

Readme

1. This Appendix presents the validation process with the transformation cycle method for all physical components of first-, second-, and third-order potential gradients in Tables 3 - 15.
2. In the following parts, the symbol “*” is omitted for simplicity.
3. Each of the following sections is independent by using the command **Clear[“Global`*”]** to clear previous arguments. The name of each section is from Table 2.
4. After running all codes, all components become themselves in the following Sects. 1-12 for 12 routes.

1. Algebraic form, Cartesian coordinates, Routes 1 → 3 → 5, and Table 3 → Table 7 → Tables 11, 12

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In[1]:= Clear["Global`*"];
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1.1 Table 3

$$In[2]:= \nabla r = (x * Vx + y * Vy) / \sqrt{x^2 + y^2};$$

$$In[3]:= \nabla \lambda = (-y * Vx + x * Vy) / \sqrt{x^2 + y^2};$$

$$In[4]:= \nabla z = Vz;$$

$$In[5]:= \nabla rr = (x^2 * Vxx + 2 * x * y * Vxy + y^2 * Vyy) / (x^2 + y^2);$$

$$In[6]:= \nabla r\lambda = (-x * y * Vxx + (x^2 - y^2) * Vxy + x * y * Vyy) / (x^2 + y^2);$$

$$In[7]:= \nabla rz = (x * Vxz + y * Vyz) / \sqrt{x^2 + y^2};$$

$$In[8]:= \nabla \lambda\lambda = (y^2 * Vxx - 2 * x * y * Vxy + x^2 * Vyy) / (x^2 + y^2);$$

$$In[9]:= \nabla \lambda z = (-y * Vxz + x * Vyz) / \sqrt{x^2 + y^2};$$

$$In[10]:= Vzz = Vzz;$$

$$In[11]:= \nabla rrr = (x^3 * Vxxx + 3 * x^2 * y * Vxxy + 3 * x * y^2 * Vyyx + y^3 * Vyyy) / (x^2 + y^2)^{3/2};$$

$$In[12]:= \nabla rr\lambda = (-x^2 * y * Vxxx + x * (x^2 - 2 * y^2) * Vxxy + y * (2 * x^2 - y^2) * Vyyx + x * y^2 * Vyyy) / (x^2 + y^2)^{3/2};$$

$$In[13]:= \nabla rrz = (x^2 * Vxxz + 2 * x * y * Vxyz + y^2 * Vyyz) / (x^2 + y^2);$$

$$In[14]:= \nabla r\lambda z = (-x * y * Vxxz + (x^2 - y^2) * Vxyz + x * y * Vyyz) / (x^2 + y^2);$$

$$In[15]:= \nabla \lambda\lambda r =$$

$$(x * y^2 * Vxxx + y * (y^2 - 2 * x^2) * Vxxy + x * (x^2 - 2 * y^2) * Vyyx + x^2 * y * Vyyy) / (x^2 + y^2)^{3/2};$$

$$In[16]:= \nabla \lambda\lambda\lambda = (-y^3 * Vxxx + 3 * x * y^2 * Vxxy - 3 * x^2 * y * Vyyx + x^3 * Vyyy) / (x^2 + y^2)^{3/2};$$

$$In[17]:= \nabla \lambda\lambda z = (y^2 * Vxxz - 2 * x * y * Vxyz + x^2 * Vyyz) / (x^2 + y^2);$$

$$\text{In[1]:= } Vzzr = (x * Vzxx + y * Vzyy) / \sqrt{x^2 + y^2};$$

$$\text{In[2]:= } Vzz\lambda = (-y * Vzxx + x * Vzyy) / \sqrt{x^2 + y^2};$$

$$\text{In[3]:= } Vzzz = Vzzz;$$

1.2 Table 7

$$\text{In[4]:= } V\lambda = V\lambda;$$

$$\text{In[5]:= } V\varphi = (-z * Vr + r * Vz) / \sqrt{r^2 + z^2};$$

$$\text{In[6]:= } V\rho = (r * Vr + z * Vz) / \sqrt{r^2 + z^2};$$

$$\text{In[7]:= } V\lambda\lambda = V\lambda\lambda;$$

$$\text{In[8]:= } V\lambda\varphi = (-z * Vr\lambda + r * V\lambda z) / \sqrt{r^2 + z^2};$$

$$\text{In[9]:= } V\lambda\rho = (r * Vr\lambda + z * V\lambda z) / \sqrt{r^2 + z^2};$$

$$\text{In[10]:= } V\varphi\varphi = (z^2 * Vrr - 2 * r * z * Vrz + r^2 * Vzz) / (r^2 + z^2);$$

$$\text{In[11]:= } V\varphi\rho = (-r * z * Vrr + (r^2 - z^2) * Vrz + r * z * Vzz) / (r^2 + z^2);$$

$$\text{In[12]:= } V\rho\rho = (r^2 * Vrr + 2 * r * z * Vrz + z^2 * Vzz) / (r^2 + z^2);$$

$$\text{In[13]:= } V\lambda\lambda\lambda = V\lambda\lambda\lambda;$$

$$\text{In[14]:= } V\lambda\lambda\varphi = (-z * V\lambda\lambda r + r * V\lambda\lambda z) / \sqrt{r^2 + z^2};$$

$$\text{In[15]:= } V\lambda\lambda\rho = (r * V\lambda\lambda r + z * V\lambda\lambda z) / \sqrt{r^2 + z^2};$$

$$\text{In[16]:= } V\lambda\varphi\rho = (-r * z * Vrr\lambda + (r^2 - z^2) * Vr\lambda z + r * z * Vzz\lambda) / (r^2 + z^2);$$

$$\text{In[17]:= } V\varphi\varphi\lambda = (z^2 * Vrr\lambda - 2 * r * z * Vr\lambda z + r^2 * Vzz\lambda) / (r^2 + z^2);$$

$$\text{In[18]:= } V\varphi\varphi\varphi = (-z^3 * Vrrr + 3 * r * z^2 * Vrrz - 3 * r^2 * z * Vzzr + r^3 * Vzzz) / (r^2 + z^2)^{3/2};$$

$$\text{In[19]:= } V\varphi\varphi\rho =$$

$$(r * z^2 * Vrrr + z * (z^2 - 2 * r^2) * Vrrz + r * (r^2 - 2 * z^2) * Vzzr + r^2 * z * Vzzz) / (r^2 + z^2)^{3/2};$$

$$\text{In[20]:= } V\rho\rho\lambda = (r^2 * Vrr\lambda + 2 * r * z * Vr\lambda z + z^2 * Vzz\lambda) / (r^2 + z^2);$$

$$\text{In[21]:= } V\rho\rho\varphi = (-r^2 * z * Vrrr + r * (r^2 - 2 * z^2) * Vrrz + z * (2 * r^2 - z^2) * Vzzr + r * z^2 * Vzzz) / (r^2 + z^2)^{3/2};$$

$$\text{In[22]:= } V\rho\rho\rho = (r^3 * Vrrr + 3 * r^2 * z * Vrrz + 3 * r * z^2 * Vzzr + z^3 * Vzzz) / (r^2 + z^2)^{3/2};$$

1.3 Tables 11, 12

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In[]:= VxRoutes135 =
  
$$\left( -y * \sqrt{x^2 + y^2 + z^2} * V\lambda - x * z * V\varphi + x * \sqrt{x^2 + y^2} * V\rho \right) / \left( \sqrt{x^2 + y^2} * \sqrt{x^2 + y^2 + z^2} \right);$$


In[]:= VyRoutes135 =
  
$$\left( x * \sqrt{x^2 + y^2 + z^2} * V\lambda - y * z * V\varphi + y * \sqrt{x^2 + y^2} * V\rho \right) / \left( \sqrt{x^2 + y^2} * \sqrt{x^2 + y^2 + z^2} \right);$$


In[]:= VzRoutes135 = 
$$\left( \sqrt{x^2 + y^2} * V\varphi + z * V\rho \right) / \sqrt{x^2 + y^2 + z^2};$$


In[]:= VxxRoutes135 = 
$$\begin{aligned} & \left( y^2 * (x^2 + y^2 + z^2) * V\lambda\lambda + 2 * x * y * z * \sqrt{x^2 + y^2 + z^2} * V\lambda\varphi - \right. \\ & \quad 2 * x * y * \sqrt{x^2 + y^2} * \sqrt{x^2 + y^2 + z^2} * V\lambda\rho + x^2 * z^2 * V\varphi\varphi - \\ & \quad \left. 2 * x^2 * z * \sqrt{x^2 + y^2} * V\varphi\rho + x^2 * (x^2 + y^2) * V\rho\rho \right) / ((x^2 + y^2) * (x^2 + y^2 + z^2)); \end{aligned}$$


In[]:= VxyRoutes135 = 
$$\begin{aligned} & \left( -x * y * (x^2 + y^2 + z^2)^{3/2} * V\lambda\lambda - \right. \\ & \quad z * (x^2 - y^2) * (x^2 + y^2 + z^2) * V\lambda\varphi + (x^2 - y^2) * \sqrt{x^2 + y^2} * (x^2 + y^2 + z^2) * V\lambda\rho + \\ & \quad x * y * z^2 * \sqrt{x^2 + y^2 + z^2} * V\varphi\varphi - 2 * x * y * z * \sqrt{x^2 + y^2} * \sqrt{x^2 + y^2 + z^2} * V\varphi\rho + \\ & \quad \left. x * y * (x^2 + y^2) * \sqrt{x^2 + y^2 + z^2} * V\rho\rho \right) / ((x^2 + y^2) * (x^2 + y^2 + z^2)^{3/2}); \end{aligned}$$


In[]:= VxzRoutes135 =
  
$$\begin{aligned} & \left( -y * \sqrt{x^2 + y^2} * \sqrt{x^2 + y^2 + z^2} * V\lambda\varphi - y * z * \sqrt{x^2 + y^2 + z^2} * V\lambda\rho - x * z * \sqrt{x^2 + y^2} * V\varphi\varphi + \right. \\ & \quad x * (x^2 + y^2 - z^2) * V\varphi\rho + x * z * \sqrt{x^2 + y^2} * V\rho\rho \left. \right) / (\sqrt{x^2 + y^2} * (x^2 + y^2 + z^2)); \end{aligned}$$


In[]:= VyyRoutes135 = 
$$\begin{aligned} & \left( x^2 * (x^2 + y^2 + z^2) * V\lambda\lambda - 2 * x * y * z * \sqrt{x^2 + y^2 + z^2} * V\lambda\varphi + \right. \\ & \quad 2 * x * y * \sqrt{x^2 + y^2} * \sqrt{x^2 + y^2 + z^2} * V\lambda\rho + y^2 * z^2 * V\varphi\varphi - \\ & \quad \left. 2 * y^2 * z * \sqrt{x^2 + y^2} * V\varphi\rho + y^2 * (x^2 + y^2) * V\rho\rho \right) / ((x^2 + y^2) * (x^2 + y^2 + z^2)); \end{aligned}$$


In[]:= VyzRoutes135 =
  
$$\begin{aligned} & \left( x * \sqrt{x^2 + y^2} * \sqrt{x^2 + y^2 + z^2} * V\lambda\varphi + x * z * \sqrt{x^2 + y^2 + z^2} * V\lambda\rho - y * z * \sqrt{x^2 + y^2} * V\varphi\varphi + \right. \\ & \quad y * (x^2 + y^2 - z^2) * V\varphi\rho + y * z * \sqrt{x^2 + y^2} * V\rho\rho \left. \right) / (\sqrt{x^2 + y^2} * (x^2 + y^2 + z^2)); \end{aligned}$$


In[]:= VzzRoutes135 = 
$$\left( (x^2 + y^2) * V\varphi\varphi + 2 * z * \sqrt{x^2 + y^2} * V\varphi\rho + z^2 * V\rho\rho \right) / (x^2 + y^2 + z^2);$$


In[]:= VxxxRoutes135 =
  
$$\begin{aligned} & \left( -y^3 * (x^2 + y^2 + z^2)^{3/2} * V\lambda\lambda\lambda - 3 * x * y^2 * z * (x^2 + y^2 + z^2) * V\lambda\lambda\varphi + 3 * x * y^2 * \right. \\ & \quad \sqrt{x^2 + y^2} * (x^2 + y^2 + z^2) * V\lambda\lambda\rho + 6 * x^2 * y * z * \sqrt{x^2 + y^2} * \sqrt{x^2 + y^2 + z^2} * V\lambda\varphi\rho - \\ & \quad 3 * x^2 * y * z^2 * \sqrt{x^2 + y^2 + z^2} * V\varphi\varphi\lambda - x^3 * z^3 * V\varphi\varphi\varphi + 3 * x^3 * z^2 * \sqrt{x^2 + y^2} * V\varphi\varphi\rho - \\ & \quad 3 * x^2 * y * (x^2 + y^2) * \sqrt{x^2 + y^2 + z^2} * V\rho\rho\lambda - 3 * x^3 * z * (x^2 + y^2) * V\rho\rho\varphi + \\ & \quad \left. x^3 * (x^2 + y^2)^{3/2} * V\rho\rho\rho \right) / ((x^2 + y^2)^{3/2} * (x^2 + y^2 + z^2)^{3/2}); \end{aligned}$$


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$$\text{In[1]:= } \text{VxxxyRoutes135} = \left(x * y^2 * (x^2 + y^2 + z^2)^{3/2} * V\lambda\lambda\lambda + y * z * (2 * x^2 - y^2) * (x^2 + y^2 + z^2) * V\lambda\lambda\varphi - y * \sqrt{x^2 + y^2} * (2 * x^2 - y^2) * (x^2 + y^2 + z^2) * V\lambda\lambda\rho - 2 * x * z * (x^2 - 2 * y^2) * \sqrt{x^2 + y^2} * \sqrt{x^2 + y^2 + z^2} * V\lambda\varphi\rho + x * z^2 * (x^2 - 2 * y^2) * \sqrt{x^2 + y^2 + z^2} * V\varphi\varphi\lambda - x^2 * y * z^3 * V\varphi\varphi\varphi + 3 * x^2 * y * z^2 * \sqrt{x^2 + y^2} * V\varphi\varphi\rho + x * (x^4 - x^2 * y^2 - 2 * y^4) * \sqrt{x^2 + y^2 + z^2} * V\rho\rho\lambda - 3 * x^2 * y * z * (x^2 + y^2) * V\rho\rho\varphi + x^2 * y * (x^2 + y^2)^{3/2} * V\rho\rho\rho \right) / \left((x^2 + y^2)^{3/2} * (x^2 + y^2 + z^2)^{3/2} \right);$$

$$\text{In[2]:= } \text{VxxzRoutes135} = \left(y^2 * \sqrt{x^2 + y^2} * (x^2 + y^2 + z^2) * V\lambda\lambda\varphi + y^2 * z * (x^2 + y^2 + z^2) * V\lambda\lambda\rho - 2 * x * y * (x^2 + y^2 - z^2) * \sqrt{x^2 + y^2 + z^2} * V\lambda\varphi\rho + 2 * x * y * z * \sqrt{x^2 + y^2} * \sqrt{x^2 + y^2 + z^2} * V\varphi\varphi\lambda + x^2 * z^2 * \sqrt{x^2 + y^2} * V\varphi\varphi\varphi - x^2 * z * (2 * x^2 + 2 * y^2 - z^2) * V\varphi\varphi\rho - 2 * x * y * z * \sqrt{x^2 + y^2} * \sqrt{x^2 + y^2 + z^2} * V\rho\rho\lambda + x^2 * \sqrt{x^2 + y^2} * (x^2 + y^2 - 2 * z^2) * V\rho\rho\varphi + x^2 * z * (x^2 + y^2) * V\rho\rho\rho \right) / \left((x^2 + y^2) * (x^2 + y^2 + z^2)^{3/2} \right);$$

$$\text{In[3]:= } \text{VxyzRoutes135} = \left(-x * y * \sqrt{x^2 + y^2} * (x^2 + y^2 + z^2) * V\lambda\lambda\varphi - x * y * z * (x^2 + y^2 + z^2) * V\lambda\lambda\rho + (x^2 - y^2) * (x^2 + y^2 - z^2) * \sqrt{x^2 + y^2 + z^2} * V\lambda\varphi\rho - z * (x^2 - y^2) * \sqrt{x^2 + y^2} * \sqrt{x^2 + y^2 + z^2} * V\varphi\varphi\lambda + x * y * z^2 * \sqrt{x^2 + y^2} * V\varphi\varphi\varphi - x * y * z * (2 * x^2 + 2 * y^2 - z^2) * V\varphi\varphi\rho + z * (x^2 - y^2) * \sqrt{x^2 + y^2} * \sqrt{x^2 + y^2 + z^2} * V\rho\rho\lambda + x * y * (x^2 + y^2 - 2 * z^2) * \sqrt{x^2 + y^2} * V\rho\rho\varphi + x * y * z * (x^2 + y^2) * V\rho\rho\rho \right) / \left((x^2 + y^2) * (x^2 + y^2 + z^2)^{3/2} \right);$$

$$\text{In[4]:= } \text{VyyxRoutes135} = \left(-x^2 * y * (x^2 + y^2 + z^2)^{3/2} * V\lambda\lambda\lambda - x * z * (x^2 - 2 * y^2) * (x^2 + y^2 + z^2) * V\lambda\lambda\varphi + x * (x^2 - 2 * y^2) * \sqrt{x^2 + y^2} * (x^2 + y^2 + z^2) * V\lambda\lambda\rho - 2 * y * z * (2 * x^2 - y^2) * \sqrt{x^2 + y^2} * \sqrt{x^2 + y^2 + z^2} * V\lambda\varphi\rho + y * z^2 * (2 * x^2 - y^2) * \sqrt{x^2 + y^2 + z^2} * V\varphi\varphi\lambda - x * y * z^3 * V\varphi\varphi\varphi + 3 * x * y^2 * z^2 * \sqrt{x^2 + y^2} * V\varphi\varphi\rho + y * (2 * x^4 + x^2 * y^2 - y^4) * \sqrt{x^2 + y^2 + z^2} * V\rho\rho\lambda - 3 * x * y^2 * z * (x^2 + y^2) * V\rho\rho\varphi + x * y^2 * (x^2 + y^2)^{3/2} * V\rho\rho\rho \right) / \left((x^2 + y^2)^{3/2} * (x^2 + y^2 + z^2)^{3/2} \right);$$

$$\text{In[5]:= } \text{VyyyyRoutes135} = \left(x^3 * (x^2 + y^2 + z^2)^{3/2} * V\lambda\lambda\lambda - 3 * x^2 * y * z * (x^2 + y^2 + z^2) * V\lambda\lambda\varphi + 3 * x^2 * y * \sqrt{x^2 + y^2} * (x^2 + y^2 + z^2) * V\lambda\lambda\rho - 6 * x * y^2 * z * \sqrt{x^2 + y^2} * \sqrt{x^2 + y^2 + z^2} * V\lambda\varphi\rho + 3 * x * y^2 * z^2 * \sqrt{x^2 + y^2 + z^2} * V\varphi\varphi\lambda - y^3 * z^3 * V\varphi\varphi\varphi + 3 * y^3 * z^2 * \sqrt{x^2 + y^2} * V\varphi\varphi\rho + 3 * x * y^2 * (x^2 + y^2) * \sqrt{x^2 + y^2 + z^2} * V\rho\rho\lambda - 3 * y^3 * z * (x^2 + y^2) * V\rho\rho\varphi + y^3 * (x^2 + y^2)^{3/2} * V\rho\rho\rho \right) / \left((x^2 + y^2)^{3/2} * (x^2 + y^2 + z^2)^{3/2} \right);$$

In[1]:= VyzzRoutes135 =

$$\left(x^2 * \sqrt{x^2 + y^2} * (x^2 + y^2 + z^2) * V\lambda\lambda\varphi + x^2 * z * (x^2 + y^2 + z^2) * V\lambda\lambda\rho + 2 * x * y * (x^2 + y^2 - z^2) * \right. \\ \left. \sqrt{x^2 + y^2 + z^2} * V\lambda\varphi\rho - 2 * x * y * z * \sqrt{x^2 + y^2} * \sqrt{x^2 + y^2 + z^2} * V\varphi\varphi\lambda + \right. \\ \left. y^2 * z^2 * \sqrt{x^2 + y^2} * V\varphi\varphi\varphi - y^2 * z * (2 * x^2 + 2 * y^2 - z^2) * V\varphi\varphi\rho + \right. \\ \left. 2 * x * y * z * \sqrt{x^2 + y^2} * \sqrt{x^2 + y^2 + z^2} * V\rho\rho\lambda + y^2 * (x^2 + y^2 - 2 * z^2) * \sqrt{x^2 + y^2} * V\rho\rho\varphi + \right. \\ \left. y^2 * z * (x^2 + y^2) * V\rho\rho\rho \right) / \left((x^2 + y^2) * (x^2 + y^2 + z^2)^{3/2} \right);$$

In[2]:= VzzxRoutes135 =

$$\left(-2 * y * z * \sqrt{x^2 + y^2} * \sqrt{x^2 + y^2 + z^2} * V\lambda\varphi\rho - y * (x^2 + y^2) * \sqrt{x^2 + y^2 + z^2} * V\varphi\varphi\lambda - \right. \\ \left. x * z * (x^2 + y^2) * V\varphi\varphi\varphi + x * (x^2 + y^2 - 2 * z^2) * \sqrt{x^2 + y^2} * V\varphi\varphi\rho - \right. \\ \left. y * z^2 * \sqrt{x^2 + y^2 + z^2} * V\rho\rho\lambda + x * z * (2 * x^2 + 2 * y^2 - z^2) * V\rho\rho\varphi + \right. \\ \left. x * z^2 * \sqrt{x^2 + y^2} * V\rho\rho\rho \right) / \left(\sqrt{x^2 + y^2} * (x^2 + y^2 + z^2)^{3/2} \right);$$

In[3]:= VzzyRoutes135 =

$$\left(2 * x * z * \sqrt{x^2 + y^2} * \sqrt{x^2 + y^2 + z^2} * V\lambda\varphi\rho + x * (x^2 + y^2) * \sqrt{x^2 + y^2 + z^2} * V\varphi\varphi\lambda - \right. \\ \left. y * z * (x^2 + y^2) * V\varphi\varphi\varphi + y * (x^2 + y^2 - 2 * z^2) * \sqrt{x^2 + y^2} * V\varphi\varphi\rho + \right. \\ \left. x * z^2 * \sqrt{x^2 + y^2 + z^2} * V\rho\rho\lambda + y * z * (2 * x^2 + 2 * y^2 - z^2) * V\rho\rho\varphi + \right. \\ \left. y * z^2 * \sqrt{x^2 + y^2} * V\rho\rho\rho \right) / \left(\sqrt{x^2 + y^2} * (x^2 + y^2 + z^2)^{3/2} \right);$$

In[4]:= VzzzRoutes135 =

$$\left((x^2 + y^2)^{3/2} * V\varphi\varphi\varphi + 3 * z * (x^2 + y^2) * V\varphi\varphi\rho + 3 * z^2 * \sqrt{x^2 + y^2} * V\rho\rho\varphi + z^3 * V\rho\rho\rho \right) / \\ (x^2 + y^2 + z^2)^{3/2};$$

1.4 Check whether they become themselves with $r = \sqrt{x^2 + y^2}$

In[1]:= FullSimplify[VxRoutes135 /. {r → √(x^2 + y^2)}]

Out[1]=

Vx

In[2]:= FullSimplify[VyRoutes135 /. {r → √(x^2 + y^2)}]

Out[2]=

Vy

In[3]:= FullSimplify[VzRoutes135 /. {r → √(x^2 + y^2)}]

Out[3]=

Vz

In[4]:= FullSimplify[VxxRoutes135 /. {r → √(x^2 + y^2)}]

Out[4]=

Vxx

In[1]:= FullSimplify[VxyRoutes135 /. {r → √(x² + y²)}]

Out[1]=

Vxy

In[2]:= FullSimplify[VxzRoutes135 /. {r → √(x² + y²)}]

Out[2]=

Vxz

In[3]:= FullSimplify[VyyRoutes135 /. {r → √(x² + y²)}]

Out[3]=

Vyy

In[4]:= FullSimplify[VyzRoutes135 /. {r → √(x² + y²)}]

Out[4]=

Vyz

In[5]:= FullSimplify[VzzRoutes135 /. {r → √(x² + y²)}]

Out[5]=

Vzz

In[6]:= FullSimplify[VxxxRoutes135 /. {r → √(x² + y²)}]

Out[6]=

Vxxx

In[7]:= FullSimplify[VxxyRoutes135 /. {r → √(x² + y²)}]

Out[7]=

Vxxy

In[8]:= FullSimplify[VxxzRoutes135 /. {r → √(x² + y²)}]

Out[8]=

Vxxz

In[9]:= FullSimplify[VxyzRoutes135 /. {r → √(x² + y²)}]

Out[9]=

Vxyz

In[10]:= FullSimplify[VyyxRoutes135 /. {r → √(x² + y²)}]

Out[10]=

Vyyx

In[11]:= FullSimplify[VyyyRoutes135 /. {r → √(x² + y²)}]

Out[11]=

Vyyy

In[12]:= FullSimplify[VyyzRoutes135 /. {r → √(x² + y²)}]

Out[12]=

Vyyz

In[1]:= FullSimplify[VzzxRoutes135 /. {r → √(x^2 + y^2)}]

Out[1]=

Vzzx

In[2]:= FullSimplify[VzzyRoutes135 /. {r → √(x^2 + y^2)}]

Out[2]=

Vzzy

In[3]:= FullSimplify[VzzzRoutes135 /. {r → √(x^2 + y^2)}]

Out[3]=

Vzzz

2. Algebraic form, Cartesian coordinates, Routes 6 → 4 → 2, and Table 14 → Table 9 → Table 5

In[1]:= Clear["Global`*"];

2.1 Table 14

In[2]:= Vλ = (-y * Vx + x * Vy) / √(x^2 + y^2);

In[3]:= Vφ = (-x * z * Vx - y * z * Vy + (x^2 + y^2) * Vz) / (√(x^2 + y^2) * √(x^2 + y^2 + z^2));

In[4]:= Vρ = (x * Vx + y * Vy + z * Vz) / √(x^2 + y^2 + z^2);

In[5]:= Vλλ = (y^2 * Vxx - 2 * x * y * Vxy + x^2 * Vyy) / (x^2 + y^2);

In[6]:= Vλφ = (x * y * z * Vxx - z * (x^2 - y^2) * Vxy - y * (x^2 + y^2) * Vxz - x * y * z * Vyy + x * (x^2 + y^2) * Vyz) / ((x^2 + y^2) * √(x^2 + y^2 + z^2));

In[7]:= Vλρ = (-x * y * Vxx + (x^2 - y^2) * Vxy - y * z * Vxz + x * z * Vyz + x * y * Vyy) / (√(x^2 + y^2) * √(x^2 + y^2 + z^2));

In[8]:= Vφφ = (x^2 * z^2 * Vxx + 2 * x * y * z^2 * Vxy - 2 * x * z * (x^2 + y^2) * Vxz + y^2 * z^2 * Vyy - 2 * y * z * (x^2 + y^2) * Vyz + (x^2 + y^2)^2 * Vzz) / ((x^2 + y^2) * (x^2 + y^2 + z^2));

In[9]:= Vφρ = (-x^2 * z * Vxx - 2 * x * y * z * Vxy + x * (x^2 + y^2 - z^2) * Vxz - y^2 * z * Vyy + y * (x^2 + y^2 - z^2) * Vyz + z * (x^2 + y^2) * Vzz) / (√(x^2 + y^2) * (x^2 + y^2 + z^2));

In[10]:= Vρρ = (x^2 * Vxx + 2 * x * y * Vxy + 2 * x * z * Vxz + 2 * y * z * Vyz + y^2 * Vyy + z^2 * Vzz) / (x^2 + y^2 + z^2);

In[11]:= Vλλλ = (-y^3 * Vxxx + 3 * x * y^2 * Vxxy - 3 * x^2 * y * Vyyx + x^3 * Vyyy) / (x^2 + y^2)^{3/2};

$$\begin{aligned}
In[1]:= & V\lambda\lambda\varphi = \left(-x * y^2 * z * Vxxx + y * z * (2 * x^2 - y^2) * Vxxxy + \right. \\
& y^2 * (x^2 + y^2) * Vxxz - 2 * x * y * (x^2 + y^2) * Vxyz + x * z * (2 * y^2 - x^2) * Vyyx - \\
& \left. x^2 * y * z * Vyyy + x^2 * (x^2 + y^2) * Vyyz \right) / \left((x^2 + y^2)^{3/2} * \sqrt{x^2 + y^2 + z^2} \right); \\
In[2]:= & V\lambda\lambda\rho = \left(x * y^2 * Vxxx + y * (y^2 - 2 * x^2) * Vxxxy + y^2 * z * Vxxz - 2 * x * y * z * Vxyz + \right. \\
& x * (x^2 - 2 * y^2) * Vyyx + x^2 * y * Vyyy + x^2 * z * Vyyz \left. \right) / \left((x^2 + y^2) * \sqrt{x^2 + y^2 + z^2} \right); \\
In[3]:= & V\lambda\varphi\rho = \left(x^2 * y * z * Vxxx + x * z * (2 * y^2 - x^2) * Vxxxy - \right. \\
& x * y * (x^2 + y^2 - z^2) * Vxxz + (x^2 - y^2) * (x^2 + y^2 - z^2) * Vxyz + \\
& y * z * (y^2 - 2 * x^2) * Vyyx - x * y^2 * z * Vyyy + x * y * (x^2 + y^2 - z^2) * Vyyz - \\
& \left. y * z * (x^2 + y^2) * Vzzx + x * z * (x^2 + y^2) * Vzzy \right) / \left((x^2 + y^2) * (x^2 + y^2 + z^2) \right); \\
In[4]:= & V\varphi\varphi\lambda = \left(-x^2 * y * z^2 * Vxxx + x * z^2 * (x^2 - 2 * y^2) * Vxxxy + \right. \\
& 2 * x * y * z * (x^2 + y^2) * Vxxz - 2 * z * (x^4 - y^4) * Vxyz + \\
& y * z^2 * (2 * x^2 - y^2) * Vyyx + x * y^2 * z^2 * Vyyy - 2 * x * y * z * (x^2 + y^2) * Vyyz - \\
& \left. y * (x^2 + y^2)^2 * Vzzx + x * (x^2 + y^2)^2 * Vzzy \right) / \left((x^2 + y^2)^{3/2} * (x^2 + y^2 + z^2) \right); \\
In[5]:= & V\varphi\varphi\varphi = \left(-x^3 * z^3 * Vxxx - 3 * x^2 * y * z^3 * Vxxxy + 3 * x^2 * z^2 * (x^2 + y^2) * Vxxz + \right. \\
& 6 * x * y * z^2 * (x^2 + y^2) * Vxyz - 3 * x * y^2 * z^3 * Vyyx - y^3 * z^3 * Vyyy + \\
& 3 * y^2 * z^2 * (x^2 + y^2) * Vyz - 3 * x * z * (x^2 + y^2)^2 * Vzzx - \\
& \left. 3 * y * z * (x^2 + y^2)^2 * Vzzy + (x^2 + y^2)^3 * Vzzz \right) / \left((x^2 + y^2)^{3/2} * (x^2 + y^2 + z^2)^{3/2} \right); \\
In[6]:= & V\varphi\varphi\rho = \left(x^3 * z^2 * Vxxx + 3 * x^2 * y * z^2 * Vxxxy + \right. \\
& x^2 * z * (z^2 - 2 * x^2 - 2 * y^2) * Vxxz + 2 * x * y * z * (z^2 - 2 * x^2 - 2 * y^2) * Vxyz + \\
& 3 * x * y^2 * z^2 * Vyyx + y^3 * z^2 * Vyyy + y^2 * z * (z^2 - 2 * x^2 - 2 * y^2) * Vyyz + \\
& x * (x^2 + y^2) * (x^2 + y^2 - 2 * z^2) * Vzzx + y * (x^2 + y^2) * (x^2 + y^2 - 2 * z^2) * Vzzy + \\
& \left. z * (x^2 + y^2)^2 * Vzzz \right) / \left((x^2 + y^2) * (x^2 + y^2 + z^2)^{3/2} \right); \\
In[7]:= & V\rho\rho\lambda = \left(-x^2 * y * Vxxx + x * (x^2 - 2 * y^2) * Vxxxy - 2 * x * y * z * Vxxz + \right. \\
& 2 * z * (x^2 - y^2) * Vxyz + y * (2 * x^2 - y^2) * Vyyx + x * y^2 * Vyyy + \\
& \left. 2 * x * y * z * Vyyz - y * z^2 * Vzzx + x * z^2 * Vzzy \right) / \left(\sqrt{x^2 + y^2} * (x^2 + y^2 + z^2) \right); \\
In[8]:= & V\rho\rho\varphi = \left(-x^3 * z * Vxxx - 3 * x^2 * y * z * Vxxxy + x^2 * (x^2 + y^2 - 2 * z^2) * Vxxz + \right. \\
& 2 * x * y * (x^2 + y^2 - 2 * z^2) * Vxyz - 3 * x * y^2 * z * Vyyx - y^3 * z * Vyyy + \\
& y^2 * (x^2 + y^2 - 2 * z^2) * Vyz + x * z * (2 * x^2 + 2 * y^2 - z^2) * Vzzx + y * z * \\
& \left. (2 * x^2 + 2 * y^2 - z^2) * Vzzy + z^2 * (x^2 + y^2) * Vzzz \right) / \left(\sqrt{x^2 + y^2} * (x^2 + y^2 + z^2)^{3/2} \right); \\
In[9]:= & V\rho\rho\rho = \left(x^3 * Vxxx + 3 * x^2 * y * Vxxxy + 3 * x^2 * z * Vxxz + \right. \\
& 6 * x * y * z * Vxyz + 3 * x * y^2 * Vyyx + y^3 * Vyyy + 3 * y^2 * z * Vyyz + \\
& \left. 3 * x * z^2 * Vzzx + 3 * y * z^2 * Vzzy + z^3 * Vzzz \right) / \left(x^2 + y^2 + z^2 \right)^{3/2};
\end{aligned}$$

