## 0. Readme

The below derivations are based on the first term ($2\pi G\rho^*\{\text{part1}\}|_{r'=r_1}^{r'=r_2}$) and second term ($2\pi G\rho^*(\text{part 2})|_{r'=r_1}^{r'=r_2}\begin{cases}+1 & r>r_2\\ -1 & r<r_1\end{cases}$ in Eq . (3) (c.f. Eq. (52) of Heck and Seitz 2007), where r3 represents r'.

## 1. Vz = Vzpart1 + Vzpart2 in Eq. (5)

In[1]:=
```
Vzpart1 = FullSimplify[D[(1/(3*r) * (Sqrt[r^2 + r3^2 - 2*r*r3*Cos[ψ]])^3 +
      1/2 * Sqrt[r^2 + r3^2 - 2*r*r3*Cos[ψ]] * Cos[ψ] * (r3 - r*Cos[ψ]) +
      1/2 * r^2 * Cos[ψ] * (Sin[ψ])^2 * Log[Sqrt[r^2 + r3^2 - 2*r*r3*Cos[ψ]] + r3 - r*Cos[ψ]]),
   r]] /. {Sqrt[r^2 + r3^2 - 2 r r3 Cos[ψ]] → l}
```

Out[1]=
$$\frac{1}{6\,r^2}\Big(l\,\left(r^2-2\,r3^2-3\,r^2\,\mathrm{Cos}[2\,\psi]\right)+ r\,\mathrm{Cos}[\psi]\,\left(-2\,l\,r3+3\,r^2\,\left(1+2\,\mathrm{Log}[l+r3-r\,\mathrm{Cos}[\psi]]\right)\,\mathrm{Sin}[\psi]^2\right)\Big)$$

In[2]:=
```
Vzpart2 = FullSimplify[D[(1/(3*r) * r3^3 - 1/2 * r3^2), r]]
```

Out[2]=
$$-\frac{r3^3}{3\,r^2}$$

## 2. Vzz= Vzzpart1 + Vzzpart2 in Eq. (7)

In[3]:=
```
Vzzpart1 = FullSimplify[D[Vzpart1 /. {l → Sqrt[r^2 + r3^2 - 2 r r3 Cos[ψ]]}, r]] /.
   {Sqrt[r^2 + r3^2 - 2 r r3 Cos[ψ]] → l}
```

Out[3]=
$$\Big(r^4+2\,r^2\,r3^2+4\,r3^4-r\,\Big(2\,r\,r3^2\,\mathrm{Cos}[\psi]^2+3\,r^3\,\mathrm{Cos}[2\,\psi]+\mathrm{Cos}[\psi]\,\Big(r3\,\left(r^2+4\,r3^2\right)-3\,r^2\,\left(r3\,\mathrm{Cos}[2\,\psi]+\left(3\,l-2\,r3+2\,l\,\mathrm{Log}[l+r3-r\,\mathrm{Cos}[\psi]]\right)\,\mathrm{Sin}[\psi]^2\right)\Big)\Big)\Big)\Big/\left(6\,r^3\,\sqrt{r^2+r3^2-2\,r\,r3\,\mathrm{Cos}[\psi]}\right)$$

In[4]:=
```
Vzzpart2 = FullSimplify[D[(1/(3*r) * r3^3 - 1/2 * r3^2), {r, 2}]]
```

Out[4]=
$$\frac{2\,r3^3}{3\,r^3}$$

## 3. Vzzz= Vzzzpart1 + Vzzzpart2 in Eq. (9)

In[5]:=
```
Vzzzpart1 =
 FullSimplify[FullSimplify[D[Vzzpart1 /. {l → Sqrt[r^2 + r3^2 - 2 r r3 Cos[ψ]]}, r]] /.
   {Sqrt[r^2 + r3^2 - 2 r r3 Cos[ψ]] → l}]
```

