Appendix D of Higher-order gravitational potential gradients by tensor analysis in spherical coordinates

1. Expressions for the 15 defining physical components of the fourth-order gravitational potential gradients in Appendix B as Eqs. (33) - Eqs. (47)

(*V(4,0,0) Eq. (33)*)

$$V400 = -\frac{3\,V_3}{r^3} - \frac{8\,\text{Sec}[\varphi]^4\,V_{11}}{r^4} + \frac{3\,V_{33}}{r^2} + \frac{6\,\text{Sec}[\varphi]^2\,V_{113}}{r^3} + \frac{\text{Sec}[\varphi]^4\,V_{1111}}{r^4} + \frac{6\,V_2\,\text{Tan}[\varphi]}{r^4} + \frac{3\,\text{Sec}[\varphi]^2\,V_2\,\text{Tan}[\varphi]}{r^4} - \frac{6\,V_{23}\,\text{Tan}[\varphi]}{r^3} - \frac{6\,\text{Sec}[\varphi]^2\,V_{112}\,\text{Tan}[\varphi]}{r^4} + \frac{3\,V_{22}\,\text{Tan}[\varphi]^2}{r^4};$$

(*V(0,4,0) Eq. (34)*)

$$V040 = -\frac{3\,V_3}{r^3} - \frac{8\,V_{22}}{r^4} + \frac{3\,V_{33}}{r^2} + \frac{6\,V_{223}}{r^3} + \frac{V_{2222}}{r^4};$$

(*V(0,0,4) Eq. (35)*)

$$V004 = V_{3333};$$

(*V(3,1,0) Eq. (36)*)

$$V310 = \frac{3\,\text{Sec}[\varphi]\,V_{12}}{r^4} - \frac{8\,\text{Sec}[\varphi]^3\,V_{12}}{r^4} + \frac{3\,\text{Sec}[\varphi]\,V_{123}}{r^3} + \frac{\text{Sec}[\varphi]^3\,V_{1112}}{r^4} - \frac{6\,\text{Sec}[\varphi]^3\,V_1\,\text{Tan}[\varphi]}{r^4} + \frac{3\,\text{Sec}[\varphi]\,V_{13}\,\text{Tan}[\varphi]}{r^3} + \frac{3\,\text{Sec}[\varphi]^3\,V_{111}\,\text{Tan}[\varphi]}{r^4} - \frac{3\,\text{Sec}[\varphi]\,V_{221}\,\text{Tan}[\varphi]}{r^4};$$

(*V(3,0,1) Eq. (37)*)

$$V301 = \frac{6\,\text{Sec}[\varphi]^3\,V_1}{r^4} - \frac{6\,\text{Sec}[\varphi]\,V_{13}}{r^3} - \frac{2\,\text{Sec}[\varphi]^3\,V_{13}}{r^3} - \frac{3\,\text{Sec}[\varphi]^3\,V_{111}}{r^4} + \frac{3\,\text{Sec}[\varphi]\,V_{331}}{r^2} + \frac{\text{Sec}[\varphi]^3\,V_{1113}}{r^3} + \frac{9\,\text{Sec}[\varphi]\,V_{12}\,\text{Tan}[\varphi]}{r^4} - \frac{3\,\text{Sec}[\varphi]\,V_{123}\,\text{Tan}[\varphi]}{r^3};$$

(*V(1,3,0) Eq. (38)*)

$$V130 = -\frac{8\,\text{Sec}[\varphi]\,V_{12}}{r^4} + \frac{6\,\text{Sec}[\varphi]^3\,V_{12}}{r^4} + \frac{3\,\text{Sec}[\varphi]\,V_{123}}{r^3} + \frac{\text{Sec}[\varphi]\,V_{1222}}{r^4} + \frac{3\,\text{Sec}[\varphi]\,V_{13}\,\text{Tan}[\varphi]}{r^3} + \frac{3\,\text{Sec}[\varphi]\,V_{221}\,\text{Tan}[\varphi]}{r^4} + \frac{6\,\text{Sec}[\varphi]\,V_1\,\text{Tan}[\varphi]^3}{r^4};$$

(*V(1,0,3) Eq. (39)*)

$$V103 = -\frac{6\,\text{Sec}[\varphi]\,V_1}{r^4} + \frac{6\,\text{Sec}[\varphi]\,V_{13}}{r^3} - \frac{3\,\text{Sec}[\varphi]\,V_{331}}{r^2} + \frac{\text{Sec}[\varphi]\,V_{1333}}{r};$$

(*V(0,3,1) Eq. (40)*)

$$V031 = \frac{6\,V_2}{r^4} - \frac{8\,V_{23}}{r^3} - \frac{3\,V_{222}}{r^4} + \frac{3\,V_{332}}{r^2} + \frac{V_{2223}}{r^3};$$

```
(*V(0,1,3) Eq. (41)*)
V013 = -6 V2/r^4 + 6 V23/r^3 - 3 V332/r^2 + V2333/r;

(*V(2,2,0) Eq. (42)*)
V220 = -V3/r^3 - 2 Sec[φ]^2 V22/r^4 + V33/r^2 + Sec[φ]^2 V113/r^3 + V223/r^3 + Sec[φ]^2 V1122/r^4 - V23 Tan[φ]/r^3 +
    4 Sec[φ]^2 V112 Tan[φ]/r^4 - V222 Tan[φ]/r^4 + 6 Sec[φ]^2 V11 Tan[φ]^2/r^4 - 2 V2 Tan[φ]^3/r^4;

(*V(2,0,2) Eq. (43)*)
V202 = 2 V3/r^3 + 6 Sec[φ]^2 V11/r^4 - 2 V33/r^2 - 4 Sec[φ]^2 V113/r^3 +
    V333/r + Sec[φ]^2 V1133/r^2 - 6 V2 Tan[φ]/r^4 + 4 V23 Tan[φ]/r^3 - V332 Tan[φ]/r^2;

(*V(0,2,2) Eq. (44)*)
V022 = 2 V3/r^3 + 6 V22/r^4 - 2 V33/r^2 - 4 V223/r^3 + V333/r + V2233/r^2;

(*V(2,1,1) Eq. (45)*)
V211 = 3 Sec[φ]^2 V2/r^4 - 2 V23/r^3 - Sec[φ]^2 V23/r^3 - 3 Sec[φ]^2 V112/r^4 + V332/r^2 + Sec[φ]^2 V1123/r^3 -
    6 Sec[φ]^2 V11 Tan[φ]/r^4 + 3 V22 Tan[φ]/r^4 + 2 Sec[φ]^2 V113 Tan[φ]/r^3 - V223 Tan[φ]/r^3;

(*V(1,2,1) Eq. (46)*)
V121 = -2 Sec[φ] V13/r^3 - 3 Sec[φ] V221/r^4 + Sec[φ] V331/r^2 + Sec[φ] V1223/r^3 - 6 Sec[φ] V12 Tan[φ]/r^4 +
    2 Sec[φ] V123 Tan[φ]/r^3 - 6 Sec[φ] V1 Tan[φ]^2/r^4 + 2 Sec[φ] V13 Tan[φ]^2/r^3;

(*V(1,1,2) Eq. (47)*)
V112 = 6 Sec[φ] V12/r^4 - 4 Sec[φ] V123/r^3 + Sec[φ] V1233/r^2 +
    6 Sec[φ] V1 Tan[φ]/r^4 - 4 Sec[φ] V13 Tan[φ]/r^3 + Sec[φ] V331 Tan[φ]/r^2;
```

2. Expressions for the kernels of the gravitational potential (V), 3 defining expressions of $\partial_i V$, 6 defining expressions of $\partial_{ij} V$, 10 defining expressions of $\partial_{ijk} V$ and 15 defining expressions of $\partial_{ijkl} V$ in Appendix C

In[•]:=
```
kernel = (r3^2 * Cos[φ3]) /
  (√(r^2 + r3^2 - 2 * r * r3 * (Sin[φ] * Sin[φ3] + Cos[φ] * Cos[φ3] * Cos[λ - λ3])))
```

Out[•]=
$$\frac{r3^2 \cos[\varphi 3]}{\sqrt{r^2 + r3^2 - 2\, r\, r3\, (\cos[\lambda - \lambda 3]\, \cos[\varphi]\, \cos[\varphi 3] + \sin[\varphi]\, \sin[\varphi 3])}}$$

In[•]:= V_1 = D[kernel, {λ, 1}]

Out[•]= $$-\frac{r\, r3^3 \cos[\varphi]\, \cos[\varphi 3]^2 \sin[\lambda-\lambda 3]}{\left(r^2+r3^2-2\, r\, r3\, (\cos[\lambda-\lambda 3]\, \cos[\varphi]\, \cos[\varphi 3]+\sin[\varphi]\, \sin[\varphi 3])\right)^{3/2}}$$

In[•]:= V_2 = D[kernel, {φ, 1}]

Out[•]= $$\frac{r\, r3^3 \cos[\varphi 3]\, (-\cos[\lambda-\lambda 3]\, \cos[\varphi 3]\, \sin[\varphi]+\cos[\varphi]\, \sin[\varphi 3])}{\left(r^2+r3^2-2\, r\, r3\, (\cos[\lambda-\lambda 3]\, \cos[\varphi]\, \cos[\varphi 3]+\sin[\varphi]\, \sin[\varphi 3])\right)^{3/2}}$$

In[•]:= V_3 = D[kernel, {r, 1}]

Out[•]= $$-\frac{r3^2 \cos[\varphi 3]\, (2\, r-2\, r3\, (\cos[\lambda-\lambda 3]\, \cos[\varphi]\, \cos[\varphi 3]+\sin[\varphi]\, \sin[\varphi 3]))}{2\left(r^2+r3^2-2\, r\, r3\, (\cos[\lambda-\lambda 3]\, \cos[\varphi]\, \cos[\varphi 3]+\sin[\varphi]\, \sin[\varphi 3])\right)^{3/2}}$$

In[•]:= V_{11} = D[kernel, {λ, 2}]

Out[•]= $$r3^2 \cos[\varphi 3]\left(\frac{3\, r^2\, r3^2 \cos[\varphi]^2\, \cos[\varphi 3]^2 \sin[\lambda-\lambda 3]^2}{\left(r^2+r3^2-2\, r\, r3\, (\cos[\lambda-\lambda 3]\, \cos[\varphi]\, \cos[\varphi 3]+\sin[\varphi]\, \sin[\varphi 3])\right)^{5/2}}-\frac{r\, r3 \cos[\lambda-\lambda 3]\, \cos[\varphi]\, \cos[\varphi 3]}{\left(r^2+r3^2-2\, r\, r3\, (\cos[\lambda-\lambda 3]\, \cos[\varphi]\, \cos[\varphi 3]+\sin[\varphi]\, \sin[\varphi 3])\right)^{3/2}}\right)$$

In[•]:= V_{12} = D[kernel, {λ, 1}, {φ, 1}]

Out[•]= $$-\frac{3\, r^2\, r3^4 \cos[\varphi]\, \cos[\varphi 3]^2 \sin[\lambda-\lambda 3]\, (-\cos[\lambda-\lambda 3]\, \cos[\varphi 3]\, \sin[\varphi]+\cos[\varphi]\, \sin[\varphi 3])}{\left(r^2+r3^2-2\, r\, r3\, (\cos[\lambda-\lambda 3]\, \cos[\varphi]\, \cos[\varphi 3]+\sin[\varphi]\, \sin[\varphi 3])\right)^{5/2}}+\frac{r\, r3^3 \cos[\varphi 3]^2 \sin[\lambda-\lambda 3]\, \sin[\varphi]}{\left(r^2+r3^2-2\, r\, r3\, (\cos[\lambda-\lambda 3]\, \cos[\varphi]\, \cos[\varphi 3]+\sin[\varphi]\, \sin[\varphi 3])\right)^{3/2}}$$

