Readme

1. This Appendix presents the validation process with Laplace's equation for physical components of second- and third-order potential gradients in Tables A1 - A13.
2. In the following parts, the symbol "*" is omitted for simplicity.
3. Each of the following sections is independent by using the command **Clear["Global`*"]** to clear previous arguments.
4. After running all codes, the related results of Laplace's equation in Appendix B are obtained.

Table A1

```
In[•]:= Clear["Global`*"];

In[•]:= Vrr = (x^2 * Vxx + 2 * x * y * Vxy + y^2 * Vyy) / (x^2 + y^2);

In[•]:= Vλλ = (y^2 * Vxx - 2 * x * y * Vxy + x^2 * Vyy) / (x^2 + y^2);

In[•]:= Vzz = Vzz;

In[•]:= Vrrr = (x^3 * Vxxx + 3 * x^2 * y * Vxxy + 3 * x * y^2 * Vyyx + y^3 * Vyyy) / (x^2 + y^2)^(3/2);

In[•]:= Vrrλ = (-x^2 * y * Vxxx + x * (x^2 - 2 * y^2) * Vxxy + y * (2 * x^2 - y^2) * Vyyx + x * y^2 * Vyyy) /
          (x^2 + y^2)^(3/2);

In[•]:= Vrrz = (x^2 * Vxxz + 2 * x * y * Vxyz + y^2 * Vyyz) / (x^2 + y^2);

In[•]:= Vλλr =
          (x * y^2 * Vxxx + y * (y^2 - 2 * x^2) * Vxxy + x * (x^2 - 2 * y^2) * Vyyx + x^2 * y * Vyyy) / (x^2 + y^2)^(3/2);

In[•]:= Vλλλ = (-y^3 * Vxxx + 3 * x * y^2 * Vxxy - 3 * x^2 * y * Vyyx + x^3 * Vyyy) / (x^2 + y^2)^(3/2);

In[•]:= Vλλz = (y^2 * Vxxz - 2 * x * y * Vxyz + x^2 * Vyyz) / (x^2 + y^2);

In[•]:= Vzzr = (x * Vzzx + y * Vzzy) / Sqrt[x^2 + y^2];

In[•]:= Vzzλ = (-y * Vzzx + x * Vzzy) / Sqrt[x^2 + y^2];

In[•]:= Vzzz = Vzzz;

In[•]:= FullSimplify[Vrr + Vλλ + Vzz]
Out[•]=
Vxx + Vyy + Vzz

In[•]:= FullSimplify[Vrrr + Vλλr + Vzzr]
Out[•]=
(Vxxx + Vyyx + Vzzx) x + (Vxxy + Vyyy + Vzzy) y
-----------------------------------------------
              Sqrt[x^2 + y^2]
```

```
In[•]:= FullSimplify[Vrrλ + Vλλλ + Vzzλ]
```

Out[•]=

$$\frac{(Vxxy + Vyyy + Vzzy)\, x - (Vxxx + Vyyx + Vzzx)\, y}{\sqrt{x^2 + y^2}}$$

```
In[•]:= FullSimplify[Vrrz + Vλλz + Vzzz]
```

Out[•]=

$$Vxxz + Vyyz + Vzzz$$

Table A2

```
In[•]:= Clear["Global`*"];

In[•]:= Vrr = (Cos[λ])^2 * Vxx + Sin[2 * λ] * Vxy + (Sin[λ])^2 * Vyy;

In[•]:= Vλλ = (Sin[λ])^2 * Vxx - Sin[2 * λ] * Vxy + (Cos[λ])^2 * Vyy;

In[•]:= Vzz = Vzz;

In[•]:= Vrrr = (Cos[λ])^3 * Vxxx + 3 * Sin[λ] * (Cos[λ])^2 * Vxxy +
          3 * Cos[λ] * (Sin[λ])^2 * Vyyx + (Sin[λ])^3 * Vyyy;

In[•]:= Vrrλ = -Sin[λ] * (Cos[λ])^2 * Vxxx + Cos[λ] * ((Cos[λ])^2 - 2 * (Sin[λ])^2) * Vxxy +
          Sin[λ] * (2 * (Cos[λ])^2 - (Sin[λ])^2) * Vyyx + (Sin[λ])^2 * Cos[λ] * Vyyy;

In[•]:= Vrrz = (Cos[λ])^2 * Vxxz + Sin[2 * λ] * Vxyz + (Sin[λ])^2 * Vyyz;

In[•]:= Vλλr = Cos[λ] * (Sin[λ])^2 * Vxxx + Sin[λ] * ((Sin[λ])^2 - 2 * (Cos[λ])^2) * Vxxy +
          Cos[λ] * ((Cos[λ])^2 - 2 * (Sin[λ])^2) * Vyyx + Sin[λ] * (Cos[λ])^2 * Vyyy;

In[•]:= Vλλλ = -(Sin[λ])^3 * Vxxx + 3 * (Sin[λ])^2 * Cos[λ] * Vxxy -
          3 * Sin[λ] * (Cos[λ])^2 * Vyyx + (Cos[λ])^3 * Vyyy;

In[•]:= Vλλz = (Sin[λ])^2 * Vxxz - Sin[2 * λ] * Vxyz + (Cos[λ])^2 * Vyyz;

In[•]:= Vzzr = Cos[λ] * Vzzx + Sin[λ] * Vzzy;

In[•]:= Vzzλ = -Sin[λ] * Vzzx + Cos[λ] * Vzzy;

In[•]:= Vzzz = Vzzz;

In[•]:= FullSimplify[Vrr + Vλλ + Vzz]
```

Out[•]=

$$Vxx + Vyy + Vzz$$

```
In[•]:= FullSimplify[Vrrr + Vλλr + Vzzr]
```

Out[•]=

$$(Vxxx + Vyyx + Vzzx)\, Cos[\lambda] + (Vxxy + Vyyy + Vzzy)\, Sin[\lambda]$$

```
In[•]:= FullSimplify[Vrrλ + Vλλλ + Vzzλ]
```

Out[•]=

$$(Vxxy + Vyyy + Vzzy)\, Cos[\lambda] - (Vxxx + Vyyx + Vzzx)\, Sin[\lambda]$$

```
In[•]:= FullSimplify[Vrrz + Vλλz + Vzzz]
```

Out[•]=

$$Vxxz + Vyyz + Vzzz$$

Table A3

```
In[ ]:= Clear["Global`*"];

In[ ]:= Vxx = (x^2 * Vrr - 2 * x * y * Vrλ + y^2 * Vλλ) / (x^2 + y^2);

In[ ]:= Vyy = (y^2 * Vrr + 2 * x * y * Vrλ + x^2 * Vλλ) / (x^2 + y^2);

In[ ]:= Vzz = Vzz;

In[ ]:= Vxxx = (x^3 * Vrrr - 3 * x^2 * y * Vrrλ + 3 * x * y^2 * Vλλr - y^3 * Vλλλ) / (x^2 + y^2)^(3/2);

In[ ]:= Vxxy =
          (x^2 * y * Vrrr + x * (x^2 - 2 * y^2) * Vrrλ + y * (y^2 - 2 * x^2) * Vλλr + x * y^2 * Vλλλ) / (x^2 + y^2)^(3/2);

In[ ]:= Vxxz = (x^2 * Vrrz - 2 * x * y * Vrλz + y^2 * Vλλz) / (x^2 + y^2);

In[ ]:= Vyyx =
          (x * y^2 * Vrrr + y * (2 * x^2 - y^2) * Vrrλ + x * (x^2 - 2 * y^2) * Vλλr - x^2 * y * Vλλλ) / (x^2 + y^2)^(3/2);

In[ ]:= Vyyy = (y^3 * Vrrr + 3 * x * y^2 * Vrrλ + 3 * x^2 * y * Vλλr + x^3 * Vλλλ) / (x^2 + y^2)^(3/2);

In[ ]:= Vyyz = (y^2 * Vrrz + 2 * x * y * Vrλz + x^2 * Vλλz) / (x^2 + y^2);

In[ ]:= Vzzx = (x * Vzzr - y * Vzzλ) / Sqrt[x^2 + y^2];

In[ ]:= Vzzy = (y * Vzzr + x * Vzzλ) / Sqrt[x^2 + y^2];

In[ ]:= Vzzz = Vzzz;

In[ ]:= FullSimplify[Vxx + Vyy + Vzz]
Out[ ]=
        Vrr + Vzz + Vλλ

In[ ]:= FullSimplify[Vxxx + Vyyx + Vzzx]
Out[ ]=
        (Vrrr + Vzzr + Vλλr) x - (Vrrλ + Vzzλ + Vλλλ) y
        ------------------------------------------------
                        Sqrt[x^2 + y^2]

In[ ]:= FullSimplify[Vxxy + Vyyy + Vzzy]
Out[ ]=
        (Vrrλ + Vzzλ + Vλλλ) x + (Vrrr + Vzzr + Vλλr) y
        ------------------------------------------------
                        Sqrt[x^2 + y^2]

In[ ]:= FullSimplify[Vxxz + Vyyz + Vzzz]
Out[ ]=
        Vrrz + Vzzz + Vλλz
```