2.2 Table 9

```

In[1]:= Vr = (-z * Vφ + r * Vρ) / √(r² + z²);
In[2]:= Vλ = Vλ;
In[3]:= Vz1 = (r * Vφ + z * Vρ) / √(r² + z²);
In[4]:= Vrr = (z² * Vφφ - 2 * r * z * Vφρ + r² * Vρρ) / (r² + z²);
In[5]:= Vrλ = (-z * Vλφ + r * Vλρ) / √(r² + z²);
In[6]:= Vrz = (-r * z * Vφφ + (r² - z²) * Vφρ + r * z * Vρρ) / (r² + z²);
In[7]:= Vλλ = Vλλ;
In[8]:= Vλz = (r * Vλφ + z * Vλρ) / √(r² + z²);
In[9]:= Vzz1 = (r² * Vφφ + 2 * r * z * Vφρ + z² * Vρρ) / (r² + z²);
In[10]:= Vrrr = (-z³ * Vφφφ + 3 * r * z² * Vφφρ - 3 * r² * z * Vφρφ + r³ * Vρρρ) / (r² + z²)^(3/2);
In[11]:= Vrrλ = (-2 * r * z * Vλφρ + z² * Vφφλ + r² * Vρρλ) / (r² + z²);
In[12]:= Vrrz =
  (r * z² * Vφφφ + z * (z² - 2 * r²) * Vφφρ + r * (r² - 2 * z²) * Vρρφ + r² * z * Vρρρ) / (r² + z²)^(3/2);
In[13]:= Vrλz = ((r² - z²) * Vλφρ - r * z * Vφφλ + r * z * Vρρλ) / (r² + z²);
In[14]:= Vλλr = (-z * Vλλφ + r * Vλλρ) / √(r² + z²);
In[15]:= Vλλλ = Vλλλ;
In[16]:= Vλλz = (r * Vλλφ + z * Vλλρ) / √(r² + z²);
In[17]:= Vzzr = (-r² * z * Vφφφ + r * (r² - 2 * z²) * Vφφρ + z * (2 * r² - z²) * Vρρφ + r * z² * Vρρρ) /
  (r² + z²)^(3/2);
In[18]:= Vzzλ = (2 * r * z * Vλφρ + r² * Vφφλ + z² * Vρρλ) / (r² + z²);
In[19]:= Vzzz1 = (r³ * Vφφφ + 3 * r² * z * Vφφρ + 3 * r * z² * Vρρφ + z³ * Vρρρ) / (r² + z²)^(3/2);

```

2.3 Table 5

```

In[1]:= VxRoutes642 = (x * Vr - y * Vλ) / √(x² + y²);
In[2]:= VyRoutes642 = (y * Vr + x * Vλ) / √(x² + y²);
In[3]:= VzRoutes642 = Vz1;
In[4]:= VxxRoutes642 = (x² * Vrr - 2 * x * y * Vrλ + y² * Vλλ) / (x² + y²);

```

```

In[1]:= VxyRoutes642 = (x * y * Vrr + (x^2 - y^2) * Vrλ - x * y * Vλλ) / (x^2 + y^2);
In[2]:= VxzRoutes642 = (x * Vrz - y * Vλz) / √(x^2 + y^2);
In[3]:= VyyRoutes642 = (y^2 * Vrr + 2 * x * y * Vrλ + x^2 * Vλλ) / (x^2 + y^2);
In[4]:= VyzRoutes642 = (y * Vrz + x * Vλz) / √(x^2 + y^2);
In[5]:= VzzRoutes642 = Vzz1;
In[6]:= VxxxRoutes642 = (x^3 * Vrrr - 3 * x^2 * y * Vrrλ + 3 * x * y^2 * Vλλr - y^3 * Vλλλ) / (x^2 + y^2)^{3/2};
In[7]:= VxxyRoutes642 =
(x^2 * y * Vrrr + x * (x^2 - 2 * y^2) * Vrrλ + y * (y^2 - 2 * x^2) * Vλλr + x * y^2 * Vλλλ) / (x^2 + y^2)^{3/2};
In[8]:= VxxzRoutes642 = (x^2 * Vrrz - 2 * x * y * Vrλz + y^2 * Vλλz) / (x^2 + y^2);
In[9]:= VxyzRoutes642 = (x * y * Vrrz + (x^2 - y^2) * Vrλz - x * y * Vλλz) / (x^2 + y^2);
In[10]:= VyxyRoutes642 =
(x * y^2 * Vrrr + y * (2 * x^2 - y^2) * Vrrλ + x * (x^2 - 2 * y^2) * Vλλr - x^2 * y * Vλλλ) / (x^2 + y^2)^{3/2};
In[11]:= VyyyRoutes642 = (y^3 * Vrrr + 3 * x * y^2 * Vrrλ + 3 * x^2 * y * Vλλr + x^3 * Vλλλ) / (x^2 + y^2)^{3/2};
In[12]:= VyzyRoutes642 = (y^2 * Vrrz + 2 * x * y * Vrλz + x^2 * Vλλz) / (x^2 + y^2);
In[13]:= VzzxRoutes642 = (x * Vzzr - y * Vzzλ) / √(x^2 + y^2);
In[14]:= VzzyRoutes642 = (y * Vzzr + x * Vzzλ) / √(x^2 + y^2);
In[15]:= VzzzRoutes642 = Vzzz1;

```

2.4 Check whether they become themselves with $r = \sqrt{x^2 + y^2}$

```

In[1]:= FullSimplify[VxRoutes642 /. {r → √(x^2 + y^2)}]
Out[1]=
Vx

In[2]:= FullSimplify[VyRoutes642 /. {r → √(x^2 + y^2)}]
Out[2]=
Vy

In[3]:= FullSimplify[VzRoutes642 /. {r → √(x^2 + y^2)}]
Out[3]=
Vz

In[4]:= FullSimplify[VxxRoutes642 /. {r → √(x^2 + y^2)}]
Out[4]=
Vxx

```

In[1]:= FullSimplify[VxyRoutes642 /. {r → √(x² + y²)}]

Out[1]=

Vxy

In[2]:= FullSimplify[VxzRoutes642 /. {r → √(x² + y²)}]

Out[2]=

Vxz

In[3]:= FullSimplify[VyyRoutes642 /. {r → √(x² + y²)}]

Out[3]=

Vyy

In[4]:= FullSimplify[VyzRoutes642 /. {r → √(x² + y²)}]

Out[4]=

Vyz

In[5]:= FullSimplify[VzzRoutes642 /. {r → √(x² + y²)}]

Out[5]=

Vzz

In[6]:= FullSimplify[VxxxxRoutes642 /. {r → √(x² + y²)}]

Out[6]=

Vxxx

In[7]:= FullSimplify[VxxyRoutes642 /. {r → √(x² + y²)}]

Out[7]=

Vxxy

In[8]:= FullSimplify[VxxzRoutes642 /. {r → √(x² + y²)}]

Out[8]=

Vxxz

In[9]:= FullSimplify[VxyzRoutes642 /. {r → √(x² + y²)}]

Out[9]=

Vxyz

In[10]:= FullSimplify[VyyxRoutes642 /. {r → √(x² + y²)}]

Out[10]=

Vyyx

In[11]:= FullSimplify[VyyyRoutes642 /. {r → √(x² + y²)}]

Out[11]=

Vyyy

In[12]:= FullSimplify[VyyzRoutes642 /. {r → √(x² + y²)}]

Out[12]=

Vyyz

In[\circ] := FullSimplify[VzzxRoutes642 /. {r → √(x² + y²)}]

Out[\circ] =

Vzzx

In[\circ] := FullSimplify[VzzyRoutes642 /. {r → √(x² + y²)}]

Out[\circ] =

Vzzy

In[\circ] := FullSimplify[VzzzRoutes642 /. {r → √(x² + y²)}]

Out[\circ] =

Vzzz

3. Algebraic form, Cylindrical coordinates, Routes 3 → 5 → 1, and Table 7 → Tables 11, 12 → Table 3

In[\circ] := Clear["Global`*"];

3.1 Table 7

In[\circ] := Vλ = Vλ;

In[\circ] := Vφ = (-z * Vr + r * Vz) / √(r² + z²);

In[\circ] := Vρ = (r * Vr + z * Vz) / √(r² + z²);

In[\circ] := Vλλ = Vλλ;

In[\circ] := Vλφ = (-z * Vrλ + r * Vλz) / √(r² + z²);

In[\circ] := Vλρ = (r * Vrλ + z * Vλz) / √(r² + z²);

In[\circ] := Vφφ = (z² * Vrr - 2 * r * z * Vrz + r² * Vzz) / (r² + z²);

In[\circ] := Vφρ = (-r * z * Vrr + (r² - z²) * Vrz + r * z * Vzz) / (r² + z²);

In[\circ] := Vρρ = (r² * Vrr + 2 * r * z * Vrz + z² * Vzz) / (r² + z²);

In[\circ] := Vλλλ = Vλλλ;

In[\circ] := Vλλφ = (-z * Vλλr + r * Vλλz) / √(r² + z²);

In[\circ] := Vλλρ = (r * Vλλr + z * Vλλz) / √(r² + z²);

In[\circ] := Vλφρ = (-r * z * Vrrλ + (r² - z²) * Vrλz + r * z * Vzzλ) / (r² + z²);

In[\circ] := Vφφλ = (z² * Vrrλ - 2 * r * z * Vrλz + r² * Vzzλ) / (r² + z²);

In[\circ] := Vφφφ = (-z³ * Vrrr + 3 * r * z² * Vrrz - 3 * r² * z * Vzzr + r³ * Vzzz) / (r² + z²)^{3/2};

```

In[8]:= Vφφρ = 
  (r * z^2 * Vrrr + z * (z^2 - 2 * r^2) * Vrrz + r * (r^2 - 2 * z^2) * Vzzr + r^2 * z * Vzzz) / (r^2 + z^2)^{3/2};

In[9]:= Vρρλ = (r^2 * Vrrλ + 2 * r * z * Vrλz + z^2 * Vzzλ) / (r^2 + z^2);

In[10]:= Vρρφ = (-r^2 * z * Vrrr + r * (r^2 - 2 * z^2) * Vrrz + z * (2 * r^2 - z^2) * Vzzr + r * z^2 * Vzzz) / 
  (r^2 + z^2)^{3/2};

In[11]:= Vρρρ = (r^3 Vrrr + 3 * r^2 * z * Vrrz + 3 * r * z^2 * Vzzr + z^3 * Vzzz) / (r^2 + z^2)^{3/2};

```

3.2 Tables 11, 12

```

In[12]:= Vx = (-y * √(x^2 + y^2 + z^2) * Vλ - x * z * Vφ + x * √(x^2 + y^2) * Vρ) / (√(x^2 + y^2) * √(x^2 + y^2 + z^2));

In[13]:= Vy = (x * √(x^2 + y^2 + z^2) * Vλ - y * z * Vφ + y * √(x^2 + y^2) * Vρ) / (√(x^2 + y^2) * √(x^2 + y^2 + z^2));

In[14]:= Vz1 = (√(x^2 + y^2) * Vφ + z * Vρ) / √(x^2 + y^2 + z^2);

In[15]:= Vxx = (y^2 * (x^2 + y^2 + z^2) * Vλλ + 2 * x * y * z * √(x^2 + y^2 + z^2) * Vλφ - 
  2 * x * y * √(x^2 + y^2) * √(x^2 + y^2 + z^2) * Vλρ + x^2 * z^2 * Vφφ - 
  2 * x^2 * z * √(x^2 + y^2) * Vφρ + x^2 * (x^2 + y^2) * Vρρ) / ((x^2 + y^2) * (x^2 + y^2 + z^2));

In[16]:= Vxy = (-x * y * (x^2 + y^2 + z^2)^{3/2} * Vλλ - 
  z * (x^2 - y^2) * (x^2 + y^2 + z^2) * Vλφ + (x^2 - y^2) * √(x^2 + y^2) * (x^2 + y^2 + z^2) * Vλρ + 
  x * y * z^2 * √(x^2 + y^2 + z^2) * Vφφ - 2 * x * y * z * √(x^2 + y^2) * √(x^2 + y^2 + z^2) * Vφρ + 
  x * y * (x^2 + y^2) * √(x^2 + y^2 + z^2) * Vρρ) / ((x^2 + y^2) * (x^2 + y^2 + z^2)^{3/2});

In[17]:= Vxz = (-y * √(x^2 + y^2) * √(x^2 + y^2 + z^2) * Vλφ - y * z * √(x^2 + y^2 + z^2) * Vλρ - x * z * √(x^2 + y^2) * Vφφ + 
  x * (x^2 + y^2 - z^2) * Vφρ + x * z * √(x^2 + y^2) * Vρρ) / (√(x^2 + y^2) * (x^2 + y^2 + z^2));

In[18]:= Vyy = (x^2 * (x^2 + y^2 + z^2) * Vλλ - 2 * x * y * z * √(x^2 + y^2 + z^2) * Vλφ + 
  2 * x * y * √(x^2 + y^2) * √(x^2 + y^2 + z^2) * Vλρ + y^2 * z^2 * Vφφ - 
  2 * y^2 * z * √(x^2 + y^2) * Vφρ + y^2 * (x^2 + y^2) * Vρρ) / ((x^2 + y^2) * (x^2 + y^2 + z^2));

In[19]:= Vyz = (x * √(x^2 + y^2) * √(x^2 + y^2 + z^2) * Vλφ + x * z * √(x^2 + y^2 + z^2) * Vλρ - y * z * √(x^2 + y^2) * Vφφ + 
  y * (x^2 + y^2 - z^2) * Vφρ + y * z * √(x^2 + y^2) * Vρρ) / (√(x^2 + y^2) * (x^2 + y^2 + z^2));

In[20]:= Vzz1 = ((x^2 + y^2) * Vφφ + 2 * z * √(x^2 + y^2) * Vφρ + z^2 * Vρρ) / (x^2 + y^2 + z^2);

```

In[8]:= Vxxx =

$$\begin{aligned} & \left(-y^3 * (x^2 + y^2 + z^2)^{3/2} * V\lambda\lambda\lambda - 3 * x * y^2 * z * (x^2 + y^2 + z^2) * V\lambda\lambda\varphi + 3 * x * y^2 * \sqrt{x^2 + y^2} * \right. \\ & \quad \left(x^2 + y^2 + z^2 \right) * V\lambda\lambda\rho + 6 * x^2 * y * z * \sqrt{x^2 + y^2} * \sqrt{x^2 + y^2 + z^2} * V\lambda\varphi\rho - \\ & \quad 3 * x^2 * y * z^2 * \sqrt{x^2 + y^2 + z^2} * V\varphi\varphi\lambda - x^3 * z^3 * V\varphi\varphi\varphi + 3 * x^3 * z^2 * \sqrt{x^2 + y^2} * V\varphi\varphi\rho - \\ & \quad 3 * x^2 * y * (x^2 + y^2) * \sqrt{x^2 + y^2 + z^2} * V\rho\rho\lambda - 3 * x^3 * z * (x^2 + y^2) * V\rho\rho\varphi + \\ & \quad \left. x^3 * (x^2 + y^2)^{3/2} * V\rho\rho\rho \right) / \left((x^2 + y^2)^{3/2} * (x^2 + y^2 + z^2)^{3/2} \right); \end{aligned}$$

$$\begin{aligned} & \text{i}n[9]:= Vxxxy = \left(x * y^2 * (x^2 + y^2 + z^2)^{3/2} * V\lambda\lambda\lambda + y * z * (2 * x^2 - y^2) * (x^2 + y^2 + z^2) * V\lambda\lambda\varphi - \right. \\ & \quad y * \sqrt{x^2 + y^2} * (2 * x^2 - y^2) * (x^2 + y^2 + z^2) * V\lambda\lambda\rho - \\ & \quad 2 * x * z * (x^2 - 2 * y^2) * \sqrt{x^2 + y^2} * \sqrt{x^2 + y^2 + z^2} * V\lambda\varphi\rho + x * z^2 * (x^2 - 2 * y^2) * \\ & \quad \left. \sqrt{x^2 + y^2 + z^2} * V\varphi\varphi\lambda - x^2 * y * z^3 * V\varphi\varphi\varphi + 3 * x^2 * y * z^2 * \sqrt{x^2 + y^2} * V\varphi\varphi\rho + \right. \\ & \quad x * (x^4 - x^2 * y^2 - 2 * y^4) * \sqrt{x^2 + y^2 + z^2} * V\rho\rho\lambda - 3 * x^2 * y * z * (x^2 + y^2) * V\rho\rho\varphi + \\ & \quad \left. x^2 * y * (x^2 + y^2)^{3/2} * V\rho\rho\rho \right) / \left((x^2 + y^2)^{3/2} * (x^2 + y^2 + z^2)^{3/2} \right); \end{aligned}$$

In[10]:= Vxxxz =

$$\begin{aligned} & \left(y^2 * \sqrt{x^2 + y^2} * (x^2 + y^2 + z^2) * V\lambda\lambda\varphi + y^2 * z * (x^2 + y^2 + z^2) * V\lambda\lambda\rho - 2 * x * y * (x^2 + y^2 - z^2) * \right. \\ & \quad \left. \sqrt{x^2 + y^2 + z^2} * V\lambda\varphi\rho + 2 * x * y * z * \sqrt{x^2 + y^2} * \sqrt{x^2 + y^2 + z^2} * V\varphi\varphi\lambda + \right. \\ & \quad x^2 * z^2 * \sqrt{x^2 + y^2} * V\varphi\varphi\varphi - x^2 * z * (2 * x^2 + 2 * y^2 - z^2) * V\varphi\varphi\rho - \\ & \quad 2 * x * y * z * \sqrt{x^2 + y^2} * \sqrt{x^2 + y^2 + z^2} * V\rho\rho\lambda + x^2 * \sqrt{x^2 + y^2} * (x^2 + y^2 - 2 * z^2) * V\rho\rho\varphi + \\ & \quad \left. x^2 * z * (x^2 + y^2) * V\rho\rho\rho \right) / \left((x^2 + y^2) * (x^2 + y^2 + z^2)^{3/2} \right); \end{aligned}$$

$$\begin{aligned} & \text{i}n[11]:= Vxyz = \left(-x * y * \sqrt{x^2 + y^2} * (x^2 + y^2 + z^2) * V\lambda\lambda\varphi - \right. \\ & \quad x * y * z * (x^2 + y^2 + z^2) * V\lambda\lambda\rho + (x^2 - y^2) * (x^2 + y^2 - z^2) * \sqrt{x^2 + y^2 + z^2} * V\lambda\varphi\rho - \\ & \quad z * (x^2 - y^2) * \sqrt{x^2 + y^2} * \sqrt{x^2 + y^2 + z^2} * V\varphi\varphi\lambda + x * y * z^2 * \sqrt{x^2 + y^2} * V\varphi\varphi\varphi - \\ & \quad x * y * z * (2 * x^2 + 2 * y^2 - z^2) * V\varphi\varphi\rho + z * (x^2 - y^2) * \sqrt{x^2 + y^2} * \sqrt{x^2 + y^2 + z^2} * V\rho\rho\lambda + \\ & \quad x * y * (x^2 + y^2 - 2 * z^2) * \sqrt{x^2 + y^2} * V\rho\rho\varphi + \\ & \quad \left. x * y * z * (x^2 + y^2) * V\rho\rho\rho \right) / \left((x^2 + y^2) * (x^2 + y^2 + z^2)^{3/2} \right); \end{aligned}$$

$$\begin{aligned} & \text{i}n[12]:= Vyyx = \left(-x^2 * y * (x^2 + y^2 + z^2)^{3/2} * V\lambda\lambda\lambda - x * z * (x^2 - 2 * y^2) * (x^2 + y^2 + z^2) * V\lambda\lambda\varphi + \right. \\ & \quad x * (x^2 - 2 * y^2) * \sqrt{x^2 + y^2} * (x^2 + y^2 + z^2) * V\lambda\lambda\rho - \\ & \quad 2 * y * z * (2 * x^2 - y^2) * \sqrt{x^2 + y^2} * \sqrt{x^2 + y^2 + z^2} * V\lambda\varphi\rho + y * z^2 * (2 * x^2 - y^2) * \\ & \quad \left. \sqrt{x^2 + y^2 + z^2} * V\varphi\varphi\lambda - x * y^2 * z^3 * V\varphi\varphi\varphi + 3 * x * y^2 * z^2 * \sqrt{x^2 + y^2} * V\varphi\varphi\rho + \right. \\ & \quad y * (2 * x^4 + x^2 * y^2 - y^4) * \sqrt{x^2 + y^2 + z^2} * V\rho\rho\lambda - 3 * x * y^2 * z * (x^2 + y^2) * V\rho\rho\varphi + \\ & \quad \left. x * y^2 * (x^2 + y^2)^{3/2} * V\rho\rho\rho \right) / \left((x^2 + y^2)^{3/2} * (x^2 + y^2 + z^2)^{3/2} \right); \end{aligned}$$

$$\text{In[1]:= } Vyyy = \left(x^3 * (x^2 + y^2 + z^2)^{3/2} * V\lambda\lambda\lambda - \right. \\ \left. 3 * x^2 * y * z * (x^2 + y^2 + z^2) * V\lambda\lambda\varphi + 3 * x^2 * y * \sqrt{x^2 + y^2} * (x^2 + y^2 + z^2) * V\lambda\lambda\rho - \right. \\ \left. 6 * x * y^2 * z * \sqrt{x^2 + y^2} * \sqrt{x^2 + y^2 + z^2} * V\lambda\varphi\rho + 3 * x * y^2 * z^2 * \sqrt{x^2 + y^2 + z^2} * V\varphi\varphi\lambda - \right. \\ \left. y^3 * z^3 * V\varphi\varphi\varphi + 3 * y^3 * z^2 * \sqrt{x^2 + y^2} * V\varphi\varphi\rho + 3 * x * y^2 * (x^2 + y^2) * \sqrt{x^2 + y^2 + z^2} * V\rho\rho\lambda - \right. \\ \left. 3 * y^3 * z * (x^2 + y^2) * V\rho\rho\varphi + y^3 * (x^2 + y^2)^{3/2} * V\rho\rho\rho \right) / \left((x^2 + y^2)^{3/2} * (x^2 + y^2 + z^2)^{3/2} \right);$$

$$\text{In[2]:= } Vyyz = \left(x^2 * \sqrt{x^2 + y^2} * (x^2 + y^2 + z^2) * V\lambda\lambda\varphi + x^2 * z * (x^2 + y^2 + z^2) * V\lambda\lambda\rho + 2 * x * y * (x^2 + y^2 - z^2) * \right. \\ \left. \sqrt{x^2 + y^2 + z^2} * V\lambda\varphi\rho - 2 * x * y * z * \sqrt{x^2 + y^2} * \sqrt{x^2 + y^2 + z^2} * V\varphi\varphi\lambda + \right. \\ \left. y^2 * z^2 * \sqrt{x^2 + y^2} * V\varphi\varphi\varphi - y^2 * z * (2 * x^2 + 2 * y^2 - z^2) * V\varphi\varphi\rho + \right. \\ \left. 2 * x * y * z * \sqrt{x^2 + y^2} * \sqrt{x^2 + y^2 + z^2} * V\rho\rho\lambda + y^2 * (x^2 + y^2 - 2 * z^2) * \sqrt{x^2 + y^2} * V\rho\rho\varphi + \right. \\ \left. y^2 * z * (x^2 + y^2) * V\rho\rho\rho \right) / \left((x^2 + y^2) * (x^2 + y^2 + z^2)^{3/2} \right);$$

$$\text{In[3]:= } Vzzx = \left(-2 * y * z * \sqrt{x^2 + y^2} * \sqrt{x^2 + y^2 + z^2} * V\lambda\varphi\rho - y * (x^2 + y^2) * \sqrt{x^2 + y^2 + z^2} * V\varphi\varphi\lambda - \right. \\ \left. x * z * (x^2 + y^2) * V\varphi\varphi\varphi + x * (x^2 + y^2 - 2 * z^2) * \sqrt{x^2 + y^2} * V\varphi\varphi\rho - \right. \\ \left. y * z^2 * \sqrt{x^2 + y^2 + z^2} * V\rho\rho\lambda + x * z * (2 * x^2 + 2 * y^2 - z^2) * V\rho\rho\varphi + \right. \\ \left. x * z^2 * \sqrt{x^2 + y^2} * V\rho\rho\rho \right) / \left(\sqrt{x^2 + y^2} * (x^2 + y^2 + z^2)^{3/2} \right);$$

$$\text{In[4]:= } Vzzy = \left(2 * x * z * \sqrt{x^2 + y^2} * \sqrt{x^2 + y^2 + z^2} * V\lambda\varphi\rho + x * (x^2 + y^2) * \sqrt{x^2 + y^2 + z^2} * V\varphi\varphi\lambda - \right. \\ \left. y * z * (x^2 + y^2) * V\varphi\varphi\varphi + y * (x^2 + y^2 - 2 * z^2) * \sqrt{x^2 + y^2} * V\varphi\varphi\rho + \right. \\ \left. x * z^2 * \sqrt{x^2 + y^2 + z^2} * V\rho\rho\lambda + y * z * (2 * x^2 + 2 * y^2 - z^2) * V\rho\rho\varphi + \right. \\ \left. y * z^2 * \sqrt{x^2 + y^2} * V\rho\rho\rho \right) / \left(\sqrt{x^2 + y^2} * (x^2 + y^2 + z^2)^{3/2} \right);$$

$$\text{In[5]:= } Vzzz1 = \left((x^2 + y^2)^{3/2} * V\varphi\varphi\varphi + 3 * z * (x^2 + y^2) * V\varphi\varphi\rho + 3 * z^2 * \sqrt{x^2 + y^2} * V\rho\rho\varphi + z^3 * V\rho\rho\rho \right) / \left((x^2 + y^2 + z^2)^{3/2} \right);$$

3.3 Table 3

$$\text{In[6]:= } VrRoutes351 = (x * Vx + y * Vy) / \sqrt{x^2 + y^2};$$

$$\text{In[7]:= } V\lambda Routes351 = (-y * Vx + x * Vy) / \sqrt{x^2 + y^2};$$

$$\text{In[8]:= } VzRoutes351 = Vz1;$$

$$\text{In[9]:= } VrrRoutes351 = (x^2 * Vxx + 2 * x * y * Vxy + y^2 * Vyy) / (x^2 + y^2);$$

$$\text{In[10]:= } Vr\lambda Routes351 = (-x * y * Vxx + (x^2 - y^2) * Vxy + x * y * Vyy) / (x^2 + y^2);$$

$$\text{In[11]:= } VrzRoutes351 = (x * Vxz + y * Vyz) / \sqrt{x^2 + y^2};$$

$$\text{In[12]:= } V\lambda\lambda Routes351 = (y^2 * Vxx - 2 * x * y * Vxy + x^2 * Vyy) / (x^2 + y^2);$$

```

In[1]:= VλzRoutes351 = (-y * Vxz + x * Vyz) / √(x^2 + y^2);

In[2]:= VzzRoutes351 = Vzz1;

In[3]:= VrrrRoutes351 = (x^3 * Vxxx + 3 * x^2 * y * Vxxy + 3 * x * y^2 * Vyyx + y^3 * Vyyy) / (x^2 + y^2)^3/2;

In[4]:= VrrλRoutes351 =
(-x^2 * y * Vxxx + x * (x^2 - 2 * y^2) * Vxxy + y * (2 * x^2 - y^2) * Vyyx + x * y^2 * Vyyy) /
(x^2 + y^2)^3/2;

In[5]:= VrrzRoutes351 = (x^2 * Vxxz + 2 * x * y * Vxyz + y^2 * Vyyz) / (x^2 + y^2);

In[6]:= VrλzRoutes351 = (-x * y * Vxxz + (x^2 - y^2) * Vxyz + x * y * Vyyz) / (x^2 + y^2);

In[7]:= VλλrRoutes351 =
(x * y^2 * Vxxx + y * (y^2 - 2 * x^2) * Vxxy + x * (x^2 - 2 * y^2) * Vyyx + x^2 * y * Vyyy) / (x^2 + y^2)^3/2;

In[8]:= VλλλRoutes351 = (-y^3 * Vxxx + 3 * x * y^2 * Vxxy - 3 * x^2 * y * Vyyx + x^3 * Vyyy) / (x^2 + y^2)^3/2;

In[9]:= VλλzRoutes351 = (y^2 * Vxxz - 2 * x * y * Vxyz + x^2 * Vyyz) / (x^2 + y^2);

In[10]:= VzzrRoutes351 = (x * Vzzx + y * Vzy) / √(x^2 + y^2);

In[11]:= VzzλRoutes351 = (-y * Vzzx + x * Vzy) / √(x^2 + y^2);

In[12]:= VzzzRoutes351 = Vzzz1;

```

3.4 Check whether they become themselves with $r = \sqrt{x^2 + y^2}$

```

In[1]:= FullSimplify[VrRoutes351 /. {r → √(x^2 + y^2)}]

Out[1]=
Vr

In[2]:= FullSimplify[VλRoutes351 /. {r → √(x^2 + y^2)}]

Out[2]=
Vλ

In[3]:= FullSimplify[VzRoutes351 /. {r → √(x^2 + y^2)}]

Out[3]=
Vz

In[4]:= FullSimplify[VrrRoutes351 /. {r → √(x^2 + y^2)}]

Out[4]=
Vrr

In[5]:= FullSimplify[VrλRoutes351 /. {r → √(x^2 + y^2)}]

Out[5]=
Vrλ

