Appendix C of Higher-order gravitational potential gradients by tensor analysis in spherical coordinates

1.Readme

1.(λ, φ, r) is the spherical longitude, latitude and radius of the field point; (λ3, φ3, r3)=(λ', φ', r') is the spherical longitude, latitude and radius of the integration point.

2. "kernel" is the kernel of the gravitational potential of a uniform tesseroid. The complete formulae should add the related triple integral symbol $G\rho\int_{\lambda_1}^{\lambda_2}\int_{\varphi_1}^{\varphi_2}\int_{r_1}^{r_2}(\text{kernel})\,dr'\,d\varphi'\,d\lambda'$

3.To run the codes, "ctrl + A" is to select all the cells, and "Shift + Enter" is to run the selected cells.

2.Kernel of the gravitational potential (V)

```
In[•]:= kernel = (r3^2 * Cos[φ3]) /
          (√(r^2 + r3^2 - 2 * r * r3 * (Sin[φ] * Sin[φ3] + Cos[φ] * Cos[φ3] * Cos[λ - λ3])))
```

$$\text{Out[•]}= \frac{r3^2\,\text{Cos}[\varphi 3]}{\sqrt{r^2 + r3^2 - 2\,r\,r3\,(\text{Cos}[\lambda - \lambda 3]\,\text{Cos}[\varphi]\,\text{Cos}[\varphi 3] + \text{Sin}[\varphi]\,\text{Sin}[\varphi 3])}}$$

3.Kernels of 3 defining expressions of $\partial_i V$:

V_1 is the kernel of $\partial_\lambda V$,

V_2 is the kernel of $\partial_\varphi V$ and

V_3 is the kernel of $\partial_r V$

```
In[•]:= V1 = D[kernel, {λ, 1}]
```

$$\text{Out[•]}= -\frac{r\,r3^3\,\text{Cos}[\varphi]\,\text{Cos}[\varphi 3]^2\,\text{Sin}[\lambda - \lambda 3]}{\left(r^2 + r3^2 - 2\,r\,r3\,(\text{Cos}[\lambda - \lambda 3]\,\text{Cos}[\varphi]\,\text{Cos}[\varphi 3] + \text{Sin}[\varphi]\,\text{Sin}[\varphi 3])\right)^{3/2}}$$

```
In[•]:= V2 = D[kernel, {φ, 1}]
```

$$\text{Out[•]}= \frac{r\,r3^3\,\text{Cos}[\varphi 3]\,(-\text{Cos}[\lambda - \lambda 3]\,\text{Cos}[\varphi 3]\,\text{Sin}[\varphi] + \text{Cos}[\varphi]\,\text{Sin}[\varphi 3])}{\left(r^2 + r3^2 - 2\,r\,r3\,(\text{Cos}[\lambda - \lambda 3]\,\text{Cos}[\varphi]\,\text{Cos}[\varphi 3] + \text{Sin}[\varphi]\,\text{Sin}[\varphi 3])\right)^{3/2}}$$

```
In[•]:= V3 = D[kernel, {r, 1}]
```

$$\text{Out[•]}= -\frac{r3^2\,\text{Cos}[\varphi 3]\,(2\,r - 2\,r3\,(\text{Cos}[\lambda - \lambda 3]\,\text{Cos}[\varphi]\,\text{Cos}[\varphi 3] + \text{Sin}[\varphi]\,\text{Sin}[\varphi 3]))}{2\left(r^2 + r3^2 - 2\,r\,r3\,(\text{Cos}[\lambda - \lambda 3]\,\text{Cos}[\varphi]\,\text{Cos}[\varphi 3] + \text{Sin}[\varphi]\,\text{Sin}[\varphi 3])\right)^{3/2}}$$

4.Kernels of 6 defining expressions of $\partial_{ij} V$:

V_{11} is the kernel of $\partial_{\lambda\lambda} V$, V_{12} is the kernel of $\partial_{\lambda\varphi} V$, V_{13} is the kernel of $\partial_{\lambda r} V$,

V_{22} is the kernel of $\partial_{\varphi\varphi} V$, V_{23} is the kernel of $\partial_{\varphi r} V$ and

V_{33} is the kernel of $\partial_{rr} V$

In[•]:= V_{11} = D[kernel, {λ, 2}]

Out[•]=
$$r3^2 \cos[\varphi 3]\left(\frac{3\, r^2\, r3^2 \cos[\varphi]^2 \cos[\varphi 3]^2 \sin[\lambda-\lambda 3]^2}{\left(r^2+r3^2-2\, r\, r3\,(\cos[\lambda-\lambda 3]\cos[\varphi]\cos[\varphi 3]+\sin[\varphi]\sin[\varphi 3])\right)^{5/2}}-\frac{r\, r3\cos[\lambda-\lambda 3]\cos[\varphi]\cos[\varphi 3]}{\left(r^2+r3^2-2\, r\, r3\,(\cos[\lambda-\lambda 3]\cos[\varphi]\cos[\varphi 3]+\sin[\varphi]\sin[\varphi 3])\right)^{3/2}}\right)$$

In[•]:= V_{12} = D[kernel, {λ, 1}, {φ, 1}]

Out[•]=
$$-\frac{3\, r^2\, r3^4 \cos[\varphi]\cos[\varphi 3]^2\sin[\lambda-\lambda 3]\,(-\cos[\lambda-\lambda 3]\cos[\varphi 3]\sin[\varphi]+\cos[\varphi]\sin[\varphi 3])}{\left(r^2+r3^2-2\, r\, r3\,(\cos[\lambda-\lambda 3]\cos[\varphi]\cos[\varphi 3]+\sin[\varphi]\sin[\varphi 3])\right)^{5/2}}+\frac{r\, r3^3\cos[\varphi 3]^2\sin[\lambda-\lambda 3]\sin[\varphi]}{\left(r^2+r3^2-2\, r\, r3\,(\cos[\lambda-\lambda 3]\cos[\varphi]\cos[\varphi 3]+\sin[\varphi]\sin[\varphi 3])\right)^{3/2}}$$

In[•]:= V_{13} = D[kernel, {λ, 1}, {r, 1}]

Out[•]=
$$\left(3\, r\, r3^3\cos[\varphi]\cos[\varphi 3]^2\sin[\lambda-\lambda 3]\,(2\, r-2\, r3\,(\cos[\lambda-\lambda 3]\cos[\varphi]\cos[\varphi 3]+\sin[\varphi]\sin[\varphi 3]))\right)\Big/\left(2\left(r^2+r3^2-2\, r\, r3\,(\cos[\lambda-\lambda 3]\cos[\varphi]\cos[\varphi 3]+\sin[\varphi]\sin[\varphi 3])\right)^{5/2}\right)-\frac{r3^3\cos[\varphi]\cos[\varphi 3]^2\sin[\lambda-\lambda 3]}{\left(r^2+r3^2-2\, r\, r3\,(\cos[\lambda-\lambda 3]\cos[\varphi]\cos[\varphi 3]+\sin[\varphi]\sin[\varphi 3])\right)^{3/2}}$$

In[•]:= V_{22} = D[kernel, {φ, 2}]

Out[•]=
$$r3^2\cos[\varphi 3]\left(\frac{3\, r^2\, r3^2\,(-\cos[\lambda-\lambda 3]\cos[\varphi 3]\sin[\varphi]+\cos[\varphi]\sin[\varphi 3])^2}{\left(r^2+r3^2-2\, r\, r3\,(\cos[\lambda-\lambda 3]\cos[\varphi]\cos[\varphi 3]+\sin[\varphi]\sin[\varphi 3])\right)^{5/2}}+\frac{r\, r3\,(-\cos[\lambda-\lambda 3]\cos[\varphi]\cos[\varphi 3]-\sin[\varphi]\sin[\varphi 3])}{\left(r^2+r3^2-2\, r\, r3\,(\cos[\lambda-\lambda 3]\cos[\varphi]\cos[\varphi 3]+\sin[\varphi]\sin[\varphi 3])\right)^{3/2}}\right)$$

In[•]:= V_{23} = D[kernel, {φ, 1}, {r, 1}]

Out[•]=
$$-\left(\left(3\, r\, r3^3\cos[\varphi 3]\,(-\cos[\lambda-\lambda 3]\cos[\varphi 3]\sin[\varphi]+\cos[\varphi]\sin[\varphi 3])\,(2\, r-2\, r3\,(\cos[\lambda-\lambda 3]\cos[\varphi]\cos[\varphi 3]+\sin[\varphi]\sin[\varphi 3]))\right)\Big/\left(2\left(r^2+r3^2-2\, r\, r3\,(\cos[\lambda-\lambda 3]\cos[\varphi]\cos[\varphi 3]+\sin[\varphi]\sin[\varphi 3])\right)^{5/2}\right)\right)+\frac{r3^3\cos[\varphi 3]\,(-\cos[\lambda-\lambda 3]\cos[\varphi 3]\sin[\varphi]+\cos[\varphi]\sin[\varphi 3])}{\left(r^2+r3^2-2\, r\, r3\,(\cos[\lambda-\lambda 3]\cos[\varphi]\cos[\varphi 3]+\sin[\varphi]\sin[\varphi 3])\right)^{3/2}}$$

In[•]:= V_{33} = D[kernel, {r, 2}]

Out[•]=
$$r3^2\cos[\varphi 3]\left(\frac{3\,(2\, r-2\, r3\,(\cos[\lambda-\lambda 3]\cos[\varphi]\cos[\varphi 3]+\sin[\varphi]\sin[\varphi 3]))^2}{4\left(r^2+r3^2-2\, r\, r3\,(\cos[\lambda-\lambda 3]\cos[\varphi]\cos[\varphi 3]+\sin[\varphi]\sin[\varphi 3])\right)^{5/2}}-\frac{1}{\left(r^2+r3^2-2\, r\, r3\,(\cos[\lambda-\lambda 3]\cos[\varphi]\cos[\varphi 3]+\sin[\varphi]\sin[\varphi 3])\right)^{3/2}}\right)$$

5.Kernels of 10 defining expressions of $\partial_{ijk} V$:

V_{111} is the kernel of $\partial_{\lambda\lambda\lambda} V$, V_{112} is the kernel of $\partial_{\lambda\lambda\varphi} V$, V_{113} is the kernel of $\partial_{\lambda\lambda r} V$,
V_{123} is the kernel of $\partial_{\lambda\varphi r} V$,
V_{221} is the kernel of $\partial_{\lambda\varphi\varphi} V$, V_{222} is the kernel of $\partial_{\varphi\varphi\varphi} V$, V_{223} is the kernel of $\partial_{\varphi\varphi r} V$,
V_{331} is the kernel of $\partial_{\lambda r r} V$, V_{332} is the kernel of $\partial_{\varphi r r} V$ and V_{333} is the kernel of $\partial_{r r r} V$

