Readme

1. This Appendix presents the validation process with the transformation cycle method for all physical components of first-, second-, and third-order potential gradients in Tables A1 - A13.
2. In the following parts, the symbol "*" is omitted for simplicity.
3. Each of the following sections is independent by using the command **Clear["Global`*"]** to clear previous arguments. The name of each section is from Table 2.
4. After running all codes, all components become themselves in the following Sects. 1-12 for 12 routes.

1. Algebraic form, Cartesian coordinates, Routes 1 → 3 → 5, and Table A1 → Table A5 → Tables A9, A10

```
In[•]:= Clear["Global`*"];
```

1.1 Table A1

```
In[•]:= Vr = (x * Vx + y * Vy) / Sqrt[x^2 + y^2];

In[•]:= Vλ = (-y * Vx + x * Vy) / Sqrt[x^2 + y^2];

In[•]:= Vz = Vz;

In[•]:= Vrr = (x^2 * Vxx + 2 * x * y * Vxy + y^2 * Vyy) / (x^2 + y^2);

In[•]:= Vrλ = (-x * y * Vxx + (x^2 - y^2) * Vxy + x * y * Vyy) / (x^2 + y^2);

In[•]:= Vrz = (x * Vxz + y * Vyz) / Sqrt[x^2 + y^2];

In[•]:= Vλλ = (y^2 * Vxx - 2 * x * y * Vxy + x^2 * Vyy) / (x^2 + y^2);

In[•]:= Vλz = (-y * Vxz + x * Vyz) / Sqrt[x^2 + y^2];

In[•]:= Vzz = Vzz;

In[•]:= Vrrr = (x^3 * Vxxx + 3 * x^2 * y * Vxxy + 3 * x * y^2 * Vyyx + y^3 * Vyyy) / (x^2 + y^2)^(3/2);

In[•]:= Vrrλ = (-x^2 * y * Vxxx + x * (x^2 - 2 * y^2) * Vxxy + y * (2 * x^2 - y^2) * Vyyx + x * y^2 * Vyyy) /
          (x^2 + y^2)^(3/2);

In[•]:= Vrrz = (x^2 * Vxxz + 2 * x * y * Vxyz + y^2 * Vyyz) / (x^2 + y^2);

In[•]:= Vrλz = (-x * y * Vxxz + (x^2 - y^2) * Vxyz + x * y * Vyyz) / (x^2 + y^2);

In[•]:= Vλλr =
          (x * y^2 * Vxxx + y * (y^2 - 2 * x^2) * Vxxy + x * (x^2 - 2 * y^2) * Vyyx + x^2 * y * Vyyy) / (x^2 + y^2)^(3/2);

In[•]:= Vλλλ = (-y^3 * Vxxx + 3 * x * y^2 * Vxxy - 3 * x^2 * y * Vyyx + x^3 * Vyyy) / (x^2 + y^2)^(3/2);

In[•]:= Vλλz = (y^2 * Vxxz - 2 * x * y * Vxyz + x^2 * Vyyz) / (x^2 + y^2);
```

```
In[•]:= Vzzr = (x * Vzzx + y * Vzzy) / Sqrt[x^2 + y^2];

In[•]:= Vzzλ = (-y * Vzzx + x * Vzzy) / Sqrt[x^2 + y^2];

In[•]:= Vzzz = Vzzz;
```

1.2 Table A5

```
In[•]:= Vλ = Vλ;

In[•]:= Vφ = (-z * Vr + r * Vz) / Sqrt[r^2 + z^2];

In[•]:= Vρ = (r * Vr + z * Vz) / Sqrt[r^2 + z^2];

In[•]:= Vλλ = Vλλ;

In[•]:= Vλφ = (-z * Vrλ + r * Vλz) / Sqrt[r^2 + z^2];

In[•]:= Vλρ = (r * Vrλ + z * Vλz) / Sqrt[r^2 + z^2];

In[•]:= Vφφ = (z^2 * Vrr - 2 * r * z * Vrz + r^2 * Vzz) / (r^2 + z^2);

In[•]:= Vφρ = (-r * z * Vrr + (r^2 - z^2) * Vrz + r * z * Vzz) / (r^2 + z^2);

In[•]:= Vρρ = (r^2 * Vrr + 2 * r * z * Vrz + z^2 * Vzz) / (r^2 + z^2);

In[•]:= Vλλλ = Vλλλ;

In[•]:= Vλλφ = (-z * Vλλr + r * Vλλz) / Sqrt[r^2 + z^2];

In[•]:= Vλλρ = (r * Vλλr + z * Vλλz) / Sqrt[r^2 + z^2];

In[•]:= Vλφρ = (-r * z * Vrrλ + (r^2 - z^2) * Vrλz + r * z * Vzzλ) / (r^2 + z^2);

In[•]:= Vφφλ = (z^2 * Vrrλ - 2 * r * z * Vrλz + r^2 * Vzzλ) / (r^2 + z^2);

In[•]:= Vφφφ = (-z^3 * Vrrr + 3 * r * z^2 * Vrrz - 3 * r^2 * z * Vzzr + r^3 * Vzzz) / (r^2 + z^2)^(3/2);

In[•]:= Vφφρ =
          (r * z^2 * Vrrr + z * (z^2 - 2 * r^2) * Vrrz + r * (r^2 - 2 * z^2) * Vzzr + r^2 * z * Vzzz) / (r^2 + z^2)^(3/2);

In[•]:= Vρρλ = (r^2 * Vrrλ + 2 * r * z * Vrλz + z^2 * Vzzλ) / (r^2 + z^2);

In[•]:= Vρρφ = (-r^2 * z * Vrrr + r * (r^2 - 2 * z^2) * Vrrz + z * (2 * r^2 - z^2) * Vzzr + r * z^2 * Vzzz) /
          (r^2 + z^2)^(3/2);

In[•]:= Vρρρ = (r^3 Vrrr + 3 * r^2 * z * Vrrz + 3 * r * z^2 * Vzzr + z^3 * Vzzz) / (r^2 + z^2)^(3/2);
```

1.3 Tables A9, A10

```
In[•]:= VxRoutes135 =
  (-y * Sqrt[x^2 + y^2 + z^2] * Vλ - x * z * Vφ + x * Sqrt[x^2 + y^2] * Vρ) / (Sqrt[x^2 + y^2] * Sqrt[x^2 + y^2 + z^2]);

In[•]:= VyRoutes135 =
  (x * Sqrt[x^2 + y^2 + z^2] * Vλ - y * z * Vφ + y * Sqrt[x^2 + y^2] * Vρ) / (Sqrt[x^2 + y^2] * Sqrt[x^2 + y^2 + z^2]);

In[•]:= VzRoutes135 = (Sqrt[x^2 + y^2] * Vφ + z * Vρ) / Sqrt[x^2 + y^2 + z^2];

In[•]:= VxxRoutes135 = (y^2 * (x^2 + y^2 + z^2) * Vλλ + 2 * x * y * z * Sqrt[x^2 + y^2 + z^2] * Vλφ -
    2 * x * y * Sqrt[x^2 + y^2] * Sqrt[x^2 + y^2 + z^2] * Vλρ + x^2 * z^2 * Vφφ -
    2 * x^2 * z * Sqrt[x^2 + y^2] * Vφρ + x^2 * (x^2 + y^2) * Vρρ) / ((x^2 + y^2) * (x^2 + y^2 + z^2));

In[•]:= VxyRoutes135 = (-x * y * (x^2 + y^2 + z^2)^(3/2) * Vλλ -
    z * (x^2 - y^2) * (x^2 + y^2 + z^2) * Vλφ + (x^2 - y^2) * Sqrt[x^2 + y^2] * (x^2 + y^2 + z^2) * Vλρ +
    x * y * z^2 * Sqrt[x^2 + y^2 + z^2] * Vφφ - 2 * x * y * z * Sqrt[x^2 + y^2] * Sqrt[x^2 + y^2 + z^2] * Vφρ +
    x * y * (x^2 + y^2) * Sqrt[x^2 + y^2 + z^2] * Vρρ) / ((x^2 + y^2) * (x^2 + y^2 + z^2)^(3/2));

In[•]:= VxzRoutes135 =
  (-y * Sqrt[x^2 + y^2] * Sqrt[x^2 + y^2 + z^2] * Vλφ - y * z * Sqrt[x^2 + y^2 + z^2] * Vλρ - x * z * Sqrt[x^2 + y^2] * Vφφ +
    x * (x^2 + y^2 - z^2) * Vφρ + x * z * Sqrt[x^2 + y^2] * Vρρ) / (Sqrt[x^2 + y^2] * (x^2 + y^2 + z^2));

In[•]:= VyyRoutes135 = (x^2 * (x^2 + y^2 + z^2) * Vλλ - 2 * x * y * z * Sqrt[x^2 + y^2 + z^2] * Vλφ +
    2 * x * y * Sqrt[x^2 + y^2] * Sqrt[x^2 + y^2 + z^2] * Vλρ + y^2 * z^2 * Vφφ -
    2 * y^2 * z * Sqrt[x^2 + y^2] * Vφρ + y^2 * (x^2 + y^2) * Vρρ) / ((x^2 + y^2) * (x^2 + y^2 + z^2));

In[•]:= VyzRoutes135 =
  (x * Sqrt[x^2 + y^2] * Sqrt[x^2 + y^2 + z^2] * Vλφ + x * z * Sqrt[x^2 + y^2 + z^2] * Vλρ - y * z * Sqrt[x^2 + y^2] * Vφφ +
    y * (x^2 + y^2 - z^2) * Vφρ + y * z * Sqrt[x^2 + y^2] * Vρρ) / (Sqrt[x^2 + y^2] * (x^2 + y^2 + z^2));

In[•]:= VzzRoutes135 = ((x^2 + y^2) * Vφφ + 2 * z * Sqrt[x^2 + y^2] * Vφρ + z^2 * Vρρ) / (x^2 + y^2 + z^2);

In[•]:= VxxxRoutes135 =
  (-y^3 * (x^2 + y^2 + z^2)^(3/2) * Vλλλ - 3 * x * y^2 * z * (x^2 + y^2 + z^2) * Vλλφ + 3 * x * y^2 *
    Sqrt[x^2 + y^2] * (x^2 + y^2 + z^2) * Vλλρ + 6 * x^2 * y * z * Sqrt[x^2 + y^2] * Sqrt[x^2 + y^2 + z^2] * Vλφρ -
    3 * x^2 * y * z^2 * Sqrt[x^2 + y^2 + z^2] * Vφφλ - x^3 * z^3 * Vφφφ + 3 * x^3 * z^2 * Sqrt[x^2 + y^2] * Vφφρ -
    3 * x^2 * y * (x^2 + y^2) * Sqrt[x^2 + y^2 + z^2] * Vρρλ - 3 * x^3 * z * (x^2 + y^2) * Vρρφ +
    x^3 * (x^2 + y^2)^(3/2) * Vρρρ) / ((x^2 + y^2)^(3/2) * (x^2 + y^2 + z^2)^(3/2));
```

In[•]:=
$$\begin{aligned}\text{VxxyRoutes135} = \Big(&x*y^2*\left(x^2+y^2+z^2\right)^{3/2}*V\lambda\lambda\lambda + y*z*\left(2*x^2-y^2\right)*\left(x^2+y^2+z^2\right)*V\lambda\lambda\varphi -\\ &y*\sqrt{x^2+y^2}*\left(2*x^2-y^2\right)*\left(x^2+y^2+z^2\right)*V\lambda\lambda\rho -\\ &2*x*z*\left(x^2-2*y^2\right)*\sqrt{x^2+y^2}*\sqrt{x^2+y^2+z^2}*V\lambda\varphi\rho + x*z^2*\left(x^2-2*y^2\right)*\\ &\sqrt{x^2+y^2+z^2}*V\varphi\varphi\lambda - x^2*y*z^3*V\varphi\varphi\varphi + 3*x^2*y*z^2*\sqrt{x^2+y^2}*V\varphi\varphi\rho +\\ &x*\left(x^4-x^2*y^2-2*y^4\right)*\sqrt{x^2+y^2+z^2}*V\rho\rho\lambda - 3*x^2*y*z*\left(x^2+y^2\right)*V\rho\rho\varphi +\\ &x^2*y*\left(x^2+y^2\right)^{3/2}*V\rho\rho\rho\Big)\Big/\left(\left(x^2+y^2\right)^{3/2}*\left(x^2+y^2+z^2\right)^{3/2}\right);\end{aligned}$$

In[•]:=
$$\begin{aligned}\text{VxxzRoutes135} = &\\ \Big(&y^2*\sqrt{x^2+y^2}*\left(x^2+y^2+z^2\right)*V\lambda\lambda\varphi + y^2*z*\left(x^2+y^2+z^2\right)*V\lambda\lambda\rho - 2*x*y*\left(x^2+y^2-z^2\right)*\\ &\sqrt{x^2+y^2+z^2}*V\lambda\varphi\rho + 2*x*y*z*\sqrt{x^2+y^2}*\sqrt{x^2+y^2+z^2}*V\varphi\varphi\lambda +\\ &x^2*z^2*\sqrt{x^2+y^2}*V\varphi\varphi\varphi - x^2*z*\left(2*x^2+2*y^2-z^2\right)*V\varphi\varphi\rho -\\ &2*x*y*z*\sqrt{x^2+y^2}*\sqrt{x^2+y^2+z^2}*V\rho\rho\lambda + x^2*\sqrt{x^2+y^2}*\left(x^2+y^2-2*z^2\right)*V\rho\rho\varphi +\\ &x^2*z*\left(x^2+y^2\right)*V\rho\rho\rho\Big)\Big/\left(\left(x^2+y^2\right)*\left(x^2+y^2+z^2\right)^{3/2}\right);\end{aligned}$$

In[•]:=
$$\begin{aligned}\text{VxyzRoutes135} = &\\ \Big(&-x*y*\sqrt{x^2+y^2}*\left(x^2+y^2+z^2\right)*V\lambda\lambda\varphi - x*y*z*\left(x^2+y^2+z^2\right)*V\lambda\lambda\rho + \left(x^2-y^2\right)*\\ &\left(x^2+y^2-z^2\right)*\sqrt{x^2+y^2+z^2}*V\lambda\varphi\rho - z*\left(x^2-y^2\right)*\sqrt{x^2+y^2}*\sqrt{x^2+y^2+z^2}*V\varphi\varphi\lambda +\\ &x*y*z^2*\sqrt{x^2+y^2}*V\varphi\varphi\varphi - x*y*z*\left(2*x^2+2*y^2-z^2\right)*V\varphi\varphi\rho +\\ &z*\left(x^2-y^2\right)*\sqrt{x^2+y^2}*\sqrt{x^2+y^2+z^2}*V\rho\rho\lambda + x*y*\left(x^2+y^2-2*z^2\right)*\\ &\sqrt{x^2+y^2}*V\rho\rho\varphi + x*y*z*\left(x^2+y^2\right)*V\rho\rho\rho\Big)\Big/\left(\left(x^2+y^2\right)*\left(x^2+y^2+z^2\right)^{3/2}\right);\end{aligned}$$

In[•]:=
$$\begin{aligned}\text{VyyxRoutes135} = \Big(&-x^2*y*\left(x^2+y^2+z^2\right)^{3/2}*V\lambda\lambda\lambda - x*z*\left(x^2-2*y^2\right)*\left(x^2+y^2+z^2\right)*V\lambda\lambda\varphi +\\ &x*\left(x^2-2*y^2\right)*\sqrt{x^2+y^2}*\left(x^2+y^2+z^2\right)*V\lambda\lambda\rho -\\ &2*y*z*\left(2*x^2-y^2\right)*\sqrt{x^2+y^2}*\sqrt{x^2+y^2+z^2}*V\lambda\varphi\rho + y*z^2*\left(2*x^2-y^2\right)*\\ &\sqrt{x^2+y^2+z^2}*V\varphi\varphi\lambda - x*y^2*z^3*V\varphi\varphi\varphi + 3*x*y^2*z^2*\sqrt{x^2+y^2}*V\varphi\varphi\rho +\\ &y*\left(2*x^4+x^2*y^2-y^4\right)*\sqrt{x^2+y^2+z^2}*V\rho\rho\lambda - 3*x*y^2*z*\left(x^2+y^2\right)*V\rho\rho\varphi +\\ &x*y^2*\left(x^2+y^2\right)^{3/2}*V\rho\rho\rho\Big)\Big/\left(\left(x^2+y^2\right)^{3/2}*\left(x^2+y^2+z^2\right)^{3/2}\right);\end{aligned}$$

In[•]:=
$$\begin{aligned}\text{VyyyRoutes135} = &\\ \Big(&x^3*\left(x^2+y^2+z^2\right)^{3/2}*V\lambda\lambda\lambda - 3*x^2*y*z*\left(x^2+y^2+z^2\right)*V\lambda\lambda\varphi + 3*x^2*y*\sqrt{x^2+y^2}*\\ &\left(x^2+y^2+z^2\right)*V\lambda\lambda\rho - 6*x*y^2*z*\sqrt{x^2+y^2}*\sqrt{x^2+y^2+z^2}*V\lambda\varphi\rho +\\ &3*x*y^2*z^2*\sqrt{x^2+y^2+z^2}*V\varphi\varphi\lambda - y^3*z^3*V\varphi\varphi\varphi + 3*y^3*z^2*\sqrt{x^2+y^2}*V\varphi\varphi\rho +\\ &3*x*y^2*\left(x^2+y^2\right)*\sqrt{x^2+y^2+z^2}*V\rho\rho\lambda - 3*y^3*z*\left(x^2+y^2\right)*V\rho\rho\varphi +\\ &y^3*\left(x^2+y^2\right)^{3/2}*V\rho\rho\rho\Big)\Big/\left(\left(x^2+y^2\right)^{3/2}*\left(x^2+y^2+z^2\right)^{3/2}\right);\end{aligned}$$

In[ ]:=
```
VyyzRoutes135 =
 (x^2 * Sqrt[x^2 + y^2] * (x^2 + y^2 + z^2) * Vλλφ + x^2 * z * (x^2 + y^2 + z^2) * Vλλρ + 2 * x * y * (x^2 + y^2 - z^2) *
    Sqrt[x^2 + y^2 + z^2] * Vλφρ - 2 * x * y * z * Sqrt[x^2 + y^2] * Sqrt[x^2 + y^2 + z^2] * Vφφλ +
  y^2 * z^2 * Sqrt[x^2 + y^2] * Vφφφ - y^2 * z * (2 * x^2 + 2 * y^2 - z^2) * Vφφρ +
  2 * x * y * z * Sqrt[x^2 + y^2] * Sqrt[x^2 + y^2 + z^2] * Vρρλ + y^2 * (x^2 + y^2 - 2 * z^2) * Sqrt[x^2 + y^2] * Vρρφ +
  y^2 * z * (x^2 + y^2) * Vρρρ) / ((x^2 + y^2) * (x^2 + y^2 + z^2)^(3/2));
```

In[ ]:=
```
VzzxRoutes135 =
 (-2 * y * z * Sqrt[x^2 + y^2] * Sqrt[x^2 + y^2 + z^2] * Vλφρ - y * (x^2 + y^2) * Sqrt[x^2 + y^2 + z^2] * Vφφλ -
  x * z * (x^2 + y^2) * Vφφφ + x * (x^2 + y^2 - 2 * z^2) * Sqrt[x^2 + y^2] * Vφφρ -
  y * z^2 * Sqrt[x^2 + y^2 + z^2] * Vρρλ + x * z * (2 * x^2 + 2 * y^2 - z^2) * Vρρφ +
  x * z^2 * Sqrt[x^2 + y^2] * Vρρρ) / (Sqrt[x^2 + y^2] * (x^2 + y^2 + z^2)^(3/2));
```

In[ ]:=
```
VzzyRoutes135 =
 (2 * x * z * Sqrt[x^2 + y^2] * Sqrt[x^2 + y^2 + z^2] * Vλφρ + x * (x^2 + y^2) * Sqrt[x^2 + y^2 + z^2] * Vφφλ -
  y * z * (x^2 + y^2) * Vφφφ + y * (x^2 + y^2 - 2 * z^2) * Sqrt[x^2 + y^2] * Vφφρ +
  x * z^2 * Sqrt[x^2 + y^2 + z^2] * Vρρλ + y * z * (2 * x^2 + 2 * y^2 - z^2) * Vρρφ +
  y * z^2 * Sqrt[x^2 + y^2] * Vρρρ) / (Sqrt[x^2 + y^2] * (x^2 + y^2 + z^2)^(3/2));
```

In[ ]:=
```
VzzzRoutes135 =
 ((x^2 + y^2)^(3/2) * Vφφφ + 3 * z * (x^2 + y^2) * Vφφρ + 3 * z^2 * Sqrt[x^2 + y^2] * Vρρφ + z^3 * Vρρρ) /
  (x^2 + y^2 + z^2)^(3/2);
```

1.4 Check whether they become themselves with r = $\sqrt{x^2+y^2}$

In[ ]:=
```
FullSimplify[VxRoutes135 /. {r → Sqrt[x^2 + y^2]}]
```

Out[ ]=
```
Vx
```

In[ ]:=
```
FullSimplify[VyRoutes135 /. {r → Sqrt[x^2 + y^2]}]
```

Out[ ]=
```
Vy
```

In[ ]:=
```
FullSimplify[VzRoutes135 /. {r → Sqrt[x^2 + y^2]}]
```

Out[ ]=
```
Vz
```

In[ ]:=
```
FullSimplify[VxxRoutes135 /. {r → Sqrt[x^2 + y^2]}]
```

Out[ ]=
```
Vxx
```

In[•]:= FullSimplify[VxyRoutes135 /. {r → √(x^2 + y^2)}]

Out[•]=

Vxy

In[•]:= FullSimplify[VxzRoutes135 /. {r → √(x^2 + y^2)}]

Out[•]=

Vxz

In[•]:= FullSimplify[VyyRoutes135 /. {r → √(x^2 + y^2)}]

Out[•]=

Vyy

In[•]:= FullSimplify[VyzRoutes135 /. {r → √(x^2 + y^2)}]

Out[•]=

Vyz

In[•]:= FullSimplify[VzzRoutes135 /. {r → √(x^2 + y^2)}]

Out[•]=

Vzz

In[•]:= FullSimplify[VxxxRoutes135 /. {r → √(x^2 + y^2)}]

Out[•]=

Vxxx

In[•]:= FullSimplify[VxxyRoutes135 /. {r → √(x^2 + y^2)}]

Out[•]=

Vxxy

In[•]:= FullSimplify[VxxzRoutes135 /. {r → √(x^2 + y^2)}]

Out[•]=

Vxxz

In[•]:= FullSimplify[VxyzRoutes135 /. {r → √(x^2 + y^2)}]

Out[•]=

Vxyz

In[•]:= FullSimplify[VyyxRoutes135 /. {r → √(x^2 + y^2)}]

Out[•]=

Vyyx

In[•]:= FullSimplify[VyyyRoutes135 /. {r → √(x^2 + y^2)}]

Out[•]=

Vyyy

In[•]:= FullSimplify[VyyzRoutes135 /. {r → √(x^2 + y^2)}]

Out[•]=

Vyyz

```
In[ ]:= FullSimplify[VzzxRoutes135 /. {r → Sqrt[x^2 + y^2]}]
Out[ ]=
Vzzx

In[ ]:= FullSimplify[VzzyRoutes135 /. {r → Sqrt[x^2 + y^2]}]
Out[ ]=
Vzzy

In[ ]:= FullSimplify[VzzzRoutes135 /. {r → Sqrt[x^2 + y^2]}]
Out[ ]=
Vzzz
```

2. Algebraic form, Cartesian coordinates, Routes 6 → 4 → 2, and Table A12 → Table A7 → Table A3

```
In[ ]:= Clear["Global`*"];
```

2.1 Table A12

```
In[ ]:= Vλ = (-y * Vx + x * Vy) / Sqrt[x^2 + y^2];

In[ ]:= Vφ = (-x * z * Vx - y * z * Vy + (x^2 + y^2) * Vz) / (Sqrt[x^2 + y^2] * Sqrt[x^2 + y^2 + z^2]);

In[ ]:= Vρ = (x * Vx + y * Vy + z * Vz) / Sqrt[x^2 + y^2 + z^2];

In[ ]:= Vλλ = (y^2 * Vxx - 2 * x * y * Vxy + x^2 * Vyy) / (x^2 + y^2);

In[ ]:= Vλφ = (x * y * z * Vxx - z * (x^2 - y^2) * Vxy - y * (x^2 + y^2) * Vxz -
          x * y * z * Vyy + x * (x^2 + y^2) * Vyz) / ((x^2 + y^2) * Sqrt[x^2 + y^2 + z^2]);

In[ ]:= Vλρ = (-x * y * Vxx + (x^2 - y^2) * Vxy - y * z * Vxz + x * z * Vyz + x * y * Vyy) /
        (Sqrt[x^2 + y^2] * Sqrt[x^2 + y^2 + z^2]);

In[ ]:= Vφφ = (x^2 * z^2 * Vxx + 2 * x * y * z^2 * Vxy - 2 * x * z * (x^2 + y^2) * Vxz + y^2 * z^2 * Vyy -
          2 * y * z * (x^2 + y^2) * Vyz + (x^2 + y^2)^2 * Vzz) / ((x^2 + y^2) * (x^2 + y^2 + z^2));

In[ ]:= Vφρ = (-x^2 * z * Vxx - 2 * x * y * z * Vxy + x * (x^2 + y^2 - z^2) * Vxz - y^2 * z * Vyy +
          y * (x^2 + y^2 - z^2) * Vyz + z * (x^2 + y^2) * Vzz) / (Sqrt[x^2 + y^2] * (x^2 + y^2 + z^2));

In[ ]:= Vρρ =
        (x^2 * Vxx + 2 * x * y * Vxy + 2 * x * z * Vxz + 2 * y * z * Vyz + y^2 * Vyy + z^2 * Vzz) / (x^2 + y^2 + z^2);

In[ ]:= Vλλλ = (-y^3 * Vxxx + 3 * x * y^2 * Vxxy - 3 * x^2 * y * Vyyx + x^3 * Vyyy) / (x^2 + y^2)^(3/2);
```

```
In[•]:= Vλλφ = (-x * y^2 * z * Vxxx + y * z * (2 * x^2 - y^2) * Vxxy +
    y^2 * (x^2 + y^2) * Vxxz - 2 * x * y * (x^2 + y^2) * Vxyz + x * z * (2 * y^2 - x^2) * Vyyx -
    x^2 * y * z * Vyyy + x^2 * (x^2 + y^2) * Vyyz) / ((x^2 + y^2)^(3/2) * Sqrt[x^2 + y^2 + z^2]);

In[•]:= Vλλρ = (x * y^2 * Vxxx + y * (y^2 - 2 * x^2) * Vxxy + y^2 * z * Vxxz - 2 * x * y * z * Vxyz +
    x * (x^2 - 2 * y^2) * Vyyx + x^2 * y * Vyyy + x^2 * z * Vyyz) / ((x^2 + y^2) * Sqrt[x^2 + y^2 + z^2]);

In[•]:= Vλφρ = (x^2 * y * z * Vxxx + x * z * (2 * y^2 - x^2) * Vxxy -
    x * y * (x^2 + y^2 - z^2) * Vxxz + (x^2 - y^2) * (x^2 + y^2 - z^2) * Vxyz +
    y * z * (y^2 - 2 * x^2) * Vyyx - x * y^2 * z * Vyyy + x * y * (x^2 + y^2 - z^2) * Vyyz -
    y * z * (x^2 + y^2) * Vzzx + x * z * (x^2 + y^2) * Vzzy) / ((x^2 + y^2) * (x^2 + y^2 + z^2));

In[•]:= Vφφλ = (-x^2 * y * z^2 * Vxxx + x * z^2 * (x^2 - 2 * y^2) * Vxxy +
    2 * x * y * z * (x^2 + y^2) * Vxxz - 2 * z * (x^4 - y^4) * Vxyz +
    y * z^2 * (2 * x^2 - y^2) * Vyyx + x * y^2 * z^2 * Vyyy - 2 * x * y * z * (x^2 + y^2) * Vyyz -
    y * (x^2 + y^2)^2 * Vzzx + x * (x^2 + y^2)^2 * Vzzy) / ((x^2 + y^2)^(3/2) * (x^2 + y^2 + z^2));

In[•]:= Vφφφ = (-x^3 * z^3 * Vxxx - 3 * x^2 * y * z^3 * Vxxy + 3 * x^2 * z^2 * (x^2 + y^2) * Vxxz +
    6 * x * y * z^2 * (x^2 + y^2) * Vxyz - 3 * x * y^2 * z^3 * Vyyx - y^3 * z^3 * Vyyy +
    3 * y^2 * z^2 * (x^2 + y^2) * Vyyz - 3 * x * z * (x^2 + y^2)^2 * Vzzx -
    3 * y * z * (x^2 + y^2)^2 * Vzzy + (x^2 + y^2)^3 * Vzzz) / ((x^2 + y^2)^(3/2) * (x^2 + y^2 + z^2)^(3/2));

In[•]:= Vφφρ = (x^3 * z^2 * Vxxx + 3 * x^2 * y * z^2 * Vxxy +
    x^2 * z * (z^2 - 2 * x^2 - 2 * y^2) * Vxxz + 2 * x * y * z * (z^2 - 2 * x^2 - 2 * y^2) * Vxyz +
    3 * x * y^2 * z^2 * Vyyx + y^3 * z^2 * Vyyy + y^2 * z * (z^2 - 2 * x^2 - 2 * y^2) * Vyyz +
    x * (x^2 + y^2) * (x^2 + y^2 - 2 * z^2) * Vzzx + y * (x^2 + y^2) * (x^2 + y^2 - 2 * z^2) * Vzzy +
    z * (x^2 + y^2)^2 * Vzzz) / ((x^2 + y^2) * (x^2 + y^2 + z^2)^(3/2));

In[•]:= Vρρλ = (-x^2 * y * Vxxx + x * (x^2 - 2 * y^2) * Vxxy - 2 * x * y * z * Vxxz +
    2 * z * (x^2 - y^2) * Vxyz + y * (2 * x^2 - y^2) * Vyyx + x * y^2 * Vyyy +
    2 * x * y * z * Vyyz - y * z^2 * Vzzx + x * z^2 * Vzzy) / (Sqrt[x^2 + y^2] * (x^2 + y^2 + z^2));

In[•]:= Vρρφ = (-x^3 * z * Vxxx - 3 * x^2 * y * z * Vxxy + x^2 * (x^2 + y^2 - 2 * z^2) * Vxxz +
    2 * x * y * (x^2 + y^2 - 2 * z^2) * Vxyz - 3 * x * y^2 * z * Vyyx - y^3 * z * Vyyy +
    y^2 * (x^2 + y^2 - 2 * z^2) * Vyyz + x * z * (2 * x^2 + 2 * y^2 - z^2) * Vzzx + y * z *
     (2 * x^2 + 2 * y^2 - z^2) * Vzzy + z^2 * (x^2 + y^2) * Vzzz) / (Sqrt[x^2 + y^2] * (x^2 + y^2 + z^2)^(3/2));

In[•]:= Vρρρ = (x^3 * Vxxx + 3 * x^2 * y * Vxxy + 3 * x^2 * z * Vxxz +
    6 * x * y * z * Vxyz + 3 * x * y^2 * Vyyx + y^3 * Vyyy + 3 * y^2 * z * Vyyz +
    3 * x * z^2 * Vzzx + 3 * y * z^2 * Vzzy + z^3 * Vzzz) / (x^2 + y^2 + z^2)^(3/2);
```

