AIM 5002 Computational Statistics and Probability
<Spring 2021>

Exam 1

The Katz School of Science and Health, Yeshiva University
(8:20 PM ~ 9:40 PM, Wednesday, March 3)

Name: ______________________________ Score: __________/15

Please answer the questions under each problem or in a white paper. Show your work while deriving the answer. Leave your answer in the final step before calculating the fractional answer if you don't have or don't want to use a calculator. Submit it in the Canvas or directly email me to wonjun.lee@yu.edu in case you have a trouble to upload your file. You can take a picture of your answer using your smart phone or mobile devices.

1. A fair **6**-sided die is rolled twice and all the possible outcomes are equally likely. Let X and Y be the result of the 1st and 2nd roll respectively. Find the **P(A|B).** The conditional probability of A given B: P(A|B) = P(A∩B)/P(B) (1 point)

   $$A = \{max(X, Y) = 3\},\ B = \{min(X, Y) = 3\}$$

   B = {(3,3), (3,4), (3,5), (3,6), (4,3), (5,3), (6,3)}, A ∩ B = {(3,3)} → P(A|B) = 1/7.

2. If an aircraft is present in a certain area, a radar detects it and generates an alarm signal with probability **0.9**. If an aircraft is not present, the radar generates a false alarm, with probability **0.2**. Assume that an aircraft is present with probability **0.3**. (3 points)

   (A) What is the probability of <u>no</u> aircraft presence and a false alarm?
   P(alarm | presence)=0.9
   P(alarm | NO_presence)=0.2
   P(presence)=0.3
   P(alarm ∩ NO_presence)=?
   P(alarm | NO_presence) = P(alarm ∩ NO_presence) / P(NO_presence)
   → P(alarm ∩ NO_presence) = P(alarm | NO_presence) * P(NO_presence)
   = 0.2*(1-0.3) = 0.2*0.7 = 0.14

   (B) What is the probability that the aircraft is present when the radar generates alarm?

   $$P(\text{presence} \mid \text{alarm}) = \frac{P(\text{presence})P(\text{alarm} \mid \text{presence})}{P(\text{alarm})} = \frac{P(\text{presence})P(\text{alarm} \mid \text{presence})}{P(\text{presence})P(\text{alarm} \mid \text{presence}) + P(\text{NO\_presence})P(\text{alarm} \mid \text{NO\_presence})}$$

   $$= \frac{0.3*0.9}{0.3*0.9+(1-0.3)0.2} = 0.6585365854$$

3. A football team plays a game against premier league and loses with probability **0.5**. When they play against the 2nd league teams, they lose with probability **0.2**. When they play against the 3rd league teams, they lose with probability **0.1**. Suppose there are only three types of leagues including premier league, 2nd league, and 3rd leagues and there **20**, **25**, **35** teams respectively. What is the probability of losing? Use total probability theorem: For any event B, $P(B) = P(A_1)P(B|A_1) + \cdots + P(A_n)P(B|A_n)$ where $A_1, \ldots, A_n$ are disjoint events that form a partition of the sample space (1.5 points).

   A_i is an event of playing with an opponent of type *i*, $P(A_1) = 1/4$, $P(A_2) = 5/16$, $P(A_3) = 7/16$,
   Let B be the event of losing. $P(B|A_1) = 0.5$, $P(B|A_2) = 0.2$, $P(B|A_3) = 0.1$.
   By the Total Probability Theorem:
   $P(B) = P(A_1)P(B|A_1) + P(A_2)P(B|A_2) + P(A_3)P(B|A_3)$
   $= (1/4)0.5 + (5/16)0.2 + (7/16)0.1 = 37/160 = 0.23125$

4. There are two coins, a white and a black one. We choose one of the two at random, each being chosen with probability **½**, and proceed with two independent tosses. The coins are biased: with the white coin, the probability of heads in any given toss is **0.01**, whereas for the black coin it is **0.99**. Suppose that X is the event that black coin was selected and Y_i is the event that *i*th toss resulted in tail. Show that the two events Y_i are dependent, even though they are conditionally independent given X (1.5 points).