```

In[1]:= FullSimplify[VrzRoutes351 /. {r → √(x² + y²)}]

Out[1]=

Vrz

In[2]:= FullSimplify[VλλRoutes351 /. {r → √(x² + y²)}]

Out[2]=

Vλλ

In[3]:= FullSimplify[VλzRoutes351 /. {r → √(x² + y²)}]

Out[3]=

Vλz

In[4]:= FullSimplify[VzzRoutes351 /. {r → √(x² + y²)}]

Out[4]=

Vzz

In[5]:= FullSimplify[VrrrRoutes351 /. {r → √(x² + y²)}]

Out[5]=

Vrrr

In[6]:= FullSimplify[VrrλRoutes351 /. {r → √(x² + y²)}]

Out[6]=

Vrrλ

In[7]:= FullSimplify[VrrzRoutes351 /. {r → √(x² + y²)}]

Out[7]=

Vrrz

In[8]:= FullSimplify[VrλzRoutes351 /. {r → √(x² + y²)}]

Out[8]=

Vrλz

In[9]:= FullSimplify[VλλrRoutes351 /. {r → √(x² + y²)}]

Out[9]=

Vλλr

In[10]:= FullSimplify[VλλλRoutes351 /. {r → √(x² + y²)}]

Out[10]=

Vλλλ

In[11]:= FullSimplify[VλλzRoutes351 /. {r → √(x² + y²)}]

Out[11]=

Vλλz

In[12]:= FullSimplify[VzzrRoutes351 /. {r → √(x² + y²)}]

Out[12]=

Vzzr

In[8]:= FullSimplify[VzzλRoutes351 /. {r → √(x^2 + y^2)}]

Out[8]=

Vzzλ

In[9]:= FullSimplify[VzzzRoutes351 /. {r → √(x^2 + y^2)}]

Out[9]=

Vzzz

4. Algebraic form, Cylindrical coordinates, Routes 2 → 6 → 4, and Table 5 → Table 14 → Table 9

In[10]:= Clear["Global`*"];

4.1 Table 5

In[11]:= Vx = (x * Vr - y * Vλ) / √(x^2 + y^2);

In[12]:= Vy = (y * Vr + x * Vλ) / √(x^2 + y^2);

In[13]:= Vz = Vz;

In[14]:= Vxx = (x^2 * Vrr - 2 * x * y * Vrλ + y^2 * Vλλ) / (x^2 + y^2);

In[15]:= Vxy = (x * y * Vrr + (x^2 - y^2) * Vrλ - x * y * Vλλ) / (x^2 + y^2);

In[16]:= Vxz = (x * Vrz - y * Vλz) / √(x^2 + y^2);

In[17]:= Vyy = (y^2 * Vrr + 2 * x * y * Vrλ + x^2 * Vλλ) / (x^2 + y^2);

In[18]:= Vyz = (y * Vrz + x * Vλz) / √(x^2 + y^2);

In[19]:= Vzz = Vzz;

In[20]:= Vxxx = (x^3 * Vrrr - 3 * x^2 * y * Vrrλ + 3 * x * y^2 * Vλλr - y^3 * Vλλλ) / (x^2 + y^2)^{3/2};

In[21]:= Vxxy =

(x^2 * y * Vrrr + x * (x^2 - 2 * y^2) * Vrrλ + y * (y^2 - 2 * x^2) * Vλλr + x * y^2 * Vλλλ) / (x^2 + y^2)^{3/2};

In[22]:= Vxxz = (x^2 * Vrrz - 2 * x * y * Vrλz + y^2 * Vλλz) / (x^2 + y^2);

In[23]:= Vxyz = (x * y * Vrrz + (x^2 - y^2) * Vrλz - x * y * Vλλz) / (x^2 + y^2);

In[24]:= Vyyx =

(x * y^2 * Vrrr + y * (2 * x^2 - y^2) * Vrrλ + x * (x^2 - 2 * y^2) * Vλλr - x^2 * y * Vλλλ) / (x^2 + y^2)^{3/2};

In[25]:= Vyyy = (y^3 * Vrrr + 3 * x * y^2 * Vrrλ + 3 * x^2 * y * Vλλr + x^3 * Vλλλ) / (x^2 + y^2)^{3/2};

In[26]:= Vyyz = (y^2 * Vrrz + 2 * x * y * Vrλz + x^2 * Vλλz) / (x^2 + y^2);

In[27]:= Vzzx = (x * Vzrz - y * Vzzλ) / √(x^2 + y^2);

$$\text{In[1]:= } \nabla_{zz}y = (y * \nabla_{zz}r + x * \nabla_{zz}\lambda) / \sqrt{x^2 + y^2};$$

$$\text{In[2]:= } \nabla_{zzz} = \nabla_{zzz};$$

4.2 Table 14

$$\text{In[3]:= } \nabla_{\lambda}1 = (-y * \nabla_x + x * \nabla_y) / \sqrt{x^2 + y^2};$$

$$\text{In[4]:= } \nabla_{\varphi} = \left(-x * z * \nabla_x - y * z * \nabla_y + (x^2 + y^2) * \nabla_z \right) / \left(\sqrt{x^2 + y^2} * \sqrt{x^2 + y^2 + z^2} \right);$$

$$\text{In[5]:= } \nabla_{\rho} = (x * \nabla_x + y * \nabla_y + z * \nabla_z) / \sqrt{x^2 + y^2 + z^2};$$

$$\text{In[6]:= } \nabla_{\lambda\lambda}1 = (y^2 * \nabla_{xx} - 2 * x * y * \nabla_{xy} + x^2 * \nabla_{yy}) / (x^2 + y^2);$$

$$\text{In[7]:= } \nabla_{\lambda\varphi} = \left(x * y * z * \nabla_{xx} - z * (x^2 - y^2) * \nabla_{xy} - y * (x^2 + y^2) * \nabla_{xz} - x * y * z * \nabla_{yy} + x * (x^2 + y^2) * \nabla_{yz} \right) / \left((x^2 + y^2) * \sqrt{x^2 + y^2 + z^2} \right);$$

$$\text{In[8]:= } \nabla_{\lambda\rho} = \left(-x * y * \nabla_{xx} + (x^2 - y^2) * \nabla_{xy} - y * z * \nabla_{xz} + x * z * \nabla_{yz} + x * y * \nabla_{yy} \right) / \left(\sqrt{x^2 + y^2} * \sqrt{x^2 + y^2 + z^2} \right);$$

$$\text{In[9]:= } \nabla_{\varphi\varphi} = \left(x^2 * z^2 * \nabla_{xx} + 2 * x * y * z^2 * \nabla_{xy} - 2 * x * z * (x^2 + y^2) * \nabla_{xz} + y^2 * z^2 * \nabla_{yy} - 2 * y * z * (x^2 + y^2) * \nabla_{yz} + (x^2 + y^2)^2 * \nabla_{zz} \right) / \left((x^2 + y^2) * (x^2 + y^2 + z^2) \right);$$

$$\text{In[10]:= } \nabla_{\varphi\rho} = \left(-x^2 * z * \nabla_{xx} - 2 * x * y * z * \nabla_{xy} + x * (x^2 + y^2 - z^2) * \nabla_{xz} - y^2 * z * \nabla_{yy} + y * (x^2 + y^2 - z^2) * \nabla_{yz} + z * (x^2 + y^2) * \nabla_{zz} \right) / \left(\sqrt{x^2 + y^2} * (x^2 + y^2 + z^2) \right);$$

$$\text{In[11]:= } \nabla_{\rho\rho} = (x^2 * \nabla_{xx} + 2 * x * y * \nabla_{xy} + 2 * x * z * \nabla_{xz} + 2 * y * z * \nabla_{yz} + y^2 * \nabla_{yy} + z^2 * \nabla_{zz}) / (x^2 + y^2 + z^2);$$

$$\text{In[12]:= } \nabla_{\lambda\lambda\lambda}1 = (-y^3 * \nabla_{xxx} + 3 * x * y^2 * \nabla_{xxy} - 3 * x^2 * y * \nabla_{yyx} + x^3 * \nabla_{yyy}) / (x^2 + y^2)^{3/2};$$

$$\text{In[13]:= } \nabla_{\lambda\lambda\varphi} = \left(-x * y^2 * z * \nabla_{xxx} + y * z * (2 * x^2 - y^2) * \nabla_{xxy} + y^2 * (x^2 + y^2) * \nabla_{xxz} - 2 * x * y * (x^2 + y^2) * \nabla_{xyz} + x * z * (2 * y^2 - x^2) * \nabla_{yyx} - x^2 * y * z * \nabla_{yyy} + x^2 * (x^2 + y^2) * \nabla_{yyz} \right) / \left((x^2 + y^2)^{3/2} * \sqrt{x^2 + y^2 + z^2} \right);$$

$$\text{In[14]:= } \nabla_{\lambda\lambda\rho} = (x * y^2 * \nabla_{xxx} + y * (y^2 - 2 * x^2) * \nabla_{xxy} + y^2 * z * \nabla_{xxz} - 2 * x * y * z * \nabla_{xyz} + x * (x^2 - 2 * y^2) * \nabla_{yyx} + x^2 * y * \nabla_{yyy} + x^2 * z * \nabla_{yyz}) / \left((x^2 + y^2) * \sqrt{x^2 + y^2 + z^2} \right);$$

$$\text{In[15]:= } \nabla_{\lambda\varphi\rho} = \left(x^2 * y * z * \nabla_{xxx} + x * z * (2 * y^2 - x^2) * \nabla_{xxy} - x * y * (x^2 + y^2 - z^2) * \nabla_{xxz} + (x^2 - y^2) * (x^2 + y^2 - z^2) * \nabla_{xyz} + y * z * (y^2 - 2 * x^2) * \nabla_{yyx} - x * y^2 * z * \nabla_{yyy} + x * y * (x^2 + y^2 - z^2) * \nabla_{yyz} - y * z * (x^2 + y^2) * \nabla_{zzx} + x * z * (x^2 + y^2) * \nabla_{zzy} \right) / \left((x^2 + y^2) * (x^2 + y^2 + z^2) \right);$$

```

In[8]:= Vφφλ = (-x^2 * y * z^2 * Vxxx + x * z^2 * (x^2 - 2 * y^2) * Vxxxy +
2 * x * y * z * (x^2 + y^2) * Vxxz - 2 * z * (x^4 - y^4) * Vxyz +
y * z^2 * (2 * x^2 - y^2) * Vyyx + x * y^2 * z^2 * Vyyy - 2 * x * y * z * (x^2 + y^2) * Vyyz -
y * (x^2 + y^2)^2 * Vzzx + x * (x^2 + y^2)^2 * Vzzy) / ((x^2 + y^2)^{3/2} * (x^2 + y^2 + z^2));

```

```

In[9]:= Vφφφ = (-x^3 * z^3 * Vxxx - 3 * x^2 * y * z^3 * Vxxxy + 3 * x^2 * z^2 * (x^2 + y^2) * Vxxz +
6 * x * y * z^2 * (x^2 + y^2) * Vxyz - 3 * x * y^2 * z^3 * Vyyx - y^3 * z^3 * Vyyy +
3 * y^2 * z^2 * (x^2 + y^2) * Vyz - 3 * x * z * (x^2 + y^2)^2 * Vzzx -
3 * y * z * (x^2 + y^2)^2 * Vzzy + (x^2 + y^2)^3 * Vzzz) / ((x^2 + y^2)^{3/2} * (x^2 + y^2 + z^2)^{3/2});

```

```

In[10]:= Vφφρ = (x^3 * z^2 * Vxxx + 3 * x^2 * y * z^2 * Vxxxy +
x^2 * z * (z^2 - 2 * x^2 - 2 * y^2) * Vxxz + 2 * x * y * z * (z^2 - 2 * x^2 - 2 * y^2) * Vxyz +
3 * x * y^2 * z^2 * Vyyx + y^3 * z^2 * Vyyy + y^2 * z * (z^2 - 2 * x^2 - 2 * y^2) * Vyz +
x * (x^2 + y^2) * (x^2 + y^2 - 2 * z^2) * Vzzx + y * (x^2 + y^2) * (x^2 + y^2 - 2 * z^2) * Vzzy +
z * (x^2 + y^2)^2 * Vzzz) / ((x^2 + y^2) * (x^2 + y^2 + z^2)^{3/2});

```

```

In[11]:= Vρρλ = (-x^2 * y * Vxxx + x * (x^2 - 2 * y^2) * Vxxxy - 2 * x * y * z * Vxxz +
2 * z * (x^2 - y^2) * Vxyz + y * (2 * x^2 - y^2) * Vyyx + x * y^2 * Vyyy +
2 * x * y * z * Vyz - y * z^2 * Vzzx + x * z^2 * Vzzy) / (sqrt(x^2 + y^2) * (x^2 + y^2 + z^2));

```

```

In[12]:= Vρρφ = (-x^3 * z * Vxxx - 3 * x^2 * y * z * Vxxxy + x^2 * (x^2 + y^2 - 2 * z^2) * Vxxz +
2 * x * y * (x^2 + y^2 - 2 * z^2) * Vxyz - 3 * x * y^2 * z * Vyyx - y^3 * z * Vyyy +
y^2 * (x^2 + y^2 - 2 * z^2) * Vyz + x * z * (2 * x^2 + 2 * y^2 - z^2) * Vzzx + y * z *
(2 * x^2 + 2 * y^2 - z^2) * Vzzy + z^2 * (x^2 + y^2) * Vzzz) / (sqrt(x^2 + y^2) * (x^2 + y^2 + z^2)^{3/2});

```

```

In[13]:= Vρρρ = (x^3 * Vxxx + 3 * x^2 * y * Vxxxy + 3 * x^2 * z * Vxxz +
6 * x * y * z * Vxyz + 3 * x * y^2 * Vyyx + y^3 * Vyyy + 3 * y^2 * z * Vyz +
3 * x * z^2 * Vzzx + 3 * y * z^2 * Vzzy + z^3 * Vzzz) / (x^2 + y^2 + z^2)^{3/2};

```

4.3 Table 9

```

In[1]:= VrRoutes264 = (-z * Vφ + r * Vρ) / sqrt(r^2 + z^2);

```

```

In[2]:= VλRoutes264 = Vλ1;

```

```

In[3]:= VzRoutes264 = (r * Vφ + z * Vρ) / sqrt(r^2 + z^2);

```

```

In[4]:= VrrRoutes264 = (z^2 * Vφφ - 2 * r * z * Vφρ + r^2 * Vρρ) / (r^2 + z^2);

```

```

In[5]:= VrλRoutes264 = (-z * Vλφ + r * Vλρ) / sqrt(r^2 + z^2);

```

```

In[6]:= VrzRoutes264 = (-r * z * Vφφ + (r^2 - z^2) * Vφρ + r * z * Vρρ) / (r^2 + z^2);

```

```

In[7]:= VλλRoutes264 = Vλλ1;

```

```

In[8]:= VλzRoutes264 = (r * Vλφ + z * Vλρ) / sqrt(r^2 + z^2);

```

```

In[1]:= VzzRoutes264 = (r^2 * Vφφ + 2 * r * z * Vφρ + z^2 * Vρρ) / (r^2 + z^2);

In[2]:= VrrrRoutes264 = (-z^3 * Vφφφ + 3 * r * z^2 * Vφφρ - 3 * r^2 * z * Vρρφ + r^3 * Vρρρ) / (r^2 + z^2)^{3/2};

In[3]:= VrrλRoutes264 = (-2 * r * z * Vλφρ + z^2 * Vφφλ + r^2 * Vρρλ) / (r^2 + z^2);

In[4]:= VrrzRoutes264 =
  (r * z^2 * Vφφφ + z * (z^2 - 2 * r^2) * Vφφρ + r * (r^2 - 2 * z^2) * Vρρφ + r^2 * z * Vρρρ) / (r^2 + z^2)^{3/2};

In[5]:= VrλzRoutes264 = ((r^2 - z^2) * Vλφρ - r * z * Vφφλ + r * z * Vρρλ) / (r^2 + z^2);

In[6]:= VλλrRoutes264 = (-z * Vλλφ + r * Vλλρ) / √(r^2 + z^2);

In[7]:= VλλλRoutes264 = Vλλλ1;

In[8]:= VλλzRoutes264 = (r * Vλλφ + z * Vλλρ) / √(r^2 + z^2);

In[9]:= VzzrRoutes264 =
  (-r^2 * z * Vφφφ + r * (r^2 - 2 * z^2) * Vφφρ + z * (2 * r^2 - z^2) * Vρρφ + r * z^2 * Vρρρ) / (r^2 + z^2)^{3/2};

In[10]:= VzzλRoutes264 = (2 * r * z * Vλφρ + r^2 * Vφφλ + z^2 * Vρρλ) / (r^2 + z^2);

In[11]:= VzzzRoutes264 = (r^3 * Vφφφ + 3 * r^2 * z * Vφφρ + 3 * r * z^2 * Vρρφ + z^3 * Vρρρ) / (r^2 + z^2)^{3/2};

```

4.4 Check whether they become themselves with $r = \sqrt{x^2 + y^2}$

```
In[1]:= FullSimplify[VrRoutes264 /. {r → √(x^2 + y^2)}]
```

Out[1]=

Vr

```
In[2]:= FullSimplify[VλRoutes264 /. {r → √(x^2 + y^2)}]
```

Out[2]=

Vλ

```
In[3]:= FullSimplify[VzRoutes264 /. {r → √(x^2 + y^2)}]
```

Out[3]=

Vz

```
In[4]:= FullSimplify[VrrRoutes264 /. {r → √(x^2 + y^2)}]
```

Out[4]=

Vrr

```
In[5]:= FullSimplify[VrλRoutes264 /. {r → √(x^2 + y^2)}]
```

Out[5]=

Vrλ

In[1]:= FullSimplify[VrzRoutes264 /. {r → √(x^2 + y^2)}]

Out[1]=

Vrz

In[2]:= FullSimplify[VλλRoutes264 /. {r → √(x^2 + y^2)}]

Out[2]=

Vλλ

In[3]:= FullSimplify[VλzRoutes264 /. {r → √(x^2 + y^2)}]

Out[3]=

Vλz

In[4]:= FullSimplify[VzzRoutes264 /. {r → √(x^2 + y^2)}]

Out[4]=

Vzz

In[5]:= FullSimplify[VrrrRoutes264 /. {r → √(x^2 + y^2)}]

Out[5]=

Vrrr

In[6]:= FullSimplify[VrrλRoutes264 /. {r → √(x^2 + y^2)}]

Out[6]=

Vrrλ

In[7]:= FullSimplify[VrrzRoutes264 /. {r → √(x^2 + y^2)}]

Out[7]=

Vrrz

In[8]:= FullSimplify[VrλzRoutes264 /. {r → √(x^2 + y^2)}]

Out[8]=

Vrλz

In[9]:= FullSimplify[VλλrRoutes264 /. {r → √(x^2 + y^2)}]

Out[9]=

Vλλr

In[10]:= FullSimplify[VλλλRoutes264 /. {r → √(x^2 + y^2)}]

Out[10]=

Vλλλ

In[11]:= FullSimplify[VλλzRoutes264 /. {r → √(x^2 + y^2)}]

Out[11]=

Vλλz

In[12]:= FullSimplify[VzzrRoutes264 /. {r → √(x^2 + y^2)}]

Out[12]=

Vzzr

In[1]:= FullSimplify[VzzRoutes264 /. {r → √(x^2 + y^2)}]

Out[1]=

Vzzλ

In[2]:= FullSimplify[VzzzRoutes264 /. {r → √(x^2 + y^2)}]

Out[2]=

Vzzz

5. Algebraic form, Spherical coordinates, Routes 5 → 1 → 3, and Tables 11, 12 → Table 3 → Table 7

In[3]:= Clear["Global`*"];

5.1 Tables 11, 12

$$\text{Vx} = \left(-y * \sqrt{x^2 + y^2 + z^2} * V\lambda - x * z * V\varphi + x * \sqrt{x^2 + y^2} * V\rho \right) / \left(\sqrt{x^2 + y^2} * \sqrt{x^2 + y^2 + z^2} \right);$$

$$\text{Vy} = \left(x * \sqrt{x^2 + y^2 + z^2} * V\lambda - y * z * V\varphi + y * \sqrt{x^2 + y^2} * V\rho \right) / \left(\sqrt{x^2 + y^2} * \sqrt{x^2 + y^2 + z^2} \right);$$

$$\text{Vz} = \left(\sqrt{x^2 + y^2} * V\varphi + z * V\rho \right) / \sqrt{x^2 + y^2 + z^2};$$

$$\begin{aligned} \text{Vxx} = & \left(y^2 * (x^2 + y^2 + z^2) * V\lambda\lambda + 2 * x * y * z * \sqrt{x^2 + y^2 + z^2} * V\lambda\varphi - \right. \\ & \left. 2 * x * y * \sqrt{x^2 + y^2} * \sqrt{x^2 + y^2 + z^2} * V\lambda\rho + x^2 * z^2 * V\varphi\varphi - \right. \\ & \left. 2 * x^2 * z * \sqrt{x^2 + y^2} * V\varphi\rho + x^2 * (x^2 + y^2) * V\rho\rho \right) / ((x^2 + y^2) * (x^2 + y^2 + z^2)); \end{aligned}$$

$$\begin{aligned} \text{Vxy} = & \left(-x * y * (x^2 + y^2 + z^2)^{3/2} * V\lambda\lambda - \right. \\ & \left. z * (x^2 - y^2) * (x^2 + y^2 + z^2) * V\lambda\varphi + (x^2 - y^2) * \sqrt{x^2 + y^2} * (x^2 + y^2 + z^2) * V\lambda\rho + \right. \\ & \left. x * y * z^2 * \sqrt{x^2 + y^2 + z^2} * V\varphi\varphi - 2 * x * y * z * \sqrt{x^2 + y^2} * \sqrt{x^2 + y^2 + z^2} * V\varphi\rho + \right. \\ & \left. x * y * (x^2 + y^2) * \sqrt{x^2 + y^2 + z^2} * V\rho\rho \right) / ((x^2 + y^2) * (x^2 + y^2 + z^2)^{3/2}); \end{aligned}$$

$$\begin{aligned} \text{Vxz} = & \left(-y * \sqrt{x^2 + y^2} * \sqrt{x^2 + y^2 + z^2} * V\lambda\varphi - y * z * \sqrt{x^2 + y^2 + z^2} * V\lambda\rho - x * z * \sqrt{x^2 + y^2} * V\varphi\varphi + \right. \\ & \left. x * (x^2 + y^2 - z^2) * V\varphi\rho + x * z * \sqrt{x^2 + y^2} * V\rho\rho \right) / \left(\sqrt{x^2 + y^2} * (x^2 + y^2 + z^2) \right); \end{aligned}$$

$$\begin{aligned} \text{Vyy} = & \left(x^2 * (x^2 + y^2 + z^2) * V\lambda\lambda - 2 * x * y * z * \sqrt{x^2 + y^2 + z^2} * V\lambda\varphi + \right. \\ & \left. 2 * x * y * \sqrt{x^2 + y^2} * \sqrt{x^2 + y^2 + z^2} * V\lambda\rho + y^2 * z^2 * V\varphi\varphi - \right. \\ & \left. 2 * y^2 * z * \sqrt{x^2 + y^2} * V\varphi\rho + y^2 * (x^2 + y^2) * V\rho\rho \right) / ((x^2 + y^2) * (x^2 + y^2 + z^2)); \end{aligned}$$

$$\begin{aligned} \text{Vyz} = & \left(x * \sqrt{x^2 + y^2} * \sqrt{x^2 + y^2 + z^2} * V\lambda\varphi + x * z * \sqrt{x^2 + y^2 + z^2} * V\lambda\rho - y * z * \sqrt{x^2 + y^2} * V\varphi\varphi + \right. \\ & \left. y * (x^2 + y^2 - z^2) * V\varphi\rho + y * z * \sqrt{x^2 + y^2} * V\rho\rho \right) / \left(\sqrt{x^2 + y^2} * (x^2 + y^2 + z^2) \right); \end{aligned}$$

$$\text{Vzz} = \left((x^2 + y^2) * V\varphi\varphi + 2 * z * \sqrt{x^2 + y^2} * V\varphi\rho + z^2 * V\rho\rho \right) / (x^2 + y^2 + z^2);$$

In[1]:= Vxxx =

$$\begin{aligned} & \left(-y^3 * (x^2 + y^2 + z^2)^{3/2} * V\lambda\lambda\lambda - 3 * x * y^2 * z * (x^2 + y^2 + z^2) * V\lambda\lambda\varphi + 3 * x * y^2 * \sqrt{x^2 + y^2} * \right. \\ & \quad \left(x^2 + y^2 + z^2 \right) * V\lambda\lambda\rho + 6 * x^2 * y * z * \sqrt{x^2 + y^2} * \sqrt{x^2 + y^2 + z^2} * V\lambda\varphi\rho - \\ & \quad 3 * x^2 * y * z^2 * \sqrt{x^2 + y^2 + z^2} * V\varphi\varphi\lambda - x^3 * z^3 * V\varphi\varphi\varphi + 3 * x^3 * z^2 * \sqrt{x^2 + y^2} * V\varphi\varphi\rho - \\ & \quad 3 * x^2 * y * (x^2 + y^2) * \sqrt{x^2 + y^2 + z^2} * V\rho\rho\lambda - 3 * x^3 * z * (x^2 + y^2) * V\rho\rho\varphi + \\ & \quad \left. x^3 * (x^2 + y^2)^{3/2} * V\rho\rho\rho \right) / \left((x^2 + y^2)^{3/2} * (x^2 + y^2 + z^2)^{3/2} \right); \end{aligned}$$

$$\begin{aligned} & \text{i}n[2]:= Vxxxy = \left(x * y^2 * (x^2 + y^2 + z^2)^{3/2} * V\lambda\lambda\lambda + y * z * (2 * x^2 - y^2) * (x^2 + y^2 + z^2) * V\lambda\lambda\varphi - \right. \\ & \quad y * \sqrt{x^2 + y^2} * (2 * x^2 - y^2) * (x^2 + y^2 + z^2) * V\lambda\lambda\rho - \\ & \quad 2 * x * z * (x^2 - 2 * y^2) * \sqrt{x^2 + y^2} * \sqrt{x^2 + y^2 + z^2} * V\lambda\varphi\rho + x * z^2 * (x^2 - 2 * y^2) * \\ & \quad \left. \sqrt{x^2 + y^2 + z^2} * V\varphi\varphi\lambda - x^2 * y * z^3 * V\varphi\varphi\varphi + 3 * x^2 * y * z^2 * \sqrt{x^2 + y^2} * V\varphi\varphi\rho + \right. \\ & \quad x * (x^4 - x^2 * y^2 - 2 * y^4) * \sqrt{x^2 + y^2 + z^2} * V\rho\rho\lambda - 3 * x^2 * y * z * (x^2 + y^2) * V\rho\rho\varphi + \\ & \quad \left. x^2 * y * (x^2 + y^2)^{3/2} * V\rho\rho\rho \right) / \left((x^2 + y^2)^{3/2} * (x^2 + y^2 + z^2)^{3/2} \right); \end{aligned}$$