In[•]:=
```
V111 = D[kernel, {λ, 3}]
```

Out[•]=
$$r3^2 \cos[\varphi 3] \left(-\frac{15\, r^3\, r3^3 \cos[\varphi]^3 \cos[\varphi 3]^3 \sin[\lambda - \lambda 3]^3}{\left(r^2 + r3^2 - 2\, r\, r3\, (\cos[\lambda - \lambda 3] \cos[\varphi] \cos[\varphi 3] + \sin[\varphi] \sin[\varphi 3])\right)^{7/2}} + \frac{9\, r^2\, r3^2 \cos[\lambda - \lambda 3] \cos[\varphi]^2 \cos[\varphi 3]^2 \sin[\lambda - \lambda 3]}{\left(r^2 + r3^2 - 2\, r\, r3\, (\cos[\lambda - \lambda 3] \cos[\varphi] \cos[\varphi 3] + \sin[\varphi] \sin[\varphi 3])\right)^{5/2}} + \frac{r\, r3 \cos[\varphi] \cos[\varphi 3] \sin[\lambda - \lambda 3]}{\left(r^2 + r3^2 - 2\, r\, r3\, (\cos[\lambda - \lambda 3] \cos[\varphi] \cos[\varphi 3] + \sin[\varphi] \sin[\varphi 3])\right)^{3/2}} \right)$$

In[•]:=
```
V112 = D[kernel, {λ, 2}, {φ, 1}]
```

Out[•]=
$$r3^2 \cos[\varphi 3] \left(\frac{15\, r^3\, r3^3 \cos[\varphi]^2 \cos[\varphi 3]^2 \sin[\lambda - \lambda 3]^2\, (-\cos[\lambda - \lambda 3] \cos[\varphi 3] \sin[\varphi] + \cos[\varphi] \sin[\varphi 3])}{\left(r^2 + r3^2 - 2\, r\, r3\, (\cos[\lambda - \lambda 3] \cos[\varphi] \cos[\varphi 3] + \sin[\varphi] \sin[\varphi 3])\right)^{7/2}} - \frac{6\, r^2\, r3^2 \cos[\varphi] \cos[\varphi 3]^2 \sin[\lambda - \lambda 3]^2 \sin[\varphi]}{\left(r^2 + r3^2 - 2\, r\, r3\, (\cos[\lambda - \lambda 3] \cos[\varphi] \cos[\varphi 3] + \sin[\varphi] \sin[\varphi 3])\right)^{5/2}} - \frac{3\, r^2\, r3^2 \cos[\lambda - \lambda 3] \cos[\varphi] \cos[\varphi 3]\, (-\cos[\lambda - \lambda 3] \cos[\varphi 3] \sin[\varphi] + \cos[\varphi] \sin[\varphi 3])}{\left(r^2 + r3^2 - 2\, r\, r3\, (\cos[\lambda - \lambda 3] \cos[\varphi] \cos[\varphi 3] + \sin[\varphi] \sin[\varphi 3])\right)^{5/2}} + \frac{r\, r3 \cos[\lambda - \lambda 3] \cos[\varphi 3] \sin[\varphi]}{\left(r^2 + r3^2 - 2\, r\, r3\, (\cos[\lambda - \lambda 3] \cos[\varphi] \cos[\varphi 3] + \sin[\varphi] \sin[\varphi 3])\right)^{3/2}} \right)$$

In[•]:=
```
V113 = D[kernel, {λ, 2}, {r, 1}]
```

Out[•]=
$$r3^2 \cos[\varphi 3] \left(-\left(\left(15\, r^2\, r3^2 \cos[\varphi]^2 \cos[\varphi 3]^2 \sin[\lambda - \lambda 3]^2 \right.\right.\right.$$
$$\left.\left.\left. (2\, r - 2\, r3\, (\cos[\lambda - \lambda 3] \cos[\varphi] \cos[\varphi 3] + \sin[\varphi] \sin[\varphi 3])) \right) \Big/ \right.$$
$$\left. \left(2 \left(r^2 + r3^2 - 2\, r\, r3\, (\cos[\lambda - \lambda 3] \cos[\varphi] \cos[\varphi 3] + \sin[\varphi] \sin[\varphi 3]) \right)^{7/2} \right) \right) +$$
$$\frac{6\, r\, r3^2 \cos[\varphi]^2 \cos[\varphi 3]^2 \sin[\lambda - \lambda 3]^2}{\left(r^2 + r3^2 - 2\, r\, r3\, (\cos[\lambda - \lambda 3] \cos[\varphi] \cos[\varphi 3] + \sin[\varphi] \sin[\varphi 3]) \right)^{5/2}} +$$
$$(3\, r\, r3 \cos[\lambda - \lambda 3] \cos[\varphi] \cos[\varphi 3]$$
$$(2\, r - 2\, r3\, (\cos[\lambda - \lambda 3] \cos[\varphi] \cos[\varphi 3] + \sin[\varphi] \sin[\varphi 3]))) \Big/$$
$$\left(2 \left(r^2 + r3^2 - 2\, r\, r3\, (\cos[\lambda - \lambda 3] \cos[\varphi] \cos[\varphi 3] + \sin[\varphi] \sin[\varphi 3]) \right)^{5/2} \right) -$$
$$\left. \frac{r3 \cos[\lambda - \lambda 3] \cos[\varphi] \cos[\varphi 3]}{\left(r^2 + r3^2 - 2\, r\, r3\, (\cos[\lambda - \lambda 3] \cos[\varphi] \cos[\varphi 3] + \sin[\varphi] \sin[\varphi 3]) \right)^{3/2}} \right)$$

In[•]:=
```
V123 = D[kernel, {λ, 1}, {φ, 1}, {r, 1}]
```

Out[•]=
$$\left(15\, r^2\, r3^4 \cos[\varphi] \cos[\varphi 3]^2 \sin[\lambda - \lambda 3]\, (-\cos[\lambda - \lambda 3] \cos[\varphi 3] \sin[\varphi] + \cos[\varphi] \sin[\varphi 3]) \right.$$
$$(2\, r - 2\, r3\, (\cos[\lambda - \lambda 3] \cos[\varphi] \cos[\varphi 3] + \sin[\varphi] \sin[\varphi 3]))) \Big/$$
$$\left(2 \left(r^2 + r3^2 - 2\, r\, r3\, (\cos[\lambda - \lambda 3] \cos[\varphi] \cos[\varphi 3] + \sin[\varphi] \sin[\varphi 3]) \right)^{7/2} \right) -$$
$$\frac{6\, r\, r3^4 \cos[\varphi] \cos[\varphi 3]^2 \sin[\lambda - \lambda 3]\, (-\cos[\lambda - \lambda 3] \cos[\varphi 3] \sin[\varphi] + \cos[\varphi] \sin[\varphi 3])}{\left(r^2 + r3^2 - 2\, r\, r3\, (\cos[\lambda - \lambda 3] \cos[\varphi] \cos[\varphi 3] + \sin[\varphi] \sin[\varphi 3]) \right)^{5/2}} -$$
$$\left(3\, r\, r3^3 \cos[\varphi 3]^2 \sin[\lambda - \lambda 3] \sin[\varphi] \right.$$
$$(2\, r - 2\, r3\, (\cos[\lambda - \lambda 3] \cos[\varphi] \cos[\varphi 3] + \sin[\varphi] \sin[\varphi 3]))) \Big/$$
$$\left(2 \left(r^2 + r3^2 - 2\, r\, r3\, (\cos[\lambda - \lambda 3] \cos[\varphi] \cos[\varphi 3] + \sin[\varphi] \sin[\varphi 3]) \right)^{5/2} \right) +$$
$$\frac{r3^3 \cos[\varphi 3]^2 \sin[\lambda - \lambda 3] \sin[\varphi]}{\left(r^2 + r3^2 - 2\, r\, r3\, (\cos[\lambda - \lambda 3] \cos[\varphi] \cos[\varphi 3] + \sin[\varphi] \sin[\varphi 3]) \right)^{3/2}}$$

In[•]:=
```
V221 = D[kernel, {λ, 1}, {φ, 2}]
```

Out[•]=
$$\frac{6\, r^2\, r3^4 \cos[\varphi 3]^2 \sin[\lambda - \lambda 3] \sin[\varphi]\, (-\cos[\lambda - \lambda 3] \cos[\varphi 3] \sin[\varphi] + \cos[\varphi] \sin[\varphi 3])}{\left(r^2 + r3^2 - 2\, r\, r3\, (\cos[\lambda - \lambda 3] \cos[\varphi] \cos[\varphi 3] + \sin[\varphi] \sin[\varphi 3]) \right)^{5/2}} +$$
$$\frac{r\, r3^3 \cos[\varphi] \cos[\varphi 3]^2 \sin[\lambda - \lambda 3]}{\left(r^2 + r3^2 - 2\, r\, r3\, (\cos[\lambda - \lambda 3] \cos[\varphi] \cos[\varphi 3] + \sin[\varphi] \sin[\varphi 3]) \right)^{3/2}} -$$
$$r\, r3^3 \cos[\varphi] \cos[\varphi 3]^2 \sin[\lambda - \lambda 3]$$
$$\left(\frac{15\, r^2\, r3^2\, (-\cos[\lambda - \lambda 3] \cos[\varphi 3] \sin[\varphi] + \cos[\varphi] \sin[\varphi 3])^2}{\left(r^2 + r3^2 - 2\, r\, r3\, (\cos[\lambda - \lambda 3] \cos[\varphi] \cos[\varphi 3] + \sin[\varphi] \sin[\varphi 3]) \right)^{7/2}} + \right.$$
$$\left. \frac{3\, r\, r3\, (-\cos[\lambda - \lambda 3] \cos[\varphi] \cos[\varphi 3] - \sin[\varphi] \sin[\varphi 3])}{\left(r^2 + r3^2 - 2\, r\, r3\, (\cos[\lambda - \lambda 3] \cos[\varphi] \cos[\varphi 3] + \sin[\varphi] \sin[\varphi 3]) \right)^{5/2}} \right)$$

In[•]:= V_{222} = D[kernel, {φ, 3}]