2.2 Table A7

```
In[•]:= Vr = (-z * Vφ + r * Vρ) / Sqrt[r^2 + z^2];

In[•]:= Vλ = Vλ;

In[•]:= Vz1 = (r * Vφ + z * Vρ) / Sqrt[r^2 + z^2];

In[•]:= Vrr = (z^2 * Vφφ - 2 * r * z * Vφρ + r^2 * Vρρ) / (r^2 + z^2);

In[•]:= Vrλ = (-z * Vλφ + r * Vλρ) / Sqrt[r^2 + z^2];

In[•]:= Vrz = (-r * z * Vφφ + (r^2 - z^2) * Vφρ + r * z * Vρρ) / (r^2 + z^2);

In[•]:= Vλλ = Vλλ;

In[•]:= Vλz = (r * Vλφ + z * Vλρ) / Sqrt[r^2 + z^2];

In[•]:= Vzz1 = (r^2 * Vφφ + 2 * r * z * Vφρ + z^2 * Vρρ) / (r^2 + z^2);

In[•]:= Vrrr = (-z^3 * Vφφφ + 3 * r * z^2 * Vφφρ - 3 * r^2 * z * Vρρφ + r^3 * Vρρρ) / (r^2 + z^2)^(3/2);

In[•]:= Vrrλ = (-2 * r * z * Vλφρ + z^2 * Vφφλ + r^2 * Vρρλ) / (r^2 + z^2);

In[•]:= Vrrz =
    (r * z^2 * Vφφφ + z * (z^2 - 2 * r^2) * Vφφρ + r * (r^2 - 2 * z^2) * Vρρφ + r^2 * z * Vρρρ) / (r^2 + z^2)^(3/2);

In[•]:= Vrλz = ((r^2 - z^2) * Vλφρ - r * z * Vφφλ + r * z * Vρρλ) / (r^2 + z^2);

In[•]:= Vλλr = (-z * Vλλφ + r * Vλλρ) / Sqrt[r^2 + z^2];

In[•]:= Vλλλ = Vλλλ;

In[•]:= Vλλz = (r * Vλλφ + z * Vλλρ) / Sqrt[r^2 + z^2];

In[•]:= Vzzr = (-r^2 * z * Vφφφ + r * (r^2 - 2 * z^2) * Vφφρ + z * (2 * r^2 - z^2) * Vρρφ + r * z^2 * Vρρρ) /
     (r^2 + z^2)^(3/2);

In[•]:= Vzzλ = (2 * r * z * Vλφρ + r^2 * Vφφλ + z^2 * Vρρλ) / (r^2 + z^2);

In[•]:= Vzzz1 = (r^3 * Vφφφ + 3 * r^2 * z * Vφφρ + 3 * r * z^2 * Vρρφ + z^3 * Vρρρ) / (r^2 + z^2)^(3/2);
```

2.3 Table A3

```
In[•]:= VxRoutes642 = (x * Vr - y * Vλ) / Sqrt[x^2 + y^2];

In[•]:= VyRoutes642 = (y * Vr + x * Vλ) / Sqrt[x^2 + y^2];

In[•]:= VzRoutes642 = Vz1;

In[•]:= VxxRoutes642 = (x^2 * Vrr - 2 * x * y * Vrλ + y^2 * Vλλ) / (x^2 + y^2);
```

```
In[•]:= VxyRoutes642 = (x * y * Vrr + (x^2 - y^2) * Vrλ - x * y * Vλλ) / (x^2 + y^2);

In[•]:= VxzRoutes642 = (x * Vrz - y * Vλz) / Sqrt[x^2 + y^2];

In[•]:= VyyRoutes642 = (y^2 * Vrr + 2 * x * y * Vrλ + x^2 * Vλλ) / (x^2 + y^2);

In[•]:= VyzRoutes642 = (y * Vrz + x * Vλz) / Sqrt[x^2 + y^2];

In[•]:= VzzRoutes642 = Vzz1;

In[•]:= VxxxRoutes642 = (x^3 * Vrrr - 3 * x^2 * y * Vrrλ + 3 * x * y^2 * Vλλr - y^3 * Vλλλ) / (x^2 + y^2)^(3/2);

In[•]:= VxxyRoutes642 =
          (x^2 * y * Vrrr + x * (x^2 - 2 * y^2) * Vrrλ + y * (y^2 - 2 * x^2) * Vλλr + x * y^2 * Vλλλ) / (x^2 + y^2)^(3/2);

In[•]:= VxxzRoutes642 = (x^2 * Vrrz - 2 * x * y * Vrλz + y^2 * Vλλz) / (x^2 + y^2);

In[•]:= VxyzRoutes642 = (x * y * Vrrz + (x^2 - y^2) * Vrλz - x * y * Vλλz) / (x^2 + y^2);

In[•]:= VyyxRoutes642 =
          (x * y^2 * Vrrr + y * (2 * x^2 - y^2) * Vrrλ + x * (x^2 - 2 * y^2) * Vλλr - x^2 * y * Vλλλ) / (x^2 + y^2)^(3/2);

In[•]:= VyyyRoutes642 = (y^3 * Vrrr + 3 * x * y^2 * Vrrλ + 3 * x^2 * y * Vλλr + x^3 * Vλλλ) / (x^2 + y^2)^(3/2);

In[•]:= VyyzRoutes642 = (y^2 * Vrrz + 2 * x * y * Vrλz + x^2 * Vλλz) / (x^2 + y^2);

In[•]:= VzzxRoutes642 = (x * Vzzr - y * Vzzλ) / Sqrt[x^2 + y^2];

In[•]:= VzzyRoutes642 = (y * Vzzr + x * Vzzλ) / Sqrt[x^2 + y^2];

In[•]:= VzzzRoutes642 = Vzzz1;
```

2.4 Check whether they become themselves with r = $\sqrt{x^2+y^2}$

```
In[•]:= FullSimplify[VxRoutes642 /. {r → Sqrt[x^2 + y^2]}]
Out[•]=
Vx

In[•]:= FullSimplify[VyRoutes642 /. {r → Sqrt[x^2 + y^2]}]
Out[•]=
Vy

In[•]:= FullSimplify[VzRoutes642 /. {r → Sqrt[x^2 + y^2]}]
Out[•]=
Vz

In[•]:= FullSimplify[VxxRoutes642 /. {r → Sqrt[x^2 + y^2]}]
Out[•]=
Vxx
```

In[•]:= FullSimplify[VxyRoutes642 /. {r → √(x^2 + y^2)}]

Out[•]=

Vxy

In[•]:= FullSimplify[VxzRoutes642 /. {r → √(x^2 + y^2)}]

Out[•]=

Vxz

In[•]:= FullSimplify[VyyRoutes642 /. {r → √(x^2 + y^2)}]

Out[•]=

Vyy

In[•]:= FullSimplify[VyzRoutes642 /. {r → √(x^2 + y^2)}]

Out[•]=

Vyz

In[•]:= FullSimplify[VzzRoutes642 /. {r → √(x^2 + y^2)}]

Out[•]=

Vzz

In[•]:= FullSimplify[VxxxRoutes642 /. {r → √(x^2 + y^2)}]

Out[•]=

Vxxx

In[•]:= FullSimplify[VxxyRoutes642 /. {r → √(x^2 + y^2)}]

Out[•]=

Vxxy

In[•]:= FullSimplify[VxxzRoutes642 /. {r → √(x^2 + y^2)}]

Out[•]=

Vxxz

In[•]:= FullSimplify[VxyzRoutes642 /. {r → √(x^2 + y^2)}]

Out[•]=

Vxyz

In[•]:= FullSimplify[VyyxRoutes642 /. {r → √(x^2 + y^2)}]

Out[•]=

Vyyx

In[•]:= FullSimplify[VyyyRoutes642 /. {r → √(x^2 + y^2)}]

Out[•]=

Vyyy

In[•]:= FullSimplify[VyyzRoutes642 /. {r → √(x^2 + y^2)}]

Out[•]=

Vyyz

In[ ]:= FullSimplify[VzzxRoutes642 /. {r → Sqrt[x^2 + y^2]}]

Out[ ]=

Vzzx

In[ ]:= FullSimplify[VzzyRoutes642 /. {r → Sqrt[x^2 + y^2]}]

Out[ ]=

Vzzy

In[ ]:= FullSimplify[VzzzRoutes642 /. {r → Sqrt[x^2 + y^2]}]

Out[ ]=

Vzzz

3. Algebraic form, Cylindrical coordinates, Routes 3 → 5 → 1, and Table A5 → Tables A9, A10 → Table A1

In[ ]:= Clear["Global`*"];

3.1 Table A5

In[ ]:= Vλ = Vλ;

In[ ]:= Vφ = (-z * Vr + r * Vz) / Sqrt[r^2 + z^2];

In[ ]:= Vρ = (r * Vr + z * Vz) / Sqrt[r^2 + z^2];

In[ ]:= Vλλ = Vλλ;

In[ ]:= Vλφ = (-z * Vrλ + r * Vλz) / Sqrt[r^2 + z^2];

In[ ]:= Vλρ = (r * Vrλ + z * Vλz) / Sqrt[r^2 + z^2];

In[ ]:= Vφφ = (z^2 * Vrr - 2 * r * z * Vrz + r^2 * Vzz) / (r^2 + z^2);

In[ ]:= Vφρ = (-r * z * Vrr + (r^2 - z^2) * Vrz + r * z * Vzz) / (r^2 + z^2);

In[ ]:= Vρρ = (r^2 * Vrr + 2 * r * z * Vrz + z^2 * Vzz) / (r^2 + z^2);

In[ ]:= Vλλλ = Vλλλ;

In[ ]:= Vλλφ = (-z * Vλλr + r * Vλλz) / Sqrt[r^2 + z^2];

In[ ]:= Vλλρ = (r * Vλλr + z * Vλλz) / Sqrt[r^2 + z^2];

In[ ]:= Vλφρ = (-r * z * Vrrλ + (r^2 - z^2) * Vrλz + r * z * Vzzλ) / (r^2 + z^2);

In[ ]:= Vφφλ = (z^2 * Vrrλ - 2 * r * z * Vrλz + r^2 * Vzzλ) / (r^2 + z^2);

In[ ]:= Vφφφ = (-z^3 * Vrrr + 3 * r * z^2 * Vrrz - 3 * r^2 * z * Vzzr + r^3 * Vzzz) / (r^2 + z^2)^(3/2);

```
In[•]:= Vφφρ =
          (r * z^2 * Vrrr + z * (z^2 - 2 * r^2) * Vrrz + r * (r^2 - 2 * z^2) * Vzzr + r^2 * z * Vzzz) / (r^2 + z^2)^(3/2);

In[•]:= Vρρλ = (r^2 * Vrrλ + 2 * r * z * Vrλz + z^2 * Vzzλ) / (r^2 + z^2);

In[•]:= Vρρφ = (-r^2 * z * Vrrr + r * (r^2 - 2 * z^2) * Vrrz + z * (2 * r^2 - z^2) * Vzzr + r * z^2 * Vzzz) /
          (r^2 + z^2)^(3/2);

In[•]:= Vρρρ = (r^3 Vrrr + 3 * r^2 * z * Vrrz + 3 * r * z^2 * Vzzr + z^3 * Vzzz) / (r^2 + z^2)^(3/2);
```

3.2 Tables A9, A10

```
In[•]:= Vx = (-y * Sqrt[x^2 + y^2 + z^2] * Vλ - x * z * Vφ + x * Sqrt[x^2 + y^2] * Vρ) / (Sqrt[x^2 + y^2] * Sqrt[x^2 + y^2 + z^2]);

In[•]:= Vy = (x * Sqrt[x^2 + y^2 + z^2] * Vλ - y * z * Vφ + y * Sqrt[x^2 + y^2] * Vρ) / (Sqrt[x^2 + y^2] * Sqrt[x^2 + y^2 + z^2]);

In[•]:= Vz1 = (Sqrt[x^2 + y^2] * Vφ + z * Vρ) / Sqrt[x^2 + y^2 + z^2];

In[•]:= Vxx = (y^2 * (x^2 + y^2 + z^2) * Vλλ + 2 * x * y * z * Sqrt[x^2 + y^2 + z^2] * Vλφ -
          2 * x * y * Sqrt[x^2 + y^2] * Sqrt[x^2 + y^2 + z^2] * Vλρ + x^2 * z^2 * Vφφ -
          2 * x^2 * z * Sqrt[x^2 + y^2] * Vφρ + x^2 * (x^2 + y^2) * Vρρ) / ((x^2 + y^2) * (x^2 + y^2 + z^2));

In[•]:= Vxy = (-x * y * (x^2 + y^2 + z^2)^(3/2) * Vλλ -
          z * (x^2 - y^2) * (x^2 + y^2 + z^2) * Vλφ + (x^2 - y^2) * Sqrt[x^2 + y^2] * (x^2 + y^2 + z^2) * Vλρ +
          x * y * z^2 * Sqrt[x^2 + y^2 + z^2] * Vφφ - 2 * x * y * z * Sqrt[x^2 + y^2] * Sqrt[x^2 + y^2 + z^2] * Vφρ +
          x * y * (x^2 + y^2) * Sqrt[x^2 + y^2 + z^2] * Vρρ) / ((x^2 + y^2) * (x^2 + y^2 + z^2)^(3/2));

In[•]:= Vxz = (-y * Sqrt[x^2 + y^2] * Sqrt[x^2 + y^2 + z^2] * Vλφ - y * z * Sqrt[x^2 + y^2 + z^2] * Vλρ - x * z * Sqrt[x^2 + y^2] * Vφφ +
          x * (x^2 + y^2 - z^2) * Vφρ + x * z * Sqrt[x^2 + y^2] * Vρρ) / (Sqrt[x^2 + y^2] * (x^2 + y^2 + z^2));

In[•]:= Vyy = (x^2 * (x^2 + y^2 + z^2) * Vλλ - 2 * x * y * z * Sqrt[x^2 + y^2 + z^2] * Vλφ +
          2 * x * y * Sqrt[x^2 + y^2] * Sqrt[x^2 + y^2 + z^2] * Vλρ + y^2 * z^2 * Vφφ -
          2 * y^2 * z * Sqrt[x^2 + y^2] * Vφρ + y^2 * (x^2 + y^2) * Vρρ) / ((x^2 + y^2) * (x^2 + y^2 + z^2));

In[•]:= Vyz = (x * Sqrt[x^2 + y^2] * Sqrt[x^2 + y^2 + z^2] * Vλφ + x * z * Sqrt[x^2 + y^2 + z^2] * Vλρ - y * z * Sqrt[x^2 + y^2] * Vφφ +
          y * (x^2 + y^2 - z^2) * Vφρ + y * z * Sqrt[x^2 + y^2] * Vρρ) / (Sqrt[x^2 + y^2] * (x^2 + y^2 + z^2));

In[•]:= Vzz1 = ((x^2 + y^2) * Vφφ + 2 * z * Sqrt[x^2 + y^2] * Vφρ + z^2 * Vρρ) / (x^2 + y^2 + z^2);
```

In[•]:= Vxxx =

$$\Big(-y^3 * \left(x^2+y^2+z^2\right)^{3/2} * V\lambda\lambda\lambda - 3 * x * y^2 * z * \left(x^2+y^2+z^2\right) * V\lambda\lambda\varphi + 3 * x * y^2 * \sqrt{x^2+y^2} * \left(x^2+y^2+z^2\right) * V\lambda\lambda\rho + 6 * x^2 * y * z \sqrt{x^2+y^2} * \sqrt{x^2+y^2+z^2} * V\lambda\varphi\rho - 3 * x^2 * y * z^2 * \sqrt{x^2+y^2+z^2} * V\varphi\varphi\lambda - x^3 * z^3 * V\varphi\varphi\varphi + 3 * x^3 * z^2 * \sqrt{x^2+y^2} * V\varphi\varphi\rho - 3 * x^2 * y * \left(x^2+y^2\right) * \sqrt{x^2+y^2+z^2} * V\rho\rho\lambda - 3 * x^3 * z * \left(x^2+y^2\right) * V\rho\rho\varphi + x^3 * \left(x^2+y^2\right)^{3/2} * V\rho\rho\rho\Big) \Big/ \left(\left(x^2+y^2\right)^{3/2} * \left(x^2+y^2+z^2\right)^{3/2}\right);$$

In[•]:= $$Vxxy = \Big(x * y^2 * \left(x^2+y^2+z^2\right)^{3/2} * V\lambda\lambda\lambda + y * z * \left(2 * x^2 - y^2\right) * \left(x^2+y^2+z^2\right) * V\lambda\lambda\varphi - y * \sqrt{x^2+y^2} * \left(2 * x^2 - y^2\right) * \left(x^2+y^2+z^2\right) * V\lambda\lambda\rho - 2 * x * z * \left(x^2 - 2 * y^2\right) * \sqrt{x^2+y^2} * \sqrt{x^2+y^2+z^2} * V\lambda\varphi\rho + x * z^2 * \left(x^2 - 2 * y^2\right) * \sqrt{x^2+y^2+z^2} * V\varphi\varphi\lambda - x^2 * y * z^3 * V\varphi\varphi\varphi + 3 * x^2 * y * z^2 * \sqrt{x^2+y^2} * V\varphi\varphi\rho + x * \left(x^4 - x^2 * y^2 - 2 * y^4\right) * \sqrt{x^2+y^2+z^2} * V\rho\rho\lambda - 3 * x^2 * y * z * \left(x^2+y^2\right) * V\rho\rho\varphi + x^2 * y * \left(x^2+y^2\right)^{3/2} * V\rho\rho\rho\Big) \Big/ \left(\left(x^2+y^2\right)^{3/2} * \left(x^2+y^2+z^2\right)^{3/2}\right);$$

In[•]:= Vxxz =

$$\Big(y^2 * \sqrt{x^2+y^2} * \left(x^2+y^2+z^2\right) * V\lambda\lambda\varphi + y^2 * z * \left(x^2+y^2+z^2\right) * V\lambda\lambda\rho - 2 * x * y * \left(x^2+y^2-z^2\right) * \sqrt{x^2+y^2+z^2} * V\lambda\varphi\rho + 2 * x * y * z * \sqrt{x^2+y^2} * \sqrt{x^2+y^2+z^2} * V\varphi\varphi\lambda + x^2 * z^2 * \sqrt{x^2+y^2} * V\varphi\varphi\varphi - x^2 * z * \left(2 * x^2 + 2 * y^2 - z^2\right) * V\varphi\varphi\rho - 2 * x * y * z * \sqrt{x^2+y^2} * \sqrt{x^2+y^2+z^2} * V\rho\rho\lambda + x^2 * \sqrt{x^2+y^2} * \left(x^2+y^2-2 * z^2\right) * V\rho\rho\varphi + x^2 * z * \left(x^2+y^2\right) * V\rho\rho\rho\Big) \Big/ \left(\left(x^2+y^2\right) * \left(x^2+y^2+z^2\right)^{3/2}\right);$$

In[•]:= $$Vxyz = \Big(-x * y * \sqrt{x^2+y^2} * \left(x^2+y^2+z^2\right) * V\lambda\lambda\varphi - x * y * z * \left(x^2+y^2+z^2\right) * V\lambda\lambda\rho + \left(x^2-y^2\right) * \left(x^2+y^2-z^2\right) * \sqrt{x^2+y^2+z^2} * V\lambda\varphi\rho - z * \left(x^2-y^2\right) * \sqrt{x^2+y^2} * \sqrt{x^2+y^2+z^2} * V\varphi\varphi\lambda + x * y * z^2 * \sqrt{x^2+y^2} * V\varphi\varphi\varphi - x * y * z * \left(2 * x^2 + 2 * y^2 - z^2\right) * V\varphi\varphi\rho + z * \left(x^2-y^2\right) * \sqrt{x^2+y^2} * \sqrt{x^2+y^2+z^2} * V\rho\rho\lambda + x * y * \left(x^2+y^2-2 * z^2\right) * \sqrt{x^2+y^2} * V\rho\rho\varphi + x * y * z * \left(x^2+y^2\right) * V\rho\rho\rho\Big) \Big/ \left(\left(x^2+y^2\right) * \left(x^2+y^2+z^2\right)^{3/2}\right);$$

In[•]:= $$Vyyx = \Big(-x^2 * y * \left(x^2+y^2+z^2\right)^{3/2} * V\lambda\lambda\lambda - x * z * \left(x^2 - 2 * y^2\right) * \left(x^2+y^2+z^2\right) * V\lambda\lambda\varphi + x * \left(x^2 - 2 * y^2\right) * \sqrt{x^2+y^2} * \left(x^2+y^2+z^2\right) * V\lambda\lambda\rho - 2 * y * z * \left(2 * x^2 - y^2\right) * \sqrt{x^2+y^2} * \sqrt{x^2+y^2+z^2} * V\lambda\varphi\rho + y * z^2 * \left(2 * x^2 - y^2\right) * \sqrt{x^2+y^2+z^2} * V\varphi\varphi\lambda - x * y^2 * z^3 * V\varphi\varphi\varphi + 3 * x * y^2 * z^2 * \sqrt{x^2+y^2} * V\varphi\varphi\rho + y * \left(2 * x^4 + x^2 * y^2 - y^4\right) * \sqrt{x^2+y^2+z^2} * V\rho\rho\lambda - 3 * x * y^2 * z * \left(x^2+y^2\right) * V\rho\rho\varphi + x * y^2 * \left(x^2+y^2\right)^{3/2} * V\rho\rho\rho\Big) \Big/ \left(\left(x^2+y^2\right)^{3/2} * \left(x^2+y^2+z^2\right)^{3/2}\right);$$

```
In[ ]:= Vyyy = (x^3 * (x^2 + y^2 + z^2)^(3/2) * Vλλλ -
    3 * x^2 * y * z * (x^2 + y^2 + z^2) * Vλλφ + 3 * x^2 * y * Sqrt[x^2 + y^2] * (x^2 + y^2 + z^2) * Vλλρ -
    6 * x * y^2 * z * Sqrt[x^2 + y^2] * Sqrt[x^2 + y^2 + z^2] * Vλφρ + 3 * x * y^2 * z^2 * Sqrt[x^2 + y^2 + z^2] * Vφφλ -
    y^3 * z^3 * Vφφφ + 3 * y^3 * z^2 * Sqrt[x^2 + y^2] * Vφφρ + 3 * x * y^2 * (x^2 + y^2) * Sqrt[x^2 + y^2 + z^2] * Vρρλ -
    3 * y^3 * z * (x^2 + y^2) * Vρρφ + y^3 * (x^2 + y^2)^(3/2) * Vρρρ) /
    ((x^2 + y^2)^(3/2) * (x^2 + y^2 + z^2)^(3/2));

In[ ]:= Vyyz =
    (x^2 * Sqrt[x^2 + y^2] * (x^2 + y^2 + z^2) * Vλλφ + x^2 * z * (x^2 + y^2 + z^2) * Vλλρ + 2 * x * y * (x^2 + y^2 - z^2) *
      Sqrt[x^2 + y^2 + z^2] * Vλφρ - 2 * x * y * z * Sqrt[x^2 + y^2] * Sqrt[x^2 + y^2 + z^2] * Vφφλ +
    y^2 * z^2 * Sqrt[x^2 + y^2] * Vφφφ - y^2 * z * (2 * x^2 + 2 * y^2 - z^2) * Vφφρ +
    2 * x * y * z * Sqrt[x^2 + y^2] * Sqrt[x^2 + y^2 + z^2] * Vρρλ + y^2 * (x^2 + y^2 - 2 * z^2) * Sqrt[x^2 + y^2] * Vρρφ +
    y^2 * z * (x^2 + y^2) * Vρρρ) / ((x^2 + y^2) * (x^2 + y^2 + z^2)^(3/2));

In[ ]:= Vzzx = (-2 * y * z * Sqrt[x^2 + y^2] * Sqrt[x^2 + y^2 + z^2] * Vλφρ - y * (x^2 + y^2) * Sqrt[x^2 + y^2 + z^2] * Vφφλ -
    x * z * (x^2 + y^2) * Vφφφ + x * (x^2 + y^2 - 2 * z^2) * Sqrt[x^2 + y^2] * Vφφρ -
    y * z^2 * Sqrt[x^2 + y^2 + z^2] * Vρρλ + x * z * (2 * x^2 + 2 * y^2 - z^2) * Vρρφ +
    x * z^2 * Sqrt[x^2 + y^2] * Vρρρ) / (Sqrt[x^2 + y^2] * (x^2 + y^2 + z^2)^(3/2));

In[ ]:= Vzzy = (2 * x * z * Sqrt[x^2 + y^2] * Sqrt[x^2 + y^2 + z^2] * Vλφρ + x * (x^2 + y^2) * Sqrt[x^2 + y^2 + z^2] * Vφφλ -
    y * z * (x^2 + y^2) * Vφφφ + y * (x^2 + y^2 - 2 * z^2) * Sqrt[x^2 + y^2] * Vφφρ +
    x * z^2 * Sqrt[x^2 + y^2 + z^2] * Vρρλ + y * z * (2 * x^2 + 2 * y^2 - z^2) * Vρρφ +
    y * z^2 * Sqrt[x^2 + y^2] * Vρρρ) / (Sqrt[x^2 + y^2] * (x^2 + y^2 + z^2)^(3/2));

In[ ]:= Vzzz1 = ((x^2 + y^2)^(3/2) * Vφφφ + 3 * z * (x^2 + y^2) * Vφφρ + 3 * z^2 * Sqrt[x^2 + y^2] * Vρρφ + z^3 * Vρρρ) /
    (x^2 + y^2 + z^2)^(3/2);
```

3.3 Table A1

```
In[ ]:= VrRoutes351 = (x * Vx + y * Vy) / Sqrt[x^2 + y^2];

In[ ]:= VλRoutes351 = (-y * Vx + x * Vy) / Sqrt[x^2 + y^2];

In[ ]:= VzRoutes351 = Vz1;

In[ ]:= VrrRoutes351 = (x^2 * Vxx + 2 * x * y * Vxy + y^2 * Vyy) / (x^2 + y^2);

In[ ]:= VrλRoutes351 = (-x * y * Vxx + (x^2 - y^2) * Vxy + x * y * Vyy) / (x^2 + y^2);

In[ ]:= VrzRoutes351 = (x * Vxz + y * Vyz) / Sqrt[x^2 + y^2];

In[ ]:= VλλRoutes351 = (y^2 * Vxx - 2 * x * y * Vxy + x^2 * Vyy) / (x^2 + y^2);
```

```
In[•]:= VλzRoutes351 = (-y * Vxz + x * Vyz) / Sqrt[x^2 + y^2];

In[•]:= VzzRoutes351 = Vzz1;

In[•]:= VrrrRoutes351 = (x^3 * Vxxx + 3 * x^2 * y * Vxxy + 3 * x * y^2 * Vyyx + y^3 * Vyyy) / (x^2 + y^2)^(3/2);

In[•]:= VrrλRoutes351 =
          (-x^2 * y * Vxxx + x * (x^2 - 2 * y^2) * Vxxy + y * (2 * x^2 - y^2) * Vyyx + x * y^2 * Vyyy) /
           (x^2 + y^2)^(3/2);

In[•]:= VrrzRoutes351 = (x^2 * Vxxz + 2 * x * y * Vxyz + y^2 * Vyyz) / (x^2 + y^2);

In[•]:= VrλzRoutes351 = (-x * y * Vxxz + (x^2 - y^2) * Vxyz + x * y * Vyyz) / (x^2 + y^2);

In[•]:= VλλrRoutes351 =
          (x * y^2 * Vxxx + y * (y^2 - 2 * x^2) * Vxxy + x * (x^2 - 2 * y^2) * Vyyx + x^2 * y * Vyyy) / (x^2 + y^2)^(3/2);

In[•]:= VλλλRoutes351 = (-y^3 * Vxxx + 3 * x * y^2 * Vxxy - 3 * x^2 * y * Vyyx + x^3 * Vyyy) / (x^2 + y^2)^(3/2);

In[•]:= VλλzRoutes351 = (y^2 * Vxxz - 2 * x * y * Vxyz + x^2 * Vyyz) / (x^2 + y^2);

In[•]:= VzzrRoutes351 = (x * Vzzx + y * Vzzy) / Sqrt[x^2 + y^2];

In[•]:= VzzλRoutes351 = (-y * Vzzx + x * Vzzy) / Sqrt[x^2 + y^2];

In[•]:= VzzzRoutes351 = Vzzz1;
```