In[•]:= V_{13} = D[kernel, {λ, 1}, {r, 1}]

Out[•]= $$\left(3\, r\, r3^3 \cos[\varphi]\, \cos[\varphi 3]^2 \sin[\lambda-\lambda 3]\, (2\, r-2\, r3\, (\cos[\lambda-\lambda 3]\, \cos[\varphi]\, \cos[\varphi 3]+\sin[\varphi]\, \sin[\varphi 3]))\right)\Big/\left(2\left(r^2+r3^2-2\, r\, r3\, (\cos[\lambda-\lambda 3]\, \cos[\varphi]\, \cos[\varphi 3]+\sin[\varphi]\, \sin[\varphi 3])\right)^{5/2}\right)-\frac{r3^3 \cos[\varphi]\, \cos[\varphi 3]^2 \sin[\lambda-\lambda 3]}{\left(r^2+r3^2-2\, r\, r3\, (\cos[\lambda-\lambda 3]\, \cos[\varphi]\, \cos[\varphi 3]+\sin[\varphi]\, \sin[\varphi 3])\right)^{3/2}}$$

In[•]:= V_{22} = D[kernel, {φ, 2}]

Out[•]= $$r3^2 \cos[\varphi 3]\left(\frac{3\, r^2\, r3^2\, (-\cos[\lambda-\lambda 3]\, \cos[\varphi 3]\, \sin[\varphi]+\cos[\varphi]\, \sin[\varphi 3])^2}{\left(r^2+r3^2-2\, r\, r3\, (\cos[\lambda-\lambda 3]\, \cos[\varphi]\, \cos[\varphi 3]+\sin[\varphi]\, \sin[\varphi 3])\right)^{5/2}}+\frac{r\, r3\, (-\cos[\lambda-\lambda 3]\, \cos[\varphi]\, \cos[\varphi 3]-\sin[\varphi]\, \sin[\varphi 3])}{\left(r^2+r3^2-2\, r\, r3\, (\cos[\lambda-\lambda 3]\, \cos[\varphi]\, \cos[\varphi 3]+\sin[\varphi]\, \sin[\varphi 3])\right)^{3/2}}\right)$$

In[•]:= **V_{23} = D[kernel, {φ, 1}, {r, 1}]**

Out[•]=
$$-\left(\left(3\,r\,r3^3\,\text{Cos}[\varphi 3]\,\left(-\text{Cos}[\lambda-\lambda 3]\,\text{Cos}[\varphi]\,\text{Sin}[\varphi]+\text{Cos}[\varphi]\,\text{Sin}[\varphi 3]\right)\right.\right.$$
$$\left.\left(2\,r-2\,r3\,\left(\text{Cos}[\lambda-\lambda 3]\,\text{Cos}[\varphi]\,\text{Cos}[\varphi 3]+\text{Sin}[\varphi]\,\text{Sin}[\varphi 3]\right)\right)\right)\Big/$$
$$\left.\left(2\left(r^2+r3^2-2\,r\,r3\,\left(\text{Cos}[\lambda-\lambda 3]\,\text{Cos}[\varphi]\,\text{Cos}[\varphi 3]+\text{Sin}[\varphi]\,\text{Sin}[\varphi 3]\right)\right)^{5/2}\right)\right)+$$
$$\frac{r3^3\,\text{Cos}[\varphi 3]\,\left(-\text{Cos}[\lambda-\lambda 3]\,\text{Cos}[\varphi 3]\,\text{Sin}[\varphi]+\text{Cos}[\varphi]\,\text{Sin}[\varphi 3]\right)}{\left(r^2+r3^2-2\,r\,r3\,\left(\text{Cos}[\lambda-\lambda 3]\,\text{Cos}[\varphi]\,\text{Cos}[\varphi 3]+\text{Sin}[\varphi]\,\text{Sin}[\varphi 3]\right)\right)^{3/2}}$$

In[•]:= **V_{33} = D[kernel, {r, 2}]**

Out[•]=
$$r3^2\,\text{Cos}[\varphi 3]\left(\frac{3\,\left(2\,r-2\,r3\,\left(\text{Cos}[\lambda-\lambda 3]\,\text{Cos}[\varphi]\,\text{Cos}[\varphi 3]+\text{Sin}[\varphi]\,\text{Sin}[\varphi 3]\right)\right)^2}{4\left(r^2+r3^2-2\,r\,r3\,\left(\text{Cos}[\lambda-\lambda 3]\,\text{Cos}[\varphi]\,\text{Cos}[\varphi 3]+\text{Sin}[\varphi]\,\text{Sin}[\varphi 3]\right)\right)^{5/2}}-\right.$$
$$\left.\frac{1}{\left(r^2+r3^2-2\,r\,r3\,\left(\text{Cos}[\lambda-\lambda 3]\,\text{Cos}[\varphi]\,\text{Cos}[\varphi 3]+\text{Sin}[\varphi]\,\text{Sin}[\varphi 3]\right)\right)^{3/2}}\right)$$

In[•]:= **V_{111} = D[kernel, {λ, 3}]**

Out[•]=
$$r3^2\,\text{Cos}[\varphi 3]\left(-\frac{15\,r^3\,r3^3\,\text{Cos}[\varphi]^3\,\text{Cos}[\varphi 3]^3\,\text{Sin}[\lambda-\lambda 3]^3}{\left(r^2+r3^2-2\,r\,r3\,\left(\text{Cos}[\lambda-\lambda 3]\,\text{Cos}[\varphi]\,\text{Cos}[\varphi 3]+\text{Sin}[\varphi]\,\text{Sin}[\varphi 3]\right)\right)^{7/2}}+\right.$$
$$\frac{9\,r^2\,r3^2\,\text{Cos}[\lambda-\lambda 3]\,\text{Cos}[\varphi]^2\,\text{Cos}[\varphi 3]^2\,\text{Sin}[\lambda-\lambda 3]}{\left(r^2+r3^2-2\,r\,r3\,\left(\text{Cos}[\lambda-\lambda 3]\,\text{Cos}[\varphi]\,\text{Cos}[\varphi 3]+\text{Sin}[\varphi]\,\text{Sin}[\varphi 3]\right)\right)^{5/2}}+$$
$$\left.\frac{r\,r3\,\text{Cos}[\varphi]\,\text{Cos}[\varphi 3]\,\text{Sin}[\lambda-\lambda 3]}{\left(r^2+r3^2-2\,r\,r3\,\left(\text{Cos}[\lambda-\lambda 3]\,\text{Cos}[\varphi]\,\text{Cos}[\varphi 3]+\text{Sin}[\varphi]\,\text{Sin}[\varphi 3]\right)\right)^{3/2}}\right)$$

In[•]:= **V_{112} = D[kernel, {λ, 2}, {φ, 1}]**

Out[•]=
$$r3^2\,\text{Cos}[\varphi 3]$$
$$\left(\frac{15\,r^3\,r3^3\,\text{Cos}[\varphi]^2\,\text{Cos}[\varphi 3]^2\,\text{Sin}[\lambda-\lambda 3]^2\,\left(-\text{Cos}[\lambda-\lambda 3]\,\text{Cos}[\varphi 3]\,\text{Sin}[\varphi]+\text{Cos}[\varphi]\,\text{Sin}[\varphi 3]\right)}{\left(r^2+r3^2-2\,r\,r3\,\left(\text{Cos}[\lambda-\lambda 3]\,\text{Cos}[\varphi]\,\text{Cos}[\varphi 3]+\text{Sin}[\varphi]\,\text{Sin}[\varphi 3]\right)\right)^{7/2}}\right.$$
$$-\frac{6\,r^2\,r3^2\,\text{Cos}[\varphi]\,\text{Cos}[\varphi 3]^2\,\text{Sin}[\lambda-\lambda 3]^2\,\text{Sin}[\varphi]}{\left(r^2+r3^2-2\,r\,r3\,\left(\text{Cos}[\lambda-\lambda 3]\,\text{Cos}[\varphi]\,\text{Cos}[\varphi 3]+\text{Sin}[\varphi]\,\text{Sin}[\varphi 3]\right)\right)^{5/2}}-$$
$$\frac{3\,r^2\,r3^2\,\text{Cos}[\lambda-\lambda 3]\,\text{Cos}[\varphi]\,\text{Cos}[\varphi 3]\,\left(-\text{Cos}[\lambda-\lambda 3]\,\text{Cos}[\varphi 3]\,\text{Sin}[\varphi]+\text{Cos}[\varphi]\,\text{Sin}[\varphi 3]\right)}{\left(r^2+r3^2-2\,r\,r3\,\left(\text{Cos}[\lambda-\lambda 3]\,\text{Cos}[\varphi]\,\text{Cos}[\varphi 3]+\text{Sin}[\varphi]\,\text{Sin}[\varphi 3]\right)\right)^{5/2}}+$$
$$\left.\frac{r\,r3\,\text{Cos}[\lambda-\lambda 3]\,\text{Cos}[\varphi 3]\,\text{Sin}[\varphi]}{\left(r^2+r3^2-2\,r\,r3\,\left(\text{Cos}[\lambda-\lambda 3]\,\text{Cos}[\varphi]\,\text{Cos}[\varphi 3]+\text{Sin}[\varphi]\,\text{Sin}[\varphi 3]\right)\right)^{3/2}}\right)$$

In[•]:=
```
V113 = D[kernel, {λ, 2}, {r, 1}]
```

Out[•]=
$$r3^2 \cos[\varphi 3] \left(-\left(\left(15\, r^2\, r3^2 \cos[\varphi]^2 \cos[\varphi 3]^2 \sin[\lambda - \lambda 3]^2 \right.\right.\right.$$
$$\left.\left.\left. (2\, r - 2\, r3\, (\cos[\lambda - \lambda 3] \cos[\varphi] \cos[\varphi 3] + \sin[\varphi] \sin[\varphi 3])) \right) \Big/ \right.$$
$$\left. \left(2 \left(r^2 + r3^2 - 2\, r\, r3\, (\cos[\lambda - \lambda 3] \cos[\varphi] \cos[\varphi 3] + \sin[\varphi] \sin[\varphi 3]) \right)^{7/2} \right) \right) +$$
$$\frac{6\, r\, r3^2 \cos[\varphi]^2 \cos[\varphi 3]^2 \sin[\lambda - \lambda 3]^2}{\left(r^2 + r3^2 - 2\, r\, r3\, (\cos[\lambda - \lambda 3] \cos[\varphi] \cos[\varphi 3] + \sin[\varphi] \sin[\varphi 3]) \right)^{5/2}} +$$
$$(3\, r\, r3 \cos[\lambda - \lambda 3] \cos[\varphi] \cos[\varphi 3]$$
$$(2\, r - 2\, r3\, (\cos[\lambda - \lambda 3] \cos[\varphi] \cos[\varphi 3] + \sin[\varphi] \sin[\varphi 3]))) \Big/$$
$$\left(2 \left(r^2 + r3^2 - 2\, r\, r3\, (\cos[\lambda - \lambda 3] \cos[\varphi] \cos[\varphi 3] + \sin[\varphi] \sin[\varphi 3]) \right)^{5/2} \right) -$$
$$\left. \frac{r3 \cos[\lambda - \lambda 3] \cos[\varphi] \cos[\varphi 3]}{\left(r^2 + r3^2 - 2\, r\, r3\, (\cos[\lambda - \lambda 3] \cos[\varphi] \cos[\varphi 3] + \sin[\varphi] \sin[\varphi 3]) \right)^{3/2}} \right)$$

