1. $P(0)=\frac{1}{2}$ $P(1)=\frac{1}{2}$

When $x \geqslant 0$

$$F(x) = P(X \le x) = \frac{1}{2}\times 1 + \frac{1}{2}(1-e^{-\lambda x})$$
$$= \frac{1}{2} + \frac{1}{2} - \frac{1}{2}e^{-\lambda x}$$
$$= 1 - \frac{1}{2}e^{-\lambda x}$$

2. (a) Since X has mean 0, var 1

so X is a standard normal distribution

According to the table,

$$P(X \le 1.5) = \Phi(1.5) = 0.93319$$

$$P(X \le -1) = 1 - \Phi(1) = 1 - 0.84134 = 0.15866$$

(b) $$P(-5 \le Y \le 1) = P\left(\frac{-5-1}{3} \le \frac{Y-1}{3} \le \frac{1-1}{3}\right)$$
$$= \Phi(0) - \Phi(-2)$$
$$= \Phi(0) - (1-\Phi(2))$$
$$= 0.5 - (1-0.97725)$$
$$= 0.5 - 0.02275$$
$$= 0.47725$$

3. $50 = 1.8 \times$ Celsius $+ 32$

Cesius $= 10$ degree

$$P(X \le 10) = P\left(\frac{X-3}{7} \le \frac{10-3}{7}\right)$$
$$= \Phi(1)$$
$$= 0.84134$$

4. (a) since it's a quarter circle

The area of this is $\frac{\pi r^2}{4}$, $\therefore$ PDF $= \frac{1}{\frac{\pi r^2}{4}} = \frac{4}{\pi r^2}$

$$\therefore f(x,y) = \begin{cases} \frac{4}{\pi r^2} & (x,y) \in \{x^2+y^2 \le r^2, x \geqslant 0, y \geqslant 0\} \\ 0 & , \text{otherwise} \end{cases}$$

(b)

$$f_Y(y) = \int_{-\infty}^{+\infty} f(x,y)\,dx$$

Sin $x^2+y^2 \le r^2$

$$x \le \sqrt{r^2-y^2}$$
$$-\sqrt{r^2-y^2} \le x \le \sqrt{r^2-y^2}$$

$$\therefore f_Y(y) = \int_{-\sqrt{r^2-y^2}}^{\sqrt{r^2-y^2}} \frac{4}{\pi r^2}\,dx$$
$$= \frac{4}{\pi r^2}\sqrt{r^2-y^2} + \frac{4}{\pi r^2}\left(\sqrt{r^2-y^2}\right) = \frac{8}{\pi r^2}\sqrt{r^2-y^2}$$

5.
(a) $E[X] = \int_1^3 x \cdot \frac{x}{4}\, dx$

$= \frac{x^3}{12}\Big|_1^3$

$= \frac{27}{12} - \frac{1}{12}$

$= \frac{26}{12} = \frac{13}{6}$

(b) $P(A) = \int_2^3 \frac{x}{4}\, dx$

$= \frac{x^2}{8}\Big|_2^3$

$= \frac{9}{8} - \frac{4}{8}$

$= \frac{5}{8}$

(c) $f_{X|A}(x) = \frac{f_X(x)}{P(A)} = \frac{\frac{x}{4}}{\frac{5}{8}} = \frac{2x}{5}$

$\therefore f_{X|A}(x) = \begin{cases} \frac{2x}{5}, & 2 \le x \le 3 \\ 0, & \text{otherwise} \end{cases}$

(d) $E[X|A] = \int_2^3 x \cdot \frac{2x}{5}\, dx$

$= \frac{2x^3}{15}\Big|_2^3$

$= \frac{54}{15} - \frac{16}{15} = \frac{38}{15}$

(e) $\because E[g(x)] = \int_{-\infty}^{\infty} g(x) f_X(x)\, dx$

$\therefore E[Y] = E[X^2] = \int_1^3 x^2 \frac{x}{4}\, dx$

$= \frac{x^4}{16}\Big|_1^3$

$= \frac{81}{16} - \frac{1}{16}$

$= \frac{80}{16} = 5$

$\because Var(X) = E[X^2] - (E[X])^2$

$\therefore Var[Y] = Var[X^2]$

$= E[X^4] - (E[X^2])^2$

$= \int_1^3 x^4 \frac{x}{4}\, dx - 25$

$= \frac{x^6}{24}\Big|_1^3 - 25$

$= \frac{729}{24} - \frac{1}{24} - 25$

$= \frac{728}{24} - 25$

$= \frac{728 - 600}{24}$

$= \frac{128}{24}$

$= \frac{16}{3}$

6. (a) For x, y area $= 2 \times 2 \times \frac{1}{2} = 2$

$\therefore f(x,y) = \begin{cases} \frac{1}{2}, & x, y \ge 0,\ x + y \le 2 \\ 0, & \text{otherwise} \end{cases}$

(b) $\because 0 \le y \le 2,\ 0 \le x \le 2 - y$

$\therefore f_Y(y) = \int_0^{2-y} \frac{1}{2}\, dx$

$= \frac{2-y}{2}$

$\therefore f_Y(y) = \begin{cases} \frac{2-y}{2}, & 0 \le y \le 2 \\ 0, & \text{otherwise} \end{cases}$

(c) $f_{X|Y}(x,y) = \frac{f(x,y)}{f_Y(y)} = \frac{\frac{1}{2}}{\frac{2-y}{2}}$

$= \frac{1}{2-y}$

$\therefore f_{X|Y}(x,y) = \begin{cases} \frac{1}{2-y}, & 0 \le y \le 2,\ 0 \le x \le 2-y \\ 0, & \text{otherwise} \end{cases}$

6.(d) since the $f_{X|Y}$ is uniform on $[0, 2-y]$

the $E[X|Y]$ is mean $\frac{2-y}{2}$

$$E[X] = \int_0^2 E[X|Y] \cdot f_Y(y)\, dy$$

$$= \int_0^2 \frac{2-y}{2} \cdot f_Y(y)\, dy$$

$$= \int_0^2 (1-\frac{y}{2}) f_Y(y)\, dy$$

$$= \int_0^2 f_Y(y) - \frac{1}{2}\int y f_Y(y)\, dy$$

$$= \int_0^2 \frac{2-y}{2} - \frac{1}{2}E[Y]$$

$$= y - \frac{y^2}{4}\Big|_0^2 - \frac{1}{2}E[Y]$$

$$= 1 - \frac{1}{2}E[Y]$$

(e) $\because$ X is symmetry of Y

thus $E[X] = E[Y]$.

$\therefore$ $E[X] = 1 - \frac{1}{2}E[X]$.

$$\frac{3}{2}E[X] = 1$$

$$E[X] = \frac{2}{3}$$