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Expressions for other third-order partial derivatives to "Accurate computation of gravitational curvatures of a tesseroid"

Xiao-Le Deng[1,*]

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1 Expressions for the third-order partial derivatives $(\partial^3 V/\partial \Phi^3)_{\Lambda,H}$

Similar to the expressions for the third-order partial derivatives $(\partial^3 V/\partial H^3)_{\Phi,\Lambda}$ in Eqs. (53) – (55) in the main text, the expressions for the $(\partial^3 V/\partial \Phi^3)_{\Lambda,H}$ can be obtained as:

(1) if $\Lambda_W \leq \Lambda \leq \Lambda_E$ **and** $H_B \leq H \leq H_T$ **and** either $\Phi_S < \Phi < \Phi_S + \Delta_3\Phi$ or $\Phi_N < \Phi < \Phi_N + \Delta_3\Phi$, based on Eq. (13) in the main text by adopting $t = \Phi$ with the positive sign it yields:

$$\begin{aligned}\left(\frac{\partial^3 V}{\partial \Phi^3}\right)_{\Lambda,H} \approx & \frac{1}{2(\Delta_3\Phi)^3}\big[-3V(\Phi+4\Delta_3\Phi,\Lambda,H)+14V(\Phi+3\Delta_3\Phi,\Lambda,H) \\ & -24V(\Phi+2\Delta_3\Phi,\Lambda,H)+18V(\Phi+\Delta_3\Phi,\Lambda,H)-5V(\Phi,\Lambda,H)\big]\end{aligned} \tag{1}$$

(2) else if $\Lambda_W \leq \Lambda \leq \Lambda_E$ **and** $H_B \leq H \leq H_T$ **and** either $\Phi_S - \Delta_3\Phi < \Phi < \Phi_S$ or $\Phi_N - \Delta_3\Phi < \Phi < \Phi_N$, based on Eq. (13) in the main text by adopting $t = \Phi$ with the negative sign it yields:

$$\begin{aligned}\left(\frac{\partial^3 V}{\partial \Phi^3}\right)_{\Lambda,H} \approx & \frac{1}{2(\Delta_3\Phi)^3}\big[3V(\Phi-4\Delta_3\Phi,\Lambda,H)-14V(\Phi-3\Delta_3\Phi,\Lambda,H) \\ & +24V(\Phi-2\Delta_3\Phi,\Lambda,H)-18V(\Phi-\Delta_3\Phi,\Lambda,H)+5V(\Phi,\Lambda,H)\big]\end{aligned} \tag{2}$$

(3) else, based on Eq. (3) in the main text by adopting $t = \Phi$ it yields:

$$\begin{aligned}\left(\frac{\partial^3 V}{\partial \Phi^3}\right)_{\Lambda,H} \approx & \frac{1}{2(\Delta_3\Phi)^3}\big[V(\Phi+2\Delta_3\Phi,\Lambda,H)-2V(\Phi+\Delta_3\Phi,\Lambda,H) \\ & +2V(\Phi-\Delta_3\Phi,\Lambda,H)-V(\Phi-2\Delta_3\Phi,\Lambda,H)\big]\end{aligned} \tag{3}$$

Corresponding author: X.L. Deng (xiaole.deng@gis.uni-stuttgart.de; xldeng@whu.edu.cn)

1 Institute of Geodesy, University of Stuttgart, 70174 Stuttgart, Germany

2 Expressions for the third-order partial derivatives $(\partial^3 V/\partial\Lambda^3)_{\Phi,H}$

Analogously, the expressions for the $(\partial^3 V/\partial\Lambda^3)_{\Phi,H}$ can be calculated by:

(1) if $\Phi_\mathrm{S} \leq \Phi \leq \Phi_\mathrm{N}$ **and** $H_\mathrm{B} \leq H \leq H_\mathrm{T}$ **and** either $\Lambda_\mathrm{W} < \Lambda < \Lambda_\mathrm{W} + \Delta_3\Lambda$ or $\Lambda_\mathrm{E} < \Lambda < \Lambda_\mathrm{E} + \Delta_3\Lambda$, based on Eq. (13) in the main text by adopting $t = \Lambda$ with the positive sign it yields:

$$\begin{aligned}\left(\frac{\partial^3 V}{\partial\Lambda^3}\right)_{\Phi,H} \approx & \frac{1}{2(\Delta_3\Lambda)^3}\big[-3V(\Phi,\Lambda+4\Delta_3\Lambda,H)+14V(\Phi,\Lambda+3\Delta_3\Lambda,H) \\ & -24V(\Phi,\Lambda+2\Delta_3\Lambda,H)+18V(\Phi,\Lambda+\Delta_3\Lambda,H)-5V(\Phi,\Lambda,H)\big]\end{aligned} \tag{4}$$