Table A4

```
In[ ]:= Clear["Global`*"];

In[ ]:= Vxx = (Cos[λ])^2 * Vrr - Sin[2 * λ] * Vrλ + (Sin[λ])^2 * Vλλ;
```

```
In[•]:= Vyy = (Sin[λ])^2 * Vrr + Sin[2 * λ] * Vrλ + (Cos[λ])^2 * Vλλ;

In[•]:= Vzz = Vzz;

In[•]:= Vxxx = (Cos[λ])^3 * Vrrr - 3 * Sin[λ] * (Cos[λ])^2 * Vrrλ +
          3 * (Sin[λ])^2 * Cos[λ] * Vλλr - (Sin[λ])^3 * Vλλλ;

In[•]:= Vxxy = Sin[λ] * (Cos[λ])^2 * Vrrr + Cos[λ] * ((Cos[λ])^2 - 2 * (Sin[λ])^2) * Vrrλ +
          Sin[λ] * ((Sin[λ])^2 - 2 * (Cos[λ])^2) * Vλλr + (Sin[λ])^2 * Cos[λ] * Vλλλ;

In[•]:= Vxxz = (Cos[λ])^2 * Vrrz - Sin[2 * λ] * Vrλz + (Sin[λ])^2 * Vλλz;

In[•]:= Vyyx = (Sin[λ])^2 * Cos[λ] * Vrrr + Sin[λ] * (2 * (Cos[λ])^2 - (Sin[λ])^2) * Vrrλ +
          Cos[λ] * ((Cos[λ])^2 - 2 * (Sin[λ])^2) * Vλλr - Sin[λ] * (Cos[λ])^2 * Vλλλ;

In[•]:= Vyyy = (Sin[λ])^3 * Vrrr + 3 * (Sin[λ])^2 * Cos[λ] * Vrrλ +
          3 * Sin[λ] * (Cos[λ])^2 * Vλλr + (Cos[λ])^3 * Vλλλ;

In[•]:= Vyyz = (Sin[λ])^2 * Vrrz + Sin[2 * λ] * Vrλz + (Cos[λ])^2 * Vλλz;

In[•]:= Vzzx = Cos[λ] * Vzzr - Sin[λ] * Vzzλ;

In[•]:= Vzzy = Sin[λ] * Vzzr + Cos[λ] * Vzzλ;

In[•]:= Vzzz = Vzzz;

In[•]:= FullSimplify[Vxx + Vyy + Vzz]
Out[•]=
        Vrr + Vzz + Vλλ

In[•]:= FullSimplify[Vxxx + Vyyx + Vzzx]
Out[•]=
        (Vrrr + Vzzr + Vλλr) Cos[λ] - (Vrrλ + Vzzλ + Vλλλ) Sin[λ]

In[•]:= FullSimplify[Vxxy + Vyyy + Vzzy]
Out[•]=
        (Vrrλ + Vzzλ + Vλλλ) Cos[λ] + (Vrrr + Vzzr + Vλλr) Sin[λ]

In[•]:= FullSimplify[Vxxz + Vyyz + Vzzz]
Out[•]=
        Vrrz + Vzzz + Vλλz
```

Table A5

```
In[•]:= Clear["Global`*"];

In[•]:= Vλλ = Vλλ;

In[•]:= Vφφ = (z^2 * Vrr - 2 * r * z * Vrz + r^2 * Vzz) / (r^2 + z^2);

In[•]:= Vρρ = (r^2 * Vrr + 2 * r * z * Vrz + z^2 * Vzz) / (r^2 + z^2);

In[•]:= Vλλλ = Vλλλ;

In[•]:= Vλλφ = (-z * Vλλr + r * Vλλz) / Sqrt[r^2 + z^2];

In[•]:= Vλλρ = (r * Vλλr + z * Vλλz) / Sqrt[r^2 + z^2];
```

```
In[•]:= Vφφλ = (z^2 * Vrrλ - 2 * r * z * Vrλz + r^2 * Vzzλ) / (r^2 + z^2);

In[•]:= Vφφφ = (-z^3 * Vrrr + 3 * r * z^2 * Vrrz - 3 * r^2 * z * Vzzr + r^3 * Vzzz) / (r^2 + z^2)^(3/2);

In[•]:= Vφφρ =
          (r * z^2 * Vrrr + z * (z^2 - 2 * r^2) * Vrrz + r * (r^2 - 2 * z^2) * Vzzr + r^2 * z * Vzzz) / (r^2 + z^2)^(3/2);

In[•]:= Vρρλ = (r^2 * Vrrλ + 2 * r * z * Vrλz + z^2 * Vzzλ) / (r^2 + z^2);

In[•]:= Vρρφ = (-r^2 * z * Vrrr + r * (r^2 - 2 * z^2) * Vrrz + z * (2 * r^2 - z^2) * Vzzr + r * z^2 * Vzzz) /
          (r^2 + z^2)^(3/2);

In[•]:= Vρρρ = (r^3 Vrrr + 3 * r^2 * z * Vrrz + 3 * r * z^2 * Vzzr + z^3 * Vzzz) / (r^2 + z^2)^(3/2);

In[•]:= FullSimplify[Vλλ + Vφφ + Vρρ]
Out[•]=
          Vrr + Vzz + Vλλ

In[•]:= FullSimplify[Vλλλ + Vφφλ + Vρρλ]
Out[•]=
          Vrrλ + Vzzλ + Vλλλ

In[•]:= FullSimplify[Vλλφ + Vφφφ + Vρρφ]
Out[•]=
          r (Vrrz + Vzzz + Vλλz) - (Vrrr + Vzzr + Vλλr) z
          ------------------------------------------------
                          √(r^2 + z^2)

In[•]:= FullSimplify[Vλλρ + Vφφρ + Vρρρ]
Out[•]=
          r (Vrrr + Vzzr + Vλλr) + (Vrrz + Vzzz + Vλλz) z
          ------------------------------------------------
                          √(r^2 + z^2)
```