Out[5]=
$$-\frac{1}{8\,r^4\,(l+r3-r\,\mathrm{Cos}[\psi])\,\left(r^2+r3^2-2\,r\,r3\,\mathrm{Cos}[\psi]\right)^{3/2}}\Big(4\,r3\,\left(r^6+3\,r^4\,r3^2+15\,r^2\,r3^4+4\,r3^6\right)+l\,\left(r^6+3\,r^4\,r3^2+36\,r^2\,r3^4+16\,r3^6\right)-2\,r\,\left(r^6+2\,r^4\,r3\,(l+2\,r3)+8\,r3^5\,(3\,l+4\,r3)+3\,r^2\,r3^3\,(3\,l+10\,r3)\right)\mathrm{Cos}[\psi]+r^2\,\Big(4\,r3^3\,\left(2\,r^2+3\,r3\,(l+3\,r3)\right)\mathrm{Cos}[2\,\psi]+r\,\left(2\,r^4+2\,(l-2\,r3)\,r3^3+3\,r^2\,r3\,(l+2\,r3)\right)\mathrm{Cos}[3\,\psi]-r^2\,\left(4\,r3\,\left(r^2+r3^2\right)+l\,\left(r^2+3\,r3^2\right)\right)\mathrm{Cos}[4\,\psi]+r^3\,r3\,(l+2\,r3)\,\mathrm{Cos}[5\,\psi]\Big)\Big)$$

In[6]:=
```
Vzzzpart2 = FullSimplify[D[(1/(3*r) * r3^3 - 1/2 * r3^2), {r, 3}]]
```

Out[6]=
$$-\frac{2\,r3^3}{r^4}$$

## 4. Analytical consistencies of the formulae between the new Vz, Vzz in this paper and old Vz, Vzz in Heck and Seitz (2007) and Lin et al. (2020) of a spherical cap

In[7]:= (*The formula of the old Vz is from Eq. (54) of Heck and Seitz (2007)*)

$$\mathrm{OldVz} = \frac{2 G \pi r1^3 \rho}{3 r^2} - \frac{2 G \pi r2^3 \rho}{3 r^2} - 2 G \pi \rho \left( \frac{\mathrm{Cos}[\psi] (r1 - r \mathrm{Cos}[\psi]) (r - r1 \mathrm{Cos}[\psi])}{2 \sqrt{r^2 + r1^2 - 2 r\, r1 \mathrm{Cos}[\psi]}} - \frac{1}{2} \mathrm{Cos}[\psi]^2 \sqrt{r^2 + r1^2 - 2 r\, r1 \mathrm{Cos}[\psi]} + \frac{(r - r1 \mathrm{Cos}[\psi]) \sqrt{r^2 + r1^2 - 2 r\, r1 \mathrm{Cos}[\psi]}}{r} - \frac{\left(r^2 + r1^2 - 2 r\, r1 \mathrm{Cos}[\psi]\right)^{3/2}}{3 r^2} + \frac{r^2 \mathrm{Cos}[\psi] \left(r - \mathrm{Cos}[\psi] \left(r1 + \sqrt{r^2 + r1^2 - 2 r\, r1 \mathrm{Cos}[\psi]}\right)\right) \mathrm{Sin}[\psi]^2}{2 \sqrt{r^2 + r1^2 - 2 r\, r1 \mathrm{Cos}[\psi]} \left(r1 - r \mathrm{Cos}[\psi] + \sqrt{r^2 + r1^2 - 2 r\, r1 \mathrm{Cos}[\psi]}\right)} + r \mathrm{Cos}[\psi] \mathrm{Log}\left[r1 - r \mathrm{Cos}[\psi] + \sqrt{r^2 + r1^2 - 2 r\, r1 \mathrm{Cos}[\psi]}\right] \mathrm{Sin}[\psi]^2 \right) + 2 G \pi \rho \left( \frac{\mathrm{Cos}[\psi] (r2 - r \mathrm{Cos}[\psi]) (r - r2 \mathrm{Cos}[\psi])}{2 \sqrt{r^2 + r2^2 - 2 r\, r2 \mathrm{Cos}[\psi]}} - \frac{1}{2} \mathrm{Cos}[\psi]^2 \sqrt{r^2 + r2^2 - 2 r\, r2 \mathrm{Cos}[\psi]} + \frac{(r - r2 \mathrm{Cos}[\psi]) \sqrt{r^2 + r2^2 - 2 r\, r2 \mathrm{Cos}[\psi]}}{r} - \frac{\left(r^2 + r2^2 - 2 r\, r2 \mathrm{Cos}[\psi]\right)^{3/2}}{3 r^2} + \frac{r^2 \mathrm{Cos}[\psi] \left(r - \mathrm{Cos}[\psi] \left(r2 + \sqrt{r^2 + r2^2 - 2 r\, r2 \mathrm{Cos}[\psi]}\right)\right) \mathrm{Sin}[\psi]^2}{2 \sqrt{r^2 + r2^2 - 2 r\, r2 \mathrm{Cos}[\psi]} \left(r2 - r \mathrm{Cos}[\psi] + \sqrt{r^2 + r2^2 - 2 r\, r2 \mathrm{Cos}[\psi]}\right)} + r \mathrm{Cos}[\psi] \mathrm{Log}\left[r2 - r \mathrm{Cos}[\psi] + \sqrt{r^2 + r2^2 - 2 r\, r2 \mathrm{Cos}[\psi]}\right] \mathrm{Sin}[\psi]^2 \right);$$