(2) else if $\Phi_\mathrm{S} \leq \Phi \leq \Phi_\mathrm{N}$ **and** $H_\mathrm{B} \leq H \leq H_\mathrm{T}$ **and** either $\Lambda_\mathrm{W} - \Delta_3\Lambda < \Lambda < \Lambda_\mathrm{W}$ or $\Lambda_\mathrm{E} - \Delta_3\Lambda < \Lambda < \Lambda_\mathrm{E}$, based on Eq. (13) in the main text by adopting $t = \Lambda$ with the negative sign it yields:

$$\begin{aligned}\left(\frac{\partial^3 V}{\partial\Lambda^3}\right)_{\Phi,H} \approx & \frac{1}{2(\Delta_3\Lambda)^3}\big[3V(\Phi,\Lambda-4\Delta_3\Lambda,H)-14V(\Phi,\Lambda-3\Delta_3\Lambda,H) \\ & +24V(\Phi,\Lambda-2\Delta_3\Lambda,H)-18V(\Phi,\Lambda-\Delta_3\Lambda,H)+5V(\Phi,\Lambda,H)\big]\end{aligned} \tag{5}$$

(3) else, based on Eq. (13) in the main text by adopting $t = \Lambda$ it yields:

$$\begin{aligned}\left(\frac{\partial^3 V}{\partial\Lambda^3}\right)_{\Phi,H} \approx & \frac{1}{2(\Delta_3\Lambda)^3}\big[V(\Phi,\Lambda+2\Delta_3\Lambda,H)-2V(\Phi,\Lambda+\Delta_3\Lambda,H) \\ & +2V(\Phi,\Lambda-\Delta_3\Lambda,H)-V(\Phi,\Lambda-2\Delta_3\Lambda,H)\big]\end{aligned} \tag{6}$$

3 Expressions for the third-order partial derivatives $(\partial^3 V/(\partial\Phi^2\partial\Lambda))_H$

Similarly, the expressions for the $(\partial^3 V/(\partial\Phi^2\partial\Lambda))_H$ are obtained by:

(1) if $H_\mathrm{B} \leq H \leq H_\mathrm{T}$ **and** either $\Phi_\mathrm{S} < \Phi < \Phi_\mathrm{S} + \Delta_3\Phi$ or $\Phi_\mathrm{N} < \Phi < \Phi_\mathrm{N} + \Delta_3\Phi$ **and** either $\Lambda_\mathrm{W} < \Lambda < \Lambda_\mathrm{W} + \Delta_3\Lambda$ or $\Lambda_\mathrm{E} < \Lambda < \Lambda_\mathrm{E} + \Delta_3\Lambda$, based on Eq. (18) in the main text by adopting $u = \Phi$ and $v = \Lambda$ with two corresponding sign factors of $+1$ and $+1$ it yields:

$$\begin{aligned}\left(\frac{\partial^3 V}{\partial\Phi^2\partial\Lambda}\right)_{H} \approx & \frac{1}{2(\Delta_3\Phi)^2(\Delta_3\Lambda)}\big[V(\Phi+3\Delta_3\Phi,\Lambda+2\Delta_3\Lambda,H)-4V(\Phi+3\Delta_3\Phi,\Lambda+\Delta_3\Lambda,H) \\ & +3V(\Phi+3\Delta_3\Phi,\Lambda,H)-4V(\Phi+2\Delta_3\Phi,\Lambda+2\Delta_3\Lambda,H) \\ & +16V(\Phi+2\Delta_3\Phi,\Lambda+\Delta_3\Lambda,H)-12V(\Phi+2\Delta_3\Phi,\Lambda,H) \\ & +5V(\Phi+\Delta_3\Phi,\Lambda+2\Delta_3\Lambda,H)-20V(\Phi+\Delta_3\Phi,\Lambda+\Delta_3\Lambda,H) \\ & +15V(\Phi+\Delta_3\Phi,\Lambda,H)-2V(\Phi,\Lambda+2\Delta_3\Lambda,H) \\ & +8V(\Phi,\Lambda+\Delta_3\Lambda,H)-6V(\Phi,\Lambda,H)\big]\end{aligned} \tag{7}$$