Out[•]=
$$r3^2 \cos[\varphi 3] \left(\frac{15\, r^3\, r3^3\, (-\cos[\lambda-\lambda 3]\cos[\varphi 3]\sin[\varphi]+\cos[\varphi]\sin[\varphi 3])^3}{\left(r^2+r3^2-2\,r\,r3\,(\cos[\lambda-\lambda 3]\cos[\varphi]\cos[\varphi 3]+\sin[\varphi]\sin[\varphi 3])\right)^{7/2}} + \right.$$
$$\left(9\, r^2\, r3^2\, (-\cos[\lambda-\lambda 3]\cos[\varphi 3]\sin[\varphi]+\cos[\varphi]\sin[\varphi 3])\right.$$
$$\left.(-\cos[\lambda-\lambda 3]\cos[\varphi]\cos[\varphi 3]-\sin[\varphi]\sin[\varphi 3])\right) \Big/$$
$$\left(r^2+r3^2-2\,r\,r3\,(\cos[\lambda-\lambda 3]\cos[\varphi]\cos[\varphi 3]+\sin[\varphi]\sin[\varphi 3])\right)^{5/2} +$$
$$\left. \frac{r\, r3\, (\cos[\lambda-\lambda 3]\cos[\varphi 3]\sin[\varphi]-\cos[\varphi]\sin[\varphi 3])}{\left(r^2+r3^2-2\,r\,r3\,(\cos[\lambda-\lambda 3]\cos[\varphi]\cos[\varphi 3]+\sin[\varphi]\sin[\varphi 3])\right)^{3/2}} \right)$$

In[•]:= V_{223} = D[kernel, {φ, 2}, {r, 1}]

Out[•]=
$$r3^2 \cos[\varphi 3] \left(-\left(\left(15\, r^2\, r3^2\, (-\cos[\lambda-\lambda 3]\cos[\varphi 3]\sin[\varphi]+\cos[\varphi]\sin[\varphi 3])^2\right.\right.\right.$$
$$\left.\left.\left.(2\,r-2\,r3\,(\cos[\lambda-\lambda 3]\cos[\varphi]\cos[\varphi 3]+\sin[\varphi]\sin[\varphi 3]))\right) \Big/ \right.$$
$$\left.\left(2\left(r^2+r3^2-2\,r\,r3\,(\cos[\lambda-\lambda 3]\cos[\varphi]\cos[\varphi 3]+\sin[\varphi]\sin[\varphi 3])\right)^{7/2}\right)\right) +$$
$$\frac{6\, r\, r3^2\, (-\cos[\lambda-\lambda 3]\cos[\varphi 3]\sin[\varphi]+\cos[\varphi]\sin[\varphi 3])^2}{\left(r^2+r3^2-2\,r\,r3\,(\cos[\lambda-\lambda 3]\cos[\varphi]\cos[\varphi 3]+\sin[\varphi]\sin[\varphi 3])\right)^{5/2}} -$$
$$(3\, r\, r3\, (-\cos[\lambda-\lambda 3]\cos[\varphi]\cos[\varphi 3]-\sin[\varphi]\sin[\varphi 3])$$
$$(2\,r-2\,r3\,(\cos[\lambda-\lambda 3]\cos[\varphi]\cos[\varphi 3]+\sin[\varphi]\sin[\varphi 3]))) \Big/$$
$$\left(2\left(r^2+r3^2-2\,r\,r3\,(\cos[\lambda-\lambda 3]\cos[\varphi]\cos[\varphi 3]+\sin[\varphi]\sin[\varphi 3])\right)^{5/2}\right) +$$
$$\left. \frac{r3\, (-\cos[\lambda-\lambda 3]\cos[\varphi]\cos[\varphi 3]-\sin[\varphi]\sin[\varphi 3])}{\left(r^2+r3^2-2\,r\,r3\,(\cos[\lambda-\lambda 3]\cos[\varphi]\cos[\varphi 3]+\sin[\varphi]\sin[\varphi 3])\right)^{3/2}} \right)$$

In[•]:= V_{331} = D[kernel, {λ, 1}, {r, 2}]

Out[•]=
$$\left(3\, r3^3 \cos[\varphi]\cos[\varphi 3]^2 \sin[\lambda-\lambda 3]\right.$$
$$\left.(2\,r-2\,r3\,(\cos[\lambda-\lambda 3]\cos[\varphi]\cos[\varphi 3]+\sin[\varphi]\sin[\varphi 3]))\right) \Big/$$
$$\left(r^2+r3^2-2\,r\,r3\,(\cos[\lambda-\lambda 3]\cos[\varphi]\cos[\varphi 3]+\sin[\varphi]\sin[\varphi 3])\right)^{5/2} -$$
$$r\, r3^3 \cos[\varphi]\cos[\varphi 3]^2 \sin[\lambda-\lambda 3]$$
$$\left(\frac{15\,(2\,r-2\,r3\,(\cos[\lambda-\lambda 3]\cos[\varphi]\cos[\varphi 3]+\sin[\varphi]\sin[\varphi 3]))^2}{4\left(r^2+r3^2-2\,r\,r3\,(\cos[\lambda-\lambda 3]\cos[\varphi]\cos[\varphi 3]+\sin[\varphi]\sin[\varphi 3])\right)^{7/2}} - \right.$$
$$\left. \frac{3}{\left(r^2+r3^2-2\,r\,r3\,(\cos[\lambda-\lambda 3]\cos[\varphi]\cos[\varphi 3]+\sin[\varphi]\sin[\varphi 3])\right)^{5/2}} \right)$$

In[•]:=
```
V332 = D[kernel, {φ, 1}, {r, 2}]
```

Out[•]=
$$-\left(\left(3\,r3^3\,\text{Cos}[\varphi 3]\,\left(-\text{Cos}[\lambda-\lambda 3]\,\text{Cos}[\varphi 3]\,\text{Sin}[\varphi]+\text{Cos}[\varphi]\,\text{Sin}[\varphi 3]\right)\right.\right.$$
$$\left.\left(2\,r-2\,r3\,\left(\text{Cos}[\lambda-\lambda 3]\,\text{Cos}[\varphi]\,\text{Cos}[\varphi 3]+\text{Sin}[\varphi]\,\text{Sin}[\varphi 3]\right)\right)\right)\Big/$$
$$\left.\left(r^2+r3^2-2\,r\,r3\,\left(\text{Cos}[\lambda-\lambda 3]\,\text{Cos}[\varphi]\,\text{Cos}[\varphi 3]+\text{Sin}[\varphi]\,\text{Sin}[\varphi 3]\right)\right)^{5/2}\right)+$$
$$r\,r3^3\,\text{Cos}[\varphi 3]\,\left(-\text{Cos}[\lambda-\lambda 3]\,\text{Cos}[\varphi 3]\,\text{Sin}[\varphi]+\text{Cos}[\varphi]\,\text{Sin}[\varphi 3]\right)$$
$$\left(\frac{15\,\left(2\,r-2\,r3\,\left(\text{Cos}[\lambda-\lambda 3]\,\text{Cos}[\varphi]\,\text{Cos}[\varphi 3]+\text{Sin}[\varphi]\,\text{Sin}[\varphi 3]\right)\right)^2}{4\,\left(r^2+r3^2-2\,r\,r3\,\left(\text{Cos}[\lambda-\lambda 3]\,\text{Cos}[\varphi]\,\text{Cos}[\varphi 3]+\text{Sin}[\varphi]\,\text{Sin}[\varphi 3]\right)\right)^{7/2}}-\right.$$
$$\left.\frac{3}{\left(r^2+r3^2-2\,r\,r3\,\left(\text{Cos}[\lambda-\lambda 3]\,\text{Cos}[\varphi]\,\text{Cos}[\varphi 3]+\text{Sin}[\varphi]\,\text{Sin}[\varphi 3]\right)\right)^{5/2}}\right)$$

In[•]:=
```
V333 = D[kernel, {r, 3}]
```

Out[•]=
$$r3^2\,\text{Cos}[\varphi 3]\,\left(-\frac{15\,\left(2\,r-2\,r3\,\left(\text{Cos}[\lambda-\lambda 3]\,\text{Cos}[\varphi]\,\text{Cos}[\varphi 3]+\text{Sin}[\varphi]\,\text{Sin}[\varphi 3]\right)\right)^3}{8\,\left(r^2+r3^2-2\,r\,r3\,\left(\text{Cos}[\lambda-\lambda 3]\,\text{Cos}[\varphi]\,\text{Cos}[\varphi 3]+\text{Sin}[\varphi]\,\text{Sin}[\varphi 3]\right)\right)^{7/2}}+\right.$$
$$\left.\frac{9\,\left(2\,r-2\,r3\,\left(\text{Cos}[\lambda-\lambda 3]\,\text{Cos}[\varphi]\,\text{Cos}[\varphi 3]+\text{Sin}[\varphi]\,\text{Sin}[\varphi 3]\right)\right)}{2\,\left(r^2+r3^2-2\,r\,r3\,\left(\text{Cos}[\lambda-\lambda 3]\,\text{Cos}[\varphi]\,\text{Cos}[\varphi 3]+\text{Sin}[\varphi]\,\text{Sin}[\varphi 3]\right)\right)^{5/2}}\right)$$

6.Kernels of 15 defining expressions of $\partial_{ijkl} V$:

V_{1111} is the kernel of $\partial_{\lambda\lambda\lambda\lambda} V$, V_{2222} is the kernel of $\partial_{\varphi\varphi\varphi\varphi} V$, V_{3333} is the kernel of $\partial_{rrrr} V$,
V_{1112} is the kernel of $\partial_{\lambda\lambda\lambda\varphi} V$, V_{1113} is the kernel of $\partial_{\lambda\lambda\lambda r} V$, V_{1222} is the kernel of $\partial_{\lambda\varphi\varphi\varphi} V$, V_{1333} is the kernel of $\partial_{\lambda rrr} V$, V_{2223} is the kernel of $\partial_{\varphi\varphi\varphi r} V$, V_{2333} is the kernel of $\partial_{\varphi rrr} V$,
V_{1122} is the kernel of $\partial_{\lambda\lambda\varphi\varphi} V$, V_{1133} is the kernel of $\partial_{\lambda\lambda rr} V$, V_{2233} is the kernel of $\partial_{\varphi\varphi rr} V$,
V_{1123} is the kernel of $\partial_{\lambda\lambda\varphi r} V$, V_{1223} is the kernel of $\partial_{\lambda\varphi\varphi r} V$ and V_{1233} is the kernel of $\partial_{\lambda\varphi rr} V$

```
In[•]:= V1111 = D[kernel, {λ, 4}]
```

```
Out[•]= r3^2 Cos[φ3] ( 105 r^4 r3^4 Cos[φ]^4 Cos[φ3]^4 Sin[λ - λ3]^4 / (r^2 + r3^2 - 2 r r3 (Cos[λ - λ3] Cos[φ] Cos[φ3] + Sin[φ] Sin[φ3]))^(9/2) -
  90 r^3 r3^3 Cos[λ - λ3] Cos[φ]^3 Cos[φ3]^3 Sin[λ - λ3]^2 / (r^2 + r3^2 - 2 r r3 (Cos[λ - λ3] Cos[φ] Cos[φ3] + Sin[φ] Sin[φ3]))^(7/2) +
  9 r^2 r3^2 Cos[λ - λ3]^2 Cos[φ]^2 Cos[φ3]^2 / (r^2 + r3^2 - 2 r r3 (Cos[λ - λ3] Cos[φ] Cos[φ3] + Sin[φ] Sin[φ3]))^(5/2) -
  12 r^2 r3^2 Cos[φ]^2 Cos[φ3]^2 Sin[λ - λ3]^2 / (r^2 + r3^2 - 2 r r3 (Cos[λ - λ3] Cos[φ] Cos[φ3] + Sin[φ] Sin[φ3]))^(5/2) +
  r r3 Cos[λ - λ3] Cos[φ] Cos[φ3] / (r^2 + r3^2 - 2 r r3 (Cos[λ - λ3] Cos[φ] Cos[φ3] + Sin[φ] Sin[φ3]))^(3/2) )
```

