AIM 5002 Computational Statistics and Probability

<Spring 2021>

Exam 2

The Katz School of Science and Health, Yeshiva University

(Take-Home, April 9 ~ 11)

Name: ________________________________ Score: __________/15

Please answer the questions under each problem or in a white paper. Show your work while deriving the answer. Leave your answer in the final step before calculating the fractional answer if you don't have or don't want to use a calculator. Submit it in the Canvas or directly email me to wonjun.lee@yu.edu in case you have a trouble to upload your file. You can take a picture of your answer using your smart phone or mobile devices. This is a take-home exam.

1. K school has a quota of 650 students for the 2022 cohort. 10,000 students applied to K school for the 650 admissions. To admit 650 students, K school decided to use only national exam that covers mathematics and verbal ability. The expected score of this exam is 65 and standard deviation is 5. Evaluate whether an applicant who received 85 score can get into the K school (3 points).

   $\mu = 65, \sigma = 5$, using Chebyshev's theorem:

   $$P(\mu - k\sigma < X < \mu + k\sigma) \geq 1 - \frac{1}{k^2}$$

   $\mu + k\sigma = 85$ → 65 + 5k = 85 → k = 4

   $$P(X < 85) \geq P(45 < X < 85) \geq 1 - \frac{1}{4^2} = 0.9375$$

   $$P(X \geq 85) \leq 1 - 0.9375 = 0.0625$$

   10,000(0.0625) = 625 < 650 → The applicant can get in K school.

2. Local supermarket K's customer arrivals are modeled by a Poisson process with λ=3 customers per minute.

   (a) Find the probability that exactly 4 customers in a 2-minute period in supermarket K (1 point).

   P(4, 2) = $\frac{e^{-3*2}(3*2)^4}{4!}$ = 0.134

   (b) Find the probability that there will be at least 2 customers in a 2-minute period (2 points).

   A = event that there will be at least 2 customers in a 2-minute period

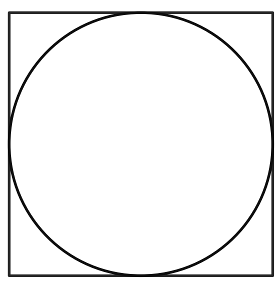
   P(A) = 1 – P(0, 2) – P(1, 2) = 1 - $\frac{e^{-3*2}(3*2)^0}{0!} - \frac{e^{-3*2}(3*2)^1}{1!}$

   = 1 – e^{-6} – $6e^{-6}$ = 1 – $7e^{-6}$ = 0.98264873476

(c) Find the conditional probability that there will be exactly 4 customers in a 2-minute period, given that there are 5 customers in the previous two minutes (1 point).
Same as (a) → 0.134 from the memorylessness.

3. You have a biased coin with probability 0.55 for head and probability 0.45 for tail. Suppose that you toss the biased coin 10,000 times and calculate the probability that you will get no more than 5,650 of heads and no less than 5,400 of heads (4 points).
Y is # of heads obtained in the 10,000 tosses
E[Y] = 10,000(0.55) = 5,500
var(Y) = 10,000(0.55)(0.45) = 2,475
$P(5{,}400 \le Y \le 5{,}650) = P\left(\frac{5{,}399.5-5{,}500}{\sqrt{2{,}475}} \le \frac{Y-5{,}500}{\sqrt{2{,}475}} \le \frac{5{,}650.5-5{,}500}{\sqrt{2{,}475}}\right)$
$= P\left(-2.02 \le \frac{Y-5{,}500}{\sqrt{2{,}475}} \le 3.03\right) \approx \Phi(3.03) - \Phi(-2.02) = 0.9771$

4. Suppose that you have a circle to play with arrow shown below. Whenever you shoot arrow, suppose that it is within the square. The probability that you shoot within the circle is PI/4. Here, X_i is the random variable that represents whether your arrow is shot within the circle or not.
Please explain how you can estimate PI and find how many shooting you need to make sure that the estimation error is within 0.01 with probability at least 95% (4 points)?



We can estimate $PI/4 = \frac{\sum_{i=1}^{n} X_i}{n}$ → $PI = M_n = 4\frac{\sum_{i=1}^{n} X_i}{n}$,
We want $P(|M_n - PI| \ge 0.01) \le 0.05$
X_i is Bernoulli RV with $P(X_i = 1) = PI/4$
$E[M_n] = E\left[4\frac{\sum_{i=1}^{n} X_i}{n}\right] = 4\frac{\sum_{i=1}^{n} E[X_i]}{n} = (4/n)n(PI/4) = PI$
$var(M_n) = var\left(4\frac{\sum_{i=1}^{n} X_i}{n}\right) = (16/n)\frac{\sum_{i=1}^{n} var(X_i)}{n} = (16/n^2)n(PI/4)(1-PI/4) = PI(4-PI)/n$
By Chebyshev's inequality: $(|M_n - PI| \ge 0.01) \le var(M_n)/0.01^2 = PI(4-PI)/(0.01^2 n)$
$PI(4-PI)/(0.01^2 n) \le 0.05$ → $n \ge PI(4-PI)/(0.01^2 \cdot 0.05) = 539353.2$ → At least you should shoot 539354 times

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