Table A6

```
In[•]:= Clear["Global`*"];

In[•]:= Vλλ = Vλλ;

In[•]:= Vφφ = ((Sin[φ])^2 * Vrr - Sin[2 * φ] * Vrz + (Cos[φ])^2 * Vzz);

In[•]:= Vρρ = ((Cos[φ])^2 * Vrr + Sin[2 * φ] * Vrz + (Sin[φ])^2 * Vzz);

In[•]:= Vλλλ = Vλλλ;

In[•]:= Vλλφ = -Sin[φ] * Vλλr + Cos[φ] * Vλλz;

In[•]:= Vλλρ = Cos[φ] * Vλλr + Sin[φ] * Vλλz;

In[•]:= Vφφλ = (Sin[φ])^2 * Vrrλ - Sin[2 * φ] * Vrλz + (Cos[φ])^2 * Vzzλ;

In[•]:= Vφφφ = -(Sin[φ])^3 * Vrrr + 3 * (Sin[φ])^2 * Cos[φ] * Vrrz -
          3 * Sin[φ] * (Cos[φ])^2 * Vzzr + (Cos[φ])^3 * Vzzz;

In[•]:= Vφφρ = (Sin[φ])^2 * Cos[φ] * Vrrr + Sin[φ] * ((Sin[φ])^2 - 2 * (Cos[φ])^2) * Vrrz +
          Cos[φ] * ((Cos[φ])^2 - 2 * (Sin[φ])^2) * Vzzr + Sin[φ] * (Cos[φ])^2 * Vzzz;
```

```
In[•]:= Vρρλ = (Cos[φ])^2 * Vrrλ + Sin[2 * φ] * Vrλz + (Sin[φ])^2 * Vzzλ;

In[•]:= Vρρφ = -Sin[φ] * (Cos[φ])^2 * Vrrr + Cos[φ] * ((Cos[φ])^2 - 2 * (Sin[φ])^2) * Vrrz +
          Sin[φ] * (2 * (Cos[φ])^2 - (Sin[φ])^2) * Vzzr + Cos[φ] * (Sin[φ])^2 * Vzzz;

In[•]:= Vρρρ = (Cos[φ])^3 * Vrrr + 3 * Sin[φ] * (Cos[φ])^2 * Vrrz +
          3 * Cos[φ] * (Sin[φ])^2 * Vzzr + (Sin[φ])^3 * Vzzz;

In[•]:= FullSimplify[Vλλ + Vφφ + Vρρ]
Out[•]=
        Vrr + Vzz + Vλλ

In[•]:= FullSimplify[Vλλλ + Vφφλ + Vρρλ]
Out[•]=
        Vrrλ + Vzzλ + Vλλλ

In[•]:= FullSimplify[Vλλφ + Vφφφ + Vρρφ]
Out[•]=
        (Vrrz + Vzzz + Vλλz) Cos[φ] - (Vrrr + Vzzr + Vλλr) Sin[φ]

In[•]:= FullSimplify[Vλλρ + Vφφρ + Vρρρ]
Out[•]=
        (Vrrr + Vzzr + Vλλr) Cos[φ] + (Vrrz + Vzzz + Vλλz) Sin[φ]
```

Table A7

```
In[•]:= Clear["Global`*"];

In[•]:= Vrr = (z^2 * Vφφ - 2 * r * z * Vφρ + r^2 * Vρρ) / (r^2 + z^2);

In[•]:= Vλλ = Vλλ;

In[•]:= Vzz = (r^2 * Vφφ + 2 * r * z * Vφρ + z^2 * Vρρ) / (r^2 + z^2);

In[•]:= Vrrr = (-z^3 * Vφφφ + 3 * r * z^2 * Vφφρ - 3 * r^2 * z * Vρρφ + r^3 * Vρρρ) / (r^2 + z^2)^(3/2);

In[•]:= Vrrλ = (-2 * r * z * Vλφρ + z^2 * Vφφλ + r^2 * Vρρλ) / (r^2 + z^2);

In[•]:= Vrrz =
          (r * z^2 * Vφφφ + z * (z^2 - 2 * r^2) * Vφφρ + r * (r^2 - 2 * z^2) * Vρρφ + r^2 * z * Vρρρ) / (r^2 + z^2)^(3/2);

In[•]:= Vλλr = (-z * Vλλφ + r * Vλλρ) / Sqrt[r^2 + z^2];

In[•]:= Vλλλ = Vλλλ;

In[•]:= Vλλz = (r * Vλλφ + z * Vλλρ) / Sqrt[r^2 + z^2];

In[•]:= Vzzr = (-r^2 * z * Vφφφ + r * (r^2 - 2 * z^2) * Vφφρ + z * (2 * r^2 - z^2) * Vρρφ + r * z^2 * Vρρρ) /
          (r^2 + z^2)^(3/2);

In[•]:= Vzzλ = (2 * r * z * Vλφρ + r^2 * Vφφλ + z^2 * Vρρλ) / (r^2 + z^2);

In[•]:= Vzzz = (r^3 * Vφφφ + 3 * r^2 * z * Vφφρ + 3 * r * z^2 * Vρρφ + z^3 * Vρρρ) / (r^2 + z^2)^(3/2);
```

```
In[•]:= FullSimplify[Vrr + Vλλ + Vzz]
Out[•]=
Vλλ + Vρρ + Vφφ

In[•]:= FullSimplify[Vrrr + Vλλr + Vzzr]
Out[•]=
r (Vλλρ + Vρρρ + Vφφρ) − (Vλλφ + Vρρφ + Vφφφ) z
√(r^2 + z^2)

In[•]:= FullSimplify[Vrrλ + Vλλλ + Vzzλ]
Out[•]=
Vλλλ + Vρρλ + Vφφλ

In[•]:= FullSimplify[Vrrz + Vλλz + Vzzz]
Out[•]=
r (Vλλφ + Vρρφ + Vφφφ) + (Vλλρ + Vρρρ + Vφφρ) z
√(r^2 + z^2)
```

Table A8

```
In[•]:= Clear["Global`*"];

In[•]:= Vrr = (Sin[φ])^2 * Vφφ - Sin[2 * φ] * Vφρ + (Cos[φ])^2 * Vρρ;

In[•]:= Vλλ = Vλλ;

In[•]:= Vzz = (Cos[φ])^2 * Vφφ + Sin[2 * φ] * Vφρ + (Sin[φ])^2 * Vρρ;

In[•]:= Vrrr = -(Sin[φ])^3 * Vφφφ + 3 * (Sin[φ])^2 * Cos[φ] * Vφφρ -
    3 * Sin[φ] * (Cos[φ])^2 * Vρρφ + (Cos[φ])^3 * Vρρρ;

In[•]:= Vrrλ = (Sin[φ])^2 * Vφφλ - Sin[2 * φ] * Vλφρ + (Cos[φ])^2 * Vρρλ;

In[•]:= Vrrz = (Sin[φ])^2 * Cos[φ] * Vφφφ + Sin[φ] * ((Sin[φ])^2 - 2 * (Cos[φ])^2) * Vφφρ +
    Cos[φ] * ((Cos[φ])^2 - 2 * (Sin[φ])^2) * Vρρφ + Sin[φ] * (Cos[φ])^2 * Vρρρ;

In[•]:= Vλλr = -Sin[φ] * Vλλφ + Cos[φ] * Vλλρ;

In[•]:= Vλλλ = Vλλλ;

In[•]:= Vλλz = Cos[φ] * Vλλφ + Sin[φ] * Vλλρ;

In[•]:= Vzzr = -Sin[φ] * (Cos[φ])^2 * Vφφφ + Cos[φ] * ((Cos[φ])^2 - 2 * (Sin[φ])^2) * Vφφρ +
    Sin[φ] * (2 * (Cos[φ])^2 - (Sin[φ])^2) * Vρρφ + (Sin[φ])^2 * Cos[φ] * Vρρρ;

In[•]:= Vzzλ = Sin[2 * φ] * Vλφρ + (Cos[φ])^2 * Vφφλ + (Sin[φ])^2 * Vρρλ;

In[•]:= Vzzz = (Cos[φ])^3 * Vφφφ + 3 * Sin[φ] * (Cos[φ])^2 * Vφφρ +
    3 * (Sin[φ])^2 * Cos[φ] * Vρρφ + (Sin[φ])^3 * Vρρρ;

In[•]:= FullSimplify[Vrr + Vλλ + Vzz]
Out[•]=
Vλλ + Vρρ + Vφφ
```

```
In[•]:= FullSimplify[Vrrr + Vλλr + Vzzr]
Out[•]=
(Vλλρ + Vρρρ + Vφφρ) Cos[φ] - (Vλλφ + Vρρφ + Vφφφ) Sin[φ]

In[•]:= FullSimplify[Vrrλ + Vλλλ + Vzzλ]
Out[•]=
Vλλλ + Vρρλ + Vφφλ

In[•]:= FullSimplify[Vrrz + Vλλz + Vzzz]
Out[•]=
(Vλλφ + Vρρφ + Vφφφ) Cos[φ] + (Vλλρ + Vρρρ + Vφφρ) Sin[φ]
```