In[8]:= (*The formula of the old Vzz is from Eqs. (19) and (20) of Lin et al. (2020)*)

$$\mathrm{OldVzz} = -\frac{4 G \pi r1^3 \rho}{3 r^3} + \frac{4 G \pi r2^3 \rho}{3 r^3} - 2 G \pi \rho \left( \frac{r1 \mathrm{Cos}[\psi] \sqrt{r^2 + r1^2 - 2 r\, r1 \mathrm{Cos}[\psi]}}{r^2} - \frac{(3 r - 3 r1 \mathrm{Cos}[\psi]) \sqrt{r^2 + r1^2 - 2 r\, r1 \mathrm{Cos}[\psi]}}{3 r^2} + \frac{2 \left(r^2 + r1^2 - 2 r\, r1 \mathrm{Cos}[\psi]\right)^{3/2}}{3 r^3} + \frac{(r - r1 \mathrm{Cos}[\psi]) \left(1 - \frac{r1 \mathrm{Cos}[\psi]}{r} - \mathrm{Cos}[\psi]^2\right)}{\sqrt{r^2 + r1^2 - 2 r\, r1 \mathrm{Cos}[\psi]}} + \frac{1}{2} \mathrm{Cos}[\psi]$$

$$(r1 - r\,\mathrm{Cos}[\psi])\left(\frac{(r - r1\,\mathrm{Cos}[\psi])\,(-r + r1\,\mathrm{Cos}[\psi])}{\left(r^2 + r1^2 - 2\,r\,r1\,\mathrm{Cos}[\psi]\right)^{3/2}} + \frac{1}{\sqrt{r^2 + r1^2 - 2\,r\,r1\,\mathrm{Cos}[\psi]}}\right) +$$

$$\frac{1}{2}\,r^2\,\mathrm{Cos}[\psi]\,\Bigg(\frac{\left(\mathrm{Cos}[\psi] - \frac{r-r1\,\mathrm{Cos}[\psi]}{\sqrt{r^2+r1^2-2\,r\,r1\,\mathrm{Cos}[\psi]}}\right)\left(-\mathrm{Cos}[\psi] + \frac{r-r1\,\mathrm{Cos}[\psi]}{\sqrt{r^2+r1^2-2\,r\,r1\,\mathrm{Cos}[\psi]}}\right)}{\left(r1 - r\,\mathrm{Cos}[\psi] + \sqrt{r^2 + r1^2 - 2\,r\,r1\,\mathrm{Cos}[\psi]}\right)^2} +$$

$$\frac{\frac{(r-r1\,\mathrm{Cos}[\psi])\,(-r+r1\,\mathrm{Cos}[\psi])}{\left(r^2+r1^2-2\,r\,r1\,\mathrm{Cos}[\psi]\right)^{3/2}} + \frac{1}{\sqrt{r^2+r1^2-2\,r\,r1\,\mathrm{Cos}[\psi]}}}{r1 - r\,\mathrm{Cos}[\psi] + \sqrt{r^2 + r1^2 - 2\,r\,r1\,\mathrm{Cos}[\psi]}}\Bigg)\,\mathrm{Sin}[\psi]^2 +$$

