1. Let x be the RV in uniform $(-1,1)$

Let $Y = e^x$

$\because f_X(x) = \frac{1}{2}$ $\therefore F_X(x) = \begin{cases} -\frac{1}{2}, & x < -1 \\ \frac{1}{2}x, & -1 \le x \le 1 \\ \frac{1}{2}, & x > 1 \end{cases}$

$\because e^x$ increase with x increase

$\therefore Y = [\frac{1}{e}, e]$

$F_Y(y) = P(Y \le y)$
$= P(e^x \le y)$
$= P(x \le \ln y)$
$= F_X(\ln y) = \frac{1}{2}\ln y, \quad \frac{1}{e} \le y \le e$

since $0 < \ln y < 1$

$\therefore F_Y(y) = \begin{cases} P(Y \le y) = -\frac{1}{2}, & y < \frac{1}{e} \\ \frac{1}{2}\ln y, & \frac{1}{e} \le y \le e \\ P(Y \le y) = \frac{1}{2}, & y > e \end{cases}$

$f_Y(y) = F_Y'(y) = \begin{cases} \frac{1}{2y}, & \frac{1}{e} \le y \le e \\ 0, & \text{othermise} \end{cases}$

2. $\sigma(x, Y) = \frac{cov(X, Y)}{\sqrt{var(X)\,var(Y)}}$

$cov(X, Y) = E[XY] - E[X]E[Y]$

$E[XY] = E[X \cdot (a + bx + cx^2)] = E[ax + bx^2 + cx^3]$
$= aE[x] + bE[x^2] + cE[x^3] = b + c$

$E[x]E[Y] = E[x]E[a + bx + cx^2] = E[x] \cdot (a + bE[x] + cE[x^2])$
$= 0$

$\therefore cov(X, Y) = b + c$

$var(X) = E[X^2] - E[x]^2 = E[x^2] = 1$

$var(Y) = E[(a + bx + cx^2)^2] - (E(a + bx + cx^2))^2$
$= (a^2 + 2abE(x) + 2acE[x^2] + 2bcE[x^3] + b^2E[x^2] + c^2E[x^4])$
$- (a + bE[x] + cE[x^2])^2$
$= a^2 + 2ac + 2bc + b^2 - (a^2 + 2ac + c^2) = 2bc + b^2 - c^2$

$\therefore \sigma(x, Y) = \frac{b + c}{\sqrt{2bc + b^2 - c^2}}$

3. (a) Let x be the expected amount of time that professor devotes to the task

$\because Y \sim u(0,4)$

$\therefore E[Y] = \frac{4}{2} = 2$

$E[X] = E[E[X|Y]] = E[5-Y] = 5 - E[Y] = 5-2=3$

(b) $9+2+3=14$

The expected time at which the task is completed is 14pm

4.

$$P_X(x) = \begin{cases} \frac{1}{3}, & \text{if } x=0 \\ \frac{1}{5}, & \text{if } x=1,3 \\ \frac{4}{15}, & \text{if } x=5 \end{cases}$$

$$M(s) = E[e^{sX}] = \frac{1}{3}e^{0s} + \frac{1}{5}e^{1s} + \frac{1}{5}e^{3s} + \frac{4}{15}e^{5s}$$

Mean: $E[X] = \frac{d}{ds}M(s)\Big|_{s=0} = \frac{1}{3}\cdot 0e^{0s} + \frac{1}{5}\cdot 1e^{1s} + \frac{1}{5}\cdot 3e^{3s} + \frac{4}{15}\cdot 5e^{5s}$

$= 0 + \frac{1}{5} + \frac{3}{5} + \frac{20}{15}$

$= \frac{3}{15} + \frac{9}{15} + \frac{20}{15} = \frac{32}{15}$

$E[X^2] = \frac{d^2}{ds^2}M(s)\Big|_{s=0} = \frac{1}{3}\cdot 0^2 e^{0s} + \frac{1}{5}\cdot 1^2 e^{1s} + \frac{1}{5}\cdot 3^2 e^{3s} + \frac{4}{15}\cdot 5^2 e^{5s}$

$= 0 + \frac{1}{5} + \frac{9}{5} + \frac{100}{15} = \frac{3}{15} + \frac{27}{15} + \frac{100}{15} = \frac{130}{15}$

$\text{Var} = E[X^2] - (E[X])^2 = \frac{130}{15} - \left(\frac{32}{15}\right)^2 = \frac{1950}{225} - \frac{1024}{225} = \frac{926}{225} \approx 4.1$

5. Let x be number of different types of pizzas

$E[X] = E[E[X|K]]$

$X_i = \begin{cases} 1, & \text{if type of pizza } i \text{ is ordered by at least one customer} \\ 0, & \text{otherwise} \end{cases}$

$X = X_1 + X_2 + \cdots + X_n$

$E[E[X|K]] = E[E[X_1 + X_2 + \cdots + X_n | K]] = n \cdot E[E[X_1|K]]$

$P(\text{customer does not order type } i) = \frac{n-1}{n}$

$\therefore E[X_1|K] = 1 - \left(\frac{n-1}{n}\right)^K$

$E[X] = n \cdot E[1-p^K]$

$= n - nE[p^K]$

$= n - nE[e^{K\log p}]$

$= n - n\cdot M_K(\log p)$

$= n - n\cdot M_K\left(\log \frac{n-1}{n}\right)$