(2) else if $H_\mathrm{B} \leq H \leq H_\mathrm{T}$ **and** either $\Phi_\mathrm{S} < \Phi < \Phi_\mathrm{S} + \Delta_3\Phi$ or $\Phi_\mathrm{N} < \Phi < \Phi_\mathrm{N} + \Delta_3\Phi$ **and** either $\Lambda_\mathrm{W} - \Delta_3\Lambda < \Lambda < \Lambda_\mathrm{W}$ or $\Lambda_\mathrm{E} - \Delta_3\Lambda < \Lambda < \Lambda_\mathrm{E}$, based on Eq. (18) in the main

text by adopting $u = \Phi$ and $v = \Lambda$ with two corresponding sign factors of $+1$ and -1 it yields:

$$\left(\frac{\partial^3 V}{\partial \Phi^2 \partial \Lambda}\right)_H \approx \frac{-1}{2(\Delta_3\Phi)^2(\Delta_3\Lambda)}\big[V(\Phi+3\Delta_3\Phi,\Lambda-2\Delta_3\Lambda,H)-4V(\Phi+3\Delta_3\Phi,\Lambda-\Delta_3\Lambda,H)$$
$$+3V(\Phi+3\Delta_3\Phi,\Lambda,H)-4V(\Phi+2\Delta_3\Phi,\Lambda-2\Delta_3\Lambda,H)$$
$$+16V(\Phi+2\Delta_3\Phi,\Lambda-\Delta_3\Lambda,H)-12V(\Phi+2\Delta_3\Phi,\Lambda,H)$$
$$+5V(\Phi+\Delta_3\Phi,\Lambda-2\Delta_3\Lambda,H)-20V(\Phi+\Delta_3\Phi,\Lambda-\Delta_3\Lambda,H)$$
$$+15V(\Phi+\Delta_3\Phi,\Lambda,H)-2V(\Phi,\Lambda-2\Delta_3\Lambda,H)$$
$$+8V(\Phi,\Lambda-\Delta_3\Lambda,H)-6V(\Phi,\Lambda,H)\big] \tag{8}$$

(3) else if $H_B \le H \le H_T$ **and** either $\Phi_S - \Delta_3\Phi < \Phi < \Phi_S$ or $\Phi_N - \Delta_3\Phi < \Phi < \Phi_N$ **and** either $\Lambda_W < \Lambda < \Lambda_W + \Delta_3\Lambda$ or $\Lambda_E < \Lambda < \Lambda_E + \Delta_3\Lambda$, based on Eq. (18) in the main text by adopting $u = \Phi$ and $v = \Lambda$ with two corresponding sign factors of -1 and $+1$ it yields:

$$\left(\frac{\partial^3 V}{\partial \Phi^2 \partial \Lambda}\right)_H \approx \frac{1}{2(\Delta_3\Phi)^2(\Delta_3\Lambda)}\big[V(\Phi-3\Delta_3\Phi,\Lambda+2\Delta_3\Lambda,H)-4V(\Phi-3\Delta_3\Phi,\Lambda+\Delta_3\Lambda,H)$$
$$+3V(\Phi-3\Delta_3\Phi,\Lambda,H)-4V(\Phi-2\Delta_3\Phi,\Lambda+2\Delta_3\Lambda,H)$$
$$+16V(\Phi-2\Delta_3\Phi,\Lambda+\Delta_3\Lambda,H)-12V(\Phi-2\Delta_3\Phi,\Lambda,H)$$
$$+5V(\Phi-\Delta_3\Phi,\Lambda+2\Delta_3\Lambda,H)-20V(\Phi-\Delta_3\Phi,\Lambda+\Delta_3\Lambda,H)$$
$$+15V(\Phi-\Delta_3\Phi,\Lambda,H)-2V(\Phi,\Lambda+2\Delta_3\Lambda,H)$$
$$+8V(\Phi,\Lambda+\Delta_3\Lambda,H)-6V(\Phi,\Lambda,H)\big] \tag{9}$$