3.4 Check whether they become themselves with r = $\sqrt{x^2+y^2}$

```
In[•]:= FullSimplify[VrRoutes351 /. {r → Sqrt[x^2 + y^2]}]
Out[•]=
        Vr

In[•]:= FullSimplify[VλRoutes351 /. {r → Sqrt[x^2 + y^2]}]
Out[•]=
        Vλ

In[•]:= FullSimplify[VzRoutes351 /. {r → Sqrt[x^2 + y^2]}]
Out[•]=
        Vz

In[•]:= FullSimplify[VrrRoutes351 /. {r → Sqrt[x^2 + y^2]}]
Out[•]=
        Vrr

In[•]:= FullSimplify[VrλRoutes351 /. {r → Sqrt[x^2 + y^2]}]
Out[•]=
        Vrλ
```

In[ ]:= FullSimplify[VrzRoutes351 /. {r → Sqrt[x^2 + y^2]}]

Out[ ]=

Vrz

In[ ]:= FullSimplify[VλλRoutes351 /. {r → Sqrt[x^2 + y^2]}]

Out[ ]=

Vλλ

In[ ]:= FullSimplify[VλzRoutes351 /. {r → Sqrt[x^2 + y^2]}]

Out[ ]=

Vλz

In[ ]:= FullSimplify[VzzRoutes351 /. {r → Sqrt[x^2 + y^2]}]

Out[ ]=

Vzz

In[ ]:= FullSimplify[VrrrRoutes351 /. {r → Sqrt[x^2 + y^2]}]

Out[ ]=

Vrrr

In[ ]:= FullSimplify[VrrλRoutes351 /. {r → Sqrt[x^2 + y^2]}]

Out[ ]=

Vrrλ

In[ ]:= FullSimplify[VrrzRoutes351 /. {r → Sqrt[x^2 + y^2]}]

Out[ ]=

Vrrz

In[ ]:= FullSimplify[VrλzRoutes351 /. {r → Sqrt[x^2 + y^2]}]

Out[ ]=

Vrλz

In[ ]:= FullSimplify[VλλrRoutes351 /. {r → Sqrt[x^2 + y^2]}]

Out[ ]=

Vλλr

In[ ]:= FullSimplify[VλλλRoutes351 /. {r → Sqrt[x^2 + y^2]}]

Out[ ]=

Vλλλ

In[ ]:= FullSimplify[VλλzRoutes351 /. {r → Sqrt[x^2 + y^2]}]

Out[ ]=

Vλλz

In[ ]:= FullSimplify[VzzrRoutes351 /. {r → Sqrt[x^2 + y^2]}]

Out[ ]=

Vzzr

```
In[ ]:= FullSimplify[VzzλRoutes351 /. {r → Sqrt[x^2 + y^2]}]
```

```
Out[ ]=
Vzzλ
```

```
In[ ]:= FullSimplify[VzzzRoutes351 /. {r → Sqrt[x^2 + y^2]}]
```

```
Out[ ]=
Vzzz
```

4. Algebraic form, Cylindrical coordinates, Routes 2 → 6 → 4, and Table A3 → Table A12 → Table A7

```
In[ ]:= Clear["Global`*"];
```

4.1 Table A3

```
In[ ]:= Vx = (x * Vr - y * Vλ) / Sqrt[x^2 + y^2];
```

```
In[ ]:= Vy = (y * Vr + x * Vλ) / Sqrt[x^2 + y^2];
```

```
In[ ]:= Vz = Vz;
```

```
In[ ]:= Vxx = (x^2 * Vrr - 2 * x * y * Vrλ + y^2 * Vλλ) / (x^2 + y^2);
```

```
In[ ]:= Vxy = (x * y * Vrr + (x^2 - y^2) * Vrλ - x * y * Vλλ) / (x^2 + y^2);
```

```
In[ ]:= Vxz = (x * Vrz - y * Vλz) / Sqrt[x^2 + y^2];
```

```
In[ ]:= Vyy = (y^2 * Vrr + 2 * x * y * Vrλ + x^2 * Vλλ) / (x^2 + y^2);
```

```
In[ ]:= Vyz = (y * Vrz + x * Vλz) / Sqrt[x^2 + y^2];
```

```
In[ ]:= Vzz = Vzz;
```

```
In[ ]:= Vxxx = (x^3 * Vrrr - 3 * x^2 * y * Vrrλ + 3 * x * y^2 * Vλλr - y^3 * Vλλλ) / (x^2 + y^2)^(3/2);
```

```
In[ ]:= Vxxy =
          (x^2 * y * Vrrr + x * (x^2 - 2 * y^2) * Vrrλ + y * (y^2 - 2 * x^2) * Vλλr + x * y^2 * Vλλλ) / (x^2 + y^2)^(3/2);
```

```
In[ ]:= Vxxz = (x^2 * Vrrz - 2 * x * y * Vrλz + y^2 * Vλλz) / (x^2 + y^2);
```

```
In[ ]:= Vxyz = (x * y * Vrrz + (x^2 - y^2) * Vrλz - x * y * Vλλz) / (x^2 + y^2);
```

```
In[ ]:= Vyyx =
          (x * y^2 * Vrrr + y * (2 * x^2 - y^2) * Vrrλ + x * (x^2 - 2 * y^2) * Vλλr - x^2 * y * Vλλλ) / (x^2 + y^2)^(3/2);
```

```
In[ ]:= Vyyy = (y^3 * Vrrr + 3 * x * y^2 * Vrrλ + 3 * x^2 * y * Vλλr + x^3 * Vλλλ) / (x^2 + y^2)^(3/2);
```

```
In[ ]:= Vyyz = (y^2 * Vrrz + 2 * x * y * Vrλz + x^2 * Vλλz) / (x^2 + y^2);
```

```
In[ ]:= Vzzx = (x * Vzzr - y * Vzzλ) / Sqrt[x^2 + y^2];
```

```
In[ ]:= Vzzy = (y * Vzzr + x * Vzzλ) / Sqrt[x^2 + y^2];

In[ ]:= Vzzz = Vzzz;
```

4.2 Table A12

```
In[ ]:= Vλ1 = (-y * Vx + x * Vy) / Sqrt[x^2 + y^2];

In[ ]:= Vφ = (-x * z * Vx - y * z * Vy + (x^2 + y^2) * Vz) / (Sqrt[x^2 + y^2] * Sqrt[x^2 + y^2 + z^2]);

In[ ]:= Vρ = (x * Vx + y * Vy + z * Vz) / Sqrt[x^2 + y^2 + z^2];

In[ ]:= Vλλ1 = (y^2 * Vxx - 2 * x * y * Vxy + x^2 * Vyy) / (x^2 + y^2);

In[ ]:= Vλφ = (x * y * z * Vxx - z * (x^2 - y^2) * Vxy - y * (x^2 + y^2) * Vxz -
          x * y * z * Vyy + x * (x^2 + y^2) * Vyz) / ((x^2 + y^2) * Sqrt[x^2 + y^2 + z^2]);

In[ ]:= Vλρ = (-x * y * Vxx + (x^2 - y^2) * Vxy - y * z * Vxz + x * z * Vyz + x * y * Vyy) /
        (Sqrt[x^2 + y^2] * Sqrt[x^2 + y^2 + z^2]);

In[ ]:= Vφφ = (x^2 * z^2 * Vxx + 2 * x * y * z^2 * Vxy - 2 * x * z * (x^2 + y^2) * Vxz + y^2 * z^2 * Vyy -
          2 * y * z * (x^2 + y^2) * Vyz + (x^2 + y^2)^2 * Vzz) / ((x^2 + y^2) * (x^2 + y^2 + z^2));

In[ ]:= Vφρ = (-x^2 * z * Vxx - 2 * x * y * z * Vxy + x * (x^2 + y^2 - z^2) * Vxz - y^2 * z * Vyy +
          y * (x^2 + y^2 - z^2) * Vyz + z * (x^2 + y^2) * Vzz) / (Sqrt[x^2 + y^2] * (x^2 + y^2 + z^2));

In[ ]:= Vρρ =
        (x^2 * Vxx + 2 * x * y * Vxy + 2 * x * z * Vxz + 2 * y * z * Vyz + y^2 * Vyy + z^2 * Vzz) / (x^2 + y^2 + z^2);

In[ ]:= Vλλλ1 = (-y^3 * Vxxx + 3 * x * y^2 * Vxxy - 3 * x^2 * y * Vyyx + x^3 * Vyyy) / (x^2 + y^2)^(3/2);

In[ ]:= Vλλφ = (-x * y^2 * z * Vxxx + y * z * (2 * x^2 - y^2) * Vxxy +
          y^2 * (x^2 + y^2) * Vxxz - 2 * x * y * (x^2 + y^2) * Vxyz + x * z * (2 * y^2 - x^2) * Vyyx -
          x^2 * y * z * Vyyy + x^2 * (x^2 + y^2) * Vyyz) / ((x^2 + y^2)^(3/2) * Sqrt[x^2 + y^2 + z^2]);

In[ ]:= Vλλρ = (x * y^2 * Vxxx + y * (y^2 - 2 * x^2) * Vxxy + y^2 * z * Vxxz - 2 * x * y * z * Vxyz +
          x * (x^2 - 2 * y^2) * Vyyx + x^2 * y * Vyyy + x^2 * z * Vyyz) / ((x^2 + y^2) * Sqrt[x^2 + y^2 + z^2]);

In[ ]:= Vλφρ = (x^2 * y * z * Vxxx + x * z * (2 * y^2 - x^2) * Vxxy -
          x * y * (x^2 + y^2 - z^2) * Vxxz + (x^2 - y^2) * (x^2 + y^2 - z^2) * Vxyz +
          y * z * (y^2 - 2 * x^2) * Vyyx - x * y^2 * z * Vyyy + x * y * (x^2 + y^2 - z^2) * Vyyz -
          y * z * (x^2 + y^2) * Vzzx + x * z * (x^2 + y^2) * Vzzy) / ((x^2 + y^2) * (x^2 + y^2 + z^2));
```

```
In[•]:= Vφφλ = (-x^2 * y * z^2 * Vxxx + x * z^2 * (x^2 - 2 * y^2) * Vxxy +
          2 * x * y * z * (x^2 + y^2) * Vxxz - 2 * z * (x^4 - y^4) * Vxyz +
          y * z^2 * (2 * x^2 - y^2) * Vyyx + x * y^2 * z^2 * Vyyy - 2 * x * y * z * (x^2 + y^2) * Vyyz -
          y * (x^2 + y^2)^2 * Vzzx + x * (x^2 + y^2)^2 * Vzzy) / ((x^2 + y^2)^(3/2) * (x^2 + y^2 + z^2));

In[•]:= Vφφφ = (-x^3 * z^3 * Vxxx - 3 * x^2 * y * z^3 * Vxxy + 3 * x^2 * z^2 * (x^2 + y^2) * Vxxz +
          6 * x * y * z^2 * (x^2 + y^2) * Vxyz - 3 * x * y^2 * z^3 * Vyyx - y^3 * z^3 * Vyyy +
          3 * y^2 * z^2 * (x^2 + y^2) * Vyyz - 3 * x * z * (x^2 + y^2)^2 * Vzzx -
          3 * y * z * (x^2 + y^2)^2 * Vzzy + (x^2 + y^2)^3 * Vzzz) / ((x^2 + y^2)^(3/2) * (x^2 + y^2 + z^2)^(3/2));

In[•]:= Vφφρ = (x^3 * z^2 * Vxxx + 3 * x^2 * y * z^2 * Vxxy +
          x^2 * z * (z^2 - 2 * x^2 - 2 * y^2) * Vxxz + 2 * x * y * z * (z^2 - 2 * x^2 - 2 * y^2) * Vxyz +
          3 * x * y^2 * z^2 * Vyyx + y^3 * z^2 * Vyyy + y^2 * z * (z^2 - 2 * x^2 - 2 * y^2) * Vyyz +
          x * (x^2 + y^2) * (x^2 + y^2 - 2 * z^2) * Vzzx + y * (x^2 + y^2) * (x^2 + y^2 - 2 * z^2) * Vzzy +
          z * (x^2 + y^2)^2 * Vzzz) / ((x^2 + y^2) * (x^2 + y^2 + z^2)^(3/2));

In[•]:= Vρρλ = (-x^2 * y * Vxxx + x * (x^2 - 2 * y^2) * Vxxy - 2 * x * y * z * Vxxz +
          2 * z * (x^2 - y^2) * Vxyz + y * (2 * x^2 - y^2) * Vyyx + x * y^2 * Vyyy +
          2 * x * y * z * Vyyz - y * z^2 * Vzzx + x * z^2 * Vzzy) / (Sqrt[x^2 + y^2] * (x^2 + y^2 + z^2));

In[•]:= Vρρφ = (-x^3 * z * Vxxx - 3 * x^2 * y * z * Vxxy + x^2 * (x^2 + y^2 - 2 * z^2) * Vxxz +
          2 * x * y * (x^2 + y^2 - 2 * z^2) * Vxyz - 3 * x * y^2 * z * Vyyx - y^3 * z * Vyyy +
          y^2 * (x^2 + y^2 - 2 * z^2) * Vyyz + x * z * (2 * x^2 + 2 * y^2 - z^2) * Vzzx + y * z *
            (2 * x^2 + 2 * y^2 - z^2) * Vzzy + z^2 * (x^2 + y^2) * Vzzz) / (Sqrt[x^2 + y^2] * (x^2 + y^2 + z^2)^(3/2));

In[•]:= Vρρρ = (x^3 * Vxxx + 3 * x^2 * y * Vxxy + 3 * x^2 * z * Vxxz +
          6 * x * y * z * Vxyz + 3 * x * y^2 * Vyyx + y^3 * Vyyy + 3 * y^2 * z * Vyyz +
          3 * x * z^2 * Vzzx + 3 * y * z^2 * Vzzy + z^3 * Vzzz) / (x^2 + y^2 + z^2)^(3/2);
```

4.3 Table A7

```
In[•]:= VrRoutes264 = (-z * Vφ + r * Vρ) / Sqrt[r^2 + z^2];

In[•]:= VλRoutes264 = Vλ1;

In[•]:= VzRoutes264 = (r * Vφ + z * Vρ) / Sqrt[r^2 + z^2];

In[•]:= VrrRoutes264 = (z^2 * Vφφ - 2 * r * z * Vφρ + r^2 * Vρρ) / (r^2 + z^2);

In[•]:= VrλRoutes264 = (-z * Vλφ + r * Vλρ) / Sqrt[r^2 + z^2];

In[•]:= VrzRoutes264 = (-r * z * Vφφ + (r^2 - z^2) * Vφρ + r * z * Vρρ) / (r^2 + z^2);

In[•]:= VλλRoutes264 = Vλλ1;

In[•]:= VλzRoutes264 = (r * Vλφ + z * Vλρ) / Sqrt[r^2 + z^2];
```

```
In[•]:= VzzRoutes264 = (r^2 * Vφφ + 2 * r * z * Vφρ + z^2 * Vρρ) / (r^2 + z^2);

In[•]:= VrrrRoutes264 = (-z^3 * Vφφφ + 3 * r * z^2 * Vφφρ - 3 * r^2 * z * Vρρφ + r^3 * Vρρρ) / (r^2 + z^2)^(3/2);

In[•]:= VrrλRoutes264 = (-2 * r * z * Vλφρ + z^2 * Vφφλ + r^2 * Vρρλ) / (r^2 + z^2);

In[•]:= VrrzRoutes264 =
          (r * z^2 * Vφφφ + z * (z^2 - 2 * r^2) * Vφφρ + r * (r^2 - 2 * z^2) * Vρρφ + r^2 * z * Vρρρ) / (r^2 + z^2)^(3/2);

In[•]:= VrλzRoutes264 = ((r^2 - z^2) * Vλφρ - r * z * Vφφλ + r * z * Vρρλ) / (r^2 + z^2);

In[•]:= VλλrRoutes264 = (-z * Vλλφ + r * Vλλρ) / Sqrt[r^2 + z^2];

In[•]:= VλλλRoutes264 = Vλλλ1;

In[•]:= VλλzRoutes264 = (r * Vλλφ + z * Vλλρ) / Sqrt[r^2 + z^2];

In[•]:= VzzrRoutes264 =
          (-r^2 * z * Vφφφ + r * (r^2 - 2 * z^2) * Vφφρ + z * (2 * r^2 - z^2) * Vρρφ + r * z^2 * Vρρρ) /
           (r^2 + z^2)^(3/2);

In[•]:= VzzλRoutes264 = (2 * r * z * Vλφρ + r^2 * Vφφλ + z^2 * Vρρλ) / (r^2 + z^2);

In[•]:= VzzzRoutes264 = (r^3 * Vφφφ + 3 * r^2 * z * Vφφρ + 3 * r * z^2 * Vρρφ + z^3 * Vρρρ) / (r^2 + z^2)^(3/2);
```

4.4 Check whether they become themselves with r = $\sqrt{x^2+y^2}$

```
In[•]:= FullSimplify[VrRoutes264 /. {r → Sqrt[x^2 + y^2]}]

Out[•]=
        Vr

In[•]:= FullSimplify[VλRoutes264 /. {r → Sqrt[x^2 + y^2]}]

Out[•]=
        Vλ

In[•]:= FullSimplify[VzRoutes264 /. {r → Sqrt[x^2 + y^2]}]

Out[•]=
        Vz

In[•]:= FullSimplify[VrrRoutes264 /. {r → Sqrt[x^2 + y^2]}]

Out[•]=
        Vrr

In[•]:= FullSimplify[VrλRoutes264 /. {r → Sqrt[x^2 + y^2]}]

Out[•]=
        Vrλ
```

In[•]:= FullSimplify[VrzRoutes264 /. {r → Sqrt[x^2 + y^2]}]

Out[•]=

Vrz

In[•]:= FullSimplify[VλλRoutes264 /. {r → Sqrt[x^2 + y^2]}]

Out[•]=

Vλλ

In[•]:= FullSimplify[VλzRoutes264 /. {r → Sqrt[x^2 + y^2]}]

Out[•]=

Vλz

In[•]:= FullSimplify[VzzRoutes264 /. {r → Sqrt[x^2 + y^2]}]

Out[•]=

Vzz

In[•]:= FullSimplify[VrrrRoutes264 /. {r → Sqrt[x^2 + y^2]}]

Out[•]=

Vrrr

In[•]:= FullSimplify[VrrλRoutes264 /. {r → Sqrt[x^2 + y^2]}]

Out[•]=

Vrrλ

In[•]:= FullSimplify[VrrzRoutes264 /. {r → Sqrt[x^2 + y^2]}]

Out[•]=

Vrrz

In[•]:= FullSimplify[VrλzRoutes264 /. {r → Sqrt[x^2 + y^2]}]

Out[•]=

Vrλz

In[•]:= FullSimplify[VλλrRoutes264 /. {r → Sqrt[x^2 + y^2]}]

Out[•]=

Vλλr

In[•]:= FullSimplify[VλλλRoutes264 /. {r → Sqrt[x^2 + y^2]}]

Out[•]=

Vλλλ

In[•]:= FullSimplify[VλλzRoutes264 /. {r → Sqrt[x^2 + y^2]}]

Out[•]=

Vλλz

In[•]:= FullSimplify[VzzrRoutes264 /. {r → Sqrt[x^2 + y^2]}]

Out[•]=

Vzzr

```
In[ ]:= FullSimplify[VzzλRoutes264 /. {r → Sqrt[x^2 + y^2]}]
Out[ ]=
Vzzλ

In[ ]:= FullSimplify[VzzzRoutes264 /. {r → Sqrt[x^2 + y^2]}]
Out[ ]=
Vzzz
```

5. Algebraic form, Spherical coordinates, Routes 5 → 1 → 3, and Tables A9, A10 → Table A1 → Table A5

```
In[ ]:= Clear["Global`*"];
```

5.1 Tables A9, A10

```
In[ ]:= Vx = (-y * Sqrt[x^2 + y^2 + z^2] * Vλ - x * z * Vφ + x * Sqrt[x^2 + y^2] * Vρ) / (Sqrt[x^2 + y^2] * Sqrt[x^2 + y^2 + z^2]);

In[ ]:= Vy = (x * Sqrt[x^2 + y^2 + z^2] * Vλ - y * z * Vφ + y * Sqrt[x^2 + y^2] * Vρ) / (Sqrt[x^2 + y^2] * Sqrt[x^2 + y^2 + z^2]);

In[ ]:= Vz = (Sqrt[x^2 + y^2] * Vφ + z * Vρ) / Sqrt[x^2 + y^2 + z^2];

In[ ]:= Vxx = (y^2 * (x^2 + y^2 + z^2) * Vλλ + 2 * x * y * z * Sqrt[x^2 + y^2 + z^2] * Vλφ -
          2 * x * y * Sqrt[x^2 + y^2] * Sqrt[x^2 + y^2 + z^2] * Vλρ + x^2 * z^2 * Vφφ -
          2 * x^2 * z * Sqrt[x^2 + y^2] * Vφρ + x^2 * (x^2 + y^2) * Vρρ) / ((x^2 + y^2) * (x^2 + y^2 + z^2));

In[ ]:= Vxy = (-x * y * (x^2 + y^2 + z^2)^(3/2) * Vλλ -
          z * (x^2 - y^2) * (x^2 + y^2 + z^2) * Vλφ + (x^2 - y^2) * Sqrt[x^2 + y^2] * (x^2 + y^2 + z^2) * Vλρ +
          x * y * z^2 * Sqrt[x^2 + y^2 + z^2] * Vφφ - 2 * x * y * z * Sqrt[x^2 + y^2] * Sqrt[x^2 + y^2 + z^2] * Vφρ +
          x * y * (x^2 + y^2) * Sqrt[x^2 + y^2 + z^2] * Vρρ) / ((x^2 + y^2) * (x^2 + y^2 + z^2)^(3/2));

In[ ]:= Vxz = (-y * Sqrt[x^2 + y^2] * Sqrt[x^2 + y^2 + z^2] * Vλφ - y * z * Sqrt[x^2 + y^2 + z^2] * Vλρ - x * z * Sqrt[x^2 + y^2] * Vφφ +
          x * (x^2 + y^2 - z^2) * Vφρ + x * z * Sqrt[x^2 + y^2] * Vρρ) / (Sqrt[x^2 + y^2] * (x^2 + y^2 + z^2));

In[ ]:= Vyy = (x^2 * (x^2 + y^2 + z^2) * Vλλ - 2 * x * y * z * Sqrt[x^2 + y^2 + z^2] * Vλφ +
          2 * x * y * Sqrt[x^2 + y^2] * Sqrt[x^2 + y^2 + z^2] * Vλρ + y^2 * z^2 * Vφφ -
          2 * y^2 * z * Sqrt[x^2 + y^2] * Vφρ + y^2 * (x^2 + y^2) * Vρρ) / ((x^2 + y^2) * (x^2 + y^2 + z^2));

In[ ]:= Vyz = (x * Sqrt[x^2 + y^2] * Sqrt[x^2 + y^2 + z^2] * Vλφ + x * z * Sqrt[x^2 + y^2 + z^2] * Vλρ - y * z * Sqrt[x^2 + y^2] * Vφφ +
          y * (x^2 + y^2 - z^2) * Vφρ + y * z * Sqrt[x^2 + y^2] * Vρρ) / (Sqrt[x^2 + y^2] * (x^2 + y^2 + z^2));

In[ ]:= Vzz = ((x^2 + y^2) * Vφφ + 2 * z * Sqrt[x^2 + y^2] * Vφρ + z^2 * Vρρ) / (x^2 + y^2 + z^2);
```

In[•]:=
$$\begin{aligned}
Vxxx = {} & \Big(-y^3 * \left(x^2+y^2+z^2\right)^{3/2} * V\lambda\lambda\lambda - 3 * x * y^2 * z * \left(x^2+y^2+z^2\right) * V\lambda\lambda\varphi + 3 * x * y^2 * \sqrt{x^2+y^2} * \\
& \left(x^2+y^2+z^2\right) * V\lambda\lambda\rho + 6 * x^2 * y * z \sqrt{x^2+y^2} * \sqrt{x^2+y^2+z^2} * V\lambda\varphi\rho - \\
& 3 * x^2 * y * z^2 * \sqrt{x^2+y^2+z^2} * V\varphi\varphi\lambda - x^3 * z^3 * V\varphi\varphi\varphi + 3 * x^3 * z^2 * \sqrt{x^2+y^2} * V\varphi\varphi\rho - \\
& 3 * x^2 * y * \left(x^2+y^2\right) * \sqrt{x^2+y^2+z^2} * V\rho\rho\lambda - 3 * x^3 * z * \left(x^2+y^2\right) * V\rho\rho\varphi + \\
& x^3 * \left(x^2+y^2\right)^{3/2} * V\rho\rho\rho\Big) \Big/ \left(\left(x^2+y^2\right)^{3/2} * \left(x^2+y^2+z^2\right)^{3/2}\right);
\end{aligned}$$

In[•]:=
$$\begin{aligned}
Vxxy = {} & \Big(x * y^2 * \left(x^2+y^2+z^2\right)^{3/2} * V\lambda\lambda\lambda + y * z * \left(2 * x^2 - y^2\right) * \left(x^2+y^2+z^2\right) * V\lambda\lambda\varphi - \\
& y * \sqrt{x^2+y^2} * \left(2 * x^2 - y^2\right) * \left(x^2+y^2+z^2\right) * V\lambda\lambda\rho - \\
& 2 * x * z * \left(x^2 - 2 * y^2\right) * \sqrt{x^2+y^2} * \sqrt{x^2+y^2+z^2} * V\lambda\varphi\rho + x * z^2 * \left(x^2 - 2 * y^2\right) * \\
& \sqrt{x^2+y^2+z^2} * V\varphi\varphi\lambda - x^2 * y * z^3 * V\varphi\varphi\varphi + 3 * x^2 * y * z^2 * \sqrt{x^2+y^2} * V\varphi\varphi\rho + \\
& x * \left(x^4 - x^2 * y^2 - 2 * y^4\right) * \sqrt{x^2+y^2+z^2} * V\rho\rho\lambda - 3 * x^2 * y * z * \left(x^2+y^2\right) * V\rho\rho\varphi + \\
& x^2 * y * \left(x^2+y^2\right)^{3/2} * V\rho\rho\rho\Big) \Big/ \left(\left(x^2+y^2\right)^{3/2} * \left(x^2+y^2+z^2\right)^{3/2}\right);
\end{aligned}$$

In[•]:=
$$\begin{aligned}
Vxxz = {} & \Big(y^2 * \sqrt{x^2+y^2} * \left(x^2+y^2+z^2\right) * V\lambda\lambda\varphi + y^2 * z * \left(x^2+y^2+z^2\right) * V\lambda\lambda\rho - 2 * x * y * \left(x^2+y^2-z^2\right) * \\
& \sqrt{x^2+y^2+z^2} * V\lambda\varphi\rho + 2 * x * y * z * \sqrt{x^2+y^2} * \sqrt{x^2+y^2+z^2} * V\varphi\varphi\lambda + \\
& x^2 * z^2 * \sqrt{x^2+y^2} * V\varphi\varphi\varphi - x^2 * z * \left(2 * x^2 + 2 * y^2 - z^2\right) * V\varphi\varphi\rho - \\
& 2 * x * y * z * \sqrt{x^2+y^2} * \sqrt{x^2+y^2+z^2} * V\rho\rho\lambda + x^2 * \sqrt{x^2+y^2} * \left(x^2+y^2-2 * z^2\right) * V\rho\rho\varphi + \\
& x^2 * z * \left(x^2+y^2\right) * V\rho\rho\rho\Big) \Big/ \left(\left(x^2+y^2\right) * \left(x^2+y^2+z^2\right)^{3/2}\right);
\end{aligned}$$

In[•]:=
$$\begin{aligned}
Vxyz = {} & \Big(-x * y * \sqrt{x^2+y^2} * \left(x^2+y^2+z^2\right) * V\lambda\lambda\varphi - \\
& x * y * z * \left(x^2+y^2+z^2\right) * V\lambda\lambda\rho + \left(x^2-y^2\right) * \left(x^2+y^2-z^2\right) * \sqrt{x^2+y^2+z^2} * V\lambda\varphi\rho - \\
& z * \left(x^2-y^2\right) * \sqrt{x^2+y^2} * \sqrt{x^2+y^2+z^2} * V\varphi\varphi\lambda + x * y * z^2 * \sqrt{x^2+y^2} * V\varphi\varphi\varphi - \\
& x * y * z * \left(2 * x^2 + 2 * y^2 - z^2\right) * V\varphi\varphi\rho + z * \left(x^2-y^2\right) * \sqrt{x^2+y^2} * \sqrt{x^2+y^2+z^2} * V\rho\rho\lambda + \\
& x * y * \left(x^2+y^2-2 * z^2\right) * \sqrt{x^2+y^2} * V\rho\rho\varphi + \\
& x * y * z * \left(x^2+y^2\right) * V\rho\rho\rho\Big) \Big/ \left(\left(x^2+y^2\right) * \left(x^2+y^2+z^2\right)^{3/2}\right);
\end{aligned}$$