Tables A9 and A10

```
In[•]:= Clear["Global`*"];

In[•]:= Vxx = (y^2 * (x^2 + y^2 + z^2) * Vλλ + 2 * x * y * z * Sqrt[x^2 + y^2 + z^2] * Vλφ -
          2 * x * y * Sqrt[x^2 + y^2] * Sqrt[x^2 + y^2 + z^2] * Vλρ + x^2 * z^2 * Vφφ -
          2 * x^2 * z * Sqrt[x^2 + y^2] * Vφρ + x^2 * (x^2 + y^2) * Vρρ) / ((x^2 + y^2) * (x^2 + y^2 + z^2));

In[•]:= Vyy = (x^2 * (x^2 + y^2 + z^2) * Vλλ - 2 * x * y * z * Sqrt[x^2 + y^2 + z^2] * Vλφ +
          2 * x * y * Sqrt[x^2 + y^2] * Sqrt[x^2 + y^2 + z^2] * Vλρ + y^2 * z^2 * Vφφ -
          2 * y^2 * z * Sqrt[x^2 + y^2] * Vφρ + y^2 * (x^2 + y^2) * Vρρ) / ((x^2 + y^2) * (x^2 + y^2 + z^2));

In[•]:= Vzz = ((x^2 + y^2) * Vφφ + 2 * z * Sqrt[x^2 + y^2] * Vφρ + z^2 * Vρρ) / (x^2 + y^2 + z^2);

In[•]:= Vxxx =
          (-y^3 * (x^2 + y^2 + z^2)^(3/2) * Vλλλ - 3 * x * y^2 * z * (x^2 + y^2 + z^2) * Vλλφ + 3 * x * y^2 * Sqrt[x^2 + y^2] *
            (x^2 + y^2 + z^2) * Vλλρ + 6 * x^2 * y * z Sqrt[x^2 + y^2] * Sqrt[x^2 + y^2 + z^2] * Vλφρ -
          3 * x^2 * y * z^2 * Sqrt[x^2 + y^2 + z^2] * Vφφλ - x^3 * z^3 * Vφφφ + 3 * x^3 * z^2 * Sqrt[x^2 + y^2] * Vφφρ -
          3 * x^2 * y * (x^2 + y^2) * Sqrt[x^2 + y^2 + z^2] * Vρρλ - 3 * x^3 * z * (x^2 + y^2) * Vρρφ +
          x^3 * (x^2 + y^2)^(3/2) * Vρρρ) / ((x^2 + y^2)^(3/2) * (x^2 + y^2 + z^2)^(3/2));

In[•]:= Vxxy = (x * y^2 * (x^2 + y^2 + z^2)^(3/2) * Vλλλ + y * z * (2 * x^2 - y^2) * (x^2 + y^2 + z^2) * Vλλφ -
          y * Sqrt[x^2 + y^2] * (2 * x^2 - y^2) * (x^2 + y^2 + z^2) * Vλλρ -
          2 * x * z * (x^2 - 2 * y^2) * Sqrt[x^2 + y^2] * Sqrt[x^2 + y^2 + z^2] * Vλφρ + x * z^2 * (x^2 - 2 * y^2) *
            Sqrt[x^2 + y^2 + z^2] * Vφφλ - x^2 * y * z^3 * Vφφφ + 3 * x^2 * y * z^2 * Sqrt[x^2 + y^2] * Vφφρ +
          x * (x^4 - x^2 * y^2 - 2 * y^4) * Sqrt[x^2 + y^2 + z^2] * Vρρλ - 3 * x^2 * y * z * (x^2 + y^2) * Vρρφ +
          x^2 * y * (x^2 + y^2)^(3/2) * Vρρρ) / ((x^2 + y^2)^(3/2) * (x^2 + y^2 + z^2)^(3/2));
```

In[ ]:=
$$Vxxz = \left(y^2 * \sqrt{x^2+y^2} * \left(x^2+y^2+z^2\right) * V\lambda\lambda\varphi + y^2 * z * \left(x^2+y^2+z^2\right) * V\lambda\lambda\rho - 2 * x * y * \left(x^2+y^2-z^2\right) * \sqrt{x^2+y^2+z^2} * V\lambda\varphi\rho + 2 * x * y * z * \sqrt{x^2+y^2} * \sqrt{x^2+y^2+z^2} * V\varphi\varphi\lambda + x^2 * z^2 * \sqrt{x^2+y^2} * V\varphi\varphi\varphi - x^2 * z * \left(2 * x^2 + 2 * y^2 - z^2\right) * V\varphi\varphi\rho - 2 * x * y * z * \sqrt{x^2+y^2} * \sqrt{x^2+y^2+z^2} * V\rho\rho\lambda + x^2 * \sqrt{x^2+y^2} * \left(x^2+y^2-2 * z^2\right) * V\rho\rho\varphi + x^2 * z * \left(x^2+y^2\right) * V\rho\rho\rho\right) \Big/ \left(\left(x^2+y^2\right) * \left(x^2+y^2+z^2\right)^{3/2}\right);$$

In[ ]:=
$$Vyyx = \left(-x^2 * y * \left(x^2+y^2+z^2\right)^{3/2} * V\lambda\lambda\lambda - x * z * \left(x^2-2 * y^2\right) * \left(x^2+y^2+z^2\right) * V\lambda\lambda\varphi + x * \left(x^2-2 * y^2\right) * \sqrt{x^2+y^2} * \left(x^2+y^2+z^2\right) * V\lambda\lambda\rho - 2 * y * z * \left(2 * x^2 - y^2\right) * \sqrt{x^2+y^2} * \sqrt{x^2+y^2+z^2} * V\lambda\varphi\rho + y * z^2 * \left(2 * x^2 - y^2\right) * \sqrt{x^2+y^2+z^2} * V\varphi\varphi\lambda - x * y^2 * z^3 * V\varphi\varphi\varphi + 3 * x * y^2 * z^2 * \sqrt{x^2+y^2} * V\varphi\varphi\rho + y * \left(2 * x^4 + x^2 * y^2 - y^4\right) * \sqrt{x^2+y^2+z^2} * V\rho\rho\lambda - 3 * x * y^2 * z * \left(x^2+y^2\right) * V\rho\rho\varphi + x * y^2 * \left(x^2+y^2\right)^{3/2} * V\rho\rho\rho\right) \Big/ \left(\left(x^2+y^2\right)^{3/2} * \left(x^2+y^2+z^2\right)^{3/2}\right);$$

In[ ]:=
$$Vyyy = \left(x^3 * \left(x^2+y^2+z^2\right)^{3/2} * V\lambda\lambda\lambda - 3 * x^2 * y * z * \left(x^2+y^2+z^2\right) * V\lambda\lambda\varphi + 3 * x^2 * y * \sqrt{x^2+y^2} * \left(x^2+y^2+z^2\right) * V\lambda\lambda\rho - 6 * x * y^2 * z * \sqrt{x^2+y^2} * \sqrt{x^2+y^2+z^2} * V\lambda\varphi\rho + 3 * x * y^2 * z^2 * \sqrt{x^2+y^2+z^2} * V\varphi\varphi\lambda - y^3 * z^3 * V\varphi\varphi\varphi + 3 * y^3 * z^2 * \sqrt{x^2+y^2} * V\varphi\varphi\rho + 3 * x * y^2 * \left(x^2+y^2\right) * \sqrt{x^2+y^2+z^2} * V\rho\rho\lambda - 3 * y^3 * z * \left(x^2+y^2\right) * V\rho\rho\varphi + y^3 * \left(x^2+y^2\right)^{3/2} * V\rho\rho\rho\right) \Big/ \left(\left(x^2+y^2\right)^{3/2} * \left(x^2+y^2+z^2\right)^{3/2}\right);$$