(4) else if $H_B \le H \le H_T$ **and** either $\Phi_S - \Delta_3\Phi < \Phi < \Phi_S$ or $\Phi_N - \Delta_3\Phi < \Phi < \Phi_N$ **and** either $\Lambda_W - \Delta_3\Lambda < \Lambda < \Lambda_W$ or $\Lambda_E - \Delta_3\Lambda < \Lambda < \Lambda_E$, based on Eq. (18) in the main text by adopting $u = \Phi$ and $v = \Lambda$ with two corresponding sign factors of -1 and -1 it yields:

$$\left(\frac{\partial^3 V}{\partial \Phi^2 \partial \Lambda}\right)_H \approx \frac{-1}{2(\Delta_3\Phi)^2(\Delta_3\Lambda)}\big[V(\Phi-3\Delta_3\Phi,\Lambda-2\Delta_3\Lambda,H)-4V(\Phi-3\Delta_3\Phi,\Lambda-\Delta_3\Lambda,H)$$
$$+3V(\Phi-3\Delta_3\Phi,\Lambda,H)-4V(\Phi-2\Delta_3\Phi,\Lambda-2\Delta_3\Lambda,H)$$
$$+16V(\Phi-2\Delta_3\Phi,\Lambda-\Delta_3\Lambda,H)-12V(\Phi-2\Delta_3\Phi,\Lambda,H)$$
$$+5V(\Phi-\Delta_3\Phi,\Lambda-2\Delta_3\Lambda,H)-20V(\Phi-\Delta_3\Phi,\Lambda-\Delta_3\Lambda,H)$$
$$+15V(\Phi-\Delta_3\Phi,\Lambda,H)-2V(\Phi,\Lambda-2\Delta_3\Lambda,H)$$
$$+8V(\Phi,\Lambda-\Delta_3\Lambda,H)-6V(\Phi,\Lambda,H)\big] \tag{10}$$

(5) else if $H_B \le H \le H_T$ **and** either $\Phi_S < \Phi < \Phi_S + \Delta_3\Phi$ or $\Phi_N < \Phi < \Phi_N + \Delta_3\Phi$ **and** either $\Lambda < \Lambda_W - \Delta_3\Lambda$ or $\Lambda_W + \Delta_3\Lambda < \Lambda < \Lambda_E - \Delta_3\Lambda$ or $\Lambda_E + \Delta_3\Lambda < \Lambda$, then based on Eq. (17) in the main text by adopting $u = \Phi$ and $v = \Lambda$ with the positive sign for $u = \Phi$

it yields:

$$\left(\frac{\partial^3 V}{\partial \Phi^2 \partial \Lambda}\right)_H \approx \frac{1}{2(\Delta_3\Phi)^2(\Delta_3\Lambda)}\big[-V(\Phi+3\Delta_3\Phi,\Lambda+\Delta_3\Lambda,H)+V(\Phi+3\Delta_3\Phi,\Lambda-\Delta_3\Lambda,H)$$
$$+4V(\Phi+2\Delta_3\Phi,\Lambda+\Delta_3\Lambda,H)-4V(\Phi+2\Delta_3\Phi,\Lambda-\Delta_3\Lambda,H)$$
$$-5V(\Phi+\Delta_3\Phi,\Lambda+\Delta_3\Lambda,H)+5V(\Phi+\Delta_3\Phi,\Lambda-\Delta_3\Lambda,H)$$
$$+2V(\Phi,\Lambda+\Delta_3\Lambda,H)-2V(\Phi,\Lambda-\Delta_3\Lambda,H)\big] \tag{11}$$

(6) else if $H_\mathrm{B} \le H \le H_\mathrm{T}$ **and** either $\Phi_\mathrm{S}-\Delta_3\Phi < \Phi < \Phi_\mathrm{S}$ or $\Phi_\mathrm{N}-\Delta_3\Phi < \Phi < \Phi_\mathrm{N}$ **and** either $\Lambda < \Lambda_\mathrm{W}-\Delta_3\Lambda$ or $\Lambda_\mathrm{W}+\Delta_3\Lambda < \Lambda < \Lambda_\mathrm{E}-\Delta_3\Lambda$ or $\Lambda_\mathrm{E}+\Delta_3\Lambda < \Lambda$, then based on Eq. (17) in the main text by adopting $u=\Phi$ and $v=\Lambda$ with the negative sign for $u=\Phi$ it yields:

$$\left(\frac{\partial^3 V}{\partial \Phi^2 \partial \Lambda}\right)_H \approx \frac{1}{2(\Delta_3\Phi)^2(\Delta_3\Lambda)}\big[-V(\Phi-3\Delta_3\Phi,\Lambda+\Delta_3\Lambda,H)+V(\Phi-3\Delta_3\Phi,\Lambda-\Delta_3\Lambda,H)$$
$$+4V(\Phi-2\Delta_3\Phi,\Lambda+\Delta_3\Lambda,H)-4V(\Phi-2\Delta_3\Phi,\Lambda-\Delta_3\Lambda,H)$$
$$-5V(\Phi-\Delta_3\Phi,\Lambda+\Delta_3\Lambda,H)+5V(\Phi-\Delta_3\Phi,\Lambda-\Delta_3\Lambda,H)$$
$$+2V(\Phi,\Lambda+\Delta_3\Lambda,H)-2V(\Phi,\Lambda-\Delta_3\Lambda,H)\big] \tag{12}$$