In[3]:= Vxxxz =

$$\begin{aligned} & \left(y^2 * \sqrt{x^2 + y^2} * (x^2 + y^2 + z^2) * V\lambda\lambda\varphi + y^2 * z * (x^2 + y^2 + z^2) * V\lambda\lambda\rho - 2 * x * y * (x^2 + y^2 - z^2) * \right. \\ & \quad \left. \sqrt{x^2 + y^2 + z^2} * V\lambda\varphi\rho + 2 * x * y * z * \sqrt{x^2 + y^2} * \sqrt{x^2 + y^2 + z^2} * V\varphi\varphi\lambda + \right. \\ & \quad x^2 * z^2 * \sqrt{x^2 + y^2} * V\varphi\varphi\varphi - x^2 * z * (2 * x^2 + 2 * y^2 - z^2) * V\varphi\varphi\rho - \\ & \quad 2 * x * y * z * \sqrt{x^2 + y^2} * \sqrt{x^2 + y^2 + z^2} * V\rho\rho\lambda + x^2 * \sqrt{x^2 + y^2} * (x^2 + y^2 - 2 * z^2) * V\rho\rho\varphi + \\ & \quad \left. x^2 * z * (x^2 + y^2) * V\rho\rho\rho \right) / \left((x^2 + y^2) * (x^2 + y^2 + z^2)^{3/2} \right); \end{aligned}$$

$$\begin{aligned} & \text{i}n[4]:= Vxyz = \left(-x * y * \sqrt{x^2 + y^2} * (x^2 + y^2 + z^2) * V\lambda\lambda\varphi - \right. \\ & \quad x * y * z * (x^2 + y^2 + z^2) * V\lambda\lambda\rho + (x^2 - y^2) * (x^2 + y^2 - z^2) * \sqrt{x^2 + y^2 + z^2} * V\lambda\varphi\rho - \\ & \quad z * (x^2 - y^2) * \sqrt{x^2 + y^2} * \sqrt{x^2 + y^2 + z^2} * V\varphi\varphi\lambda + x * y * z^2 * \sqrt{x^2 + y^2} * V\varphi\varphi\varphi - \\ & \quad x * y * z * (2 * x^2 + 2 * y^2 - z^2) * V\varphi\varphi\rho + z * (x^2 - y^2) * \sqrt{x^2 + y^2} * \sqrt{x^2 + y^2 + z^2} * V\rho\rho\lambda + \\ & \quad x * y * (x^2 + y^2 - 2 * z^2) * \sqrt{x^2 + y^2} * V\rho\rho\varphi + \\ & \quad \left. x * y * z * (x^2 + y^2) * V\rho\rho\rho \right) / \left((x^2 + y^2) * (x^2 + y^2 + z^2)^{3/2} \right); \end{aligned}$$

$$\begin{aligned} & \text{i}n[5]:= Vyyx = \left(-x^2 * y * (x^2 + y^2 + z^2)^{3/2} * V\lambda\lambda\lambda - x * z * (x^2 - 2 * y^2) * (x^2 + y^2 + z^2) * V\lambda\lambda\varphi + \right. \\ & \quad x * (x^2 - 2 * y^2) * \sqrt{x^2 + y^2} * (x^2 + y^2 + z^2) * V\lambda\lambda\rho - \\ & \quad 2 * y * z * (2 * x^2 - y^2) * \sqrt{x^2 + y^2} * \sqrt{x^2 + y^2 + z^2} * V\lambda\varphi\rho + y * z^2 * (2 * x^2 - y^2) * \\ & \quad \left. \sqrt{x^2 + y^2 + z^2} * V\varphi\varphi\lambda - x * y^2 * z^3 * V\varphi\varphi\varphi + 3 * x * y^2 * z^2 * \sqrt{x^2 + y^2} * V\varphi\varphi\rho + \right. \\ & \quad y * (2 * x^4 + x^2 * y^2 - y^4) * \sqrt{x^2 + y^2 + z^2} * V\rho\rho\lambda - 3 * x * y^2 * z * (x^2 + y^2) * V\rho\rho\varphi + \\ & \quad \left. x * y^2 * (x^2 + y^2)^{3/2} * V\rho\rho\rho \right) / \left((x^2 + y^2)^{3/2} * (x^2 + y^2 + z^2)^{3/2} \right); \end{aligned}$$

$$\text{In[1]:= } Vyyy = \left(x^3 * (x^2 + y^2 + z^2)^{3/2} * V\lambda\lambda\lambda - \right. \\ \left. 3 * x^2 * y * z * (x^2 + y^2 + z^2) * V\lambda\lambda\varphi + 3 * x^2 * y * \sqrt{x^2 + y^2} * (x^2 + y^2 + z^2) * V\lambda\lambda\rho - \right. \\ \left. 6 * x * y^2 * z * \sqrt{x^2 + y^2} * \sqrt{x^2 + y^2 + z^2} * V\lambda\varphi\rho + 3 * x * y^2 * z^2 * \sqrt{x^2 + y^2 + z^2} * V\varphi\varphi\lambda - \right. \\ \left. y^3 * z^3 * V\varphi\varphi\varphi + 3 * y^3 * z^2 * \sqrt{x^2 + y^2} * V\varphi\varphi\rho + 3 * x * y^2 * (x^2 + y^2) * \sqrt{x^2 + y^2 + z^2} * V\rho\rho\lambda - \right. \\ \left. 3 * y^3 * z * (x^2 + y^2) * V\rho\rho\varphi + y^3 * (x^2 + y^2)^{3/2} * V\rho\rho\rho \right) / \left((x^2 + y^2)^{3/2} * (x^2 + y^2 + z^2)^{3/2} \right);$$

$$\text{In[2]:= } Vyyz = \left(x^2 * \sqrt{x^2 + y^2} * (x^2 + y^2 + z^2) * V\lambda\lambda\varphi + x^2 * z * (x^2 + y^2 + z^2) * V\lambda\lambda\rho + 2 * x * y * (x^2 + y^2 - z^2) * \right. \\ \left. \sqrt{x^2 + y^2 + z^2} * V\lambda\varphi\rho - 2 * x * y * z * \sqrt{x^2 + y^2} * \sqrt{x^2 + y^2 + z^2} * V\varphi\varphi\lambda + \right. \\ \left. y^2 * z^2 * \sqrt{x^2 + y^2} * V\varphi\varphi\varphi - y^2 * z * (2 * x^2 + 2 * y^2 - z^2) * V\varphi\varphi\rho + \right. \\ \left. 2 * x * y * z * \sqrt{x^2 + y^2} * \sqrt{x^2 + y^2 + z^2} * V\rho\rho\lambda + y^2 * (x^2 + y^2 - 2 * z^2) * \sqrt{x^2 + y^2} * V\rho\rho\varphi + \right. \\ \left. y^2 * z * (x^2 + y^2) * V\rho\rho\rho \right) / \left((x^2 + y^2) * (x^2 + y^2 + z^2)^{3/2} \right);$$

$$\text{In[3]:= } Vzzx = \left(-2 * y * z * \sqrt{x^2 + y^2} * \sqrt{x^2 + y^2 + z^2} * V\lambda\varphi\rho - y * (x^2 + y^2) * \sqrt{x^2 + y^2 + z^2} * V\varphi\varphi\lambda - \right. \\ \left. x * z * (x^2 + y^2) * V\varphi\varphi\varphi + x * (x^2 + y^2 - 2 * z^2) * \sqrt{x^2 + y^2} * V\varphi\varphi\rho - \right. \\ \left. y * z^2 * \sqrt{x^2 + y^2 + z^2} * V\rho\rho\lambda + x * z * (2 * x^2 + 2 * y^2 - z^2) * V\rho\rho\varphi + \right. \\ \left. x * z^2 * \sqrt{x^2 + y^2} * V\rho\rho\rho \right) / \left(\sqrt{x^2 + y^2} * (x^2 + y^2 + z^2)^{3/2} \right);$$

$$\text{In[4]:= } Vzzy = \left(2 * x * z * \sqrt{x^2 + y^2} * \sqrt{x^2 + y^2 + z^2} * V\lambda\varphi\rho + x * (x^2 + y^2) * \sqrt{x^2 + y^2 + z^2} * V\varphi\varphi\lambda - \right. \\ \left. y * z * (x^2 + y^2) * V\varphi\varphi\varphi + y * (x^2 + y^2 - 2 * z^2) * \sqrt{x^2 + y^2} * V\varphi\varphi\rho + \right. \\ \left. x * z^2 * \sqrt{x^2 + y^2 + z^2} * V\rho\rho\lambda + y * z * (2 * x^2 + 2 * y^2 - z^2) * V\rho\rho\varphi + \right. \\ \left. y * z^2 * \sqrt{x^2 + y^2} * V\rho\rho\rho \right) / \left(\sqrt{x^2 + y^2} * (x^2 + y^2 + z^2)^{3/2} \right);$$

$$\text{In[5]:= } Vzzz = \left((x^2 + y^2)^{3/2} * V\varphi\varphi\varphi + 3 * z * (x^2 + y^2) * V\varphi\varphi\rho + 3 * z^2 * \sqrt{x^2 + y^2} * V\rho\rho\varphi + z^3 * V\rho\rho\rho \right) / \\ (x^2 + y^2 + z^2)^{3/2};$$

5.2 Table 3

$$\text{In[6]:= } Vr = (x * Vx + y * Vy) / \sqrt{x^2 + y^2};$$

$$\text{In[7]:= } V\lambda 1 = (-y * Vx + x * Vy) / \sqrt{x^2 + y^2};$$

$$\text{In[8]:= } Vz = Vz;$$

$$\text{In[9]:= } Vrr = (x^2 * Vxx + 2 * x * y * Vxy + y^2 * Vyy) / (x^2 + y^2);$$

$$\text{In[10]:= } Vr\lambda = (-x * y * Vxx + (x^2 - y^2) * Vxy + x * y * Vyy) / (x^2 + y^2);$$

$$\text{In[11]:= } Vrz = (x * Vxz + y * Vyz) / \sqrt{x^2 + y^2};$$

$$\text{In[12]:= } V\lambda\lambda 1 = (y^2 * Vxx - 2 * x * y * Vxy + x^2 * Vyy) / (x^2 + y^2);$$

```

In[6]:= Vλz = (-y * Vxz + x * Vy়) / √(x^2 + y^2) ;
In[6]:= Vzz = Vzz;
In[6]:= Vrrr = (x^3 * Vxxx + 3 * x^2 * y * Vxxy + 3 * x * y^2 * Vyyx + y^3 * Vyyy) / (x^2 + y^2)^{3/2} ;
In[6]:= Vrrλ = (-x^2 * y * Vxxx + x * (x^2 - 2 * y^2) * Vxxy + y * (2 * x^2 - y^2) * Vyyx + x * y^2 * Vyyy) / (x^2 + y^2)^{3/2} ;
In[6]:= Vrrz = (x^2 * Vxxz + 2 * x * y * Vxyz + y^2 * Vyyz) / (x^2 + y^2) ;
In[6]:= Vrλz = (-x * y * Vxxz + (x^2 - y^2) * Vxyz + x * y * Vyyz) / (x^2 + y^2) ;
In[6]:= Vλλr = (x * y^2 * Vxxx + y * (y^2 - 2 * x^2) * Vxxy + x * (x^2 - 2 * y^2) * Vyyx + x^2 * y * Vyyy) / (x^2 + y^2)^{3/2} ;
In[6]:= Vλλλ1 = (-y^3 * Vxxx + 3 * x * y^2 * Vxxy - 3 * x^2 * y * Vyyx + x^3 * Vyyy) / (x^2 + y^2)^{3/2} ;
In[6]:= Vλλz = (y^2 * Vxxz - 2 * x * y * Vxyz + x^2 * Vyyz) / (x^2 + y^2) ;
In[6]:= Vzzr = (x * Vzzx + y * Vzy) / √(x^2 + y^2) ;
In[6]:= Vzzλ = (-y * Vzzx + x * Vzy) / √(x^2 + y^2) ;
In[6]:= Vzzz = Vzzz;

```

5.3 Table 7

```

In[6]:= VλRoutes513 = Vλ1;
In[6]:= VφRoutes513 = (-z * Vr + r * Vz) / √(r^2 + z^2) ;
In[6]:= VρRoutes513 = (r * Vr + z * Vz) / √(r^2 + z^2) ;
In[6]:= VλλRoutes513 = Vλλ1;
In[6]:= VλφRoutes513 = (-z * Vrλ + r * Vλz) / √(r^2 + z^2) ;
In[6]:= VλρRoutes513 = (r * Vrλ + z * Vλz) / √(r^2 + z^2) ;
In[6]:= VφφRoutes513 = (z^2 * Vrr - 2 * r * z * Vrz + r^2 * Vzz) / (r^2 + z^2) ;
In[6]:= VφρRoutes513 = (-r * z * Vrr + (r^2 - z^2) * Vrz + r * z * Vzz) / (r^2 + z^2) ;
In[6]:= VρρRoutes513 = (r^2 * Vrr + 2 * r * z * Vrz + z^2 * Vzz) / (r^2 + z^2) ;
In[6]:= VλλλRoutes513 = Vλλλ1;
In[6]:= VλλφRoutes513 = (-z * Vλλr + r * Vλλz) / √(r^2 + z^2) ;
In[6]:= VλλρRoutes513 = (r * Vλλr + z * Vλλz) / √(r^2 + z^2) ;

```

```

In[1]:= VλφρRoutes513 = (-r * z * Vrrλ + (r^2 - z^2) * Vrλz + r * z * Vzzλ) / (r^2 + z^2);
In[2]:= VφφλRoutes513 = (z^2 * Vrrλ - 2 * r * z * Vrλz + r^2 * Vzzλ) / (r^2 + z^2);
In[3]:= VφφφRoutes513 = (-z^3 * Vrrr + 3 * r * z^2 * Vrrz - 3 * r^2 * z * Vzrz + r^3 * Vzzz) / (r^2 + z^2)^{3/2};
In[4]:= VφρρRoutes513 =
(r * z^2 * Vrrr + z * (z^2 - 2 * r^2) * Vrrz + r * (r^2 - 2 * z^2) * Vzrz + r^2 * z * Vzzz) / (r^2 + z^2)^{3/2};
In[5]:= VρρλRoutes513 = (r^2 * Vrrλ + 2 * r * z * Vrλz + z^2 * Vzzλ) / (r^2 + z^2);
In[6]:= VρρφRoutes513 =
(-r^2 * z * Vrrr + r * (r^2 - 2 * z^2) * Vrrz + z * (2 * r^2 - z^2) * Vzrz + r * z^2 * Vzzz) /
(r^2 + z^2)^{3/2};
In[7]:= VρρρRoutes513 = (r^3 * Vrrr + 3 * r^2 * z * Vrrz + 3 * r * z^2 * Vzrz + z^3 * Vzzz) / (r^2 + z^2)^{3/2};

```

5.4 Check whether they become themselves with $r = \sqrt{x^2 + y^2}$

```
In[1]:= FullSimplify[VλRoutes513 /. {r → √(x^2 + y^2)}]
```

```
Out[1]= Vλ
```

```
In[2]:= FullSimplify[VφRoutes513 /. {r → √(x^2 + y^2)}]
```

```
Out[2]= Vφ
```

```
In[3]:= FullSimplify[VρRoutes513 /. {r → √(x^2 + y^2)}]
```

```
Out[3]= Vρ
```

```
In[4]:= FullSimplify[VλλRoutes513 /. {r → √(x^2 + y^2)}]
```

```
Out[4]= Vλλ
```

```
In[5]:= FullSimplify[VλφRoutes513 /. {r → √(x^2 + y^2)}]
```

```
Out[5]= Vλφ
```

```
In[6]:= FullSimplify[VλρRoutes513 /. {r → √(x^2 + y^2)}]
```

```
Out[6]= Vλρ
```

```
In[7]:= FullSimplify[VφφRoutes513 /. {r → √(x^2 + y^2)}]
```

```
Out[7]= Vφφ
```

In[1]:= FullSimplify[V $\varphi\rho$ Routes513 /. {r $\rightarrow \sqrt{x^2 + y^2}$ }]

Out[1]=

V $\varphi\rho$

In[2]:= FullSimplify[V $\rho\rho$ Routes513 /. {r $\rightarrow \sqrt{x^2 + y^2}$ }]

Out[2]=

V $\rho\rho$

In[3]:= FullSimplify[V $\lambda\lambda\lambda$ Routes513 /. {r $\rightarrow \sqrt{x^2 + y^2}$ }]

Out[3]=

V $\lambda\lambda\lambda$

In[4]:= FullSimplify[V $\lambda\lambda\varphi$ Routes513 /. {r $\rightarrow \sqrt{x^2 + y^2}$ }]

Out[4]=

V $\lambda\lambda\varphi$

In[5]:= FullSimplify[V $\lambda\lambda\rho$ Routes513 /. {r $\rightarrow \sqrt{x^2 + y^2}$ }]

Out[5]=

V $\lambda\lambda\rho$

In[6]:= FullSimplify[V $\lambda\varphi\rho$ Routes513 /. {r $\rightarrow \sqrt{x^2 + y^2}$ }]

Out[6]=

V $\lambda\varphi\rho$

In[7]:= FullSimplify[V $\varphi\varphi\lambda$ Routes513 /. {r $\rightarrow \sqrt{x^2 + y^2}$ }]

Out[7]=

V $\varphi\varphi\lambda$

In[8]:= FullSimplify[V $\varphi\varphi\varphi$ Routes513 /. {r $\rightarrow \sqrt{x^2 + y^2}$ }]

Out[8]=

V $\varphi\varphi\varphi$

In[9]:= FullSimplify[V $\varphi\varphi\rho$ Routes513 /. {r $\rightarrow \sqrt{x^2 + y^2}$ }]

Out[9]=

V $\varphi\varphi\rho$

In[10]:= FullSimplify[V $\rho\rho\lambda$ Routes513 /. {r $\rightarrow \sqrt{x^2 + y^2}$ }]

Out[10]=

V $\rho\rho\lambda$

In[11]:= FullSimplify[V $\rho\rho\varphi$ Routes513 /. {r $\rightarrow \sqrt{x^2 + y^2}$ }]

Out[11]=

V $\rho\rho\varphi$

In[12]:= FullSimplify[V $\rho\rho\rho$ Routes513 /. {r $\rightarrow \sqrt{x^2 + y^2}$ }]

Out[12]=

V $\rho\rho\rho$

6. Algebraic form, Spherical coordinates, Routes 4 → 2 → 6, and Table 9 → Table 5 → Table 14

```
In[1]:= Clear["Global`*"];
```

6.1 Table 9

$$\text{In[1]:= } \nabla r = (-z * \nabla \varphi + r * \nabla \rho) / \sqrt{r^2 + z^2};$$

$$\text{In[2]:= } \nabla \lambda = \nabla \lambda;$$

$$\text{In[3]:= } \nabla z = (r * \nabla \varphi + z * \nabla \rho) / \sqrt{r^2 + z^2};$$

$$\text{In[4]:= } \nabla rr = (z^2 * \nabla \varphi \varphi - 2 * r * z * \nabla \varphi \rho + r^2 * \nabla \rho \rho) / (r^2 + z^2);$$

$$\text{In[5]:= } \nabla r \lambda = (-z * \nabla \lambda \varphi + r * \nabla \lambda \rho) / \sqrt{r^2 + z^2};$$

$$\text{In[6]:= } \nabla r z = (-r * z * \nabla \varphi \varphi + (r^2 - z^2) * \nabla \varphi \rho + r * z * \nabla \rho \rho) / (r^2 + z^2);$$

$$\text{In[7]:= } \nabla \lambda \lambda = \nabla \lambda \lambda;$$

$$\text{In[8]:= } \nabla \lambda z = (r * \nabla \lambda \varphi + z * \nabla \lambda \rho) / \sqrt{r^2 + z^2};$$

$$\text{In[9]:= } \nabla zz = (r^2 * \nabla \varphi \varphi + 2 * r * z * \nabla \varphi \rho + z^2 * \nabla \rho \rho) / (r^2 + z^2);$$

$$\text{In[10]:= } \nabla rrr = (-z^3 * \nabla \varphi \varphi \varphi + 3 * r * z^2 * \nabla \varphi \varphi \rho - 3 * r^2 * z * \nabla \rho \rho \varphi + r^3 * \nabla \rho \rho \rho) / (r^2 + z^2)^{3/2};$$

$$\text{In[11]:= } \nabla rr \lambda = (-2 * r * z * \nabla \lambda \varphi \rho + z^2 * \nabla \varphi \varphi \lambda + r^2 * \nabla \rho \rho \lambda) / (r^2 + z^2);$$

$$\text{In[12]:= } \nabla rr z =$$

$$(r * z^2 * \nabla \varphi \varphi \varphi + z * (z^2 - 2 * r^2) * \nabla \varphi \varphi \rho + r * (r^2 - 2 * z^2) * \nabla \rho \rho \varphi + r^2 * z * \nabla \rho \rho \rho) / (r^2 + z^2)^{3/2};$$

$$\text{In[13]:= } \nabla r \lambda z = ((r^2 - z^2) * \nabla \lambda \varphi \rho - r * z * \nabla \varphi \varphi \lambda + r * z * \nabla \rho \rho \lambda) / (r^2 + z^2);$$

$$\text{In[14]:= } \nabla \lambda \lambda r = (-z * \nabla \lambda \lambda \varphi + r * \nabla \lambda \lambda \rho) / \sqrt{r^2 + z^2};$$

$$\text{In[15]:= } \nabla \lambda \lambda \lambda = \nabla \lambda \lambda \lambda;$$

$$\text{In[16]:= } \nabla \lambda \lambda z = (r * \nabla \lambda \lambda \varphi + z * \nabla \lambda \lambda \rho) / \sqrt{r^2 + z^2};$$

$$\text{In[17]:= } \nabla zzr = (-r^2 * z * \nabla \varphi \varphi \varphi + r * (r^2 - 2 * z^2) * \nabla \varphi \varphi \rho + z * (2 * r^2 - z^2) * \nabla \rho \rho \varphi + r * z^2 * \nabla \rho \rho \rho) / (r^2 + z^2)^{3/2};$$

$$\text{In[18]:= } \nabla zz \lambda = (2 * r * z * \nabla \lambda \varphi \rho + r^2 * \nabla \varphi \varphi \lambda + z^2 * \nabla \rho \rho \lambda) / (r^2 + z^2);$$

$$\text{In[19]:= } \nabla zzz = (r^3 * \nabla \varphi \varphi \varphi + 3 * r^2 * z * \nabla \varphi \varphi \rho + 3 * r * z^2 * \nabla \rho \rho \varphi + z^3 * \nabla \rho \rho \rho) / (r^2 + z^2)^{3/2};$$

6.2 Table 5

$$\text{In[1]:= } \nabla x = (x * \nabla r - y * \nabla \lambda) / \sqrt{x^2 + y^2};$$

```

In[6]:= Vy = (y * Vr + x * Vlambda) / Sqrt[x^2 + y^2];
In[6]:= Vz = Vz;
In[6]:= Vxx = (x^2 * Vrr - 2 * x * y * Vrlambda + y^2 * Vlambda^2) / (x^2 + y^2);
In[6]:= Vxy = (x * y * Vrr + (x^2 - y^2) * Vrlambda - x * y * Vlambda^2) / (x^2 + y^2);
In[6]:= Vxz = (x * Vrz - y * Vlambda z) / Sqrt[x^2 + y^2];
In[6]:= Vyy = (y^2 * Vrr + 2 * x * y * Vrlambda + x^2 * Vlambda^2) / (x^2 + y^2);
In[6]:= Vyz = (y * Vrz + x * Vlambda z) / Sqrt[x^2 + y^2];
In[6]:= Vzz = Vzz;
In[6]:= Vxxx = (x^3 * Vrrr - 3 * x^2 * y * Vrrlambda + 3 * x * y^2 * Vlambda r - y^3 * Vlambda^3) / (x^2 + y^2)^{3/2};
In[6]:= Vxxy =
  (x^2 * y * Vrrr + x * (x^2 - 2 * y^2) * Vrrlambda + y * (y^2 - 2 * x^2) * Vlambda r + x * y^2 * Vlambda^2) / (x^2 + y^2)^{3/2};
In[6]:= Vxxz = (x^2 * Vrrz - 2 * x * y * Vrlambda z + y^2 * Vlambda z) / (x^2 + y^2);
In[6]:= Vxyz = (x * y * Vrrz + (x^2 - y^2) * Vrlambda z - x * y * Vlambda z) / (x^2 + y^2);
In[6]:= Vyyx =
  (x * y^2 * Vrrr + y * (2 * x^2 - y^2) * Vrrlambda + x * (x^2 - 2 * y^2) * Vlambda r - x^2 * y * Vlambda^2) / (x^2 + y^2)^{3/2};
In[6]:= Vyyy = (y^3 * Vrrr + 3 * x * y^2 * Vrrlambda + 3 * x^2 * y * Vlambda r + x^3 * Vlambda^3) / (x^2 + y^2)^{3/2};
In[6]:= Vyyz = (y^2 * Vrrz + 2 * x * y * Vrlambda z + x^2 * Vlambda z) / (x^2 + y^2);
In[6]:= Vzzx = (x * Vzzr - y * Vzzlambda) / Sqrt[x^2 + y^2];
In[6]:= Vzzy = (y * Vzzr + x * Vzzlambda) / Sqrt[x^2 + y^2];
In[6]:= Vzzz = Vzzz;

```

6.3 Table 14

```

In[6]:= VlambdaRoutes426 = (-y * Vx + x * Vy) / Sqrt[x^2 + y^2];
In[6]:= VphiRoutes426 = (-x * z * Vx - y * z * Vy + (x^2 + y^2) * Vz) / (Sqrt[x^2 + y^2] * Sqrt[x^2 + y^2 + z^2]);
In[6]:= VrhoRoutes426 = (x * Vx + y * Vy + z * Vz) / Sqrt[x^2 + y^2 + z^2];
In[6]:= VlambdaRoutes426 = (y^2 * Vxx - 2 * x * y * Vxy + x^2 * Vyy) / (x^2 + y^2);
In[6]:= VlambdaPhiRoutes426 = (x * y * z * Vxx - z * (x^2 - y^2) * Vxy - y * (x^2 + y^2) * Vxz -
  x * y * z * Vyy + x * (x^2 + y^2) * Vyz) / ((x^2 + y^2) * Sqrt[x^2 + y^2 + z^2]);

```

```

In[]:= VλρRoutes426 = (-x * y * Vxx + (x2 - y2) * Vxy - y * z * Vxz + x * z * Vyz + x * y * Vyy) /
  (sqrt(x2 + y2) * sqrt(x2 + y2 + z2));

```

```

In[]:= VφψRoutes426 = (x2 * z2 * Vxx + 2 * x * y * z2 * Vxy - 2 * x * z * (x2 + y2) * Vxz + y2 * z2 * Vyy -
  2 * y * z * (x2 + y2) * Vyz + (x2 + y2)2 * Vzz) / ((x2 + y2) * (x2 + y2 + z2));

```

```

In[]:= VφρRoutes426 = (-x2 * z * Vxx - 2 * x * y * z * Vxy + x * (x2 + y2 - z2) * Vxz - y2 * z * Vyy +
  y * (x2 + y2 - z2) * Vyz + z * (x2 + y2) * Vzz) / (sqrt(x2 + y2) * (x2 + y2 + z2));

```

```

In[]:= VρρRoutes426 =
  (x2 * Vxx + 2 * x * y * Vxy + 2 * x * z * Vxz + 2 * y * z * Vyz + y2 * Vyy + z2 * Vzz) / (x2 + y2 + z2);

```

```

In[]:= VλλλRoutes426 = (-y3 * Vxxx + 3 * x * y2 * Vxxy - 3 * x2 * y * Vyyx + x3 * Vyyy) / (x2 + y2)3/2;

```

```

In[]:= VλλφRoutes426 = (-x * y2 * z * Vxxx + y * z * (2 * x2 - y2) * Vxxy +
  y2 * (x2 + y2) * Vxxz - 2 * x * y * (x2 + y2) * Vxyz + x * z * (2 * y2 - x2) * Vyyx -
  x2 * y * z * Vyyy + x2 * (x2 + y2) * Vyyz) / ((x2 + y2)3/2 * sqrt(x2 + y2 + z2));

```

```

In[]:= VλλρRoutes426 = (x * y2 * Vxxx + y * (y2 - 2 * x2) * Vxxy + y2 * z * Vxxz - 2 * x * y * z * Vxyz +
  x * (x2 - 2 * y2) * Vyyx + x2 * y * Vyyy + x2 * z * Vyyz) / ((x2 + y2) * sqrt(x2 + y2 + z2));

```

```

In[]:= VλφρRoutes426 = (x2 * y * z * Vxxx + x * z * (2 * y2 - x2) * Vxxy - x * y * (x2 + y2 - z2) * Vxxz +
  y * z * (y2 - 2 * x2) * Vyyx - x * y2 * z * Vyyy + x * y * (x2 + y2 - z2) * Vyyz -
  y * z * (x2 + y2) * Vzzx + x * z * (x2 + y2) * Vzzy) / ((x2 + y2) * (x2 + y2 + z2));

```

```

In[]:= VφφλRoutes426 =
  (-x2 * y * z2 * Vxxx + x * z2 * (x2 - 2 * y2) * Vxxy + 2 * x * y * z * (x2 + y2) * Vxxz -
  2 * z * (x4 - y4) * Vxyz + y * z2 * (2 * x2 - y2) * Vyyx + x * y2 * z2 * Vyyy -
  2 * x * y * z * (x2 + y2) * Vyyz - y * (x2 + y2)2 * Vzzx +
  x * (x2 + y2)2 * Vzzy) / ((x2 + y2)3/2 * (x2 + y2 + z2));

```

```

In[]:= VφφφRoutes426 = (-x3 * z3 * Vxxx - 3 * x2 * y * z3 * Vxxy +
  3 * x2 * z2 * (x2 + y2) * Vxxz + 6 * x * y * z2 * (x2 + y2) * Vxyz - 3 * x * y2 * z3 * Vyyx -
  y3 * z3 * Vyyy + 3 * y2 * z2 * (x2 + y2) * Vyyz - 3 * x * z * (x2 + y2)2 * Vzzx -
  3 * y * z * (x2 + y2)2 * Vzzy + (x2 + y2)3 * Vzzz) / ((x2 + y2)3/2 * (x2 + y2 + z2)3/2);

```

```

In[]:= VφφρRoutes426 = (x3 * z2 * Vxxx + 3 * x2 * y * z2 * Vxxy +
  x2 * z * (z2 - 2 * x2 - 2 * y2) * Vxxz + 2 * x * y * z * (z2 - 2 * x2 - 2 * y2) * Vxyz +
  3 * x * y2 * z2 * Vyyx + y3 * z2 * Vyyy + y2 * z * (z2 - 2 * x2 - 2 * y2) * Vyyz +
  x * (x2 + y2) * (x2 + y2 - 2 * z2) * Vzzx + y * (x2 + y2) * (x2 + y2 - 2 * z2) * Vzzy +
  z * (x2 + y2)2 * Vzzz) / ((x2 + y2) * (x2 + y2 + z2)3/2);

```

```
In[1]:= VρρλRoutes426 =
(-x^2 * y * Vxxxx + x * (x^2 - 2 * y^2) * Vxxxy - 2 * x * y * z * Vxxz + 2 * z * (x^2 - y^2) * Vxyz +
y * (2 * x^2 - y^2) * Vyyx + x * y^2 * Vyyy + 2 * x * y * z * Vyyz -
y * z^2 * Vzzx + x * z^2 * Vzzy) / (sqrt(x^2 + y^2) * (x^2 + y^2 + z^2));

In[2]:= VρρφRoutes426 = (-x^3 * z * Vxxxx - 3 * x^2 * y * z * Vxxxy + x^2 * (x^2 + y^2 - 2 * z^2) * Vxxz +
2 * x * y * (x^2 + y^2 - 2 * z^2) * Vxyz - 3 * x * y^2 * z * Vyx - y^3 * z * Vyyy +
y^2 * (x^2 + y^2 - 2 * z^2) * Vyz + x * z * (2 * x^2 + 2 * y^2 - z^2) * Vzzx + y * z *
(2 * x^2 + 2 * y^2 - z^2) * Vzzy + z^2 * (x^2 + y^2) * Vzzz) / (sqrt(x^2 + y^2) * (x^2 + y^2 + z^2)^{3/2});

In[3]:= VρρρRoutes426 = (x^3 * Vxxxx + 3 * x^2 * y * Vxxxy + 3 * x^2 * z * Vxxz +
6 * x * y * z * Vxyz + 3 * x * y^2 * Vyyx + y^3 * Vyyy + 3 * y^2 * z * Vyyz +
3 * x * z^2 * Vzzx + 3 * y * z^2 * Vzzy + z^3 * Vzzz) / (x^2 + y^2 + z^2)^{3/2};
```

6.4 Check whether they become themselves with $r = \sqrt{x^2 + y^2}$

```
In[1]:= FullSimplify[VλRoutes426 /. {r → sqrt(x^2 + y^2)}]
```

```
Out[1]= Vλ
```

```
In[2]:= FullSimplify[VφRoutes426 /. {r → sqrt(x^2 + y^2)}]
```

```
Out[2]= Vφ
```

```
In[3]:= FullSimplify[VρRoutes426 /. {r → sqrt(x^2 + y^2)}]
```

```
Out[3]= Vρ
```

```
In[4]:= FullSimplify[VλλRoutes426 /. {r → sqrt(x^2 + y^2)}]
```

```
Out[4]= Vλλ
```

```
In[5]:= FullSimplify[VλφRoutes426 /. {r → sqrt(x^2 + y^2)}]
```

```
Out[5]= Vλφ
```

```
In[6]:= FullSimplify[VλρRoutes426 /. {r → sqrt(x^2 + y^2)}]
```

```
Out[6]= Vλρ
```

```
In[7]:= FullSimplify[VφφRoutes426 /. {r → sqrt(x^2 + y^2)}]
```

```
Out[7]= Vφφ
```

In[1]:= FullSimplify[V $\varphi\rho$ Routes426 /. {r $\rightarrow \sqrt{x^2 + y^2}$ }]

Out[1]=

V $\varphi\rho$

In[2]:= FullSimplify[V $\rho\rho$ Routes426 /. {r $\rightarrow \sqrt{x^2 + y^2}$ }]

Out[2]=

V $\rho\rho$

In[3]:= FullSimplify[V $\lambda\lambda\lambda$ Routes426 /. {r $\rightarrow \sqrt{x^2 + y^2}$ }]

Out[3]=

V $\lambda\lambda\lambda$

In[4]:= FullSimplify[V $\lambda\lambda\varphi$ Routes426 /. {r $\rightarrow \sqrt{x^2 + y^2}$ }]

Out[4]=

V $\lambda\lambda\varphi$

In[5]:= FullSimplify[V $\lambda\lambda\rho$ Routes426 /. {r $\rightarrow \sqrt{x^2 + y^2}$ }]

Out[5]=

V $\lambda\lambda\rho$

In[6]:= FullSimplify[V $\lambda\varphi\rho$ Routes426 /. {r $\rightarrow \sqrt{x^2 + y^2}$ }]

Out[6]=

V $\lambda\varphi\rho$

In[7]:= FullSimplify[V $\varphi\varphi\lambda$ Routes426 /. {r $\rightarrow \sqrt{x^2 + y^2}$ }]

Out[7]=

V $\varphi\varphi\lambda$

In[8]:= FullSimplify[V $\varphi\varphi\varphi$ Routes426 /. {r $\rightarrow \sqrt{x^2 + y^2}$ }]

Out[8]=

V $\varphi\varphi\varphi$

In[9]:= FullSimplify[V $\varphi\varphi\rho$ Routes426 /. {r $\rightarrow \sqrt{x^2 + y^2}$ }]

Out[9]=

V $\varphi\varphi\rho$

In[10]:= FullSimplify[V $\rho\rho\lambda$ Routes426 /. {r $\rightarrow \sqrt{x^2 + y^2}$ }]