In[ ]:=
$$Vyyz = \left(x^2 * \sqrt{x^2+y^2} * \left(x^2+y^2+z^2\right) * V\lambda\lambda\varphi + x^2 * z * \left(x^2+y^2+z^2\right) * V\lambda\lambda\rho + 2 * x * y * \left(x^2+y^2-z^2\right) * \sqrt{x^2+y^2+z^2} * V\lambda\varphi\rho - 2 * x * y * z * \sqrt{x^2+y^2} * \sqrt{x^2+y^2+z^2} * V\varphi\varphi\lambda + y^2 * z^2 * \sqrt{x^2+y^2} * V\varphi\varphi\varphi - y^2 * z * \left(2 * x^2 + 2 * y^2 - z^2\right) * V\varphi\varphi\rho + 2 * x * y * z * \sqrt{x^2+y^2} * \sqrt{x^2+y^2+z^2} * V\rho\rho\lambda + y^2 * \left(x^2+y^2-2 * z^2\right) * \sqrt{x^2+y^2} * V\rho\rho\varphi + y^2 * z * \left(x^2+y^2\right) * V\rho\rho\rho\right) \Big/ \left(\left(x^2+y^2\right) * \left(x^2+y^2+z^2\right)^{3/2}\right);$$

In[ ]:=
$$Vzzx = \left(-2 * y * z * \sqrt{x^2+y^2} * \sqrt{x^2+y^2+z^2} * V\lambda\varphi\rho - y * \left(x^2+y^2\right) * \sqrt{x^2+y^2+z^2} * V\varphi\varphi\lambda - x * z * \left(x^2+y^2\right) * V\varphi\varphi\varphi + x * \left(x^2+y^2-2 * z^2\right) * \sqrt{x^2+y^2} * V\varphi\varphi\rho - y * z^2 * \sqrt{x^2+y^2+z^2} * V\rho\rho\lambda + x * z * \left(2 * x^2 + 2 * y^2 - z^2\right) * V\rho\rho\varphi + x * z^2 * \sqrt{x^2+y^2} * V\rho\rho\rho\right) \Big/ \left(\sqrt{x^2+y^2} * \left(x^2+y^2+z^2\right)^{3/2}\right);$$

In[ ]:=
$$Vzzy = \left(2 * x * z * \sqrt{x^2+y^2} * \sqrt{x^2+y^2+z^2} * V\lambda\varphi\rho + x * \left(x^2+y^2\right) * \sqrt{x^2+y^2+z^2} * V\varphi\varphi\lambda - y * z * \left(x^2+y^2\right) * V\varphi\varphi\varphi + y * \left(x^2+y^2-2 * z^2\right) * \sqrt{x^2+y^2} * V\varphi\varphi\rho + x * z^2 * \sqrt{x^2+y^2+z^2} * V\rho\rho\lambda + y * z * \left(2 * x^2 + 2 * y^2 - z^2\right) * V\rho\rho\varphi + y * z^2 * \sqrt{x^2+y^2} * V\rho\rho\rho\right) \Big/ \left(\sqrt{x^2+y^2} * \left(x^2+y^2+z^2\right)^{3/2}\right);$$

```
In[•]:= Vzzz = ((x^2 + y^2)^(3/2) * Vφφφ + 3 * z * (x^2 + y^2) * Vφφρ + 3 * z^2 * Sqrt[x^2 + y^2] * Vρρφ + z^3 * Vρρρ) /
          (x^2 + y^2 + z^2)^(3/2);

In[•]:= FullSimplify[Vxx + Vyy + Vzz]
Out[•]=
        Vλλ + Vρρ + Vφφ

In[•]:= FullSimplify[Vxxx + Vyyx + Vzzx]
Out[•]=
        1/(Sqrt[x^2 + y^2] Sqrt[x^2 + y^2 + z^2]) ((Vλλρ + Vρρρ + Vφφρ) x Sqrt[x^2 + y^2] -
          (Vλλφ + Vρρφ + Vφφφ) x z - (Vλλλ + Vρρλ + Vφφλ) y Sqrt[x^2 + y^2 + z^2])

In[•]:= FullSimplify[Vxxy + Vyyy + Vzzy]
Out[•]=
        1/(Sqrt[x^2 + y^2] Sqrt[x^2 + y^2 + z^2]) ((Vλλρ + Vρρρ + Vφφρ) y Sqrt[x^2 + y^2] -
          (Vλλφ + Vρρφ + Vφφφ) y z + (Vλλλ + Vρρλ + Vφφλ) x Sqrt[x^2 + y^2 + z^2])

In[•]:= FullSimplify[Vxxz + Vyyz + Vzzz]
Out[•]=
        ((Vλλφ + Vρρφ + Vφφφ) Sqrt[x^2 + y^2] + (Vλλρ + Vρρρ + Vφφρ) z) / Sqrt[x^2 + y^2 + z^2]
```