In[•]:=
```
V123 = D[kernel, {λ, 1}, {φ, 1}, {r, 1}]
```

Out[•]=
$$\left(15\, r^2\, r3^4 \cos[\varphi] \cos[\varphi 3]^2 \sin[\lambda - \lambda 3]\, (-\cos[\lambda - \lambda 3] \cos[\varphi 3] \sin[\varphi] + \cos[\varphi] \sin[\varphi 3]) \right.$$
$$(2\, r - 2\, r3\, (\cos[\lambda - \lambda 3] \cos[\varphi] \cos[\varphi 3] + \sin[\varphi] \sin[\varphi 3])) \Big) \Big/$$
$$\left(2 \left(r^2 + r3^2 - 2\, r\, r3\, (\cos[\lambda - \lambda 3] \cos[\varphi] \cos[\varphi 3] + \sin[\varphi] \sin[\varphi 3]) \right)^{7/2} \right) -$$
$$\frac{6\, r\, r3^4 \cos[\varphi] \cos[\varphi 3]^2 \sin[\lambda - \lambda 3]\, (-\cos[\lambda - \lambda 3] \cos[\varphi 3] \sin[\varphi] + \cos[\varphi] \sin[\varphi 3])}{\left(r^2 + r3^2 - 2\, r\, r3\, (\cos[\lambda - \lambda 3] \cos[\varphi] \cos[\varphi 3] + \sin[\varphi] \sin[\varphi 3]) \right)^{5/2}} -$$
$$\left(3\, r\, r3^3 \cos[\varphi 3]^2 \sin[\lambda - \lambda 3] \sin[\varphi] \right.$$
$$(2\, r - 2\, r3\, (\cos[\lambda - \lambda 3] \cos[\varphi] \cos[\varphi 3] + \sin[\varphi] \sin[\varphi 3])) \Big) \Big/$$
$$\left(2 \left(r^2 + r3^2 - 2\, r\, r3\, (\cos[\lambda - \lambda 3] \cos[\varphi] \cos[\varphi 3] + \sin[\varphi] \sin[\varphi 3]) \right)^{5/2} \right) +$$
$$\frac{r3^3 \cos[\varphi 3]^2 \sin[\lambda - \lambda 3] \sin[\varphi]}{\left(r^2 + r3^2 - 2\, r\, r3\, (\cos[\lambda - \lambda 3] \cos[\varphi] \cos[\varphi 3] + \sin[\varphi] \sin[\varphi 3]) \right)^{3/2}}$$

In[•]:=
```
V221 = D[kernel, {λ, 1}, {φ, 2}]
```

Out[•]=
$$\frac{6\, r^2\, r3^4 \cos[\varphi 3]^2 \sin[\lambda - \lambda 3] \sin[\varphi]\, (-\cos[\lambda - \lambda 3] \cos[\varphi 3] \sin[\varphi] + \cos[\varphi] \sin[\varphi 3])}{\left(r^2 + r3^2 - 2\, r\, r3\, (\cos[\lambda - \lambda 3] \cos[\varphi] \cos[\varphi 3] + \sin[\varphi] \sin[\varphi 3]) \right)^{5/2}} +$$
$$\frac{r\, r3^3 \cos[\varphi] \cos[\varphi 3]^2 \sin[\lambda - \lambda 3]}{\left(r^2 + r3^2 - 2\, r\, r3\, (\cos[\lambda - \lambda 3] \cos[\varphi] \cos[\varphi 3] + \sin[\varphi] \sin[\varphi 3]) \right)^{3/2}} -$$
$$r\, r3^3 \cos[\varphi] \cos[\varphi 3]^2 \sin[\lambda - \lambda 3]$$
$$\left(\frac{15\, r^2\, r3^2\, (-\cos[\lambda - \lambda 3] \cos[\varphi 3] \sin[\varphi] + \cos[\varphi] \sin[\varphi 3])^2}{\left(r^2 + r3^2 - 2\, r\, r3\, (\cos[\lambda - \lambda 3] \cos[\varphi] \cos[\varphi 3] + \sin[\varphi] \sin[\varphi 3]) \right)^{7/2}} + \right.$$
$$\left. \frac{3\, r\, r3\, (-\cos[\lambda - \lambda 3] \cos[\varphi] \cos[\varphi 3] - \sin[\varphi] \sin[\varphi 3])}{\left(r^2 + r3^2 - 2\, r\, r3\, (\cos[\lambda - \lambda 3] \cos[\varphi] \cos[\varphi 3] + \sin[\varphi] \sin[\varphi 3]) \right)^{5/2}} \right)$$

In[•]:= V_{222} = D[kernel, {φ, 3}]

Out[•]=
$$r3^2 \cos[\varphi 3]\left(\frac{15\, r^3\, r3^3\, (-\cos[\lambda-\lambda 3]\cos[\varphi 3]\sin[\varphi]+\cos[\varphi]\sin[\varphi 3])^3}{\left(r^2+r3^2-2\,r\,r3\,(\cos[\lambda-\lambda 3]\cos[\varphi]\cos[\varphi 3]+\sin[\varphi]\sin[\varphi 3])\right)^{7/2}}+\right.$$
$$\left(9\, r^2\, r3^2\, (-\cos[\lambda-\lambda 3]\cos[\varphi 3]\sin[\varphi]+\cos[\varphi]\sin[\varphi 3])\right.$$
$$(-\cos[\lambda-\lambda 3]\cos[\varphi]\cos[\varphi 3]-\sin[\varphi]\sin[\varphi 3])\Big)\Big/$$
$$\left(r^2+r3^2-2\,r\,r3\,(\cos[\lambda-\lambda 3]\cos[\varphi]\cos[\varphi 3]+\sin[\varphi]\sin[\varphi 3])\right)^{5/2}+$$
$$\left.\frac{r\,r3\,(\cos[\lambda-\lambda 3]\cos[\varphi 3]\sin[\varphi]-\cos[\varphi]\sin[\varphi 3])}{\left(r^2+r3^2-2\,r\,r3\,(\cos[\lambda-\lambda 3]\cos[\varphi]\cos[\varphi 3]+\sin[\varphi]\sin[\varphi 3])\right)^{3/2}}\right)$$

In[•]:= V_{223} = D[kernel, {φ, 2}, {r, 1}]

Out[•]=
$$r3^2 \cos[\varphi 3]\left(-\left(\left(15\, r^2\, r3^2\, (-\cos[\lambda-\lambda 3]\cos[\varphi 3]\sin[\varphi]+\cos[\varphi]\sin[\varphi 3])^2\right.\right.\right.$$
$$(2\,r-2\,r3\,(\cos[\lambda-\lambda 3]\cos[\varphi]\cos[\varphi 3]+\sin[\varphi]\sin[\varphi 3]))\Big)\Big/$$
$$\left.\left.\left(2\left(r^2+r3^2-2\,r\,r3\,(\cos[\lambda-\lambda 3]\cos[\varphi]\cos[\varphi 3]+\sin[\varphi]\sin[\varphi 3])\right)^{7/2}\right)\right)+$$
$$\frac{6\,r\,r3^2\,(-\cos[\lambda-\lambda 3]\cos[\varphi 3]\sin[\varphi]+\cos[\varphi]\sin[\varphi 3])^2}{\left(r^2+r3^2-2\,r\,r3\,(\cos[\lambda-\lambda 3]\cos[\varphi]\cos[\varphi 3]+\sin[\varphi]\sin[\varphi 3])\right)^{5/2}}-$$
$$(3\,r\,r3\,(-\cos[\lambda-\lambda 3]\cos[\varphi]\cos[\varphi 3]-\sin[\varphi]\sin[\varphi 3])$$
$$(2\,r-2\,r3\,(\cos[\lambda-\lambda 3]\cos[\varphi]\cos[\varphi 3]+\sin[\varphi]\sin[\varphi 3])))\Big/$$
$$\left(2\left(r^2+r3^2-2\,r\,r3\,(\cos[\lambda-\lambda 3]\cos[\varphi]\cos[\varphi 3]+\sin[\varphi]\sin[\varphi 3])\right)^{5/2}\right)+$$
$$\left.\frac{r3\,(-\cos[\lambda-\lambda 3]\cos[\varphi]\cos[\varphi 3]-\sin[\varphi]\sin[\varphi 3])}{\left(r^2+r3^2-2\,r\,r3\,(\cos[\lambda-\lambda 3]\cos[\varphi]\cos[\varphi 3]+\sin[\varphi]\sin[\varphi 3])\right)^{3/2}}\right)$$

In[•]:= V_{331} = D[kernel, {λ, 1}, {r, 2}]

Out[•]=
$$\left(3\,r3^3\cos[\varphi]\cos[\varphi 3]^2\sin[\lambda-\lambda 3]\right.$$
$$(2\,r-2\,r3\,(\cos[\lambda-\lambda 3]\cos[\varphi]\cos[\varphi 3]+\sin[\varphi]\sin[\varphi 3]))\Big)\Big/$$
$$\left(r^2+r3^2-2\,r\,r3\,(\cos[\lambda-\lambda 3]\cos[\varphi]\cos[\varphi 3]+\sin[\varphi]\sin[\varphi 3])\right)^{5/2}-$$
$$r\,r3^3\cos[\varphi]\cos[\varphi 3]^2\sin[\lambda-\lambda 3]$$
$$\left(\frac{15\,(2\,r-2\,r3\,(\cos[\lambda-\lambda 3]\cos[\varphi]\cos[\varphi 3]+\sin[\varphi]\sin[\varphi 3]))^2}{4\left(r^2+r3^2-2\,r\,r3\,(\cos[\lambda-\lambda 3]\cos[\varphi]\cos[\varphi 3]+\sin[\varphi]\sin[\varphi 3])\right)^{7/2}}-\right.$$
$$\left.\frac{3}{\left(r^2+r3^2-2\,r\,r3\,(\cos[\lambda-\lambda 3]\cos[\varphi]\cos[\varphi 3]+\sin[\varphi]\sin[\varphi 3])\right)^{5/2}}\right)$$

In[•]:= V_{332} = D[kernel, {φ, 1}, {r, 2}]

