AIM 5002 Computational Statistics and Probability

(Spring 2021)

Assignment 4

Name:______________________ Score:________/5

Submit your assignment at CANVAS by uploading your file.
Due date: Tuesday, 2nd of the March, 2021 by 11:59 PM

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1. If X is a random variable that is uniformly distributed between -1 and 1, find the PDF of e^X.

   $f_X(x)$ = ½ if $-1 \leq x \leq 1$, 0 otherwise
   Y = e^X.
   $F_Y(y) = P(Y \leq y) = P(e^X \leq y) = P(X \leq \ln y)$ = ½ + ½$\ln y$, if y > 0, 0 otherwise
   $f_Y(y)$ =1/(2y) for $\frac{1}{e} \leq y \leq e$, 0 otherwise

2. Suppose that a random variable X satisfies

   $$E[X] = 0, E[X^2] = 1, E[X^3] = 1, E[X^4] = 0$$

   and let

   $$Y = a + bX + cX^2$$

   Find the correlation coefficient $\sigma(X, Y)$.

   cov(X, Y) = E[XY] – E[X]E[Y]
   $= E[aX + bX^2 + cX^3] = aE[X] + bE[X^2] + cE[X^3] = b + c$
   $var(Y) = var(a + bX + cX^2) = E[(a + bX + cX^2)^2] - (E[a + bX + cX^2])^2$
   $= b^2 + 2bc - c^2$
   var(X) = 1

   $$\sigma(X, Y) = \frac{b + c}{\sqrt{\text{b2 + 2bc} - \text{c2}}}$$

3. A retired professor comes to the office at a time which is uniformly distributed between 9 a.m. and 1 p.m., performs a single task, and leaves when the task is completed. The duration of the task is exponentially distributed with parameter λ(y) = 1/(5-y), where y is the length of the time interval between 9 a.m. and the time of his arrival.
   (a) What is the expected amount of time that the professor devotes to the task?
   X = amount of time that professor spent for the task
   Y = length of time 9 am and the time of professor's arrival

$E[Y] = 2$ $(9 \sim 13) \rightarrow E[X|Y = y] = 1/\lambda(y) = 5 - y \rightarrow E[X|Y] = 5 - Y$
$E[X] = E[E[X|Y]] = E[5 - Y] = 5 - E[Y] = 5 - 2 = 3$

(b) What is the expected time at which the task is completed?
Z = length of time from 9 am to time when professor completes the task
$Z = X + Y \rightarrow E[Z] = E[X] + E[Y] = 3 + 2 = 5$

4. Let X be a random variable that takes the values 0, 1, 3, 5:

   $P(X=0) = 1/3$, $P(X=1) = 1/5$, $P(X= 3) = 1/5$, $P(X=5) = 4/15$.

   Find the transform associated with X and use it to obtain the mean and variance.

   $M(s) = E[e^{sX}] = 1/3 + (1/5)e^{s} + (1/5)e^{3s} + (4/15)e^{5s}$
   $E[X] = 1/5 + 3/5 + 4/3 = 32/15 = 2.13333333$
   $E[X^2] = 1/5 + 9/5 +20/3 = 26/3$
   $var(X) = 26/3 - (32/15)^2 = 926/225 =4.1155555556$

5. A pizza parlor serves n different types of pizza, and is visited by a number of K of customers in a given period of time, where K is a nonnegative integer random variable with a known associated transform $M_K(s) = E[e^{sK}]$. Each customer orders a single pizza with all types of pizza being equally likely, independent of the number of other customers and the types of pizza they order. Give a formula, in terms of $M_K(*)$ for the expected number of different types of pizzas ordered.

   X_i = 1, if a type i pizza is ordered by at least one customer, 0, otherwise
   $Y = X_1 + *** + X_n \rightarrow E[Y] = E[E[Y|K]] = E[E[X_1 + *** + X_n|K]] = nE[E[X_1|K]]$
   $E[X_1|K=k] = 1 - (1\text{-}1/n)^k \Rightarrow E[X_1|K] = 1 - (1\text{-}1/n)^K$ where 1-1/n is a probability that a customer does not order a pizza of type 1 (p).
   $E[Y] = nE[E[X_1|K]] = nE[1 - (1\text{-}1/n)^K] = nE[1] - nE[(1\text{-}1/n)^K] = n - nE[p^K]$
   $= n - nE[e^{K\log P}] = n - nM_K(\log p)$