In[•]:=
$$\begin{aligned}
Vyyx = {} & \Big(-x^2 * y * \left(x^2+y^2+z^2\right)^{3/2} * V\lambda\lambda\lambda - x * z * \left(x^2 - 2 * y^2\right) * \left(x^2+y^2+z^2\right) * V\lambda\lambda\varphi + \\
& x * \left(x^2 - 2 * y^2\right) * \sqrt{x^2+y^2} * \left(x^2+y^2+z^2\right) * V\lambda\lambda\rho - \\
& 2 * y * z * \left(2 * x^2 - y^2\right) * \sqrt{x^2+y^2} * \sqrt{x^2+y^2+z^2} * V\lambda\varphi\rho + y * z^2 * \left(2 * x^2 - y^2\right) * \\
& \sqrt{x^2+y^2+z^2} * V\varphi\varphi\lambda - x * y^2 * z^3 * V\varphi\varphi\varphi + 3 * x * y^2 * z^2 * \sqrt{x^2+y^2} * V\varphi\varphi\rho + \\
& y * \left(2 * x^4 + x^2 * y^2 - y^4\right) * \sqrt{x^2+y^2+z^2} * V\rho\rho\lambda - 3 * x * y^2 * z * \left(x^2+y^2\right) * V\rho\rho\varphi + \\
& x * y^2 * \left(x^2+y^2\right)^{3/2} * V\rho\rho\rho\Big) \Big/ \left(\left(x^2+y^2\right)^{3/2} * \left(x^2+y^2+z^2\right)^{3/2}\right);
\end{aligned}$$

```
In[ ]:= Vyyy = (x^3 * (x^2 + y^2 + z^2)^(3/2) * Vλλλ -
    3 * x^2 * y * z * (x^2 + y^2 + z^2) * Vλλφ + 3 * x^2 * y * Sqrt[x^2 + y^2] * (x^2 + y^2 + z^2) * Vλλρ -
    6 * x * y^2 * z * Sqrt[x^2 + y^2] * Sqrt[x^2 + y^2 + z^2] * Vλφρ + 3 * x * y^2 * z^2 * Sqrt[x^2 + y^2 + z^2] * Vφφλ -
    y^3 * z^3 * Vφφφ + 3 * y^3 * z^2 * Sqrt[x^2 + y^2] * Vφφρ + 3 * x * y^2 * (x^2 + y^2) * Sqrt[x^2 + y^2 + z^2] * Vρρλ -
    3 * y^3 * z * (x^2 + y^2) * Vρρφ + y^3 * (x^2 + y^2)^(3/2) * Vρρρ) / ((x^2 + y^2)^(3/2) * (x^2 + y^2 + z^2)^(3/2));

In[ ]:= Vyyz =
  (x^2 * Sqrt[x^2 + y^2] * (x^2 + y^2 + z^2) * Vλλφ + x^2 * z * (x^2 + y^2 + z^2) * Vλλρ + 2 * x * y * (x^2 + y^2 - z^2) *
     Sqrt[x^2 + y^2 + z^2] * Vλφρ - 2 * x * y * z * Sqrt[x^2 + y^2] * Sqrt[x^2 + y^2 + z^2] * Vφφλ +
    y^2 * z^2 * Sqrt[x^2 + y^2] * Vφφφ - y^2 * z * (2 * x^2 + 2 * y^2 - z^2) * Vφφρ +
    2 * x * y * z * Sqrt[x^2 + y^2] * Sqrt[x^2 + y^2 + z^2] * Vρρλ + y^2 * (x^2 + y^2 - 2 * z^2) * Sqrt[x^2 + y^2] * Vρρφ +
    y^2 * z * (x^2 + y^2) * Vρρρ) / ((x^2 + y^2) * (x^2 + y^2 + z^2)^(3/2));

In[ ]:= Vzzx = (-2 * y * z * Sqrt[x^2 + y^2] * Sqrt[x^2 + y^2 + z^2] * Vλφρ - y * (x^2 + y^2) * Sqrt[x^2 + y^2 + z^2] * Vφφλ -
    x * z * (x^2 + y^2) * Vφφφ + x * (x^2 + y^2 - 2 * z^2) * Sqrt[x^2 + y^2] * Vφφρ -
    y * z^2 * Sqrt[x^2 + y^2 + z^2] * Vρρλ + x * z * (2 * x^2 + 2 * y^2 - z^2) * Vρρφ +
    x * z^2 * Sqrt[x^2 + y^2] * Vρρρ) / (Sqrt[x^2 + y^2] * (x^2 + y^2 + z^2)^(3/2));

In[ ]:= Vzzy = (2 * x * z * Sqrt[x^2 + y^2] * Sqrt[x^2 + y^2 + z^2] * Vλφρ + x * (x^2 + y^2) * Sqrt[x^2 + y^2 + z^2] * Vφφλ -
    y * z * (x^2 + y^2) * Vφφφ + y * (x^2 + y^2 - 2 * z^2) * Sqrt[x^2 + y^2] * Vφφρ +
    x * z^2 * Sqrt[x^2 + y^2 + z^2] * Vρρλ + y * z * (2 * x^2 + 2 * y^2 - z^2) * Vρρφ +
    y * z^2 * Sqrt[x^2 + y^2] * Vρρρ) / (Sqrt[x^2 + y^2] * (x^2 + y^2 + z^2)^(3/2));

In[ ]:= Vzzz = ((x^2 + y^2)^(3/2) * Vφφφ + 3 * z * (x^2 + y^2) * Vφφρ + 3 * z^2 * Sqrt[x^2 + y^2] * Vρρφ + z^3 * Vρρρ) /
    (x^2 + y^2 + z^2)^(3/2);
```

5.2 Table A1

```
In[ ]:= Vr = (x * Vx + y * Vy) / Sqrt[x^2 + y^2];

In[ ]:= Vλ1 = (-y * Vx + x * Vy) / Sqrt[x^2 + y^2];

In[ ]:= Vz = Vz;

In[ ]:= Vrr = (x^2 * Vxx + 2 * x * y * Vxy + y^2 * Vyy) / (x^2 + y^2);

In[ ]:= Vrλ = (-x * y * Vxx + (x^2 - y^2) * Vxy + x * y * Vyy) / (x^2 + y^2);

In[ ]:= Vrz = (x * Vxz + y * Vyz) / Sqrt[x^2 + y^2];

In[ ]:= Vλλ1 = (y^2 * Vxx - 2 * x * y * Vxy + x^2 * Vyy) / (x^2 + y^2);
```

```
In[•]:= Vλz = (-y * Vxz + x * Vyz) / √(x^2 + y^2);

In[•]:= Vzz = Vzz;

In[•]:= Vrrr = (x^3 * Vxxx + 3 * x^2 * y * Vxxy + 3 * x * y^2 * Vyyx + y^3 * Vyyy) / (x^2 + y^2)^(3/2);

In[•]:= Vrrλ = (-x^2 * y * Vxxx + x * (x^2 - 2 * y^2) * Vxxy + y * (2 * x^2 - y^2) * Vyyx + x * y^2 * Vyyy) /
          (x^2 + y^2)^(3/2);

In[•]:= Vrrz = (x^2 * Vxxz + 2 * x * y * Vxyz + y^2 * Vyyz) / (x^2 + y^2);

In[•]:= Vrλz = (-x * y * Vxxz + (x^2 - y^2) * Vxyz + x * y * Vyyz) / (x^2 + y^2);

In[•]:= Vλλr =
          (x * y^2 * Vxxx + y * (y^2 - 2 * x^2) * Vxxy + x * (x^2 - 2 * y^2) * Vyyx + x^2 * y * Vyyy) / (x^2 + y^2)^(3/2);

In[•]:= Vλλλ1 = (-y^3 * Vxxx + 3 * x * y^2 * Vxxy - 3 * x^2 * y * Vyyx + x^3 * Vyyy) / (x^2 + y^2)^(3/2);

In[•]:= Vλλz = (y^2 * Vxxz - 2 * x * y * Vxyz + x^2 * Vyyz) / (x^2 + y^2);

In[•]:= Vzzr = (x * Vzzx + y * Vzzy) / √(x^2 + y^2);

In[•]:= Vzzλ = (-y * Vzzx + x * Vzzy) / √(x^2 + y^2);

In[•]:= Vzzz = Vzzz;
```

5.3 Table A5

```
In[•]:= VλRoutes513 = Vλ1;

In[•]:= VφRoutes513 = (-z * Vr + r * Vz) / √(r^2 + z^2);

In[•]:= VρRoutes513 = (r * Vr + z * Vz) / √(r^2 + z^2);

In[•]:= VλλRoutes513 = Vλλ1;

In[•]:= VλφRoutes513 = (-z * Vrλ + r * Vλz) / √(r^2 + z^2);

In[•]:= VλρRoutes513 = (r * Vrλ + z * Vλz) / √(r^2 + z^2);

In[•]:= VφφRoutes513 = (z^2 * Vrr - 2 * r * z * Vrz + r^2 * Vzz) / (r^2 + z^2);

In[•]:= VφρRoutes513 = (-r * z * Vrr + (r^2 - z^2) * Vrz + r * z * Vzz) / (r^2 + z^2);

In[•]:= VρρRoutes513 = (r^2 * Vrr + 2 * r * z * Vrz + z^2 * Vzz) / (r^2 + z^2);

In[•]:= VλλλRoutes513 = Vλλλ1;

In[•]:= VλλφRoutes513 = (-z * Vλλr + r * Vλλz) / √(r^2 + z^2);

In[•]:= VλλρRoutes513 = (r * Vλλr + z * Vλλz) / √(r^2 + z^2);
```

```
In[•]:= Vλφρ Routes513 = (-r * z * Vrrλ + (r^2 - z^2) * Vrλz + r * z * Vzzλ) / (r^2 + z^2);

In[•]:= Vφφλ Routes513 = (z^2 * Vrrλ - 2 * r * z * Vrλz + r^2 * Vzzλ) / (r^2 + z^2);

In[•]:= Vφφφ Routes513 = (-z^3 * Vrrr + 3 * r * z^2 * Vrrz - 3 * r^2 * z * Vzzr + r^3 * Vzzz) / (r^2 + z^2)^(3/2);

In[•]:= Vφφρ Routes513 =
          (r * z^2 * Vrrr + z * (z^2 - 2 * r^2) * Vrrz + r * (r^2 - 2 * z^2) * Vzzr + r^2 * z * Vzzz) / (r^2 + z^2)^(3/2);

In[•]:= Vρρλ Routes513 = (r^2 * Vrrλ + 2 * r * z * Vrλz + z^2 * Vzzλ) / (r^2 + z^2);

In[•]:= Vρρφ Routes513 =
          (-r^2 * z * Vrrr + r * (r^2 - 2 * z^2) * Vrrz + z * (2 * r^2 - z^2) * Vzzr + r * z^2 * Vzzz) /
           (r^2 + z^2)^(3/2);

In[•]:= Vρρρ Routes513 = (r^3 Vrrr + 3 * r^2 * z * Vrrz + 3 * r * z^2 * Vzzr + z^3 * Vzzz) / (r^2 + z^2)^(3/2);
```

5.4 Check whether they become themselves with r = $\sqrt{x^2+y^2}$

```
In[•]:= FullSimplify[Vλ Routes513 /. {r → Sqrt[x^2 + y^2]}]
Out[•]=
Vλ

In[•]:= FullSimplify[Vφ Routes513 /. {r → Sqrt[x^2 + y^2]}]
Out[•]=
Vφ

In[•]:= FullSimplify[Vρ Routes513 /. {r → Sqrt[x^2 + y^2]}]
Out[•]=
Vρ

In[•]:= FullSimplify[Vλλ Routes513 /. {r → Sqrt[x^2 + y^2]}]
Out[•]=
Vλλ

In[•]:= FullSimplify[Vλφ Routes513 /. {r → Sqrt[x^2 + y^2]}]
Out[•]=
Vλφ

In[•]:= FullSimplify[Vλρ Routes513 /. {r → Sqrt[x^2 + y^2]}]
Out[•]=
Vλρ

In[•]:= FullSimplify[Vφφ Routes513 /. {r → Sqrt[x^2 + y^2]}]
Out[•]=
Vφφ
```

In[•]:= FullSimplify[Vφρ Routes513 /. {r → √(x^2 + y^2)}]

Out[•]=

$V_{\varphi\rho}$

In[•]:= FullSimplify[Vρρ Routes513 /. {r → √(x^2 + y^2)}]

Out[•]=

$V_{\rho\rho}$

In[•]:= FullSimplify[Vλλλ Routes513 /. {r → √(x^2 + y^2)}]

Out[•]=

$V_{\lambda\lambda\lambda}$

In[•]:= FullSimplify[Vλλφ Routes513 /. {r → √(x^2 + y^2)}]

Out[•]=

$V_{\lambda\lambda\varphi}$

In[•]:= FullSimplify[Vλλρ Routes513 /. {r → √(x^2 + y^2)}]

Out[•]=

$V_{\lambda\lambda\rho}$

In[•]:= FullSimplify[Vλφρ Routes513 /. {r → √(x^2 + y^2)}]

Out[•]=

$V_{\lambda\varphi\rho}$

In[•]:= FullSimplify[Vφφλ Routes513 /. {r → √(x^2 + y^2)}]

Out[•]=

$V_{\varphi\varphi\lambda}$

In[•]:= FullSimplify[Vφφφ Routes513 /. {r → √(x^2 + y^2)}]

Out[•]=

$V_{\varphi\varphi\varphi}$

In[•]:= FullSimplify[Vφφρ Routes513 /. {r → √(x^2 + y^2)}]

Out[•]=

$V_{\varphi\varphi\rho}$

In[•]:= FullSimplify[Vρρλ Routes513 /. {r → √(x^2 + y^2)}]

Out[•]=

$V_{\rho\rho\lambda}$

In[•]:= FullSimplify[Vρρφ Routes513 /. {r → √(x^2 + y^2)}]

Out[•]=

$V_{\rho\rho\varphi}$

In[•]:= FullSimplify[Vρρρ Routes513 /. {r → √(x^2 + y^2)}]

Out[•]=

$V_{\rho\rho\rho}$

6. Algebraic form, Spherical coordinates, Routes 4 → 2 → 6, and Table A7 → Table A3 → Table A12

```
In[•]:= Clear["Global`*"];
```

6.1 Table A7

```
In[•]:= Vr = (-z * Vφ + r * Vρ) / Sqrt[r^2 + z^2];

In[•]:= Vλ = Vλ;

In[•]:= Vz = (r * Vφ + z * Vρ) / Sqrt[r^2 + z^2];

In[•]:= Vrr = (z^2 * Vφφ - 2 * r * z * Vφρ + r^2 * Vρρ) / (r^2 + z^2);

In[•]:= Vrλ = (-z * Vλφ + r * Vλρ) / Sqrt[r^2 + z^2];

In[•]:= Vrz = (-r * z * Vφφ + (r^2 - z^2) * Vφρ + r * z * Vρρ) / (r^2 + z^2);

In[•]:= Vλλ = Vλλ;

In[•]:= Vλz = (r * Vλφ + z * Vλρ) / Sqrt[r^2 + z^2];

In[•]:= Vzz = (r^2 * Vφφ + 2 * r * z * Vφρ + z^2 * Vρρ) / (r^2 + z^2);

In[•]:= Vrrr = (-z^3 * Vφφφ + 3 * r * z^2 * Vφφρ - 3 * r^2 * z * Vρρφ + r^3 * Vρρρ) / (r^2 + z^2)^(3/2);

In[•]:= Vrrλ = (-2 * r * z * Vλφρ + z^2 * Vφφλ + r^2 * Vρρλ) / (r^2 + z^2);

In[•]:= Vrrz =
    (r * z^2 * Vφφφ + z * (z^2 - 2 * r^2) * Vφφρ + r * (r^2 - 2 * z^2) * Vρρφ + r^2 * z * Vρρρ) / (r^2 + z^2)^(3/2);

In[•]:= Vrλz = ((r^2 - z^2) * Vλφρ - r * z * Vφφλ + r * z * Vρρλ) / (r^2 + z^2);

In[•]:= Vλλr = (-z * Vλλφ + r * Vλλρ) / Sqrt[r^2 + z^2];

In[•]:= Vλλλ = Vλλλ;

In[•]:= Vλλz = (r * Vλλφ + z * Vλλρ) / Sqrt[r^2 + z^2];

In[•]:= Vzzr = (-r^2 * z * Vφφφ + r * (r^2 - 2 * z^2) * Vφφρ + z * (2 * r^2 - z^2) * Vρρφ + r * z^2 * Vρρρ) /
    (r^2 + z^2)^(3/2);

In[•]:= Vzzλ = (2 * r * z * Vλφρ + r^2 * Vφφλ + z^2 * Vρρλ) / (r^2 + z^2);

In[•]:= Vzzz = (r^3 * Vφφφ + 3 * r^2 * z * Vφφρ + 3 * r * z^2 * Vρρφ + z^3 * Vρρρ) / (r^2 + z^2)^(3/2);
```

6.2 Table A3

```
In[•]:= Vx = (x * Vr - y * Vλ) / Sqrt[x^2 + y^2];
```

```
In[•]:= Vy = (y * Vr + x * Vλ) / √(x^2 + y^2);

In[•]:= Vz = Vz;

In[•]:= Vxx = (x^2 * Vrr - 2 * x * y * Vrλ + y^2 * Vλλ) / (x^2 + y^2);

In[•]:= Vxy = (x * y * Vrr + (x^2 - y^2) * Vrλ - x * y * Vλλ) / (x^2 + y^2);

In[•]:= Vxz = (x * Vrz - y * Vλz) / √(x^2 + y^2);

In[•]:= Vyy = (y^2 * Vrr + 2 * x * y * Vrλ + x^2 * Vλλ) / (x^2 + y^2);

In[•]:= Vyz = (y * Vrz + x * Vλz) / √(x^2 + y^2);

In[•]:= Vzz = Vzz;

In[•]:= Vxxx = (x^3 * Vrrr - 3 * x^2 * y * Vrrλ + 3 * x * y^2 * Vλλr - y^3 * Vλλλ) / (x^2 + y^2)^(3/2);

In[•]:= Vxxy =
          (x^2 * y * Vrrr + x * (x^2 - 2 * y^2) * Vrrλ + y * (y^2 - 2 * x^2) * Vλλr + x * y^2 * Vλλλ) / (x^2 + y^2)^(3/2);

In[•]:= Vxxz = (x^2 * Vrrz - 2 * x * y * Vrλz + y^2 * Vλλz) / (x^2 + y^2);

In[•]:= Vxyz = (x * y * Vrrz + (x^2 - y^2) * Vrλz - x * y * Vλλz) / (x^2 + y^2);

In[•]:= Vyyx =
          (x * y^2 * Vrrr + y * (2 * x^2 - y^2) * Vrrλ + x * (x^2 - 2 * y^2) * Vλλr - x^2 * y * Vλλλ) / (x^2 + y^2)^(3/2);

In[•]:= Vyyy = (y^3 * Vrrr + 3 * x * y^2 * Vrrλ + 3 * x^2 * y * Vλλr + x^3 * Vλλλ) / (x^2 + y^2)^(3/2);

In[•]:= Vyyz = (y^2 * Vrrz + 2 * x * y * Vrλz + x^2 * Vλλz) / (x^2 + y^2);

In[•]:= Vzzx = (x * Vzzr - y * Vzzλ) / √(x^2 + y^2);

In[•]:= Vzzy = (y * Vzzr + x * Vzzλ) / √(x^2 + y^2);

In[•]:= Vzzz = Vzzz;
```

6.3 Table A12

```
In[•]:= VλRoutes426 = (-y * Vx + x * Vy) / √(x^2 + y^2);

In[•]:= VφRoutes426 = (-x * z * Vx - y * z * Vy + (x^2 + y^2) * Vz) / (√(x^2 + y^2) * √(x^2 + y^2 + z^2));

In[•]:= VρRoutes426 = (x * Vx + y * Vy + z * Vz) / √(x^2 + y^2 + z^2);

In[•]:= VλλRoutes426 = (y^2 * Vxx - 2 * x * y * Vxy + x^2 * Vyy) / (x^2 + y^2);

In[•]:= VλφRoutes426 = (x * y * z * Vxx - z * (x^2 - y^2) * Vxy - y * (x^2 + y^2) * Vxz -
            x * y * z * Vyy + x * (x^2 + y^2) * Vyz) / ((x^2 + y^2) * √(x^2 + y^2 + z^2));
```

```
In[•]:= Vλρ𝚁𝚘𝚞𝚝𝚎𝚜426 = (-x * y * Vxx + (x^2 - y^2) * Vxy - y * z * Vxz + x * z * Vyz + x * y * Vyy) /
          (Sqrt[x^2 + y^2] * Sqrt[x^2 + y^2 + z^2]);

In[•]:= Vφφ𝚁𝚘𝚞𝚝𝚎𝚜426 = (x^2 * z^2 * Vxx + 2 * x * y * z^2 * Vxy - 2 * x * z * (x^2 + y^2) * Vxz + y^2 * z^2 * Vyy -
          2 * y * z * (x^2 + y^2) * Vyz + (x^2 + y^2)^2 * Vzz) / ((x^2 + y^2) * (x^2 + y^2 + z^2));

In[•]:= Vφρ𝚁𝚘𝚞𝚝𝚎𝚜426 = (-x^2 * z * Vxx - 2 * x * y * z * Vxy + x * (x^2 + y^2 - z^2) * Vxz - y^2 * z * Vyy +
          y * (x^2 + y^2 - z^2) * Vyz + z * (x^2 + y^2) * Vzz) / (Sqrt[x^2 + y^2] * (x^2 + y^2 + z^2));

In[•]:= Vρρ𝚁𝚘𝚞𝚝𝚎𝚜426 =
          (x^2 * Vxx + 2 * x * y * Vxy + 2 * x * z * Vxz + 2 * y * z * Vyz + y^2 * Vyy + z^2 * Vzz) / (x^2 + y^2 + z^2);

In[•]:= Vλλλ𝚁𝚘𝚞𝚝𝚎𝚜426 = (-y^3 * Vxxx + 3 * x * y^2 * Vxxy - 3 * x^2 * y * Vyyx + x^3 * Vyyy) / (x^2 + y^2)^(3/2);

In[•]:= Vλλφ𝚁𝚘𝚞𝚝𝚎𝚜426 = (-x * y^2 * z * Vxxx + y * z * (2 * x^2 - y^2) * Vxxy +
          y^2 * (x^2 + y^2) * Vxxz - 2 * x * y * (x^2 + y^2) * Vxyz + x * z * (2 * y^2 - x^2) * Vyyx -
          x^2 * y * z * Vyyy + x^2 * (x^2 + y^2) * Vyyz) / ((x^2 + y^2)^(3/2) * Sqrt[x^2 + y^2 + z^2]);

In[•]:= Vλλρ𝚁𝚘𝚞𝚝𝚎𝚜426 = (x * y^2 * Vxxx + y * (y^2 - 2 * x^2) * Vxxy + y^2 * z * Vxxz - 2 * x * y * z * Vxyz +
          x * (x^2 - 2 * y^2) * Vyyx + x^2 * y * Vyyy + x^2 * z * Vyyz) / ((x^2 + y^2) * Sqrt[x^2 + y^2 + z^2]);

In[•]:= Vλφρ𝚁𝚘𝚞𝚝𝚎𝚜426 = (x^2 * y * z * Vxxx + x * z * (2 * y^2 - x^2) * Vxxy -
          x * y * (x^2 + y^2 - z^2) * Vxxz + (x^2 - y^2) * (x^2 + y^2 - z^2) * Vxyz +
          y * z * (y^2 - 2 * x^2) * Vyyx - x * y^2 * z * Vyyy + x * y * (x^2 + y^2 - z^2) * Vyyz -
          y * z * (x^2 + y^2) * Vzzx + x * z * (x^2 + y^2) * Vzzy) / ((x^2 + y^2) * (x^2 + y^2 + z^2));

In[•]:= Vφφλ𝚁𝚘𝚞𝚝𝚎𝚜426 =
          (-x^2 * y * z^2 * Vxxx + x * z^2 * (x^2 - 2 * y^2) * Vxxy + 2 * x * y * z * (x^2 + y^2) * Vxxz -
          2 * z * (x^4 - y^4) * Vxyz + y * z^2 * (2 * x^2 - y^2) * Vyyx + x * y^2 * z^2 * Vyyy -
          2 * x * y * z * (x^2 + y^2) * Vyyz - y * (x^2 + y^2)^2 * Vzzx +
          x * (x^2 + y^2)^2 * Vzzy) / ((x^2 + y^2)^(3/2) * (x^2 + y^2 + z^2));

In[•]:= Vφφφ𝚁𝚘𝚞𝚝𝚎𝚜426 = (-x^3 * z^3 * Vxxx - 3 * x^2 * y * z^3 * Vxxy +
          3 * x^2 * z^2 * (x^2 + y^2) * Vxxz + 6 * x * y * z^2 * (x^2 + y^2) * Vxyz - 3 * x * y^2 * z^3 * Vyyx -
          y^3 * z^3 * Vyyy + 3 * y^2 * z^2 * (x^2 + y^2) * Vyyz - 3 * x * z * (x^2 + y^2)^2 * Vzzx -
          3 * y * z * (x^2 + y^2)^2 * Vzzy + (x^2 + y^2)^3 * Vzzz) / ((x^2 + y^2)^(3/2) * (x^2 + y^2 + z^2)^(3/2));

In[•]:= Vφφρ𝚁𝚘𝚞𝚝𝚎𝚜426 = (x^3 * z^2 * Vxxx + 3 * x^2 * y * z^2 * Vxxy +
          x^2 * z * (z^2 - 2 * x^2 - 2 * y^2) * Vxxz + 2 * x * y * z * (z^2 - 2 * x^2 - 2 * y^2) * Vxyz +
          3 * x * y^2 * z^2 * Vyyx + y^3 * z^2 * Vyyy + y^2 * z * (z^2 - 2 * x^2 - 2 * y^2) * Vyyz +
          x * (x^2 + y^2) * (x^2 + y^2 - 2 * z^2) * Vzzx + y * (x^2 + y^2) * (x^2 + y^2 - 2 * z^2) * Vzzy +
          z * (x^2 + y^2)^2 * Vzzz) / ((x^2 + y^2) * (x^2 + y^2 + z^2)^(3/2));
```

```
In[•]:= VρρλRoutes426 =
          (-x^2 * y * Vxxx + x * (x^2 - 2 * y^2) * Vxxy - 2 * x * y * z * Vxxz + 2 * z * (x^2 - y^2) * Vxyz +
            y * (2 * x^2 - y^2) * Vyyx + x * y^2 * Vyyy + 2 * x * y * z * Vyyz -
            y * z^2 * Vzzx + x * z^2 * Vzzy) / (Sqrt[x^2 + y^2] * (x^2 + y^2 + z^2));

In[•]:= VρρφRoutes426 = (-x^3 * z * Vxxx - 3 * x^2 * y * z * Vxxy + x^2 * (x^2 + y^2 - 2 * z^2) * Vxxz +
            2 * x * y * (x^2 + y^2 - 2 * z^2) * Vxyz - 3 * x * y^2 * z * Vyyx - y^3 * z * Vyyy +
            y^2 * (x^2 + y^2 - 2 * z^2) * Vyyz + x * z * (2 * x^2 + 2 * y^2 - z^2) * Vzzx + y * z *
             (2 * x^2 + 2 * y^2 - z^2) * Vzzy + z^2 * (x^2 + y^2) * Vzzz) / (Sqrt[x^2 + y^2] * (x^2 + y^2 + z^2)^(3/2));

In[•]:= VρρρRoutes426 = (x^3 * Vxxx + 3 * x^2 * y * Vxxy + 3 * x^2 * z * Vxxz +
            6 * x * y * z * Vxyz + 3 * x * y^2 * Vyyx + y^3 * Vyyy + 3 * y^2 * z * Vyyz +
            3 * x * z^2 * Vzzx + 3 * y * z^2 * Vzzy + z^3 * Vzzz) / (x^2 + y^2 + z^2)^(3/2);
```

6.4 Check whether they become themselves with r = $\sqrt{x^2+y^2}$

```
In[•]:= FullSimplify[VλRoutes426 /. {r → Sqrt[x^2 + y^2]}]
Out[•]=
Vλ

In[•]:= FullSimplify[VφRoutes426 /. {r → Sqrt[x^2 + y^2]}]
Out[•]=
Vφ

In[•]:= FullSimplify[VρRoutes426 /. {r → Sqrt[x^2 + y^2]}]
Out[•]=
Vρ

In[•]:= FullSimplify[VλλRoutes426 /. {r → Sqrt[x^2 + y^2]}]
Out[•]=
Vλλ

In[•]:= FullSimplify[VλφRoutes426 /. {r → Sqrt[x^2 + y^2]}]
Out[•]=
Vλφ

In[•]:= FullSimplify[VλρRoutes426 /. {r → Sqrt[x^2 + y^2]}]
Out[•]=
Vλρ

In[•]:= FullSimplify[VφφRoutes426 /. {r → Sqrt[x^2 + y^2]}]
Out[•]=
Vφφ
```

In[ ]:= FullSimplify[Vφρ Routes426 /. {r → √(x^2 + y^2)}]

Out[ ]=

$V_{\varphi\rho}$

In[ ]:= FullSimplify[Vρρ Routes426 /. {r → √(x^2 + y^2)}]

Out[ ]=

$V_{\rho\rho}$

In[ ]:= FullSimplify[Vλλλ Routes426 /. {r → √(x^2 + y^2)}]

Out[ ]=

$V_{\lambda\lambda\lambda}$

In[ ]:= FullSimplify[Vλλφ Routes426 /. {r → √(x^2 + y^2)}]

Out[ ]=

$V_{\lambda\lambda\varphi}$

In[ ]:= FullSimplify[Vλλρ Routes426 /. {r → √(x^2 + y^2)}]