Out[•]=
$$-\left(\left(3\,r3^3\,\text{Cos}[\varphi 3]\,\left(-\text{Cos}[\lambda-\lambda 3]\,\text{Cos}[\varphi 3]\,\text{Sin}[\varphi]+\text{Cos}[\varphi]\,\text{Sin}[\varphi 3]\right)\right.\right.$$
$$\left.\left(2\,r-2\,r3\,\left(\text{Cos}[\lambda-\lambda 3]\,\text{Cos}[\varphi]\,\text{Cos}[\varphi 3]+\text{Sin}[\varphi]\,\text{Sin}[\varphi 3]\right)\right)\right)\Big/$$
$$\left.\left(r^2+r3^2-2\,r\,r3\,\left(\text{Cos}[\lambda-\lambda 3]\,\text{Cos}[\varphi]\,\text{Cos}[\varphi 3]+\text{Sin}[\varphi]\,\text{Sin}[\varphi 3]\right)\right)^{5/2}\right)+$$
$$r\,r3^3\,\text{Cos}[\varphi 3]\,\left(-\text{Cos}[\lambda-\lambda 3]\,\text{Cos}[\varphi 3]\,\text{Sin}[\varphi]+\text{Cos}[\varphi]\,\text{Sin}[\varphi 3]\right)$$
$$\left(\frac{15\,\left(2\,r-2\,r3\,\left(\text{Cos}[\lambda-\lambda 3]\,\text{Cos}[\varphi]\,\text{Cos}[\varphi 3]+\text{Sin}[\varphi]\,\text{Sin}[\varphi 3]\right)\right)^2}{4\,\left(r^2+r3^2-2\,r\,r3\,\left(\text{Cos}[\lambda-\lambda 3]\,\text{Cos}[\varphi]\,\text{Cos}[\varphi 3]+\text{Sin}[\varphi]\,\text{Sin}[\varphi 3]\right)\right)^{7/2}}-\right.$$
$$\left.\frac{3}{\left(r^2+r3^2-2\,r\,r3\,\left(\text{Cos}[\lambda-\lambda 3]\,\text{Cos}[\varphi]\,\text{Cos}[\varphi 3]+\text{Sin}[\varphi]\,\text{Sin}[\varphi 3]\right)\right)^{5/2}}\right)$$

In[•]:= V_{333} = D[kernel, {r, 3}]

Out[•]=
$$r3^2\,\text{Cos}[\varphi 3]\,\left(-\frac{15\,\left(2\,r-2\,r3\,\left(\text{Cos}[\lambda-\lambda 3]\,\text{Cos}[\varphi]\,\text{Cos}[\varphi 3]+\text{Sin}[\varphi]\,\text{Sin}[\varphi 3]\right)\right)^3}{8\,\left(r^2+r3^2-2\,r\,r3\,\left(\text{Cos}[\lambda-\lambda 3]\,\text{Cos}[\varphi]\,\text{Cos}[\varphi 3]+\text{Sin}[\varphi]\,\text{Sin}[\varphi 3]\right)\right)^{7/2}}+\right.$$
$$\left.\frac{9\,\left(2\,r-2\,r3\,\left(\text{Cos}[\lambda-\lambda 3]\,\text{Cos}[\varphi]\,\text{Cos}[\varphi 3]+\text{Sin}[\varphi]\,\text{Sin}[\varphi 3]\right)\right)}{2\,\left(r^2+r3^2-2\,r\,r3\,\left(\text{Cos}[\lambda-\lambda 3]\,\text{Cos}[\varphi]\,\text{Cos}[\varphi 3]+\text{Sin}[\varphi]\,\text{Sin}[\varphi 3]\right)\right)^{5/2}}\right)$$

In[•]:= V_{1111} = D[kernel, {λ, 4}]

Out[•]=
$$r3^2\,\text{Cos}[\varphi 3]\,\left(\frac{105\,r^4\,r3^4\,\text{Cos}[\varphi]^4\,\text{Cos}[\varphi 3]^4\,\text{Sin}[\lambda-\lambda 3]^4}{\left(r^2+r3^2-2\,r\,r3\,\left(\text{Cos}[\lambda-\lambda 3]\,\text{Cos}[\varphi]\,\text{Cos}[\varphi 3]+\text{Sin}[\varphi]\,\text{Sin}[\varphi 3]\right)\right)^{9/2}}-\right.$$
$$\frac{90\,r^3\,r3^3\,\text{Cos}[\lambda-\lambda 3]\,\text{Cos}[\varphi]^3\,\text{Cos}[\varphi 3]^3\,\text{Sin}[\lambda-\lambda 3]^2}{\left(r^2+r3^2-2\,r\,r3\,\left(\text{Cos}[\lambda-\lambda 3]\,\text{Cos}[\varphi]\,\text{Cos}[\varphi 3]+\text{Sin}[\varphi]\,\text{Sin}[\varphi 3]\right)\right)^{7/2}}+$$
$$\frac{9\,r^2\,r3^2\,\text{Cos}[\lambda-\lambda 3]^2\,\text{Cos}[\varphi]^2\,\text{Cos}[\varphi 3]^2}{\left(r^2+r3^2-2\,r\,r3\,\left(\text{Cos}[\lambda-\lambda 3]\,\text{Cos}[\varphi]\,\text{Cos}[\varphi 3]+\text{Sin}[\varphi]\,\text{Sin}[\varphi 3]\right)\right)^{5/2}}-$$
$$\frac{12\,r^2\,r3^2\,\text{Cos}[\varphi]^2\,\text{Cos}[\varphi 3]^2\,\text{Sin}[\lambda-\lambda 3]^2}{\left(r^2+r3^2-2\,r\,r3\,\left(\text{Cos}[\lambda-\lambda 3]\,\text{Cos}[\varphi]\,\text{Cos}[\varphi 3]+\text{Sin}[\varphi]\,\text{Sin}[\varphi 3]\right)\right)^{5/2}}+$$
$$\left.\frac{r\,r3\,\text{Cos}[\lambda-\lambda 3]\,\text{Cos}[\varphi]\,\text{Cos}[\varphi 3]}{\left(r^2+r3^2-2\,r\,r3\,\left(\text{Cos}[\lambda-\lambda 3]\,\text{Cos}[\varphi]\,\text{Cos}[\varphi 3]+\text{Sin}[\varphi]\,\text{Sin}[\varphi 3]\right)\right)^{3/2}}\right)$$

In[•]:= V_{2222} = D[kernel, {φ, 4}]

Out[•]=
$$r3^2 \cos[\varphi 3]\left(\frac{105\, r^4\, r3^4\, (-\cos[\lambda-\lambda 3]\cos[\varphi 3]\sin[\varphi]+\cos[\varphi]\sin[\varphi 3])^4}{\left(r^2+r3^2-2\,r\,r3\,(\cos[\lambda-\lambda 3]\cos[\varphi]\cos[\varphi 3]+\sin[\varphi]\sin[\varphi 3])\right)^{9/2}}+\right.$$
$$\left(90\, r^3\, r3^3\,(-\cos[\lambda-\lambda 3]\cos[\varphi 3]\sin[\varphi]+\cos[\varphi]\sin[\varphi 3])^2\right.$$
$$\left.(-\cos[\lambda-\lambda 3]\cos[\varphi]\cos[\varphi 3]-\sin[\varphi]\sin[\varphi 3])\right)\Big/$$
$$\left(r^2+r3^2-2\,r\,r3\,(\cos[\lambda-\lambda 3]\cos[\varphi]\cos[\varphi 3]+\sin[\varphi]\sin[\varphi 3])\right)^{7/2}+$$
$$\left(12\, r^2\, r3^2\,(\cos[\lambda-\lambda 3]\cos[\varphi 3]\sin[\varphi]-\cos[\varphi]\sin[\varphi 3])\right.$$
$$\left.(-\cos[\lambda-\lambda 3]\cos[\varphi 3]\sin[\varphi]+\cos[\varphi]\sin[\varphi 3])\right)\Big/$$
$$\left(r^2+r3^2-2\,r\,r3\,(\cos[\lambda-\lambda 3]\cos[\varphi]\cos[\varphi 3]+\sin[\varphi]\sin[\varphi 3])\right)^{5/2}+$$
$$\frac{9\, r^2\, r3^2\,(-\cos[\lambda-\lambda 3]\cos[\varphi]\cos[\varphi 3]-\sin[\varphi]\sin[\varphi 3])^2}{\left(r^2+r3^2-2\,r\,r3\,(\cos[\lambda-\lambda 3]\cos[\varphi]\cos[\varphi 3]+\sin[\varphi]\sin[\varphi 3])\right)^{5/2}}+$$
$$\left.\frac{r\,r3\,(\cos[\lambda-\lambda 3]\cos[\varphi]\cos[\varphi 3]+\sin[\varphi]\sin[\varphi 3])}{\left(r^2+r3^2-2\,r\,r3\,(\cos[\lambda-\lambda 3]\cos[\varphi]\cos[\varphi 3]+\sin[\varphi]\sin[\varphi 3])\right)^{3/2}}\right)$$

In[•]:= V_{3333} = D[kernel, {r, 4}]

Out[•]=
$$r3^2 \cos[\varphi 3]\left(\frac{105\,(2\,r-2\,r3\,(\cos[\lambda-\lambda 3]\cos[\varphi]\cos[\varphi 3]+\sin[\varphi]\sin[\varphi 3]))^4}{16\left(r^2+r3^2-2\,r\,r3\,(\cos[\lambda-\lambda 3]\cos[\varphi]\cos[\varphi 3]+\sin[\varphi]\sin[\varphi 3])\right)^{9/2}}-\right.$$
$$\frac{45\,(2\,r-2\,r3\,(\cos[\lambda-\lambda 3]\cos[\varphi]\cos[\varphi 3]+\sin[\varphi]\sin[\varphi 3]))^2}{2\left(r^2+r3^2-2\,r\,r3\,(\cos[\lambda-\lambda 3]\cos[\varphi]\cos[\varphi 3]+\sin[\varphi]\sin[\varphi 3])\right)^{7/2}}+$$
$$\left.\frac{9}{\left(r^2+r3^2-2\,r\,r3\,(\cos[\lambda-\lambda 3]\cos[\varphi]\cos[\varphi 3]+\sin[\varphi]\sin[\varphi 3])\right)^{5/2}}\right)$$

In[•]:= V_{1112} = D[kernel, {λ, 3}, {φ, 1}]

Out[•]=

$$r3^2 \cos[\varphi 3] \left(-\left(\left(105\, r^4\, r3^4 \cos[\varphi]^3 \cos[\varphi 3]^3 \sin[\lambda-\lambda 3]^3 \left(-\cos[\lambda-\lambda 3]\cos[\varphi 3]\sin[\varphi]+\cos[\varphi]\sin[\varphi 3]\right)\right) \Big/ \left(r^2+r3^2-2\,r\,r3\left(\cos[\lambda-\lambda 3]\cos[\varphi]\cos[\varphi 3]+\sin[\varphi]\sin[\varphi 3]\right)\right)^{9/2}\right) + \frac{45\, r^3\, r3^3 \cos[\varphi]^2 \cos[\varphi 3]^3 \sin[\lambda-\lambda 3]^3 \sin[\varphi]}{\left(r^2+r3^2-2\,r\,r3\left(\cos[\lambda-\lambda 3]\cos[\varphi]\cos[\varphi 3]+\sin[\varphi]\sin[\varphi 3]\right)\right)^{7/2}} + \left(45\, r^3\, r3^3 \cos[\lambda-\lambda 3]\cos[\varphi]^2\cos[\varphi 3]^2\sin[\lambda-\lambda 3]\left(-\cos[\lambda-\lambda 3]\cos[\varphi 3]\sin[\varphi]+\cos[\varphi]\sin[\varphi 3]\right)\right) \Big/ \left(r^2+r3^2-2\,r\,r3\left(\cos[\lambda-\lambda 3]\cos[\varphi]\cos[\varphi 3]+\sin[\varphi]\sin[\varphi 3]\right)\right)^{7/2} - \frac{18\, r^2\, r3^2 \cos[\lambda-\lambda 3]\cos[\varphi]\cos[\varphi 3]^2\sin[\lambda-\lambda 3]\sin[\varphi]}{\left(r^2+r3^2-2\,r\,r3\left(\cos[\lambda-\lambda 3]\cos[\varphi]\cos[\varphi 3]+\sin[\varphi]\sin[\varphi 3]\right)\right)^{5/2}} + \frac{3\, r^2\, r3^2 \cos[\varphi]\cos[\varphi 3]\sin[\lambda-\lambda 3]\left(-\cos[\lambda-\lambda 3]\cos[\varphi 3]\sin[\varphi]+\cos[\varphi]\sin[\varphi 3]\right)}{\left(r^2+r3^2-2\,r\,r3\left(\cos[\lambda-\lambda 3]\cos[\varphi]\cos[\varphi 3]+\sin[\varphi]\sin[\varphi 3]\right)\right)^{5/2}} - \frac{r\, r3 \cos[\varphi 3]\sin[\lambda-\lambda 3]\sin[\varphi]}{\left(r^2+r3^2-2\,r\,r3\left(\cos[\lambda-\lambda 3]\cos[\varphi]\cos[\varphi 3]+\sin[\varphi]\sin[\varphi 3]\right)\right)^{3/2}} \right)$$