Out[10]=

V $\rho\rho\lambda$

In[11]:= FullSimplify[V $\rho\rho\varphi$ Routes426 /. {r $\rightarrow \sqrt{x^2 + y^2}$ }]

Out[11]=

V $\rho\rho\varphi$

In[12]:= FullSimplify[V $\rho\rho\rho$ Routes426 /. {r $\rightarrow \sqrt{x^2 + y^2}$ }]

Out[12]=

V $\rho\rho\rho$

7. Trigonometric form, Cartesian coordinates, Routes 1 → 3 → 5, and Table 4 → Table 8 → Table 13

In[1]:= Clear["Global`*"];

7.1 Table 4

In[2]:= $V_r = \cos[\lambda] * V_x + \sin[\lambda] * V_y;$

In[3]:= $V_\lambda = -\sin[\lambda] * V_x + \cos[\lambda] * V_y;$

In[4]:= $V_z = V_z;$

In[5]:= $V_{rr} = (\cos[\lambda])^2 * V_{xx} + \sin[2*\lambda] * V_{xy} + (\sin[\lambda])^2 * V_{yy};$

In[6]:= $V_{r\lambda} = -\sin[\lambda] * \cos[\lambda] * V_{xx} + \cos[2*\lambda] * V_{xy} + \sin[\lambda] * \cos[\lambda] * V_{yy};$

In[7]:= $V_{rz} = \cos[\lambda] * V_{xz} + \sin[\lambda] * V_{yz};$

In[8]:= $V_{\lambda\lambda} = (\sin[\lambda])^2 * V_{xx} - \sin[2*\lambda] * V_{xy} + (\cos[\lambda])^2 * V_{yy};$

In[9]:= $V_{\lambda z} = -\sin[\lambda] * V_{xz} + \cos[\lambda] * V_{yz};$

In[10]:= $V_{zz} = V_{zz};$

In[11]:= $V_{rrr} = (\cos[\lambda])^3 * V_{xxx} + 3 * \sin[\lambda] * (\cos[\lambda])^2 * V_{xxy} + 3 * \cos[\lambda] * (\sin[\lambda])^2 * V_{yyx} + (\sin[\lambda])^3 * V_{yyy};$

In[12]:= $V_{rr\lambda} = -\sin[\lambda] * (\cos[\lambda])^2 * V_{xxx} + \cos[\lambda] * ((\cos[\lambda])^2 - 2 * (\sin[\lambda])^2) * V_{xxy} + \sin[\lambda] * (2 * (\cos[\lambda])^2 - (\sin[\lambda])^2) * V_{yyx} + (\sin[\lambda])^2 * \cos[\lambda] * V_{yyy};$

In[13]:= $V_{rrz} = (\cos[\lambda])^2 * V_{xxz} + \sin[2*\lambda] * V_{xyz} + (\sin[\lambda])^2 * V_{yyz};$

In[14]:= $V_{r\lambda z} = -\sin[\lambda] * \cos[\lambda] * V_{xxz} + \cos[2*\lambda] * V_{xyz} + \sin[\lambda] * \cos[\lambda] * V_{yyz};$

In[15]:= $V_{\lambda\lambda r} = \cos[\lambda] * (\sin[\lambda])^2 * V_{xxx} + \sin[\lambda] * ((\sin[\lambda])^2 - 2 * (\cos[\lambda])^2) * V_{xxy} + \cos[\lambda] * ((\cos[\lambda])^2 - 2 * (\sin[\lambda])^2) * V_{yyx} + \sin[\lambda] * (\cos[\lambda])^2 * V_{yyy};$

In[16]:= $V_{\lambda\lambda\lambda} = -(\sin[\lambda])^3 * V_{xxx} + 3 * (\sin[\lambda])^2 * \cos[\lambda] * V_{xxy} - 3 * \sin[\lambda] * (\cos[\lambda])^2 * V_{yyx} + (\cos[\lambda])^3 * V_{yyy};$

In[17]:= $V_{\lambda\lambda z} = (\sin[\lambda])^2 * V_{xxz} - \sin[2*\lambda] * V_{xyz} + (\cos[\lambda])^2 * V_{yyz};$

In[18]:= $V_{zzr} = \cos[\lambda] * V_{zzx} + \sin[\lambda] * V_{zzy};$

In[19]:= $V_{zz\lambda} = -\sin[\lambda] * V_{zzx} + \cos[\lambda] * V_{zzy};$

In[20]:= $V_{zzz} = V_{zzz};$

7.2 Table 8

In[1]:= $V_\lambda = V_\lambda;$

In[2]:= $V_\varphi = -\sin[\varphi] * V_r + \cos[\varphi] * V_z;$

In[3]:= $V_\rho = \cos[\varphi] * V_r + \sin[\varphi] * V_z;$

In[4]:= $V_{\lambda\lambda} = V_{\lambda\lambda};$

```

In[1]:= Vλφ = -Sin[φ] * Vrλ + Cos[φ] * Vλz;
In[2]:= Vλρ = Cos[φ] * Vrλ + Sin[φ] * Vλz;
In[3]:= Vφφ = ((Sin[φ])2 * Vrr - Sin[2 * φ] * Vrz + (Cos[φ])2 * Vzz);
In[4]:= Vφρ = -Sin[φ] * Cos[φ] * Vrr + Cos[2 * φ] * Vrz + Sin[φ] * Cos[φ] * Vzz;
In[5]:= Vρρ = ((Cos[φ])2 * Vrr + Sin[2 * φ] * Vrz + (Sin[φ])2 * Vzz);
In[6]:= Vλλλ = Vλλλ;
In[7]:= Vλλφ = -Sin[φ] * Vλλr + Cos[φ] * Vλλz;
In[8]:= Vλλρ = Cos[φ] * Vλλr + Sin[φ] * Vλλz;
In[9]:= Vλφρ = -Sin[φ] * Cos[φ] * Vrrλ + Cos[2 * φ] * Vrλz + Sin[φ] * Cos[φ] * Vzzλ;
In[10]:= Vφφλ = (Sin[φ])2 * Vrrλ - Sin[2 * φ] * Vrλz + (Cos[φ])2 * Vzzλ;
In[11]:= Vφφφ = -(Sin[φ])3 * Vrrr + 3 * (Sin[φ])2 * Cos[φ] * Vrrz -
          3 * Sin[φ] * (Cos[φ])2 * Vzr + (Cos[φ])3 * Vzzz;
In[12]:= Vφφρ = (Sin[φ])2 * Cos[φ] * Vrrr + Sin[φ] * ((Sin[φ])2 - 2 * (Cos[φ])2) * Vrrz +
          Cos[φ] * ((Cos[φ])2 - 2 * (Sin[φ])2) * Vzr + Sin[φ] * (Cos[φ])2 * Vzzz;
In[13]:= Vρρλ = (Cos[φ])2 * Vrrλ + Sin[2 * φ] * Vrλz + (Sin[φ])2 * Vzzλ;
In[14]:= Vρρφ = -Sin[φ] * (Cos[φ])2 * Vrrr + Cos[φ] * ((Cos[φ])2 - 2 * (Sin[φ])2) * Vrrz +
          Sin[φ] * (2 * (Cos[φ])2 - (Sin[φ])2) * Vzr + Cos[φ] * (Sin[φ])2 * Vzzz;
In[15]:= Vρρρ = (Cos[φ])3 * Vrrr + 3 * Sin[φ] * (Cos[φ])2 * Vrrz +
          3 * Cos[φ] * (Sin[φ])2 * Vzr + (Sin[φ])3 * Vzzz;

```

7.4 Table 13

```

In[1]:= VxRoutes135 = -Sin[λ] * Vλ - Sin[φ] * Cos[λ] * Vφ + Cos[φ] * Cos[λ] * Vρ;
In[2]:= VyRoutes135 = Cos[λ] * Vλ - Sin[φ] * Sin[λ] * Vφ + Cos[φ] * Sin[λ] * Vρ;
In[3]:= VzRoutes135 = Cos[φ] * Vφ + Sin[φ] * Vρ;
In[4]:= VxxRoutes135 = (Sin[λ])2 * Vλλ + Sin[φ] * Sin[2 * λ] * Vλφ -
          Cos[φ] * Sin[2 * λ] * Vλρ + (Sin[φ])2 * (Cos[λ])2 * Vφφ -
          Sin[2 * φ] * (Cos[λ])2 * Vφρ + (Cos[φ])2 * (Cos[λ])2 * Vρρ;
In[5]:= VxyRoutes135 = -Sin[λ] * Cos[λ] * Vλλ - Sin[φ] * Cos[2 * λ] * Vλφ +
          Cos[φ] * Cos[2 * λ] * Vλρ + (Sin[φ])2 * Sin[λ] * Cos[λ] * Vφφ -
          Sin[2 * φ] * Sin[λ] * Cos[λ] * Vφρ + (Cos[φ])2 * Sin[λ] * Cos[λ] * Vρρ;
In[6]:= VxzRoutes135 =
          -Cos[φ] * Sin[λ] * Vλφ - Sin[φ] * Sin[λ] * Vλρ - Sin[φ] * Cos[φ] * Cos[λ] * Vφφ +
          Cos[2 * φ] * Cos[λ] * Vφρ + Sin[φ] * Cos[φ] * Cos[λ] * Vρρ;
In[7]:= VyyRoutes135 = (Cos[λ])2 * Vλλ - Sin[φ] * Sin[2 * λ] * Vλφ +
          Cos[φ] * Sin[2 * λ] * Vλρ + (Sin[φ])2 * (Sin[λ])2 * Vφφ -
          Sin[2 * φ] * (Sin[λ])2 * Vφρ + (Cos[φ])2 * (Sin[λ])2 * Vρρ;

```

```

In[1]:= VyzRoutes135 =
Cos[\varphi] * Cos[\lambda] * V\lambda\varphi + Sin[\varphi] * Cos[\lambda] * V\lambda\rho - Sin[\varphi] * Cos[\varphi] * Sin[\lambda] * V\varphi\varphi +
Cos[2 * \varphi] * Sin[\lambda] * V\varphi\rho + Sin[\varphi] * Cos[\varphi] * Sin[\lambda] * V\rho\rho;

In[2]:= VzzRoutes135 = (Cos[\varphi])^2 * V\varphi\varphi + Sin[2 * \varphi] * V\varphi\rho + (Sin[\varphi])^2 * V\rho\rho;

In[3]:= VxxxRoutes135 = - (Sin[\lambda])^3 * V\lambda\lambda\lambda + 3 * Cos[\varphi] * (Sin[\lambda])^2 * Cos[\lambda] * V\lambda\lambda\rho -
3 * Sin[\varphi] * (Sin[\lambda])^2 * Cos[\lambda] * V\lambda\lambda\varphi + 3 * Sin[2 * \varphi] * Sin[\lambda] * (Cos[\lambda])^2 * V\lambda\varphi\rho -
3 * (Sin[\varphi])^2 * Sin[\lambda] * (Cos[\lambda])^2 * V\varphi\varphi\lambda - (Sin[\varphi])^3 * (Cos[\lambda])^3 * V\varphi\varphi\varphi +
3 * (Sin[\varphi])^2 * Cos[\varphi] * (Cos[\lambda])^3 * V\varphi\varphi\rho - 3 * (Cos[\varphi])^2 * Sin[\lambda] * (Cos[\lambda])^2 * V\rho\rho\lambda -
3 * Sin[\varphi] * (Cos[\varphi])^2 * (Cos[\lambda])^3 * V\rho\rho\varphi + (Cos[\varphi])^3 * (Cos[\lambda])^3 * V\rho\rho\rho;

In[4]:= VxxyRoutes135 =
(Sin[\lambda])^2 * Cos[\lambda] * V\lambda\lambda\lambda + Sin[\varphi] * Sin[\lambda] * (2 * (Cos[\lambda])^2 - (Sin[\lambda])^2) * V\lambda\lambda\varphi +
Cos[\varphi] * Sin[\lambda] * ((Sin[\lambda])^2 - 2 * (Cos[\lambda])^2) * V\lambda\lambda\rho +
Sin[2 * \varphi] * Cos[\lambda] * (2 * (Sin[\lambda])^2 - (Cos[\lambda])^2) * V\lambda\varphi\rho +
(Sin[\varphi])^2 * Cos[\lambda] * ((Cos[\lambda])^2 - 2 * (Sin[\lambda])^2) * V\varphi\varphi\lambda - (Sin[\varphi])^3 * Sin[\lambda] *
(Cos[\lambda])^2 * V\varphi\varphi\varphi + 3 * (Sin[\varphi])^2 * Cos[\varphi] * Sin[\lambda] * (Cos[\lambda])^2 * V\varphi\varphi\rho +
(Cos[\varphi])^2 * Cos[\lambda] * ((Cos[\lambda])^2 - 2 * (Sin[\lambda])^2) * V\rho\rho\lambda - 3 * Sin[\varphi] *
(Cos[\varphi])^2 * Sin[\lambda] * (Cos[\lambda])^2 * V\rho\rho\varphi + (Cos[\varphi])^3 * Sin[\lambda] * (Cos[\lambda])^2 * V\rho\rho\rho;

In[5]:= VxxzRoutes135 =
Cos[\varphi] * (Sin[\lambda])^2 * V\lambda\lambda\varphi + Sin[\varphi] * (Sin[\lambda])^2 * V\lambda\lambda\rho - Cos[2 * \varphi] * Sin[2 * \lambda] * V\lambda\varphi\rho +
Sin[\varphi] * Cos[\varphi] * Sin[2 * \lambda] * V\varphi\varphi\lambda + (Sin[\varphi])^2 * Cos[\varphi] * (Cos[\lambda])^2 * V\varphi\varphi\varphi +
Sin[\varphi] * (Cos[\lambda])^2 * ((Sin[\varphi])^2 - 2 * (Cos[\varphi])^2) * V\varphi\varphi\rho -
Sin[2 * \varphi] * Sin[\lambda] * Cos[\lambda] * V\rho\rho\lambda + Cos[\varphi] * (Cos[\lambda])^2 *
((Cos[\varphi])^2 - 2 * (Sin[\varphi])^2) * V\rho\rho\varphi + Sin[\varphi] * (Cos[\varphi])^2 * (Cos[\lambda])^2 * V\rho\rho\rho;

In[6]:= VxyzRoutes135 = - Cos[\varphi] * Sin[\lambda] * Cos[\lambda] * V\lambda\lambda\varphi -
Sin[\varphi] * Sin[\lambda] * Cos[\lambda] * V\lambda\lambda\rho + Cos[2 * \varphi] * Cos[2 * \lambda] * V\lambda\varphi\rho -
Sin[\varphi] * Cos[\varphi] * Cos[2 * \lambda] * V\varphi\varphi\lambda + (Sin[\varphi])^2 * Cos[\varphi] * Sin[\lambda] * Cos[\lambda] * V\varphi\varphi\varphi -
Sin[\varphi] * Sin[\lambda] * Cos[\lambda] * (2 * (Cos[\varphi])^2 - (Sin[\varphi])^2) * V\varphi\varphi\rho +
Sin[\varphi] * Cos[\varphi] * Cos[2 * \lambda] * V\rho\rho\lambda + Cos[\varphi] * Sin[\lambda] * Cos[\lambda] *
((Cos[\varphi])^2 - 2 * (Sin[\varphi])^2) * V\rho\rho\varphi + Sin[\varphi] * (Cos[\varphi])^2 * Sin[\lambda] * Cos[\lambda] * V\rho\rho\rho;

In[7]:= VyyxRoutes135 =
-Sin[\lambda] * (Cos[\lambda])^2 * V\lambda\lambda\lambda - Sin[\varphi] * Cos[\lambda] * ((Cos[\lambda])^2 - 2 * (Sin[\lambda])^2) * V\lambda\lambda\varphi +
Cos[\varphi] * Cos[\lambda] * ((Cos[\lambda])^2 - 2 * (Sin[\lambda])^2) * V\lambda\lambda\rho -
Sin[2 * \varphi] * Sin[\lambda] * (2 * (Cos[\lambda])^2 - (Sin[\lambda])^2) * V\lambda\varphi\rho +
(Sin[\varphi])^2 * Sin[\lambda] * (2 * (Cos[\lambda])^2 - (Sin[\lambda])^2) * V\varphi\varphi\lambda - (Sin[\varphi])^3 * (Sin[\lambda])^2 * V\varphi\varphi\varphi +
(Cos[\varphi])^2 * Cos[\lambda] * V\varphi\varphi\varphi + 3 * (Sin[\varphi])^2 * Cos[\varphi] * (Sin[\lambda])^2 * Cos[\lambda] * V\varphi\varphi\rho +
(Cos[\varphi])^2 * Sin[\lambda] * (2 * (Cos[\lambda])^2 - (Sin[\lambda])^2) * V\rho\rho\lambda - 3 * Sin[\varphi] *
(Cos[\varphi])^2 * (Sin[\lambda])^2 * Cos[\lambda] * V\rho\rho\varphi + (Cos[\varphi])^3 * (Sin[\lambda])^2 * Cos[\lambda] * V\rho\rho\rho;

In[8]:= VyyyRoutes135 = (Cos[\lambda])^3 * V\lambda\lambda\lambda - 3 * Sin[\varphi] * Sin[\lambda] * (Cos[\lambda])^2 * V\lambda\lambda\varphi +
3 * Cos[\varphi] * Sin[\lambda] * (Cos[\lambda])^2 * V\lambda\lambda\rho - 3 * Sin[2 * \varphi] * (Sin[\lambda])^2 * Cos[\lambda] * V\lambda\varphi\rho +
3 * (Sin[\varphi])^2 * (Sin[\lambda])^2 * Cos[\lambda] * V\varphi\varphi\lambda - (Sin[\varphi])^3 * (Sin[\lambda])^3 * V\varphi\varphi\varphi +
3 * (Sin[\varphi])^2 * Cos[\varphi] * (Sin[\lambda])^3 * V\varphi\varphi\rho + 3 * (Cos[\varphi])^2 * (Sin[\lambda])^2 * Cos[\lambda] * V\rho\rho\lambda -
3 * Sin[\varphi] * (Cos[\varphi])^2 * (Sin[\lambda])^3 * V\rho\rho\varphi + (Cos[\varphi])^3 * (Sin[\lambda])^3 * V\rho\rho\rho;

```

```
In[1]:= VyzzRoutes135 =
Cos[\varphi] * (Cos[\lambda])^2 * V\lambda\lambda\varphi + Sin[\varphi] * (Cos[\lambda])^2 * V\lambda\lambda\rho + Cos[2*\varphi] * Sin[2*\lambda] * V\lambda\varphi\rho -
Sin[2*\varphi] * Sin[\lambda] * Cos[\lambda] * V\varphi\varphi\lambda + (Sin[\varphi])^2 * Cos[\varphi] * (Sin[\lambda])^2 * V\varphi\varphi\varphi +
Sin[\varphi] * (Sin[\lambda])^2 * ((Sin[\varphi])^2 - 2 * (Cos[\varphi])^2) * V\varphi\varphi\rho +
Sin[2*\varphi] * Sin[\lambda] * Cos[\lambda] * V\rho\rho\lambda + Cos[\varphi] * (Sin[\lambda])^2 *
((Cos[\varphi])^2 - 2 * (Sin[\varphi])^2) * V\rho\rho\varphi + Sin[\varphi] * (Cos[\varphi])^2 * (Sin[\lambda])^2 * V\rho\rho\rho;

In[2]:= VzzxRoutes135 = -Sin[2*\varphi] * Sin[\lambda] * V\lambda\varphi\rho -
(Cos[\varphi])^2 * Sin[\lambda] * V\varphi\varphi\lambda - Sin[\varphi] * (Cos[\varphi])^2 * Cos[\lambda] * V\varphi\varphi\varphi +
Cos[\varphi] * Cos[\lambda] * ((Cos[\varphi])^2 - 2 * (Sin[\varphi])^2) * V\varphi\varphi\rho -
(Sin[\varphi])^2 * Sin[\lambda] * V\rho\rho\lambda + Sin[\varphi] * Cos[\lambda] * (2 * (Cos[\varphi])^2 - (Sin[\varphi])^2) * V\rho\rho\varphi +
(Sin[\varphi])^2 * Cos[\varphi] * Cos[\lambda] * V\rho\rho\rho;

In[3]:= VzzyRoutes135 = Sin[2*\varphi] * Cos[\lambda] * V\lambda\varphi\rho +
(Cos[\varphi])^2 * Cos[\lambda] * V\varphi\varphi\lambda - Sin[\varphi] * (Cos[\varphi])^2 * Sin[\lambda] * V\varphi\varphi\varphi +
Cos[\varphi] * Sin[\lambda] * ((Cos[\varphi])^2 - 2 * (Sin[\varphi])^2) * V\varphi\varphi\rho +
(Sin[\varphi])^2 * Cos[\lambda] * V\rho\rho\lambda + Sin[\varphi] * Sin[\lambda] * (2 * (Cos[\varphi])^2 - (Sin[\varphi])^2) * V\rho\rho\varphi +
(Sin[\varphi])^2 * Cos[\varphi] * Sin[\lambda] * V\rho\rho\rho;

In[4]:= VzzzRoutes135 = (Cos[\varphi])^3 * V\varphi\varphi\varphi +
3 * Sin[\varphi] * (Cos[\varphi])^2 * V\varphi\varphi\rho + 3 * (Sin[\varphi])^2 * Cos[\varphi] * V\rho\rho\varphi + (Sin[\varphi])^3 * V\rho\rho\rho;
```

7.4 Check whether they become themselves

In[1]:= FullSimplify[VxRoutes135]

Out[1]=
Vx

In[2]:= FullSimplify[VyRoutes135]

Out[2]=
Vy

In[3]:= FullSimplify[VzRoutes135]

Out[3]=
Vz

In[4]:= FullSimplify[VxxRoutes135]

Out[4]=
Vxx

In[5]:= FullSimplify[VxyRoutes135]

Out[5]=
Vxy

In[6]:= FullSimplify[VxzRoutes135]

Out[6]=
Vxz

In[7]:= FullSimplify[VyyRoutes135]

Out[7]=
Vyy

```

In[°]:= FullSimplify[VyzRoutes135]
Out[°]= Vyz

In[°]:= FullSimplify[VzzRoutes135]
Out[°]= Vzz

In[°]:= FullSimplify[VxxxRoutes135]
Out[°]= Vxxx

In[°]:= FullSimplify[VxxyRoutes135]
Out[°]= Vxxy

In[°]:= FullSimplify[VxxzRoutes135]
Out[°]= Vxxz

In[°]:= FullSimplify[VxyzRoutes135]
Out[°]= Vxyz

In[°]:= FullSimplify[VyyxRoutes135]
Out[°]= Vyyx

In[°]:= FullSimplify[VyyyRoutes135]
Out[°]= Vyyy

In[°]:= FullSimplify[VyyzRoutes135]
Out[°]= Vyyz

In[°]:= FullSimplify[VzzxRoutes135]
Out[°]= Vzzx

In[°]:= FullSimplify[VzzyRoutes135]
Out[°]= Vzzy

In[°]:= FullSimplify[VzzzRoutes135]
Out[°]= Vzzz

```

8. Trigonometric form, Cartesian coordinates, Routes 6 → 4 → 2, and Table 15 → Table 10 → Table 6

```
In[°]:= Clear["Global`*"];
```

8.1 Table 15

```

In[1]:= Vλ = -Sin[λ] * Vx + Cos[λ] * Vy;
In[2]:= Vφ = -Sin[φ] * Cos[λ] * Vx - Sin[φ] * Sin[λ] * Vy + Cos[φ] * Vz;
In[3]:= Vρ = Cos[φ] * Cos[λ] * Vx + Cos[φ] * Sin[λ] * Vy + Sin[φ] * Vz;
In[4]:= Vλλ = (Sin[λ])^2 * Vxx - Sin[2*λ] * Vxy + (Cos[λ])^2 * Vyy;
In[5]:= Vλφ = Sin[φ] * Sin[λ] * Cos[λ] * Vxx - Sin[φ] * Cos[2*λ] * Vxy -
          Cos[φ] * Sin[λ] * Vxz - Sin[φ] * Sin[λ] * Cos[λ] * Vyy + Cos[φ] * Cos[λ] * Vyz;
In[6]:= Vλρ = -Cos[φ] * Sin[λ] * Cos[λ] * Vxx + Cos[φ] * Cos[2*λ] * Vxy -
          Sin[φ] * Sin[λ] * Vxz + Cos[φ] * Sin[λ] * Cos[λ] * Vyy + Sin[φ] * Cos[λ] * Vyz;
In[7]:= Vφφ = (Sin[φ])^2 * (Cos[λ])^2 * Vxx + (Sin[φ])^2 * Sin[2*λ] * Vxy - Sin[2*φ] * Cos[λ] * Vxz +
          (Sin[φ])^2 * (Sin[λ])^2 * Vyy - Sin[2*φ] * Sin[λ] * Vyz + (Cos[φ])^2 * Vzz;
In[8]:= Vφρ = -Sin[φ] * Cos[φ] * (Cos[λ])^2 * Vxx -
          Sin[2*φ] * Sin[λ] * Cos[λ] * Vxy + Cos[2*φ] * Cos[λ] * Vxz -
          Sin[φ] * Cos[φ] * (Sin[λ])^2 * Vyy + Cos[2*φ] * Sin[λ] * Vyz + Sin[φ] * Cos[φ] * Vzz;
In[9]:= Vρρ = (Cos[φ])^2 * (Cos[λ])^2 * Vxx + (Cos[φ])^2 * Sin[2*λ] * Vxy + Sin[2*φ] * Cos[λ] * Vxz +
          (Cos[φ])^2 * (Sin[λ])^2 * Vyy + Sin[2*φ] * Sin[λ] * Vyz + (Sin[φ])^2 * Vzz;
In[10]:= Vλλλ = -(Sin[λ])^3 * Vxxx + 3 * (Sin[λ])^2 * Cos[λ] * Vxxy -
           3 * Sin[λ] * (Cos[λ])^2 * Vyyx + (Cos[λ])^3 * Vyyy;
In[11]:= Vλλφ = -Sin[φ] * (Sin[λ])^2 * Cos[λ] * Vxxx +
           Sin[φ] * Sin[λ] * (2 * (Cos[λ])^2 - (Sin[λ])^2) * Vxxy + Cos[φ] * (Sin[λ])^2 * Vxxz -
           Cos[φ] * Sin[2*λ] * Vxyz + Sin[φ] * Cos[λ] * (2 * (Sin[λ])^2 - (Cos[λ])^2) * Vyyx -
           Sin[φ] * Sin[λ] * (Cos[λ])^2 * Vyyy + Cos[φ] * (Cos[λ])^2 * Vyyz;
In[12]:= Vλλρ = Cos[φ] * (Sin[λ])^2 * Cos[λ] * Vxxx +
           Cos[φ] * Sin[λ] * ((Sin[λ])^2 - 2 * (Cos[λ])^2) * Vxxy + Sin[φ] * (Sin[λ])^2 * Vxxz -
           Sin[φ] * Sin[2*λ] * Vxyz + Cos[φ] * Cos[λ] * ((Cos[λ])^2 - 2 * (Sin[λ])^2) * Vyyx +
           Cos[φ] * Sin[λ] * (Cos[λ])^2 * Vyyy + Sin[φ] * (Cos[λ])^2 * Vyyz;
In[13]:= Vλφρ = Sin[φ] * Cos[φ] * Sin[λ] * (Cos[λ])^2 * Vxxx +
           Sin[φ] * Cos[φ] * Cos[λ] * (2 * (Sin[λ])^2 - (Cos[λ])^2) * Vxxy -
           Cos[2*φ] * Sin[λ] * Cos[λ] * Vxxz + Cos[2*φ] * Cos[λ] * Vxyz +
           Sin[φ] * Cos[φ] * Sin[λ] * ((Sin[λ])^2 - 2 * (Cos[λ])^2) * Vyyx -
           Sin[φ] * Cos[φ] * (Sin[λ])^2 * Cos[λ] * Vyyy + Cos[2*φ] * Sin[λ] * Cos[λ] * Vyyz -
           Sin[φ] * Cos[φ] * Sin[λ] * Vzzx + Sin[φ] * Cos[φ] * Cos[λ] * Vzzy;
In[14]:= Vφφλ = -(Sin[φ])^2 * Sin[λ] * (Cos[λ])^2 * Vxxx +
           (Sin[φ])^2 * Cos[λ] * ((Cos[λ])^2 - 2 * (Sin[λ])^2) * Vxxy +
           Sin[2*φ] * Sin[λ] * Cos[λ] * Vxxz - Sin[2*φ] * Cos[2*λ] * Vxyz +
           (Sin[φ])^2 * Sin[λ] * (2 * (Cos[λ])^2 - (Sin[λ])^2) * Vyyx +
           (Sin[φ])^2 * (Sin[λ])^2 * Cos[λ] * Vyyy - Sin[2*φ] * Sin[λ] * Cos[λ] * Vyyz -
           (Cos[φ])^2 * Sin[λ] * Vzzx + (Cos[φ])^2 * Cos[λ] * Vzzy;

```

```

In[8]:= Vφφφ = - (Sin[φ])3 * (Cos[λ])3 * Vxxx - 3 * (Sin[φ])3 * Sin[λ] * (Cos[λ])2 * Vxxy +
           3 * (Sin[φ])2 * Cos[φ] * (Cos[λ])2 * Vxxz + 3 * (Sin[φ])2 * Cos[φ] * Sin[2*λ] * Vxyz -
           3 * (Sin[φ])3 * (Sin[λ])2 * Cos[λ] * Vyxy - (Sin[φ])3 * (Sin[λ])3 * Vyyy +
           3 * (Sin[φ])2 * Cos[φ] * (Sin[λ])2 * Vyzy - 3 * Sin[φ] * (Cos[φ])2 * Cos[λ] * Vzzx -
           3 * Sin[φ] * (Cos[φ])2 * Sin[λ] * Vzzy + (Cos[φ])3 * Vzzz;

In[9]:= Vφφρ = (Sin[φ])2 * Cos[φ] * (Cos[λ])3 * Vxxx + 3 * (Sin[φ])2 * Cos[φ] * Sin[λ] *
           (Cos[λ])2 * Vxxy + Sin[φ] * (Cos[λ])2 * ((Sin[φ])2 - 2 * (Cos[φ])2) * Vxxz +
           Sin[φ] * Sin[2*λ] * ((Sin[φ])2 - 2 * (Cos[φ])2) * Vxyz + 3 * (Sin[φ])2 *
           Cos[φ] * (Sin[λ])2 * Cos[λ] * Vyxy + (Sin[φ])2 * Cos[φ] * (Sin[λ])3 * Vyyy +
           Sin[φ] * (Sin[λ])2 * ((Sin[φ])2 - 2 * (Cos[φ])2) * Vyzy +
           Cos[φ] * Cos[λ] * ((Cos[φ])2 - 2 * (Sin[φ])2) * Vzzx +
           Cos[φ] * Sin[λ] * ((Cos[φ])2 - 2 * (Sin[φ])2) * Vzzy + Sin[φ] * (Cos[φ])2 * Vzzz;

In[10]:= Vρρλ = - (Cos[φ])2 * Sin[λ] * (Cos[λ])2 * Vxxx +
           (Cos[φ])2 * Cos[λ] * ((Cos[λ])2 - 2 * (Sin[λ])2) * Vxxy -
           Sin[2*φ] * Sin[λ] * Cos[λ] * Vxxz + Sin[2*φ] * Cos[2*λ] * Vxyz +
           (Cos[φ])2 * Sin[λ] * (2 * (Cos[λ])2 - (Sin[λ])2) * Vyxy +
           (Cos[φ])2 * (Sin[λ])2 * Cos[λ] * Vyyy + Sin[2*φ] * Sin[λ] * Cos[λ] * Vyzy -
           (Sin[φ])2 * Sin[λ] * Vzzx + (Sin[φ])2 * Cos[λ] * Vzzy;

In[11]:= Vρρφ = - Sin[φ] * (Cos[φ])2 * (Cos[λ])3 * Vxxx - 3 * Sin[φ] * (Cos[φ])2 * Sin[λ] *
           (Cos[λ])2 * Vxxy + Cos[φ] * (Cos[λ])2 * ((Cos[φ])2 - 2 * (Sin[φ])2) * Vxxz +
           Cos[φ] * Sin[2*λ] * ((Cos[φ])2 - 2 * (Sin[φ])2) * Vxyz -
           3 * Sin[φ] * (Cos[φ])2 * (Sin[λ])2 * Cos[λ] * Vyxy - Sin[φ] * (Cos[φ])2 *
           (Sin[λ])3 * Vyyy + Cos[φ] * (Sin[λ])2 * ((Cos[φ])2 - 2 * (Sin[φ])2) * Vyzy +
           Sin[φ] * Cos[λ] * (2 * (Cos[φ])2 - (Sin[φ])2) * Vzzx +
           Sin[φ] * Sin[λ] * (2 * (Cos[φ])2 - (Sin[φ])2) * Vzzy + (Sin[φ])2 * Cos[φ] * Vzzz;

In[12]:= Vρρρ = (Cos[φ])3 * (Cos[λ])3 * Vxxx +
           3 * (Cos[φ])3 * Sin[λ] * (Cos[λ])2 * Vxxy + 3 * Sin[φ] * (Cos[φ])2 * (Cos[λ])2 * Vxxz +
           3 * Sin[2*φ] * Cos[φ] * Sin[λ] * Cos[λ] * Vxyz +
           3 * (Cos[φ])3 * (Sin[λ])2 * Cos[λ] * Vyxy + (Cos[φ])3 * (Sin[λ])3 * Vyyy +
           3 * Sin[φ] * (Cos[φ])2 * (Sin[λ])2 * Vyzy + 3 * (Sin[φ])2 * Cos[φ] * Cos[λ] * Vzzx +
           3 * (Sin[φ])2 * Cos[φ] * Sin[λ] * Vzzy + (Sin[φ])3 * Vzzz;