$$\mathrm{Cos}[\psi]\,\Bigg(\frac{2\,r\left(r - \mathrm{Cos}[\psi]\left(r1 + \sqrt{r^2 + r1^2 - 2\,r\,r1\,\mathrm{Cos}[\psi]}\right)\right)}{\sqrt{r^2 + r1^2 - 2\,r\,r1\,\mathrm{Cos}[\psi]}\left(r1 - r\,\mathrm{Cos}[\psi] + \sqrt{r^2 + r1^2 - 2\,r\,r1\,\mathrm{Cos}[\psi]}\right)} +$$

$$\mathrm{Log}\left[r1 - r\,\mathrm{Cos}[\psi] + \sqrt{r^2 + r1^2 - 2\,r\,r1\,\mathrm{Cos}[\psi]}\right]\Bigg)\,\mathrm{Sin}[\psi]^2\Bigg) + 2\,G\,\pi\,\rho$$

$$\Bigg(\frac{r2\,\mathrm{Cos}[\psi]\,\sqrt{r^2 + r2^2 - 2\,r\,r2\,\mathrm{Cos}[\psi]}}{r^2} - \frac{(3\,r - 3\,r2\,\mathrm{Cos}[\psi])\,\sqrt{r^2 + r2^2 - 2\,r\,r2\,\mathrm{Cos}[\psi]}}{3\,r^2} +$$

$$\frac{2\left(r^2 + r2^2 - 2\,r\,r2\,\mathrm{Cos}[\psi]\right)^{3/2}}{3\,r^3} + \frac{(r - r2\,\mathrm{Cos}[\psi])\left(1 - \frac{r2\,\mathrm{Cos}[\psi]}{r} - \mathrm{Cos}[\psi]^2\right)}{\sqrt{r^2 + r2^2 - 2\,r\,r2\,\mathrm{Cos}[\psi]}} + \frac{1}{2}\,\mathrm{Cos}[\psi]$$

$$(r2 - r\,\mathrm{Cos}[\psi])\left(\frac{(r - r2\,\mathrm{Cos}[\psi])\,(-r + r2\,\mathrm{Cos}[\psi])}{\left(r^2 + r2^2 - 2\,r\,r2\,\mathrm{Cos}[\psi]\right)^{3/2}} + \frac{1}{\sqrt{r^2 + r2^2 - 2\,r\,r2\,\mathrm{Cos}[\psi]}}\right) +$$

$$\frac{1}{2}\,r^2\,\mathrm{Cos}[\psi]\,\Bigg(\frac{\left(\mathrm{Cos}[\psi] - \frac{r-r2\,\mathrm{Cos}[\psi]}{\sqrt{r^2+r2^2-2\,r\,r2\,\mathrm{Cos}[\psi]}}\right)\left(-\mathrm{Cos}[\psi] + \frac{r-r2\,\mathrm{Cos}[\psi]}{\sqrt{r^2+r2^2-2\,r\,r2\,\mathrm{Cos}[\psi]}}\right)}{\left(r2 - r\,\mathrm{Cos}[\psi] + \sqrt{r^2 + r2^2 - 2\,r\,r2\,\mathrm{Cos}[\psi]}\right)^2} +$$

$$\frac{\frac{(r-r2\,\mathrm{Cos}[\psi])\,(-r+r2\,\mathrm{Cos}[\psi])}{\left(r^2+r2^2-2\,r\,r2\,\mathrm{Cos}[\psi]\right)^{3/2}} + \frac{1}{\sqrt{r^2+r2^2-2\,r\,r2\,\mathrm{Cos}[\psi]}}}{r2 - r\,\mathrm{Cos}[\psi] + \sqrt{r^2 + r2^2 - 2\,r\,r2\,\mathrm{Cos}[\psi]}}\Bigg)\,\mathrm{Sin}[\psi]^2 +$$

