_1(n - y_1), & n - D_{21} < y_1, \end{cases}$$

which is a decreasing concave function of y_1. Thus, it can be concluded that $C(y_1) = EC_1(y_1) + EC_2(y_1)$ is a concave function of y_1 since the expectation operator preserves the concavity.

For ease of exposition, we write $\partial H/\partial x$ as $\nabla_x H$. The optimal solution to $V_1^n(x_1)$ can be characterized by two control limits. Define

$$\begin{cases} x^l = \sup\{\{0\} \cup \{y_1 : \nabla_{y_1} C(y_1) \ge k\}\} \\ x^h = \inf\{\{\infty\} \cup \{y_1 : \nabla_{y_1} C(y_1) \le -k\}\}. \end{cases} \tag{15}$$

By the concavity of $C(y_1)$, it is easy to see that $x^l \le x^h$ and that

$$y^* = \begin{cases} x^l, & x_1 < x^l, \\ x_1, & x^l \le x_1 \le x^h, \\ x^h, & x_1 > x^h \end{cases} \tag{16}$$

is optimal for Eq. (3). □

Proof of Lemma 1. Suppose that $a_i^* > 0$ for $i = 1, 2$. Without loss of generality, assume that $a_1^* \ge a_2^*$. We construct an alternative solution as the following:

(i) transfer $a_1^* - a_2^*$ cars from city 1 to city 2;
(ii) in both cities, reserve a_2^* cars for one day (e.g., we deliberately keep the reserved cars from being rented);
(iii) in the next day, resume the optimal policy from then on.

One can see that such a transfer policy will lead to a higher one-day reward (due to the reduction in vehicle transfer cost) and in the next evening, the number of cars in either city remains the same as it would be under the original policy. Clearly, our alternative policy improves the profit, thus, a transfer policy with simultaneous two-way transfer is not optimal. □

Proof of Lemma 2. The proof is by induction. When $t = 1$, the truncated optimality equation is

$$\begin{aligned} V_1(x_1, n) &= \min_{0 \le y_1 \le n} \left\{ \begin{array}{l} k|y_1 - x_1| + (b_1 - b_2)E[(D_{11} - y_1)^+ \\ \quad + (D_{21} - n + y_1)^+] \\ \quad + b_2 E[(D_{11} + D_{12} - y_1)^+ \\ \quad + (D_{21} + D_{22} - n + y_1)^+] \end{array} \right\} \\ &= \min_{0 \le y_1 \le n} \{k|y_1 - x_1| + H_1(y_1, n)\}. \end{aligned}$$

It is well known that if $f_1(x)$ and $f_2(x)$ are continuous and convex in x, then $g(x) = \max\{f_1(x), f_2(x)\}$ is also convex in x. Note that $(D_{11} - y_1)^+ = \max\{0, D_{11} - y_1\}$, we see that $(D_{11} - y_1)^+$ is convex in y_1. Similarly, we see that $(D_{21} - n + y_1)^+$ and $(D_{21} + D_{22} - n + y_1)^+$ are jointly convex in n and y_1 and that $(D_{11} + D_{12} - y_1)^+$ is convex in y_1. As the expectation sign preserves the convexity and $b_1 \ge b_2$, we conclude that $H_1(y_1, n)$ is jointly convex in y_1 and n.

Notice that $k|y_1 - x_1|$ is also convex in y_1. The optimal solution to $V_1(x_1, n)$ can be characterized by two control limits that are given as the following:

$$x_1^l = \sup\{\{0\} \cup \{y_1 : \nabla_{y_1} H_1(y_1, n) \le -k\}\} \quad \text{and}$$
$$x_1^h = \inf\{\{\infty\} \cup \{y_1 : \nabla_{y_1} H_1(y_1, n) \ge k\}\}.$$

Due to the convexity of $H_1(y_1, n)$, we see that $x_1^l \le x_1^h$. We find

$$V_1(x_1, n) = \begin{cases} k(x_1^l - x_1) + H_1(x_1^l, n), & x_1 < x_1^l, \\ H_1(x_1, n), & x_1^l \le x_1 \le x_1^h, \\ k(x_1 - x_1^h) + H_1(x_1^h, n), & x_1 > x_1^h. \end{cases}$$

Then it can be verified that $V_1(x_1, n)$ is jointly convex in x_1 and n. Consequently, when x_1 is fixed, $V_1(x_1, n)$ is convex in n.