In[•]:=
```
V2222 = D[kernel, {φ, 4}]
```

Out[•]=
$$r3^2 \cos[\varphi 3]\left(\frac{105\, r^4\, r3^4\, (-\cos[\lambda-\lambda 3]\cos[\varphi 3]\sin[\varphi]+\cos[\varphi]\sin[\varphi 3])^4}{\left(r^2+r3^2-2\,r\,r3\,(\cos[\lambda-\lambda 3]\cos[\varphi]\cos[\varphi 3]+\sin[\varphi]\sin[\varphi 3])\right)^{9/2}}+\right.$$
$$\left(90\, r^3\, r3^3\,(-\cos[\lambda-\lambda 3]\cos[\varphi 3]\sin[\varphi]+\cos[\varphi]\sin[\varphi 3])^2\right.$$
$$\left.(-\cos[\lambda-\lambda 3]\cos[\varphi]\cos[\varphi 3]-\sin[\varphi]\sin[\varphi 3])\right)\Big/$$
$$\left(r^2+r3^2-2\,r\,r3\,(\cos[\lambda-\lambda 3]\cos[\varphi]\cos[\varphi 3]+\sin[\varphi]\sin[\varphi 3])\right)^{7/2}+$$
$$\left(12\, r^2\, r3^2\,(\cos[\lambda-\lambda 3]\cos[\varphi 3]\sin[\varphi]-\cos[\varphi]\sin[\varphi 3])\right.$$
$$\left.(-\cos[\lambda-\lambda 3]\cos[\varphi 3]\sin[\varphi]+\cos[\varphi]\sin[\varphi 3])\right)\Big/$$
$$\left(r^2+r3^2-2\,r\,r3\,(\cos[\lambda-\lambda 3]\cos[\varphi]\cos[\varphi 3]+\sin[\varphi]\sin[\varphi 3])\right)^{5/2}+$$
$$\frac{9\, r^2\, r3^2\,(-\cos[\lambda-\lambda 3]\cos[\varphi]\cos[\varphi 3]-\sin[\varphi]\sin[\varphi 3])^2}{\left(r^2+r3^2-2\,r\,r3\,(\cos[\lambda-\lambda 3]\cos[\varphi]\cos[\varphi 3]+\sin[\varphi]\sin[\varphi 3])\right)^{5/2}}+$$
$$\left.\frac{r\,r3\,(\cos[\lambda-\lambda 3]\cos[\varphi]\cos[\varphi 3]+\sin[\varphi]\sin[\varphi 3])}{\left(r^2+r3^2-2\,r\,r3\,(\cos[\lambda-\lambda 3]\cos[\varphi]\cos[\varphi 3]+\sin[\varphi]\sin[\varphi 3])\right)^{3/2}}\right)$$

In[•]:=
```
V3333 = D[kernel, {r, 4}]
```

Out[•]=
$$r3^2 \cos[\varphi 3]\left(\frac{105\,(2\,r-2\,r3\,(\cos[\lambda-\lambda 3]\cos[\varphi]\cos[\varphi 3]+\sin[\varphi]\sin[\varphi 3]))^4}{16\left(r^2+r3^2-2\,r\,r3\,(\cos[\lambda-\lambda 3]\cos[\varphi]\cos[\varphi 3]+\sin[\varphi]\sin[\varphi 3])\right)^{9/2}}-\right.$$
$$\frac{45\,(2\,r-2\,r3\,(\cos[\lambda-\lambda 3]\cos[\varphi]\cos[\varphi 3]+\sin[\varphi]\sin[\varphi 3]))^2}{2\left(r^2+r3^2-2\,r\,r3\,(\cos[\lambda-\lambda 3]\cos[\varphi]\cos[\varphi 3]+\sin[\varphi]\sin[\varphi 3])\right)^{7/2}}+$$
$$\left.\frac{9}{\left(r^2+r3^2-2\,r\,r3\,(\cos[\lambda-\lambda 3]\cos[\varphi]\cos[\varphi 3]+\sin[\varphi]\sin[\varphi 3])\right)^{5/2}}\right)$$

In[•]:= V_{1112} = D[kernel, {λ, 3}, {φ, 1}]

Out[•]=

$$r3^2 \cos[\varphi 3] \left(-\left(\left(105\, r^4\, r3^4 \cos[\varphi]^3 \cos[\varphi 3]^3 \sin[\lambda-\lambda 3]^3 \left(-\cos[\lambda-\lambda 3]\cos[\varphi 3]\sin[\varphi]+\cos[\varphi]\sin[\varphi 3]\right)\right) \Big/ \left(r^2+r3^2-2\,r\,r3\left(\cos[\lambda-\lambda 3]\cos[\varphi]\cos[\varphi 3]+\sin[\varphi]\sin[\varphi 3]\right)\right)^{9/2}\right) + \frac{45\, r^3\, r3^3 \cos[\varphi]^2 \cos[\varphi 3]^3 \sin[\lambda-\lambda 3]^3 \sin[\varphi]}{\left(r^2+r3^2-2\,r\,r3\left(\cos[\lambda-\lambda 3]\cos[\varphi]\cos[\varphi 3]+\sin[\varphi]\sin[\varphi 3]\right)\right)^{7/2}} + \left(45\, r^3\, r3^3 \cos[\lambda-\lambda 3]\cos[\varphi]^2\cos[\varphi 3]^2\sin[\lambda-\lambda 3]\left(-\cos[\lambda-\lambda 3]\cos[\varphi 3]\sin[\varphi]+\cos[\varphi]\sin[\varphi 3]\right)\right) \Big/ \left(r^2+r3^2-2\,r\,r3\left(\cos[\lambda-\lambda 3]\cos[\varphi]\cos[\varphi 3]+\sin[\varphi]\sin[\varphi 3]\right)\right)^{7/2} - \frac{18\, r^2\, r3^2 \cos[\lambda-\lambda 3]\cos[\varphi]\cos[\varphi 3]^2\sin[\lambda-\lambda 3]\sin[\varphi]}{\left(r^2+r3^2-2\,r\,r3\left(\cos[\lambda-\lambda 3]\cos[\varphi]\cos[\varphi 3]+\sin[\varphi]\sin[\varphi 3]\right)\right)^{5/2}} + \frac{3\, r^2\, r3^2 \cos[\varphi]\cos[\varphi 3]\sin[\lambda-\lambda 3]\left(-\cos[\lambda-\lambda 3]\cos[\varphi 3]\sin[\varphi]+\cos[\varphi]\sin[\varphi 3]\right)}{\left(r^2+r3^2-2\,r\,r3\left(\cos[\lambda-\lambda 3]\cos[\varphi]\cos[\varphi 3]+\sin[\varphi]\sin[\varphi 3]\right)\right)^{5/2}} - \frac{r\, r3 \cos[\varphi 3]\sin[\lambda-\lambda 3]\sin[\varphi]}{\left(r^2+r3^2-2\,r\,r3\left(\cos[\lambda-\lambda 3]\cos[\varphi]\cos[\varphi 3]+\sin[\varphi]\sin[\varphi 3]\right)\right)^{3/2}} \right)$$

In[•]:= V_{1113} = D[kernel, {λ, 3}, {r, 1}]

Out[•]=

$$r3^2 \cos[\varphi 3] \left(\left(105\, r^3\, r3^3 \cos[\varphi]^3 \cos[\varphi 3]^3 \sin[\lambda-\lambda 3]^3 \left(2\,r-2\,r3\left(\cos[\lambda-\lambda 3]\cos[\varphi]\cos[\varphi 3]+\sin[\varphi]\sin[\varphi 3]\right)\right)\right) \Big/ \left(2\left(r^2+r3^2-2\,r\,r3\left(\cos[\lambda-\lambda 3]\cos[\varphi]\cos[\varphi 3]+\sin[\varphi]\sin[\varphi 3]\right)\right)^{9/2}\right) - \frac{45\, r^2\, r3^3 \cos[\varphi]^3 \cos[\varphi 3]^3 \sin[\lambda-\lambda 3]^3}{\left(r^2+r3^2-2\,r\,r3\left(\cos[\lambda-\lambda 3]\cos[\varphi]\cos[\varphi 3]+\sin[\varphi]\sin[\varphi 3]\right)\right)^{7/2}} - \left(45\, r^2\, r3^2 \cos[\lambda-\lambda 3]\cos[\varphi]^2\cos[\varphi 3]^2\sin[\lambda-\lambda 3]\left(2\,r-2\,r3\left(\cos[\lambda-\lambda 3]\cos[\varphi]\cos[\varphi 3]+\sin[\varphi]\sin[\varphi 3]\right)\right)\right) \Big/ \left(2\left(r^2+r3^2-2\,r\,r3\left(\cos[\lambda-\lambda 3]\cos[\varphi]\cos[\varphi 3]+\sin[\varphi]\sin[\varphi 3]\right)\right)^{7/2}\right) + \frac{18\, r\, r3^2 \cos[\lambda-\lambda 3]\cos[\varphi]^2\cos[\varphi 3]^2\sin[\lambda-\lambda 3]}{\left(r^2+r3^2-2\,r\,r3\left(\cos[\lambda-\lambda 3]\cos[\varphi]\cos[\varphi 3]+\sin[\varphi]\sin[\varphi 3]\right)\right)^{5/2}} - \left(3\, r\, r3\cos[\varphi]\cos[\varphi 3]\sin[\lambda-\lambda 3]\left(2\,r-2\,r3\left(\cos[\lambda-\lambda 3]\cos[\varphi]\cos[\varphi 3]+\sin[\varphi]\sin[\varphi 3]\right)\right)\right) \Big/ \left(2\left(r^2+r3^2-2\,r\,r3\left(\cos[\lambda-\lambda 3]\cos[\varphi]\cos[\varphi 3]+\sin[\varphi]\sin[\varphi 3]\right)\right)^{5/2}\right) + \frac{r3\cos[\varphi]\cos[\varphi 3]\sin[\lambda-\lambda 3]}{\left(r^2+r3^2-2\,r\,r3\left(\cos[\lambda-\lambda 3]\cos[\varphi]\cos[\varphi 3]+\sin[\varphi]\sin[\varphi 3]\right)\right)^{3/2}} \right)$$