Out[ ]=

$V_{\lambda\lambda\rho}$

In[ ]:= FullSimplify[Vλφρ Routes426 /. {r → √(x^2 + y^2)}]

Out[ ]=

$V_{\lambda\varphi\rho}$

In[ ]:= FullSimplify[Vφφλ Routes426 /. {r → √(x^2 + y^2)}]

Out[ ]=

$V_{\varphi\varphi\lambda}$

In[ ]:= FullSimplify[Vφφφ Routes426 /. {r → √(x^2 + y^2)}]

Out[ ]=

$V_{\varphi\varphi\varphi}$

In[ ]:= FullSimplify[Vφφρ Routes426 /. {r → √(x^2 + y^2)}]

Out[ ]=

$V_{\varphi\varphi\rho}$

In[ ]:= FullSimplify[Vρρλ Routes426 /. {r → √(x^2 + y^2)}]

Out[ ]=

$V_{\rho\rho\lambda}$

In[ ]:= FullSimplify[Vρρφ Routes426 /. {r → √(x^2 + y^2)}]

Out[ ]=

$V_{\rho\rho\varphi}$

In[ ]:= FullSimplify[Vρρρ Routes426 /. {r → √(x^2 + y^2)}]

Out[ ]=

$V_{\rho\rho\rho}$

7. Trigonometric form, Cartesian coordinates, Routes 1 → 3 → 5, and Table A2 → Table A6→ Table A11

```
In[ ]:= Clear["Global`*"];
```

7.1 Table A2

```
In[ ]:= Vr = Cos[λ] * Vx + Sin[λ] * Vy;

In[ ]:= Vλ = -Sin[λ] * Vx + Cos[λ] * Vy;

In[ ]:= Vz = Vz;

In[ ]:= Vrr = (Cos[λ])^2 * Vxx + Sin[2 * λ] * Vxy + (Sin[λ])^2 * Vyy;

In[ ]:= Vrλ = -Sin[λ] * Cos[λ] * Vxx + Cos[2 * λ] * Vxy + Sin[λ] * Cos[λ] * Vyy;

In[ ]:= Vrz = Cos[λ] * Vxz + Sin[λ] * Vyz;

In[ ]:= Vλλ = (Sin[λ])^2 * Vxx - Sin[2 * λ] * Vxy + (Cos[λ])^2 * Vyy;

In[ ]:= Vλz = -Sin[λ] * Vxz + Cos[λ] * Vyz;

In[ ]:= Vzz = Vzz;

In[ ]:= Vrrr = (Cos[λ])^3 * Vxxx + 3 * Sin[λ] * (Cos[λ])^2 * Vxxy +
          3 * Cos[λ] * (Sin[λ])^2 * Vyyx + (Sin[λ])^3 * Vyyy;

In[ ]:= Vrrλ = -Sin[λ] * (Cos[λ])^2 * Vxxx + Cos[λ] * ((Cos[λ])^2 - 2 * (Sin[λ])^2) * Vxxy +
          Sin[λ] * (2 * (Cos[λ])^2 - (Sin[λ])^2) * Vyyx + (Sin[λ])^2 * Cos[λ] * Vyyy;

In[ ]:= Vrrz = (Cos[λ])^2 * Vxxz + Sin[2 * λ] * Vxyz + (Sin[λ])^2 * Vyyz;

In[ ]:= Vrλz = -Sin[λ] * Cos[λ] * Vxxz + Cos[2 * λ] * Vxyz + Sin[λ] * Cos[λ] * Vyyz;

In[ ]:= Vλλr = Cos[λ] * (Sin[λ])^2 * Vxxx + Sin[λ] * ((Sin[λ])^2 - 2 * (Cos[λ])^2) * Vxxy +
          Cos[λ] * ((Cos[λ])^2 - 2 * (Sin[λ])^2) * Vyyx + Sin[λ] * (Cos[λ])^2 * Vyyy;

In[ ]:= Vλλλ = -(Sin[λ])^3 * Vxxx + 3 * (Sin[λ])^2 * Cos[λ] * Vxxy -
          3 * Sin[λ] * (Cos[λ])^2 * Vyyx + (Cos[λ])^3 * Vyyy;

In[ ]:= Vλλz = (Sin[λ])^2 * Vxxz - Sin[2 * λ] * Vxyz + (Cos[λ])^2 * Vyyz;

In[ ]:= Vzzr = Cos[λ] * Vzzx + Sin[λ] * Vzzy;

In[ ]:= Vzzλ = -Sin[λ] * Vzzx + Cos[λ] * Vzzy;

In[ ]:= Vzzz = Vzzz;
```

7.2 Table A6

```
In[ ]:= Vλ = Vλ;

In[ ]:= Vφ = -Sin[φ] * Vr + Cos[φ] * Vz;

In[ ]:= Vρ = Cos[φ] * Vr + Sin[φ] * Vz;

In[ ]:= Vλλ = Vλλ;
```

```
In[•]:= Vλφ = -Sin[φ] * Vrλ + Cos[φ] * Vλz;

In[•]:= Vλρ = Cos[φ] * Vrλ + Sin[φ] * Vλz;

In[•]:= Vφφ = ((Sin[φ])^2 * Vrr - Sin[2 * φ] * Vrz + (Cos[φ])^2 * Vzz);

In[•]:= Vφρ = -Sin[φ] * Cos[φ] * Vrr + Cos[2 * φ] * Vrz + Sin[φ] * Cos[φ] * Vzz;

In[•]:= Vρρ = ((Cos[φ])^2 * Vrr + Sin[2 * φ] * Vrz + (Sin[φ])^2 * Vzz);

In[•]:= Vλλλ = Vλλλ;

In[•]:= Vλλφ = -Sin[φ] * Vλλr + Cos[φ] * Vλλz;

In[•]:= Vλλρ = Cos[φ] * Vλλr + Sin[φ] * Vλλz;

In[•]:= Vλφρ = -Sin[φ] * Cos[φ] * Vrrλ + Cos[2 * φ] * Vrλz + Sin[φ] * Cos[φ] * Vzzλ;

In[•]:= Vφφλ = (Sin[φ])^2 * Vrrλ - Sin[2 * φ] * Vrλz + (Cos[φ])^2 * Vzzλ;

In[•]:= Vφφφ = -(Sin[φ])^3 * Vrrr + 3 * (Sin[φ])^2 * Cos[φ] * Vrrz -
          3 * Sin[φ] * (Cos[φ])^2 * Vzzr + (Cos[φ])^3 * Vzzz;

In[•]:= Vφφρ = (Sin[φ])^2 * Cos[φ] * Vrrr + Sin[φ] * ((Sin[φ])^2 - 2 * (Cos[φ])^2) * Vrrz +
          Cos[φ] * ((Cos[φ])^2 - 2 * (Sin[φ])^2) * Vzzr + Sin[φ] * (Cos[φ])^2 * Vzzz;

In[•]:= Vρρλ = (Cos[φ])^2 * Vrrλ + Sin[2 * φ] * Vrλz + (Sin[φ])^2 * Vzzλ;

In[•]:= Vρρφ = -Sin[φ] * (Cos[φ])^2 * Vrrr + Cos[φ] * ((Cos[φ])^2 - 2 * (Sin[φ])^2) * Vrrz +
          Sin[φ] * (2 * (Cos[φ])^2 - (Sin[φ])^2) * Vzzr + Cos[φ] * (Sin[φ])^2 * Vzzz;

In[•]:= Vρρρ = (Cos[φ])^3 * Vrrr + 3 * Sin[φ] * (Cos[φ])^2 * Vrrz +
          3 * Cos[φ] * (Sin[φ])^2 * Vzzr + (Sin[φ])^3 * Vzzz;
```

7.4 Table A11

```
In[•]:= VxRoutes135 = -Sin[λ] * Vλ - Sin[φ] * Cos[λ] * Vφ + Cos[φ] * Cos[λ] * Vρ;

In[•]:= VyRoutes135 = Cos[λ] * Vλ - Sin[φ] * Sin[λ] * Vφ + Cos[φ] * Sin[λ] * Vρ;

In[•]:= VzRoutes135 = Cos[φ] * Vφ + Sin[φ] * Vρ;

In[•]:= VxxRoutes135 = (Sin[λ])^2 * Vλλ + Sin[φ] * Sin[2 * λ] * Vλφ -
          Cos[φ] * Sin[2 * λ] * Vλρ + (Sin[φ])^2 * (Cos[λ])^2 * Vφφ -
          Sin[2 * φ] * (Cos[λ])^2 * Vφρ + (Cos[φ])^2 * (Cos[λ])^2 * Vρρ;

In[•]:= VxyRoutes135 = -Sin[λ] * Cos[λ] * Vλλ - Sin[φ] * Cos[2 * λ] * Vλφ +
          Cos[φ] * Cos[2 * λ] * Vλρ + (Sin[φ])^2 * Sin[λ] * Cos[λ] * Vφφ -
          Sin[2 * φ] * Sin[λ] * Cos[λ] * Vφρ + (Cos[φ])^2 * Sin[λ] * Cos[λ] * Vρρ;

In[•]:= VxzRoutes135 =
        -Cos[φ] * Sin[λ] * Vλφ - Sin[φ] * Sin[λ] * Vλρ - Sin[φ] * Cos[φ] * Cos[λ] * Vφφ +
          Cos[2 * φ] * Cos[λ] * Vφρ + Sin[φ] * Cos[φ] * Cos[λ] * Vρρ;

In[•]:= VyyRoutes135 = (Cos[λ])^2 * Vλλ - Sin[φ] * Sin[2 * λ] * Vλφ +
          Cos[φ] * Sin[2 * λ] * Vλρ + (Sin[φ])^2 * (Sin[λ])^2 * Vφφ -
          Sin[2 * φ] * (Sin[λ])^2 * Vφρ + (Cos[φ])^2 * (Sin[λ])^2 * Vρρ;
```

```
In[•]:= VyzRoutes135 =
          Cos[φ] * Cos[λ] * Vλφ + Sin[φ] * Cos[λ] * Vλρ - Sin[φ] * Cos[φ] * Sin[λ] * Vφφ +
           Cos[2 * φ] * Sin[λ] * Vφρ + Sin[φ] * Cos[φ] * Sin[λ] * Vρρ;

In[•]:= VzzRoutes135 = (Cos[φ])^2 * Vφφ + Sin[2 * φ] * Vφρ + (Sin[φ])^2 * Vρρ;

In[•]:= VxxxRoutes135 = - (Sin[λ])^3 * Vλλλ + 3 * Cos[φ] * (Sin[λ])^2 * Cos[λ] * Vλλρ -
          3 * Sin[φ] * (Sin[λ])^2 * Cos[λ] * Vλλφ + 3 * Sin[2 * φ] * Sin[λ] * (Cos[λ])^2 * Vλφρ -
          3 * (Sin[φ])^2 * Sin[λ] * (Cos[λ])^2 * Vφφλ - (Sin[φ])^3 * (Cos[λ])^3 * Vφφφ +
          3 * (Sin[φ])^2 * Cos[φ] * (Cos[λ])^3 * Vφφρ - 3 * (Cos[φ])^2 * Sin[λ] * (Cos[λ])^2 * Vρρλ -
          3 * Sin[φ] * (Cos[φ])^2 * (Cos[λ])^3 * Vρρφ + (Cos[φ])^3 * (Cos[λ])^3 * Vρρρ;

In[•]:= VxxyRoutes135 =
          (Sin[λ])^2 * Cos[λ] * Vλλλ + Sin[φ] * Sin[λ] * (2 * (Cos[λ])^2 - (Sin[λ])^2) * Vλλφ +
           Cos[φ] * Sin[λ] * ((Sin[λ])^2 - 2 * (Cos[λ])^2) * Vλλρ +
           Sin[2 * φ] * Cos[λ] * (2 * (Sin[λ])^2 - (Cos[λ])^2) * Vλφρ +
           (Sin[φ])^2 * Cos[λ] * ((Cos[λ])^2 - 2 * (Sin[λ])^2) * Vφφλ - (Sin[φ])^3 * Sin[λ] *
            (Cos[λ])^2 * Vφφφ + 3 * (Sin[φ])^2 * Cos[φ] * Sin[λ] * (Cos[λ])^2 * Vφφρ +
           (Cos[φ])^2 * Cos[λ] * ((Cos[λ])^2 - 2 * (Sin[λ])^2) * Vρρλ - 3 * Sin[φ] *
            (Cos[φ])^2 * Sin[λ] * (Cos[λ])^2 * Vρρφ + (Cos[φ])^3 * Sin[λ] * (Cos[λ])^2 * Vρρρ;

In[•]:= VxxzRoutes135 =
          Cos[φ] * (Sin[λ])^2 * Vλλφ + Sin[φ] * (Sin[λ])^2 * Vλλρ - Cos[2 * φ] * Sin[2 * λ] * Vλφρ +
           Sin[φ] * Cos[φ] * Sin[2 * λ] * Vφφλ + (Sin[φ])^2 * Cos[φ] * (Cos[λ])^2 * Vφφφ +
           Sin[φ] * (Cos[λ])^2 * ((Sin[φ])^2 - 2 * (Cos[φ])^2) * Vφφρ -
           Sin[2 * φ] * Sin[λ] * Cos[λ] * Vρρλ + Cos[φ] * (Cos[λ])^2 *
            ((Cos[φ])^2 - 2 * (Sin[φ])^2) * Vρρφ + Sin[φ] * (Cos[φ])^2 * (Cos[λ])^2 * Vρρρ;

In[•]:= VxyzRoutes135 = - Cos[φ] * Sin[λ] * Cos[λ] * Vλλφ -
          Sin[φ] * Sin[λ] * Cos[λ] * Vλλρ + Cos[2 * φ] * Cos[2 * λ] * Vλφρ -
          Sin[φ] * Cos[φ] * Cos[2 * λ] * Vφφλ + (Sin[φ])^2 * Cos[φ] * Sin[λ] * Cos[λ] * Vφφφ -
          Sin[φ] * Sin[λ] * Cos[λ] * (2 * (Cos[φ])^2 - (Sin[φ])^2) * Vφφρ +
          Sin[φ] * Cos[φ] * Cos[2 * λ] * Vρρλ + Cos[φ] * Sin[λ] * Cos[λ] *
            ((Cos[φ])^2 - 2 * (Sin[φ])^2) * Vρρφ + Sin[φ] * (Cos[φ])^2 * Sin[λ] * Cos[λ] * Vρρρ;

In[•]:= VyyxRoutes135 =
          - Sin[λ] * (Cos[λ])^2 * Vλλλ - Sin[φ] * Cos[λ] * ((Cos[λ])^2 - 2 * (Sin[λ])^2) * Vλλφ +
           Cos[φ] * Cos[λ] * ((Cos[λ])^2 - 2 * (Sin[λ])^2) * Vλλρ -
           Sin[2 * φ] * Sin[λ] * (2 * (Cos[λ])^2 - (Sin[λ])^2) * Vλφρ +
           (Sin[φ])^2 * Sin[λ] * (2 * (Cos[λ])^2 - (Sin[λ])^2) * Vφφλ - (Sin[φ])^3 *
            (Sin[λ])^2 * Cos[λ] * Vφφφ + 3 * (Sin[φ])^2 * Cos[φ] * (Sin[λ])^2 * Cos[λ] * Vφφρ +
           (Cos[φ])^2 * Sin[λ] * (2 * (Cos[λ])^2 - (Sin[λ])^2) * Vρρλ - 3 * Sin[φ] *
            (Cos[φ])^2 * (Sin[λ])^2 * Cos[λ] * Vρρφ + (Cos[φ])^3 * (Sin[λ])^2 * Cos[λ] * Vρρρ;

In[•]:= VyyyRoutes135 = (Cos[λ])^3 * Vλλλ - 3 * Sin[φ] * Sin[λ] * (Cos[λ])^2 * Vλλφ +
          3 * Cos[φ] * Sin[λ] * (Cos[λ])^2 * Vλλρ - 3 * Sin[2 * φ] * (Sin[λ])^2 * Cos[λ] * Vλφρ +
          3 * (Sin[φ])^2 * (Sin[λ])^2 * Cos[λ] * Vφφλ - (Sin[φ])^3 * (Sin[λ])^3 * Vφφφ +
          3 * (Sin[φ])^2 * Cos[φ] * (Sin[λ])^3 * Vφφρ + 3 * (Cos[φ])^2 * (Sin[λ])^2 * Cos[λ] * Vρρλ -
          3 * Sin[φ] * (Cos[φ])^2 * (Sin[λ])^3 * Vρρφ + (Cos[φ])^3 * (Sin[λ])^3 * Vρρρ;
```

```
In[•]:= VyyzRoutes135 =
  Cos[φ] * (Cos[λ])^2 * Vλλφ + Sin[φ] * (Cos[λ])^2 * Vλλρ + Cos[2 * φ] * Sin[2 * λ] * Vλφρ -
   Sin[2 * φ] * Sin[λ] * Cos[λ] * Vφφλ + (Sin[φ])^2 * Cos[φ] * (Sin[λ])^2 * Vφφφ +
   Sin[φ] * (Sin[λ])^2 * ((Sin[φ])^2 - 2 * (Cos[φ])^2) * Vφφρ +
   Sin[2 * φ] * Sin[λ] * Cos[λ] * Vρρλ + Cos[φ] * (Sin[λ])^2 *
    ((Cos[φ])^2 - 2 * (Sin[φ])^2) * Vρρφ + Sin[φ] * (Cos[φ])^2 * (Sin[λ])^2 * Vρρρ;

In[•]:= VzzxRoutes135 = -Sin[2 * φ] * Sin[λ] * Vλφρ -
   (Cos[φ])^2 * Sin[λ] * Vφφλ - Sin[φ] * (Cos[φ])^2 * Cos[λ] * Vφφφ +
   Cos[φ] * Cos[λ] * ((Cos[φ])^2 - 2 * (Sin[φ])^2) * Vφφρ -
   (Sin[φ])^2 * Sin[λ] * Vρρλ + Sin[φ] * Cos[λ] * (2 * (Cos[φ])^2 - (Sin[φ])^2) * Vρρφ +
   (Sin[φ])^2 * Cos[φ] * Cos[λ] * Vρρρ;

In[•]:= VzzyRoutes135 = Sin[2 * φ] * Cos[λ] * Vλφρ +
   (Cos[φ])^2 * Cos[λ] * Vφφλ - Sin[φ] * (Cos[φ])^2 * Sin[λ] * Vφφφ +
   Cos[φ] * Sin[λ] * ((Cos[φ])^2 - 2 * (Sin[φ])^2) * Vφφρ +
   (Sin[φ])^2 * Cos[λ] * Vρρλ + Sin[φ] * Sin[λ] * (2 * (Cos[φ])^2 - (Sin[φ])^2) * Vρρφ +
   (Sin[φ])^2 * Cos[φ] * Sin[λ] * Vρρρ;

In[•]:= VzzzRoutes135 = (Cos[φ])^3 * Vφφφ +
   3 * Sin[φ] * (Cos[φ])^2 * Vφφρ + 3 * (Sin[φ])^2 * Cos[φ] * Vρρφ + (Sin[φ])^3 * Vρρρ;
```

7.4 Check whether they become themselves

```
In[•]:= FullSimplify[VxRoutes135]
Out[•]=
Vx

In[•]:= FullSimplify[VyRoutes135]
Out[•]=
Vy

In[•]:= FullSimplify[VzRoutes135]
Out[•]=
Vz

In[•]:= FullSimplify[VxxRoutes135]
Out[•]=
Vxx

In[•]:= FullSimplify[VxyRoutes135]
Out[•]=
Vxy

In[•]:= FullSimplify[VxzRoutes135]
Out[•]=
Vxz

In[•]:= FullSimplify[VyyRoutes135]
Out[•]=
Vyy
```

In[ ]:=
```
FullSimplify[VyzRoutes135]
```
Out[ ]=
```
Vyz
```

In[ ]:=
```
FullSimplify[VzzRoutes135]
```
Out[ ]=
```
Vzz
```

In[ ]:=
```
FullSimplify[VxxxRoutes135]
```
Out[ ]=
```
Vxxx
```

In[ ]:=
```
FullSimplify[VxxyRoutes135]
```
Out[ ]=
```
Vxxy
```

In[ ]:=
```
FullSimplify[VxxzRoutes135]
```
Out[ ]=
```
Vxxz
```

In[ ]:=
```
FullSimplify[VxyzRoutes135]
```
Out[ ]=
```
Vxyz
```

In[ ]:=
```
FullSimplify[VyyxRoutes135]
```
Out[ ]=
```
Vyyx
```

In[ ]:=
```
FullSimplify[VyyyRoutes135]
```
Out[ ]=
```
Vyyy
```

In[ ]:=
```
FullSimplify[VyyzRoutes135]
```
Out[ ]=
```
Vyyz
```

In[ ]:=
```
FullSimplify[VzzxRoutes135]
```
Out[ ]=
```
Vzzx
```

In[ ]:=
```
FullSimplify[VzzyRoutes135]
```
Out[ ]=
```
Vzzy
```

In[ ]:=
```
FullSimplify[VzzzRoutes135]
```
Out[ ]=
```
Vzzz
```

8. Trigonometric form, Cartesian coordinates, Routes 6 → 4 → 2, and Table A13 → Table A8 → Table A4

In[ ]:=
```
Clear["Global`*"];
```

8.1 Table A13

```
In[ ]:= Vλ = -Sin[λ] * Vx + Cos[λ] * Vy;

In[ ]:= Vφ = -Sin[φ] * Cos[λ] * Vx - Sin[φ] * Sin[λ] * Vy + Cos[φ] * Vz;

In[ ]:= Vρ = Cos[φ] * Cos[λ] * Vx + Cos[φ] * Sin[λ] * Vy + Sin[φ] * Vz;

In[ ]:= Vλλ = (Sin[λ])^2 * Vxx - Sin[2 * λ] * Vxy + (Cos[λ])^2 * Vyy;

In[ ]:= Vλφ = Sin[φ] * Sin[λ] * Cos[λ] * Vxx - Sin[φ] * Cos[2 * λ] * Vxy -
          Cos[φ] * Sin[λ] * Vxz - Sin[φ] * Sin[λ] * Cos[λ] * Vyy + Cos[φ] * Cos[λ] * Vyz;

In[ ]:= Vλρ = -Cos[φ] * Sin[λ] * Cos[λ] * Vxx + Cos[φ] * Cos[2 * λ] * Vxy -
          Sin[φ] * Sin[λ] * Vxz + Cos[φ] * Sin[λ] * Cos[λ] * Vyy + Sin[φ] * Cos[λ] * Vyz;

In[ ]:= Vφφ = (Sin[φ])^2 * (Cos[λ])^2 * Vxx + (Sin[φ])^2 * Sin[2 * λ] * Vxy - Sin[2 * φ] * Cos[λ] * Vxz +
          (Sin[φ])^2 * (Sin[λ])^2 * Vyy - Sin[2 * φ] * Sin[λ] * Vyz + (Cos[φ])^2 * Vzz;

In[ ]:= Vφρ = -Sin[φ] * Cos[φ] * (Cos[λ])^2 * Vxx -
          Sin[2 * φ] * Sin[λ] * Cos[λ] * Vxy + Cos[2 * φ] * Cos[λ] * Vxz -
          Sin[φ] * Cos[φ] * (Sin[λ])^2 * Vyy + Cos[2 * φ] * Sin[λ] * Vyz + Sin[φ] * Cos[φ] * Vzz;

In[ ]:= Vρρ = (Cos[φ])^2 * (Cos[λ])^2 * Vxx + (Cos[φ])^2 * Sin[2 * λ] * Vxy + Sin[2 * φ] * Cos[λ] * Vxz +
          (Cos[φ])^2 * (Sin[λ])^2 * Vyy + Sin[2 * φ] * Sin[λ] * Vyz + (Sin[φ])^2 * Vzz;

In[ ]:= Vλλλ = -(Sin[λ])^3 * Vxxx + 3 * (Sin[λ])^2 * Cos[λ] * Vxxy -
          3 * Sin[λ] * (Cos[λ])^2 * Vyyx + (Cos[λ])^3 * Vyyy;

In[ ]:= Vλλφ = -Sin[φ] * (Sin[λ])^2 * Cos[λ] * Vxxx +
          Sin[φ] * Sin[λ] * (2 * (Cos[λ])^2 - (Sin[λ])^2) * Vxxy + Cos[φ] * (Sin[λ])^2 * Vxxz -
          Cos[φ] * Sin[2 * λ] * Vxyz + Sin[φ] * Cos[λ] * (2 * (Sin[λ])^2 - (Cos[λ])^2) * Vyyx -
          Sin[φ] * Sin[λ] * (Cos[λ])^2 * Vyyy + Cos[φ] * (Cos[λ])^2 * Vyyz;

In[ ]:= Vλλρ = Cos[φ] * (Sin[λ])^2 * Cos[λ] * Vxxx +
          Cos[φ] * Sin[λ] * ((Sin[λ])^2 - 2 * (Cos[λ])^2) * Vxxy + Sin[φ] * (Sin[λ])^2 * Vxxz -
          Sin[φ] * Sin[2 * λ] * Vxyz + Cos[φ] * Cos[λ] * ((Cos[λ])^2 - 2 * (Sin[λ])^2) * Vyyx +
          Cos[φ] * Sin[λ] * (Cos[λ])^2 * Vyyy + Sin[φ] * (Cos[λ])^2 * Vyyz;

In[ ]:= Vλφρ = Sin[φ] * Cos[φ] * Sin[λ] * (Cos[λ])^2 * Vxxx +
          Sin[φ] * Cos[φ] * Cos[λ] * (2 * (Sin[λ])^2 - (Cos[λ])^2) * Vxxy -
          Cos[2 * φ] * Sin[λ] * Cos[λ] * Vxxz + Cos[2 * φ] * Cos[2 * λ] * Vxyz +
          Sin[φ] * Cos[φ] * Sin[λ] * ((Sin[λ])^2 - 2 * (Cos[λ])^2) * Vyyx -
          Sin[φ] * Cos[φ] * (Sin[λ])^2 * Cos[λ] * Vyyy + Cos[2 * φ] * Sin[λ] * Cos[λ] * Vyyz -
          Sin[φ] * Cos[φ] * Sin[λ] * Vzzx + Sin[φ] * Cos[φ] * Cos[λ] * Vzzy;

In[ ]:= Vφφλ = -(Sin[φ])^2 * Sin[λ] * (Cos[λ])^2 * Vxxx +
          (Sin[φ])^2 * Cos[λ] * ((Cos[λ])^2 - 2 * (Sin[λ])^2) * Vxxy +
          Sin[2 * φ] * Sin[λ] * Cos[λ] * Vxxz - Sin[2 * φ] * Cos[2 * λ] * Vxyz +
          (Sin[φ])^2 * Sin[λ] * (2 * (Cos[λ])^2 - (Sin[λ])^2) * Vyyx +
          (Sin[φ])^2 * (Sin[λ])^2 * Cos[λ] * Vyyy - Sin[2 * φ] * Sin[λ] * Cos[λ] * Vyyz -
          (Cos[φ])^2 * Sin[λ] * Vzzx + (Cos[φ])^2 * Cos[λ] * Vzzy;
```

```
In[•]:= Vφφφ = -(Sin[φ])^3 * (Cos[λ])^3 * Vxxx - 3 * (Sin[φ])^3 * Sin[λ] * (Cos[λ])^2 * Vxxy +
    3 * (Sin[φ])^2 * Cos[φ] * (Cos[λ])^2 * Vxxz + 3 * (Sin[φ])^2 * Cos[φ] * Sin[2 * λ] * Vxyz -
    3 * (Sin[φ])^3 * (Sin[λ])^2 * Cos[λ] * Vyyx - (Sin[φ])^3 * (Sin[λ])^3 * Vyyy +
    3 * (Sin[φ])^2 * Cos[φ] * (Sin[λ])^2 * Vyyz - 3 * Sin[φ] * (Cos[φ])^2 * Cos[λ] * Vzzx -
    3 * Sin[φ] * (Cos[φ])^2 * Sin[λ] * Vzzy + (Cos[φ])^3 * Vzzz;

In[•]:= Vφφρ = (Sin[φ])^2 * Cos[φ] * (Cos[λ])^3 * Vxxx + 3 * (Sin[φ])^2 * Cos[φ] * Sin[λ] *
     (Cos[λ])^2 * Vxxy + Sin[φ] * (Cos[λ])^2 * ((Sin[φ])^2 - 2 * (Cos[φ])^2) * Vxxz +
    Sin[φ] * Sin[2 * λ] * ((Sin[φ])^2 - 2 * (Cos[φ])^2) * Vxyz + 3 * (Sin[φ])^2 *
     Cos[φ] * (Sin[λ])^2 * Cos[λ] * Vyyx + (Sin[φ])^2 * Cos[φ] * (Sin[λ])^3 * Vyyy +
    Sin[φ] * (Sin[λ])^2 * ((Sin[φ])^2 - 2 * (Cos[φ])^2) * Vyyz +
    Cos[φ] * Cos[λ] * ((Cos[φ])^2 - 2 * (Sin[φ])^2) * Vzzx +
    Cos[φ] * Sin[λ] * ((Cos[φ])^2 - 2 * (Sin[φ])^2) * Vzzy + Sin[φ] * (Cos[φ])^2 * Vzzz;

In[•]:= Vρρλ = -(Cos[φ])^2 * Sin[λ] * (Cos[λ])^2 * Vxxx +
    (Cos[φ])^2 * Cos[λ] * ((Cos[λ])^2 - 2 * (Sin[λ])^2) * Vxxy -
    Sin[2 * φ] * Sin[λ] * Cos[λ] * Vxxz + Sin[2 * φ] * Cos[2 * λ] * Vxyz +
    (Cos[φ])^2 * Sin[λ] * (2 * (Cos[λ])^2 - (Sin[λ])^2) * Vyyx +
    (Cos[φ])^2 * (Sin[λ])^2 * Cos[λ] * Vyyy + Sin[2 * φ] * Sin[λ] * Cos[λ] * Vyyz -
    (Sin[φ])^2 * Sin[λ] * Vzzx + (Sin[φ])^2 * Cos[λ] * Vzzy;

In[•]:= Vρρφ = -Sin[φ] * (Cos[φ])^2 * (Cos[λ])^3 * Vxxx - 3 * Sin[φ] * (Cos[φ])^2 * Sin[λ] *
     (Cos[λ])^2 * Vxxy + Cos[φ] * (Cos[λ])^2 * ((Cos[φ])^2 - 2 * (Sin[φ])^2) * Vxxz +
    Cos[φ] * Sin[2 * λ] * ((Cos[φ])^2 - 2 * (Sin[φ])^2) * Vxyz -
    3 * Sin[φ] * (Cos[φ])^2 * (Sin[λ])^2 * Cos[λ] * Vyyx - Sin[φ] * (Cos[φ])^2 *
     (Sin[λ])^3 * Vyyy + Cos[φ] * (Sin[λ])^2 * ((Cos[φ])^2 - 2 * (Sin[φ])^2) * Vyyz +
    Sin[φ] * Cos[λ] * (2 * (Cos[φ])^2 - (Sin[φ])^2) * Vzzx +
    Sin[φ] * Sin[λ] * (2 * (Cos[φ])^2 - (Sin[φ])^2) * Vzzy + (Sin[φ])^2 * Cos[φ] * Vzzz;

In[•]:= Vρρρ = (Cos[φ])^3 * (Cos[λ])^3 * Vxxx +
    3 * (Cos[φ])^3 * Sin[λ] * (Cos[λ])^2 * Vxxy + 3 * Sin[φ] * (Cos[φ])^2 * (Cos[λ])^2 * Vxxz +
    3 * Sin[2 * φ] * Cos[φ] * Sin[λ] * Cos[λ] * Vxyz +
    3 * (Cos[φ])^3 * (Sin[λ])^2 * Cos[λ] * Vyyx + (Cos[φ])^3 * (Sin[λ])^3 * Vyyy +
    3 * Sin[φ] * (Cos[φ])^2 * (Sin[λ])^2 * Vyyz + 3 * (Sin[φ])^2 * Cos[φ] * Cos[λ] * Vzzx +
    3 * (Sin[φ])^2 * Cos[φ] * Sin[λ] * Vzzy + (Sin[φ])^3 * Vzzz;
```