In[•]:= V_{1113} = D[kernel, {λ, 3}, {r, 1}]

Out[•]=

$$r3^2 \cos[\varphi 3] \left(\left(105\, r^3\, r3^3 \cos[\varphi]^3 \cos[\varphi 3]^3 \sin[\lambda-\lambda 3]^3 \left(2\,r-2\,r3\left(\cos[\lambda-\lambda 3]\cos[\varphi]\cos[\varphi 3]+\sin[\varphi]\sin[\varphi 3]\right)\right)\right) \Big/ \left(2\left(r^2+r3^2-2\,r\,r3\left(\cos[\lambda-\lambda 3]\cos[\varphi]\cos[\varphi 3]+\sin[\varphi]\sin[\varphi 3]\right)\right)^{9/2}\right) - \frac{45\, r^2\, r3^3 \cos[\varphi]^3 \cos[\varphi 3]^3 \sin[\lambda-\lambda 3]^3}{\left(r^2+r3^2-2\,r\,r3\left(\cos[\lambda-\lambda 3]\cos[\varphi]\cos[\varphi 3]+\sin[\varphi]\sin[\varphi 3]\right)\right)^{7/2}} - \left(45\, r^2\, r3^2 \cos[\lambda-\lambda 3]\cos[\varphi]^2\cos[\varphi 3]^2\sin[\lambda-\lambda 3]\left(2\,r-2\,r3\left(\cos[\lambda-\lambda 3]\cos[\varphi]\cos[\varphi 3]+\sin[\varphi]\sin[\varphi 3]\right)\right)\right) \Big/ \left(2\left(r^2+r3^2-2\,r\,r3\left(\cos[\lambda-\lambda 3]\cos[\varphi]\cos[\varphi 3]+\sin[\varphi]\sin[\varphi 3]\right)\right)^{7/2}\right) + \frac{18\, r\, r3^2 \cos[\lambda-\lambda 3]\cos[\varphi]^2\cos[\varphi 3]^2\sin[\lambda-\lambda 3]}{\left(r^2+r3^2-2\,r\,r3\left(\cos[\lambda-\lambda 3]\cos[\varphi]\cos[\varphi 3]+\sin[\varphi]\sin[\varphi 3]\right)\right)^{5/2}} - \left(3\, r\, r3 \cos[\varphi]\cos[\varphi 3]\sin[\lambda-\lambda 3]\left(2\,r-2\,r3\left(\cos[\lambda-\lambda 3]\cos[\varphi]\cos[\varphi 3]+\sin[\varphi]\sin[\varphi 3]\right)\right)\right) \Big/ \left(2\left(r^2+r3^2-2\,r\,r3\left(\cos[\lambda-\lambda 3]\cos[\varphi]\cos[\varphi 3]+\sin[\varphi]\sin[\varphi 3]\right)\right)^{5/2}\right) + \frac{r3\cos[\varphi]\cos[\varphi 3]\sin[\lambda-\lambda 3]}{\left(r^2+r3^2-2\,r\,r3\left(\cos[\lambda-\lambda 3]\cos[\varphi]\cos[\varphi 3]+\sin[\varphi]\sin[\varphi 3]\right)\right)^{3/2}} \right)$$

In[•]:= V_{1222} = D[kernel, {λ, 1}, {φ, 3}]

Out[•]=
$$\frac{9\, r^2\, r3^4 \cos[\varphi]\, \cos[\varphi 3]^2 \sin[\lambda-\lambda 3]\, (-\cos[\lambda-\lambda 3]\, \cos[\varphi 3]\, \sin[\varphi]+\cos[\varphi]\, \sin[\varphi 3])}{\left(r^2+r3^2-2\, r\, r3\, (\cos[\lambda-\lambda 3]\, \cos[\varphi]\, \cos[\varphi 3]+\sin[\varphi]\, \sin[\varphi 3])\right)^{5/2}}-$$
$$\frac{r\, r3^3 \cos[\varphi 3]^2 \sin[\lambda-\lambda 3]\, \sin[\varphi]}{\left(r^2+r3^2-2\, r\, r3\, (\cos[\lambda-\lambda 3]\, \cos[\varphi]\, \cos[\varphi 3]+\sin[\varphi]\, \sin[\varphi 3])\right)^{3/2}}-$$
$$r\, r3^3 \cos[\varphi]\, \cos[\varphi 3]^2 \sin[\lambda-\lambda 3]$$
$$\left(\frac{105\, r^3\, r3^3\, (-\cos[\lambda-\lambda 3]\, \cos[\varphi 3]\, \sin[\varphi]+\cos[\varphi]\, \sin[\varphi 3])^3}{\left(r^2+r3^2-2\, r\, r3\, (\cos[\lambda-\lambda 3]\, \cos[\varphi]\, \cos[\varphi 3]+\sin[\varphi]\, \sin[\varphi 3])\right)^{9/2}}+\right.$$
$$\left(45\, r^2\, r3^2\, (-\cos[\lambda-\lambda 3]\, \cos[\varphi 3]\, \sin[\varphi]+\cos[\varphi]\, \sin[\varphi 3])\right.$$
$$\left.(-\cos[\lambda-\lambda 3]\, \cos[\varphi]\, \cos[\varphi 3]-\sin[\varphi]\, \sin[\varphi 3])\right)\Big/$$
$$\left(r^2+r3^2-2\, r\, r3\, (\cos[\lambda-\lambda 3]\, \cos[\varphi]\, \cos[\varphi 3]+\sin[\varphi]\, \sin[\varphi 3])\right)^{7/2}+$$
$$\left.\frac{3\, r\, r3\, (\cos[\lambda-\lambda 3]\, \cos[\varphi 3]\, \sin[\varphi]-\cos[\varphi]\, \sin[\varphi 3])}{\left(r^2+r3^2-2\, r\, r3\, (\cos[\lambda-\lambda 3]\, \cos[\varphi]\, \cos[\varphi 3]+\sin[\varphi]\, \sin[\varphi 3])\right)^{5/2}}\right)+$$
$$3\, r\, r3^3 \cos[\varphi 3]^2 \sin[\lambda-\lambda 3]\, \sin[\varphi]$$
$$\left(\frac{15\, r^2\, r3^2\, (-\cos[\lambda-\lambda 3]\, \cos[\varphi 3]\, \sin[\varphi]+\cos[\varphi]\, \sin[\varphi 3])^2}{\left(r^2+r3^2-2\, r\, r3\, (\cos[\lambda-\lambda 3]\, \cos[\varphi]\, \cos[\varphi 3]+\sin[\varphi]\, \sin[\varphi 3])\right)^{7/2}}+\right.$$
$$\left.\frac{3\, r\, r3\, (-\cos[\lambda-\lambda 3]\, \cos[\varphi]\, \cos[\varphi 3]-\sin[\varphi]\, \sin[\varphi 3])}{\left(r^2+r3^2-2\, r\, r3\, (\cos[\lambda-\lambda 3]\, \cos[\varphi]\, \cos[\varphi 3]+\sin[\varphi]\, \sin[\varphi 3])\right)^{5/2}}\right)$$

In[•]:= V_{1333} = D[kernel, {λ, 1}, {r, 3}]

Out[•]=
$$-r\, r3^3 \cos[\varphi]\, \cos[\varphi 3]^2 \sin[\lambda-\lambda 3]$$
$$\left(-\frac{105\, (2\, r-2\, r3\, (\cos[\lambda-\lambda 3]\, \cos[\varphi]\, \cos[\varphi 3]+\sin[\varphi]\, \sin[\varphi 3]))^3}{8\left(r^2+r3^2-2\, r\, r3\, (\cos[\lambda-\lambda 3]\, \cos[\varphi]\, \cos[\varphi 3]+\sin[\varphi]\, \sin[\varphi 3])\right)^{9/2}}+\right.$$
$$\left.\frac{45\, (2\, r-2\, r3\, (\cos[\lambda-\lambda 3]\, \cos[\varphi]\, \cos[\varphi 3]+\sin[\varphi]\, \sin[\varphi 3]))}{2\left(r^2+r3^2-2\, r\, r3\, (\cos[\lambda-\lambda 3]\, \cos[\varphi]\, \cos[\varphi 3]+\sin[\varphi]\, \sin[\varphi 3])\right)^{7/2}}\right)-$$
$$3\, r3^3 \cos[\varphi]\, \cos[\varphi 3]^2 \sin[\lambda-\lambda 3]$$
$$\left(\frac{15\, (2\, r-2\, r3\, (\cos[\lambda-\lambda 3]\, \cos[\varphi]\, \cos[\varphi 3]+\sin[\varphi]\, \sin[\varphi 3]))^2}{4\left(r^2+r3^2-2\, r\, r3\, (\cos[\lambda-\lambda 3]\, \cos[\varphi]\, \cos[\varphi 3]+\sin[\varphi]\, \sin[\varphi 3])\right)^{7/2}}-\right.$$
$$\left.\frac{3}{\left(r^2+r3^2-2\, r\, r3\, (\cos[\lambda-\lambda 3]\, \cos[\varphi]\, \cos[\varphi 3]+\sin[\varphi]\, \sin[\varphi 3])\right)^{5/2}}\right)$$