Table A11

```
In[•]:= Clear["Global`*"];

In[•]:= Vxx = (Sin[λ])^2 * Vλλ + Sin[φ] * Sin[2 * λ] * Vλφ -
          Cos[φ] * Sin[2 * λ] * Vλρ + (Sin[φ])^2 * (Cos[λ])^2 * Vφφ -
          Sin[2 * φ] * (Cos[λ])^2 * Vφρ + (Cos[φ])^2 * (Cos[λ])^2 * Vρρ;

In[•]:= Vyy = (Cos[λ])^2 * Vλλ - Sin[φ] * Sin[2 * λ] * Vλφ +
          Cos[φ] * Sin[2 * λ] * Vλρ + (Sin[φ])^2 * (Sin[λ])^2 * Vφφ -
          Sin[2 * φ] * (Sin[λ])^2 * Vφρ + (Cos[φ])^2 * (Sin[λ])^2 * Vρρ;

In[•]:= Vzz = (Cos[φ])^2 * Vφφ + Sin[2 * φ] * Vφρ + (Sin[φ])^2 * Vρρ;

In[•]:= Vxxx = -(Sin[λ])^3 * Vλλλ + 3 * Cos[φ] * (Sin[λ])^2 * Cos[λ] * Vλλρ -
          3 * Sin[φ] * (Sin[λ])^2 * Cos[λ] * Vλλφ + 3 * Sin[2 * φ] * Sin[λ] * (Cos[λ])^2 * Vλφρ -
          3 * (Sin[φ])^2 * Sin[λ] * (Cos[λ])^2 * Vφφλ - (Sin[φ])^3 * (Cos[λ])^3 * Vφφφ +
          3 * (Sin[φ])^2 * Cos[φ] * (Cos[λ])^3 * Vφφρ - 3 * (Cos[φ])^2 * Sin[λ] * (Cos[λ])^2 * Vρρλ -
          3 * Sin[φ] * (Cos[φ])^2 * (Cos[λ])^3 * Vρρφ + (Cos[φ])^3 * (Cos[λ])^3 * Vρρρ;
```

```
In[•]:= Vxxy = (Sin[λ])^2 * Cos[λ] * Vλλλ + Sin[φ] * Sin[λ] * (2 * (Cos[λ])^2 - (Sin[λ])^2) * Vλλφ +
    Cos[φ] * Sin[λ] * ((Sin[λ])^2 - 2 * (Cos[λ])^2) * Vλλρ +
    Sin[2 * φ] * Cos[λ] * (2 * (Sin[λ])^2 - (Cos[λ])^2) * Vλφρ +
    (Sin[φ])^2 * Cos[λ] * ((Cos[λ])^2 - 2 * (Sin[λ])^2) * Vφφλ - (Sin[φ])^3 * Sin[λ] *
     (Cos[λ])^2 * Vφφφ + 3 * (Sin[φ])^2 * Cos[φ] * Sin[λ] * (Cos[λ])^2 * Vφφρ +
    (Cos[φ])^2 * Cos[λ] * ((Cos[λ])^2 - 2 * (Sin[λ])^2) * Vρρλ - 3 * Sin[φ] *
     (Cos[φ])^2 * Sin[λ] * (Cos[λ])^2 * Vρρφ + (Cos[φ])^3 * Sin[λ] * (Cos[λ])^2 * Vρρρ;

In[•]:= Vxxz =
   Cos[φ] * (Sin[λ])^2 * Vλλφ + Sin[φ] * (Sin[λ])^2 * Vλλρ - Cos[2 * φ] * Sin[2 * λ] * Vλφρ +
    Sin[φ] * Cos[φ] * Sin[2 * λ] * Vφφλ + (Sin[φ])^2 * Cos[φ] * (Cos[λ])^2 * Vφφφ +
    Sin[φ] * (Cos[λ])^2 * ((Sin[φ])^2 - 2 * (Cos[φ])^2) * Vφφρ -
    Sin[2 * φ] * Sin[λ] * Cos[λ] * Vρρλ + Cos[φ] * (Cos[λ])^2 *
     ((Cos[φ])^2 - 2 * (Sin[φ])^2) * Vρρφ + Sin[φ] * (Cos[φ])^2 * (Cos[λ])^2 * Vρρρ;

In[•]:= Vyyx = -Sin[λ] * (Cos[λ])^2 * Vλλλ - Sin[φ] * Cos[λ] * ((Cos[λ])^2 - 2 * (Sin[λ])^2) * Vλλφ +
    Cos[φ] * Cos[λ] * ((Cos[λ])^2 - 2 * (Sin[λ])^2) * Vλλρ -
    Sin[2 * φ] * Sin[λ] * (2 * (Cos[λ])^2 - (Sin[λ])^2) * Vλφρ +
    (Sin[φ])^2 * Sin[λ] * (2 * (Cos[λ])^2 - (Sin[λ])^2) * Vφφλ - (Sin[φ])^3 *
     (Sin[λ])^2 * Cos[λ] * Vφφφ + 3 * (Sin[φ])^2 * Cos[φ] * (Sin[λ])^2 * Cos[λ] * Vφφρ +
    (Cos[φ])^2 * Sin[λ] * (2 * (Cos[λ])^2 - (Sin[λ])^2) * Vρρλ - 3 * Sin[φ] *
     (Cos[φ])^2 * (Sin[λ])^2 * Cos[λ] * Vρρφ + (Cos[φ])^3 * (Sin[λ])^2 * Cos[λ] * Vρρρ;

In[•]:= Vyyy = (Cos[λ])^3 * Vλλλ - 3 * Sin[φ] * Sin[λ] * (Cos[λ])^2 * Vλλφ +
    3 * Cos[φ] * Sin[λ] * (Cos[λ])^2 * Vλλρ - 3 * Sin[2 * φ] * (Sin[λ])^2 * Cos[λ] * Vλφρ +
    3 * (Sin[φ])^2 * (Sin[λ])^2 * Cos[λ] * Vφφλ - (Sin[φ])^3 * (Sin[λ])^3 * Vφφφ +
    3 * (Sin[φ])^2 * Cos[φ] * (Sin[λ])^3 * Vφφρ + 3 * (Cos[φ])^2 * (Sin[λ])^2 * Cos[λ] * Vρρλ -
    3 * Sin[φ] * (Cos[φ])^2 * (Sin[λ])^3 * Vρρφ + (Cos[φ])^3 * (Sin[λ])^3 * Vρρρ;

In[•]:= Vyyz =
   Cos[φ] * (Cos[λ])^2 * Vλλφ + Sin[φ] * (Cos[λ])^2 * Vλλρ + Cos[2 * φ] * Sin[2 * λ] * Vλφρ -
    Sin[2 * φ] * Sin[λ] * Cos[λ] * Vφφλ + (Sin[φ])^2 * Cos[φ] * (Sin[λ])^2 * Vφφφ +
    Sin[φ] * (Sin[λ])^2 * ((Sin[φ])^2 - 2 * (Cos[φ])^2) * Vφφρ +
    Sin[2 * φ] * Sin[λ] * Cos[λ] * Vρρλ + Cos[φ] * (Sin[λ])^2 *
     ((Cos[φ])^2 - 2 * (Sin[φ])^2) * Vρρφ + Sin[φ] * (Cos[φ])^2 * (Sin[λ])^2 * Vρρρ;

In[•]:= Vzzx = -Sin[2 * φ] * Sin[λ] * Vλφρ -
    (Cos[φ])^2 * Sin[λ] * Vφφλ - Sin[φ] * (Cos[φ])^2 * Cos[λ] * Vφφφ +
    Cos[φ] * Cos[λ] * ((Cos[φ])^2 - 2 * (Sin[φ])^2) * Vφφρ - (Sin[φ])^2 * Sin[λ] * Vρρλ +
    Sin[φ] * Cos[λ] * (2 * (Cos[φ])^2 - (Sin[φ])^2) * Vρρφ +
    (Sin[φ])^2 * Cos[φ] * Cos[λ] * Vρρρ;

In[•]:= Vzzy = Sin[2 * φ] * Cos[λ] * Vλφρ +
    (Cos[φ])^2 * Cos[λ] * Vφφλ - Sin[φ] * (Cos[φ])^2 * Sin[λ] * Vφφφ +
    Cos[φ] * Sin[λ] * ((Cos[φ])^2 - 2 * (Sin[φ])^2) * Vφφρ + (Sin[φ])^2 * Cos[λ] * Vρρλ +
    Sin[φ] * Sin[λ] * (2 * (Cos[φ])^2 - (Sin[φ])^2) * Vρρφ +
    (Sin[φ])^2 * Cos[φ] * Sin[λ] * Vρρρ;
```

```
In[•]:= Vzzz = (Cos[φ])^3 * Vφφφ + 3 * Sin[φ] * (Cos[φ])^2 * Vφφρ +
          3 * (Sin[φ])^2 * Cos[φ] * Vρρφ + (Sin[φ])^3 * Vρρρ;

In[•]:= FullSimplify[Vxx + Vyy + Vzz]
Out[•]=
        Vλλ + Vρρ + Vφφ

In[•]:= FullSimplify[Vxxx + Vyyx + Vzzx]
Out[•]=
        -((Vλλλ + Vρρλ + Vφφλ) Sin[λ]) +
         Cos[λ] ((Vλλρ + Vρρρ + Vφφρ) Cos[φ] - (Vλλφ + Vρρφ + Vφφφ) Sin[φ])

In[•]:= FullSimplify[Vxxy + Vyyy + Vzzy]
Out[•]=
        (Vλλλ + Vρρλ + Vφφλ) Cos[λ] +
         Sin[λ] ((Vλλρ + Vρρρ + Vφφρ) Cos[φ] - (Vλλφ + Vρρφ + Vφφφ) Sin[φ])

In[•]:= FullSimplify[Vxxz + Vyyz + Vzzz]
Out[•]=
        (Vλλφ + Vρρφ + Vφφφ) Cos[φ] + (Vλλρ + Vρρρ + Vφφρ) Sin[φ]
```