(7) else if $H_\mathrm{B} \le H \le H_\mathrm{T}$ **and** either $\Phi < \Phi_\mathrm{S}-\Delta_3\Phi$ or $\Phi_\mathrm{S}+\Delta_3\Phi < \Phi < \Phi_\mathrm{N}-\Delta_3\Phi$ or $\Phi_\mathrm{N}+\Delta_3\Phi < \Phi$ **and** either $\Lambda_\mathrm{W} < \Lambda < \Lambda_\mathrm{W}+\Delta_3\Lambda$ or $\Lambda_\mathrm{E} < \Lambda < \Lambda_\mathrm{E}+\Delta_3\Lambda$, then based on Eq. (16) in the main text by adopting $u=\Phi$ and $v=\Lambda$ with the positive sign for $v=\Lambda$ it yields:

$$\left(\frac{\partial^3 V}{\partial \Phi^2 \partial \Lambda}\right)_H \approx \frac{1}{2(\Delta_3\Phi)^2(\Delta_3\Lambda)}\big[-V(\Phi+\Delta_3\Phi,\Lambda+2\Delta_3\Lambda,H)+4V(\Phi+\Delta_3\Phi,\Lambda+\Delta_3\Lambda,H)$$
$$-3V(\Phi+\Delta_3\Phi,\Lambda,H)+2V(\Phi,\Lambda+2\Delta_3\Lambda,H)$$
$$-8V(\Phi,\Lambda+\Delta_3\Lambda,H)+6V(\Phi,\Lambda,H)-V(\Phi-\Delta_3\Phi,\Lambda+2\Delta_3\Lambda,H)$$
$$+4V(\Phi-\Delta_3\Phi,\Lambda+\Delta_3\Lambda,H)-3V(\Phi-\Delta_3\Phi,\Lambda,H)\big] \tag{13}$$

(8) else if $H_\mathrm{B} \le H \le H_\mathrm{T}$ **and** either $\Phi < \Phi_\mathrm{S}-\Delta_3\Phi$ or $\Phi_\mathrm{S}+\Delta_3\Phi < \Phi < \Phi_\mathrm{N}-\Delta_3\Phi$ or $\Phi_\mathrm{N}+\Delta_3\Phi < \Phi$ **and** either $\Lambda_\mathrm{W}-\Delta_3\Lambda < \Lambda < \Lambda_\mathrm{W}$ or $\Lambda_\mathrm{E}-\Delta_3\Lambda < \Lambda < \Lambda_\mathrm{E}$, then based on Eq. (16) in the main text by adopting $u=\Phi$ and $v=\Lambda$ with the negative sign for $v=\Lambda$ it yields:

$$\left(\frac{\partial^3 V}{\partial \Phi^2 \partial \Lambda}\right)_H \approx \frac{-1}{2(\Delta_3\Phi)^2(\Delta_3\Lambda)}\big[-V(\Phi+\Delta_3\Phi,\Lambda-2\Delta_3\Lambda,H)+4V(\Phi+\Delta_3\Phi,\Lambda-\Delta_3\Lambda,H)$$
$$-3V(\Phi+\Delta_3\Phi,\Lambda,H)+2V(\Phi,\Lambda-2\Delta_3\Lambda,H)$$
$$-8V(\Phi,\Lambda-\Delta_3\Lambda,H)+6V(\Phi,\Lambda,H)-V(\Phi-\Delta_3\Phi,\Lambda-2\Delta_3\Lambda,H)$$
$$+4V(\Phi-\Delta_3\Phi,\Lambda-\Delta_3\Lambda,H)-3V(\Phi-\Delta_3\Phi,\Lambda,H)\big] \tag{14}$$