In[•]:=
```
V2223 = D[kernel, {φ, 3}, {r, 1}]
```

Out[•]=
$$\begin{aligned}
&r3^2 \cos[\varphi 3] \left(-\left(\left(105\, r^3\, r3^3\, (-\cos[\lambda-\lambda 3]\cos[\varphi 3]\sin[\varphi]+\cos[\varphi]\sin[\varphi 3])^3 \right.\right.\right.\\
&\qquad \left.\left.\left. (2r-2\,r3\,(\cos[\lambda-\lambda 3]\cos[\varphi]\cos[\varphi 3]+\sin[\varphi]\sin[\varphi 3])) \right) \Big/ \right.\\
&\qquad \left. \left(2\left(r^2+r3^2-2\,r\,r3\,(\cos[\lambda-\lambda 3]\cos[\varphi]\cos[\varphi 3]+\sin[\varphi]\sin[\varphi 3])\right)^{9/2} \right) \right) +\\
&\qquad \frac{45\, r^2\, r3^3\, (-\cos[\lambda-\lambda 3]\cos[\varphi 3]\sin[\varphi]+\cos[\varphi]\sin[\varphi 3])^3}{\left(r^2+r3^2-2\,r\,r3\,(\cos[\lambda-\lambda 3]\cos[\varphi]\cos[\varphi 3]+\sin[\varphi]\sin[\varphi 3])\right)^{7/2}} -\\
&\qquad \left(45\, r^2\, r3^2\, (-\cos[\lambda-\lambda 3]\cos[\varphi 3]\sin[\varphi]+\cos[\varphi]\sin[\varphi 3])\right.\\
&\qquad\quad (-\cos[\lambda-\lambda 3]\cos[\varphi]\cos[\varphi 3]-\sin[\varphi]\sin[\varphi 3])\\
&\qquad\quad \left.(2r-2\,r3\,(\cos[\lambda-\lambda 3]\cos[\varphi]\cos[\varphi 3]+\sin[\varphi]\sin[\varphi 3]))\right) \Big/\\
&\qquad \left(2\left(r^2+r3^2-2\,r\,r3\,(\cos[\lambda-\lambda 3]\cos[\varphi]\cos[\varphi 3]+\sin[\varphi]\sin[\varphi 3])\right)^{7/2}\right) +\\
&\qquad \left(18\, r\, r3^2\, (-\cos[\lambda-\lambda 3]\cos[\varphi 3]\sin[\varphi]+\cos[\varphi]\sin[\varphi 3])\right.\\
&\qquad\quad \left.(-\cos[\lambda-\lambda 3]\cos[\varphi]\cos[\varphi 3]-\sin[\varphi]\sin[\varphi 3])\right) \Big/\\
&\qquad \left(r^2+r3^2-2\,r\,r3\,(\cos[\lambda-\lambda 3]\cos[\varphi]\cos[\varphi 3]+\sin[\varphi]\sin[\varphi 3])\right)^{5/2} -\\
&\qquad \left(3\, r\, r3\,(\cos[\lambda-\lambda 3]\cos[\varphi 3]\sin[\varphi]-\cos[\varphi]\sin[\varphi 3])\right.\\
&\qquad\quad \left.(2r-2\,r3\,(\cos[\lambda-\lambda 3]\cos[\varphi]\cos[\varphi 3]+\sin[\varphi]\sin[\varphi 3]))\right) \Big/\\
&\qquad \left(2\left(r^2+r3^2-2\,r\,r3\,(\cos[\lambda-\lambda 3]\cos[\varphi]\cos[\varphi 3]+\sin[\varphi]\sin[\varphi 3])\right)^{5/2}\right) +\\
&\qquad \left. \frac{r3\,(\cos[\lambda-\lambda 3]\cos[\varphi 3]\sin[\varphi]-\cos[\varphi]\sin[\varphi 3])}{\left(r^2+r3^2-2\,r\,r3\,(\cos[\lambda-\lambda 3]\cos[\varphi]\cos[\varphi 3]+\sin[\varphi]\sin[\varphi 3])\right)^{3/2}} \right)
\end{aligned}$$

In[•]:=
```
V2333 = D[kernel, {φ, 1}, {r, 3}]
```

Out[•]=
$$\begin{aligned}
&r\, r3^3 \cos[\varphi 3]\,(-\cos[\lambda-\lambda 3]\cos[\varphi 3]\sin[\varphi]+\cos[\varphi]\sin[\varphi 3])\\
&\qquad \left(-\frac{105\,(2r-2\,r3\,(\cos[\lambda-\lambda 3]\cos[\varphi]\cos[\varphi 3]+\sin[\varphi]\sin[\varphi 3]))^3}{8\left(r^2+r3^2-2\,r\,r3\,(\cos[\lambda-\lambda 3]\cos[\varphi]\cos[\varphi 3]+\sin[\varphi]\sin[\varphi 3])\right)^{9/2}} + \right.\\
&\qquad \left. \frac{45\,(2r-2\,r3\,(\cos[\lambda-\lambda 3]\cos[\varphi]\cos[\varphi 3]+\sin[\varphi]\sin[\varphi 3]))}{2\left(r^2+r3^2-2\,r\,r3\,(\cos[\lambda-\lambda 3]\cos[\varphi]\cos[\varphi 3]+\sin[\varphi]\sin[\varphi 3])\right)^{7/2}} \right) +\\
&\quad 3\, r3^3 \cos[\varphi 3]\,(-\cos[\lambda-\lambda 3]\cos[\varphi 3]\sin[\varphi]+\cos[\varphi]\sin[\varphi 3])\\
&\qquad \left(\frac{15\,(2r-2\,r3\,(\cos[\lambda-\lambda 3]\cos[\varphi]\cos[\varphi 3]+\sin[\varphi]\sin[\varphi 3]))^2}{4\left(r^2+r3^2-2\,r\,r3\,(\cos[\lambda-\lambda 3]\cos[\varphi]\cos[\varphi 3]+\sin[\varphi]\sin[\varphi 3])\right)^{7/2}} - \right.\\
&\qquad \left. \frac{3}{\left(r^2+r3^2-2\,r\,r3\,(\cos[\lambda-\lambda 3]\cos[\varphi]\cos[\varphi 3]+\sin[\varphi]\sin[\varphi 3])\right)^{5/2}} \right)
\end{aligned}$$

In[•]:= V_{1122} = D[kernel, {λ, 2}, {φ, 2}]

Out[•]=

$$r3^2 \cos[\varphi 3] \left(-\left(\left(60\, r^3\, r3^3 \cos[\varphi] \cos[\varphi 3]^2 \sin[\lambda - \lambda 3]^2 \right.\right.\right.$$

$$\left.\left.\left. \sin[\varphi]\, (-\cos[\lambda - \lambda 3] \cos[\varphi 3] \sin[\varphi] + \cos[\varphi] \sin[\varphi 3]) \right) \Big/ \right.$$

$$\left. \left(r^2 + r3^2 - 2\, r\, r3\, (\cos[\lambda - \lambda 3] \cos[\varphi] \cos[\varphi 3] + \sin[\varphi] \sin[\varphi 3]) \right)^{7/2} \right) +$$

$$\frac{3\, r^2\, r3^2 \cos[\varphi 3]^2 \sin[\lambda - \lambda 3]^2 \left(-2 \cos[\varphi]^2 + 2 \sin[\varphi]^2 \right)}{\left(r^2 + r3^2 - 2\, r\, r3\, (\cos[\lambda - \lambda 3] \cos[\varphi] \cos[\varphi 3] + \sin[\varphi] \sin[\varphi 3]) \right)^{5/2}} +$$

$$3\, r^2\, r3^2 \cos[\varphi]^2 \cos[\varphi 3]^2 \sin[\lambda - \lambda 3]^2$$

$$\left(\frac{35\, r^2\, r3^2\, (-\cos[\lambda - \lambda 3] \cos[\varphi 3] \sin[\varphi] + \cos[\varphi] \sin[\varphi 3])^2}{\left(r^2 + r3^2 - 2\, r\, r3\, (\cos[\lambda - \lambda 3] \cos[\varphi] \cos[\varphi 3] + \sin[\varphi] \sin[\varphi 3]) \right)^{9/2}} + \right.$$

$$\left. \frac{5\, r\, r3\, (-\cos[\lambda - \lambda 3] \cos[\varphi] \cos[\varphi 3] - \sin[\varphi] \sin[\varphi 3])}{\left(r^2 + r3^2 - 2\, r\, r3\, (\cos[\lambda - \lambda 3] \cos[\varphi] \cos[\varphi 3] + \sin[\varphi] \sin[\varphi 3]) \right)^{7/2}} \right) - r\, r3 \cos[$$

$$\lambda - \lambda 3] \left(-\frac{6\, r\, r3 \cos[\varphi 3] \sin[\varphi]\, (-\cos[\lambda - \lambda 3] \cos[\varphi 3] \sin[\varphi] + \cos[\varphi] \sin[\varphi 3])}{\left(r^2 + r3^2 - 2\, r\, r3\, (\cos[\lambda - \lambda 3] \cos[\varphi] \cos[\varphi 3] + \sin[\varphi] \sin[\varphi 3]) \right)^{5/2}} - \right.$$

$$\frac{\cos[\varphi] \cos[\varphi 3]}{\left(r^2 + r3^2 - 2\, r\, r3\, (\cos[\lambda - \lambda 3] \cos[\varphi] \cos[\varphi 3] + \sin[\varphi] \sin[\varphi 3]) \right)^{3/2}} + \cos[\varphi]$$

$$\cos[\varphi 3] \left(\frac{15\, r^2\, r3^2\, (-\cos[\lambda - \lambda 3] \cos[\varphi 3] \sin[\varphi] + \cos[\varphi] \sin[\varphi 3])^2}{\left(r^2 + r3^2 - 2\, r\, r3\, (\cos[\lambda - \lambda 3] \cos[\varphi] \cos[\varphi 3] + \sin[\varphi] \sin[\varphi 3]) \right)^{7/2}} + \right.$$

$$\left. \left. \left. \frac{3\, r\, r3\, (-\cos[\lambda - \lambda 3] \cos[\varphi] \cos[\varphi 3] - \sin[\varphi] \sin[\varphi 3])}{\left(r^2 + r3^2 - 2\, r\, r3\, (\cos[\lambda - \lambda 3] \cos[\varphi] \cos[\varphi 3] + \sin[\varphi] \sin[\varphi 3]) \right)^{5/2}} \right) \right) \right)$$

In[•]:=
```
V1133 = D[kernel, {λ, 2}, {r, 2}]
```

Out[•]=
$$
\begin{aligned}
&r3^2 \cos[\varphi 3] \left(-\left(\left(30\, r\, r3^2 \cos[\varphi]^2 \cos[\varphi 3]^2 \sin[\lambda - \lambda 3]^2 \right.\right.\right. \\
&\qquad \left.\left.\left. (2\, r - 2\, r3\, (\cos[\lambda - \lambda 3] \cos[\varphi] \cos[\varphi 3] + \sin[\varphi] \sin[\varphi 3])) \right) \Big/ \right. \\
&\qquad \left. \left(r^2 + r3^2 - 2\, r\, r3\, (\cos[\lambda - \lambda 3] \cos[\varphi] \cos[\varphi 3] + \sin[\varphi] \sin[\varphi 3]) \right)^{7/2} \right) + \\
&\qquad \frac{6\, r3^2 \cos[\varphi]^2 \cos[\varphi 3]^2 \sin[\lambda - \lambda 3]^2}{\left(r^2 + r3^2 - 2\, r\, r3\, (\cos[\lambda - \lambda 3] \cos[\varphi] \cos[\varphi 3] + \sin[\varphi] \sin[\varphi 3]) \right)^{5/2}} + \\
&\qquad (3\, r3 \cos[\lambda - \lambda 3] \cos[\varphi] \cos[\varphi 3] \\
&\qquad\quad (2\, r - 2\, r3\, (\cos[\lambda - \lambda 3] \cos[\varphi] \cos[\varphi 3] + \sin[\varphi] \sin[\varphi 3]))) \Big/ \\
&\qquad \left(r^2 + r3^2 - 2\, r\, r3\, (\cos[\lambda - \lambda 3] \cos[\varphi] \cos[\varphi 3] + \sin[\varphi] \sin[\varphi 3]) \right)^{5/2} + \\
&\qquad 3\, r^2\, r3^2 \cos[\varphi]^2 \cos[\varphi 3]^2 \sin[\lambda - \lambda 3]^2 \\
&\qquad\quad \left(\frac{35\, (2\, r - 2\, r3\, (\cos[\lambda - \lambda 3] \cos[\varphi] \cos[\varphi 3] + \sin[\varphi] \sin[\varphi 3]))^2}{4\, \left(r^2 + r3^2 - 2\, r\, r3\, (\cos[\lambda - \lambda 3] \cos[\varphi] \cos[\varphi 3] + \sin[\varphi] \sin[\varphi 3]) \right)^{9/2}} - \right. \\
&\qquad\qquad \left. \frac{5}{\left(r^2 + r3^2 - 2\, r\, r3\, (\cos[\lambda - \lambda 3] \cos[\varphi] \cos[\varphi 3] + \sin[\varphi] \sin[\varphi 3]) \right)^{7/2}} \right) - \\
&\qquad r\, r3 \cos[\lambda - \lambda 3] \cos[\varphi] \cos[\varphi 3] \\
&\qquad\quad \left(\frac{15\, (2\, r - 2\, r3\, (\cos[\lambda - \lambda 3] \cos[\varphi] \cos[\varphi 3] + \sin[\varphi] \sin[\varphi 3]))^2}{4\, \left(r^2 + r3^2 - 2\, r\, r3\, (\cos[\lambda - \lambda 3] \cos[\varphi] \cos[\varphi 3] + \sin[\varphi] \sin[\varphi 3]) \right)^{7/2}} - \right. \\
&\qquad\qquad \left. \left. \frac{3}{\left(r^2 + r3^2 - 2\, r\, r3\, (\cos[\lambda - \lambda 3] \cos[\varphi] \cos[\varphi 3] + \sin[\varphi] \sin[\varphi 3]) \right)^{5/2}} \right) \right)
\end{aligned}
$$