$$\mathrm{Cos}[\psi]\,\Bigg(\frac{2\,r\left(r - \mathrm{Cos}[\psi]\left(r2 + \sqrt{r^2 + r2^2 - 2\,r\,r2\,\mathrm{Cos}[\psi]}\right)\right)}{\sqrt{r^2 + r2^2 - 2\,r\,r2\,\mathrm{Cos}[\psi]}\left(r2 - r\,\mathrm{Cos}[\psi] + \sqrt{r^2 + r2^2 - 2\,r\,r2\,\mathrm{Cos}[\psi]}\right)} +$$

$$\mathrm{Log}\left[r2 - r\,\mathrm{Cos}[\psi] + \sqrt{r^2 + r2^2 - 2\,r\,r2\,\mathrm{Cos}[\psi]}\right]\Bigg)\,\mathrm{Sin}[\psi]^2\Bigg);$$

```
In[9]:= (*The formula of the new Vz is from Eq. (5) of this paper*)
NewVz = (2 G π r1^3 ρ)/(3 r^2) - (2 G π r2^3 ρ)/(3 r^2) -
   1/(3 r^2) G π ρ (Sqrt[r^2 + r1^2 - 2 r r1 Cos[ψ]] (r^2 - 2 r1^2 - 3 r^2 Cos[2 ψ]) +
      r Cos[ψ] (-2 r1 Sqrt[r^2 + r1^2 - 2 r r1 Cos[ψ]] +
         3 r^2 (1 + 2 Log[r1 - r Cos[ψ] + Sqrt[r^2 + r1^2 - 2 r r1 Cos[ψ]]]) Sin[ψ]^2)) +
   1/(3 r^2) G π ρ (Sqrt[r^2 + r2^2 - 2 r r2 Cos[ψ]] (r^2 - 2 r2^2 - 3 r^2 Cos[2 ψ]) +
      r Cos[ψ] (-2 r2 Sqrt[r^2 + r2^2 - 2 r r2 Cos[ψ]] +
         3 r^2 (1 + 2 Log[r2 - r Cos[ψ] + Sqrt[r^2 + r2^2 - 2 r r2 Cos[ψ]]]) Sin[ψ]^2));

In[10]:= (*The formula of the new Vzz is from Eq. (7) of this paper*)
NewVzz =
  -(4 G π r1^3 ρ)/(3 r^3) + (4 G π r2^3 ρ)/(3 r^3) - 1/(3 r^3 Sqrt[r^2 + r1^2 - 2 r r1 Cos[ψ]]) G π ρ (r^4 + 2 r^2 r1^2 + 4 r1^4 -
      2 r^2 r1^2 Cos[ψ]^2 - 3 r^4 Cos[2 ψ] - r Cos[ψ] (r^2 r1 + 4 r1^3 - 3 r^2 r1 Cos[2 ψ] -
         3 r^2 (-2 r1 + 3 Sqrt[r^2 + r1^2 - 2 r r1 Cos[ψ]] + 2 Sqrt[r^2 + r1^2 - 2 r r1 Cos[ψ]]
            Log[r1 - r Cos[ψ] + Sqrt[r^2 + r1^2 - 2 r r1 Cos[ψ]]]) Sin[ψ]^2)) +
   1/(3 r^3 Sqrt[r^2 + r2^2 - 2 r r2 Cos[ψ]]) G π ρ (r^4 + 2 r^2 r2^2 + 4 r2^4 - 2 r^2 r2^2 Cos[ψ]^2 -
      3 r^4 Cos[2 ψ] - r Cos[ψ] (r^2 r2 + 4 r2^3 - 3 r^2 r2 Cos[2 ψ] -
         3 r^2 (-2 r2 + 3 Sqrt[r^2 + r2^2 - 2 r r2 Cos[ψ]] + 2 Sqrt[r^2 + r2^2 - 2 r r2 Cos[ψ]]
            Log[r2 - r Cos[ψ] + Sqrt[r^2 + r2^2 - 2 r r2 Cos[ψ]]]) Sin[ψ]^2));

In[11]:= FullSimplify[NewVz - OldVz]
Out[11]= 0

In[12]:= FullSimplify[NewVzz - OldVzz]
Out[12]= 0

In[13]:= NotebookSave[EvaluationNotebook[]];
```