In[•]:=
```
V1222 = D[kernel, {λ, 1}, {φ, 3}]
```

Out[•]=
$$\frac{9\, r^2\, r3^4 \cos[\varphi]\, \cos[\varphi 3]^2 \sin[\lambda-\lambda 3]\, (-\cos[\lambda-\lambda 3]\, \cos[\varphi 3]\, \sin[\varphi]+\cos[\varphi]\, \sin[\varphi 3])}{\left(r^2+r3^2-2\, r\, r3\, (\cos[\lambda-\lambda 3]\, \cos[\varphi]\, \cos[\varphi 3]+\sin[\varphi]\, \sin[\varphi 3])\right)^{5/2}}-$$
$$\frac{r\, r3^3 \cos[\varphi 3]^2 \sin[\lambda-\lambda 3]\, \sin[\varphi]}{\left(r^2+r3^2-2\, r\, r3\, (\cos[\lambda-\lambda 3]\, \cos[\varphi]\, \cos[\varphi 3]+\sin[\varphi]\, \sin[\varphi 3])\right)^{3/2}}-$$
$$r\, r3^3 \cos[\varphi]\, \cos[\varphi 3]^2 \sin[\lambda-\lambda 3]$$
$$\left(\frac{105\, r^3\, r3^3\, (-\cos[\lambda-\lambda 3]\, \cos[\varphi 3]\, \sin[\varphi]+\cos[\varphi]\, \sin[\varphi 3])^3}{\left(r^2+r3^2-2\, r\, r3\, (\cos[\lambda-\lambda 3]\, \cos[\varphi]\, \cos[\varphi 3]+\sin[\varphi]\, \sin[\varphi 3])\right)^{9/2}}+\right.$$
$$\left(45\, r^2\, r3^2\, (-\cos[\lambda-\lambda 3]\, \cos[\varphi 3]\, \sin[\varphi]+\cos[\varphi]\, \sin[\varphi 3])\right.$$
$$\left.(-\cos[\lambda-\lambda 3]\, \cos[\varphi]\, \cos[\varphi 3]-\sin[\varphi]\, \sin[\varphi 3])\right)\Big/$$
$$\left(r^2+r3^2-2\, r\, r3\, (\cos[\lambda-\lambda 3]\, \cos[\varphi]\, \cos[\varphi 3]+\sin[\varphi]\, \sin[\varphi 3])\right)^{7/2}+$$
$$\left.\frac{3\, r\, r3\, (\cos[\lambda-\lambda 3]\, \cos[\varphi 3]\, \sin[\varphi]-\cos[\varphi]\, \sin[\varphi 3])}{\left(r^2+r3^2-2\, r\, r3\, (\cos[\lambda-\lambda 3]\, \cos[\varphi]\, \cos[\varphi 3]+\sin[\varphi]\, \sin[\varphi 3])\right)^{5/2}}\right)+$$
$$3\, r\, r3^3 \cos[\varphi 3]^2 \sin[\lambda-\lambda 3]\, \sin[\varphi]$$
$$\left(\frac{15\, r^2\, r3^2\, (-\cos[\lambda-\lambda 3]\, \cos[\varphi 3]\, \sin[\varphi]+\cos[\varphi]\, \sin[\varphi 3])^2}{\left(r^2+r3^2-2\, r\, r3\, (\cos[\lambda-\lambda 3]\, \cos[\varphi]\, \cos[\varphi 3]+\sin[\varphi]\, \sin[\varphi 3])\right)^{7/2}}+\right.$$
$$\left.\frac{3\, r\, r3\, (-\cos[\lambda-\lambda 3]\, \cos[\varphi]\, \cos[\varphi 3]-\sin[\varphi]\, \sin[\varphi 3])}{\left(r^2+r3^2-2\, r\, r3\, (\cos[\lambda-\lambda 3]\, \cos[\varphi]\, \cos[\varphi 3]+\sin[\varphi]\, \sin[\varphi 3])\right)^{5/2}}\right)$$

In[•]:=
```
V1333 = D[kernel, {λ, 1}, {r, 3}]
```

Out[•]=
$$-r\, r3^3 \cos[\varphi]\, \cos[\varphi 3]^2 \sin[\lambda-\lambda 3]$$
$$\left(-\frac{105\, (2\, r-2\, r3\, (\cos[\lambda-\lambda 3]\, \cos[\varphi]\, \cos[\varphi 3]+\sin[\varphi]\, \sin[\varphi 3]))^3}{8\left(r^2+r3^2-2\, r\, r3\, (\cos[\lambda-\lambda 3]\, \cos[\varphi]\, \cos[\varphi 3]+\sin[\varphi]\, \sin[\varphi 3])\right)^{9/2}}+\right.$$
$$\left.\frac{45\, (2\, r-2\, r3\, (\cos[\lambda-\lambda 3]\, \cos[\varphi]\, \cos[\varphi 3]+\sin[\varphi]\, \sin[\varphi 3]))}{2\left(r^2+r3^2-2\, r\, r3\, (\cos[\lambda-\lambda 3]\, \cos[\varphi]\, \cos[\varphi 3]+\sin[\varphi]\, \sin[\varphi 3])\right)^{7/2}}\right)-$$
$$3\, r3^3 \cos[\varphi]\, \cos[\varphi 3]^2 \sin[\lambda-\lambda 3]$$
$$\left(\frac{15\, (2\, r-2\, r3\, (\cos[\lambda-\lambda 3]\, \cos[\varphi]\, \cos[\varphi 3]+\sin[\varphi]\, \sin[\varphi 3]))^2}{4\left(r^2+r3^2-2\, r\, r3\, (\cos[\lambda-\lambda 3]\, \cos[\varphi]\, \cos[\varphi 3]+\sin[\varphi]\, \sin[\varphi 3])\right)^{7/2}}-\right.$$
$$\left.\frac{3}{\left(r^2+r3^2-2\, r\, r3\, (\cos[\lambda-\lambda 3]\, \cos[\varphi]\, \cos[\varphi 3]+\sin[\varphi]\, \sin[\varphi 3])\right)^{5/2}}\right)$$

In[•]:= V_{2223} = D[kernel, {φ, 3}, {r, 1}]

Out[•]=
$$r3^2 \cos[\varphi 3]\left(-\left(\left(105\, r^3\, r3^3\, (-\cos[\lambda-\lambda 3]\cos[\varphi 3]\sin[\varphi]+\cos[\varphi]\sin[\varphi 3])^3\right.\right.\right.$$
$$\left.\left.\left.(2r-2\,r3\,(\cos[\lambda-\lambda 3]\cos[\varphi]\cos[\varphi 3]+\sin[\varphi]\sin[\varphi 3]))\right)\Big/\right.$$
$$\left.\left(2\left(r^2+r3^2-2\,r\,r3\,(\cos[\lambda-\lambda 3]\cos[\varphi]\cos[\varphi 3]+\sin[\varphi]\sin[\varphi 3])\right)^{9/2}\right)\right)+$$
$$\frac{45\, r^2\, r3^3\,(-\cos[\lambda-\lambda 3]\cos[\varphi 3]\sin[\varphi]+\cos[\varphi]\sin[\varphi 3])^3}{\left(r^2+r3^2-2\,r\,r3\,(\cos[\lambda-\lambda 3]\cos[\varphi]\cos[\varphi 3]+\sin[\varphi]\sin[\varphi 3])\right)^{7/2}}-$$
$$\left(45\, r^2\, r3^2\,(-\cos[\lambda-\lambda 3]\cos[\varphi 3]\sin[\varphi]+\cos[\varphi]\sin[\varphi 3])\right.$$
$$(-\cos[\lambda-\lambda 3]\cos[\varphi]\cos[\varphi 3]-\sin[\varphi]\sin[\varphi 3])$$
$$\left.(2r-2\,r3\,(\cos[\lambda-\lambda 3]\cos[\varphi]\cos[\varphi 3]+\sin[\varphi]\sin[\varphi 3]))\right)\Big/$$
$$\left(2\left(r^2+r3^2-2\,r\,r3\,(\cos[\lambda-\lambda 3]\cos[\varphi]\cos[\varphi 3]+\sin[\varphi]\sin[\varphi 3])\right)^{7/2}\right)+$$
$$\left(18\, r\, r3^2\,(-\cos[\lambda-\lambda 3]\cos[\varphi 3]\sin[\varphi]+\cos[\varphi]\sin[\varphi 3])\right.$$
$$\left.(-\cos[\lambda-\lambda 3]\cos[\varphi]\cos[\varphi 3]-\sin[\varphi]\sin[\varphi 3])\right)\Big/$$
$$\left(r^2+r3^2-2\,r\,r3\,(\cos[\lambda-\lambda 3]\cos[\varphi]\cos[\varphi 3]+\sin[\varphi]\sin[\varphi 3])\right)^{5/2}-$$
$$(3\, r\, r3\,(\cos[\lambda-\lambda 3]\cos[\varphi 3]\sin[\varphi]-\cos[\varphi]\sin[\varphi 3])$$
$$\left.(2r-2\,r3\,(\cos[\lambda-\lambda 3]\cos[\varphi]\cos[\varphi 3]+\sin[\varphi]\sin[\varphi 3]))\right)\Big/$$
$$\left(2\left(r^2+r3^2-2\,r\,r3\,(\cos[\lambda-\lambda 3]\cos[\varphi]\cos[\varphi 3]+\sin[\varphi]\sin[\varphi 3])\right)^{5/2}\right)+$$
$$\left.\frac{r3\,(\cos[\lambda-\lambda 3]\cos[\varphi 3]\sin[\varphi]-\cos[\varphi]\sin[\varphi 3])}{\left(r^2+r3^2-2\,r\,r3\,(\cos[\lambda-\lambda 3]\cos[\varphi]\cos[\varphi 3]+\sin[\varphi]\sin[\varphi 3])\right)^{3/2}}\right)$$

In[•]:= V_{2333} = D[kernel, {φ, 1}, {r, 3}]