In[•]:=
$$V_{2233} = D[\text{kernel}, \{\varphi, 2\}, \{r, 2\}]$$

Out[•]=
$$r3^2 \cos[\varphi 3] \left(-\left(\left(30\, r\, r3^2\, (-\cos[\lambda - \lambda 3]\cos[\varphi 3]\sin[\varphi] + \cos[\varphi]\sin[\varphi 3])^2 \right.\right.\right.$$
$$\left.\left.\left. (2\, r - 2\, r3\, (\cos[\lambda - \lambda 3]\cos[\varphi]\cos[\varphi 3] + \sin[\varphi]\sin[\varphi 3])) \right) \Big/ \right.$$
$$\left. \left(r^2 + r3^2 - 2\, r\, r3\, (\cos[\lambda - \lambda 3]\cos[\varphi]\cos[\varphi 3] + \sin[\varphi]\sin[\varphi 3]) \right)^{7/2} \right) +$$
$$\frac{6\, r3^2\, (-\cos[\lambda - \lambda 3]\cos[\varphi 3]\sin[\varphi] + \cos[\varphi]\sin[\varphi 3])^2}{\left(r^2 + r3^2 - 2\, r\, r3\, (\cos[\lambda - \lambda 3]\cos[\varphi]\cos[\varphi 3] + \sin[\varphi]\sin[\varphi 3]) \right)^{5/2}} -$$
$$(3\, r3\, (-\cos[\lambda - \lambda 3]\cos[\varphi]\cos[\varphi 3] - \sin[\varphi]\sin[\varphi 3])$$
$$(2\, r - 2\, r3\, (\cos[\lambda - \lambda 3]\cos[\varphi]\cos[\varphi 3] + \sin[\varphi]\sin[\varphi 3]))) \Big/$$
$$\left(r^2 + r3^2 - 2\, r\, r3\, (\cos[\lambda - \lambda 3]\cos[\varphi]\cos[\varphi 3] + \sin[\varphi]\sin[\varphi 3]) \right)^{5/2} +$$
$$3\, r^2\, r3^2\, (-\cos[\lambda - \lambda 3]\cos[\varphi 3]\sin[\varphi] + \cos[\varphi]\sin[\varphi 3])^2$$
$$\left(\frac{35\, (2\, r - 2\, r3\, (\cos[\lambda - \lambda 3]\cos[\varphi]\cos[\varphi 3] + \sin[\varphi]\sin[\varphi 3]))^2}{4\, \left(r^2 + r3^2 - 2\, r\, r3\, (\cos[\lambda - \lambda 3]\cos[\varphi]\cos[\varphi 3] + \sin[\varphi]\sin[\varphi 3]) \right)^{9/2}} - \right.$$
$$\left. \frac{5}{\left(r^2 + r3^2 - 2\, r\, r3\, (\cos[\lambda - \lambda 3]\cos[\varphi]\cos[\varphi 3] + \sin[\varphi]\sin[\varphi 3]) \right)^{7/2}} \right) +$$
$$r\, r3\, (-\cos[\lambda - \lambda 3]\cos[\varphi]\cos[\varphi 3] - \sin[\varphi]\sin[\varphi 3])$$
$$\left(\frac{15\, (2\, r - 2\, r3\, (\cos[\lambda - \lambda 3]\cos[\varphi]\cos[\varphi 3] + \sin[\varphi]\sin[\varphi 3]))^2}{4\, \left(r^2 + r3^2 - 2\, r\, r3\, (\cos[\lambda - \lambda 3]\cos[\varphi]\cos[\varphi 3] + \sin[\varphi]\sin[\varphi 3]) \right)^{7/2}} - \right.$$
$$\left. \left. \frac{3}{\left(r^2 + r3^2 - 2\, r\, r3\, (\cos[\lambda - \lambda 3]\cos[\varphi]\cos[\varphi 3] + \sin[\varphi]\sin[\varphi 3]) \right)^{5/2}} \right) \right)$$

In[•]:=
```
V1123 = D[kernel, {λ, 2}, {φ, 1}, {r, 1}]
```

Out[•]=
```
r3^2 Cos[φ3]
```

$$\Bigg(-\Bigg(\Big(105\, r^3\, r3^3 \cos[\varphi]^2 \cos[\varphi 3]^2 \sin[\lambda-\lambda 3]^2 \,(-\cos[\lambda-\lambda 3]\cos[\varphi 3]\sin[\varphi]+\cos[\varphi]\sin[\varphi 3])\,(2\,r-2\,r3\,(\cos[\lambda-\lambda 3]\cos[\varphi]\cos[\varphi 3]+\sin[\varphi]\sin[\varphi 3]))\Big)\Big/\Big(2\left(r^2+r3^2-2\,r\,r3\,(\cos[\lambda-\lambda 3]\cos[\varphi]\cos[\varphi 3]+\sin[\varphi]\sin[\varphi 3])\right)^{9/2}\Big)\Bigg)+$$

$$\Big(45\, r^2\, r3^3 \cos[\varphi]^2 \cos[\varphi 3]^2 \sin[\lambda-\lambda 3]^2\,(-\cos[\lambda-\lambda 3]\cos[\varphi 3]\sin[\varphi]+\cos[\varphi]\sin[\varphi 3])\Big)\Big/\Big(r^2+r3^2-2\,r\,r3\,(\cos[\lambda-\lambda 3]\cos[\varphi]\cos[\varphi 3]+\sin[\varphi]\sin[\varphi 3])\Big)^{7/2}+$$

$$\Big(15\, r^2\, r3^2 \cos[\varphi]\cos[\varphi 3]^2 \sin[\lambda-\lambda 3]^2 \sin[\varphi]\,(2\,r-2\,r3\,(\cos[\lambda-\lambda 3]\cos[\varphi]\cos[\varphi 3]+\sin[\varphi]\sin[\varphi 3]))\Big)\Big/\Big(r^2+r3^2-2\,r\,r3\,(\cos[\lambda-\lambda 3]\cos[\varphi]\cos[\varphi 3]+\sin[\varphi]\sin[\varphi 3])\Big)^{7/2}+$$

$$\Big(15\, r^2\, r3^2 \cos[\lambda-\lambda 3]\cos[\varphi]\cos[\varphi 3]\,(-\cos[\lambda-\lambda 3]\cos[\varphi 3]\sin[\varphi]+\cos[\varphi]\sin[\varphi 3])\,(2\,r-2\,r3\,(\cos[\lambda-\lambda 3]\cos[\varphi]\cos[\varphi 3]+\sin[\varphi]\sin[\varphi 3]))\Big)\Big/\Big(2\left(r^2+r3^2-2\,r\,r3\,(\cos[\lambda-\lambda 3]\cos[\varphi]\cos[\varphi 3]+\sin[\varphi]\sin[\varphi 3])\right)^{7/2}\Big)-$$

$$\frac{12\, r\, r3^2 \cos[\varphi]\cos[\varphi 3]^2 \sin[\lambda-\lambda 3]^2 \sin[\varphi]}{\left(r^2+r3^2-2\,r\,r3\,(\cos[\lambda-\lambda 3]\cos[\varphi]\cos[\varphi 3]+\sin[\varphi]\sin[\varphi 3])\right)^{5/2}}-$$

$$\frac{6\, r\, r3^2 \cos[\lambda-\lambda 3]\cos[\varphi]\cos[\varphi 3]\,(-\cos[\lambda-\lambda 3]\cos[\varphi 3]\sin[\varphi]+\cos[\varphi]\sin[\varphi 3])}{\left(r^2+r3^2-2\,r\,r3\,(\cos[\lambda-\lambda 3]\cos[\varphi]\cos[\varphi 3]+\sin[\varphi]\sin[\varphi 3])\right)^{5/2}}-$$

$$\Big(3\, r\, r3 \cos[\lambda-\lambda 3]\cos[\varphi 3]\sin[\varphi]\,(2\,r-2\,r3\,(\cos[\lambda-\lambda 3]\cos[\varphi]\cos[\varphi 3]+\sin[\varphi]\sin[\varphi 3]))\Big)\Big/\Big(2\left(r^2+r3^2-2\,r\,r3\,(\cos[\lambda-\lambda 3]\cos[\varphi]\cos[\varphi 3]+\sin[\varphi]\sin[\varphi 3])\right)^{5/2}\Big)+$$

$$\frac{r3\cos[\lambda-\lambda 3]\cos[\varphi 3]\sin[\varphi]}{\left(r^2+r3^2-2\,r\,r3\,(\cos[\lambda-\lambda 3]\cos[\varphi]\cos[\varphi 3]+\sin[\varphi]\sin[\varphi 3])\right)^{3/2}}\Bigg)$$

In[•]:=
$$V_{1223} = D[\text{kernel}, \{\lambda, 1\}, \{\varphi, 2\}, \{r, 1\}]$$