(9) else, then based on Eq. (15) in the main text by adopting $u = \Phi$ and $v = \Lambda$ it yields:

$$\left(\frac{\partial^3 V}{\partial \Phi^2 \partial \Lambda}\right)_H \approx \frac{1}{2(\Delta_3\Phi)^2(\Delta_3\Lambda)}\big[V(\Phi+\Delta_3\Phi,\Lambda+\Delta_3\Lambda,H)-V(\Phi+\Delta_3\Phi,\Lambda-\Delta_3\Lambda,H) \\ -2V(\Phi,\Lambda+\Delta_3\Lambda,H)+2V(\Phi,\Lambda-\Delta_3\Lambda,H) \\ +V(\Phi-\Delta_3\Phi,\Lambda+\Delta_3\Lambda,H)-V(\Phi-\Delta_3\Phi,\Lambda-\Delta_3\Lambda,H)\big] \tag{15}$$

4 Expressions for the third-order partial derivatives $(\partial^3 V/(\partial \Phi^2 \partial H))_\Lambda$

Analogously, the expressions for the $(\partial^3 V/(\partial \Phi^2 \partial H))_\Lambda$ can be calculated by:

(1) if $\Lambda_W \le \Lambda \le \Lambda_E$ **and** either $\Phi_S < \Phi < \Phi_S + \Delta_3\Phi$ or $\Phi_N < \Phi < \Phi_N + \Delta_3\Phi$ **and** either $H_B < H < H_B + \Delta_3 H$ or $H_T < H < H_T + \Delta_3 H$, based on Eq. (18) in the main text by adopting $u = \Phi$ and $v = H$ with two corresponding sign factors of $+1$ and $+1$ it yields:

$$\left(\frac{\partial^3 V}{\partial \Phi^2 \partial H}\right)_\Lambda \approx \frac{1}{2(\Delta_3\Phi)^2(\Delta_3 H)}\big[V(\Phi+3\Delta_3\Phi,\Lambda,H+2\Delta_3 H)-4V(\Phi+3\Delta_3\Phi,\Lambda,H+\Delta_3 H) \\ +3V(\Phi+3\Delta_3\Phi,\Lambda,H)-4V(\Phi+2\Delta_3\Phi,\Lambda,H+2\Delta_3 H) \\ +16V(\Phi+2\Delta_3\Phi,\Lambda,H+\Delta_3 H)-12V(\Phi+2\Delta_3\Phi,\Lambda,H) \\ +5V(\Phi+\Delta_3\Phi,\Lambda,H+2\Delta_3 H)-20V(\Phi+\Delta_3\Phi,\Lambda,H+\Delta_3 H) \\ +15V(\Phi+\Delta_3\Phi,\Lambda,H)-2V(\Phi,\Lambda,H+2\Delta_3 H) \\ +8V(\Phi,\Lambda,H+\Delta_3 H)-6V(\Phi,\Lambda,H)\big] \tag{16}$$

(2) else if $\Lambda_W \le \Lambda \le \Lambda_E$ **and** either $\Phi_S < \Phi < \Phi_S + \Delta_3\Phi$ or $\Phi_N < \Phi < \Phi_N + \Delta_3\Phi$ **and** either $H_B - \Delta_3 H < H < H_B$ or $H_T - \Delta_3 H < H < H_T$, based on Eq. (18) in the main text by adopting $u = \Phi$ and $v = H$ with two corresponding sign factors of $+1$ and -1 it yields:

$$\left(\frac{\partial^3 V}{\partial \Phi^2 \partial H}\right)_\Lambda \approx \frac{-1}{2(\Delta_3\Phi)^2(\Delta_3 H)}\big[V(\Phi+3\Delta_3\Phi,\Lambda,H-2\Delta_3 H)-4V(\Phi+3\Delta_3\Phi,\Lambda,H-\Delta_3 H) \\ +3V(\Phi+3\Delta_3\Phi,\Lambda,H)-4V(\Phi+2\Delta_3\Phi,\Lambda,H-2\Delta_3 H) \\ +16V(\Phi+2\Delta_3\Phi,\Lambda,H-\Delta_3 H)-12V(\Phi+2\Delta_3\Phi,\Lambda,H) \\ +5V(\Phi+\Delta_3\Phi,\Lambda,H-2\Delta_3 H)-20V(\Phi+\Delta_3\Phi,\Lambda,H-\Delta_3 H) \\ +15V(\Phi+\Delta_3\Phi,\Lambda,H)-2V(\Phi,\Lambda,H-2\Delta_3 H) \\ +8V(\Phi,\Lambda,H-\Delta_3 H)-6V(\Phi,\Lambda,H)\big] \tag{17}$$

