

0. Readme

The below derivations are based on the first term ($2\pi G\rho^* \{ \text{part1} \} |_{r'=r_1}^{r'=r_2}$) and second term ($2\pi G\rho^* \{ \text{part2} \} |_{r'=r_1}^{r'=r_2} {}_{r < r_1}^{+1}$) in Eq . (3) (c.f. Eq. (52) of Heck and Seitz 2007), where r_3 represents r' .

1. $V_z = V_z \text{part1} + V_z \text{part2}$ in Eq. (5)

$$\text{In[1]:= } V_z \text{part1} = \text{FullSimplify}\left[D\left[\left(\frac{1}{3 * r} * \left(\sqrt{r^2 + r3^2 - 2 * r * r3 * \cos[\psi]} * \cos[\psi] * (r3 - r * \cos[\psi]) + \frac{1}{2} * \sqrt{r^2 + r3^2 - 2 * r * r3 * \cos[\psi]} * \cos[\psi] * (\sin[\psi])^2 * \log\left[\sqrt{r^2 + r3^2 - 2 * r * r3 * \cos[\psi]} + r3 - r * \cos[\psi]\right]\right)^3 + \frac{1}{2} * r^2 * \cos[\psi] * (\sin[\psi])^2 * \log\left[\sqrt{r^2 + r3^2 - 2 * r * r3 * \cos[\psi]} + r3 - r * \cos[\psi]\right]\right], \{ \sqrt{r^2 + r3^2 - 2 * r * r3 * \cos[\psi]} \rightarrow l \}]$$

$$\text{Out[1]= } \frac{1}{6 r^2} (l (r^2 - 2 r3^2 - 3 r^2 \cos[2 \psi]) + r \cos[\psi] (-2 l r3 + 3 r^2 (1 + 2 \log[l + r3 - r \cos[\psi]]) \sin[\psi]^2))$$

$$\text{In[2]:= } V_z \text{part2} = \text{FullSimplify}\left[D\left[\left(\frac{1}{3 * r} * r3^3 - \frac{1}{2} * r3^2\right), r\right]\right]$$

$$\text{Out[2]= } -\frac{r3^3}{3 r^2}$$

2. $V_{zz} = V_{zz} \text{part1} + V_{zz} \text{part2}$ in Eq. (7)

$$\text{In[3]:= } V_{zz} \text{part1} = \text{FullSimplify}\left[D\left[V_z \text{part1} /., \{ l \rightarrow \sqrt{r^2 + r3^2 - 2 * r * r3 * \cos[\psi]}, r \}\right] /., \{ \sqrt{r^2 + r3^2 - 2 * r * r3 * \cos[\psi]} \rightarrow l \}\right]$$

$$\text{Out[3]= } \frac{(r^4 + 2 r^2 r3^2 + 4 r3^4 - r (2 r r3^2 \cos[\psi]^2 + 3 r^3 \cos[2 \psi] + \cos[\psi] (r3 (r^2 + 4 r3^2) - 3 r^2 (r3 \cos[2 \psi] + (3 l - 2 r3 + 2 l \log[l + r3 - r \cos[\psi]])) \sin[\psi]^2)))}{(6 r^3 \sqrt{r^2 + r3^2 - 2 * r * r3 * \cos[\psi]})}$$

$$\text{In[4]:= } V_{zz} \text{part2} = \text{FullSimplify}\left[D\left[\left(\frac{1}{3 * r} * r3^3 - \frac{1}{2} * r3^2\right), \{ r, 2 \}\right]\right]$$

$$\text{Out[4]= } \frac{2 r3^3}{3 r^3}$$

3. $V_{zzz} = V_{zzzpart1} + V_{zzzpart2}$ in Eq. (9)

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In[5]:= Vzzzpart1 =
FullSimplify[FullSimplify[D[Vzzzpart1 /. {l -> Sqrt[r^2 + r3^2 - 2 r r3 Cos[\psi]], r} /. {Sqrt[r^2 + r3^2 - 2 r r3 Cos[\psi]] -> l}], r]
Out[5]= - 1
          8 r^4 (l + r3 - r Cos[\psi]) (r^2 + r3^2 - 2 r r3 Cos[\psi])^{3/2}
          (4 r3 (r^6 + 3 r^4 r3^2 + 15 r^2 r3^4 + 4 r3^6) + l (r^6 + 3 r^4 r3^2 + 36 r^2 r3^4 + 16 r3^6) -
           2 r (r^6 + 2 r^4 r3 (l + 2 r3) + 8 r3^5 (3 l + 4 r3) + 3 r^2 r3^3 (3 l + 10 r3)) Cos[\psi] +
           r^2 (4 r3^3 (2 r^2 + 3 r3 (l + 3 r3)) Cos[2 \psi] + r (2 r^4 + 2 (l - 2 r3) r3^3 + 3 r^2 r3 (l + 2 r3))
           Cos[3 \psi] - r^2 (4 r3 (r^2 + r3^2) + l (r^2 + 3 r3^2)) Cos[4 \psi] + r^3 r3 (l + 2 r3) Cos[5 \psi])]