Out[•]=
$$r\, r3^3\cos[\varphi 3]\,(-\cos[\lambda-\lambda 3]\cos[\varphi 3]\sin[\varphi]+\cos[\varphi]\sin[\varphi 3])$$
$$\left(-\frac{105\,(2r-2\,r3\,(\cos[\lambda-\lambda 3]\cos[\varphi]\cos[\varphi 3]+\sin[\varphi]\sin[\varphi 3]))^3}{8\left(r^2+r3^2-2\,r\,r3\,(\cos[\lambda-\lambda 3]\cos[\varphi]\cos[\varphi 3]+\sin[\varphi]\sin[\varphi 3])\right)^{9/2}}+\right.$$
$$\left.\frac{45\,(2r-2\,r3\,(\cos[\lambda-\lambda 3]\cos[\varphi]\cos[\varphi 3]+\sin[\varphi]\sin[\varphi 3]))}{2\left(r^2+r3^2-2\,r\,r3\,(\cos[\lambda-\lambda 3]\cos[\varphi]\cos[\varphi 3]+\sin[\varphi]\sin[\varphi 3])\right)^{7/2}}\right)+$$
$$3\, r3^3\cos[\varphi 3]\,(-\cos[\lambda-\lambda 3]\cos[\varphi 3]\sin[\varphi]+\cos[\varphi]\sin[\varphi 3])$$
$$\left(\frac{15\,(2r-2\,r3\,(\cos[\lambda-\lambda 3]\cos[\varphi]\cos[\varphi 3]+\sin[\varphi]\sin[\varphi 3]))^2}{4\left(r^2+r3^2-2\,r\,r3\,(\cos[\lambda-\lambda 3]\cos[\varphi]\cos[\varphi 3]+\sin[\varphi]\sin[\varphi 3])\right)^{7/2}}-\right.$$
$$\left.\frac{3}{\left(r^2+r3^2-2\,r\,r3\,(\cos[\lambda-\lambda 3]\cos[\varphi]\cos[\varphi 3]+\sin[\varphi]\sin[\varphi 3])\right)^{5/2}}\right)$$

In[•]:= V_{1122} = D[kernel, {λ, 2}, {φ, 2}]

Out[•]=

$$r3^2 \cos[\varphi 3] \left(-\left(\left(60\, r^3\, r3^3 \cos[\varphi] \cos[\varphi 3]^2 \sin[\lambda - \lambda 3]^2 \right.\right.\right.$$

$$\left.\left.\left. \sin[\varphi]\, (-\cos[\lambda - \lambda 3] \cos[\varphi 3] \sin[\varphi] + \cos[\varphi] \sin[\varphi 3]) \right) \Big/ \right.$$

$$\left. \left(r^2 + r3^2 - 2\, r\, r3\, (\cos[\lambda - \lambda 3] \cos[\varphi] \cos[\varphi 3] + \sin[\varphi] \sin[\varphi 3]) \right)^{7/2} \right) +$$

$$\frac{3\, r^2\, r3^2 \cos[\varphi 3]^2 \sin[\lambda - \lambda 3]^2 \left(-2 \cos[\varphi]^2 + 2 \sin[\varphi]^2 \right)}{\left(r^2 + r3^2 - 2\, r\, r3\, (\cos[\lambda - \lambda 3] \cos[\varphi] \cos[\varphi 3] + \sin[\varphi] \sin[\varphi 3]) \right)^{5/2}} +$$

$$3\, r^2\, r3^2 \cos[\varphi]^2 \cos[\varphi 3]^2 \sin[\lambda - \lambda 3]^2$$

$$\left(\frac{35\, r^2\, r3^2\, (-\cos[\lambda - \lambda 3] \cos[\varphi 3] \sin[\varphi] + \cos[\varphi] \sin[\varphi 3])^2}{\left(r^2 + r3^2 - 2\, r\, r3\, (\cos[\lambda - \lambda 3] \cos[\varphi] \cos[\varphi 3] + \sin[\varphi] \sin[\varphi 3]) \right)^{9/2}} + \right.$$

$$\left. \frac{5\, r\, r3\, (-\cos[\lambda - \lambda 3] \cos[\varphi] \cos[\varphi 3] - \sin[\varphi] \sin[\varphi 3])}{\left(r^2 + r3^2 - 2\, r\, r3\, (\cos[\lambda - \lambda 3] \cos[\varphi] \cos[\varphi 3] + \sin[\varphi] \sin[\varphi 3]) \right)^{7/2}} \right) - r\, r3 \cos[$$

$$\lambda - \lambda 3] \left(-\frac{6\, r\, r3 \cos[\varphi 3] \sin[\varphi]\, (-\cos[\lambda - \lambda 3] \cos[\varphi 3] \sin[\varphi] + \cos[\varphi] \sin[\varphi 3])}{\left(r^2 + r3^2 - 2\, r\, r3\, (\cos[\lambda - \lambda 3] \cos[\varphi] \cos[\varphi 3] + \sin[\varphi] \sin[\varphi 3]) \right)^{5/2}} - \right.$$

$$\frac{\cos[\varphi] \cos[\varphi 3]}{\left(r^2 + r3^2 - 2\, r\, r3\, (\cos[\lambda - \lambda 3] \cos[\varphi] \cos[\varphi 3] + \sin[\varphi] \sin[\varphi 3]) \right)^{3/2}} + \cos[\varphi]$$

$$\cos[\varphi 3] \left(\frac{15\, r^2\, r3^2\, (-\cos[\lambda - \lambda 3] \cos[\varphi 3] \sin[\varphi] + \cos[\varphi] \sin[\varphi 3])^2}{\left(r^2 + r3^2 - 2\, r\, r3\, (\cos[\lambda - \lambda 3] \cos[\varphi] \cos[\varphi 3] + \sin[\varphi] \sin[\varphi 3]) \right)^{7/2}} + \right.$$

$$\left. \left. \left. \frac{3\, r\, r3\, (-\cos[\lambda - \lambda 3] \cos[\varphi] \cos[\varphi 3] - \sin[\varphi] \sin[\varphi 3])}{\left(r^2 + r3^2 - 2\, r\, r3\, (\cos[\lambda - \lambda 3] \cos[\varphi] \cos[\varphi 3] + \sin[\varphi] \sin[\varphi 3]) \right)^{5/2}} \right) \right) \right)$$

In[•]:=
```
V1133 = D[kernel, {λ, 2}, {r, 2}]
```

Out[•]=
$$
\begin{aligned}
&\mathrm{r3}^2 \cos[\varphi 3] \left(-\left(\left(30\, \mathrm{r}\, \mathrm{r3}^2 \cos[\varphi]^2 \cos[\varphi 3]^2 \sin[\lambda - \lambda 3]^2 \right.\right.\right. \\
&\qquad \left.\left.\left. (2\, \mathrm{r} - 2\, \mathrm{r3}\, (\cos[\lambda - \lambda 3] \cos[\varphi] \cos[\varphi 3] + \sin[\varphi] \sin[\varphi 3])) \right) \Big/ \right. \\
&\qquad \left. \left(\mathrm{r}^2 + \mathrm{r3}^2 - 2\, \mathrm{r}\, \mathrm{r3}\, (\cos[\lambda - \lambda 3] \cos[\varphi] \cos[\varphi 3] + \sin[\varphi] \sin[\varphi 3]) \right)^{7/2} \right) + \\
&\qquad \frac{6\, \mathrm{r3}^2 \cos[\varphi]^2 \cos[\varphi 3]^2 \sin[\lambda - \lambda 3]^2}{\left(\mathrm{r}^2 + \mathrm{r3}^2 - 2\, \mathrm{r}\, \mathrm{r3}\, (\cos[\lambda - \lambda 3] \cos[\varphi] \cos[\varphi 3] + \sin[\varphi] \sin[\varphi 3]) \right)^{5/2}} + \\
&\qquad (3\, \mathrm{r3} \cos[\lambda - \lambda 3] \cos[\varphi] \cos[\varphi 3] \\
&\qquad\quad (2\, \mathrm{r} - 2\, \mathrm{r3}\, (\cos[\lambda - \lambda 3] \cos[\varphi] \cos[\varphi 3] + \sin[\varphi] \sin[\varphi 3]))) \Big/ \\
&\qquad \left(\mathrm{r}^2 + \mathrm{r3}^2 - 2\, \mathrm{r}\, \mathrm{r3}\, (\cos[\lambda - \lambda 3] \cos[\varphi] \cos[\varphi 3] + \sin[\varphi] \sin[\varphi 3]) \right)^{5/2} + \\
&\qquad 3\, \mathrm{r}^2\, \mathrm{r3}^2 \cos[\varphi]^2 \cos[\varphi 3]^2 \sin[\lambda - \lambda 3]^2 \\
&\qquad\quad \left(\frac{35\, (2\, \mathrm{r} - 2\, \mathrm{r3}\, (\cos[\lambda - \lambda 3] \cos[\varphi] \cos[\varphi 3] + \sin[\varphi] \sin[\varphi 3]))^2}{4 \left(\mathrm{r}^2 + \mathrm{r3}^2 - 2\, \mathrm{r}\, \mathrm{r3}\, (\cos[\lambda - \lambda 3] \cos[\varphi] \cos[\varphi 3] + \sin[\varphi] \sin[\varphi 3]) \right)^{9/2}} - \right. \\
&\qquad\qquad \left. \frac{5}{\left(\mathrm{r}^2 + \mathrm{r3}^2 - 2\, \mathrm{r}\, \mathrm{r3}\, (\cos[\lambda - \lambda 3] \cos[\varphi] \cos[\varphi 3] + \sin[\varphi] \sin[\varphi 3]) \right)^{7/2}} \right) - \\
&\qquad \mathrm{r}\, \mathrm{r3} \cos[\lambda - \lambda 3] \cos[\varphi] \cos[\varphi 3] \\
&\qquad\quad \left(\frac{15\, (2\, \mathrm{r} - 2\, \mathrm{r3}\, (\cos[\lambda - \lambda 3] \cos[\varphi] \cos[\varphi 3] + \sin[\varphi] \sin[\varphi 3]))^2}{4 \left(\mathrm{r}^2 + \mathrm{r3}^2 - 2\, \mathrm{r}\, \mathrm{r3}\, (\cos[\lambda - \lambda 3] \cos[\varphi] \cos[\varphi 3] + \sin[\varphi] \sin[\varphi 3]) \right)^{7/2}} - \right. \\
&\qquad\qquad \left. \left. \frac{3}{\left(\mathrm{r}^2 + \mathrm{r3}^2 - 2\, \mathrm{r}\, \mathrm{r3}\, (\cos[\lambda - \lambda 3] \cos[\varphi] \cos[\varphi 3] + \sin[\varphi] \sin[\varphi 3]) \right)^{5/2}} \right) \right)
\end{aligned}
$$

