AIM 5002 Computational Statistics and Probability

(Spring 2021)

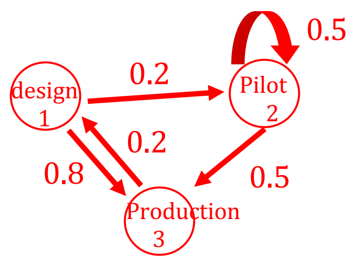
Assignment 6

Name:______________________________ Score:_________/8

Submit your assignment at CANVAS by uploading your file.
Due date: Sunday, 2nd of the May, 2021 by 11:59 PM

---

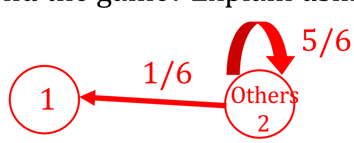
1. A machine part in the manufacturing process is composed of design, pilot, and production and has different probabilities between three phases. If the parts are developed, 20% go to the pilot phase, the others directly go to the production phase. The parts in pilot either go to the same phase or production phase with the same probability. If there is a critical defect in the part in production phase, the parts go back to design phase. 20% of the parts in production phase come back to design phase to correct. Develop Markov chain and find the steady-state probabilities (2 points).



$$\begin{bmatrix} 0 & 0.2 & 0.8 \\ 0 & 0.5 & 0.5 \\ 0.1 & 0 & 0 \end{bmatrix}$$

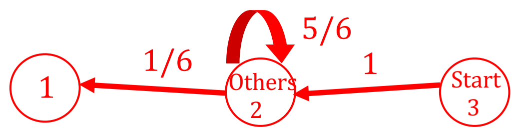
$$\pi_1 = 0.2\pi_3$$
$$\pi_2 = 0.2\pi_1 + 0.5\pi_2$$
$$\pi_3 = 0.5\pi_2 + 0.8\pi_3$$
$$\pi_1 + \pi_2 + \pi_3 = 1$$
$$\pi_1 = 0.2 * 0.625 = 0.125$$
$$\pi_2 = 0.4 * 0.625 = 0.25$$
$$\pi_3 = 1/1.6 = 0.625$$

2. You are given a fair 6-sided die and play a game by rolling repeatedly.
   (A) Suppose that if you see 1, the game stops. What is the expected number of rolling until you end the game? Explain using Markov chain (1 point).



$$\mu_1 = 0$$
$$\mu_2 = 1 + P_{21}\mu_1 + P_{22}\mu_2 = 1 + (5/6)\,\mu_2 \rightarrow \mu_2 = 6$$

(B) What is the expected number of rolling until you see the sum of the last two rolled numbers is 7? Use Markov chain (1 point).



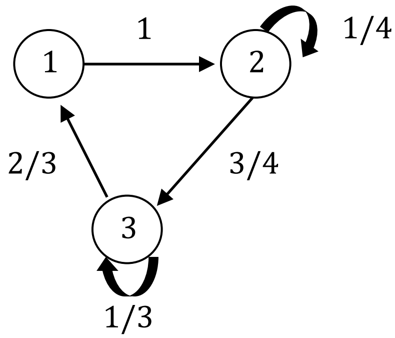
Paired numbers that make 7 are (1, 6), (2, 5), (3, 4).
No matter what you roll, the prob of rolling that makes the sum of the last two 7 is 1/6. The probability that you will not see the final number that makes 7 is 5/6 (1-1/6). For example, if you rolled 1, 4, 4, 2, 1, 3, 6, 1, the final two 6 and 1 satisfies the condition. In this case, the probability that you see the last 1 after 6 is 1/6 and prob that 1 is not seen is 5/6 after rolling 6. It would be same for all other pairs.

$$\mu_1 = \mu_3 = 0$$
$$\mu_2 = 6$$
$$\mu_3 = 1 + P_{31}\mu_1 + P_{32}\mu_2 + P_{33}\mu_3 = 1 + 6 = 7$$

3. You are playing with another person using a coin.
   (A) Suppose that the coin is fair meaning the probability of head is equal to ½ and probability of tail is equal to ½. You get one point at each flipping if you see the head and loses one point if you see the tail. The flippings are independent from each other. You play continuously until either you accumulate 10,000 points or lose all your points. What is the probability of eventually accumulating the 10,000 points when you start the game with 100 points (1 point)?
   From slide no. 38 of lecture 7 (Ex 7.11 – Gambler's Ruin) :
   $$a_{100} = 100/10{,}000 = 0.01$$

   (B) Suppose that the coin is unfair in that the probability of head is equal to 1/3 and probability of tail is equal to 2/3. You get one point at each flipping if you see the head and loses one point if you see the tail. The flippings are independent from each other. You play continuously until either you accumulate 10,000 points or lose all your points. What is the probability of eventually accumulating the 10,000 points when you start the game with 100 points (1 point)?
   From slide no. 38 of lecture 7 (Ex 7.11 – Gambler's Ruin) :
   $$\rho = \frac{1-1/3}{1/3} = \frac{2/3}{1/3} = 2$$
   $$a_i = (1-2^{100})/(1-2^{10000}) \approx 0$$

4. See the following Markov chain below. Assume that the process is in state 1 just before the first transition (2 points).



(A) What is the probability that the process will be in state 1 just after the fifth transition?

r11(5) = (3/4)(2/3)(1/16 + 1/9 + 1/12) = 37/288 = 0.128472

(B) Determine the expected value and variance of the number of transitions up to and including the next transition during which the process returns to state 1.

Distributions from 2 to 3 and from 3 to 1 are geometric with p=3/4 and 2/3 respectively.

Mean = 1/p and variance = $(1\text{-}p)/p^2$ for geometric RV.

Thus, when let RV X be the time until the process returns to state 1,

E[X] = 1 + 4/3 + 3/2 = 23/6 = 3.83333

Variance(X) = $(1\text{-}p)/p^2 = (1-3/4)(4/3)^2 + (1-2/3)(3/2)^2 = (1/4)(16/9) + (1/3)(9/4) = 43/36 = 1.1944$