In[6]:= Vzzzpart2 = FullSimplify[D[(1/(3*r)*r3^3 - 1/2*r3^2), {r, 3}]]
Out[6]= - 2 r3^3
          r^4
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4. Analytical consistencies of the formulae between the new Vz, Vzz in this paper and old Vz, Vzz in Heck and Seitz (2007) and Lin et al. (2020) of a spherical cap

In[7]:= (*The formula of the old Vz is from Eq. (54) of Heck and Seitz (2007)*)

$$\begin{aligned} \text{OldVz} = & \frac{2 G \pi r1^3 \rho}{3 r^2} - \frac{2 G \pi r2^3 \rho}{3 r^2} - \\ & 2 G \pi \rho \left(\frac{\cos[\psi] (r1 - r \cos[\psi]) (r - r1 \cos[\psi])}{2 \sqrt{r^2 + r1^2 - 2 r r1 \cos[\psi]}} - \frac{1}{2} \cos[\psi]^2 \sqrt{r^2 + r1^2 - 2 r r1 \cos[\psi]} + \right. \\ & \frac{(r - r1 \cos[\psi]) \sqrt{r^2 + r1^2 - 2 r r1 \cos[\psi]}}{r} - \frac{(r^2 + r1^2 - 2 r r1 \cos[\psi])^{3/2}}{3 r^2} + \\ & \frac{r^2 \cos[\psi] (\left(r - \cos[\psi] (r1 + \sqrt{r^2 + r1^2 - 2 r r1 \cos[\psi]})\right) \sin[\psi]^2}{2 \sqrt{r^2 + r1^2 - 2 r r1 \cos[\psi]} (r1 - r \cos[\psi] + \sqrt{r^2 + r1^2 - 2 r r1 \cos[\psi]})} + \\ & \left. r \cos[\psi] \log[r1 - r \cos[\psi] + \sqrt{r^2 + r1^2 - 2 r r1 \cos[\psi]}] \sin[\psi]^2 \right) + \\ & 2 G \pi \rho \left(\frac{\cos[\psi] (r2 - r \cos[\psi]) (r - r2 \cos[\psi])}{2 \sqrt{r^2 + r2^2 - 2 r r2 \cos[\psi]}} - \frac{1}{2} \cos[\psi]^2 \sqrt{r^2 + r2^2 - 2 r r2 \cos[\psi]} + \right. \\ & \frac{(r - r2 \cos[\psi]) \sqrt{r^2 + r2^2 - 2 r r2 \cos[\psi]}}{r} - \frac{(r^2 + r2^2 - 2 r r2 \cos[\psi])^{3/2}}{3 r^2} + \\ & \frac{r^2 \cos[\psi] (\left(r - \cos[\psi] (r2 + \sqrt{r^2 + r2^2 - 2 r r2 \cos[\psi]})\right) \sin[\psi]^2}{2 \sqrt{r^2 + r2^2 - 2 r r2 \cos[\psi]} (r2 - r \cos[\psi] + \sqrt{r^2 + r2^2 - 2 r r2 \cos[\psi]})} + \\ & \left. r \cos[\psi] \log[r2 - r \cos[\psi] + \sqrt{r^2 + r2^2 - 2 r r2 \cos[\psi]}] \sin[\psi]^2 \right); \end{aligned}$$

In[8]:= (*The formula of the old Vzz is from Eqs. (19) and (20) of Lin et al. (2020)*)

$$\begin{aligned} \text{OldVzz} = & - \frac{4 G \pi r1^3 \rho}{3 r^3} + \frac{4 G \pi r2^3 \rho}{3 r^3} - 2 G \pi \rho \\ & \left(\frac{r1 \cos[\psi] \sqrt{r^2 + r1^2 - 2 r r1 \cos[\psi]}}{r^2} - \frac{(3 r - 3 r1 \cos[\psi]) \sqrt{r^2 + r1^2 - 2 r r1 \cos[\psi]}}{3 r^2} + \right. \\ & \frac{2 (r^2 + r1^2 - 2 r r1 \cos[\psi])^{3/2}}{3 r^3} + \frac{(r - r1 \cos[\psi]) \left(1 - \frac{r1 \cos[\psi]}{r} - \cos[\psi]^2\right)}{\sqrt{r^2 + r1^2 - 2 r r1 \cos[\psi]}} + \frac{1}{2} \cos[\psi] \left. \right) \end{aligned}$$