8.2 Table A8

```
In[•]:= Vr = -Sin[φ] * Vφ + Cos[φ] * Vρ;

In[•]:= Vλ = Vλ;

In[•]:= Vz1 = Cos[φ] * Vφ + Sin[φ] * Vρ;

In[•]:= Vrr = (Sin[φ])^2 * Vφφ - Sin[2 * φ] * Vφρ + (Cos[φ])^2 * Vρρ;

In[•]:= Vrλ = -Sin[φ] * Vλφ + Cos[φ] * Vλρ;

In[•]:= Vrz = -Sin[φ] * Cos[φ] * Vφφ + Cos[2 * φ] * Vφρ + Sin[φ] * Cos[φ] * Vρρ;

In[•]:= Vλλ = Vλλ;
```

```
In[•]:= Vλz = Cos[φ] * Vλφ + Sin[φ] * Vλρ;

In[•]:= Vzz1 = (Cos[φ])^2 * Vφφ + Sin[2 * φ] * Vφρ + (Sin[φ])^2 * Vρρ;

In[•]:= Vrrr = -(Sin[φ])^3 * Vφφφ + 3 * (Sin[φ])^2 * Cos[φ] * Vφφρ -
          3 * Sin[φ] * (Cos[φ])^2 * Vρρφ + (Cos[φ])^3 * Vρρρ;

In[•]:= Vrrλ = (Sin[φ])^2 * Vφφλ - Sin[2 * φ] * Vλφρ + (Cos[φ])^2 * Vρρλ;

In[•]:= Vrrz = (Sin[φ])^2 * Cos[φ] * Vφφφ + Sin[φ] * ((Sin[φ])^2 - 2 * (Cos[φ])^2) * Vφφρ +
          Cos[φ] * ((Cos[φ])^2 - 2 * (Sin[φ])^2) * Vρρφ + Sin[φ] * (Cos[φ])^2 * Vρρρ;

In[•]:= Vrλz = Cos[2 * φ] * Vλφρ - Sin[φ] * Cos[φ] * Vφφλ + Sin[φ] * Cos[φ] * Vρρλ;

In[•]:= Vλλr = -Sin[φ] * Vλλφ + Cos[φ] * Vλλρ;

In[•]:= Vλλλ = Vλλλ;

In[•]:= Vλλz = Cos[φ] * Vλλφ + Sin[φ] * Vλλρ;

In[•]:= Vzzr = -Sin[φ] * (Cos[φ])^2 * Vφφφ + Cos[φ] * ((Cos[φ])^2 - 2 * (Sin[φ])^2) * Vφφρ +
          Sin[φ] * (2 * (Cos[φ])^2 - (Sin[φ])^2) * Vρρφ + (Sin[φ])^2 * Cos[φ] * Vρρρ;

In[•]:= Vzzλ = Sin[2 * φ] * Vλφρ + (Cos[φ])^2 * Vφφλ + (Sin[φ])^2 * Vρρλ;

In[•]:= Vzzz1 = (Cos[φ])^3 * Vφφφ + 3 * Sin[φ] * (Cos[φ])^2 * Vφφρ +
          3 * (Sin[φ])^2 * Cos[φ] * Vρρφ + (Sin[φ])^3 * Vρρρ;
```

8.3 Table A4

```
In[•]:= VxRoutes642 = Cos[λ] * Vr - Sin[λ] * Vλ;

In[•]:= VyRoutes642 = Sin[λ] * Vr + Cos[λ] * Vλ;

In[•]:= VzRoutes642 = Vz1;

In[•]:= VxxRoutes642 = (Cos[λ])^2 * Vrr - Sin[2 * λ] * Vrλ + (Sin[λ])^2 * Vλλ;

In[•]:= VxyRoutes642 = Sin[λ] * Cos[λ] * Vrr + Cos[2 * λ] * Vrλ - Sin[λ] * Cos[λ] * Vλλ;

In[•]:= VxzRoutes642 = Cos[λ] * Vrz - Sin[λ] * Vλz;

In[•]:= VyyRoutes642 = (Sin[λ])^2 * Vrr + Sin[2 * λ] * Vrλ + (Cos[λ])^2 * Vλλ;

In[•]:= VyzRoutes642 = Sin[λ] * Vrz + Cos[λ] * Vλz;

In[•]:= VzzRoutes642 = Vzz1;

In[•]:= VxxxRoutes642 = (Cos[λ])^3 * Vrrr -
          3 * Sin[λ] * (Cos[λ])^2 * Vrrλ + 3 * (Sin[λ])^2 * Cos[λ] * Vλλr - (Sin[λ])^3 * Vλλλ;

In[•]:= VxxyRoutes642 = Sin[λ] * (Cos[λ])^2 * Vrrr + Cos[λ] * ((Cos[λ])^2 - 2 * (Sin[λ])^2) * Vrrλ +
          Sin[λ] * ((Sin[λ])^2 - 2 * (Cos[λ])^2) * Vλλr + (Sin[λ])^2 * Cos[λ] * Vλλλ;

In[•]:= VxxzRoutes642 = (Cos[λ])^2 * Vrrz - Sin[2 * λ] * Vrλz + (Sin[λ])^2 * Vλλz;

In[•]:= VxyzRoutes642 = Sin[λ] * Cos[λ] * Vrrz + Cos[2 * λ] * Vrλz - Sin[λ] * Cos[λ] * Vλλz;

In[•]:= VyyxRoutes642 = (Sin[λ])^2 * Cos[λ] * Vrrr + Sin[λ] * (2 * (Cos[λ])^2 - (Sin[λ])^2) * Vrrλ +
          Cos[λ] * ((Cos[λ])^2 - 2 * (Sin[λ])^2) * Vλλr - Sin[λ] * (Cos[λ])^2 * Vλλλ;
```

```
In[•]:= VyyyRoutes642 = (Sin[λ])^3 * Vrrr +
          3 * (Sin[λ])^2 * Cos[λ] * Vrrλ + 3 * Sin[λ] * (Cos[λ])^2 * Vλλr + (Cos[λ])^3 * Vλλλ;

In[•]:= VyyzRoutes642 = (Sin[λ])^2 * Vrrz + Sin[2 * λ] * Vrλz + (Cos[λ])^2 * Vλλz;

In[•]:= VzzxRoutes642 = Cos[λ] * Vzzr - Sin[λ] * Vzzλ;

In[•]:= VzzyRoutes642 = Sin[λ] * Vzzr + Cos[λ] * Vzzλ;

In[•]:= VzzzRoutes642 = Vzzz1;
```

8.4 Check whether they become themselves

```
In[•]:= FullSimplify[VxRoutes642]
Out[•]=
Vx

In[•]:= FullSimplify[VyRoutes642]
Out[•]=
Vy

In[•]:= FullSimplify[VzRoutes642]
Out[•]=
Vz

In[•]:= FullSimplify[VxxRoutes642]
Out[•]=
Vxx

In[•]:= FullSimplify[VxyRoutes642]
Out[•]=
Vxy

In[•]:= FullSimplify[VxzRoutes642]
Out[•]=
Vxz

In[•]:= FullSimplify[VyyRoutes642]
Out[•]=
Vyy

In[•]:= FullSimplify[VyzRoutes642]
Out[•]=
Vyz

In[•]:= FullSimplify[VzzRoutes642]
Out[•]=
Vzz

In[•]:= FullSimplify[VxxxRoutes642]
Out[•]=
Vxxx

In[•]:= FullSimplify[VxxyRoutes642]
Out[•]=
Vxxy
```

```
In[•]:= FullSimplify[VxxzRoutes642]
Out[•]=
        Vxxz

In[•]:= FullSimplify[VxyzRoutes642]
Out[•]=
        Vxyz

In[•]:= FullSimplify[VyyxRoutes642]
Out[•]=
        Vyyx

In[•]:= FullSimplify[VyyyRoutes642]
Out[•]=
        Vyyy

In[•]:= FullSimplify[VyyzRoutes642]
Out[•]=
        Vyyz

In[•]:= FullSimplify[VzzxRoutes642]
Out[•]=
        Vzzx

In[•]:= FullSimplify[VzzyRoutes642]
Out[•]=
        Vzzy

In[•]:= FullSimplify[VzzzRoutes642]
Out[•]=
        Vzzz
```

9. Trigonometric form, Cylindrical coordinates, Routes 3 → 5 → 1, and Table A6 → Table A11 → Table A2

```
In[•]:= Clear["Global`*"];
```

9. 1 Table A6

```
In[•]:= Vλ = Vλ;

In[•]:= Vφ = -Sin[φ] * Vr + Cos[φ] * Vz;

In[•]:= Vρ = Cos[φ] * Vr + Sin[φ] * Vz;

In[•]:= Vλλ = Vλλ;

In[•]:= Vλφ = -Sin[φ] * Vrλ + Cos[φ] * Vλz;

In[•]:= Vλρ = Cos[φ] * Vrλ + Sin[φ] * Vλz;

In[•]:= Vφφ = ((Sin[φ])^2 * Vrr - Sin[2 * φ] * Vrz + (Cos[φ])^2 * Vzz);

In[•]:= Vφρ = -Sin[φ] * Cos[φ] * Vrr + Cos[2 * φ] * Vrz + Sin[φ] * Cos[φ] * Vzz;

In[•]:= Vρρ = ((Cos[φ])^2 * Vrr + Sin[2 * φ] * Vrz + (Sin[φ])^2 * Vzz);
```

```
In[•]:= Vλλλ = Vλλλ;

In[•]:= Vλλφ = -Sin[φ] * Vλλr + Cos[φ] * Vλλz;

In[•]:= Vλλρ = Cos[φ] * Vλλr + Sin[φ] * Vλλz;

In[•]:= Vλφρ = -Sin[φ] * Cos[φ] * Vrrλ + Cos[2 * φ] * Vrλz + Sin[φ] * Cos[φ] * Vzzλ;

In[•]:= Vφφλ = (Sin[φ])^2 * Vrrλ - Sin[2 * φ] * Vrλz + (Cos[φ])^2 * Vzzλ;

In[•]:= Vφφφ = -(Sin[φ])^3 * Vrrr + 3 * (Sin[φ])^2 * Cos[φ] * Vrrz -
        3 * Sin[φ] * (Cos[φ])^2 * Vzzr + (Cos[φ])^3 * Vzzz;

In[•]:= Vφφρ = (Sin[φ])^2 * Cos[φ] * Vrrr + Sin[φ] * ((Sin[φ])^2 - 2 * (Cos[φ])^2) * Vrrz +
        Cos[φ] * ((Cos[φ])^2 - 2 * (Sin[φ])^2) * Vzzr + Sin[φ] * (Cos[φ])^2 * Vzzz;

In[•]:= Vρρλ = (Cos[φ])^2 * Vrrλ + Sin[2 * φ] * Vrλz + (Sin[φ])^2 * Vzzλ;

In[•]:= Vρρφ = -Sin[φ] * (Cos[φ])^2 * Vrrr + Cos[φ] * ((Cos[φ])^2 - 2 * (Sin[φ])^2) * Vrrz +
        Sin[φ] * (2 * (Cos[φ])^2 - (Sin[φ])^2) * Vzzr + Cos[φ] * (Sin[φ])^2 * Vzzz;

In[•]:= Vρρρ = (Cos[φ])^3 * Vrrr + 3 * Sin[φ] * (Cos[φ])^2 * Vrrz +
        3 * Cos[φ] * (Sin[φ])^2 * Vzzr + (Sin[φ])^3 * Vzzz;
```

9.2 Table A11

```
In[•]:= Vx = -Sin[λ] * Vλ - Sin[φ] * Cos[λ] * Vφ + Cos[φ] * Cos[λ] * Vρ;

In[•]:= Vy = Cos[λ] * Vλ - Sin[φ] * Sin[λ] * Vφ + Cos[φ] * Sin[λ] * Vρ;

In[•]:= Vz1 = Cos[φ] * Vφ + Sin[φ] * Vρ;

In[•]:= Vxx = (Sin[λ])^2 * Vλλ + Sin[φ] * Sin[2 * λ] * Vλφ -
        Cos[φ] * Sin[2 * λ] * Vλρ + (Sin[φ])^2 * (Cos[λ])^2 * Vφφ -
        Sin[2 * φ] * (Cos[λ])^2 * Vφρ + (Cos[φ])^2 * (Cos[λ])^2 * Vρρ;

In[•]:= Vxy = -Sin[λ] * Cos[λ] * Vλλ - Sin[φ] * Cos[2 * λ] * Vλφ +
        Cos[φ] * Cos[2 * λ] * Vλρ + (Sin[φ])^2 * Sin[λ] * Cos[λ] * Vφφ -
        Sin[2 * φ] * Sin[λ] * Cos[λ] * Vφρ + (Cos[φ])^2 * Sin[λ] * Cos[λ] * Vρρ;

In[•]:= Vxz = -Cos[φ] * Sin[λ] * Vλφ - Sin[φ] * Sin[λ] * Vλρ - Sin[φ] * Cos[φ] * Cos[λ] * Vφφ +
        Cos[2 * φ] * Cos[λ] * Vφρ + Sin[φ] * Cos[φ] * Cos[λ] * Vρρ;

In[•]:= Vyy = (Cos[λ])^2 * Vλλ - Sin[φ] * Sin[2 * λ] * Vλφ +
        Cos[φ] * Sin[2 * λ] * Vλρ + (Sin[φ])^2 * (Sin[λ])^2 * Vφφ -
        Sin[2 * φ] * (Sin[λ])^2 * Vφρ + (Cos[φ])^2 * (Sin[λ])^2 * Vρρ;

In[•]:= Vyz = Cos[φ] * Cos[λ] * Vλφ + Sin[φ] * Cos[λ] * Vλρ - Sin[φ] * Cos[φ] * Sin[λ] * Vφφ +
        Cos[2 * φ] * Sin[λ] * Vφρ + Sin[φ] * Cos[φ] * Sin[λ] * Vρρ;

In[•]:= Vzz1 = (Cos[φ])^2 * Vφφ + Sin[2 * φ] * Vφρ + (Sin[φ])^2 * Vρρ;

In[•]:= Vxxx = -(Sin[λ])^3 * Vλλλ + 3 * Cos[φ] * (Sin[λ])^2 * Cos[λ] * Vλλρ -
        3 * Sin[φ] * (Sin[λ])^2 * Cos[λ] * Vλλφ + 3 * Sin[2 * φ] * Sin[λ] * (Cos[λ])^2 * Vλφρ -
        3 * (Sin[φ])^2 * Sin[λ] * (Cos[λ])^2 * Vφφλ - (Sin[φ])^3 * (Cos[λ])^3 * Vφφφ +
        3 * (Sin[φ])^2 * Cos[φ] * (Cos[λ])^3 * Vφφρ - 3 * (Cos[φ])^2 * Sin[λ] * (Cos[λ])^2 * Vρρλ -
        3 * Sin[φ] * (Cos[φ])^2 * (Cos[λ])^3 * Vρρφ + (Cos[φ])^3 * (Cos[λ])^3 * Vρρρ;
```

```
In[•]:= Vxxy = (Sin[λ])^2 * Cos[λ] * Vλλλ + Sin[φ] * Sin[λ] * (2 * (Cos[λ])^2 - (Sin[λ])^2) * Vλλφ +
    Cos[φ] * Sin[λ] * ((Sin[λ])^2 - 2 * (Cos[λ])^2) * Vλλρ +
    Sin[2 * φ] * Cos[λ] * (2 * (Sin[λ])^2 - (Cos[λ])^2) * Vλφρ +
    (Sin[φ])^2 * Cos[λ] * ((Cos[λ])^2 - 2 * (Sin[λ])^2) * Vφφλ - (Sin[φ])^3 * Sin[λ] *
     (Cos[λ])^2 * Vφφφ + 3 * (Sin[φ])^2 * Cos[φ] * Sin[λ] * (Cos[λ])^2 * Vφφρ +
    (Cos[φ])^2 * Cos[λ] * ((Cos[λ])^2 - 2 * (Sin[λ])^2) * Vρρλ - 3 * Sin[φ] *
     (Cos[φ])^2 * Sin[λ] * (Cos[λ])^2 * Vρρφ + (Cos[φ])^3 * Sin[λ] * (Cos[λ])^2 * Vρρρ;

In[•]:= Vxxz =
   Cos[φ] * (Sin[λ])^2 * Vλλφ + Sin[φ] * (Sin[λ])^2 * Vλλρ - Cos[2 * φ] * Sin[2 * λ] * Vλφρ +
    Sin[φ] * Cos[φ] * Sin[2 * λ] * Vφφλ + (Sin[φ])^2 * Cos[φ] * (Cos[λ])^2 * Vφφφ +
    Sin[φ] * (Cos[λ])^2 * ((Sin[φ])^2 - 2 * (Cos[φ])^2) * Vφφρ -
    Sin[2 * φ] * Sin[λ] * Cos[λ] * Vρρλ + Cos[φ] * (Cos[λ])^2 *
     ((Cos[φ])^2 - 2 * (Sin[φ])^2) * Vρρφ + Sin[φ] * (Cos[φ])^2 * (Cos[λ])^2 * Vρρρ;

In[•]:= Vxyz = -Cos[φ] * Sin[λ] * Cos[λ] * Vλλφ -
    Sin[φ] * Sin[λ] * Cos[λ] * Vλλρ + Cos[2 * φ] * Cos[2 * λ] * Vλφρ -
    Sin[φ] * Cos[φ] * Cos[2 * λ] * Vφφλ + (Sin[φ])^2 * Cos[φ] * Sin[λ] * Cos[λ] * Vφφφ -
    Sin[φ] * Sin[λ] * Cos[λ] * (2 * (Cos[φ])^2 - (Sin[φ])^2) * Vφφρ +
    Sin[φ] * Cos[φ] * Cos[2 * λ] * Vρρλ + Cos[φ] * Sin[λ] * Cos[λ] *
     ((Cos[φ])^2 - 2 * (Sin[φ])^2) * Vρρφ + Sin[φ] * (Cos[φ])^2 * Sin[λ] * Cos[λ] * Vρρρ;

In[•]:= Vyyx = -Sin[λ] * (Cos[λ])^2 * Vλλλ - Sin[φ] * Cos[λ] * ((Cos[λ])^2 - 2 * (Sin[λ])^2) * Vλλφ +
    Cos[φ] * Cos[λ] * ((Cos[λ])^2 - 2 * (Sin[λ])^2) * Vλλρ -
    Sin[2 * φ] * Sin[λ] * (2 * (Cos[λ])^2 - (Sin[λ])^2) * Vλφρ +
    (Sin[φ])^2 * Sin[λ] * (2 * (Cos[λ])^2 - (Sin[λ])^2) * Vφφλ - (Sin[φ])^3 *
     (Sin[λ])^2 * Cos[λ] * Vφφφ + 3 * (Sin[φ])^2 * Cos[φ] * (Sin[λ])^2 * Cos[λ] * Vφφρ +
    (Cos[φ])^2 * Sin[λ] * (2 * (Cos[λ])^2 - (Sin[λ])^2) * Vρρλ - 3 * Sin[φ] *
     (Cos[φ])^2 * (Sin[λ])^2 * Cos[λ] * Vρρφ + (Cos[φ])^3 * (Sin[λ])^2 * Cos[λ] * Vρρρ;

In[•]:= Vyyy = (Cos[λ])^3 * Vλλλ - 3 * Sin[φ] * Sin[λ] * (Cos[λ])^2 * Vλλφ +
    3 * Cos[φ] * Sin[λ] * (Cos[λ])^2 * Vλλρ - 3 * Sin[2 * φ] * (Sin[λ])^2 * Cos[λ] * Vλφρ +
    3 * (Sin[φ])^2 * (Sin[λ])^2 * Cos[λ] * Vφφλ - (Sin[φ])^3 * (Sin[λ])^3 * Vφφφ +
    3 * (Sin[φ])^2 * Cos[φ] * (Sin[λ])^3 * Vφφρ + 3 * (Cos[φ])^2 * (Sin[λ])^2 * Cos[λ] * Vρρλ -
    3 * Sin[φ] * (Cos[φ])^2 * (Sin[λ])^3 * Vρρφ + (Cos[φ])^3 * (Sin[λ])^3 * Vρρρ;

In[•]:= Vyyz =
   Cos[φ] * (Cos[λ])^2 * Vλλφ + Sin[φ] * (Cos[λ])^2 * Vλλρ + Cos[2 * φ] * Sin[2 * λ] * Vλφρ -
    Sin[2 * φ] * Sin[λ] * Cos[λ] * Vφφλ + (Sin[φ])^2 * Cos[φ] * (Sin[λ])^2 * Vφφφ +
    Sin[φ] * (Sin[λ])^2 * ((Sin[φ])^2 - 2 * (Cos[φ])^2) * Vφφρ +
    Sin[2 * φ] * Sin[λ] * Cos[λ] * Vρρλ + Cos[φ] * (Sin[λ])^2 *
     ((Cos[φ])^2 - 2 * (Sin[φ])^2) * Vρρφ + Sin[φ] * (Cos[φ])^2 * (Sin[λ])^2 * Vρρρ;

In[•]:= Vzzx = -Sin[2 * φ] * Sin[λ] * Vλφρ -
    (Cos[φ])^2 * Sin[λ] * Vφφλ - Sin[φ] * (Cos[φ])^2 * Cos[λ] * Vφφφ +
    Cos[φ] * Cos[λ] * ((Cos[φ])^2 - 2 * (Sin[φ])^2) * Vφφρ - (Sin[φ])^2 * Sin[λ] * Vρρλ +
    Sin[φ] * Cos[λ] * (2 * (Cos[φ])^2 - (Sin[φ])^2) * Vρρφ +
    (Sin[φ])^2 * Cos[φ] * Cos[λ] * Vρρρ;
```

```
In[ ]:= Vzzy = Sin[2 * φ] * Cos[λ] * Vλφρ +
    (Cos[φ])^2 * Cos[λ] * Vφφλ - Sin[φ] * (Cos[φ])^2 * Sin[λ] * Vφφφ +
    Cos[φ] * Sin[λ] * ((Cos[φ])^2 - 2 * (Sin[φ])^2) * Vφφρ + (Sin[φ])^2 * Cos[λ] * Vρρλ +
    Sin[φ] * Sin[λ] * (2 * (Cos[φ])^2 - (Sin[φ])^2) * Vρρφ +
    (Sin[φ])^2 * Cos[φ] * Sin[λ] * Vρρρ;

In[ ]:= Vzzz1 = (Cos[φ])^3 * Vφφφ + 3 * Sin[φ] * (Cos[φ])^2 * Vφφρ +
    3 * (Sin[φ])^2 * Cos[φ] * Vρρφ + (Sin[φ])^3 * Vρρρ;
```

9.3 Table A2

```
In[ ]:= VrRoutes351 = Cos[λ] * Vx + Sin[λ] * Vy;

In[ ]:= VλRoutes351 = -Sin[λ] * Vx + Cos[λ] * Vy;

In[ ]:= VzRoutes351 = Vz1;

In[ ]:= VrrRoutes351 = (Cos[λ])^2 * Vxx + Sin[2 * λ] * Vxy + (Sin[λ])^2 * Vyy;

In[ ]:= VrλRoutes351 = -Sin[λ] * Cos[λ] * Vxx + Cos[2 * λ] * Vxy + Sin[λ] * Cos[λ] * Vyy;

In[ ]:= VrzRoutes351 = Cos[λ] * Vxz + Sin[λ] * Vyz;

In[ ]:= VλλRoutes351 = (Sin[λ])^2 * Vxx - Sin[2 * λ] * Vxy + (Cos[λ])^2 * Vyy;

In[ ]:= VλzRoutes351 = -Sin[λ] * Vxz + Cos[λ] * Vyz;

In[ ]:= VzzRoutes351 = Vzz1;

In[ ]:= VrrrRoutes351 = (Cos[λ])^3 * Vxxx +
    3 * Sin[λ] * (Cos[λ])^2 * Vxxy + 3 * Cos[λ] * (Sin[λ])^2 * Vyyx + (Sin[λ])^3 * Vyyy;

In[ ]:= VrrλRoutes351 = -Sin[λ] * (Cos[λ])^2 * Vxxx + Cos[λ] * ((Cos[λ])^2 - 2 * (Sin[λ])^2) * Vxxy +
    Sin[λ] * (2 * (Cos[λ])^2 - (Sin[λ])^2) * Vyyx + (Sin[λ])^2 * Cos[λ] * Vyyy;

In[ ]:= VrrzRoutes351 = (Cos[λ])^2 * Vxxz + Sin[2 * λ] * Vxyz + (Sin[λ])^2 * Vyyz;

In[ ]:= VrλzRoutes351 = -Sin[λ] * Cos[λ] * Vxxz + Cos[2 * λ] * Vxyz + Sin[λ] * Cos[λ] * Vyyz;

In[ ]:= VλλrRoutes351 = Cos[λ] * (Sin[λ])^2 * Vxxx + Sin[λ] * ((Sin[λ])^2 - 2 * (Cos[λ])^2) * Vxxy +
    Cos[λ] * ((Cos[λ])^2 - 2 * (Sin[λ])^2) * Vyyx + Sin[λ] * (Cos[λ])^2 * Vyyy;

In[ ]:= VλλλRoutes351 = -(Sin[λ])^3 * Vxxx +
    3 * (Sin[λ])^2 * Cos[λ] * Vxxy - 3 * Sin[λ] * (Cos[λ])^2 * Vyyx + (Cos[λ])^3 * Vyyy;

In[ ]:= VλλzRoutes351 = (Sin[λ])^2 * Vxxz - Sin[2 * λ] * Vxyz + (Cos[λ])^2 * Vyyz;

In[ ]:= VzzrRoutes351 = Cos[λ] * Vzzx + Sin[λ] * Vzzy;

In[ ]:= VzzλRoutes351 = -Sin[λ] * Vzzx + Cos[λ] * Vzzy;

In[ ]:= VzzzRoutes351 = Vzzz1;
```

9.4 Check whether they become themselves

```
In[ ]:= FullSimplify[VrRoutes351]
Out[ ]=
    Vr
```

In[•]:= **FullSimplify[Vλ Routes351]**

Out[•]=

Vλ

In[•]:= **FullSimplify[Vz Routes351]**

Out[•]=

Vz

In[•]:= **FullSimplify[Vrr Routes351]**

Out[•]=

Vrr

In[•]:= **FullSimplify[Vrλ Routes351]**

Out[•]=

Vrλ

In[•]:= **FullSimplify[Vrz Routes351]**

Out[•]=

Vrz

In[•]:= **FullSimplify[Vλλ Routes351]**

Out[•]=

Vλλ

In[•]:= **FullSimplify[Vλz Routes351]**

Out[•]=

Vλz

In[•]:= **FullSimplify[Vzz Routes351]**

Out[•]=

Vzz

In[•]:= **FullSimplify[Vrrr Routes351]**

Out[•]=

Vrrr

In[•]:= **FullSimplify[Vrrλ Routes351]**

Out[•]=

Vrrλ

In[•]:= **FullSimplify[Vrrz Routes351]**

Out[•]=

Vrrz

In[•]:= **FullSimplify[Vrλz Routes351]**

Out[•]=

Vrλz

In[•]:= **FullSimplify[Vλλr Routes351]**

Out[•]=

Vλλr

In[•]:= **FullSimplify[Vλλλ Routes351]**

Out[•]=

Vλλλ

```
In[ ]:= FullSimplify[Vλλz Routes351]
Out[ ]=
        Vλλz

In[ ]:= FullSimplify[Vzr Routes351]
Out[ ]=
        Vzr

In[ ]:= FullSimplify[Vzλ Routes351]
Out[ ]=
        Vzλ

In[ ]:= FullSimplify[Vzz Routes351]
Out[ ]=
        Vzz
```