Out[•]=
$$-\left(\left(15\, r^2\, r3^4\, \text{Cos}[\varphi 3]^2\, \text{Sin}[\lambda-\lambda 3]\, \text{Sin}[\varphi]\, (-\text{Cos}[\lambda-\lambda 3]\, \text{Cos}[\varphi 3]\, \text{Sin}[\varphi]+\text{Cos}[\varphi]\, \text{Sin}[\varphi 3])\right.\right.$$
$$\left.\left.(2\, r-2\, r3\, (\text{Cos}[\lambda-\lambda 3]\, \text{Cos}[\varphi]\, \text{Cos}[\varphi 3]+\text{Sin}[\varphi]\, \text{Sin}[\varphi 3]))\right)\Big/$$
$$\left(r^2+r3^2-2\, r\, r3\, (\text{Cos}[\lambda-\lambda 3]\, \text{Cos}[\varphi]\, \text{Cos}[\varphi 3]+\text{Sin}[\varphi]\, \text{Sin}[\varphi 3])\right)^{7/2}\Big)+$$
$$\frac{12\, r\, r3^4\, \text{Cos}[\varphi 3]^2\, \text{Sin}[\lambda-\lambda 3]\, \text{Sin}[\varphi]\, (-\text{Cos}[\lambda-\lambda 3]\, \text{Cos}[\varphi 3]\, \text{Sin}[\varphi]+\text{Cos}[\varphi]\, \text{Sin}[\varphi 3])}{\left(r^2+r3^2-2\, r\, r3\, (\text{Cos}[\lambda-\lambda 3]\, \text{Cos}[\varphi]\, \text{Cos}[\varphi 3]+\text{Sin}[\varphi]\, \text{Sin}[\varphi 3])\right)^{5/2}}-$$
$$\left(3\, r\, r3^3\, \text{Cos}[\varphi]\, \text{Cos}[\varphi 3]^2\, \text{Sin}[\lambda-\lambda 3]\right.$$
$$\left.(2\, r-2\, r3\, (\text{Cos}[\lambda-\lambda 3]\, \text{Cos}[\varphi]\, \text{Cos}[\varphi 3]+\text{Sin}[\varphi]\, \text{Sin}[\varphi 3]))\right)\Big/$$
$$\left(2\left(r^2+r3^2-2\, r\, r3\, (\text{Cos}[\lambda-\lambda 3]\, \text{Cos}[\varphi]\, \text{Cos}[\varphi 3]+\text{Sin}[\varphi]\, \text{Sin}[\varphi 3])\right)^{5/2}\right)+$$
$$\frac{r3^3\, \text{Cos}[\varphi]\, \text{Cos}[\varphi 3]^2\, \text{Sin}[\lambda-\lambda 3]}{\left(r^2+r3^2-2\, r\, r3\, (\text{Cos}[\lambda-\lambda 3]\, \text{Cos}[\varphi]\, \text{Cos}[\varphi 3]+\text{Sin}[\varphi]\, \text{Sin}[\varphi 3])\right)^{3/2}}-$$
$$r\, r3^3\, \text{Cos}[\varphi]\, \text{Cos}[\varphi 3]^2\, \text{Sin}[\lambda-\lambda 3]$$
$$\left(-\left(\left(105\, r^2\, r3^2\, (-\text{Cos}[\lambda-\lambda 3]\, \text{Cos}[\varphi 3]\, \text{Sin}[\varphi]+\text{Cos}[\varphi]\, \text{Sin}[\varphi 3])^2\right.\right.\right.$$
$$\left.\left.(2\, r-2\, r3\, (\text{Cos}[\lambda-\lambda 3]\, \text{Cos}[\varphi]\, \text{Cos}[\varphi 3]+\text{Sin}[\varphi]\, \text{Sin}[\varphi 3]))\right)\Big/$$
$$\left.\left(2\left(r^2+r3^2-2\, r\, r3\, (\text{Cos}[\lambda-\lambda 3]\, \text{Cos}[\varphi]\, \text{Cos}[\varphi 3]+\text{Sin}[\varphi]\, \text{Sin}[\varphi 3])\right)^{9/2}\right)\right)+$$
$$\frac{30\, r\, r3^2\, (-\text{Cos}[\lambda-\lambda 3]\, \text{Cos}[\varphi 3]\, \text{Sin}[\varphi]+\text{Cos}[\varphi]\, \text{Sin}[\varphi 3])^2}{\left(r^2+r3^2-2\, r\, r3\, (\text{Cos}[\lambda-\lambda 3]\, \text{Cos}[\varphi]\, \text{Cos}[\varphi 3]+\text{Sin}[\varphi]\, \text{Sin}[\varphi 3])\right)^{7/2}}-$$
$$\left(15\, r\, r3\, (-\text{Cos}[\lambda-\lambda 3]\, \text{Cos}[\varphi]\, \text{Cos}[\varphi 3]-\text{Sin}[\varphi]\, \text{Sin}[\varphi 3])\right.$$
$$\left.(2\, r-2\, r3\, (\text{Cos}[\lambda-\lambda 3]\, \text{Cos}[\varphi]\, \text{Cos}[\varphi 3]+\text{Sin}[\varphi]\, \text{Sin}[\varphi 3]))\right)\Big/$$
$$\left(2\left(r^2+r3^2-2\, r\, r3\, (\text{Cos}[\lambda-\lambda 3]\, \text{Cos}[\varphi]\, \text{Cos}[\varphi 3]+\text{Sin}[\varphi]\, \text{Sin}[\varphi 3])\right)^{7/2}\right)+$$
$$\left.\left.\frac{3\, r3\, (-\text{Cos}[\lambda-\lambda 3]\, \text{Cos}[\varphi]\, \text{Cos}[\varphi 3]-\text{Sin}[\varphi]\, \text{Sin}[\varphi 3])}{\left(r^2+r3^2-2\, r\, r3\, (\text{Cos}[\lambda-\lambda 3]\, \text{Cos}[\varphi]\, \text{Cos}[\varphi 3]+\text{Sin}[\varphi]\, \text{Sin}[\varphi 3])\right)^{5/2}}\right)-\right.$$
$$r3^3\, \text{Cos}[\varphi]\, \text{Cos}[\varphi 3]^2\, \text{Sin}[\lambda-\lambda 3]$$
$$\left(\frac{15\, r^2\, r3^2\, (-\text{Cos}[\lambda-\lambda 3]\, \text{Cos}[\varphi 3]\, \text{Sin}[\varphi]+\text{Cos}[\varphi]\, \text{Sin}[\varphi 3])^2}{\left(r^2+r3^2-2\, r\, r3\, (\text{Cos}[\lambda-\lambda 3]\, \text{Cos}[\varphi]\, \text{Cos}[\varphi 3]+\text{Sin}[\varphi]\, \text{Sin}[\varphi 3])\right)^{7/2}}+\right.$$
$$\left.\frac{3\, r\, r3\, (-\text{Cos}[\lambda-\lambda 3]\, \text{Cos}[\varphi]\, \text{Cos}[\varphi 3]-\text{Sin}[\varphi]\, \text{Sin}[\varphi 3])}{\left(r^2+r3^2-2\, r\, r3\, (\text{Cos}[\lambda-\lambda 3]\, \text{Cos}[\varphi]\, \text{Cos}[\varphi 3]+\text{Sin}[\varphi]\, \text{Sin}[\varphi 3])\right)^{5/2}}\right)$$

In[•]:=
```
V1233 = D[kernel, {λ, 1}, {φ, 1}, {r, 2}]
```

Out[•]=
$$\left(30\, r\, r3^4 \cos[\varphi] \cos[\varphi 3]^2 \sin[\lambda-\lambda 3]\, (-\cos[\lambda-\lambda 3] \cos[\varphi 3] \sin[\varphi] + \cos[\varphi] \sin[\varphi 3])\right.$$
$$\left.(2\, r - 2\, r3\, (\cos[\lambda-\lambda 3] \cos[\varphi] \cos[\varphi 3] + \sin[\varphi] \sin[\varphi 3]))\right) \Big/$$
$$\left(r^2 + r3^2 - 2\, r\, r3\, (\cos[\lambda-\lambda 3] \cos[\varphi] \cos[\varphi 3] + \sin[\varphi] \sin[\varphi 3])\right)^{7/2} -$$
$$\frac{6\, r3^4 \cos[\varphi] \cos[\varphi 3]^2 \sin[\lambda-\lambda 3]\, (-\cos[\lambda-\lambda 3] \cos[\varphi 3] \sin[\varphi] + \cos[\varphi] \sin[\varphi 3])}{\left(r^2 + r3^2 - 2\, r\, r3\, (\cos[\lambda-\lambda 3] \cos[\varphi] \cos[\varphi 3] + \sin[\varphi] \sin[\varphi 3])\right)^{5/2}} -$$
$$\left(3\, r3^3 \cos[\varphi 3]^2 \sin[\lambda-\lambda 3] \sin[\varphi]\right.$$
$$\left.(2\, r - 2\, r3\, (\cos[\lambda-\lambda 3] \cos[\varphi] \cos[\varphi 3] + \sin[\varphi] \sin[\varphi 3]))\right) \Big/$$
$$\left(r^2 + r3^2 - 2\, r\, r3\, (\cos[\lambda-\lambda 3] \cos[\varphi] \cos[\varphi 3] + \sin[\varphi] \sin[\varphi 3])\right)^{5/2} -$$
$$3\, r^2\, r3^4 \cos[\varphi] \cos[\varphi 3]^2 \sin[\lambda-\lambda 3]\, (-\cos[\lambda-\lambda 3] \cos[\varphi 3] \sin[\varphi] + \cos[\varphi] \sin[\varphi 3])$$
$$\left(\frac{35\, (2\, r - 2\, r3\, (\cos[\lambda-\lambda 3] \cos[\varphi] \cos[\varphi 3] + \sin[\varphi] \sin[\varphi 3]))^2}{4\, \left(r^2 + r3^2 - 2\, r\, r3\, (\cos[\lambda-\lambda 3] \cos[\varphi] \cos[\varphi 3] + \sin[\varphi] \sin[\varphi 3])\right)^{9/2}} -\right.$$
$$\left.\frac{5}{\left(r^2 + r3^2 - 2\, r\, r3\, (\cos[\lambda-\lambda 3] \cos[\varphi] \cos[\varphi 3] + \sin[\varphi] \sin[\varphi 3])\right)^{7/2}}\right) +$$
$$r\, r3^3 \cos[\varphi 3]^2 \sin[\lambda-\lambda 3] \sin[\varphi]$$
$$\left(\frac{15\, (2\, r - 2\, r3\, (\cos[\lambda-\lambda 3] \cos[\varphi] \cos[\varphi 3] + \sin[\varphi] \sin[\varphi 3]))^2}{4\, \left(r^2 + r3^2 - 2\, r\, r3\, (\cos[\lambda-\lambda 3] \cos[\varphi] \cos[\varphi 3] + \sin[\varphi] \sin[\varphi 3])\right)^{7/2}} -\right.$$
$$\left.\frac{3}{\left(r^2 + r3^2 - 2\, r\, r3\, (\cos[\lambda-\lambda 3] \cos[\varphi] \cos[\varphi 3] + \sin[\varphi] \sin[\varphi 3])\right)^{5/2}}\right)$$

3. Expressions for Laplace equations for physical components of the fourth-order gravitational potential gradients of a uniform tesseroid in spherical integral kernels as Eqs. (27) - (32)

```
Laplace1 = FullSimplify[V400 + V220 + V202] (*Eq. (27)*)
```

Out[•]= 0

```
Laplace2 = FullSimplify[V310 + V130 + V112] (*Eq. (28)*)
```

Out[•]= 0

```
Laplace3 = FullSimplify[V301 + V121 + V103] (*Eq. (29)*)
```

Out[•]= 0

```
Laplace4 = FullSimplify[V220 + V040 + V022] (*Eq. (30)*)
```

Out[•]= 0

```
Laplace5 = FullSimplify[V211 + V031 + V013] (*Eq. (31)*)
```

Out[•]= 0

```
Laplace6 = FullSimplify[V202 + V022 + V004] (*Eq. (32)*)
```

Out[•]= 0