In[•]:=
```
V2233 = D[kernel, {φ, 2}, {r, 2}]
```

Out[•]=

$$
\begin{aligned}
&\mathrm{r3}^2 \cos[\varphi 3] \left(- \left(\left(30\, \mathrm{r}\, \mathrm{r3}^2 \left(-\cos[\lambda - \lambda 3] \cos[\varphi 3] \sin[\varphi] + \cos[\varphi] \sin[\varphi 3] \right)^2 \right.\right.\right. \\
&\qquad \left.\left.\left. \left(2\,\mathrm{r} - 2\,\mathrm{r3} \left(\cos[\lambda - \lambda 3] \cos[\varphi] \cos[\varphi 3] + \sin[\varphi] \sin[\varphi 3] \right) \right) \right) \Big/ \right. \\
&\qquad \left. \left(\mathrm{r}^2 + \mathrm{r3}^2 - 2\,\mathrm{r}\,\mathrm{r3} \left(\cos[\lambda - \lambda 3] \cos[\varphi] \cos[\varphi 3] + \sin[\varphi] \sin[\varphi 3] \right) \right)^{7/2} \right) + \\
&\qquad \frac{6\, \mathrm{r3}^2 \left(-\cos[\lambda - \lambda 3] \cos[\varphi 3] \sin[\varphi] + \cos[\varphi] \sin[\varphi 3] \right)^2}{\left(\mathrm{r}^2 + \mathrm{r3}^2 - 2\,\mathrm{r}\,\mathrm{r3} \left(\cos[\lambda - \lambda 3] \cos[\varphi] \cos[\varphi 3] + \sin[\varphi] \sin[\varphi 3] \right) \right)^{5/2}} - \\
&\qquad \left(3\,\mathrm{r3} \left(-\cos[\lambda - \lambda 3] \cos[\varphi] \cos[\varphi 3] - \sin[\varphi] \sin[\varphi 3] \right) \right. \\
&\qquad \left. \left(2\,\mathrm{r} - 2\,\mathrm{r3} \left(\cos[\lambda - \lambda 3] \cos[\varphi] \cos[\varphi 3] + \sin[\varphi] \sin[\varphi 3] \right) \right) \right) \Big/ \\
&\qquad \left(\mathrm{r}^2 + \mathrm{r3}^2 - 2\,\mathrm{r}\,\mathrm{r3} \left(\cos[\lambda - \lambda 3] \cos[\varphi] \cos[\varphi 3] + \sin[\varphi] \sin[\varphi 3] \right) \right)^{5/2} + \\
&\qquad 3\,\mathrm{r}^2\,\mathrm{r3}^2 \left(-\cos[\lambda - \lambda 3] \cos[\varphi 3] \sin[\varphi] + \cos[\varphi] \sin[\varphi 3] \right)^2 \\
&\qquad \left(\frac{35 \left(2\,\mathrm{r} - 2\,\mathrm{r3} \left(\cos[\lambda - \lambda 3] \cos[\varphi] \cos[\varphi 3] + \sin[\varphi] \sin[\varphi 3] \right) \right)^2}{4 \left(\mathrm{r}^2 + \mathrm{r3}^2 - 2\,\mathrm{r}\,\mathrm{r3} \left(\cos[\lambda - \lambda 3] \cos[\varphi] \cos[\varphi 3] + \sin[\varphi] \sin[\varphi 3] \right) \right)^{9/2}} - \right. \\
&\qquad \left. \frac{5}{\left(\mathrm{r}^2 + \mathrm{r3}^2 - 2\,\mathrm{r}\,\mathrm{r3} \left(\cos[\lambda - \lambda 3] \cos[\varphi] \cos[\varphi 3] + \sin[\varphi] \sin[\varphi 3] \right) \right)^{7/2}} \right) + \\
&\qquad \mathrm{r}\,\mathrm{r3} \left(-\cos[\lambda - \lambda 3] \cos[\varphi] \cos[\varphi 3] - \sin[\varphi] \sin[\varphi 3] \right) \\
&\qquad \left(\frac{15 \left(2\,\mathrm{r} - 2\,\mathrm{r3} \left(\cos[\lambda - \lambda 3] \cos[\varphi] \cos[\varphi 3] + \sin[\varphi] \sin[\varphi 3] \right) \right)^2}{4 \left(\mathrm{r}^2 + \mathrm{r3}^2 - 2\,\mathrm{r}\,\mathrm{r3} \left(\cos[\lambda - \lambda 3] \cos[\varphi] \cos[\varphi 3] + \sin[\varphi] \sin[\varphi 3] \right) \right)^{7/2}} - \right. \\
&\qquad \left. \left. \frac{3}{\left(\mathrm{r}^2 + \mathrm{r3}^2 - 2\,\mathrm{r}\,\mathrm{r3} \left(\cos[\lambda - \lambda 3] \cos[\varphi] \cos[\varphi 3] + \sin[\varphi] \sin[\varphi 3] \right) \right)^{5/2}} \right) \right)
\end{aligned}
$$

In[•]:= V_{1123} = D[kernel, {λ, 2}, {φ, 1}, {r, 1}]

Out[•]=

$$r3^2 \cos[\varphi 3]$$

$$\Bigg(-\Bigg(\Big(105\, r^3\, r3^3 \cos[\varphi]^2 \cos[\varphi 3]^2 \sin[\lambda-\lambda 3]^2 \,(-\cos[\lambda-\lambda 3]\cos[\varphi 3]\sin[\varphi]+\cos[\varphi]\sin[\varphi 3])\,(2\,r-2\,r3\,(\cos[\lambda-\lambda 3]\cos[\varphi]\cos[\varphi 3]+\sin[\varphi]\sin[\varphi 3]))\Big)\Big/\Big(2\left(r^2+r3^2-2\,r\,r3\,(\cos[\lambda-\lambda 3]\cos[\varphi]\cos[\varphi 3]+\sin[\varphi]\sin[\varphi 3])\right)^{9/2}\Big)\Big)+$$

$$\Big(45\, r^2\, r3^3 \cos[\varphi]^2 \cos[\varphi 3]^2 \sin[\lambda-\lambda 3]^2\,(-\cos[\lambda-\lambda 3]\cos[\varphi 3]\sin[\varphi]+\cos[\varphi]\sin[\varphi 3])\Big)\Big/\Big(r^2+r3^2-2\,r\,r3\,(\cos[\lambda-\lambda 3]\cos[\varphi]\cos[\varphi 3]+\sin[\varphi]\sin[\varphi 3])\Big)^{7/2}+$$

$$\Big(15\, r^2\, r3^2 \cos[\varphi]\cos[\varphi 3]^2 \sin[\lambda-\lambda 3]^2 \sin[\varphi]\,(2\,r-2\,r3\,(\cos[\lambda-\lambda 3]\cos[\varphi]\cos[\varphi 3]+\sin[\varphi]\sin[\varphi 3]))\Big)\Big/\Big(r^2+r3^2-2\,r\,r3\,(\cos[\lambda-\lambda 3]\cos[\varphi]\cos[\varphi 3]+\sin[\varphi]\sin[\varphi 3])\Big)^{7/2}+$$

$$\Big(15\, r^2\, r3^2 \cos[\lambda-\lambda 3]\cos[\varphi]\cos[\varphi 3]\,(-\cos[\lambda-\lambda 3]\cos[\varphi 3]\sin[\varphi]+\cos[\varphi]\sin[\varphi 3])\,(2\,r-2\,r3\,(\cos[\lambda-\lambda 3]\cos[\varphi]\cos[\varphi 3]+\sin[\varphi]\sin[\varphi 3]))\Big)\Big/\Big(2\left(r^2+r3^2-2\,r\,r3\,(\cos[\lambda-\lambda 3]\cos[\varphi]\cos[\varphi 3]+\sin[\varphi]\sin[\varphi 3])\right)^{7/2}\Big)-$$

$$\frac{12\, r\, r3^2 \cos[\varphi]\cos[\varphi 3]^2 \sin[\lambda-\lambda 3]^2 \sin[\varphi]}{\left(r^2+r3^2-2\,r\,r3\,(\cos[\lambda-\lambda 3]\cos[\varphi]\cos[\varphi 3]+\sin[\varphi]\sin[\varphi 3])\right)^{5/2}}-$$

$$\frac{6\, r\, r3^2 \cos[\lambda-\lambda 3]\cos[\varphi]\cos[\varphi 3]\,(-\cos[\lambda-\lambda 3]\cos[\varphi 3]\sin[\varphi]+\cos[\varphi]\sin[\varphi 3])}{\left(r^2+r3^2-2\,r\,r3\,(\cos[\lambda-\lambda 3]\cos[\varphi]\cos[\varphi 3]+\sin[\varphi]\sin[\varphi 3])\right)^{5/2}}-$$

$$\Big(3\, r\, r3 \cos[\lambda-\lambda 3]\cos[\varphi 3]\sin[\varphi]\,(2\,r-2\,r3\,(\cos[\lambda-\lambda 3]\cos[\varphi]\cos[\varphi 3]+\sin[\varphi]\sin[\varphi 3]))\Big)\Big/\Big(2\left(r^2+r3^2-2\,r\,r3\,(\cos[\lambda-\lambda 3]\cos[\varphi]\cos[\varphi 3]+\sin[\varphi]\sin[\varphi 3])\right)^{5/2}\Big)+$$

$$\frac{r3\cos[\lambda-\lambda 3]\cos[\varphi 3]\sin[\varphi]}{\left(r^2+r3^2-2\,r\,r3\,(\cos[\lambda-\lambda 3]\cos[\varphi]\cos[\varphi 3]+\sin[\varphi]\sin[\varphi 3])\right)^{3/2}}\Bigg)$$