$$\begin{aligned}
& (r1 - r \cos[\psi]) \left(\frac{(r - r1 \cos[\psi]) (-r + r1 \cos[\psi])}{(r^2 + r1^2 - 2 r r1 \cos[\psi])^{3/2}} + \frac{1}{\sqrt{r^2 + r1^2 - 2 r r1 \cos[\psi]}} \right) + \\
& \frac{1}{2} r^2 \cos[\psi] \left(\frac{\left(\cos[\psi] - \frac{r - r1 \cos[\psi]}{\sqrt{r^2 + r1^2 - 2 r r1 \cos[\psi]}} \right) \left(-\cos[\psi] + \frac{r - r1 \cos[\psi]}{\sqrt{r^2 + r1^2 - 2 r r1 \cos[\psi]}} \right)}{(r1 - r \cos[\psi] + \sqrt{r^2 + r1^2 - 2 r r1 \cos[\psi]})^2} + \right. \\
& \left. \frac{\frac{(r - r1 \cos[\psi]) (-r + r1 \cos[\psi])}{(r^2 + r1^2 - 2 r r1 \cos[\psi])^{3/2}} + \frac{1}{\sqrt{r^2 + r1^2 - 2 r r1 \cos[\psi]}}}{r1 - r \cos[\psi] + \sqrt{r^2 + r1^2 - 2 r r1 \cos[\psi]}} \right) \sin[\psi]^2 + \\
& \cos[\psi] \left(\frac{2 r \left(r - \cos[\psi] \left(r1 + \sqrt{r^2 + r1^2 - 2 r r1 \cos[\psi]} \right) \right)}{\sqrt{r^2 + r1^2 - 2 r r1 \cos[\psi]} \left(r1 - r \cos[\psi] + \sqrt{r^2 + r1^2 - 2 r r1 \cos[\psi]} \right)} + \right. \\
& \left. \log[r1 - r \cos[\psi] + \sqrt{r^2 + r1^2 - 2 r r1 \cos[\psi]}] \right) \sin[\psi]^2 + 2 G \pi \rho \\
& \left(\frac{r2 \cos[\psi] \sqrt{r^2 + r2^2 - 2 r r2 \cos[\psi]}}{r^2} - \frac{(3 r - 3 r2 \cos[\psi]) \sqrt{r^2 + r2^2 - 2 r r2 \cos[\psi]}}{3 r^2} + \right. \\
& \left. \frac{2 (r^2 + r2^2 - 2 r r2 \cos[\psi])^{3/2}}{3 r^3} + \frac{(r - r2 \cos[\psi]) \left(1 - \frac{r2 \cos[\psi]}{r} - \cos[\psi]^2 \right)}{\sqrt{r^2 + r2^2 - 2 r r2 \cos[\psi]}} + \frac{1}{2} \cos[\psi] \right. \\
& \left. (r2 - r \cos[\psi]) \left(\frac{(r - r2 \cos[\psi]) (-r + r2 \cos[\psi])}{(r^2 + r2^2 - 2 r r2 \cos[\psi])^{3/2}} + \frac{1}{\sqrt{r^2 + r2^2 - 2 r r2 \cos[\psi]}} \right) + \right. \\
& \left. \frac{1}{2} r^2 \cos[\psi] \left(\frac{\left(\cos[\psi] - \frac{r - r2 \cos[\psi]}{\sqrt{r^2 + r2^2 - 2 r r2 \cos[\psi]}} \right) \left(-\cos[\psi] + \frac{r - r2 \cos[\psi]}{\sqrt{r^2 + r2^2 - 2 r r2 \cos[\psi]}} \right)}{(r2 - r \cos[\psi] + \sqrt{r^2 + r2^2 - 2 r r2 \cos[\psi]})^2} + \right. \right. \\
& \left. \left. \frac{\frac{(r - r2 \cos[\psi]) (-r + r2 \cos[\psi])}{(r^2 + r2^2 - 2 r r2 \cos[\psi])^{3/2}} + \frac{1}{\sqrt{r^2 + r2^2 - 2 r r2 \cos[\psi]}}}{r2 - r \cos[\psi] + \sqrt{r^2 + r2^2 - 2 r r2 \cos[\psi]}} \right) \sin[\psi]^2 + \right. \\
& \left. \cos[\psi] \left(\frac{2 r \left(r - \cos[\psi] \left(r2 + \sqrt{r^2 + r2^2 - 2 r r2 \cos[\psi]} \right) \right)}{\sqrt{r^2 + r2^2 - 2 r r2 \cos[\psi]} \left(r2 - r \cos[\psi] + \sqrt{r^2 + r2^2 - 2 r r2 \cos[\psi]} \right)} + \right. \right. \\
& \left. \left. \log[r2 - r \cos[\psi] + \sqrt{r^2 + r2^2 - 2 r r2 \cos[\psi]}] \right) \sin[\psi]^2 \right);
\end{aligned}$$