Next, we assume that $V_t(x_t, n)$ is jointly convex in x_t and n. When there are $t+1$ days remaining, the optimality equation is

$$\begin{aligned} V_{t+1}(x_{t+1}, n) &= \min_{0 \le y_{t+1} \le n} \{k|y_{t+1} - x_{t+1}| + H_{t+1}(y_{t+1}, n)\} \\ &= \min_{0 \le y_{t+1} \le n} \{k|y_{t+1} - x_{t+1}| + H_1(y_{t+1}, n) \\ &\qquad + EV_t(y_{t+1} - D_{11} + D_{21}, n)\}. \end{aligned}$$

It is well known that $f(ax+b)$ is convex in x if $f(x)$ is convex in x and a and b are constants. Since $y_{t+1} - D_{11} + D_{21}$ is linear in y_{t+1} and $V_t(x_t, n)$ is jointly convex in x_t and n, we see that $EV_t(y_{t+1} - D_{11} + D_{21}, n)$ is jointly convex in y_{t+1} and n. As $H_1(y_{t+1}, n)$ is jointly convex in y_{t+1} and n, we conclude that $H_{t+1}(y_{t+1}, n)$ is jointly convex in y_{t+1} and n.

The optimal solution to $V_{t+1}(x_{t+1}, n)$ can be characterized by two control limits that are defined as the following:

$$x_{t+1}^l = \sup\{\{0\} \cup \{y_{t+1} : \nabla_{y_{t+1}} H_{t+1}(y_{t+1}, n) \le -k\}\} \quad \text{and}$$
$$x_{t+1}^h = \inf\{\{\infty\} \cup \{y_{t+1} : \nabla_{y_{t+1}} H_{t+1}(y_{t+1}, n) \ge k\}\}.$$

It is obvious that $x_{t+1}^l \le x_{t+1}^h$. One can verify that $V_{t+1}(x_{t+1}, n)$ is jointly convex in x_{t+1} and n and that $V_{t+1}(x_{t+1}, n)$ is convex in n when x_{t+1} is fixed. □

References

[1] Archibald TW, Sassen AE, Thomas LC. An optimal policy for a two-depot inventory problem with stock transfer. Management Science 1997;43(2):173–83.
[2] Avriel M. Nonlinear programming: analysis and methods. Eaglewood Cliffs, NJ, USA: Prentice-Hall; 1976.
[3] Beaujon GJ, Turnquist MA. A model for fleet sizing and allocation. Transportation Science 1991;25(1):19–45.
[4] Dong L, Rudi N. Who benefits from transshipment? Exogenous vs. endogenous wholesale prices. Management Science 2004;50(5):645–57.
[5] Eberly J, Van Mieghem JA. Multi-factor dynamic investment under uncertainty. Journal of Economic Theory 1997;75:345–87.
[6] Ernst AT, Jiang H, Krishnamoorthy M, Nott H, Sier D. An integrated optimization model for train crew management. Annals of Operations Research 2001;108: 211–24.
[7] Hutzler C. On the road in China–soon for rent–tourism options widen as Hertz and Avis plan to offer cars for hire. Wall Street Journal, New York, USA, March 7, 2002. p. A18.
[8] Lev B, Geraghty M, Johnston E. Revenue management saves national car rental. Interfaces 1997;27(1):107–27.
[9] Luss H. Operations research and capacity expansion problems: a survey. Operations Research 1982;30(5):907–47.
[10] Mahajan S, van Ryzin G. Retail inventories and consumer choice. In: Tayur et al., editor. Quantitative models for supply chain management. Dordrecht, The Netherlands: Kluwer Academic Publishers; 1999.

[11] Pachon J, Iakovou B, Ip C, Aboudi R. A synthesis of tactical fleet planning models for the car rental industry. IIE Transactions 2003;35(9):907–16.
[12] Powell WB, Carvalho TA. Dynamic control of logistics queuing network for large-scale fleet management. Transportation Science 1998;32(2):90–109.
[13] Robinson L. Optimal and approximate policies in multi-period, multi-location inventory models with transshipment. Operations Research 1990;38(2): 278–95.
[14] Rudi N, Kapur S, Pyke DF. A two-location inventory model with transshipment and local decision making. Management Science 2001;47(12):1668–80.
[15] Savin SV, Cohen MA, Gans N, Katalan Z. Capacity management in rental businesses with two customer bases. Operations Research 2005;53(4):617–31.
[16] Sayarshad HR, Ghoseiri K. A simulated annealing approach for the multi-periodic rail-car fleet sizing problem. Computers and Operations Research 2009;36: 1789–99.
[17] Sherali HD, Maguire LW. Determining rail fleet sizes for shipping automobiles. Interfaces 2000;30(6):80–90.
[18] Sherali HD, Tuncbilek CH. Static and dynamic time-space strategic models and algorithms for multi-level rail-car fleet management. Management Science 1997;43(2):235–50.
[19] Song DP, Carter J. Optimal empty vehicle redistribution for hub-and-spoke transportation systems. Naval Research Logistics 2008;55(2):156–71.
[20] Song DP, Earl CF. Optimal empty vehicle repositioning and fleet-sizing for two-depot service systems. European Journal of Operational Research 2008;185: 760–77.
[21] Van Mieghem JA. Capacity management, investment, and hedging: review and recent developments. Manufacturing & Service Operations Management 2003;5(4):269–302.
[22] Wu P, Hartman JC, Wilson GR. An integrated model and solution approach for fleet sizing with heterogeneous assets. Transportation Science 2005;39(1): 87–103.