In[•]:=
```
V1223 = D[kernel, {λ, 1}, {φ, 2}, {r, 1}]
```

Out[•]=

$$
-\left(\left(15\, r^2\, r3^4\, \text{Cos}[\varphi 3]^2\, \text{Sin}[\lambda-\lambda 3]\, \text{Sin}[\varphi]\, (-\text{Cos}[\lambda-\lambda 3]\, \text{Cos}[\varphi 3]\, \text{Sin}[\varphi]+\text{Cos}[\varphi]\, \text{Sin}[\varphi 3])\right.\right.
$$
$$
\left.\left.(2\, r-2\, r3\, (\text{Cos}[\lambda-\lambda 3]\, \text{Cos}[\varphi]\, \text{Cos}[\varphi 3]+\text{Sin}[\varphi]\, \text{Sin}[\varphi 3]))\right)\Big/
$$
$$
\left(r^2+r3^2-2\, r\, r3\, (\text{Cos}[\lambda-\lambda 3]\, \text{Cos}[\varphi]\, \text{Cos}[\varphi 3]+\text{Sin}[\varphi]\, \text{Sin}[\varphi 3])\right)^{7/2}\Big)+
$$
$$
\frac{12\, r\, r3^4\, \text{Cos}[\varphi 3]^2\, \text{Sin}[\lambda-\lambda 3]\, \text{Sin}[\varphi]\, (-\text{Cos}[\lambda-\lambda 3]\, \text{Cos}[\varphi 3]\, \text{Sin}[\varphi]+\text{Cos}[\varphi]\, \text{Sin}[\varphi 3])}{\left(r^2+r3^2-2\, r\, r3\, (\text{Cos}[\lambda-\lambda 3]\, \text{Cos}[\varphi]\, \text{Cos}[\varphi 3]+\text{Sin}[\varphi]\, \text{Sin}[\varphi 3])\right)^{5/2}}-
$$
$$
\left(3\, r\, r3^3\, \text{Cos}[\varphi]\, \text{Cos}[\varphi 3]^2\, \text{Sin}[\lambda-\lambda 3]\right.
$$
$$
\left.(2\, r-2\, r3\, (\text{Cos}[\lambda-\lambda 3]\, \text{Cos}[\varphi]\, \text{Cos}[\varphi 3]+\text{Sin}[\varphi]\, \text{Sin}[\varphi 3]))\right)\Big/
$$
$$
\left(2\left(r^2+r3^2-2\, r\, r3\, (\text{Cos}[\lambda-\lambda 3]\, \text{Cos}[\varphi]\, \text{Cos}[\varphi 3]+\text{Sin}[\varphi]\, \text{Sin}[\varphi 3])\right)^{5/2}\right)+
$$
$$
\frac{r3^3\, \text{Cos}[\varphi]\, \text{Cos}[\varphi 3]^2\, \text{Sin}[\lambda-\lambda 3]}{\left(r^2+r3^2-2\, r\, r3\, (\text{Cos}[\lambda-\lambda 3]\, \text{Cos}[\varphi]\, \text{Cos}[\varphi 3]+\text{Sin}[\varphi]\, \text{Sin}[\varphi 3])\right)^{3/2}}-
$$
$$
r\, r3^3\, \text{Cos}[\varphi]\, \text{Cos}[\varphi 3]^2\, \text{Sin}[\lambda-\lambda 3]
$$
$$
\left(-\left(\left(105\, r^2\, r3^2\, (-\text{Cos}[\lambda-\lambda 3]\, \text{Cos}[\varphi 3]\, \text{Sin}[\varphi]+\text{Cos}[\varphi]\, \text{Sin}[\varphi 3])^2\right.\right.\right.
$$
$$
\left.\left.(2\, r-2\, r3\, (\text{Cos}[\lambda-\lambda 3]\, \text{Cos}[\varphi]\, \text{Cos}[\varphi 3]+\text{Sin}[\varphi]\, \text{Sin}[\varphi 3]))\right)\Big/
$$
$$
\left.\left(2\left(r^2+r3^2-2\, r\, r3\, (\text{Cos}[\lambda-\lambda 3]\, \text{Cos}[\varphi]\, \text{Cos}[\varphi 3]+\text{Sin}[\varphi]\, \text{Sin}[\varphi 3])\right)^{9/2}\right)\right)+
$$
$$
\frac{30\, r\, r3^2\, (-\text{Cos}[\lambda-\lambda 3]\, \text{Cos}[\varphi 3]\, \text{Sin}[\varphi]+\text{Cos}[\varphi]\, \text{Sin}[\varphi 3])^2}{\left(r^2+r3^2-2\, r\, r3\, (\text{Cos}[\lambda-\lambda 3]\, \text{Cos}[\varphi]\, \text{Cos}[\varphi 3]+\text{Sin}[\varphi]\, \text{Sin}[\varphi 3])\right)^{7/2}}-
$$
$$
\left(15\, r\, r3\, (-\text{Cos}[\lambda-\lambda 3]\, \text{Cos}[\varphi]\, \text{Cos}[\varphi 3]-\text{Sin}[\varphi]\, \text{Sin}[\varphi 3])\right.
$$
$$
\left.(2\, r-2\, r3\, (\text{Cos}[\lambda-\lambda 3]\, \text{Cos}[\varphi]\, \text{Cos}[\varphi 3]+\text{Sin}[\varphi]\, \text{Sin}[\varphi 3]))\right)\Big/
$$
$$
\left(2\left(r^2+r3^2-2\, r\, r3\, (\text{Cos}[\lambda-\lambda 3]\, \text{Cos}[\varphi]\, \text{Cos}[\varphi 3]+\text{Sin}[\varphi]\, \text{Sin}[\varphi 3])\right)^{7/2}\right)+
$$
$$
\left.\left.\frac{3\, r3\, (-\text{Cos}[\lambda-\lambda 3]\, \text{Cos}[\varphi]\, \text{Cos}[\varphi 3]-\text{Sin}[\varphi]\, \text{Sin}[\varphi 3])}{\left(r^2+r3^2-2\, r\, r3\, (\text{Cos}[\lambda-\lambda 3]\, \text{Cos}[\varphi]\, \text{Cos}[\varphi 3]+\text{Sin}[\varphi]\, \text{Sin}[\varphi 3])\right)^{5/2}}\right)-
$$
$$
r3^3\, \text{Cos}[\varphi]\, \text{Cos}[\varphi 3]^2\, \text{Sin}[\lambda-\lambda 3]
$$
$$
\left(\frac{15\, r^2\, r3^2\, (-\text{Cos}[\lambda-\lambda 3]\, \text{Cos}[\varphi 3]\, \text{Sin}[\varphi]+\text{Cos}[\varphi]\, \text{Sin}[\varphi 3])^2}{\left(r^2+r3^2-2\, r\, r3\, (\text{Cos}[\lambda-\lambda 3]\, \text{Cos}[\varphi]\, \text{Cos}[\varphi 3]+\text{Sin}[\varphi]\, \text{Sin}[\varphi 3])\right)^{7/2}}+\right.
$$
$$
\left.\frac{3\, r\, r3\, (-\text{Cos}[\lambda-\lambda 3]\, \text{Cos}[\varphi]\, \text{Cos}[\varphi 3]-\text{Sin}[\varphi]\, \text{Sin}[\varphi 3])}{\left(r^2+r3^2-2\, r\, r3\, (\text{Cos}[\lambda-\lambda 3]\, \text{Cos}[\varphi]\, \text{Cos}[\varphi 3]+\text{Sin}[\varphi]\, \text{Sin}[\varphi 3])\right)^{5/2}}\right)
$$

In[•]:=
```
V1233 = D[kernel, {λ, 1}, {φ, 1}, {r, 2}]
```

Out[•]=
$$\left(30\, r\, r3^4 \cos[\varphi]\, \cos[\varphi 3]^2 \sin[\lambda-\lambda 3]\, (-\cos[\lambda-\lambda 3]\, \cos[\varphi 3]\, \sin[\varphi]+\cos[\varphi]\, \sin[\varphi 3])\right.$$
$$\left.(2\, r-2\, r3\, (\cos[\lambda-\lambda 3]\, \cos[\varphi]\, \cos[\varphi 3]+\sin[\varphi]\, \sin[\varphi 3]))\right)\Big/$$
$$\left(r^2+r3^2-2\, r\, r3\, (\cos[\lambda-\lambda 3]\, \cos[\varphi]\, \cos[\varphi 3]+\sin[\varphi]\, \sin[\varphi 3])\right)^{7/2}-$$
$$\frac{6\, r3^4 \cos[\varphi]\, \cos[\varphi 3]^2 \sin[\lambda-\lambda 3]\, (-\cos[\lambda-\lambda 3]\, \cos[\varphi 3]\, \sin[\varphi]+\cos[\varphi]\, \sin[\varphi 3])}{\left(r^2+r3^2-2\, r\, r3\, (\cos[\lambda-\lambda 3]\, \cos[\varphi]\, \cos[\varphi 3]+\sin[\varphi]\, \sin[\varphi 3])\right)^{5/2}}-$$
$$\left(3\, r3^3 \cos[\varphi 3]^2 \sin[\lambda-\lambda 3]\, \sin[\varphi]\right.$$
$$\left.(2\, r-2\, r3\, (\cos[\lambda-\lambda 3]\, \cos[\varphi]\, \cos[\varphi 3]+\sin[\varphi]\, \sin[\varphi 3]))\right)\Big/$$
$$\left(r^2+r3^2-2\, r\, r3\, (\cos[\lambda-\lambda 3]\, \cos[\varphi]\, \cos[\varphi 3]+\sin[\varphi]\, \sin[\varphi 3])\right)^{5/2}-$$
$$3\, r^2\, r3^4 \cos[\varphi]\, \cos[\varphi 3]^2 \sin[\lambda-\lambda 3]\, (-\cos[\lambda-\lambda 3]\, \cos[\varphi 3]\, \sin[\varphi]+\cos[\varphi]\, \sin[\varphi 3])$$
$$\left(\frac{35\, (2\, r-2\, r3\, (\cos[\lambda-\lambda 3]\, \cos[\varphi]\, \cos[\varphi 3]+\sin[\varphi]\, \sin[\varphi 3]))^2}{4\, \left(r^2+r3^2-2\, r\, r3\, (\cos[\lambda-\lambda 3]\, \cos[\varphi]\, \cos[\varphi 3]+\sin[\varphi]\, \sin[\varphi 3])\right)^{9/2}}-\right.$$
$$\left.\frac{5}{\left(r^2+r3^2-2\, r\, r3\, (\cos[\lambda-\lambda 3]\, \cos[\varphi]\, \cos[\varphi 3]+\sin[\varphi]\, \sin[\varphi 3])\right)^{7/2}}\right)+$$
$$r\, r3^3 \cos[\varphi 3]^2 \sin[\lambda-\lambda 3]\, \sin[\varphi]$$
$$\left(\frac{15\, (2\, r-2\, r3\, (\cos[\lambda-\lambda 3]\, \cos[\varphi]\, \cos[\varphi 3]+\sin[\varphi]\, \sin[\varphi 3]))^2}{4\, \left(r^2+r3^2-2\, r\, r3\, (\cos[\lambda-\lambda 3]\, \cos[\varphi]\, \cos[\varphi 3]+\sin[\varphi]\, \sin[\varphi 3])\right)^{7/2}}-\right.$$
$$\left.\frac{3}{\left(r^2+r3^2-2\, r\, r3\, (\cos[\lambda-\lambda 3]\, \cos[\varphi]\, \cos[\varphi 3]+\sin[\varphi]\, \sin[\varphi 3])\right)^{5/2}}\right)$$