10. Trigonometric form, Cylindrical coordinates, Routes 2 → 6 → 4, and Table A4 → Table A13 → Table A8

```
In[ ]:= Clear["Global`*"];
```

10.1 Table A4

```
In[ ]:= Vx = Cos[λ] * Vr - Sin[λ] * Vλ;

In[ ]:= Vy = Sin[λ] * Vr + Cos[λ] * Vλ;

In[ ]:= Vz = Vz;

In[ ]:= Vxx = (Cos[λ])^2 * Vrr - Sin[2 * λ] * Vrλ + (Sin[λ])^2 * Vλλ;

In[ ]:= Vxy = Sin[λ] * Cos[λ] * Vrr + Cos[2 * λ] * Vrλ - Sin[λ] * Cos[λ] * Vλλ;

In[ ]:= Vxz = Cos[λ] * Vrz - Sin[λ] * Vλz;

In[ ]:= Vyy = (Sin[λ])^2 * Vrr + Sin[2 * λ] * Vrλ + (Cos[λ])^2 * Vλλ;

In[ ]:= Vyz = Sin[λ] * Vrz + Cos[λ] * Vλz;

In[ ]:= Vzz = Vzz;

In[ ]:= Vxxx = (Cos[λ])^3 * Vrrr - 3 * Sin[λ] * (Cos[λ])^2 * Vrrλ +
          3 * (Sin[λ])^2 * Cos[λ] * Vλλr - (Sin[λ])^3 * Vλλλ;

In[ ]:= Vxxy = Sin[λ] * (Cos[λ])^2 * Vrrr + Cos[λ] * ((Cos[λ])^2 - 2 * (Sin[λ])^2) * Vrrλ +
          Sin[λ] * ((Sin[λ])^2 - 2 * (Cos[λ])^2) * Vλλr + (Sin[λ])^2 * Cos[λ] * Vλλλ;

In[ ]:= Vxxz = (Cos[λ])^2 * Vrrz - Sin[2 * λ] * Vrλz + (Sin[λ])^2 * Vλλz;

In[ ]:= Vxyz = Sin[λ] * Cos[λ] * Vrrz + Cos[2 * λ] * Vrλz - Sin[λ] * Cos[λ] * Vλλz;

In[ ]:= Vyyx = (Sin[λ])^2 * Cos[λ] * Vrrr + Sin[λ] * (2 * (Cos[λ])^2 - (Sin[λ])^2) * Vrrλ +
          Cos[λ] * ((Cos[λ])^2 - 2 * (Sin[λ])^2) * Vλλr - Sin[λ] * (Cos[λ])^2 * Vλλλ;

In[ ]:= Vyyy = (Sin[λ])^3 * Vrrr + 3 * (Sin[λ])^2 * Cos[λ] * Vrrλ +
          3 * Sin[λ] * (Cos[λ])^2 * Vλλr + (Cos[λ])^3 * Vλλλ;
```

```
In[•]:= Vyyz = (Sin[λ])^2 * Vrrz + Sin[2 * λ] * Vrλz + (Cos[λ])^2 * Vλλz;

In[•]:= Vzzx = Cos[λ] * Vzzr - Sin[λ] * Vzzλ;

In[•]:= Vzzy = Sin[λ] * Vzzr + Cos[λ] * Vzzλ;

In[•]:= Vzzz = Vzzz;
```

10.2 Table A13

```
In[•]:= Vλ1 = -Sin[λ] * Vx + Cos[λ] * Vy;

In[•]:= Vφ = -Sin[φ] * Cos[λ] * Vx - Sin[φ] * Sin[λ] * Vy + Cos[φ] * Vz;

In[•]:= Vρ = Cos[φ] * Cos[λ] * Vx + Cos[φ] * Sin[λ] * Vy + Sin[φ] * Vz;

In[•]:= Vλλ1 = (Sin[λ])^2 * Vxx - Sin[2 * λ] * Vxy + (Cos[λ])^2 * Vyy;

In[•]:= Vλφ = Sin[φ] * Sin[λ] * Cos[λ] * Vxx - Sin[φ] * Cos[2 * λ] * Vxy -
        Cos[φ] * Sin[λ] * Vxz - Sin[φ] * Sin[λ] * Cos[λ] * Vyy + Cos[φ] * Cos[λ] * Vyz;

In[•]:= Vλρ = -Cos[φ] * Sin[λ] * Cos[λ] * Vxx + Cos[φ] * Cos[2 * λ] * Vxy -
        Sin[φ] * Sin[λ] * Vxz + Cos[φ] * Sin[λ] * Cos[λ] * Vyy + Sin[φ] * Cos[λ] * Vyz;

In[•]:= Vφφ = (Sin[φ])^2 * (Cos[λ])^2 * Vxx + (Sin[φ])^2 * Sin[2 * λ] * Vxy - Sin[2 * φ] * Cos[λ] * Vxz +
        (Sin[φ])^2 * (Sin[λ])^2 * Vyy - Sin[2 * φ] * Sin[λ] * Vyz + (Cos[φ])^2 * Vzz;

In[•]:= Vφρ = -Sin[φ] * Cos[φ] * (Cos[λ])^2 * Vxx -
        Sin[2 * φ] * Sin[λ] * Cos[λ] * Vxy + Cos[2 * φ] * Cos[λ] * Vxz -
        Sin[φ] * Cos[φ] * (Sin[λ])^2 * Vyy + Cos[2 * φ] * Sin[λ] * Vyz + Sin[φ] * Cos[φ] * Vzz;

In[•]:= Vρρ = (Cos[φ])^2 * (Cos[λ])^2 * Vxx + (Cos[φ])^2 * Sin[2 * λ] * Vxy + Sin[2 * φ] * Cos[λ] * Vxz +
        (Cos[φ])^2 * (Sin[λ])^2 * Vyy + Sin[2 * φ] * Sin[λ] * Vyz + (Sin[φ])^2 * Vzz;

In[•]:= Vλλλ1 = -(Sin[λ])^3 * Vxxx + 3 * (Sin[λ])^2 * Cos[λ] * Vxxy -
        3 * Sin[λ] * (Cos[λ])^2 * Vyyx + (Cos[λ])^3 * Vyyy;

In[•]:= Vλλφ = -Sin[φ] * (Sin[λ])^2 * Cos[λ] * Vxxx +
        Sin[φ] * Sin[λ] * (2 * (Cos[λ])^2 - (Sin[λ])^2) * Vxxy + Cos[φ] * (Sin[λ])^2 * Vxxz -
        Cos[φ] * Sin[2 * λ] * Vxyz + Sin[φ] * Cos[λ] * (2 * (Sin[λ])^2 - (Cos[λ])^2) * Vyyx -
        Sin[φ] * Sin[λ] * (Cos[λ])^2 * Vyyy + Cos[φ] * (Cos[λ])^2 * Vyyz;

In[•]:= Vλλρ = Cos[φ] * (Sin[λ])^2 * Cos[λ] * Vxxx +
        Cos[φ] * Sin[λ] * ((Sin[λ])^2 - 2 * (Cos[λ])^2) * Vxxy + Sin[φ] * (Sin[λ])^2 * Vxxz -
        Sin[φ] * Sin[2 * λ] * Vxyz + Cos[φ] * Cos[λ] * ((Cos[λ])^2 - 2 * (Sin[λ])^2) * Vyyx +
        Cos[φ] * Sin[λ] * (Cos[λ])^2 * Vyyy + Sin[φ] * (Cos[λ])^2 * Vyyz;

In[•]:= Vλφρ = Sin[φ] * Cos[φ] * Sin[λ] * (Cos[λ])^2 * Vxxx +
        Sin[φ] * Cos[φ] * Cos[λ] * (2 * (Sin[λ])^2 - (Cos[λ])^2) * Vxxy -
        Cos[2 * φ] * Sin[λ] * Cos[λ] * Vxxz + Cos[2 * φ] * Cos[2 * λ] * Vxyz +
        Sin[φ] * Cos[φ] * Sin[λ] * ((Sin[λ])^2 - 2 * (Cos[λ])^2) * Vyyx -
        Sin[φ] * Cos[φ] * (Sin[λ])^2 * Cos[λ] * Vyyy + Cos[2 * φ] * Sin[λ] * Cos[λ] * Vyyz -
        Sin[φ] * Cos[φ] * Sin[λ] * Vzzx + Sin[φ] * Cos[φ] * Cos[λ] * Vzzy;
```

```
In[•]:= Vφφλ = -(Sin[φ])^2 * Sin[λ] * (Cos[λ])^2 * Vxxx +
          (Sin[φ])^2 * Cos[λ] * ((Cos[λ])^2 - 2 * (Sin[λ])^2) * Vxxy +
          Sin[2 * φ] * Sin[λ] * Cos[λ] * Vxxz - Sin[2 * φ] * Cos[2 * λ] * Vxyz +
          (Sin[φ])^2 * Sin[λ] * (2 * (Cos[λ])^2 - (Sin[λ])^2) * Vyyx +
          (Sin[φ])^2 * (Sin[λ])^2 * Cos[λ] * Vyyy - Sin[2 * φ] * Sin[λ] * Cos[λ] * Vyyz -
          (Cos[φ])^2 * Sin[λ] * Vzzx + (Cos[φ])^2 * Cos[λ] * Vzzy;

In[•]:= Vφφφ = -(Sin[φ])^3 * (Cos[λ])^3 * Vxxx - 3 * (Sin[φ])^3 * Sin[λ] * (Cos[λ])^2 * Vxxy +
          3 * (Sin[φ])^2 * Cos[φ] * (Cos[λ])^2 * Vxxz + 3 * (Sin[φ])^2 * Cos[φ] * Sin[2 * λ] * Vxyz -
          3 * (Sin[φ])^3 * (Sin[λ])^2 * Cos[λ] * Vyyx - (Sin[φ])^3 * (Sin[λ])^3 * Vyyy +
          3 * (Sin[φ])^2 * Cos[φ] * (Sin[λ])^2 * Vyyz - 3 * Sin[φ] * (Cos[φ])^2 * Cos[λ] * Vzzx -
          3 * Sin[φ] * (Cos[φ])^2 * Sin[λ] * Vzzy + (Cos[φ])^3 * Vzzz;

In[•]:= Vφφρ = (Sin[φ])^2 * Cos[φ] * (Cos[λ])^3 * Vxxx + 3 * (Sin[φ])^2 * Cos[φ] * Sin[λ] *
           (Cos[λ])^2 * Vxxy + Sin[φ] * (Cos[λ])^2 * ((Sin[φ])^2 - 2 * (Cos[φ])^2) * Vxxz +
          Sin[φ] * Sin[2 * λ] * ((Sin[φ])^2 - 2 * (Cos[φ])^2) * Vxyz + 3 * (Sin[φ])^2 *
           Cos[φ] * (Sin[λ])^2 * Cos[λ] * Vyyx + (Sin[φ])^2 * Cos[φ] * (Sin[λ])^3 * Vyyy +
          Sin[φ] * (Sin[λ])^2 * ((Sin[φ])^2 - 2 * (Cos[φ])^2) * Vyyz +
          Cos[φ] * Cos[λ] * ((Cos[φ])^2 - 2 * (Sin[φ])^2) * Vzzx +
          Cos[φ] * Sin[λ] * ((Cos[φ])^2 - 2 * (Sin[φ])^2) * Vzzy + Sin[φ] * (Cos[φ])^2 * Vzzz;

In[•]:= Vρρλ = -(Cos[φ])^2 * Sin[λ] * (Cos[λ])^2 * Vxxx +
          (Cos[φ])^2 * Cos[λ] * ((Cos[λ])^2 - 2 * (Sin[λ])^2) * Vxxy -
          Sin[2 * φ] * Sin[λ] * Cos[λ] * Vxxz + Sin[2 * φ] * Cos[2 * λ] * Vxyz +
          (Cos[φ])^2 * Sin[λ] * (2 * (Cos[λ])^2 - (Sin[λ])^2) * Vyyx +
          (Cos[φ])^2 * (Sin[λ])^2 * Cos[λ] * Vyyy + Sin[2 * φ] * Sin[λ] * Cos[λ] * Vyyz -
          (Sin[φ])^2 * Sin[λ] * Vzzx + (Sin[φ])^2 * Cos[λ] * Vzzy;

In[•]:= Vρρφ = -Sin[φ] * (Cos[φ])^2 * (Cos[λ])^3 * Vxxx - 3 * Sin[φ] * (Cos[φ])^2 * Sin[λ] *
           (Cos[λ])^2 * Vxxy + Cos[φ] * (Cos[λ])^2 * ((Cos[φ])^2 - 2 * (Sin[φ])^2) * Vxxz +
          Cos[φ] * Sin[2 * λ] * ((Cos[φ])^2 - 2 * (Sin[φ])^2) * Vxyz -
          3 * Sin[φ] * (Cos[φ])^2 * (Sin[λ])^2 * Cos[λ] * Vyyx - Sin[φ] * (Cos[φ])^2 *
           (Sin[λ])^3 * Vyyy + Cos[φ] * (Sin[λ])^2 * ((Cos[φ])^2 - 2 * (Sin[φ])^2) * Vyyz +
          Sin[φ] * Cos[λ] * (2 * (Cos[φ])^2 - (Sin[φ])^2) * Vzzx +
          Sin[φ] * Sin[λ] * (2 * (Cos[φ])^2 - (Sin[φ])^2) * Vzzy + (Sin[φ])^2 * Cos[φ] * Vzzz;

In[•]:= Vρρρ = (Cos[φ])^3 * (Cos[λ])^3 * Vxxx +
          3 * (Cos[φ])^3 * Sin[λ] * (Cos[λ])^2 * Vxxy + 3 * Sin[φ] * (Cos[φ])^2 * (Cos[λ])^2 * Vxxz +
          3 * Sin[2 * φ] * Cos[φ] * Sin[λ] * Cos[λ] * Vxyz +
          3 * (Cos[φ])^3 * (Sin[λ])^2 * Cos[λ] * Vyyx + (Cos[φ])^3 * (Sin[λ])^3 * Vyyy +
          3 * Sin[φ] * (Cos[φ])^2 * (Sin[λ])^2 * Vyyz + 3 * (Sin[φ])^2 * Cos[φ] * Cos[λ] * Vzzx +
          3 * (Sin[φ])^2 * Cos[φ] * Sin[λ] * Vzzy + (Sin[φ])^3 * Vzzz;
```

10.3 Table A8

```
In[•]:= VrRoutes264 = -Sin[φ] * Vφ + Cos[φ] * Vρ;

In[•]:= VλRoutes264 = Vλ1;
```

```
In[•]:= VzRoutes264 = Cos[φ] * Vφ + Sin[φ] * Vρ;

In[•]:= VrrRoutes264 = (Sin[φ])^2 * Vφφ - Sin[2 * φ] * Vφρ + (Cos[φ])^2 * Vρρ;

In[•]:= VrλRoutes264 = -Sin[φ] * Vλφ + Cos[φ] * Vλρ;

In[•]:= VrzRoutes264 = -Sin[φ] * Cos[φ] * Vφφ + Cos[2 * φ] * Vφρ + Sin[φ] * Cos[φ] * Vρρ;

In[•]:= VλλRoutes264 = Vλλ1;

In[•]:= VλzRoutes264 = Cos[φ] * Vλφ + Sin[φ] * Vλρ;

In[•]:= VzzRoutes264 = (Cos[φ])^2 * Vφφ + Sin[2 * φ] * Vφρ + (Sin[φ])^2 * Vρρ;

In[•]:= VrrrRoutes264 = -(Sin[φ])^3 * Vφφφ +
          3 * (Sin[φ])^2 * Cos[φ] * Vφφρ - 3 * Sin[φ] * (Cos[φ])^2 * Vρρφ + (Cos[φ])^3 * Vρρρ;

In[•]:= VrrλRoutes264 = (Sin[φ])^2 * Vφφλ - Sin[2 * φ] * Vλφρ + (Cos[φ])^2 * Vρρλ;

In[•]:= VrrzRoutes264 = (Sin[φ])^2 * Cos[φ] * Vφφφ + Sin[φ] * ((Sin[φ])^2 - 2 * (Cos[φ])^2) * Vφφρ +
          Cos[φ] * ((Cos[φ])^2 - 2 * (Sin[φ])^2) * Vρρφ + Sin[φ] * (Cos[φ])^2 * Vρρρ;

In[•]:= VrλzRoutes264 = Cos[2 * φ] * Vλφρ - Sin[φ] * Cos[φ] * Vφφλ + Sin[φ] * Cos[φ] * Vρρλ;

In[•]:= VλλrRoutes264 = -Sin[φ] * Vλλφ + Cos[φ] * Vλλρ;

In[•]:= VλλλRoutes264 = Vλλλ1;

In[•]:= VλλzRoutes264 = Cos[φ] * Vλλφ + Sin[φ] * Vλλρ;

In[•]:= VzzrRoutes264 = -Sin[φ] * (Cos[φ])^2 * Vφφφ + Cos[φ] * ((Cos[φ])^2 - 2 * (Sin[φ])^2) * Vφφρ +
          Sin[φ] * (2 * (Cos[φ])^2 - (Sin[φ])^2) * Vρρφ + (Sin[φ])^2 * Cos[φ] * Vρρρ;

In[•]:= VzzλRoutes264 = Sin[2 * φ] * Vλφρ + (Cos[φ])^2 * Vφφλ + (Sin[φ])^2 * Vρρλ;

In[•]:= VzzzRoutes264 = (Cos[φ])^3 * Vφφφ +
          3 * Sin[φ] * (Cos[φ])^2 * Vφφρ + 3 * (Sin[φ])^2 * Cos[φ] * Vρρφ + (Sin[φ])^3 * Vρρρ;
```

10.4 Check whether they become themselves

```
In[•]:= FullSimplify[VrRoutes264]
Out[•]=
Vr

In[•]:= FullSimplify[VλRoutes264]
Out[•]=
Vλ

In[•]:= FullSimplify[VzRoutes264]
Out[•]=
Vz

In[•]:= FullSimplify[VrrRoutes264]
Out[•]=
Vrr
```

In[•]:= **FullSimplify[VrλRoutes264]**

Out[•]=

Vrλ

In[•]:= **FullSimplify[VrzRoutes264]**

Out[•]=

Vrz

In[•]:= **FullSimplify[VλλRoutes264]**

Out[•]=

Vλλ

In[•]:= **FullSimplify[VλzRoutes264]**

Out[•]=

Vλz

In[•]:= **FullSimplify[VzzRoutes264]**

Out[•]=

Vzz

In[•]:= **FullSimplify[VrrrRoutes264]**

Out[•]=

Vrrr

In[•]:= **FullSimplify[VrrλRoutes264]**

Out[•]=

Vrrλ

In[•]:= **FullSimplify[VrrzRoutes264]**

Out[•]=

Vrrz

In[•]:= **FullSimplify[VrλzRoutes264]**

Out[•]=

Vrλz

In[•]:= **FullSimplify[VλλrRoutes264]**

Out[•]=

Vλλr

In[•]:= **FullSimplify[VλλλRoutes264]**

Out[•]=

Vλλλ

In[•]:= **FullSimplify[VλλzRoutes264]**

Out[•]=

Vλλz

In[•]:= **FullSimplify[VzzrRoutes264]**

Out[•]=

Vzzr

In[•]:= **FullSimplify[VzzλRoutes264]**

Out[•]=

Vzzλ

```
In[ ]:= FullSimplify[VzzzRoutes264]
Out[ ]=
Vzzz
```

11. Trigonometric form, Spherical coordinates, Routes 5 → 1 → 3, and Table A11 → Table A2 → Table A6

```
In[ ]:= Clear["Global`*"];
```

11.1 Table A11

```
In[ ]:= Vx = -Sin[λ] * Vλ - Sin[φ] * Cos[λ] * Vφ + Cos[φ] * Cos[λ] * Vρ;

In[ ]:= Vy = Cos[λ] * Vλ - Sin[φ] * Sin[λ] * Vφ + Cos[φ] * Sin[λ] * Vρ;

In[ ]:= Vz = Cos[φ] * Vφ + Sin[φ] * Vρ;

In[ ]:= Vxx = (Sin[λ])^2 * Vλλ + Sin[φ] * Sin[2 * λ] * Vλφ -
          Cos[φ] * Sin[2 * λ] * Vλρ + (Sin[φ])^2 * (Cos[λ])^2 * Vφφ -
          Sin[2 * φ] * (Cos[λ])^2 * Vφρ + (Cos[φ])^2 * (Cos[λ])^2 * Vρρ;

In[ ]:= Vxy = -Sin[λ] * Cos[λ] * Vλλ - Sin[φ] * Cos[2 * λ] * Vλφ +
          Cos[φ] * Cos[2 * λ] * Vλρ + (Sin[φ])^2 * Sin[λ] * Cos[λ] * Vφφ -
          Sin[2 * φ] * Sin[λ] * Cos[λ] * Vφρ + (Cos[φ])^2 * Sin[λ] * Cos[λ] * Vρρ;

In[ ]:= Vxz = -Cos[φ] * Sin[λ] * Vλφ - Sin[φ] * Sin[λ] * Vλρ - Sin[φ] * Cos[φ] * Cos[λ] * Vφφ +
          Cos[2 * φ] * Cos[λ] * Vφρ + Sin[φ] * Cos[φ] * Cos[λ] * Vρρ;

In[ ]:= Vyy = (Cos[λ])^2 * Vλλ - Sin[φ] * Sin[2 * λ] * Vλφ +
          Cos[φ] * Sin[2 * λ] * Vλρ + (Sin[φ])^2 * (Sin[λ])^2 * Vφφ -
          Sin[2 * φ] * (Sin[λ])^2 * Vφρ + (Cos[φ])^2 * (Sin[λ])^2 * Vρρ;

In[ ]:= Vyz = Cos[φ] * Cos[λ] * Vλφ + Sin[φ] * Cos[λ] * Vλρ - Sin[φ] * Cos[φ] * Sin[λ] * Vφφ +
          Cos[2 * φ] * Sin[λ] * Vφρ + Sin[φ] * Cos[φ] * Sin[λ] * Vρρ;

In[ ]:= Vzz = (Cos[φ])^2 * Vφφ + Sin[2 * φ] * Vφρ + (Sin[φ])^2 * Vρρ;

In[ ]:= Vxxx = -(Sin[λ])^3 * Vλλλ + 3 * Cos[φ] * (Sin[λ])^2 * Cos[λ] * Vλλρ -
          3 * Sin[φ] * (Sin[λ])^2 * Cos[λ] * Vλλφ + 3 * Sin[2 * φ] * Sin[λ] * (Cos[λ])^2 * Vλφρ -
          3 * (Sin[φ])^2 * Sin[λ] * (Cos[λ])^2 * Vφφλ - (Sin[φ])^3 * (Cos[λ])^3 * Vφφφ +
          3 * (Sin[φ])^2 * Cos[φ] * (Cos[λ])^3 * Vφφρ - 3 * (Cos[φ])^2 * Sin[λ] * (Cos[λ])^2 * Vρρλ -
          3 * Sin[φ] * (Cos[φ])^2 * (Cos[λ])^3 * Vρρφ + (Cos[φ])^3 * (Cos[λ])^3 * Vρρρ;

In[ ]:= Vxxy = (Sin[λ])^2 * Cos[λ] * Vλλλ + Sin[φ] * Sin[λ] * (2 * (Cos[λ])^2 - (Sin[λ])^2) * Vλλφ +
          Cos[φ] * Sin[λ] * ((Sin[λ])^2 - 2 * (Cos[λ])^2) * Vλλρ +
          Sin[2 * φ] * Cos[λ] * (2 * (Sin[λ])^2 - (Cos[λ])^2) * Vλφρ +
          (Sin[φ])^2 * Cos[λ] * ((Cos[λ])^2 - 2 * (Sin[λ])^2) * Vφφλ - (Sin[φ])^3 * Sin[λ] *
           (Cos[λ])^2 * Vφφφ + 3 * (Sin[φ])^2 * Cos[φ] * Sin[λ] * (Cos[λ])^2 * Vφφρ +
          (Cos[φ])^2 * Cos[λ] * ((Cos[λ])^2 - 2 * (Sin[λ])^2) * Vρρλ - 3 * Sin[φ] *
           (Cos[φ])^2 * Sin[λ] * (Cos[λ])^2 * Vρρφ + (Cos[φ])^3 * Sin[λ] * (Cos[λ])^2 * Vρρρ;
```

```
In[ ]:= Vxxz =
  Cos[φ] * (Sin[λ])^2 * Vλλφ + Sin[φ] * (Sin[λ])^2 * Vλλρ - Cos[2 * φ] * Sin[2 * λ] * Vλφρ +
   Sin[φ] * Cos[φ] * Sin[2 * λ] * Vφφλ + (Sin[φ])^2 * Cos[φ] * (Cos[λ])^2 * Vφφφ +
   Sin[φ] * (Cos[λ])^2 * ((Sin[φ])^2 - 2 * (Cos[φ])^2) * Vφφρ -
   Sin[2 * φ] * Sin[λ] * Cos[λ] * Vφρλ + Cos[φ] * (Cos[λ])^2 *
    ((Cos[φ])^2 - 2 * (Sin[φ])^2) * Vφρφ + Sin[φ] * (Cos[φ])^2 * (Cos[λ])^2 * Vφρρ;

In[ ]:= Vxyz = -Cos[φ] * Sin[λ] * Cos[λ] * Vλλφ -
   Sin[φ] * Sin[λ] * Cos[λ] * Vλλρ + Cos[2 * φ] * Cos[2 * λ] * Vλφρ -
   Sin[φ] * Cos[φ] * Cos[2 * λ] * Vφφλ + (Sin[φ])^2 * Cos[φ] * Sin[λ] * Cos[λ] * Vφφφ -
   Sin[φ] * Sin[λ] * Cos[λ] * (2 * (Cos[φ])^2 - (Sin[φ])^2) * Vφφρ +
   Sin[φ] * Cos[φ] * Cos[2 * λ] * Vφρλ + Cos[φ] * Sin[λ] * Cos[λ] *
    ((Cos[φ])^2 - 2 * (Sin[φ])^2) * Vφρφ + Sin[φ] * (Cos[φ])^2 * Sin[λ] * Cos[λ] * Vφρρ;

In[ ]:= Vyyx = -Sin[λ] * (Cos[λ])^2 * Vλλλ - Sin[φ] * Cos[λ] * ((Cos[λ])^2 - 2 * (Sin[λ])^2) * Vλλφ +
   Cos[φ] * Cos[λ] * ((Cos[λ])^2 - 2 * (Sin[λ])^2) * Vλλρ -
   Sin[2 * φ] * Sin[λ] * (2 * (Cos[λ])^2 - (Sin[λ])^2) * Vλφρ +
   (Sin[φ])^2 * Sin[λ] * (2 * (Cos[λ])^2 - (Sin[λ])^2) * Vφφλ - (Sin[φ])^3 *
    (Sin[λ])^2 * Cos[λ] * Vφφφ + 3 * (Sin[φ])^2 * Cos[φ] * (Sin[λ])^2 * Cos[λ] * Vφφρ +
   (Cos[φ])^2 * Sin[λ] * (2 * (Cos[λ])^2 - (Sin[λ])^2) * Vφρλ - 3 * Sin[φ] *
    (Cos[φ])^2 * (Sin[λ])^2 * Cos[λ] * Vφρφ + (Cos[φ])^3 * (Sin[λ])^2 * Cos[λ] * Vφρρ;

In[ ]:= Vyyy = (Cos[λ])^3 * Vλλλ - 3 * Sin[φ] * Sin[λ] * (Cos[λ])^2 * Vλλφ +
   3 * Cos[φ] * Sin[λ] * (Cos[λ])^2 * Vλλρ - 3 * Sin[2 * φ] * (Sin[λ])^2 * Cos[λ] * Vλφρ +
   3 * (Sin[φ])^2 * (Sin[λ])^2 * Cos[λ] * Vφφλ - (Sin[φ])^3 * (Sin[λ])^3 * Vφφφ +
   3 * (Sin[φ])^2 * Cos[φ] * (Sin[λ])^3 * Vφφρ + 3 * (Cos[φ])^2 * (Sin[λ])^2 * Cos[λ] * Vφρλ -
   3 * Sin[φ] * (Cos[φ])^2 * (Sin[λ])^3 * Vφρφ + (Cos[φ])^3 * (Sin[λ])^3 * Vφρρ;

In[ ]:= Vyyz =
  Cos[φ] * (Cos[λ])^2 * Vλλφ + Sin[φ] * (Cos[λ])^2 * Vλλρ + Cos[2 * φ] * Sin[2 * λ] * Vλφρ -
   Sin[2 * φ] * Sin[λ] * Cos[λ] * Vφφλ + (Sin[φ])^2 * Cos[φ] * (Sin[λ])^2 * Vφφφ +
   Sin[φ] * (Sin[λ])^2 * ((Sin[φ])^2 - 2 * (Cos[φ])^2) * Vφφρ +
   Sin[2 * φ] * Sin[λ] * Cos[λ] * Vφρλ + Cos[φ] * (Sin[λ])^2 *
    ((Cos[φ])^2 - 2 * (Sin[φ])^2) * Vφρφ + Sin[φ] * (Cos[φ])^2 * (Sin[λ])^2 * Vφρρ;

In[ ]:= Vzzx = -Sin[2 * φ] * Sin[λ] * Vλφρ -
   (Cos[φ])^2 * Sin[λ] * Vφφλ - Sin[φ] * (Cos[φ])^2 * Cos[λ] * Vφφφ +
   Cos[φ] * Cos[λ] * ((Cos[φ])^2 - 2 * (Sin[φ])^2) * Vφφρ - (Sin[φ])^2 * Sin[λ] * Vφρλ +
   Sin[φ] * Cos[λ] * (2 * (Cos[φ])^2 - (Sin[φ])^2) * Vφρφ +
   (Sin[φ])^2 * Cos[φ] * Cos[λ] * Vφρρ;

In[ ]:= Vzzy = Sin[2 * φ] * Cos[λ] * Vλφρ +
   (Cos[φ])^2 * Cos[λ] * Vφφλ - Sin[φ] * (Cos[φ])^2 * Sin[λ] * Vφφφ +
   Cos[φ] * Sin[λ] * ((Cos[φ])^2 - 2 * (Sin[φ])^2) * Vφφρ + (Sin[φ])^2 * Cos[λ] * Vφρλ +
   Sin[φ] * Sin[λ] * (2 * (Cos[φ])^2 - (Sin[φ])^2) * Vφρφ +
   (Sin[φ])^2 * Cos[φ] * Sin[λ] * Vφρρ;

In[ ]:= Vzzz = (Cos[φ])^3 * Vφφφ + 3 * Sin[φ] * (Cos[φ])^2 * Vφφρ +
   3 * (Sin[φ])^2 * Cos[φ] * Vφρφ + (Sin[φ])^3 * Vφρρ;
```