(3) else if $\Lambda_W \le \Lambda \le \Lambda_E$ **and** either $\Phi_S - \Delta_3\Phi < \Phi < \Phi_S$ or $\Phi_N - \Delta_3\Phi < \Phi < \Phi_N$ **and** either $H_B < H < H_B + \Delta_3 H$ or $H_T < H < H_T + \Delta_3 H$, based on Eq. (18) in the main text by adopting $u = \Phi$ and $v = H$ with two corresponding sign factors of -1 and $+1$ it

yields:

$$\left(\frac{\partial^3 V}{\partial \Phi^2 \partial H}\right)_\Lambda \approx \frac{1}{2(\Delta_3\Phi)^2(\Delta_3 H)}\big[V(\Phi-3\Delta_3\Phi,\Lambda,H+2\Delta_3H)-4V(\Phi-3\Delta_3\Phi,\Lambda,H+\Delta_3H)$$
$$+3V(\Phi-3\Delta_3\Phi,\Lambda,H)-4V(\Phi-2\Delta_3\Phi,\Lambda,H+2\Delta_3H)$$
$$+16V(\Phi-2\Delta_3\Phi,\Lambda,H+\Delta_3H)-12V(\Phi-2\Delta_3\Phi,\Lambda,H)$$
$$+5V(\Phi-\Delta_3\Phi,\Lambda,H+2\Delta_3H)-20V(\Phi-\Delta_3\Phi,\Lambda,H+\Delta_3H)$$
$$+15V(\Phi-\Delta_3\Phi,\Lambda,H)-2V(\Phi,\Lambda,H+2\Delta_3H)$$
$$+8V(\Phi,\Lambda,H+\Delta_3H)-6V(\Phi,\Lambda,H)\big] \tag{18}$$

(4) else if $\Lambda_W \le \Lambda \le \Lambda_E$ **and** either $\Phi_S - \Delta_3\Phi < \Phi < \Phi_S$ or $\Phi_N - \Delta_3\Phi < \Phi < \Phi_N$ **and** either $H_B - \Delta_3 H < H < H_B$ or $H_T - \Delta_3 H < H < H_T$, based on Eq. (18) in the main text by adopting $u = \Phi$ and $v = H$ with two corresponding sign factors of -1 and -1 it yields:

$$\left(\frac{\partial^3 V}{\partial \Phi^2 \partial H}\right)_\Lambda \approx \frac{-1}{2(\Delta_3\Phi)^2(\Delta_3 H)}\big[V(\Phi-3\Delta_3\Phi,\Lambda,H-2\Delta_3H)-4V(\Phi-3\Delta_3\Phi,\Lambda,H-\Delta_3H)$$
$$+3V(\Phi-3\Delta_3\Phi,\Lambda,H)-4V(\Phi-2\Delta_3\Phi,\Lambda,H-2\Delta_3H)$$
$$+16V(\Phi-2\Delta_3\Phi,\Lambda,H-\Delta_3H)-12V(\Phi-2\Delta_3\Phi,\Lambda,H)$$
$$+5V(\Phi-\Delta_3\Phi,\Lambda,H-2\Delta_3H)-20V(\Phi-\Delta_3\Phi,\Lambda,H-\Delta_3H)$$
$$+15V(\Phi-\Delta_3\Phi,\Lambda,H)-2V(\Phi,\Lambda,H-2\Delta_3H)$$
$$+8V(\Phi,\Lambda,H-\Delta_3H)-6V(\Phi,\Lambda,H)\big] \tag{19}$$

(5) else if $\Lambda_W \le \Lambda \le \Lambda_E$ **and** either $\Phi_S < \Phi < \Phi_S + \Delta_3\Phi$ or $\Phi_N < \Phi < \Phi_N + \Delta_3\Phi$ **and** either $H < H_B - \Delta_3 H$ or $H_B + \Delta_3 H < H < H_T - \Delta_3 H$ or $H_T + \Delta_3 H < H$, then based on Eq. (17) in the main text by adopting $u = \Phi$ and $v = H$ with the positive sign for $u = \Phi$ it yields:

$$\left(\frac{\partial^3 V}{\partial \Phi^2 \partial H}\right)_\Lambda \approx \frac{1}{2(\Delta_3\Phi)^2(\Delta_3 H)}\big[-V(\Phi+3\Delta_3\Phi,\Lambda,H+\Delta_3H)+V(\Phi+3\Delta_3\Phi,\Lambda,H-\Delta_3H)$$
$$+4V(\Phi+2\Delta_3\Phi,\Lambda,H+\Delta_3H)-4V(\Phi+2\Delta_3\Phi,\Lambda,H-\Delta_3H)$$
$$-5V(\Phi+\Delta_3\Phi,\Lambda,H+\Delta_3H)+5V(\Phi+\Delta_3\Phi,\Lambda,H-\Delta_3H)$$
$$+2V(\Phi,\Lambda,H+\Delta_3H)-2V(\Phi,\Lambda,H-\Delta_3H)\big] \tag{20}$$

