1. (a):

$f_{\Theta}(\theta) = \begin{cases} 10\theta, & \theta \in [0, \frac{1}{\sqrt{5}}] \\ 0, & \text{otherwise} \end{cases}$

$30 \mid X=\theta \sim \text{exponential}(\theta)$

$\therefore f_{30|\theta}(30|\theta) = \theta e^{-30\theta}, \quad \theta \in [0, \frac{1}{\sqrt{5}}]$

$$f_{\theta|30}(\theta|30) = \frac{f(30|\theta) \cdot f(\theta)}{\int_0^{\frac{1}{\sqrt{5}}} f(30|\theta') f(\theta')\, d\theta'}$$

$$= \frac{10\theta^2 e^{-30\theta}}{\int_0^{\frac{1}{\sqrt{5}}} \theta' e^{-30\theta'} \cdot 10\theta'\, d\theta'}$$

posterior PDF $= \begin{cases} \dfrac{\theta^2 e^{-30\theta}}{\int_0^{\frac{1}{\sqrt{5}}} \theta'^2 e^{-30\theta'} d\theta'} & \theta \in [0, \frac{1}{\sqrt{5}}] \\ 0, & \text{otherwise} \end{cases}$

conditional expectation:

$$\hat{\theta}_M = E[\theta \mid 30] = \int_0^{\frac{1}{\sqrt{5}}} f_{\theta|30}(\theta|30)\, d\theta$$

$$= \int_0^{\frac{1}{\sqrt{5}}} \frac{\theta^2 e^{-30\theta}}{\int_0^{\frac{1}{\sqrt{5}}} \theta'^2 e^{-30\theta'} d\theta'}\, d\theta$$

MAP: Find the value of $\theta \in [0, \frac{1}{\sqrt{5}}]$ that maximizes

$f_{30|\theta}(30|\theta) \cdot f_\theta(\theta) = 10\theta^2 e^{-30\theta}$

we obtain $\frac{d}{d\theta}[\theta^2 e^{-30\theta}]$

$= 2\theta \cdot e^{-30\theta} - 30\theta^2 e^{-30\theta}$

$= (2 - 30\theta)\theta \cdot e^{-30\theta} = 0$

solving for θ, $\hat{\theta} = \frac{2}{30}$

(b).

Posterior PDF $f_{\Theta|X}(\theta \mid X=(x_1, x_2, x_3, x_4, x_5)) \Rightarrow \begin{cases} x_1 = 30 \\ x_2 = 25 \\ x_3 = 15 \\ x_4 = 40 \\ x_5 = 20 \end{cases}$

Based on multiparameter problem

$f_{X|\theta}(x|\theta) = \theta e^{-x_1\theta} \cdot \theta e^{-x_2\theta} \cdot \theta e^{-x_3\theta} \cdot \theta e^{-x_4\theta} \cdot \theta e^{-x_5\theta}$

$= \theta^5 e^{-(x_1+x_2+x_3+x_4+x_5)\theta}$

$= \theta^5 e^{-130\theta}, \quad \theta \in [0, \frac{1}{\sqrt{5}}]$

$$\therefore f_{\theta|X}(\theta|x) = \frac{f(x|\theta) \cdot f(\theta)}{\int_0^{\frac{1}{\sqrt{5}}} f(x|\theta') \cdot f(\theta')\, d\theta'}$$

PDF $= \begin{cases} \dfrac{\theta^6 e^{-130\theta}}{\int_0^{\frac{1}{\sqrt{5}}} \theta'^6 e^{-130\theta'} d\theta'}, & \theta \in [0, \frac{1}{\sqrt{5}}] \\ 0, & \text{otherwise} \end{cases}$

MAP: Find the value of $\theta \in [0, \frac{1}{\sqrt{5}}]$ that maximizes $f_{X|\theta}(x|\theta) \cdot f(\theta) = 10\theta^6 e^{-130\theta}$

we obtain $\frac{d}{d\theta}[\theta^6 e^{-130\theta}]$

$= 6\theta^5 e^{-130\theta} - 130\theta^6 e^{-130\theta}$

$= (6 - 130\theta) \cdot \theta^5 e^{-130\theta} = 0$

solving for θ, $\hat{\theta} = \frac{6}{130} = \frac{3}{65}$

1.(b):

conditional expectation

$E[\theta \mid X = x_1, x_2, x_3, x_4, x_5]$

$= \int_0^{\frac{1}{\sqrt{5}}} f_{\theta|X}(\theta \mid x)$

$$= \int_0^{\frac{1}{\sqrt{5}}} \frac{\theta^6 e^{-130\theta}}{\int_0^{\frac{1}{\sqrt{5}}} \theta'^6 e^{-130\theta'} d\theta'}$$

2.(a) $\because \; 1 = \int_5^{60} c_1 \cdot e^{-0.04x} dx$

$$\therefore c_1 = \frac{1}{\int_5^{60} e^{-0.04x} dx} \approx \frac{1}{18.2005} \approx 0.055$$

$$1 = \int_5^{60} c_2 e^{-0.16x} dx$$

$$c_2 = \frac{1}{\int_5^{60} e^{-0.16x} dx} \approx \frac{1}{2.808} \approx 0.356$$