   $P(Y_1 \cap Y_2 \mid X) = P(Y_1|X)\, P(Y_2|X) \rightarrow 0.99 * 0.99 = 0.99 * 0.99$
   $P(Y_1 \cap Y_2) \neq P(Y_1)\, P(Y_1)$

   $P(Y_1 \cap Y_2) = P(X)P(Y_1 \cap Y_2 \mid X) + P(X^c)P(Y_1 \cap Y_2 \mid X^c) = (1/2)0.99*0.99 + (1/2)0.01*0.01 = 0.4901$
   $P(Y_1) = P(X)P(Y_1|X) + P(X^c)P(Y_1|X^c) = (1/2)0.99 + (1/2)0.01 = 0.5$
   $P(Y_2) = P(X)P(Y_2|X) + P(X^c)P(Y_2|X^c) = (1/2)0.99 + (1/2)0.01 = 0.5$
   $0.4901 \neq 0.25$

5. Find Probability Mass Function (PMF) and mean (3 points)
   (A) Two independent coin tosses with **2/3** probability of a head. Let X be the number of heads obtained. This is a binomial random variable with $n = 2$ and $p = 2/3$. Find the **PMF** and **mean**.

$$P_X(x) = \begin{cases} (\frac{1}{3})^2, if\ x = 0 \\ 2 * \left(\frac{1}{3}\right) * \left(\frac{2}{3}\right), if\ x = 1 \\ (\frac{2}{3})^2, if\ x = 2 \\ 0, otherwise \end{cases}$$

E[X] = 0*(1/3)2 + 1*(2*2/9) + 2*(4/9) = 4/3

(B) Let X and Y be the integers in $0 \le x \le 2, 0 \le y - x \le 1$ respectively, changing over a certain time period. Assume that the joint PMF of X and Y is uniform over the set of integers. Find the marginal **PMF** of Y and **mean** of Y.

(X, Y) = { (0, 0), (0, 1), (1, 1), (1, 2), (2, 2), (2, 3) }

$$P_Y(y) = \begin{cases} \frac{1}{6}, if\ y = 0, 3 \\ \frac{2}{6}, if\ y = 1, 2 \\ 0, otherwise \end{cases}$$

E[Y] =(1/6)(0+3) + (2/6)(1+2) = 3/2

6. **1/5** is a probability that you each time shot a basketball over and over, correctly independent of previous attempts. Find a **mean** of X, which is a number of tries until you shot correctly into the basket. This is a geometric distribution: $p_X(x) = (1-p)^{k-1}p$, k = 1, 2, …. . Use the total expectation theorem and show your work (1.5 points).

A_1={X=1}={first try is a success}, A_2={X > 1} = {first try is a failure}
If the first try is success → E[X | X = 1] = 1
If the first try fails (X>1), you wasted one, and expected # of remaining tries is E[X] → E[X | X > 1] = 1 + E[X]
Thus, E[X] = P(X = 1)E[X | X = 1] + P(X > 1)E[X | X > 1] = p + (1-p)(1 + E[X]) →
E[X] = 1/p = 1/(1/5) = 5

7. Let the random variables X and Y have a joint PDF which is uniform over the triangle with vertices at (0, 0), (1, 0), and (1, 1). Find **E[X]** in terms of E[Y] (2 points).

$f_{X,Y}(x, y) = 2$

$f_Y(y) = \int_{-\infty}^{\infty} f_{X,Y}(x, y)dx = \int_y^1 2dx = 2(1\text{-}y),\ 0 \leq y \leq 1,\ y \leq x \leq 1$

$f_{X|Y}(x|y) = f_{X,Y}(x,y) / f_Y(y) = 2/[2(1\text{-}y)] = 1/(1\text{-}y),\ 0 \leq y < 1,$

$E[X|Y = y] = \int_y^1 x[\frac{1}{1-y}]dx = (1+y)/2,\ 0 \leq y \leq 1$

$E[X] = \int_0^1 \frac{(1+y)}{2} f_Y(y)dy = (1/2) + (1/2) \int_0^1 y f_Y(y)dy = ½ + E[Y]/2$

8. Time T until a new light bulb burns out is an exponential RV with parameter λ. Ariadne turns the light on, leaves the room, and when she returns, t time units later and finds that the light bulb is still on, which corresponds to Event A = {T > t}. X is additional time until the light bulb burns out. Show that **Memoryless property** of exponential random variable, which means that regardless of the t that elapsed between lighting and arrival, the conditional CDF of X is exponential with λ. The PDF of exponential random variable, $f_X(x) = \begin{cases} \lambda e^{-\lambda x}, if\ x \geq 0, \\ 0, otherwise \end{cases}$
(1.5 points)

$P(X > x \mid A) = P(T > t+x \mid T > t) = \frac{P(T > t+x \text{ and } T > t)}{P(T>t)} = \frac{P(T > t+x)}{P(T>t)} = \frac{e^{-\lambda(t+x)}}{e^{-\lambda t}} = e^{-\lambda x}$