11.2 Table A2

```
In[•]:= Vr = Cos[λ] * Vx + Sin[λ] * Vy;

In[•]:= Vλ1 = -Sin[λ] * Vx + Cos[λ] * Vy;

In[•]:= Vz = Vz;

In[•]:= Vrr = (Cos[λ])^2 * Vxx + Sin[2 * λ] * Vxy + (Sin[λ])^2 * Vyy;

In[•]:= Vrλ = -Sin[λ] * Cos[λ] * Vxx + Cos[2 * λ] * Vxy + Sin[λ] * Cos[λ] * Vyy;

In[•]:= Vrz = Cos[λ] * Vxz + Sin[λ] * Vyz;

In[•]:= Vλλ1 = (Sin[λ])^2 * Vxx - Sin[2 * λ] * Vxy + (Cos[λ])^2 * Vyy;

In[•]:= Vλz = -Sin[λ] * Vxz + Cos[λ] * Vyz;

In[•]:= Vzz = Vzz;

In[•]:= Vrrr = (Cos[λ])^3 * Vxxx + 3 * Sin[λ] * (Cos[λ])^2 * Vxxy +
          3 * Cos[λ] * (Sin[λ])^2 * Vyyx + (Sin[λ])^3 * Vyyy;

In[•]:= Vrrλ = -Sin[λ] * (Cos[λ])^2 * Vxxx + Cos[λ] * ((Cos[λ])^2 - 2 * (Sin[λ])^2) * Vxxy +
          Sin[λ] * (2 * (Cos[λ])^2 - (Sin[λ])^2) * Vyyx + (Sin[λ])^2 * Cos[λ] * Vyyy;

In[•]:= Vrrz = (Cos[λ])^2 * Vxxz + Sin[2 * λ] * Vxyz + (Sin[λ])^2 * Vyyz;

In[•]:= Vrλz = -Sin[λ] * Cos[λ] * Vxxz + Cos[2 * λ] * Vxyz + Sin[λ] * Cos[λ] * Vyyz;

In[•]:= Vλλr = Cos[λ] * (Sin[λ])^2 * Vxxx + Sin[λ] * ((Sin[λ])^2 - 2 * (Cos[λ])^2) * Vxxy +
          Cos[λ] * ((Cos[λ])^2 - 2 * (Sin[λ])^2) * Vyyx + Sin[λ] * (Cos[λ])^2 * Vyyy;

In[•]:= Vλλλ1 = -(Sin[λ])^3 * Vxxx + 3 * (Sin[λ])^2 * Cos[λ] * Vxxy -
          3 * Sin[λ] * (Cos[λ])^2 * Vyyx + (Cos[λ])^3 * Vyyy;

In[•]:= Vλλz = (Sin[λ])^2 * Vxxz - Sin[2 * λ] * Vxyz + (Cos[λ])^2 * Vyyz;

In[•]:= Vzzr = Cos[λ] * Vzzx + Sin[λ] * Vzzy;

In[•]:= Vzzλ = -Sin[λ] * Vzzx + Cos[λ] * Vzzy;

In[•]:= Vzzz = Vzzz;
```

11.3 Table A6

```
In[•]:= VλRoutes513 = Vλ1;

In[•]:= VφRoutes513 = -Sin[φ] * Vr + Cos[φ] * Vz;

In[•]:= VρRoutes513 = Cos[φ] * Vr + Sin[φ] * Vz;

In[•]:= VλλRoutes513 = Vλλ1;

In[•]:= VλφRoutes513 = -Sin[φ] * Vrλ + Cos[φ] * Vλz;

In[•]:= VλρRoutes513 = Cos[φ] * Vrλ + Sin[φ] * Vλz;

In[•]:= VφφRoutes513 = ((Sin[φ])^2 * Vrr - Sin[2 * φ] * Vrz + (Cos[φ])^2 * Vzz);

In[•]:= VφρRoutes513 = -Sin[φ] * Cos[φ] * Vrr + Cos[2 * φ] * Vrz + Sin[φ] * Cos[φ] * Vzz;
```

```
In[•]:= VρρRoutes513 = ((Cos[φ])^2 * Vrr + Sin[2 * φ] * Vrz + (Sin[φ])^2 * Vzz);

In[•]:= VλλλRoutes513 = Vλλλ1;

In[•]:= VλλφRoutes513 = -Sin[φ] * Vλλr + Cos[φ] * Vλλz;

In[•]:= VλλρRoutes513 = Cos[φ] * Vλλr + Sin[φ] * Vλλz;

In[•]:= VλφρRoutes513 = -Sin[φ] * Cos[φ] * Vrrλ + Cos[2 * φ] * Vrλz + Sin[φ] * Cos[φ] * Vzzλ;

In[•]:= VφφλRoutes513 = (Sin[φ])^2 * Vrrλ - Sin[2 * φ] * Vrλz + (Cos[φ])^2 * Vzzλ;

In[•]:= VφφφRoutes513 = -(Sin[φ])^3 * Vrrr +
          3 * (Sin[φ])^2 * Cos[φ] * Vrrz - 3 * Sin[φ] * (Cos[φ])^2 * Vzzr + (Cos[φ])^3 * Vzzz;

In[•]:= VφφρRoutes513 = (Sin[φ])^2 * Cos[φ] * Vrrr + Sin[φ] * ((Sin[φ])^2 - 2 * (Cos[φ])^2) * Vrrz +
          Cos[φ] * ((Cos[φ])^2 - 2 * (Sin[φ])^2) * Vzzr + Sin[φ] * (Cos[φ])^2 * Vzzz;

In[•]:= VρρλRoutes513 = (Cos[φ])^2 * Vrrλ + Sin[2 * φ] * Vrλz + (Sin[φ])^2 * Vzzλ;

In[•]:= VρρφRoutes513 = -Sin[φ] * (Cos[φ])^2 * Vrrr + Cos[φ] * ((Cos[φ])^2 - 2 * (Sin[φ])^2) * Vrrz +
          Sin[φ] * (2 * (Cos[φ])^2 - (Sin[φ])^2) * Vzzr + Cos[φ] * (Sin[φ])^2 * Vzzz;

In[•]:= VρρρRoutes513 = (Cos[φ])^3 * Vrrr +
          3 * Sin[φ] * (Cos[φ])^2 * Vrrz + 3 * Cos[φ] * (Sin[φ])^2 * Vzzr + (Sin[φ])^3 * Vzzz;
```

11.4 Check whether they become themselves

```
In[•]:= FullSimplify[VλRoutes513]
Out[•]=
Vλ

In[•]:= FullSimplify[VφRoutes513]
Out[•]=
Vφ

In[•]:= FullSimplify[VρRoutes513]
Out[•]=
Vρ

In[•]:= FullSimplify[VλλRoutes513]
Out[•]=
Vλλ

In[•]:= FullSimplify[VλφRoutes513]
Out[•]=
Vλφ

In[•]:= FullSimplify[VλρRoutes513]
Out[•]=
Vλρ

In[•]:= FullSimplify[VφφRoutes513]
Out[•]=
Vφφ
```

```
In[•]:= FullSimplify[Vφρ Routes513]
Out[•]=
```

$V_{\varphi\rho}$

```
In[•]:= FullSimplify[Vρρ Routes513]
Out[•]=
```

$V_{\rho\rho}$

```
In[•]:= FullSimplify[Vλλλ Routes513]
Out[•]=
```

$V_{\lambda\lambda\lambda}$

```
In[•]:= FullSimplify[Vλλφ Routes513]
Out[•]=
```

$V_{\lambda\lambda\varphi}$

```
In[•]:= FullSimplify[Vλλρ Routes513]
Out[•]=
```

$V_{\lambda\lambda\rho}$

```
In[•]:= FullSimplify[Vλφρ Routes513]
Out[•]=
```

$V_{\lambda\varphi\rho}$

```
In[•]:= FullSimplify[Vφφλ Routes513]
Out[•]=
```

$V_{\varphi\varphi\lambda}$

```
In[•]:= FullSimplify[Vφφφ Routes513]
Out[•]=
```

$V_{\varphi\varphi\varphi}$

```
In[•]:= FullSimplify[Vφφρ Routes513]
Out[•]=
```

$V_{\varphi\varphi\rho}$

```
In[•]:= FullSimplify[Vρρλ Routes513]
Out[•]=
```

$V_{\rho\rho\lambda}$

```
In[•]:= FullSimplify[Vρρφ Routes513]
Out[•]=
```

$V_{\rho\rho\varphi}$

```
In[•]:= FullSimplify[Vρρρ Routes513]
Out[•]=
```

$V_{\rho\rho\rho}$

12. Trigonometric form, Spherical coordinates, Routes 4 → 2 → 6, and Table A8 → Table A4 → Table A13

```
In[•]:= Clear["Global`*"];
```

12.1 Table A8

```
In[•]:= Vr = -Sin[φ] * Vφ + Cos[φ] * Vρ;

In[•]:= Vλ = Vλ;

In[•]:= Vz = Cos[φ] * Vφ + Sin[φ] * Vρ;

In[•]:= Vrr = (Sin[φ])^2 * Vφφ - Sin[2 * φ] * Vφρ + (Cos[φ])^2 * Vρρ;

In[•]:= Vrλ = -Sin[φ] * Vλφ + Cos[φ] * Vλρ;

In[•]:= Vrz = -Sin[φ] * Cos[φ] * Vφφ + Cos[2 * φ] * Vφρ + Sin[φ] * Cos[φ] * Vρρ;

In[•]:= Vλλ = Vλλ;

In[•]:= Vλz = Cos[φ] * Vλφ + Sin[φ] * Vλρ;

In[•]:= Vzz = (Cos[φ])^2 * Vφφ + Sin[2 * φ] * Vφρ + (Sin[φ])^2 * Vρρ;

In[•]:= Vrrr = -(Sin[φ])^3 * Vφφφ + 3 * (Sin[φ])^2 * Cos[φ] * Vφφρ -
          3 * Sin[φ] * (Cos[φ])^2 * Vρρφ + (Cos[φ])^3 * Vρρρ;

In[•]:= Vrrλ = (Sin[φ])^2 * Vφφλ - Sin[2 * φ] * Vλφρ + (Cos[φ])^2 * Vρρλ;

In[•]:= Vrrz = (Sin[φ])^2 * Cos[φ] * Vφφφ + Sin[φ] * ((Sin[φ])^2 - 2 * (Cos[φ])^2) * Vφφρ +
          Cos[φ] * ((Cos[φ])^2 - 2 * (Sin[φ])^2) * Vρρφ + Sin[φ] * (Cos[φ])^2 * Vρρρ;

In[•]:= Vrλz = Cos[2 * φ] * Vλφρ - Sin[φ] * Cos[φ] * Vφφλ + Sin[φ] * Cos[φ] * Vρρλ;

In[•]:= Vλλr = -Sin[φ] * Vλλφ + Cos[φ] * Vλλρ;

In[•]:= Vλλλ = Vλλλ;

In[•]:= Vλλz = Cos[φ] * Vλλφ + Sin[φ] * Vλλρ;

In[•]:= Vzzr = -Sin[φ] * (Cos[φ])^2 * Vφφφ + Cos[φ] * ((Cos[φ])^2 - 2 * (Sin[φ])^2) * Vφφρ +
          Sin[φ] * (2 * (Cos[φ])^2 - (Sin[φ])^2) * Vρρφ + (Sin[φ])^2 * Cos[φ] * Vρρρ;

In[•]:= Vzzλ = Sin[2 * φ] * Vλφρ + (Cos[φ])^2 * Vφφλ + (Sin[φ])^2 * Vρρλ;

In[•]:= Vzzz = (Cos[φ])^3 * Vφφφ + 3 * Sin[φ] * (Cos[φ])^2 * Vφφρ +
          3 * (Sin[φ])^2 * Cos[φ] * Vρρφ + (Sin[φ])^3 * Vρρρ;
```

12.2 Table A4

```
In[•]:= Vx = Cos[λ] * Vr - Sin[λ] * Vλ;

In[•]:= Vy = Sin[λ] * Vr + Cos[λ] * Vλ;

In[•]:= Vz = Vz;

In[•]:= Vxx = (Cos[λ])^2 * Vrr - Sin[2 * λ] * Vrλ + (Sin[λ])^2 * Vλλ;

In[•]:= Vxy = Sin[λ] * Cos[λ] * Vrr + Cos[2 * λ] * Vrλ - Sin[λ] * Cos[λ] * Vλλ;

In[•]:= Vxz = Cos[λ] * Vrz - Sin[λ] * Vλz;

In[•]:= Vyy = (Sin[λ])^2 * Vrr + Sin[2 * λ] * Vrλ + (Cos[λ])^2 * Vλλ;

In[•]:= Vyz = Sin[λ] * Vrz + Cos[λ] * Vλz;
```

```
In[•]:= Vzz = Vzz;

In[•]:= Vxxx = (Cos[λ])^3 * Vrrr - 3 * Sin[λ] * (Cos[λ])^2 * Vrrλ +
          3 * (Sin[λ])^2 * Cos[λ] * Vλλr - (Sin[λ])^3 * Vλλλ;

In[•]:= Vxxy = Sin[λ] * (Cos[λ])^2 * Vrrr + Cos[λ] * ((Cos[λ])^2 - 2 * (Sin[λ])^2) * Vrrλ +
          Sin[λ] * ((Sin[λ])^2 - 2 * (Cos[λ])^2) * Vλλr + (Sin[λ])^2 * Cos[λ] * Vλλλ;

In[•]:= Vxxz = (Cos[λ])^2 * Vrrz - Sin[2 * λ] * Vrλz + (Sin[λ])^2 * Vλλz;

In[•]:= Vxyz = Sin[λ] * Cos[λ] * Vrrz + Cos[2 * λ] * Vrλz - Sin[λ] * Cos[λ] * Vλλz;

In[•]:= Vyyx = (Sin[λ])^2 * Cos[λ] * Vrrr + Sin[λ] * (2 * (Cos[λ])^2 - (Sin[λ])^2) * Vrrλ +
          Cos[λ] * ((Cos[λ])^2 - 2 * (Sin[λ])^2) * Vλλr - Sin[λ] * (Cos[λ])^2 * Vλλλ;

In[•]:= Vyyy = (Sin[λ])^3 * Vrrr + 3 * (Sin[λ])^2 * Cos[λ] * Vrrλ +
          3 * Sin[λ] * (Cos[λ])^2 * Vλλr + (Cos[λ])^3 * Vλλλ;

In[•]:= Vyyz = (Sin[λ])^2 * Vrrz + Sin[2 * λ] * Vrλz + (Cos[λ])^2 * Vλλz;

In[•]:= Vzzx = Cos[λ] * Vzzr - Sin[λ] * Vzzλ;

In[•]:= Vzzy = Sin[λ] * Vzzr + Cos[λ] * Vzzλ;

In[•]:= Vzzz = Vzzz;
```

12.3 Table A13

```
In[•]:= VλRoutes426 = -Sin[λ] * Vx + Cos[λ] * Vy;

In[•]:= VφRoutes426 = -Sin[φ] * Cos[λ] * Vx - Sin[φ] * Sin[λ] * Vy + Cos[φ] * Vz;

In[•]:= VρRoutes426 = Cos[φ] * Cos[λ] * Vx + Cos[φ] * Sin[λ] * Vy + Sin[φ] * Vz;

In[•]:= VλλRoutes426 = (Sin[λ])^2 * Vxx - Sin[2 * λ] * Vxy + (Cos[λ])^2 * Vyy;

In[•]:= VλφRoutes426 = Sin[φ] * Sin[λ] * Cos[λ] * Vxx - Sin[φ] * Cos[2 * λ] * Vxy -
          Cos[φ] * Sin[λ] * Vxz - Sin[φ] * Sin[λ] * Cos[λ] * Vyy + Cos[φ] * Cos[λ] * Vyz;

In[•]:= VλρRoutes426 = -Cos[φ] * Sin[λ] * Cos[λ] * Vxx + Cos[φ] * Cos[2 * λ] * Vxy -
          Sin[φ] * Sin[λ] * Vxz + Cos[φ] * Sin[λ] * Cos[λ] * Vyy + Sin[φ] * Cos[λ] * Vyz;

In[•]:= VφφRoutes426 =
          (Sin[φ])^2 * (Cos[λ])^2 * Vxx + (Sin[φ])^2 * Sin[2 * λ] * Vxy - Sin[2 * φ] * Cos[λ] * Vxz +
          (Sin[φ])^2 * (Sin[λ])^2 * Vyy - Sin[2 * φ] * Sin[λ] * Vyz + (Cos[φ])^2 * Vzz;

In[•]:= VφρRoutes426 = -Sin[φ] * Cos[φ] * (Cos[λ])^2 * Vxx -
          Sin[2 * φ] * Sin[λ] * Cos[λ] * Vxy + Cos[2 * φ] * Cos[λ] * Vxz -
          Sin[φ] * Cos[φ] * (Sin[λ])^2 * Vyy + Cos[2 * φ] * Sin[λ] * Vyz + Sin[φ] * Cos[φ] * Vzz;

In[•]:= VρρRoutes426 =
          (Cos[φ])^2 * (Cos[λ])^2 * Vxx + (Cos[φ])^2 * Sin[2 * λ] * Vxy + Sin[2 * φ] * Cos[λ] * Vxz +
          (Cos[φ])^2 * (Sin[λ])^2 * Vyy + Sin[2 * φ] * Sin[λ] * Vyz + (Sin[φ])^2 * Vzz;

In[•]:= VλλλRoutes426 = -(Sin[λ])^3 * Vxxx +
          3 * (Sin[λ])^2 * Cos[λ] * Vxxy - 3 * Sin[λ] * (Cos[λ])^2 * Vyyx + (Cos[λ])^3 * Vyyy;
```

```
In[•]:= Vλλφ Routes426 = -Sin[φ] * (Sin[λ])^2 * Cos[λ] * Vxxx +
    Sin[φ] * Sin[λ] * (2 * (Cos[λ])^2 - (Sin[λ])^2) * Vxxy + Cos[φ] * (Sin[λ])^2 * Vxxz -
    Cos[φ] * Sin[2 * λ] * Vxyz + Sin[φ] * Cos[λ] * (2 * (Sin[λ])^2 - (Cos[λ])^2) * Vyyx -
    Sin[φ] * Sin[λ] * (Cos[λ])^2 * Vyyy + Cos[φ] * (Cos[λ])^2 * Vyyz;

In[•]:= Vλλρ Routes426 = Cos[φ] * (Sin[λ])^2 * Cos[λ] * Vxxx +
    Cos[φ] * Sin[λ] * ((Sin[λ])^2 - 2 * (Cos[λ])^2) * Vxxy + Sin[φ] * (Sin[λ])^2 * Vxxz -
    Sin[φ] * Sin[2 * λ] * Vxyz + Cos[φ] * Cos[λ] * ((Cos[λ])^2 - 2 * (Sin[λ])^2) * Vyyx +
    Cos[φ] * Sin[λ] * (Cos[λ])^2 * Vyyy + Sin[φ] * (Cos[λ])^2 * Vyyz;

In[•]:= Vλφρ Routes426 = Sin[φ] * Cos[φ] * Sin[λ] * (Cos[λ])^2 * Vxxx +
    Sin[φ] * Cos[φ] * Cos[λ] * (2 * (Sin[λ])^2 - (Cos[λ])^2) * Vxxy -
    Cos[2 * φ] * Sin[λ] * Cos[λ] * Vxxz + Cos[2 * φ] * Cos[2 * λ] * Vxyz +
    Sin[φ] * Cos[φ] * Sin[λ] * ((Sin[λ])^2 - 2 * (Cos[λ])^2) * Vyyx -
    Sin[φ] * Cos[φ] * (Sin[λ])^2 * Cos[λ] * Vyyy + Cos[2 * φ] * Sin[λ] * Cos[λ] * Vyyz -
    Sin[φ] * Cos[φ] * Sin[λ] * Vzzx + Sin[φ] * Cos[φ] * Cos[λ] * Vzzy;

In[•]:= Vφφλ Routes426 = -(Sin[φ])^2 * Sin[λ] * (Cos[λ])^2 * Vxxx +
    (Sin[φ])^2 * Cos[λ] * ((Cos[λ])^2 - 2 * (Sin[λ])^2) * Vxxy +
    Sin[2 * φ] * Sin[λ] * Cos[λ] * Vxxz - Sin[2 * φ] * Cos[2 * λ] * Vxyz +
    (Sin[φ])^2 * Sin[λ] * (2 * (Cos[λ])^2 - (Sin[λ])^2) * Vyyx +
    (Sin[φ])^2 * (Sin[λ])^2 * Cos[λ] * Vyyy - Sin[2 * φ] * Sin[λ] * Cos[λ] * Vyyz -
    (Cos[φ])^2 * Sin[λ] * Vzzx + (Cos[φ])^2 * Cos[λ] * Vzzy;

In[•]:= Vφφφ Routes426 =
  -(Sin[φ])^3 * (Cos[λ])^3 * Vxxx - 3 * (Sin[φ])^3 * Sin[λ] * (Cos[λ])^2 * Vxxy +
  3 * (Sin[φ])^2 * Cos[φ] * (Cos[λ])^2 * Vxxz + 3 * (Sin[φ])^2 * Cos[φ] * Sin[2 * λ] * Vxyz -
  3 * (Sin[φ])^3 * (Sin[λ])^2 * Cos[λ] * Vyyx - (Sin[φ])^3 * (Sin[λ])^3 * Vyyy +
  3 * (Sin[φ])^2 * Cos[φ] * (Sin[λ])^2 * Vyyz - 3 * Sin[φ] * (Cos[φ])^2 * Cos[λ] * Vzzx -
  3 * Sin[φ] * (Cos[φ])^2 * Sin[λ] * Vzzy + (Cos[φ])^3 * Vzzz;

In[•]:= Vφφρ Routes426 = (Sin[φ])^2 * Cos[φ] * (Cos[λ])^3 * Vxxx +
    3 * (Sin[φ])^2 * Cos[φ] * Sin[λ] * (Cos[λ])^2 * Vxxy +
    Sin[φ] * (Cos[λ])^2 * ((Sin[φ])^2 - 2 * (Cos[φ])^2) * Vxxz +
    Sin[φ] * Sin[2 * λ] * ((Sin[φ])^2 - 2 * (Cos[φ])^2) * Vxyz +
    3 * (Sin[φ])^2 * Cos[φ] * (Sin[λ])^2 * Cos[λ] * Vyyx + (Sin[φ])^2 * Cos[φ] *
     (Sin[λ])^3 * Vyyy + Sin[φ] * (Sin[λ])^2 * ((Sin[φ])^2 - 2 * (Cos[φ])^2) * Vyyz +
    Cos[φ] * Cos[λ] * ((Cos[φ])^2 - 2 * (Sin[φ])^2) * Vzzx +
    Cos[φ] * Sin[λ] * ((Cos[φ])^2 - 2 * (Sin[φ])^2) * Vzzy + Sin[φ] * (Cos[φ])^2 * Vzzz;

In[•]:= Vρρλ Routes426 = -(Cos[φ])^2 * Sin[λ] * (Cos[λ])^2 * Vxxx +
    (Cos[φ])^2 * Cos[λ] * ((Cos[λ])^2 - 2 * (Sin[λ])^2) * Vxxy -
    Sin[2 * φ] * Sin[λ] * Cos[λ] * Vxxz + Sin[2 * φ] * Cos[2 * λ] * Vxyz +
    (Cos[φ])^2 * Sin[λ] * (2 * (Cos[λ])^2 - (Sin[λ])^2) * Vyyx +
    (Cos[φ])^2 * (Sin[λ])^2 * Cos[λ] * Vyyy + Sin[2 * φ] * Sin[λ] * Cos[λ] * Vyyz -
    (Sin[φ])^2 * Sin[λ] * Vzzx + (Sin[φ])^2 * Cos[λ] * Vzzy;
```

```
In[•]:= Vρρφ Routes426 = -Sin[φ] * (Cos[φ])^2 * (Cos[λ])^3 * Vxxx -
          3 * Sin[φ] * (Cos[φ])^2 * Sin[λ] * (Cos[λ])^2 * Vxxy +
          Cos[φ] * (Cos[λ])^2 * ((Cos[φ])^2 - 2 * (Sin[φ])^2) * Vxxz +
          Cos[φ] * Sin[2 * λ] * ((Cos[φ])^2 - 2 * (Sin[φ])^2) * Vxyz -
          3 * Sin[φ] * (Cos[φ])^2 * (Sin[λ])^2 * Cos[λ] * Vyyx - Sin[φ] * (Cos[φ])^2 *
           (Sin[λ])^3 * Vyyy + Cos[φ] * (Sin[λ])^2 * ((Cos[φ])^2 - 2 * (Sin[φ])^2) * Vyyz +
          Sin[φ] * Cos[λ] * (2 * (Cos[φ])^2 - (Sin[φ])^2) * Vzzx +
          Sin[φ] * Sin[λ] * (2 * (Cos[φ])^2 - (Sin[φ])^2) * Vzzy + (Sin[φ])^2 * Cos[φ] * Vzzz;

In[•]:= Vρρρ Routes426 = (Cos[φ])^3 * (Cos[λ])^3 * Vxxx +
          3 * (Cos[φ])^3 * Sin[λ] * (Cos[λ])^2 * Vxxy + 3 * Sin[φ] * (Cos[φ])^2 * (Cos[λ])^2 * Vxxz +
          3 * Sin[2 * φ] * Cos[φ] * Sin[λ] * Cos[λ] * Vxyz +
          3 * (Cos[φ])^3 * (Sin[λ])^2 * Cos[λ] * Vyyx + (Cos[φ])^3 * (Sin[λ])^3 * Vyyy +
          3 * Sin[φ] * (Cos[φ])^2 * (Sin[λ])^2 * Vyyz + 3 * (Sin[φ])^2 * Cos[φ] * Cos[λ] * Vzzx +
          3 * (Sin[φ])^2 * Cos[φ] * Sin[λ] * Vzzy + (Sin[φ])^3 * Vzzz;
```

12.4 Check whether they become themselves

```
In[•]:= FullSimplify[Vλ Routes426]
Out[•]=
Vλ

In[•]:= FullSimplify[Vφ Routes426]
Out[•]=
Vφ

In[•]:= FullSimplify[Vρ Routes426]
Out[•]=
Vρ

In[•]:= FullSimplify[Vλλ Routes426]
Out[•]=
Vλλ

In[•]:= FullSimplify[Vλφ Routes426]
Out[•]=
Vλφ

In[•]:= FullSimplify[Vλρ Routes426]
Out[•]=
Vλρ

In[•]:= FullSimplify[Vφφ Routes426]
Out[•]=
Vφφ

In[•]:= FullSimplify[Vφρ Routes426]
Out[•]=
Vφρ
```

In[•]:= **FullSimplify[Vρρ Routes426]**

Out[•]=

$V_{\rho\rho}$

In[•]:= **FullSimplify[Vλλλ Routes426]**

Out[•]=

$V_{\lambda\lambda\lambda}$

In[•]:= **FullSimplify[Vλλφ Routes426]**

Out[•]=

$V_{\lambda\lambda\varphi}$

In[•]:= **FullSimplify[Vλλρ Routes426]**

Out[•]=

$V_{\lambda\lambda\rho}$

In[•]:= **FullSimplify[Vλφρ Routes426]**

Out[•]=

$V_{\lambda\varphi\rho}$

In[•]:= **FullSimplify[Vφφλ Routes426]**

Out[•]=

$V_{\varphi\varphi\lambda}$

In[•]:= **FullSimplify[Vφφφ Routes426]**

Out[•]=

$V_{\varphi\varphi\varphi}$

In[•]:= **FullSimplify[Vφφρ Routes426]**

Out[•]=

$V_{\varphi\varphi\rho}$

In[•]:= **FullSimplify[Vρρλ Routes426]**

Out[•]=

$V_{\rho\rho\lambda}$

In[•]:= **FullSimplify[Vρρφ Routes426]**

Out[•]=

$V_{\rho\rho\varphi}$

In[•]:= **FullSimplify[Vρρρ Routes426]**

Out[•]=

$V_{\rho\rho\rho}$

In[•]:= **NotebookSave[EvaluationNotebook[]];**