```

8.2 Table 10

```

In[1]:= Vr = - Sin[φ] * Vφ + Cos[φ] * Vρ;

In[2]:= Vλ = Vλ;

In[3]:= Vz1 = Cos[φ] * Vφ + Sin[φ] * Vρ;

In[4]:= Vrr = (Sin[φ])2 * Vφφ - Sin[2*φ] * Vφρ + (Cos[φ])2 * Vρρ;

In[5]:= Vrλ = - Sin[φ] * Vλφ + Cos[φ] * Vλρ;

In[6]:= Vrz = - Sin[φ] * Cos[φ] * Vφφ + Cos[2*φ] * Vφρ + Sin[φ] * Cos[φ] * Vρρ;

In[7]:= Vλλ = Vλλ;

```

```

In[1]:= Vλz = Cos[φ] * Vλφ + Sin[φ] * Vλρ;
In[2]:= Vzz1 = (Cos[φ])2 * Vφφ + Sin[2 * φ] * Vφρ + (Sin[φ])2 * Vρρ;
In[3]:= Vrrr = - (Sin[φ])3 * Vφφφ + 3 * (Sin[φ])2 * Cos[φ] * Vφφρ -
          3 * Sin[φ] * (Cos[φ])2 * Vρρφ + (Cos[φ])3 * Vρρρ;
In[4]:= Vrrλ = (Sin[φ])2 * Vφφλ - Sin[2 * φ] * Vλφρ + (Cos[φ])2 * Vρρλ;
In[5]:= Vrrz = (Sin[φ])2 * Cos[φ] * Vφφφ + Sin[φ] * ((Sin[φ])2 - 2 * (Cos[φ])2) * Vφφρ +
          Cos[φ] * ((Cos[φ])2 - 2 * (Sin[φ])2) * Vρρφ + Sin[φ] * (Cos[φ])2 * Vρρρ;
In[6]:= Vrλz = Cos[2 * φ] * Vλφρ - Sin[φ] * Cos[φ] * Vφφλ + Sin[φ] * Cos[φ] * Vρρλ;
In[7]:= Vλλr = - Sin[φ] * Vλλφ + Cos[φ] * Vλλρ;
In[8]:= Vλλλ = Vλλλ;
In[9]:= Vλλz = Cos[φ] * Vλλφ + Sin[φ] * Vλλρ;
In[10]:= Vzzr = - Sin[φ] * (Cos[φ])2 * Vφφφ + Cos[φ] * ((Cos[φ])2 - 2 * (Sin[φ])2) * Vφφρ +
           Sin[φ] * (2 * (Cos[φ])2 - (Sin[φ])2) * Vρρφ + (Sin[φ])2 * Cos[φ] * Vρρρ;
In[11]:= Vzzλ = Sin[2 * φ] * Vλφρ + (Cos[φ])2 * Vφφλ + (Sin[φ])2 * Vρρλ;
In[12]:= Vzzz1 = (Cos[φ])3 * Vφφφ + 3 * Sin[φ] * (Cos[φ])2 * Vφφρ +
           3 * (Sin[φ])2 * Cos[φ] * Vρρφ + (Sin[φ])3 * Vρρρ;

```

8.3 Table 6

```

In[1]:= VxRoutes642 = Cos[λ] * Vr - Sin[λ] * Vλ;
In[2]:= VyRoutes642 = Sin[λ] * Vr + Cos[λ] * Vλ;
In[3]:= VzRoutes642 = Vz1;
In[4]:= VxxRoutes642 = (Cos[λ])2 * Vrr - Sin[2 * λ] * Vrλ + (Sin[λ])2 * Vλλ;
In[5]:= VxyRoutes642 = Sin[λ] * Cos[λ] * Vrr + Cos[2 * λ] * Vrλ - Sin[λ] * Cos[λ] * Vλλ;
In[6]:= VxzRoutes642 = Cos[λ] * Vrz - Sin[λ] * Vλz;
In[7]:= VyyRoutes642 = (Sin[λ])2 * Vrr + Sin[2 * λ] * Vrλ + (Cos[λ])2 * Vλλ;
In[8]:= VyzRoutes642 = Sin[λ] * Vrz + Cos[λ] * Vλz;
In[9]:= VzzRoutes642 = Vzz1;
In[10]:= VxxxRoutes642 = (Cos[λ])3 * Vrrr -
            3 * Sin[λ] * (Cos[λ])2 * Vrrλ + 3 * (Sin[λ])2 * Cos[λ] * Vλλr - (Sin[λ])3 * Vλλλ;
In[11]:= VxxyRoutes642 = Sin[λ] * (Cos[λ])2 * Vrrr + Cos[λ] * ((Cos[λ])2 - 2 * (Sin[λ])2) * Vrrλ +
            Sin[λ] * ((Sin[λ])2 - 2 * (Cos[λ])2) * Vλλr + (Sin[λ])2 * Cos[λ] * Vλλλ;
In[12]:= VxxzRoutes642 = (Cos[λ])2 * Vrrz - Sin[2 * λ] * Vrλz + (Sin[λ])2 * Vλλz;
In[13]:= VxyzRoutes642 = Sin[λ] * Cos[λ] * Vrrz + Cos[2 * λ] * Vrλz - Sin[λ] * Cos[λ] * Vλλz;
In[14]:= VyyxRoutes642 = (Sin[λ])2 * Cos[λ] * Vrrr + Sin[λ] * (2 * (Cos[λ])2 - (Sin[λ])2) * Vrrλ +
            Cos[λ] * ((Cos[λ])2 - 2 * (Sin[λ])2) * Vλλr - Sin[λ] * (Cos[λ])2 * Vλλλ;

```

```
In[°]:= VyyyRoutes642 = (Sin[λ])3*Vrrr +
          3*(Sin[λ])2*Cos[λ]*Vrrλ + 3*Sin[λ]*(Cos[λ])2*Vλλr + (Cos[λ])3*Vλλλ;
In[°]:= VyyzRoutes642 = (Sin[λ])2*Vrrz + Sin[2*λ]*Vrλz + (Cos[λ])2*Vλλz;
In[°]:= VzzxRoutes642 = Cos[λ]*Vzzr - Sin[λ]*Vzzλ;
In[°]:= VzyRoutes642 = Sin[λ]*Vzzr + Cos[λ]*Vzzλ;
In[°]:= VzzzRoutes642 = Vzzz1;
```

8.4 Check whether they become themselves

```
In[°]:= FullSimplify[VxRoutes642]
```

```
Out[°]=
Vx
```

```
In[°]:= FullSimplify[VyRoutes642]
```

```
Out[°]=
Vy
```

```
In[°]:= FullSimplify[VzRoutes642]
```

```
Out[°]=
Vz
```

```
In[°]:= FullSimplify[VxxRoutes642]
```

```
Out[°]=
Vxx
```

```
In[°]:= FullSimplify[VxyRoutes642]
```

```
Out[°]=
Vxy
```

```
In[°]:= FullSimplify[VxzRoutes642]
```

```
Out[°]=
Vxz
```

```
In[°]:= FullSimplify[VyyRoutes642]
```

```
Out[°]=
Vyy
```

```
In[°]:= FullSimplify[VyzRoutes642]
```

```
Out[°]=
Vyz
```

```
In[°]:= FullSimplify[VzzRoutes642]
```

```
Out[°]=
Vzz
```

```
In[°]:= FullSimplify[VxxxRoutes642]
```

```
Out[°]=
Vxxx
```

```
In[°]:= FullSimplify[VxxyRoutes642]
```

```
Out[°]=
Vxxy
```

In[1]:= FullSimplify[VxxzRoutes642]

Out[1]=

Vxxz

In[2]:= FullSimplify[VxyzRoutes642]

Out[2]=

Vxyz

In[3]:= FullSimplify[VyyxRoutes642]

Out[3]=

Vyyx

In[4]:= FullSimplify[VyyyRoutes642]

Out[4]=

Vyyy

In[5]:= FullSimplify[VyyzRoutes642]

Out[5]=

Vyyz

In[6]:= FullSimplify[VzzxRoutes642]

Out[6]=

Vzzx

In[7]:= FullSimplify[VzzyRoutes642]

Out[7]=

Vzzy

In[8]:= FullSimplify[VzzzRoutes642]

Out[8]=

Vzzz

9. Trigonometric form, Cylindrical coordinates, Routes 3 → 5 → 1, and Table 8 → Table 13 → Table 4

In[1]:= Clear["Global`*"];

9.1 Table 8

In[2]:= Vλ = Vλ;

In[3]:= Vφ = -Sin[φ] * Vr + Cos[φ] * Vz;

In[4]:= Vρ = Cos[φ] * Vr + Sin[φ] * Vz;

In[5]:= Vλλ = Vλλ;

In[6]:= Vλφ = -Sin[φ] * Vrλ + Cos[φ] * Vλz;

In[7]:= Vλρ = Cos[φ] * Vrλ + Sin[φ] * Vλz;

In[8]:= Vφφ = ((Sin[φ])^2 * Vrr - Sin[2 * φ] * Vrz + (Cos[φ])^2 * Vzz);

In[9]:= Vφρ = -Sin[φ] * Cos[φ] * Vrr + Cos[2 * φ] * Vrz + Sin[φ] * Cos[φ] * Vzz;

In[10]:= Vρρ = ((Cos[φ])^2 * Vrr + Sin[2 * φ] * Vrz + (Sin[φ])^2 * Vzz);

```

In[1]:= Vλλλ = Vλλλ;
In[2]:= Vλλφ = -Sin[φ] * Vλλr + Cos[φ] * Vλλz;
In[3]:= Vλλρ = Cos[φ] * Vλλr + Sin[φ] * Vλλz;
In[4]:= Vλφρ = -Sin[φ] * Cos[φ] * Vrrλ + Cos[2 * φ] * Vrλz + Sin[φ] * Cos[φ] * Vzzλ;
In[5]:= Vφφλ = (Sin[φ])2 * Vrrλ - Sin[2 * φ] * Vrλz + (Cos[φ])2 * Vzzλ;
In[6]:= Vφφφ = -(Sin[φ])3 * Vrrr + 3 * (Sin[φ])2 * Cos[φ] * Vrrz -
          3 * Sin[φ] * (Cos[φ])2 * Vzzr + (Cos[φ])3 * Vzzz;
In[7]:= Vφφρ = (Sin[φ])2 * Cos[φ] * Vrrr + Sin[φ] * ((Sin[φ])2 - 2 * (Cos[φ])2) * Vrrz +
          Cos[φ] * ((Cos[φ])2 - 2 * (Sin[φ])2) * Vzzr + Sin[φ] * (Cos[φ])2 * Vzzz;
In[8]:= Vρρλ = (Cos[φ])2 * Vrrλ + Sin[2 * φ] * Vrλz + (Sin[φ])2 * Vzzλ;
In[9]:= Vρρφ = -Sin[φ] * (Cos[φ])2 * Vrrr + Cos[φ] * ((Cos[φ])2 - 2 * (Sin[φ])2) * Vrrz +
          Sin[φ] * (2 * (Cos[φ])2 - (Sin[φ])2) * Vzzr + Cos[φ] * (Sin[φ])2 * Vzzz;
In[10]:= Vρρρ = (Cos[φ])3 * Vrrr + 3 * Sin[φ] * (Cos[φ])2 * Vrrz +
           3 * Cos[φ] * (Sin[φ])2 * Vzzr + (Sin[φ])3 * Vzzz;

```

9.2 Table 13

```

In[1]:= Vx = -Sin[λ] * Vλ - Sin[φ] * Cos[λ] * Vφ + Cos[φ] * Cos[λ] * Vρ;
In[2]:= Vy = Cos[λ] * Vλ - Sin[φ] * Sin[λ] * Vφ + Cos[φ] * Sin[λ] * Vρ;
In[3]:= Vz1 = Cos[φ] * Vφ + Sin[φ] * Vρ;
In[4]:= Vxx = (Sin[λ])2 * Vλλ + Sin[φ] * Sin[2 * λ] * Vλφ -
          Cos[φ] * Sin[2 * λ] * Vλρ + (Sin[φ])2 * (Cos[λ])2 * Vφφ -
          Sin[2 * φ] * (Cos[λ])2 * Vφρ + (Cos[φ])2 * (Cos[λ])2 * Vρρ;
In[5]:= Vxy = -Sin[λ] * Cos[λ] * Vλλ - Sin[φ] * Cos[2 * λ] * Vλφ +
          Cos[φ] * Cos[2 * λ] * Vλρ + (Sin[φ])2 * Sin[λ] * Cos[λ] * Vφφ -
          Sin[2 * φ] * Sin[λ] * Cos[λ] * Vφρ + (Cos[φ])2 * Sin[λ] * Cos[λ] * Vρρ;
In[6]:= Vxz = -Cos[φ] * Sin[λ] * Vλφ - Sin[φ] * Sin[λ] * Vλρ - Sin[φ] * Cos[φ] * Cos[λ] * Vφφ +
          Cos[2 * φ] * Cos[λ] * Vφρ + Sin[φ] * Cos[φ] * Cos[λ] * Vρρ;
In[7]:= Vyy = (Cos[λ])2 * Vλλ - Sin[φ] * Sin[2 * λ] * Vλφ +
          Cos[φ] * Sin[2 * λ] * Vλρ + (Sin[φ])2 * (Sin[λ])2 * Vφφ -
          Sin[2 * φ] * (Sin[λ])2 * Vφρ + (Cos[φ])2 * (Sin[λ])2 * Vρρ;
In[8]:= Vyz = Cos[φ] * Cos[λ] * Vλφ + Sin[φ] * Cos[λ] * Vλρ - Sin[φ] * Cos[φ] * Sin[λ] * Vφφ +
          Cos[2 * φ] * Sin[λ] * Vφρ + Sin[φ] * Cos[φ] * Sin[λ] * Vρρ;
In[9]:= Vzz1 = (Cos[φ])2 * Vφφ + Sin[2 * φ] * Vφρ + (Sin[φ])2 * Vρρ;
In[10]:= Vxxx = -(Sin[λ])3 * Vλλλ + 3 * Cos[φ] * (Sin[λ])2 * Cos[λ] * Vλλρ -
           3 * Sin[φ] * (Sin[λ])2 * Cos[λ] * Vλλφ + 3 * Sin[2 * φ] * Sin[λ] * (Cos[λ])2 * Vλφρ -
           3 * (Sin[φ])2 * Sin[λ] * (Cos[λ])2 * Vφφλ - (Sin[φ])3 * (Cos[λ])3 * Vφφφ +
           3 * (Sin[φ])2 * Cos[φ] * (Cos[λ])3 * Vφφρ - 3 * (Cos[φ])2 * Sin[λ] * (Cos[λ])2 * Vρρλ -
           3 * Sin[φ] * (Cos[φ])2 * (Cos[λ])3 * Vρρφ + (Cos[φ])3 * (Cos[λ])3 * Vρρρ;

```

```

In[8]:= Vxxxy = (Sin[\lambda])^2 * Cos[\lambda] * V\lambda\lambda\lambda + Sin[\varphi] * Sin[\lambda] * (2 * (Cos[\lambda])^2 - (Sin[\lambda])^2) * V\lambda\lambda\varphi +
          Cos[\varphi] * Sin[\lambda] * ((Sin[\lambda])^2 - 2 * (Cos[\lambda])^2) * V\lambda\lambda\rho +
          Sin[2 * \varphi] * Cos[\lambda] * (2 * (Sin[\lambda])^2 - (Cos[\lambda])^2) * V\lambda\varphi\rho +
          (Sin[\varphi])^2 * Cos[\lambda] * ((Cos[\lambda])^2 - 2 * (Sin[\lambda])^2) * V\varphi\varphi\lambda - (Sin[\varphi])^3 * Sin[\lambda] *
          (Cos[\lambda])^2 * V\varphi\varphi\varphi + 3 * (Sin[\varphi])^2 * Cos[\varphi] * Sin[\lambda] * (Cos[\lambda])^2 * V\varphi\varphi\rho +
          (Cos[\varphi])^2 * Cos[\lambda] * ((Cos[\lambda])^2 - 2 * (Sin[\lambda])^2) * V\rho\rho\lambda - 3 * Sin[\varphi] *
          (Cos[\varphi])^2 * Sin[\lambda] * (Cos[\lambda])^2 * V\rho\rho\varphi + (Cos[\varphi])^3 * Sin[\lambda] * (Cos[\lambda])^2 * V\rho\rho\rho;

In[9]:= Vxxxz =
          Cos[\varphi] * (Sin[\lambda])^2 * V\lambda\lambda\varphi + Sin[\varphi] * (Sin[\lambda])^2 * V\lambda\lambda\rho - Cos[2 * \varphi] * Sin[2 * \lambda] * V\lambda\varphi\rho +
          Sin[\varphi] * Cos[\varphi] * Sin[2 * \lambda] * V\varphi\varphi\lambda + (Sin[\varphi])^2 * Cos[\varphi] * (Cos[\lambda])^2 * V\varphi\varphi\varphi +
          Sin[\varphi] * (Cos[\lambda])^2 * ((Sin[\varphi])^2 - 2 * (Cos[\varphi])^2) * V\varphi\varphi\rho -
          Sin[2 * \varphi] * Sin[\lambda] * Cos[\lambda] * V\rho\rho\lambda + Cos[\varphi] * (Cos[\lambda])^2 *
          ((Cos[\varphi])^2 - 2 * (Sin[\varphi])^2) * V\rho\rho\varphi + Sin[\varphi] * (Cos[\varphi])^2 * (Cos[\lambda])^2 * V\rho\rho\rho;

In[10]:= Vxyz = -Cos[\varphi] * Sin[\lambda] * Cos[\lambda] * V\lambda\lambda\varphi -
          Sin[\varphi] * Sin[\lambda] * Cos[\lambda] * V\lambda\lambda\rho + Cos[2 * \varphi] * Cos[2 * \lambda] * V\lambda\varphi\rho -
          Sin[\varphi] * Cos[\varphi] * Cos[2 * \lambda] * V\varphi\varphi\lambda + (Sin[\varphi])^2 * Cos[\varphi] * Sin[\lambda] * Cos[\lambda] * V\varphi\varphi\varphi -
          Sin[\varphi] * Sin[\lambda] * Cos[\lambda] * (2 * (Cos[\varphi])^2 - (Sin[\varphi])^2) * V\varphi\varphi\rho +
          Sin[\varphi] * Cos[\varphi] * Cos[2 * \lambda] * V\rho\rho\lambda + Cos[\varphi] * Sin[\lambda] * Cos[\lambda] *
          ((Cos[\varphi])^2 - 2 * (Sin[\varphi])^2) * V\rho\rho\varphi + Sin[\varphi] * (Cos[\varphi])^2 * Sin[\lambda] * Cos[\lambda] * V\rho\rho\rho;

In[11]:= Vyyyx = -Sin[\lambda] * (Cos[\lambda])^2 * V\lambda\lambda\lambda - Sin[\varphi] * Cos[\lambda] * ((Cos[\lambda])^2 - 2 * (Sin[\lambda])^2) * V\lambda\lambda\varphi +
          Cos[\varphi] * Cos[\lambda] * ((Cos[\lambda])^2 - 2 * (Sin[\lambda])^2) * V\lambda\lambda\rho -
          Sin[2 * \varphi] * Sin[\lambda] * (2 * (Cos[\lambda])^2 - (Sin[\lambda])^2) * V\lambda\varphi\rho +
          (Sin[\varphi])^2 * Sin[\lambda] * (2 * (Cos[\lambda])^2 - (Sin[\lambda])^2) * V\varphi\varphi\lambda - (Sin[\varphi])^3 *
          (Sin[\lambda])^2 * Cos[\lambda] * V\varphi\varphi\varphi + 3 * (Sin[\varphi])^2 * Cos[\varphi] * (Sin[\lambda])^2 * Cos[\lambda] * V\varphi\varphi\rho +
          (Cos[\varphi])^2 * Sin[\lambda] * (2 * (Cos[\lambda])^2 - (Sin[\lambda])^2) * V\rho\rho\lambda - 3 * Sin[\varphi] *
          (Cos[\varphi])^2 * (Sin[\lambda])^2 * Cos[\lambda] * V\rho\rho\varphi + (Cos[\varphi])^3 * (Sin[\lambda])^2 * Cos[\lambda] * V\rho\rho\rho;

In[12]:= Vy়yy = (Cos[\lambda])^3 * V\lambda\lambda\lambda - 3 * Sin[\varphi] * Sin[\lambda] * (Cos[\lambda])^2 * V\lambda\lambda\varphi +
          3 * Cos[\varphi] * Sin[\lambda] * (Cos[\lambda])^2 * V\lambda\lambda\rho - 3 * Sin[2 * \varphi] * (Sin[\lambda])^2 * Cos[\lambda] * V\lambda\varphi\rho +
          3 * (Sin[\varphi])^2 * (Sin[\lambda])^2 * Cos[\lambda] * V\varphi\varphi\lambda - (Sin[\varphi])^3 * (Sin[\lambda])^3 * V\varphi\varphi\varphi +
          3 * (Sin[\varphi])^2 * Cos[\varphi] * (Sin[\lambda])^3 * V\varphi\varphi\rho + 3 * (Cos[\varphi])^2 * (Sin[\lambda])^2 * Cos[\lambda] * V\rho\rho\lambda -
          3 * Sin[\varphi] * (Cos[\varphi])^2 * (Sin[\lambda])^3 * V\rho\rho\varphi + (Cos[\varphi])^3 * (Sin[\lambda])^3 * V\rho\rho\rho;

In[13]:= Vy়yz =
          Cos[\varphi] * (Cos[\lambda])^2 * V\lambda\lambda\varphi + Sin[\varphi] * (Cos[\lambda])^2 * V\lambda\lambda\rho + Cos[2 * \varphi] * Sin[2 * \lambda] * V\lambda\varphi\rho -
          Sin[2 * \varphi] * Sin[\lambda] * Cos[\lambda] * V\varphi\varphi\lambda + (Sin[\varphi])^2 * Cos[\varphi] * (Sin[\lambda])^2 * V\varphi\varphi\varphi +
          Sin[\varphi] * (Sin[\lambda])^2 * ((Sin[\varphi])^2 - 2 * (Cos[\varphi])^2) * V\varphi\varphi\rho +
          Sin[2 * \varphi] * Sin[\lambda] * Cos[\lambda] * V\rho\rho\lambda + Cos[\varphi] * (Sin[\lambda])^2 *
          ((Cos[\varphi])^2 - 2 * (Sin[\varphi])^2) * V\rho\rho\varphi + Sin[\varphi] * (Cos[\varphi])^2 * (Sin[\lambda])^2 * V\rho\rho\rho;

In[14]:= Vzzx = -Sin[2 * \varphi] * Sin[\lambda] * V\lambda\varphi\rho -
          (Cos[\varphi])^2 * Sin[\lambda] * V\varphi\varphi\lambda - Sin[\varphi] * (Cos[\varphi])^2 * Cos[\lambda] * V\varphi\varphi\varphi +
          Cos[\varphi] * Cos[\lambda] * ((Cos[\varphi])^2 - 2 * (Sin[\varphi])^2) * V\varphi\varphi\rho - (Sin[\varphi])^2 * Sin[\lambda] * V\rho\rho\lambda +
          Sin[\varphi] * Cos[\lambda] * (2 * (Cos[\varphi])^2 - (Sin[\varphi])^2) * V\rho\rho\varphi +
          (Sin[\varphi])^2 * Cos[\varphi] * Cos[\lambda] * V\rho\rho\rho;

```

```
In[1]:= Vzzy = Sin[2 * φ] * Cos[λ] * Vλφρ +
          (Cos[φ])2 * Cos[λ] * Vφφλ - Sin[φ] * (Cos[φ])2 * Sin[λ] * Vφφφ +
          Cos[φ] * Sin[λ] * ((Cos[φ])2 - 2 * (Sin[φ])2) * Vφφρ + (Sin[φ])2 * Cos[λ] * Vρρλ +
          Sin[φ] * Sin[λ] * (2 * (Cos[φ])2 - (Sin[φ])2) * Vρρφ +
          (Sin[φ])2 * Cos[φ] * Sin[λ] * Vρρρ;

In[2]:= Vzzz1 = (Cos[φ])3 * Vφφφ + 3 * Sin[φ] * (Cos[φ])2 * Vφφρ +
          3 * (Sin[φ])2 * Cos[φ] * Vρρφ + (Sin[φ])3 * Vρρρ;
```

9.3 Table 4

```
In[1]:= VrRoutes351 = Cos[λ] * Vx + Sin[λ] * Vy;
In[2]:= VλRoutes351 = -Sin[λ] * Vx + Cos[λ] * Vy;
In[3]:= VzRoutes351 = Vz1;
In[4]:= VrrRoutes351 = (Cos[λ])2 * Vxx + Sin[2 * λ] * Vxy + (Sin[λ])2 * Vyy;
In[5]:= VrλRoutes351 = -Sin[λ] * Cos[λ] * Vxx + Cos[2 * λ] * Vxy + Sin[λ] * Cos[λ] * Vyy;
In[6]:= VrzRoutes351 = Cos[λ] * Vxz + Sin[λ] * Vyz;
In[7]:= VλλRoutes351 = (Sin[λ])2 * Vxx - Sin[2 * λ] * Vxy + (Cos[λ])2 * Vyy;
In[8]:= VλzRoutes351 = -Sin[λ] * Vxz + Cos[λ] * Vyz;
In[9]:= VzzRoutes351 = Vzz1;
In[10]:= VrrrRoutes351 = (Cos[λ])3 * Vxxx +
           3 * Sin[λ] * (Cos[λ])2 * Vxxy + 3 * Cos[λ] * (Sin[λ])2 * Vyyx + (Sin[λ])3 * Vyyy;
In[11]:= VrrλRoutes351 = -Sin[λ] * (Cos[λ])2 * Vxxx + Cos[λ] * ((Cos[λ])2 - 2 * (Sin[λ])2) * Vxxy +
           Sin[λ] * (2 * (Cos[λ])2 - (Sin[λ])2) * Vyyx + (Sin[λ])2 * Cos[λ] * Vyyy;
In[12]:= VrrzRoutes351 = (Cos[λ])2 * Vxxz + Sin[2 * λ] * Vxyz + (Sin[λ])2 * Vyyz;
In[13]:= VrλzRoutes351 = -Sin[λ] * Cos[λ] * Vxxz + Cos[2 * λ] * Vxyz + Sin[λ] * Cos[λ] * Vyyz;
In[14]:= VλλrRoutes351 = Cos[λ] * (Sin[λ])2 * Vxxx + Sin[λ] * ((Sin[λ])2 - 2 * (Cos[λ])2) * Vxxy +
           Cos[λ] * ((Cos[λ])2 - 2 * (Sin[λ])2) * Vyyx + Sin[λ] * (Cos[λ])2 * Vyyy;
In[15]:= VλλλRoutes351 = - (Sin[λ])3 * Vxxx +
           3 * (Sin[λ])2 * Cos[λ] * Vxxy - 3 * Sin[λ] * (Cos[λ])2 * Vyyx + (Cos[λ])3 * Vyyy;
In[16]:= VλλzRoutes351 = (Sin[λ])2 * Vxxz - Sin[2 * λ] * Vxyz + (Cos[λ])2 * Vyyz;
In[17]:= VzzrRoutes351 = Cos[λ] * Vzzx + Sin[λ] * Vzzy;
In[18]:= VzzλRoutes351 = -Sin[λ] * Vzzx + Cos[λ] * Vzzy;
In[19]:= VzzzRoutes351 = Vzzz1;
```

9.4 Check whether they become themselves

```
In[1]:= FullSimplify[VrRoutes351]
```

```
Out[1]=
```

Vr

In[1]:= FullSimplify[VλRoutes351]

Out[1]=

V λ

In[2]:= FullSimplify[VzRoutes351]

Out[2]=

V z

In[3]:= FullSimplify[VrrRoutes351]

Out[3]=

V rr

In[4]:= FullSimplify[VrλRoutes351]

Out[4]=

V $r\lambda$

In[5]:= FullSimplify[VrzRoutes351]

Out[5]=

V rz

In[6]:= FullSimplify[VλλRoutes351]

Out[6]=

V $\lambda\lambda$

In[7]:= FullSimplify[VλzRoutes351]

Out[7]=

V λz

In[8]:= FullSimplify[VzzRoutes351]

Out[8]=

V zz

In[9]:= FullSimplify[VrrrRoutes351]

Out[9]=

V rrr

In[10]:= FullSimplify[VrrλRoutes351]

Out[10]=

V $rr\lambda$

In[11]:= FullSimplify[VrrzRoutes351]

Out[11]=

V rrz

In[12]:= FullSimplify[VrλzRoutes351]

Out[12]=

V $r\lambda z$

In[13]:= FullSimplify[VλλrRoutes351]

Out[13]=

V $\lambda\lambda r$

In[14]:= FullSimplify[VλλλRoutes351]

Out[14]=

V $\lambda\lambda\lambda$

```
In[1]:= FullSimplify[VλλzRoutes351]
```

```
Out[1]=
```

Vλλz

```
In[2]:= FullSimplify[VzzrRoutes351]
```

```
Out[2]=
```

Vzzr

```
In[3]:= FullSimplify[VzzλRoutes351]
```

```
Out[3]=
```

Vzzλ

```
In[4]:= FullSimplify[VzzzRoutes351]
```

```
Out[4]=
```

Vzzz

10. Trigonometric form, Cylindrical coordinates, Routes 2 → 6 → 4, and Table 6 → Table 15 → Table 10

```
In[1]:= Clear["Global`*"];
```