2.(a) Thus

$$f_{T|\theta}(x \mid \theta=1) = \begin{cases} 0.055 e^{-0.04x}, & \text{if } x \in [5,60] \\ x, & \text{otherwise} \end{cases}$$

$$f_{T|\theta}(x \mid \theta=2) = \begin{cases} 0.356 e^{-0.16x}, & \text{if } x \in [5,60] \\ x, & \text{otherwise} \end{cases}$$

when $x = 20$,

$$P_{\theta|20}(\theta=1 \mid 20) = \frac{f(20|\theta=1) \cdot P(\theta=1)}{f(20|\theta=1) \cdot P(\theta=1) + f(20|\theta=2) \cdot P(\theta=2)}$$

$$= \frac{0.055 e^{-0.04 \cdot 20} \cdot 0.3}{0.055 e^{-0.04 \cdot 20} \cdot 0.3 + 0.356 e^{-0.16 \cdot 20} \cdot 0.7}$$

$$\approx 0.42$$

$$P_{\theta|20}(\theta=2 \mid 20) = \frac{f(20|\theta=2) \cdot P(\theta=2)}{f(20|\theta=2) \cdot P(\theta=2) + f(20|\theta=1) \cdot P(\theta=1)}$$

$$= \frac{0.356 e^{-0.16 \cdot 20} \cdot 0.7}{0.365 e^{-0.16 \cdot 20} \cdot 0.7 + 0.055 e^{-0.04 \cdot 20} \cdot 0.3}$$

$$\approx 0.58$$

the hypothesis of $\theta = 2$ she will a ccept

The probability of the error is $P_{\theta|20}(\theta=1 \mid 20) = 0.42$

2.(b)

$P_{\theta|T}(\theta=1 \mid T=T_1, T_2, T_3, T_4, T_5) \Rightarrow \begin{cases} T_1=20 \\ T_2=10 \\ T_3=25 \\ T_4=15 \\ T_5=35 \end{cases}$

$T=T_1+T_2+T_3+T_4+T_5=105$

$$= \frac{0.05^5 e^{-0.04\cdot 105}\cdot 0.3}{0.05^5 e^{-0.04\cdot 105}\cdot 0.3 + 0.35^5 e^{-0.16\cdot 105}\cdot 0.7}$$

≈ 0.92

$P_{\theta|T}(\theta=2 \mid T)$

$$= \frac{0.35^5 e^{-0.16\cdot 105}\cdot 0.7}{0.35^5 e^{-0.16\cdot 105}\cdot 0.7 + 0.05^5 e^{-0.04\cdot 105}\cdot 0.3}$$

≈ 0.08

Thus, the hypothesis she will accept is $\theta=1$

The probability of the error is $P_{\theta|T}(\theta=2 \mid T)=0.08$

3. We know w is radar's overestimate $\in[0,2]$

θ is the car speed $\in[50,80]$

$\sim$ uniform

$$f_\theta(\theta)=\begin{cases}\frac{1}{30}, & \theta\in[50,80] \\ 0, & \text{otherwise}\end{cases}$$

$$f_W(w)=\begin{cases}\frac{1}{2}, & x\in[0,2] \\ 0, & \text{otherwise}\end{cases}$$

$X=\theta+w \quad \in[\theta, \theta+2]$

In interval 50 to 80, x is uniform distriuted

$$f_X(x)=\int_{52}^{80}\frac{1}{30}\cdot\frac{1}{2}\,d\theta=\frac{1}{2}, \quad x\in[\theta,\theta+2]$$

$$f_{X|\theta}(x|\theta)=\begin{cases}\frac{1}{2}, & \\ 0, & \text{otherwise}\end{cases}$$

$\because 50\le\theta\le 80$, $x-2\le\theta\le x$

$\therefore 52\le x\le 80$

$$f_{\theta|X}(\theta|x)=\frac{f_\theta(\theta)f_{X|\theta}(x|\theta)}{\int f_\theta(\theta')f_{X|\theta}(x|\theta')\,d\theta'}=\frac{\frac{1}{30}\cdot\frac{1}{2}}{\int_{x-2}^{x}\frac{1}{30}\cdot\frac{1}{2}\cdot d\theta'}=\frac{1}{2}$$

3. Thus $E[\theta \mid X = x] = \int_{x-2}^{x} \frac{1}{2} \theta \, d\theta = x-1$

LMS $\hat{\theta} = x-1$, $x \in [52, 80]$

$\hat{\theta} = \frac{x}{2} + 25$, $x \in [50, 52]$

$\hat{\theta} = \frac{x}{2} + 39$, $x \in [80, 82]$

4. x is the number of detected photons

$P(on) = P$

on $\theta \sim$ Possion (λ)

off $\theta = 0$

if on $X = \theta + N$

off $X = N$

$\theta + N \sim$ Possion $(\lambda + \mu)$

Thus $P(on \mid X = h) = \dfrac{P(X = h \mid on) \cdot P(on)}{P(X = h)}$

$= \dfrac{P(\theta + N = h) \cdot P}{P(N = h) \cdot (1-P) + P(\theta + N = h) \cdot P}$

PMF $\cdot$ $P_\theta(\theta) = \dfrac{\lambda^\theta e^{-\lambda}}{\theta!}$

$P_{\theta+N}(n) = \dfrac{(\lambda+\mu)^n e^{-(\lambda+n)}}{n!}$

$\therefore P(on \mid X = h) = \dfrac{P \cdot \dfrac{(\lambda+\mu)^h e^{-(\lambda+\mu)}}{h!}}{P \dfrac{(\lambda+\mu)^h e^{-(\lambda+\mu)}}{h!} + (1+P) \cdot \dfrac{\mu^h e^{-\mu}}{h!}}$

$= \dfrac{P(\lambda+\mu)^h e^{-\lambda}}{P(\lambda+\mu)^h + (1-P)\mu^h}$