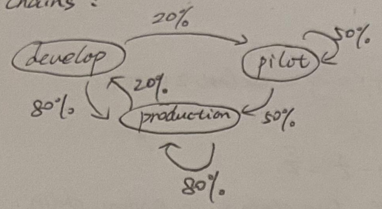
1. chains :



P:

	de	pi	pr
de	0	0.2	0.8
pi	0	0.5	0.5
pr	0.2	0	0.8

$\because \pi P = \pi$

$\therefore (\pi_1, \pi_2, \pi_3) \cdot P = \pi_1, \pi_2, \pi_3$

$$\therefore \left.\begin{aligned} \pi_1 &= 0.2\pi_3 \\ \pi_2 &= 0.2\pi_1 + 0.5\pi_2 \\ \pi_3 &= 0.8\pi_1 + 0.5\pi_2 + 0.8\pi_3 \end{aligned}\right\} \Rightarrow \begin{cases} \pi_1 = 0.15625 \\ \pi_2 = 0.0625 \\ \pi_3 = 0.78125 \end{cases}$$

$\because \pi_1 + \pi_2 + \pi_3 = 1$

2.(a)

$$P \quad \begin{array}{c|cccccc} & 1 & 2 & 3 & 4 & 5 & 6 \\ \hline 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 2 & 1/6 & 1/6 & 1/6 & 1/6 & 1/6 & 1/6 \\ 3 & 1/6 & 1/6 & 1/6 & 1/6 & 1/6 & 1/6 \\ 4 & 1/6 & 1/6 & 1/6 & 1/6 & 1/6 & 1/6 \\ 5 & 1/6 & 1/6 & 1/6 & 1/6 & 1/6 & 1/6 \\ 6 & 1/6 & 1/6 & 1/6 & 1/6 & 1/6 & 1/6 \end{array}$$

From unique solution $\mu_i = 1 + \sum_j P_{ij}\mu_j$

Because state 1 is absorption, $\mu_1 = 0$

$$\text{same} \begin{cases} \mu_2 = 1 + \frac{1}{6}\mu_2 + \frac{1}{6}\mu_3 + \frac{1}{6}\mu_4 + \frac{1}{6}\mu_5 + \frac{1}{6}\mu_6 \\ \mu_3 = 1 + \frac{1}{6}\mu_2 + \frac{1}{6}\mu_3 + \frac{1}{6}\mu_4 + \frac{1}{6}\mu_5 + \frac{1}{6}\mu_6 \\ \vdots \\ \mu_6 = 1 + \frac{1}{6}\mu_2 + \frac{1}{6}\mu_3 + \frac{1}{6}\mu_4 + \frac{1}{6}\mu_5 + \frac{1}{6}\mu_6 \end{cases}$$

Thus $\mu_2 = \mu_3 = \mu_4 = \mu_5 = \mu_6$

Take μ_2 as the start state case,

$\mu_2 = 1 + \frac{5}{6}\mu_2 \quad \therefore \mu_2 = 6$

Starting from other states has same expected result. Thus $\mu_2 = \mu_3 = \mu_4 = \mu_5 = \mu_6 = 6$

Overall, it takes expectation time 6 to the absorption state 1

2.(B) Since there are 6 combination to sum 2 rolls to 7

1-6, 2-5, 3-4

6-1, 5-2, 4,3

Chains:

$$\begin{tikzcd} \underset{1}{\text{current}} \arrow[loop left, "\frac{5}{6}"] \arrow[r, "\frac{1}{6}"] & \underset{2}{\text{success}} \arrow[loop right, "1"] \end{tikzcd}$$

We assume the first state we are at is either one of $\left\{\begin{matrix}1\\2\\3\\4\\5\\6\end{matrix}\right.$

Take we are at 1 for example, here we rolled one time.

The success state is to roll 6, Prob $= \frac{1}{6}$;

Or else we don't move and still at current state, Prob $= \frac{5}{6}$

$$P: \begin{array}{c|cc} & 1 & 2 \\ \hline 1 & \frac{5}{6} & \frac{1}{6} \\ 2 & 0 & 1 \end{array}$$

From state 1 to 2

$t_2 = 0$

$t_1 = 1 + \frac{5}{6} t_1$

$t_1 = 6$

Thus there are two states in Markov chain after first roll.

The 2nd state "success" is absorption

In other 5 cases, it has same result.

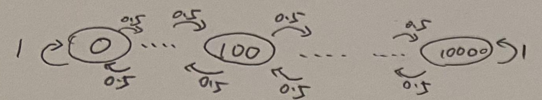
Thus there are 6 times to reach the sum of 7 from current state to sucess state

Since we have to roll one time to be in the current state

Totally $6+1=7$, the expectation times is 7.