10.1 Table 6

```
In[2]:= Vx = Cos[λ] * Vr - Sin[λ] * Vλ;
```

```
In[3]:= Vy = Sin[λ] * Vr + Cos[λ] * Vλ;
```

```
In[4]:= Vz = Vz;
```

```
In[5]:= Vxx = (Cos[λ])^2 * Vrr - Sin[2 * λ] * Vrλ + (Sin[λ])^2 * Vλλ;
```

```
In[6]:= Vxy = Sin[λ] * Cos[λ] * Vrr + Cos[2 * λ] * Vrλ - Sin[λ] * Cos[λ] * Vλλ;
```

```
In[7]:= Vxz = Cos[λ] * Vrz - Sin[λ] * Vλz;
```

```
In[8]:= Vyy = (Sin[λ])^2 * Vrr + Sin[2 * λ] * Vrλ + (Cos[λ])^2 * Vλλ;
```

```
In[9]:= Vyz = Sin[λ] * Vrz + Cos[λ] * Vλz;
```

```
In[10]:= Vzz = Vzz;
```

```
In[11]:= Vxxx = (Cos[λ])^3 * Vrrr - 3 * Sin[λ] * (Cos[λ])^2 * Vrrλ +  
3 * (Sin[λ])^2 * Cos[λ] * Vλλr - (Sin[λ])^3 * Vλλλ;
```

```
In[12]:= Vxxy = Sin[λ] * (Cos[λ])^2 * Vrrr + Cos[λ] * ((Cos[λ])^2 - 2 * (Sin[λ])^2) * Vrrλ +  
Sin[λ] * ((Sin[λ])^2 - 2 * (Cos[λ])^2) * Vλλr + (Sin[λ])^2 * Cos[λ] * Vλλλ;
```

```
In[13]:= Vxxz = (Cos[λ])^2 * Vrrz - Sin[2 * λ] * Vrλz + (Sin[λ])^2 * Vλλz;
```

```
In[14]:= Vxyz = Sin[λ] * Cos[λ] * Vrrz + Cos[2 * λ] * Vrλz - Sin[λ] * Cos[λ] * Vλλz;
```

```
In[15]:= Vyyx = (Sin[λ])^2 * Cos[λ] * Vrrr + Sin[λ] * (2 * (Cos[λ])^2 - (Sin[λ])^2) * Vrrλ +  
Cos[λ] * ((Cos[λ])^2 - 2 * (Sin[λ])^2) * Vλλr - Sin[λ] * (Cos[λ])^2 * Vλλλ;
```

```
In[16]:= Vyyy = (Sin[λ])^3 * Vrrr + 3 * (Sin[λ])^2 * Cos[λ] * Vrrλ +  
3 * Sin[λ] * (Cos[λ])^2 * Vλλr + (Cos[λ])^3 * Vλλλ;
```

```
In[6]:= Vyyz = (Sin[\lambda])^2 * Vrrz + Sin[2 * \lambda] * Vr\lambda z + (Cos[\lambda])^2 * V\lambda\lambda z;
In[6]:= Vzzx = Cos[\lambda] * Vzzr - Sin[\lambda] * Vzz\lambda;
In[6]:= Vzzy = Sin[\lambda] * Vzzr + Cos[\lambda] * Vzz\lambda;
In[6]:= Vzzz = Vzzz;
```

10.2 Table 15

```
In[6]:= V\lambda 1 = -Sin[\lambda] * Vx + Cos[\lambda] * Vy;
In[6]:= V\varphi = -Sin[\varphi] * Cos[\lambda] * Vx - Sin[\varphi] * Sin[\lambda] * Vy + Cos[\varphi] * Vz;
In[6]:= V\rho = Cos[\varphi] * Cos[\lambda] * Vx + Cos[\varphi] * Sin[\lambda] * Vy + Sin[\varphi] * Vz;
In[6]:= V\lambda\lambda 1 = (Sin[\lambda])^2 * Vxx - Sin[2 * \lambda] * Vxy + (Cos[\lambda])^2 * Vyy;
In[6]:= V\lambda\varphi = Sin[\varphi] * Sin[\lambda] * Cos[\lambda] * Vxx - Sin[\varphi] * Cos[2 * \lambda] * Vxy -
          Cos[\varphi] * Sin[\lambda] * Vxz - Sin[\varphi] * Sin[\lambda] * Cos[\lambda] * Vyy + Cos[\varphi] * Cos[\lambda] * Vyz;
In[6]:= V\lambda\rho = -Cos[\varphi] * Sin[\lambda] * Cos[\lambda] * Vxx + Cos[\varphi] * Cos[2 * \lambda] * Vxy -
          Sin[\varphi] * Sin[\lambda] * Vxz + Cos[\varphi] * Sin[\lambda] * Cos[\lambda] * Vyy + Sin[\varphi] * Cos[\lambda] * Vyz;
In[6]:= V\varphi\varphi = (Sin[\varphi])^2 * (Cos[\lambda])^2 * Vxx + (Sin[\varphi])^2 * Sin[2 * \lambda] * Vxy - Sin[2 * \varphi] * Cos[\lambda] * Vxz +
          (Sin[\varphi])^2 * (Sin[\lambda])^2 * Vyy - Sin[2 * \varphi] * Sin[\lambda] * Vyz + (Cos[\varphi])^2 * Vzz;
In[6]:= V\varphi\rho = -Sin[\varphi] * Cos[\varphi] * (Cos[\lambda])^2 * Vxx -
          Sin[2 * \varphi] * Sin[\lambda] * Cos[\lambda] * Vxy + Cos[2 * \varphi] * Cos[\lambda] * Vxz -
          Sin[\varphi] * Cos[\varphi] * (Sin[\lambda])^2 * Vyy + Cos[2 * \varphi] * Sin[\lambda] * Vyz + Sin[\varphi] * Cos[\varphi] * Vzz;
In[6]:= V\rho\rho = (Cos[\varphi])^2 * (Cos[\lambda])^2 * Vxx + (Cos[\varphi])^2 * Sin[2 * \lambda] * Vxy + Sin[2 * \varphi] * Cos[\lambda] * Vxz +
          (Cos[\varphi])^2 * (Sin[\lambda])^2 * Vyy + Sin[2 * \varphi] * Sin[\lambda] * Vyz + (Sin[\varphi])^2 * Vzz;
In[6]:= V\lambda\lambda\lambda 1 = - (Sin[\lambda])^3 * Vxxx + 3 * (Sin[\lambda])^2 * Cos[\lambda] * Vxxy -
          3 * Sin[\lambda] * (Cos[\lambda])^2 * Vyyx + (Cos[\lambda])^3 * Vyyy;
In[6]:= V\lambda\lambda\varphi = -Sin[\varphi] * (Sin[\lambda])^2 * Cos[\lambda] * Vxxx +
          Sin[\varphi] * Sin[\lambda] * (2 * (Cos[\lambda])^2 - (Sin[\lambda])^2) * Vxxy + Cos[\varphi] * (Sin[\lambda])^2 * Vxxxz -
          Cos[\varphi] * Sin[2 * \lambda] * Vxyz + Sin[\varphi] * Cos[\lambda] * (2 * (Sin[\lambda])^2 - (Cos[\lambda])^2) * Vyyx -
          Sin[\varphi] * Sin[\lambda] * (Cos[\lambda])^2 * Vyyy + Cos[\varphi] * (Cos[\lambda])^2 * Vyyz;
In[6]:= V\lambda\lambda\rho = Cos[\varphi] * (Sin[\lambda])^2 * Cos[\lambda] * Vxxx +
          Cos[\varphi] * Sin[\lambda] * ((Sin[\lambda])^2 - 2 * (Cos[\lambda])^2) * Vxxy + Sin[\varphi] * (Sin[\lambda])^2 * Vxxxz -
          Sin[\varphi] * Sin[2 * \lambda] * Vxyz + Cos[\varphi] * Cos[\lambda] * ((Cos[\lambda])^2 - 2 * (Sin[\lambda])^2) * Vyyx +
          Cos[\varphi] * Sin[\lambda] * (Cos[\lambda])^2 * Vyyy + Sin[\varphi] * (Cos[\lambda])^2 * Vyyz;
In[6]:= V\lambda\varphi\rho = Sin[\varphi] * Cos[\varphi] * Sin[\lambda] * (Cos[\lambda])^2 * Vxxx +
          Sin[\varphi] * Cos[\varphi] * Cos[\lambda] * (2 * (Sin[\lambda])^2 - (Cos[\lambda])^2) * Vxxy -
          Cos[2 * \varphi] * Sin[\lambda] * Cos[\lambda] * Vxxxz + Cos[2 * \varphi] * Cos[2 * \lambda] * Vxyz +
          Sin[\varphi] * Cos[\varphi] * Sin[\lambda] * ((Sin[\lambda])^2 - 2 * (Cos[\lambda])^2) * Vyyx -
          Sin[\varphi] * Cos[\varphi] * (Sin[\lambda])^2 * Cos[\lambda] * Vyyy + Cos[2 * \varphi] * Sin[\lambda] * Cos[\lambda] * Vyyz -
          Sin[\varphi] * Cos[\varphi] * Sin[\lambda] * Vzzx + Sin[\varphi] * Cos[\varphi] * Cos[\lambda] * Vzzy;
```

```

In[1]:= Vφφλ = - (Sin[φ])2 * Sin[λ] * (Cos[λ])2 * Vxxx +
           (Sin[φ])2 * Cos[λ] * ((Cos[λ])2 - 2 * (Sin[λ])2) * Vxxy +
           Sin[2 * φ] * Sin[λ] * Cos[λ] * Vxxz - Sin[2 * φ] * Cos[2 * λ] * Vxyz +
           (Sin[φ])2 * Sin[λ] * (2 * (Cos[λ])2 - (Sin[λ])2) * Vyvx +
           (Sin[φ])2 * (Sin[λ])2 * Cos[λ] * Vyyy - Sin[2 * φ] * Sin[λ] * Cos[λ] * Vyyz -
           (Cos[φ])2 * Sin[λ] * Vzzx + (Cos[φ])2 * Cos[λ] * Vzzy;

In[2]:= Vφφφ = - (Sin[φ])3 * (Cos[λ])3 * Vxxx - 3 * (Sin[φ])3 * Sin[λ] * (Cos[λ])2 * Vxxy +
           3 * (Sin[φ])2 * Cos[φ] * (Cos[λ])2 * Vxxz + 3 * (Sin[φ])2 * Cos[φ] * Sin[2 * λ] * Vxyz -
           3 * (Sin[φ])3 * (Sin[λ])2 * Cos[λ] * Vyvx - (Sin[φ])3 * (Sin[λ])3 * Vyyy +
           3 * (Sin[φ])2 * Cos[φ] * (Sin[λ])2 * Vyyz - 3 * Sin[φ] * (Cos[φ])2 * Cos[λ] * Vzzx -
           3 * Sin[φ] * (Cos[φ])2 * Sin[λ] * Vzzy + (Cos[φ])3 * Vzzz;

In[3]:= Vφφρ = (Sin[φ])2 * Cos[φ] * (Cos[λ])3 * Vxxx + 3 * (Sin[φ])2 * Cos[φ] * Sin[λ] *
           (Cos[λ])2 * Vxxy + Sin[φ] * (Cos[λ])2 * ((Sin[φ])2 - 2 * (Cos[φ])2) * Vxxz +
           Sin[φ] * Sin[2 * λ] * ((Sin[φ])2 - 2 * (Cos[φ])2) * Vxyz + 3 * (Sin[φ])2 *
           Cos[φ] * (Sin[λ])2 * Cos[λ] * Vyvx + (Sin[φ])2 * Cos[φ] * (Sin[λ])3 * Vyyy +
           Sin[φ] * (Sin[λ])2 * ((Sin[φ])2 - 2 * (Cos[φ])2) * Vyyz +
           Cos[φ] * Cos[λ] * ((Cos[φ])2 - 2 * (Sin[φ])2) * Vzzx +
           Cos[φ] * Sin[λ] * ((Cos[φ])2 - 2 * (Sin[φ])2) * Vzzy + Sin[φ] * (Cos[φ])2 * Vzzz;

In[4]:= Vρρλ = - (Cos[φ])2 * Sin[λ] * (Cos[λ])2 * Vxxx +
           (Cos[φ])2 * Cos[λ] * ((Cos[λ])2 - 2 * (Sin[λ])2) * Vxxy -
           Sin[2 * φ] * Sin[λ] * Cos[λ] * Vxxz + Sin[2 * φ] * Cos[2 * λ] * Vxyz +
           (Cos[φ])2 * Sin[λ] * (2 * (Cos[λ])2 - (Sin[λ])2) * Vyvx +
           (Cos[φ])2 * (Sin[λ])2 * Cos[λ] * Vyyy + Sin[2 * φ] * Sin[λ] * Cos[λ] * Vyyz -
           (Sin[φ])2 * Sin[λ] * Vzzx + (Sin[φ])2 * Cos[λ] * Vzzy;

In[5]:= Vρρφ = - Sin[φ] * (Cos[φ])2 * (Cos[λ])3 * Vxxx - 3 * Sin[φ] * (Cos[φ])2 * Sin[λ] *
           (Cos[λ])2 * Vxxy + Cos[φ] * (Cos[λ])2 * ((Cos[φ])2 - 2 * (Sin[φ])2) * Vxxz +
           Cos[φ] * Sin[2 * λ] * ((Cos[φ])2 - 2 * (Sin[φ])2) * Vxyz -
           3 * Sin[φ] * (Cos[φ])2 * (Sin[λ])2 * Cos[λ] * Vyvx - Sin[φ] * (Cos[φ])2 *
           (Sin[λ])3 * Vyyy + Cos[φ] * (Sin[λ])2 * ((Cos[φ])2 - 2 * (Sin[φ])2) * Vyyz +
           Sin[φ] * Cos[λ] * (2 * (Cos[φ])2 - (Sin[φ])2) * Vzzx +
           Sin[φ] * Sin[λ] * (2 * (Cos[φ])2 - (Sin[φ])2) * Vzzy + (Sin[φ])2 * Cos[φ] * Vzzz;

In[6]:= Vρρρ = (Cos[φ])3 * (Cos[λ])3 * Vxxx +
           3 * (Cos[φ])3 * Sin[λ] * (Cos[λ])2 * Vxxy + 3 * Sin[φ] * (Cos[φ])2 * (Cos[λ])2 * Vxxz +
           3 * Sin[2 * φ] * Cos[φ] * Sin[λ] * Cos[λ] * Vxyz +
           3 * (Cos[φ])3 * (Sin[λ])2 * Cos[λ] * Vyvx + (Cos[φ])3 * (Sin[λ])3 * Vyyy +
           3 * Sin[φ] * (Cos[φ])2 * (Sin[λ])2 * Vyyz + 3 * (Sin[φ])2 * Cos[φ] * Cos[λ] * Vzzx +
           3 * (Sin[φ])2 * Cos[φ] * Sin[λ] * Vzzy + (Sin[φ])3 * Vzzz;

```

10.3 Table 10

```
In[1]:= VrRoutes264 = - Sin[φ] * Vφ + Cos[φ] * Vρ;
```

```
In[2]:= VλRoutes264 = Vλ1;
```

```

In[1]:= VzRoutes264 = Cos[\varphi] * V\varphi + Sin[\varphi] * V\rho;
In[2]:= VrrRoutes264 = (Sin[\varphi])^2 * V\varphi\varphi - Sin[2 * \varphi] * V\varphi\rho + (Cos[\varphi])^2 * V\rho\rho;
In[3]:= Vr\lambda Routes264 = -Sin[\varphi] * V\lambda\varphi + Cos[\varphi] * V\lambda\rho;
In[4]:= VrzRoutes264 = -Sin[\varphi] * Cos[\varphi] * V\varphi\varphi + Cos[2 * \varphi] * V\varphi\rho + Sin[\varphi] * Cos[\varphi] * V\rho\rho;
In[5]:= V\lambda\lambda Routes264 = V\lambda\lambda1;
In[6]:= V\lambda z Routes264 = Cos[\varphi] * V\lambda\varphi + Sin[\varphi] * V\lambda\rho;
In[7]:= VzzRoutes264 = (Cos[\varphi])^2 * V\varphi\varphi + Sin[2 * \varphi] * V\varphi\rho + (Sin[\varphi])^2 * V\rho\rho;
In[8]:= VrrrRoutes264 = - (Sin[\varphi])^3 * V\varphi\varphi\varphi +
3 * (Sin[\varphi])^2 * Cos[\varphi] * V\varphi\varphi\rho - 3 * Sin[\varphi] * (Cos[\varphi])^2 * V\varphi\rho\varphi + (Cos[\varphi])^3 * V\rho\rho\rho;
In[9]:= Vrr\lambda Routes264 = (Sin[\varphi])^2 * V\varphi\varphi\lambda - Sin[2 * \varphi] * V\lambda\varphi\rho + (Cos[\varphi])^2 * V\rho\rho\lambda;
In[10]:= VrrzRoutes264 = (Sin[\varphi])^2 * Cos[\varphi] * V\varphi\varphi\varphi + Sin[\varphi] * ((Sin[\varphi])^2 - 2 * (Cos[\varphi])^2) * V\varphi\varphi\rho +
Cos[\varphi] * ((Cos[\varphi])^2 - 2 * (Sin[\varphi])^2) * V\varphi\rho\varphi + Sin[\varphi] * (Cos[\varphi])^2 * V\rho\rho\rho;
In[11]:= Vr\lambda z Routes264 = Cos[2 * \varphi] * V\lambda\varphi\rho - Sin[\varphi] * Cos[\varphi] * V\varphi\varphi\lambda + Sin[\varphi] * Cos[\varphi] * V\rho\rho\lambda;
In[12]:= V\lambda\lambda r Routes264 = -Sin[\varphi] * V\lambda\lambda\varphi + Cos[\varphi] * V\lambda\lambda\rho;
In[13]:= V\lambda\lambda\lambda Routes264 = V\lambda\lambda\lambda1;
In[14]:= V\lambda\lambda z Routes264 = Cos[\varphi] * V\lambda\lambda\varphi + Sin[\varphi] * V\lambda\lambda\rho;
In[15]:= VzzrRoutes264 = -Sin[\varphi] * (Cos[\varphi])^2 * V\varphi\varphi\varphi + Cos[\varphi] * ((Cos[\varphi])^2 - 2 * (Sin[\varphi])^2) * V\varphi\varphi\rho +
Sin[\varphi] * (2 * (Cos[\varphi])^2 - (Sin[\varphi])^2) * V\varphi\rho\varphi + (Sin[\varphi])^2 * Cos[\varphi] * V\rho\rho\rho;
In[16]:= Vzz\lambda Routes264 = Sin[2 * \varphi] * V\lambda\varphi\rho + (Cos[\varphi])^2 * V\varphi\varphi\lambda + (Sin[\varphi])^2 * V\rho\rho\lambda;
In[17]:= VzzzRoutes264 = (Cos[\varphi])^3 * V\varphi\varphi\varphi +
3 * Sin[\varphi] * (Cos[\varphi])^2 * V\varphi\varphi\rho + 3 * (Sin[\varphi])^2 * Cos[\varphi] * V\varphi\rho\varphi + (Sin[\varphi])^3 * V\rho\rho\rho;

```

10.4 Check whether they become themselves

In[1]:= **FullSimplify**[VrRoutes264]

Out[1]=

Vr

In[2]:= **FullSimplify**[V\lambda Routes264]

Out[2]=

V\lambda

In[3]:= **FullSimplify**[VzRoutes264]

Out[3]=

Vz

In[4]:= **FullSimplify**[VrrRoutes264]

Out[4]=

Vrr

In[1]:= FullSimplify[VrλRoutes264]

Out[1]=

Vrλ

In[2]:= FullSimplify[VrzRoutes264]

Out[2]=

Vrz

In[3]:= FullSimplify[VλλRoutes264]

Out[3]=

Vλλ

In[4]:= FullSimplify[VλzRoutes264]

Out[4]=

Vλz

In[5]:= FullSimplify[VzzRoutes264]

Out[5]=

Vzz

In[6]:= FullSimplify[VrrrRoutes264]

Out[6]=

Vrrr

In[7]:= FullSimplify[VrrλRoutes264]

Out[7]=

Vrrλ

In[8]:= FullSimplify[VrrzRoutes264]

Out[8]=

Vrrz

In[9]:= FullSimplify[VrλzRoutes264]

Out[9]=

Vrλz

In[10]:= FullSimplify[VλλrRoutes264]

Out[10]=

Vλλr

In[11]:= FullSimplify[VλλλRoutes264]

Out[11]=

Vλλλ

In[12]:= FullSimplify[VλλzRoutes264]

Out[12]=

Vλλz

In[13]:= FullSimplify[VzzrRoutes264]

Out[13]=

Vzzr

In[14]:= FullSimplify[VzzλRoutes264]

Out[14]=

Vzzλ

```
In[1]:= FullSimplify[VzzzRoutes264]
Out[1]= Vzzz
```

11. Trigonometric form, Spherical coordinates, Routes 5 → 1 → 3, and Table 13 → Table 4 → Table 8

```
In[2]:= Clear["Global`*"];
```

11.1 Table 13

```
In[3]:= Vx = -Sin[λ] * Vλ - Sin[φ] * Cos[λ] * Vφ + Cos[φ] * Cos[λ] * Vρ;
In[4]:= Vy = Cos[λ] * Vλ - Sin[φ] * Sin[λ] * Vφ + Cos[φ] * Sin[λ] * Vρ;
In[5]:= Vz = Cos[φ] * Vφ + Sin[φ] * Vρ;
In[6]:= Vxx = (Sin[λ])^2 * Vλλ + Sin[φ] * Sin[2 * λ] * Vλφ -
          Cos[φ] * Sin[2 * λ] * Vλρ + (Sin[φ])^2 * (Cos[λ])^2 * Vφφ -
          Sin[2 * φ] * (Cos[λ])^2 * Vφρ + (Cos[φ])^2 * (Cos[λ])^2 * Vρρ;
In[7]:= Vxy = -Sin[λ] * Cos[λ] * Vλλ - Sin[φ] * Cos[2 * λ] * Vλφ +
          Cos[φ] * Cos[2 * λ] * Vλρ + (Sin[φ])^2 * Sin[λ] * Cos[λ] * Vφφ -
          Sin[2 * φ] * Sin[λ] * Cos[λ] * Vφρ + (Cos[φ])^2 * Sin[λ] * Cos[λ] * Vρρ;
In[8]:= Vxz = -Cos[φ] * Sin[λ] * Vλφ - Sin[φ] * Sin[λ] * Vλρ - Sin[φ] * Cos[φ] * Cos[λ] * Vφφ +
          Cos[2 * φ] * Cos[λ] * Vφρ + Sin[φ] * Cos[φ] * Cos[λ] * Vρρ;
In[9]:= Vyy = (Cos[λ])^2 * Vλλ - Sin[φ] * Sin[2 * λ] * Vλφ +
          Cos[φ] * Sin[2 * λ] * Vλρ + (Sin[φ])^2 * (Sin[λ])^2 * Vφφ -
          Sin[2 * φ] * (Sin[λ])^2 * Vφρ + (Cos[φ])^2 * (Sin[λ])^2 * Vρρ;
In[10]:= Vyz = Cos[φ] * Cos[λ] * Vλφ + Sin[φ] * Cos[λ] * Vλρ - Sin[φ] * Cos[φ] * Sin[λ] * Vφφ +
           Cos[2 * φ] * Sin[λ] * Vφρ + Sin[φ] * Cos[φ] * Sin[λ] * Vρρ;
In[11]:= Vzz = (Cos[φ])^2 * Vφφ + Sin[2 * φ] * Vφρ + (Sin[φ])^2 * Vρρ;
In[12]:= Vxxx = -(Sin[λ])^3 * Vλλλ + 3 * Cos[φ] * (Sin[λ])^2 * Cos[λ] * Vλλρ -
          3 * Sin[φ] * (Sin[λ])^2 * Cos[λ] * Vλλφ + 3 * Sin[2 * φ] * Sin[λ] * (Cos[λ])^2 * Vλφρ -
          3 * (Sin[φ])^2 * Sin[λ] * (Cos[λ])^2 * Vφφλ - (Sin[φ])^3 * (Cos[λ])^3 * Vφφφ +
          3 * (Sin[φ])^2 * Cos[φ] * (Cos[λ])^3 * Vφφρ - 3 * (Cos[φ])^2 * Sin[λ] * (Cos[λ])^2 * Vρρλ -
          3 * Sin[φ] * (Cos[φ])^2 * (Cos[λ])^3 * Vρρφ + (Cos[φ])^3 * (Cos[λ])^3 * Vρρρ;
In[13]:= Vxxy = (Sin[λ])^2 * Cos[λ] * Vλλλ + Sin[φ] * Sin[λ] * (2 * (Cos[λ])^2 - (Sin[λ])^2) * Vλλφ +
           Cos[φ] * Sin[λ] * ((Sin[λ])^2 - 2 * (Cos[λ])^2) * Vλλρ +
           Sin[2 * φ] * Cos[λ] * (2 * (Sin[λ])^2 - (Cos[λ])^2) * Vλφρ +
           (Sin[φ])^2 * Cos[λ] * ((Cos[λ])^2 - 2 * (Sin[λ])^2) * Vφφλ - (Sin[φ])^3 * Sin[λ] *
           (Cos[λ])^2 * Vφφφ + 3 * (Sin[φ])^2 * Cos[φ] * Sin[λ] * (Cos[λ])^2 * Vφφρ +
           (Cos[φ])^2 * Cos[λ] * ((Cos[λ])^2 - 2 * (Sin[λ])^2) * Vρρλ - 3 * Sin[φ] *
           (Cos[φ])^2 * Sin[λ] * (Cos[λ])^2 * Vρρφ + (Cos[φ])^3 * Sin[λ] * (Cos[λ])^2 * Vρρρ;
```

```

In[5]:= Vxxxz =
Cos[\varphi] * (Sin[\lambda])^2 * V\lambda\lambda\varphi + Sin[\varphi] * (Sin[\lambda])^2 * V\lambda\lambda\rho - Cos[2*\varphi] * Sin[2*\lambda] * V\lambda\varphi\rho +
Sin[\varphi] * Cos[\varphi] * Sin[2*\lambda] * V\varphi\varphi\lambda + (Sin[\varphi])^2 * Cos[\varphi] * (Cos[\lambda])^2 * V\varphi\varphi\varphi +
Sin[\varphi] * (Cos[\lambda])^2 * ((Sin[\varphi])^2 - 2 * (Cos[\varphi])^2) * V\varphi\varphi\rho -
Sin[2*\varphi] * Sin[\lambda] * Cos[\lambda] * V\rho\rho\lambda + Cos[\varphi] * (Cos[\lambda])^2 *
((Cos[\varphi])^2 - 2 * (Sin[\varphi])^2) * V\rho\rho\varphi + Sin[\varphi] * (Cos[\varphi])^2 * (Cos[\lambda])^2 * V\rho\rho\rho;

In[6]:= Vxyxz = -Cos[\varphi] * Sin[\lambda] * Cos[\lambda] * V\lambda\lambda\varphi -
Sin[\varphi] * Sin[\lambda] * Cos[\lambda] * V\lambda\lambda\rho + Cos[2*\varphi] * Cos[2*\lambda] * V\lambda\varphi\rho -
Sin[\varphi] * Cos[\varphi] * Cos[2*\lambda] * V\varphi\varphi\lambda + (Sin[\varphi])^2 * Cos[\varphi] * Sin[\lambda] * Cos[\lambda] * V\varphi\varphi\varphi -
Sin[\varphi] * Sin[\lambda] * Cos[\lambda] * (2 * (Cos[\varphi])^2 - (Sin[\varphi])^2) * V\varphi\varphi\rho +
Sin[\varphi] * Cos[\varphi] * Cos[2*\lambda] * V\rho\rho\lambda + Cos[\varphi] * Sin[\lambda] * Cos[\lambda] *
((Cos[\varphi])^2 - 2 * (Sin[\varphi])^2) * V\rho\rho\varphi + Sin[\varphi] * (Cos[\varphi])^2 * Sin[\lambda] * Cos[\lambda] * V\rho\rho\rho;

In[7]:= Vyxy = -Sin[\lambda] * (Cos[\lambda])^2 * V\lambda\lambda\lambda - Sin[\varphi] * Cos[\lambda] * ((Cos[\lambda])^2 - 2 * (Sin[\lambda])^2) * V\lambda\lambda\varphi +
Cos[\varphi] * Cos[\lambda] * ((Cos[\lambda])^2 - 2 * (Sin[\lambda])^2) * V\lambda\lambda\rho -
Sin[2*\varphi] * Sin[\lambda] * (2 * (Cos[\lambda])^2 - (Sin[\lambda])^2) * V\lambda\varphi\rho +
(Sin[\varphi])^2 * Sin[\lambda] * (2 * (Cos[\lambda])^2 - (Sin[\lambda])^2) * V\varphi\varphi\lambda - (Sin[\varphi])^3 *
(Sin[\lambda])^2 * Cos[\lambda] * V\varphi\varphi\varphi + 3 * (Sin[\varphi])^2 * Cos[\varphi] * (Sin[\lambda])^2 * Cos[\lambda] * V\varphi\varphi\rho +
(Cos[\varphi])^2 * Sin[\lambda] * (2 * (Cos[\lambda])^2 - (Sin[\lambda])^2) * V\rho\rho\lambda - 3 * Sin[\varphi] *
(Cos[\varphi])^2 * (Sin[\lambda])^2 * Cos[\lambda] * V\rho\rho\varphi + (Cos[\varphi])^3 * (Sin[\lambda])^2 * Cos[\lambda] * V\rho\rho\rho;

In[8]:= Vyyy = (Cos[\lambda])^3 * V\lambda\lambda\lambda - 3 * Sin[\varphi] * Sin[\lambda] * (Cos[\lambda])^2 * V\lambda\lambda\varphi +
3 * Cos[\varphi] * Sin[\lambda] * (Cos[\lambda])^2 * V\lambda\lambda\rho - 3 * Sin[2*\varphi] * (Sin[\lambda])^2 * Cos[\lambda] * V\lambda\varphi\rho +
3 * (Sin[\varphi])^2 * (Sin[\lambda])^2 * Cos[\lambda] * V\varphi\varphi\lambda - (Sin[\varphi])^3 * (Sin[\lambda])^3 * V\varphi\varphi\varphi +
3 * (Sin[\varphi])^2 * Cos[\varphi] * (Sin[\lambda])^3 * V\varphi\varphi\rho + 3 * (Cos[\varphi])^2 * (Sin[\lambda])^2 * Cos[\lambda] * V\rho\rho\lambda -
3 * Sin[\varphi] * (Cos[\varphi])^2 * (Sin[\lambda])^3 * V\rho\rho\varphi + (Cos[\varphi])^3 * (Sin[\lambda])^3 * V\rho\rho\rho;

In[9]:= Vyyz =
Cos[\varphi] * (Cos[\lambda])^2 * V\lambda\lambda\varphi + Sin[\varphi] * (Cos[\lambda])^2 * V\lambda\lambda\rho + Cos[2*\varphi] * Sin[2*\lambda] * V\lambda\varphi\rho -
Sin[2*\varphi] * Sin[\lambda] * Cos[\lambda] * V\varphi\varphi\lambda + (Sin[\varphi])^2 * Cos[\varphi] * (Sin[\lambda])^2 * V\varphi\varphi\varphi +
Sin[\varphi] * (Sin[\lambda])^2 * ((Sin[\varphi])^2 - 2 * (Cos[\varphi])^2) * V\varphi\varphi\rho +
Sin[2*\varphi] * Sin[\lambda] * Cos[\lambda] * V\rho\rho\lambda + Cos[\varphi] * (Sin[\lambda])^2 *
((Cos[\varphi])^2 - 2 * (Sin[\varphi])^2) * V\rho\rho\varphi + Sin[\varphi] * (Cos[\varphi])^2 * (Sin[\lambda])^2 * V\rho\rho\rho;

In[10]:= Vzzx = -Sin[2*\varphi] * Sin[\lambda] * V\lambda\varphi\rho -
(Cos[\varphi])^2 * Sin[\lambda] * V\varphi\varphi\lambda - Sin[\varphi] * (Cos[\varphi])^2 * Cos[\lambda] * V\varphi\varphi\varphi +
Cos[\varphi] * Cos[\lambda] * ((Cos[\varphi])^2 - 2 * (Sin[\varphi])^2) * V\varphi\varphi\rho - (Sin[\varphi])^2 * Sin[\lambda] * V\rho\rho\lambda +
Sin[\varphi] * Cos[\lambda] * (2 * (Cos[\varphi])^2 - (Sin[\varphi])^2) * V\rho\rho\varphi +
(Sin[\varphi])^2 * Cos[\varphi] * Cos[\lambda] * V\rho\rho\rho;

In[11]:= Vzzy = Sin[2*\varphi] * Cos[\lambda] * V\lambda\varphi\rho +
(Cos[\varphi])^2 * Cos[\lambda] * V\varphi\varphi\lambda - Sin[\varphi] * (Cos[\varphi])^2 * Sin[\lambda] * V\varphi\varphi\varphi +
Cos[\varphi] * Sin[\lambda] * ((Cos[\varphi])^2 - 2 * (Sin[\varphi])^2) * V\varphi\varphi\rho + (Sin[\varphi])^2 * Cos[\lambda] * V\rho\rho\lambda +
Sin[\varphi] * Sin[\lambda] * (2 * (Cos[\varphi])^2 - (Sin[\varphi])^2) * V\rho\rho\varphi +
(Sin[\varphi])^2 * Cos[\varphi] * Sin[\lambda] * V\rho\rho\rho;

In[12]:= Vzzz = (Cos[\varphi])^3 * V\varphi\varphi\varphi + 3 * Sin[\varphi] * (Cos[\varphi])^2 * V\varphi\varphi\rho +
3 * (Sin[\varphi])^2 * Cos[\varphi] * V\rho\rho\varphi + (Sin[\varphi])^3 * V\rho\rho\rho;