Table A12

```
In[•]:= Clear["Global`*"];

In[•]:= Vλλ = (y^2 * Vxx - 2 * x * y * Vxy + x^2 * Vyy) / (x^2 + y^2);

In[•]:= Vφφ = (x^2 * z^2 * Vxx + 2 * x * y * z^2 * Vxy - 2 * x * z * (x^2 + y^2) * Vxz + y^2 * z^2 * Vyy -
          2 * y * z * (x^2 + y^2) * Vyz + (x^2 + y^2)^2 * Vzz) / ((x^2 + y^2) * (x^2 + y^2 + z^2));

In[•]:= Vρρ =
          (x^2 * Vxx + 2 * x * y * Vxy + 2 * x * z * Vxz + 2 * y * z * Vyz + y^2 * Vyy + z^2 * Vzz) / (x^2 + y^2 + z^2);

In[•]:= Vλλλ = (-y^3 * Vxxx + 3 * x * y^2 * Vxxy - 3 * x^2 * y * Vyyx + x^3 * Vyyy) / (x^2 + y^2)^(3/2);

In[•]:= Vλλφ = (-x * y^2 * z * Vxxx + y * z * (2 * x^2 - y^2) * Vxxy +
          y^2 * (x^2 + y^2) * Vxxz - 2 * x * y * (x^2 + y^2) * Vxyz + x * z * (2 * y^2 - x^2) * Vyyx -
          x^2 * y * z * Vyyy + x^2 * (x^2 + y^2) * Vyyz) / ((x^2 + y^2)^(3/2) * Sqrt[x^2 + y^2 + z^2]);

In[•]:= Vλλρ = (x * y^2 * Vxxx + y * (y^2 - 2 * x^2) * Vxxy + y^2 * z * Vxxz - 2 * x * y * z * Vxyz +
          x * (x^2 - 2 * y^2) * Vyyx + x^2 * y * Vyyy + x^2 * z * Vyyz) / ((x^2 + y^2) * Sqrt[x^2 + y^2 + z^2]);

In[•]:= Vφφλ = (-x^2 * y * z^2 * Vxxx + x * z^2 * (x^2 - 2 * y^2) * Vxxy +
          2 * x * y * z * (x^2 + y^2) * Vxxz - 2 * z * (x^4 - y^4) * Vxyz +
          y * z^2 * (2 * x^2 - y^2) * Vyyx + x * y^2 * z^2 * Vyyy - 2 * x * y * z * (x^2 + y^2) * Vyyz -
          y * (x^2 + y^2)^2 * Vzzx + x * (x^2 + y^2)^2 * Vzzy) / ((x^2 + y^2)^(3/2) * (x^2 + y^2 + z^2));
```

```
In[•]:= Vφφφ = (-x^3 * z^3 * Vxxx - 3 * x^2 * y * z^3 * Vxxy + 3 * x^2 * z^2 * (x^2 + y^2) * Vxxz +
          6 * x * y * z^2 * (x^2 + y^2) * Vxyz - 3 * x * y^2 * z^3 * Vyyx - y^3 * z^3 * Vyyy +
          3 * y^2 * z^2 * (x^2 + y^2) * Vyyz - 3 * x * z * (x^2 + y^2)^2 * Vzzx -
          3 * y * z * (x^2 + y^2)^2 * Vzzy + (x^2 + y^2)^3 * Vzzz) / ((x^2 + y^2)^(3/2) * (x^2 + y^2 + z^2)^(3/2));

In[•]:= Vφφρ = (x^3 * z^2 * Vxxx + 3 * x^2 * y * z^2 * Vxxy +
          x^2 * z * (z^2 - 2 * x^2 - 2 * y^2) * Vxxz + 2 * x * y * z * (z^2 - 2 * x^2 - 2 * y^2) * Vxyz +
          3 * x * y^2 * z^2 * Vyyx + y^3 * z^2 * Vyyy + y^2 * z * (z^2 - 2 * x^2 - 2 * y^2) * Vyyz +
          x * (x^2 + y^2) * (x^2 + y^2 - 2 * z^2) * Vzzx + y * (x^2 + y^2) * (x^2 + y^2 - 2 * z^2) * Vzzy +
          z * (x^2 + y^2)^2 * Vzzz) / ((x^2 + y^2) * (x^2 + y^2 + z^2)^(3/2));

In[•]:= Vρρλ = (-x^2 * y * Vxxx + x * (x^2 - 2 * y^2) * Vxxy - 2 * x * y * z * Vxxz +
          2 * z * (x^2 - y^2) * Vxyz + y * (2 * x^2 - y^2) * Vyyx + x * y^2 * Vyyy +
          2 * x * y * z * Vyyz - y * z^2 * Vzzx + x * z^2 * Vzzy) / (Sqrt[x^2 + y^2] * (x^2 + y^2 + z^2));

In[•]:= Vρρφ = (-x^3 * z * Vxxx - 3 * x^2 * y * z * Vxxy + x^2 * (x^2 + y^2 - 2 * z^2) * Vxxz +
          2 * x * y * (x^2 + y^2 - 2 * z^2) * Vxyz - 3 * x * y^2 * z * Vyyx - y^3 * z * Vyyy +
          y^2 * (x^2 + y^2 - 2 * z^2) * Vyyz + x * z * (2 * x^2 + 2 * y^2 - z^2) * Vzzx + y * z *
           (2 * x^2 + 2 * y^2 - z^2) * Vzzy + z^2 * (x^2 + y^2) * Vzzz) / (Sqrt[x^2 + y^2] * (x^2 + y^2 + z^2)^(3/2));

In[•]:= Vρρρ = (x^3 * Vxxx + 3 * x^2 * y * Vxxy + 3 * x^2 * z * Vxxz +
          6 * x * y * z * Vxyz + 3 * x * y^2 * Vyyx + y^3 * Vyyy + 3 * y^2 * z * Vyyz +
          3 * x * z^2 * Vzzx + 3 * y * z^2 * Vzzy + z^3 * Vzzz) / (x^2 + y^2 + z^2)^(3/2);

In[•]:= FullSimplify[Vλλ + Vφφ + Vρρ]

Out[•]=
Vxx + Vyy + Vzz

In[•]:= FullSimplify[Vλλλ + Vφφλ + Vρρλ]

Out[•]=
((Vxxy + Vyyy + Vzzy) x - (Vxxx + Vyyx + Vzzx) y) / Sqrt[x^2 + y^2]

In[•]:= FullSimplify[Vλλφ + Vφφφ + Vρρφ]

Out[•]=
((Vxxz + Vyyz + Vzzz) (x^2 + y^2) - ((Vxxx + Vyyx + Vzzx) x + (Vxxy + Vyyy + Vzzy) y) z) / (Sqrt[x^2 + y^2] Sqrt[x^2 + y^2 + z^2])

In[•]:= FullSimplify[Vλλρ + Vφφρ + Vρρρ]

Out[•]=
((Vxxx + Vyyx + Vzzx) x + (Vxxy + Vyyy + Vzzy) y + (Vxxz + Vyyz + Vzzz) z) / Sqrt[x^2 + y^2 + z^2]
```

