# AIM 5002 Computational Statistics and Probability

## (Spring 2021)

### Assignment 3

Name:______________________________ Score:_________/5

Submit your assignment at CANVAS by uploading your file.
**Due date: Tuesday, 2nd of the March, 2021 by 11:59 PM**

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1. In the bank, when you deposit your money, you equally likely find 0 or 1 customer ahead of you. The service time of the customer ahead, if present, is exponentially distributed with parameter $\lambda$. Find the CDF of your waiting time (0.5 point).

   X = {waiting time}, Y = {# of customers found}.
   $F_x(x) = 0$ for $x < 0$. For $x \geq 0$,
   $F_x(x) = P(X \leq x) = (1/2)P(X \leq x \mid Y = 0) + (1/2)P(X \leq x \mid Y = 1)$
   $P(X \leq x \mid Y = 0) = 1$,
   $P(X \leq x \mid Y = 1) = 1 - e^{-\lambda x}$,
   Thus, $F_x(x) = (1/2)(2 - e^{-\lambda x})$, if $x \geq 0$, 0, otherwise

2. Let X and Y be normal random variables with means 0 and 1, respectively, and variances 1 and 9, respectively (0.5 point)
   (a) Find $P(X \leq 1.5)$ and $P(X \leq -1)$
      $\Phi(1.5) = 0.9332$, $1 - \Phi(1) = 1 - 0.8413 = 0.1587$
   (b) Find $P(-5 \leq Y \leq 1)$
      $P((-5-1)/3 \leq (Y-1)/3 \leq (1-1)/3) = P(-2 \leq (Y-1)/3 \leq 0) = \Phi(2) - \Phi(0) =$
      $0.9772 - 0.5 = 0.4772$

3. A city's temperature is modeled as a normal random variable with mean and standard deviation equal to 3 degree Celsius and 7 degree Celsius respectively. What is the probability that the temperature at a randomly chosen time will be less than or equal to 50 degrees Fahrenheit (Fahrenheit = 1.8 * Celsius + 32) (0.4 points)?
   $E[C] = 3$, std $(C) = 7$,
   $P(F \leq 50) = P(C \leq 10) = P(Z \leq (10-3)/7) = P(Z \leq 1) = \Phi(1) = 0.8413$

4. A point is chosen at random (according to a uniform PDF) within a quarter circle of the form $\{(x, y) \mid x^2 + y^2 \leq r^2, x \geq 0, y \geq 0\}$, for some given $r > 0$ (0.6 points).
   (a) Find the joint PDF of the coordinates X and Y of the chosen point
      $f_{X,Y}(x, y) = 4/\pi r^2$ , $x^2 + y^2 \leq r^2$, $x \geq 0$, $y \geq 0$, 0, otherwise.

(b) Find the marginal PDF of Y

$f_Y(y) = \int_0^{\sqrt{r^2-y^2}} f_{X,Y}(x, y)dx = \int_0^{\sqrt{r^2-y^2}} (4/\pi r2)dx$

$= 4\sqrt{r^2 - y^2}/\pi r^2$ for $0 \le y \le r$, 0, otherwise

5. Let X be a random variable with PDF (1.5 points)

$$f_X(x) = \begin{cases} x/4, & if\ 1\ < x \le 3 \\ 0, & otherwise \end{cases}$$

And let A be the event {X ≥ 2}.

(a) Find E[X]

$E[X] = \int_1^3 \frac{x^2}{4} dx = 13/6$

(b) Find P(A)

$P(A) = \int_2^3 \frac{x}{4} dx = 5/8$

(c) Find $f_{X|A}(x)$

$f_{X|A}(x) = \begin{cases} \frac{f_X(x)}{P(A)}, if\ x \in A \\ 0, otherwise \end{cases} = 2x/5$, if $2 \le x \le 3$, 0, otherwise

(d) Find E[X|A]

$E[X|A] = \int_2^3 x \frac{2x}{5} dx = 38/15$

(e) Let Y = $X^2$, Find E[Y] and var(Y)

$E[Y] = E[X^2] = \int_1^3 \frac{x^3}{4} dx = 5$

$E[Y^2] = E[X^4] = \int_1^3 \frac{x^5}{4} dx = 91/3$

$var(Y) = E[Y^2] – (E[Y])^2 = 91/3 – 5^2 = 16/3$

6. Let the random variables X and Y have a joint PDF which is uniform over the triangle with vertices at (0, 0), (0, 2), and (2, 0) (1.5 points).

(a) Find the joint PDF of X and Y

$f_{X,Y}(x, y) = 1/2$

(b) Find the marginal PDF of Y

$f_Y(y) = \int_{-\infty}^{\infty} f_{X,Y}(x, y)dx = f_Y(y) = \int_0^{2-y} (\frac{1}{2})dx = 1 - \frac{y}{2}, 0 \le y \le 2$

(c) Find the conditional PDF of X given Y

$f_{X|Y}(x|y) = f_{X,Y}(x,y) / f_Y(y) = (1/2) / (1- \frac{y}{2}) = 1/(2-y), 0 \le y < 2, 0 \le x \le 2-y$

(d) Find E[X|Y = y], and use the total expectation theorem to find E[X] in terms of E[Y]

$E[X|Y = y] = \int_{-\infty}^{\infty} x f_{X|Y}(x \mid y)dx = [1/(2-y)] \int_0^{2-y} x dx = [1/(2-y)](2-y)^2/2$

$= (2-y)/2, 0 \le y \le 2$

$E[X] = \int_0^2 \frac{2-y}{2} f_Y(y)dy = 1-(1/2) \int_0^2 y f_Y(y)dy = (2-E[Y])/2$

(e) Use the symmetry of the problem to find the value of E[X]

E[X] = E[Y]

E[X] = E[Y] = 2/3