```

11.2 Table 4

```

In[1]:= Vr = Cos[\lambda] * Vx + Sin[\lambda] * Vy;
In[2]:= V\lambda1 = -Sin[\lambda] * Vx + Cos[\lambda] * Vy;
In[3]:= Vz = Vz;
In[4]:= Vrr = (Cos[\lambda])^2 * Vxx + Sin[2*\lambda] * Vxy + (Sin[\lambda])^2 * Vyy;
In[5]:= Vr\lambda = -Sin[\lambda] * Cos[\lambda] * Vxx + Cos[2*\lambda] * Vxy + Sin[\lambda] * Cos[\lambda] * Vyy;
In[6]:= Vrz = Cos[\lambda] * Vxz + Sin[\lambda] * Vyz;
In[7]:= V\lambda\lambda1 = (Sin[\lambda])^2 * Vxx - Sin[2*\lambda] * Vxy + (Cos[\lambda])^2 * Vyy;
In[8]:= V\lambda z = -Sin[\lambda] * Vxz + Cos[\lambda] * Vyz;
In[9]:= Vzz = Vzz;
In[10]:= Vrrr = (Cos[\lambda])^3 * Vxxx + 3 * Sin[\lambda] * (Cos[\lambda])^2 * Vxxy +
          3 * Cos[\lambda] * (Sin[\lambda])^2 * Vyyx + (Sin[\lambda])^3 * Vyyy;
In[11]:= Vrr\lambda = -Sin[\lambda] * (Cos[\lambda])^2 * Vxxx + Cos[\lambda] * ((Cos[\lambda])^2 - 2 * (Sin[\lambda])^2) * Vxxy +
          Sin[\lambda] * (2 * (Cos[\lambda])^2 - (Sin[\lambda])^2) * Vyyx + (Sin[\lambda])^2 * Cos[\lambda] * Vyyy;
In[12]:= Vrrz = (Cos[\lambda])^2 * Vxxz + Sin[2*\lambda] * Vxyz + (Sin[\lambda])^2 * Vyyz;
In[13]:= Vr\lambda z = -Sin[\lambda] * Cos[\lambda] * Vxxz + Cos[2*\lambda] * Vxyz + Sin[\lambda] * Cos[\lambda] * Vyyz;
In[14]:= V\lambda\lambda r = Cos[\lambda] * (Sin[\lambda])^2 * Vxxx + Sin[\lambda] * ((Sin[\lambda])^2 - 2 * (Cos[\lambda])^2) * Vxxy +
          Cos[\lambda] * ((Cos[\lambda])^2 - 2 * (Sin[\lambda])^2) * Vyyx + Sin[\lambda] * (Cos[\lambda])^2 * Vyyy;
In[15]:= V\lambda\lambda\lambda1 = - (Sin[\lambda])^3 * Vxxx + 3 * (Sin[\lambda])^2 * Cos[\lambda] * Vxxy -
          3 * Sin[\lambda] * (Cos[\lambda])^2 * Vyyx + (Cos[\lambda])^3 * Vyyy;
In[16]:= V\lambda\lambda z = (Sin[\lambda])^2 * Vxxz - Sin[2*\lambda] * Vxyz + (Cos[\lambda])^2 * Vyyz;
In[17]:= Vzzr = Cos[\lambda] * Vzzx + Sin[\lambda] * Vzzy;
In[18]:= Vzz\lambda = -Sin[\lambda] * Vzzx + Cos[\lambda] * Vzzy;
In[19]:= Vzzz = Vzzz;

```

11.3 Table 8

```

In[1]:= V\lambda Routes513 = V\lambda1;
In[2]:= V\phi Routes513 = -Sin[\varphi] * Vr + Cos[\varphi] * Vz;
In[3]:= V\rho Routes513 = Cos[\varphi] * Vr + Sin[\varphi] * Vz;
In[4]:= V\lambda\lambda Routes513 = V\lambda\lambda1;
In[5]:= V\lambda\varphi Routes513 = -Sin[\varphi] * Vr\lambda + Cos[\varphi] * V\lambda z;
In[6]:= V\lambda\rho Routes513 = Cos[\varphi] * Vr\lambda + Sin[\varphi] * V\lambda z;
In[7]:= V\varphi\varphi Routes513 = ((Sin[\varphi])^2 * Vrr - Sin[2*\varphi] * Vrz + (Cos[\varphi])^2 * Vzz);
In[8]:= V\varphi\rho Routes513 = -Sin[\varphi] * Cos[\varphi] * Vrr + Cos[2*\varphi] * Vrz + Sin[\varphi] * Cos[\varphi] * Vzz;

```

```
In[1]:= VρρRoutes513 = ((Cos[φ])^2 * Vrr + Sin[2 * φ] * Vrz + (Sin[φ])^2 * Vzz);  
In[2]:= VλλλRoutes513 = Vλλλ1;  
In[3]:= VλλφRoutes513 = -Sin[φ] * Vλλr + Cos[φ] * Vλλz;  
In[4]:= VλλρRoutes513 = Cos[φ] * Vλλr + Sin[φ] * Vλλz;  
In[5]:= VλφρRoutes513 = -Sin[φ] * Cos[φ] * Vrrλ + Cos[2 * φ] * Vrλz + Sin[φ] * Cos[φ] * Vzzλ;  
In[6]:= VφφλRoutes513 = (Sin[φ])^2 * Vrrλ - Sin[2 * φ] * Vrλz + (Cos[φ])^2 * Vzzλ;  
In[7]:= VφφφRoutes513 = - (Sin[φ])^3 * Vrrr +  
3 * (Sin[φ])^2 * Cos[φ] * Vrrz - 3 * Sin[φ] * (Cos[φ])^2 * Vzzr + (Cos[φ])^3 * Vzzz;  
In[8]:= VφφρRoutes513 = (Sin[φ])^2 * Cos[φ] * Vrrr + Sin[φ] * ((Sin[φ])^2 - 2 * (Cos[φ])^2) * Vrrz +  
Cos[φ] * ((Cos[φ])^2 - 2 * (Sin[φ])^2) * Vzzr + Sin[φ] * (Cos[φ])^2 * Vzzz;  
In[9]:= VρρλRoutes513 = (Cos[φ])^2 * Vrrλ + Sin[2 * φ] * Vrλz + (Sin[φ])^2 * Vzzλ;  
In[10]:= VρρφRoutes513 = -Sin[φ] * (Cos[φ])^2 * Vrrr + Cos[φ] * ((Cos[φ])^2 - 2 * (Sin[φ])^2) * Vrrz +  
Sin[φ] * (2 * (Cos[φ])^2 - (Sin[φ])^2) * Vzzr + Cos[φ] * (Sin[φ])^2 * Vzzz;  
In[11]:= VρρρRoutes513 = (Cos[φ])^3 * Vrrr +  
3 * Sin[φ] * (Cos[φ])^2 * Vrrz + 3 * Cos[φ] * (Sin[φ])^2 * Vzzr + (Sin[φ])^3 * Vzzz;
```

11.4 Check whether they become themselves

```
In[1]:= FullSimplify[VλRoutes513]
```

```
Out[1]=  
Vλ
```

```
In[2]:= FullSimplify[VφRoutes513]
```

```
Out[2]=  
Vφ
```

```
In[3]:= FullSimplify[VρRoutes513]
```

```
Out[3]=  
Vρ
```

```
In[4]:= FullSimplify[VλλRoutes513]
```

```
Out[4]=  
Vλλ
```

```
In[5]:= FullSimplify[VλφRoutes513]
```

```
Out[5]=  
Vλφ
```

```
In[6]:= FullSimplify[VλρRoutes513]
```

```
Out[6]=  
Vλρ
```

```
In[7]:= FullSimplify[VφφRoutes513]
```

```
Out[7]=  
Vφφ
```

```

In[]:= FullSimplify[VφρRoutes513]
Out[]= Vφρ

In[]:= FullSimplify[VρρRoutes513]
Out[]= Vρρ

In[]:= FullSimplify[VλλλRoutes513]
Out[]= Vλλλ

In[]:= FullSimplify[VλλφRoutes513]
Out[]= Vλλφ

In[]:= FullSimplify[VλλρRoutes513]
Out[]= Vλλρ

In[]:= FullSimplify[VλφρRoutes513]
Out[]= Vλφρ

In[]:= FullSimplify[VφφλRoutes513]
Out[]= Vφφλ

In[]:= FullSimplify[VφφφRoutes513]
Out[]= Vφφφ

In[]:= FullSimplify[VφφρRoutes513]
Out[]= Vφφρ

In[]:= FullSimplify[VρρλRoutes513]
Out[]= Vρρλ

In[]:= FullSimplify[VρρφRoutes513]
Out[]= Vρρφ

In[]:= FullSimplify[VρρρRoutes513]
Out[]= Vρρρ

```

12. Trigonometric form, Spherical coordinates, Routes 4 → 2 → 6, and Table 10 → Table 6 → Table 15

```
In[]:= Clear["Global`*"];
```

12.1 Table 10

```

In[1]:= Vr = -Sin[\varphi] * V\varphi + Cos[\varphi] * V\rho;
In[2]:= V\lambda = V\lambda;
In[3]:= Vz = Cos[\varphi] * V\varphi + Sin[\varphi] * V\rho;
In[4]:= Vrr = (Sin[\varphi])^2 * V\varphi\varphi - Sin[2*\varphi] * V\varphi\rho + (Cos[\varphi])^2 * V\rho\rho;
In[5]:= Vr\lambda = -Sin[\varphi] * V\lambda\varphi + Cos[\varphi] * V\lambda\rho;
In[6]:= Vrz = -Sin[\varphi] * Cos[\varphi] * V\varphi\varphi + Cos[2*\varphi] * V\varphi\rho + Sin[\varphi] * Cos[\varphi] * V\rho\rho;
In[7]:= V\lambda\lambda = V\lambda\lambda;
In[8]:= V\lambda z = Cos[\varphi] * V\lambda\varphi + Sin[\varphi] * V\lambda\rho;
In[9]:= Vzz = (Cos[\varphi])^2 * V\varphi\varphi + Sin[2*\varphi] * V\varphi\rho + (Sin[\varphi])^2 * V\rho\rho;
In[10]:= Vrrr = -(Sin[\varphi])^3 * V\varphi\varphi\varphi + 3 * (Sin[\varphi])^2 * Cos[\varphi] * V\varphi\varphi\rho -
          3 * Sin[\varphi] * (Cos[\varphi])^2 * V\varphi\rho\varphi + (Cos[\varphi])^3 * V\rho\rho\rho;
In[11]:= Vrr\lambda = (Sin[\varphi])^2 * V\varphi\varphi\lambda - Sin[2*\varphi] * V\lambda\varphi\rho + (Cos[\varphi])^2 * V\rho\rho\lambda;
In[12]:= Vrrz = (Sin[\varphi])^2 * Cos[\varphi] * V\varphi\varphi\varphi + Sin[\varphi] * ((Sin[\varphi])^2 - 2 * (Cos[\varphi])^2) * V\varphi\varphi\rho +
          Cos[\varphi] * ((Cos[\varphi])^2 - 2 * (Sin[\varphi])^2) * V\varphi\rho\varphi + Sin[\varphi] * (Cos[\varphi])^2 * V\rho\rho\rho;
In[13]:= Vr\lambda z = Cos[2*\varphi] * V\lambda\varphi\rho - Sin[\varphi] * Cos[\varphi] * V\varphi\varphi\lambda + Sin[\varphi] * Cos[\varphi] * V\rho\rho\lambda;
In[14]:= V\lambda\lambda r = -Sin[\varphi] * V\lambda\lambda\varphi + Cos[\varphi] * V\lambda\lambda\rho;
In[15]:= V\lambda\lambda\lambda = V\lambda\lambda\lambda;
In[16]:= V\lambda\lambda z = Cos[\varphi] * V\lambda\lambda\varphi + Sin[\varphi] * V\lambda\lambda\rho;
In[17]:= Vzzr = -Sin[\varphi] * (Cos[\varphi])^2 * V\varphi\varphi\varphi + Cos[\varphi] * ((Cos[\varphi])^2 - 2 * (Sin[\varphi])^2) * V\varphi\varphi\rho +
          Sin[\varphi] * (2 * (Cos[\varphi])^2 - (Sin[\varphi])^2) * V\varphi\rho\varphi + (Sin[\varphi])^2 * Cos[\varphi] * V\rho\rho\rho;
In[18]:= Vzz\lambda = Sin[2*\varphi] * V\lambda\varphi\rho + (Cos[\varphi])^2 * V\varphi\varphi\lambda + (Sin[\varphi])^2 * V\rho\rho\lambda;
In[19]:= Vzzz = (Cos[\varphi])^3 * V\varphi\varphi\varphi + 3 * Sin[\varphi] * (Cos[\varphi])^2 * V\varphi\varphi\rho +
          3 * (Sin[\varphi])^2 * Cos[\varphi] * V\varphi\rho\varphi + (Sin[\varphi])^3 * V\rho\rho\rho;

```

12.2 Table 6

```

In[1]:= Vx = Cos[\lambda] * Vr - Sin[\lambda] * V\lambda;
In[2]:= Vy = Sin[\lambda] * Vr + Cos[\lambda] * V\lambda;
In[3]:= Vz = Vz;
In[4]:= Vxx = (Cos[\lambda])^2 * Vrr - Sin[2*\lambda] * Vr\lambda + (Sin[\lambda])^2 * V\lambda\lambda;
In[5]:= Vxy = Sin[\lambda] * Cos[\lambda] * Vrr + Cos[2*\lambda] * Vr\lambda - Sin[\lambda] * Cos[\lambda] * V\lambda\lambda;
In[6]:= Vxz = Cos[\lambda] * Vrz - Sin[\lambda] * V\lambda z;
In[7]:= Vyy = (Sin[\lambda])^2 * Vrr + Sin[2*\lambda] * Vr\lambda + (Cos[\lambda])^2 * V\lambda\lambda;
In[8]:= Vyz = Sin[\lambda] * Vrz + Cos[\lambda] * V\lambda z;

```

```

In[1]:= Vzz = Vzz;

In[2]:= Vxxx = (Cos[\lambda])^3 * Vrrr - 3 * Sin[\lambda] * (Cos[\lambda])^2 * Vrr\lambda +
3 * (Sin[\lambda])^2 * Cos[\lambda] * V\lambda\lambda r - (Sin[\lambda])^3 * V\lambda\lambda\lambda;

In[3]:= Vxxy = Sin[\lambda] * (Cos[\lambda])^2 * Vrrr + Cos[\lambda] * ((Cos[\lambda])^2 - 2 * (Sin[\lambda])^2) * Vrr\lambda +
Sin[\lambda] * ((Sin[\lambda])^2 - 2 * (Cos[\lambda])^2) * V\lambda\lambda r + (Sin[\lambda])^2 * Cos[\lambda] * V\lambda\lambda\lambda;

In[4]:= Vxxz = (Cos[\lambda])^2 * Vrrz - Sin[2*\lambda] * Vr\lambda z + (Sin[\lambda])^2 * V\lambda\lambda z;

In[5]:= Vxyz = Sin[\lambda] * Cos[\lambda] * Vrrz + Cos[2*\lambda] * Vr\lambda z - Sin[\lambda] * Cos[\lambda] * V\lambda\lambda z;

In[6]:= Vyiy = (Sin[\lambda])^2 * Cos[\lambda] * Vrrr + Sin[\lambda] * (2 * (Cos[\lambda])^2 - (Sin[\lambda])^2) * Vrr\lambda +
Cos[\lambda] * ((Cos[\lambda])^2 - 2 * (Sin[\lambda])^2) * V\lambda\lambda r - Sin[\lambda] * (Cos[\lambda])^2 * V\lambda\lambda\lambda;

In[7]:= Vyiy = (Sin[\lambda])^3 * Vrrr + 3 * (Sin[\lambda])^2 * Cos[\lambda] * Vrr\lambda +
3 * Sin[\lambda] * (Cos[\lambda])^2 * V\lambda\lambda r + (Cos[\lambda])^3 * V\lambda\lambda\lambda;

In[8]:= Vyiy = (Sin[\lambda])^2 * Vrrz + Sin[2*\lambda] * Vr\lambda z + (Cos[\lambda])^2 * V\lambda\lambda z;

In[9]:= Vzzx = Cos[\lambda] * Vzzr - Sin[\lambda] * Vzz\lambda;

In[10]:= Vzzy = Sin[\lambda] * Vzzr + Cos[\lambda] * Vzz\lambda;

In[11]:= Vzzz = Vzzz;

```

12.3 Table 15

```

In[1]:= V\lambda Routes426 = -Sin[\lambda] * Vx + Cos[\lambda] * Vy;

In[2]:= V\varphi Routes426 = -Sin[\varphi] * Cos[\lambda] * Vx - Sin[\varphi] * Sin[\lambda] * Vy + Cos[\varphi] * Vz;

In[3]:= V\rho Routes426 = Cos[\varphi] * Cos[\lambda] * Vx + Cos[\varphi] * Sin[\lambda] * Vy + Sin[\varphi] * Vz;

In[4]:= V\lambda\lambda Routes426 = (Sin[\lambda])^2 * Vxx - Sin[2*\lambda] * Vxy + (Cos[\lambda])^2 * Vyy;

In[5]:= V\lambda\varphi Routes426 = Sin[\varphi] * Sin[\lambda] * Cos[\lambda] * Vxx - Sin[\varphi] * Cos[2*\lambda] * Vxy -
Cos[\varphi] * Sin[\lambda] * Vxz - Sin[\varphi] * Sin[\lambda] * Cos[\lambda] * Vyy + Cos[\varphi] * Cos[\lambda] * Vyz;

In[6]:= V\lambda\rho Routes426 = -Cos[\varphi] * Sin[\lambda] * Cos[\lambda] * Vxx + Cos[\varphi] * Cos[2*\lambda] * Vxy -
Sin[\varphi] * Sin[\lambda] * Vxz + Cos[\varphi] * Sin[\lambda] * Cos[\lambda] * Vyy + Sin[\varphi] * Cos[\lambda] * Vyz;

In[7]:= V\varphi\varphi Routes426 =
(Sin[\varphi])^2 * (Cos[\lambda])^2 * Vxx + (Sin[\varphi])^2 * Sin[2*\lambda] * Vxy - Sin[2*\varphi] * Cos[\lambda] * Vxz +
(Sin[\varphi])^2 * (Sin[\lambda])^2 * Vyy - Sin[2*\varphi] * Sin[\lambda] * Vyz + (Cos[\varphi])^2 * Vzz;

In[8]:= V\varphi\rho Routes426 = -Sin[\varphi] * Cos[\varphi] * (Cos[\lambda])^2 * Vxx -
Sin[2*\varphi] * Sin[\lambda] * Cos[\lambda] * Vxy + Cos[2*\varphi] * Cos[\lambda] * Vxz -
Sin[\varphi] * Cos[\varphi] * (Sin[\lambda])^2 * Vyy + Cos[2*\varphi] * Sin[\lambda] * Vyz + Sin[\varphi] * Cos[\varphi] * Vzz;

In[9]:= V\rho\rho Routes426 =
(Cos[\varphi])^2 * (Cos[\lambda])^2 * Vxx + (Cos[\varphi])^2 * Sin[2*\lambda] * Vxy + Sin[2*\varphi] * Cos[\lambda] * Vxz +
(Cos[\varphi])^2 * (Sin[\lambda])^2 * Vyy + Sin[2*\varphi] * Sin[\lambda] * Vyz + (Sin[\varphi])^2 * Vzz;

In[10]:= V\lambda\lambda\lambda Routes426 = - (Sin[\lambda])^3 * Vxxx +
3 * (Sin[\lambda])^2 * Cos[\lambda] * Vxxy - 3 * Sin[\lambda] * (Cos[\lambda])^2 * Vyiy + (Cos[\lambda])^3 * Vyiy;

```

```

In[5]:= VλλφRoutes426 = -Sin[φ] * (Sin[λ])2 * Cos[λ] * Vxxx +
          Sin[φ] * Sin[λ] * (2 * (Cos[λ])2 - (Sin[λ])2) * Vxxy + Cos[φ] * (Sin[λ])2 * Vxxz -
          Cos[φ] * Sin[2 * λ] * Vxyz + Sin[φ] * Cos[λ] * (2 * (Sin[λ])2 - (Cos[λ])2) * Vyvx -
          Sin[φ] * Sin[λ] * (Cos[λ])2 * Vyyy + Cos[φ] * (Cos[λ])2 * Vyyz;

In[6]:= VλλρRoutes426 = Cos[φ] * (Sin[λ])2 * Cos[λ] * Vxxx +
          Cos[φ] * Sin[λ] * ((Sin[λ])2 - 2 * (Cos[λ])2) * Vxxy + Sin[φ] * (Sin[λ])2 * Vxxz -
          Sin[φ] * Sin[2 * λ] * Vxyz + Cos[φ] * Cos[λ] * ((Cos[λ])2 - 2 * (Sin[λ])2) * Vyvx +
          Cos[φ] * Sin[λ] * (Cos[λ])2 * Vyyy + Sin[φ] * (Cos[λ])2 * Vyyz;

In[7]:= VλφρRoutes426 = Sin[φ] * Cos[φ] * Sin[λ] * (Cos[λ])2 * Vxxx +
          Sin[φ] * Cos[φ] * Cos[λ] * (2 * (Sin[λ])2 - (Cos[λ])2) * Vxxy -
          Cos[2 * φ] * Sin[λ] * Cos[λ] * Vxxz + Cos[2 * φ] * Cos[2 * λ] * Vxyz +
          Sin[φ] * Cos[φ] * Sin[λ] * ((Sin[λ])2 - 2 * (Cos[λ])2) * Vyvx -
          Sin[φ] * Cos[φ] * (Sin[λ])2 * Cos[λ] * Vyyy + Cos[2 * φ] * Sin[λ] * Cos[λ] * Vyyz -
          Sin[φ] * Cos[φ] * Sin[λ] * Vzzx + Sin[φ] * Cos[φ] * Cos[λ] * Vzzy;

In[8]:= VφφλRoutes426 = - (Sin[φ])2 * Sin[λ] * (Cos[λ])2 * Vxxx +
          (Sin[φ])2 * Cos[λ] * ((Cos[λ])2 - 2 * (Sin[λ])2) * Vxxy +
          Sin[2 * φ] * Sin[λ] * Cos[λ] * Vxxz - Sin[2 * φ] * Cos[2 * λ] * Vxyz +
          (Sin[φ])2 * Sin[λ] * (2 * (Cos[λ])2 - (Sin[λ])2) * Vyvx +
          (Sin[φ])2 * (Sin[λ])2 * Cos[λ] * Vyyy - Sin[2 * φ] * Sin[λ] * Cos[λ] * Vyyz -
          (Cos[φ])2 * Sin[λ] * Vzzx + (Cos[φ])2 * Cos[λ] * Vzzy;

In[9]:= VφφφRoutes426 =
          - (Sin[φ])3 * (Cos[λ])3 * Vxxx - 3 * (Sin[φ])3 * Sin[λ] * (Cos[λ])2 * Vxxy +
          3 * (Sin[φ])2 * Cos[φ] * (Cos[λ])2 * Vxxz + 3 * (Sin[φ])2 * Cos[φ] * Sin[2 * λ] * Vxyz -
          3 * (Sin[φ])3 * (Sin[λ])2 * Cos[λ] * Vyyx - (Sin[φ])3 * (Sin[λ])3 * Vyyy +
          3 * (Sin[φ])2 * Cos[φ] * (Sin[λ])2 * Vyyz - 3 * Sin[φ] * (Cos[φ])2 * Cos[λ] * Vzzx -
          3 * Sin[φ] * (Cos[φ])2 * Sin[λ] * Vzzy + (Cos[φ])3 * Vzzz;

In[10]:= VφφρRoutes426 = (Sin[φ])2 * Cos[φ] * (Cos[λ])3 * Vxxx +
           3 * (Sin[φ])2 * Cos[φ] * Sin[λ] * (Cos[λ])2 * Vxxy +
           Sin[φ] * (Cos[λ])2 * ((Sin[φ])2 - 2 * (Cos[φ])2) * Vxxz +
           Sin[φ] * Sin[2 * λ] * ((Sin[φ])2 - 2 * (Cos[φ])2) * Vxyz +
           3 * (Sin[φ])2 * Cos[φ] * (Sin[λ])2 * Cos[λ] * Vyyx + (Sin[φ])2 * Cos[φ] *
           (Sin[λ])3 * Vyyy + Sin[φ] * (Sin[λ])2 * ((Sin[φ])2 - 2 * (Cos[φ])2) * Vyyz +
           Cos[φ] * Cos[λ] * ((Cos[φ])2 - 2 * (Sin[φ])2) * Vzzx +
           Cos[φ] * Sin[λ] * ((Cos[φ])2 - 2 * (Sin[φ])2) * Vzzy + Sin[φ] * (Cos[φ])2 * Vzzz;

In[11]:= VρρλRoutes426 = - (Cos[φ])2 * Sin[λ] * (Cos[λ])2 * Vxxx +
           (Cos[φ])2 * Cos[λ] * ((Cos[λ])2 - 2 * (Sin[λ])2) * Vxxy -
           Sin[2 * φ] * Sin[λ] * Cos[λ] * Vxxz + Sin[2 * φ] * Cos[2 * λ] * Vxyz +
           (Cos[φ])2 * Sin[λ] * (2 * (Cos[λ])2 - (Sin[λ])2) * Vyvx +
           (Cos[φ])2 * (Sin[λ])2 * Cos[λ] * Vyyy + Sin[2 * φ] * Sin[λ] * Cos[λ] * Vyyz -
           (Sin[φ])2 * Sin[λ] * Vzzx + (Sin[φ])2 * Cos[λ] * Vzzy;

```

```
In[1]:= VρρρRoutes426 = -Sin[φ] * (Cos[φ])2 * (Cos[λ])3 * Vxxx -
          3 * Sin[φ] * (Cos[φ])2 * Sin[λ] * (Cos[λ])2 * Vxxy +
          Cos[φ] * (Cos[λ])2 * ((Cos[φ])2 - 2 * (Sin[φ])2) * Vxxz +
          Cos[φ] * Sin[2 * λ] * ((Cos[φ])2 - 2 * (Sin[φ])2) * Vxyz -
          3 * Sin[φ] * (Cos[φ])2 * (Sin[λ])2 * Cos[λ] * Vyyx - Sin[φ] * (Cos[φ])2 *
          (Sin[λ])3 * Vyyy + Cos[φ] * (Sin[λ])2 * ((Cos[φ])2 - 2 * (Sin[φ])2) * Vyyz +
          Sin[φ] * Cos[λ] * (2 * (Cos[φ])2 - (Sin[φ])2) * Vzzx +
          Sin[φ] * Sin[λ] * (2 * (Cos[φ])2 - (Sin[φ])2) * Vzzy + (Sin[φ])2 * Cos[φ] * Vzzz;

In[2]:= VρρρRoutes426 = (Cos[φ])3 * (Cos[λ])3 * Vxxx +
          3 * (Cos[φ])3 * Sin[λ] * (Cos[λ])2 * Vxxy + 3 * Sin[φ] * (Cos[φ])2 * (Cos[λ])2 * Vxxz +
          3 * Sin[2 * φ] * Cos[φ] * Sin[λ] * Cos[λ] * Vxyz +
          3 * (Cos[φ])3 * (Sin[λ])2 * Cos[λ] * Vyyx + (Cos[φ])3 * (Sin[λ])3 * Vyyy +
          3 * Sin[φ] * (Cos[φ])2 * (Sin[λ])2 * Vyyz + 3 * (Sin[φ])2 * Cos[φ] * Cos[λ] * Vzzx +
          3 * (Sin[φ])2 * Cos[φ] * Sin[λ] * Vzzy + (Sin[φ])3 * Vzzz;
```

12.4 Check whether they become themselves

In[1]:= FullSimplify[VλRoutes426]

Out[1]=
Vλ

In[2]:= FullSimplify[VφRoutes426]

Out[2]=
Vφ

In[3]:= FullSimplify[VρRoutes426]

Out[3]=
Vρ

In[4]:= FullSimplify[VλλRoutes426]

Out[4]=
Vλλ

In[5]:= FullSimplify[VλφRoutes426]

Out[5]=
Vλφ

In[6]:= FullSimplify[VλρRoutes426]

Out[6]=
Vλρ

In[7]:= FullSimplify[VφφRoutes426]

Out[7]=
Vφφ

In[8]:= FullSimplify[VφρRoutes426]

Out[8]=
Vφρ

```

In[]:= FullSimplify[VρρRoutes426]
Out[]= Vρρ

In[]:= FullSimplify[VλλλRoutes426]
Out[=] Vλλλ

In[]:= FullSimplify[VλλφRoutes426]
Out[=] Vλλφ

In[]:= FullSimplify[VλλρRoutes426]
Out[=] Vλλρ

In[]:= FullSimplify[VλφρRoutes426]
Out[=] Vλφρ

In[]:= FullSimplify[VφφλRoutes426]
Out[=] Vφφλ

In[]:= FullSimplify[VφφφRoutes426]
Out[=] Vφφφ

In[]:= FullSimplify[VφφρRoutes426]
Out[=] Vφφρ

In[]:= FullSimplify[VρρλRoutes426]
Out[=] Vρρλ

In[]:= FullSimplify[VρρφRoutes426]
Out[=] Vρρφ

In[]:= FullSimplify[VρρρRoutes426]
Out[=] Vρρρ

In[]:= NotebookSave[EvaluationNotebook[]];

```