Table A13

```
In[•]:= Clear["Global`*"];
```

```
In[•]:= Vλλ = (Sin[λ])^2 * Vxx - Sin[2 * λ] * Vxy + (Cos[λ])^2 * Vyy;

In[•]:= Vφφ = (Sin[φ])^2 * (Cos[λ])^2 * Vxx + (Sin[φ])^2 * Sin[2 * λ] * Vxy - Sin[2 * φ] * Cos[λ] * Vxz +
          (Sin[φ])^2 * (Sin[λ])^2 * Vyy - Sin[2 * φ] * Sin[λ] * Vyz + (Cos[φ])^2 * Vzz;

In[•]:= Vρρ = (Cos[φ])^2 * (Cos[λ])^2 * Vxx + (Cos[φ])^2 * Sin[2 * λ] * Vxy + Sin[2 * φ] * Cos[λ] * Vxz +
          (Cos[φ])^2 * (Sin[λ])^2 * Vyy + Sin[2 * φ] * Sin[λ] * Vyz + (Sin[φ])^2 * Vzz;

In[•]:= Vλλλ = - (Sin[λ])^3 * Vxxx + 3 * (Sin[λ])^2 * Cos[λ] * Vxxy -
          3 * Sin[λ] * (Cos[λ])^2 * Vyyx + (Cos[λ])^3 * Vyyy;

In[•]:= Vλλφ = -Sin[φ] * (Sin[λ])^2 * Cos[λ] * Vxxx +
          Sin[φ] * Sin[λ] * (2 * (Cos[λ])^2 - (Sin[λ])^2) * Vxxy + Cos[φ] * (Sin[λ])^2 * Vxxz -
          Cos[φ] * Sin[2 * λ] * Vxyz + Sin[φ] * Cos[λ] * (2 * (Sin[λ])^2 - (Cos[λ])^2) * Vyyx -
          Sin[φ] * Sin[λ] * (Cos[λ])^2 * Vyyy + Cos[φ] * (Cos[λ])^2 * Vyyz;

In[•]:= Vλλρ = Cos[φ] * (Sin[λ])^2 * Cos[λ] * Vxxx +
          Cos[φ] * Sin[λ] * ((Sin[λ])^2 - 2 * (Cos[λ])^2) * Vxxy + Sin[φ] * (Sin[λ])^2 * Vxxz -
          Sin[φ] * Sin[2 * λ] * Vxyz + Cos[φ] * Cos[λ] * ((Cos[λ])^2 - 2 * (Sin[λ])^2) * Vyyx +
          Cos[φ] * Sin[λ] * (Cos[λ])^2 * Vyyy + Sin[φ] * (Cos[λ])^2 * Vyyz;

In[•]:= Vφφλ = - (Sin[φ])^2 * Sin[λ] * (Cos[λ])^2 * Vxxx +
          (Sin[φ])^2 * Cos[λ] * ((Cos[λ])^2 - 2 * (Sin[λ])^2) * Vxxy +
          Sin[2 * φ] * Sin[λ] * Cos[λ] * Vxxz - Sin[2 * φ] * Cos[2 * λ] * Vxyz +
          (Sin[φ])^2 * Sin[λ] * (2 * (Cos[λ])^2 - (Sin[λ])^2) * Vyyx +
          (Sin[φ])^2 * (Sin[λ])^2 * Cos[λ] * Vyyy - Sin[2 * φ] * Sin[λ] * Cos[λ] * Vyyz -
          (Cos[φ])^2 * Sin[λ] * Vzzx + (Cos[φ])^2 * Cos[λ] * Vzzy;

In[•]:= Vφφφ = - (Sin[φ])^3 * (Cos[λ])^3 * Vxxx - 3 * (Sin[φ])^3 * Sin[λ] * (Cos[λ])^2 * Vxxy +
          3 * (Sin[φ])^2 * Cos[φ] * (Cos[λ])^2 * Vxxz + 3 * (Sin[φ])^2 * Cos[φ] * Sin[2 * λ] * Vxyz -
          3 * (Sin[φ])^3 * (Sin[λ])^2 * Cos[λ] * Vyyx - (Sin[φ])^3 * (Sin[λ])^3 * Vyyy +
          3 * (Sin[φ])^2 * Cos[φ] * (Sin[λ])^2 * Vyyz - 3 * Sin[φ] * (Cos[φ])^2 * Cos[λ] * Vzzx -
          3 * Sin[φ] * (Cos[φ])^2 * Sin[λ] * Vzzy + (Cos[φ])^3 * Vzzz;

In[•]:= Vφφρ = (Sin[φ])^2 * Cos[φ] * (Cos[λ])^3 * Vxxx + 3 * (Sin[φ])^2 * Cos[φ] * Sin[λ] *
           (Cos[λ])^2 * Vxxy + Sin[φ] * (Cos[λ])^2 * ((Sin[φ])^2 - 2 * (Cos[φ])^2) * Vxxz +
          Sin[φ] * Sin[2 * λ] * ((Sin[φ])^2 - 2 * (Cos[φ])^2) * Vxyz + 3 * (Sin[φ])^2 *
           Cos[φ] * (Sin[λ])^2 * Cos[λ] * Vyyx + (Sin[φ])^2 * Cos[φ] * (Sin[λ])^3 * Vyyy +
          Sin[φ] * (Sin[λ])^2 * ((Sin[φ])^2 - 2 * (Cos[φ])^2) * Vyyz +
          Cos[φ] * Cos[λ] * ((Cos[φ])^2 - 2 * (Sin[φ])^2) * Vzzx +
          Cos[φ] * Sin[λ] * ((Cos[φ])^2 - 2 * (Sin[φ])^2) * Vzzy + Sin[φ] * (Cos[φ])^2 * Vzzz;

In[•]:= Vρρλ = - (Cos[φ])^2 * Sin[λ] * (Cos[λ])^2 * Vxxx +
          (Cos[φ])^2 * Cos[λ] * ((Cos[λ])^2 - 2 * (Sin[λ])^2) * Vxxy -
          Sin[2 * φ] * Sin[λ] * Cos[λ] * Vxxz + Sin[2 * φ] * Cos[2 * λ] * Vxyz +
          (Cos[φ])^2 * Sin[λ] * (2 * (Cos[λ])^2 - (Sin[λ])^2) * Vyyx +
          (Cos[φ])^2 * (Sin[λ])^2 * Cos[λ] * Vyyy + Sin[2 * φ] * Sin[λ] * Cos[λ] * Vyyz -
          (Sin[φ])^2 * Sin[λ] * Vzzx + (Sin[φ])^2 * Cos[λ] * Vzzy;
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In[•]:= Vρφφ = -Sin[φ] * (Cos[φ])^2 * (Cos[λ])^3 * Vxxx - 3 * Sin[φ] * (Cos[φ])^2 * Sin[λ] *
          (Cos[λ])^2 * Vxxy + Cos[φ] * (Cos[λ])^2 * ((Cos[φ])^2 - 2 * (Sin[φ])^2) * Vxxz +
        Cos[φ] * Sin[2 * λ] * ((Cos[φ])^2 - 2 * (Sin[φ])^2) * Vxyz -
        3 * Sin[φ] * (Cos[φ])^2 * (Sin[λ])^2 * Cos[λ] * Vyyx - Sin[φ] * (Cos[φ])^2 *
          (Sin[λ])^3 * Vyyy + Cos[φ] * (Sin[λ])^2 * ((Cos[φ])^2 - 2 * (Sin[φ])^2) * Vyyz +
        Sin[φ] * Cos[λ] * (2 * (Cos[φ])^2 - (Sin[φ])^2) * Vzzx +
        Sin[φ] * Sin[λ] * (2 * (Cos[φ])^2 - (Sin[φ])^2) * Vzzy + (Sin[φ])^2 * Cos[φ] * Vzzz;

In[•]:= Vρρρ = (Cos[φ])^3 * (Cos[λ])^3 * Vxxx +
        3 * (Cos[φ])^3 * Sin[λ] * (Cos[λ])^2 * Vxxy + 3 * Sin[φ] * (Cos[φ])^2 * (Cos[λ])^2 * Vxxz +
        3 * Sin[2 * φ] * Cos[φ] * Sin[λ] * Cos[λ] * Vxyz +
        3 * (Cos[φ])^3 * (Sin[λ])^2 * Cos[λ] * Vyyx + (Cos[φ])^3 * (Sin[λ])^3 * Vyyy +
        3 * Sin[φ] * (Cos[φ])^2 * (Sin[λ])^2 * Vyyz + 3 * (Sin[φ])^2 * Cos[φ] * Cos[λ] * Vzzx +
        3 * (Sin[φ])^2 * Cos[φ] * Sin[λ] * Vzzy + (Sin[φ])^3 * Vzzz;

In[•]:= FullSimplify[Vλλ + Vφφ + Vρρ]

Out[•]=

        Vxx + Vyy + Vzz

In[•]:= FullSimplify[Vλλλ + Vφφλ + Vρρλ]

Out[•]=

        (Vxxy + Vyyy + Vzzy) Cos[λ] - (Vxxx + Vyyx + Vzzx) Sin[λ]

In[•]:= FullSimplify[Vλλφ + Vφφφ + Vρρφ]

Out[•]=

        (Vxxz + Vyyz + Vzzz) Cos[φ] -
          ((Vxxx + Vyyx + Vzzx) Cos[λ] + (Vxxy + Vyyy + Vzzy) Sin[λ]) Sin[φ]

In[•]:= FullSimplify[Vλλρ + Vφφρ + Vρρρ]

Out[•]=

        (Vxxx + Vyyx + Vzzx) Cos[λ] Cos[φ] +
          (Vxxy + Vyyy + Vzzy) Cos[φ] Sin[λ] + (Vxxz + Vyyz + Vzzz) Sin[φ]

In[•]:= NotebookSave[EvaluationNotebook[]];
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