3. (A) This is an expectation time to absorbtion state problem in gambler ruin case

Markov chain:



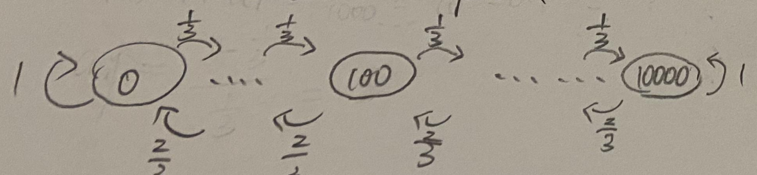
As we know: $a_0 = 0$, $a_{10000} = 1$

$a_{100} = (1-0.5)a_{99} + 0.5a_{101}$, $i = 1, 2 \cdots 9999$

$\because a_i = \begin{cases} \frac{1-p^i}{1-p^m}, & \text{if } p \neq 1 \\ \frac{i}{m}, & \text{if } p = 1 \end{cases}$, $p = \frac{0.5}{0.5} = 1$

$\therefore a_i = \frac{100}{10000} = 0.01$

(B) Since the coin is unfair:



$\therefore p = \frac{\frac{2}{3}}{\frac{1}{3}} = 2$ $\therefore a_i = \frac{1-2^{100}}{1-2^{10000}} \approx \frac{2^{100}}{2^{10000}}$

$= \frac{2^{100}}{2^{100} \cdot 2^{9900}} = \frac{1}{2^{9900}}$

4. (A)

$$P: \begin{array}{c|ccc} & 1 & 2 & 3 \\ \hline 1 & 0 & 1 & 0 \\ 2 & 0 & \frac{1}{4} & \frac{3}{4} \\ 3 & \frac{2}{3} & 0 & \frac{1}{3} \end{array}$$

since the trasition is start from 1, P_{11}^5 is the prob return to 1

Here I use calculator to calculate P^5

Thus

$$P^5: \begin{array}{c|ccc} & 1 & 2 & 3 \\ \hline 1 & 37/288 & 323/768 & 1039/2304 \\ 2 & 1039/3456 & 2153/9216 & 12877/27648 \\ 3 & 305/972 & 1039/2592 & 2219/7776 \end{array}$$

Thus $P_{11}^5 = \frac{37}{288} \approx 0.1285$

4. (B) Mean Recurrence time :

It's a mean first message problem from state 1 to state 1

$$\begin{array}{c|ccc} P & 1 & 2 & 3 \\ \hline 1 & 0 & 1 & 0 \\ 2 & 0 & \frac{1}{4} & \frac{3}{4} \\ 3 & \frac{2}{3} & 0 & \frac{1}{3} \end{array}$$

$$t_s^* = 1 + \sum_{j=1}^{m} P_{sj} t_j$$

$$t_1^* = 1 + t_2$$

$$\left.\begin{array}{l} t_2 = 1 + \frac{1}{4} t_2 + \frac{3}{4} t_3 \\ t_3 = 1 + \frac{2}{3} t_1 + \frac{1}{3} t_3 \end{array}\right\} \because t_1 = 0 \Rightarrow \left\{\begin{array}{l} t_2 = \frac{17}{6} \\ t_3 = \frac{3}{2} \end{array}\right. \Rightarrow t_1^* = \frac{23}{6}$$

$$\therefore E[t_1^*] = \frac{23}{6}$$

According to steady state behavior we have Prob π_1, π_2, π_3 for each states

$$(\pi_1, \pi_2, \pi_3) \cdot P = (\pi_1, \pi_2, \pi_3)$$

Thus,

$$\left\{\begin{array}{l} \pi_1 = \frac{2}{3}\pi_3 \\ \pi_2 = \frac{1}{3}\pi_1 + \frac{1}{4}\pi_2 \\ \pi_3 = \frac{3}{4}\pi_2 + \frac{1}{3}\pi_3 \\ \pi_1 + \pi_2 + \pi_3 = 1 \end{array}\right. \Rightarrow \left\{\begin{array}{l} \pi_1 = \frac{6}{23} = P_{11} \\ \pi_2 = \frac{8}{23} \\ \pi_3 = \frac{9}{23} \end{array}\right.$$

since it's a geometric problem to reach first success

$$\text{Thus } Var(P_{11}) = \frac{1-P}{P^2} = \frac{1 - \frac{6}{23}}{\left(\frac{6}{23}\right)^2} = \frac{0.739}{0.068} \approx 10.87$$