1. Let x be the RV in uniform $(-1,1)$

Let $Y = e^x$

$\because \frac{x-(-1)}{2}$ $\therefore F_X(x) = \begin{cases} 0, & x<-1 \\ \frac{x+1}{2}, & -1 \le x \le 1 \\ 1, & x>1 \end{cases}$

$\because e^x$ increase with x increase

$\therefore Y = [\frac{1}{e}, e]$

$$\begin{aligned} F_Y(y) &= P(Y \le y) \\ &= P(e^x \le y) \\ &= P(x \le \ln y) \\ &= F_X(\ln y) = \tfrac{1}{2}\ln y + \tfrac{1}{2}, \quad \tfrac{1}{e} \le y \le e \end{aligned}$$

since $0 < \ln y < 1$

$$\therefore F_Y(y) = \begin{cases} P(Y \le y) = 0, & y < \frac{1}{e} \\ \frac{1}{2}\ln y + \frac{1}{2}, & \frac{1}{e} \le y \le e \\ P(Y \le y) = 1, & y > e \end{cases}$$

PDF: $f_Y(y) = F'_Y(y) = \begin{cases} \frac{1}{2y}, & \frac{1}{e} \le y \le e \\ 0, & \text{othermise} \end{cases}$

2. $\sigma(X,Y) = \frac{\text{cov}(X,Y)}{\sqrt{\text{var}(X)\,\text{var}(Y)}}$

$\text{cov}(X,Y) = E[XY] - E[X]E[Y]$

$$\begin{aligned} E[XY] &= E[X\cdot(a+bX+cX^2)] = E[aX+bX^2+cX^3] \\ &= aE[X] + bE[X^2] + cE[X^3] = b+c \end{aligned}$$

$$E[X]E[Y] = E[X]E[a+bX+cX^2] = E[X](a+bE[X]+cE[X^2]) = 0$$

$\therefore \text{cov}[X,Y] = b+c$

$$\begin{aligned} \text{var}(X) &= E[X^2] - E[X]^2 = E[X^2] = 1 \\ \text{var}(Y) &= E[(a+bX+cX^2)^2] - (E(a+bX+cX^2))^2 \\ &= (a^2 + 2abE[X] + 2acE[X^2] + 2bcE[X^3] + b^2E[X^2] + c^2E[X^4]) \\ &\quad - (a + bE[X] + cE[X^2])^2 \\ &= a^2 + 2ac + 2bc + b^2 - (a^2 + 2ac + c^2) = 2bc + b^2 - c^2 \end{aligned}$$

$\therefore \sigma(X,Y) = \frac{b+c}{\sqrt{2bc+b^2-c^2}}$

3. (a) Let x be the expected amount of time that professor devotes to the task

$\because Y \sim u(0,4)$

$\therefore E[Y] = \frac{4}{2} = 2$

$E[X] = E[E[X|Y]] = E[5-Y] = 5 - E[Y] = 5-2=3$

(b) $9+2+3=14$

The expected time at which the task is completed is 14pm

4.

$$P_X(x) = \begin{cases} \frac{1}{3}, & \text{if } x=0 \\ \frac{1}{5}, & \text{if } x=1,3 \\ \frac{4}{15}, & \text{if } x=5 \end{cases}$$

$$M(s) = E[e^{sx}] = \frac{1}{3}e^{0s} + \frac{1}{5}e^{1s} + \frac{1}{5}e^{3s} + \frac{4}{15}e^{5s}$$

Mean: $E[X] = \frac{d}{ds}M(s)\Big|_{s=0} = \frac{1}{3}\cdot 0e^{0s} + \frac{1}{5}\cdot 1e^{1s} + \frac{1}{5}\cdot 3e^{3s} + \frac{4}{15}\cdot 5e^{5s}$

$$= 0 + \frac{1}{5} + \frac{3}{5} + \frac{20}{15}$$

$$= \frac{3}{15} + \frac{9}{15} + \frac{20}{15} = \frac{32}{15}$$

$E[X^2] = \frac{d^2}{ds^2}M(s)\Big|_{s=0} = \frac{1}{3}\cdot 0^2 e^{0s} + \frac{1}{5}\cdot 1^2 e^{1s} + \frac{1}{5}\cdot 3^2 e^{3s} + \frac{4}{15}\cdot 5^2 e^{5s}$

$$= 0 + \frac{1}{5} + \frac{9}{5} + \frac{100}{15} = \frac{3}{15} + \frac{27}{15} + \frac{100}{15} = \frac{130}{15}$$

$$\text{var} = E[X^2] - (E[X])^2 = \frac{130}{15} - \left(\frac{32}{15}\right)^2 = \frac{1950}{225} - \frac{1024}{225} = \frac{926}{225} \approx 4.1$$

5. Let x be number of different types of pizzas

$E[X] = E[E[X|K]]$

$X_i = \begin{cases} 1, & \text{if type of pizza i is ordered by at least one customer} \\ 0, & \text{otherwise} \end{cases}$

$X = X_1 + X_2 + \cdots + X_n$

$E[E[X|K]] = E[E[X_1 + X_2 + \cdots + X_n | K]] = n \cdot E[E[X_1|K]]$

$P(\text{customer does not order type } i) = \frac{n-1}{n}$

$\therefore E[X_1|K] = 1 - \left(\frac{n-1}{n}\right)^K$

$E[X] = n \cdot E[1-p^K]$

$= n - nE[p^K]$

$= n - nE[e^{K\log P}]$

$= n - n\cdot M_K(\log P)$

$= n - n\cdot M_K(\log \frac{n-1}{n})$