In[9]:= (*The formula of the new Vz is from Eq. (5) of this paper*)

$$\begin{aligned} \text{NewVz} = & \frac{2 G \pi r1^3 \rho}{3 r^2} - \frac{2 G \pi r2^3 \rho}{3 r^2} - \\ & \frac{1}{3 r^2} G \pi \rho \left(\sqrt{r^2 + r1^2 - 2 r r1 \cos[\psi]} (r^2 - 2 r1^2 - 3 r^2 \cos[2\psi]) + \right. \\ & r \cos[\psi] \left(-2 r1 \sqrt{r^2 + r1^2 - 2 r r1 \cos[\psi]} + \right. \\ & \left. \left. 3 r^2 \left(1 + 2 \log[r1 - r \cos[\psi] + \sqrt{r^2 + r1^2 - 2 r r1 \cos[\psi]}] \right) \sin[\psi]^2 \right) \right) + \\ & \frac{1}{3 r^2} G \pi \rho \left(\sqrt{r^2 + r2^2 - 2 r r2 \cos[\psi]} (r^2 - 2 r2^2 - 3 r^2 \cos[2\psi]) + \right. \\ & r \cos[\psi] \left(-2 r2 \sqrt{r^2 + r2^2 - 2 r r2 \cos[\psi]} + \right. \\ & \left. \left. 3 r^2 \left(1 + 2 \log[r2 - r \cos[\psi] + \sqrt{r^2 + r2^2 - 2 r r2 \cos[\psi]}] \right) \sin[\psi]^2 \right) \right); \end{aligned}$$

In[10]:= (*The formula of the new Vz is from Eq. (7) of this paper*)

$$\begin{aligned} \text{NewVzz} = & - \frac{4 G \pi r1^3 \rho}{3 r^3} + \frac{4 G \pi r2^3 \rho}{3 r^3} - \frac{1}{3 r^3 \sqrt{r^2 + r1^2 - 2 r r1 \cos[\psi]}} G \pi \rho \left(r^4 + 2 r^2 r1^2 + 4 r1^4 - \right. \\ & 2 r^2 r1^2 \cos[\psi]^2 - 3 r^4 \cos[2\psi] - r \cos[\psi] \left(r^2 r1 + 4 r1^3 - 3 r^2 r1 \cos[2\psi] - \right. \\ & \left. \left. 3 r^2 \left(-2 r1 + 3 \sqrt{r^2 + r1^2 - 2 r r1 \cos[\psi]} + 2 \sqrt{r^2 + r1^2 - 2 r r1 \cos[\psi]} \right. \right. \right. \\ & \left. \left. \left. \log[r1 - r \cos[\psi] + \sqrt{r^2 + r1^2 - 2 r r1 \cos[\psi]}] \right) \sin[\psi]^2 \right) + \\ & \frac{1}{3 r^3 \sqrt{r^2 + r2^2 - 2 r r2 \cos[\psi]}} G \pi \rho \left(r^4 + 2 r^2 r2^2 + 4 r2^4 - 2 r^2 r2^2 \cos[\psi]^2 - \right. \\ & 3 r^4 \cos[2\psi] - r \cos[\psi] \left(r^2 r2 + 4 r2^3 - 3 r^2 r2 \cos[2\psi] - \right. \\ & \left. \left. 3 r^2 \left(-2 r2 + 3 \sqrt{r^2 + r2^2 - 2 r r2 \cos[\psi]} + 2 \sqrt{r^2 + r2^2 - 2 r r2 \cos[\psi]} \right. \right. \right. \\ & \left. \left. \left. \log[r2 - r \cos[\psi] + \sqrt{r^2 + r2^2 - 2 r r2 \cos[\psi]}] \right) \sin[\psi]^2 \right) \right); \end{aligned}$$

In[11]:= FullSimplify[NewVz - OldVz]

Out[11]= 0

In[12]:= FullSimplify[NewVzz - OldVzz]

Out[12]= 0

In[13]:= NotebookSave[EvaluationNotebook[]];