(6) else if $\Lambda_W \le \Lambda \le \Lambda_E$ **and** either $\Phi_S - \Delta_3\Phi < \Phi < \Phi_S$ or $\Phi_N - \Delta_3\Phi < \Phi < \Phi_N$ **and** either $H < H_B - \Delta_3 H$ or $H_B + \Delta_3 H < H < H_T - \Delta_3 H$ or $H_T + \Delta_3 H < H$, then based on Eq. (17) in the main text by adopting $u = \Phi$ and $v = H$ with the negative sign for $u = \Phi$ it yields:

$$\left(\frac{\partial^3 V}{\partial \Phi^2 \partial H}\right)_\Lambda \approx \frac{1}{2(\Delta_3\Phi)^2(\Delta_3 H)}\big[-V(\Phi-3\Delta_3\Phi,\Lambda,H+\Delta_3H)+V(\Phi-3\Delta_3\Phi,\Lambda,H-\Delta_3H)$$
$$+4V(\Phi-2\Delta_3\Phi,\Lambda,H+\Delta_3H)-4V(\Phi-2\Delta_3\Phi,\Lambda,H-\Delta_3H)$$
$$-5V(\Phi-\Delta_3\Phi,\Lambda,H+\Delta_3H)+5V(\Phi-\Delta_3\Phi,\Lambda,H-\Delta_3H)$$
$$+2V(\Phi,\Lambda,H+\Delta_3H)-2V(\Phi,\Lambda,H-\Delta_3H)\big] \tag{21}$$

(7) else if $\Lambda_\mathrm{W} \leq \Lambda \leq \Lambda_\mathrm{E}$ **and** either $\Phi < \Phi_\mathrm{S} - \Delta_3\Phi$ or $\Phi_\mathrm{S} + \Delta_3\Phi < \Phi < \Phi_\mathrm{N} - \Delta_3\Phi$ or $\Phi_\mathrm{N} + \Delta_3\Phi < \Phi$ **and** either $H_\mathrm{B} < H < H_\mathrm{B} + \Delta_3 H$ or $H_\mathrm{T} < H < H_\mathrm{T} + \Delta_3 H$, then based on Eq. (16) in the main text by adopting $u = \Phi$ and $v = H$ with the positive sign for $v = H$ it yields:

$$\begin{aligned}\left(\frac{\partial^3 V}{\partial \Phi^2 \partial H}\right)_\Lambda \approx & \frac{1}{2(\Delta_3\Phi)^2(\Delta_3 H)}\big[-V(\Phi+\Delta_3\Phi,\Lambda,H+2\Delta_3 H)+4V(\Phi+\Delta_3\Phi,\Lambda,H+\Delta_3 H)\\ & -3V(\Phi+\Delta_3\Phi,\Lambda,H)+2V(\Phi,\Lambda,H+2\Delta_3 H)\\ & -8V(\Phi,\Lambda,H+\Delta_3 H)+6V(\Phi,\Lambda,H)-V(\Phi-\Delta_3\Phi,\Lambda,H+2\Delta_3 H)\\ & +4V(\Phi-\Delta_3\Phi,\Lambda,H+\Delta_3 H)-3V(\Phi-\Delta_3\Phi,\Lambda,H)\big]\end{aligned} \tag{22}$$

(8) else if $\Lambda_\mathrm{W} \leq \Lambda \leq \Lambda_\mathrm{E}$ **and** either $\Phi < \Phi_\mathrm{S} - \Delta_3\Phi$ or $\Phi_\mathrm{S} + \Delta_3\Phi < \Phi < \Phi_\mathrm{N} - \Delta_3\Phi$ or $\Phi_\mathrm{N} + \Delta_3\Phi < \Phi$ **and** either either $H_\mathrm{B} - \Delta_3 H < H < H_\mathrm{B}$ or $H_\mathrm{T} - \Delta_3 H < H < H_\mathrm{T}$, then based on Eq. (16) in the main text by adopting $u = \Phi$ and $v = H$ with the negative sign for $v = H$ it yields:

$$\begin{aligned}\left(\frac{\partial^3 V}{\partial \Phi^2 \partial H}\right)_\Lambda \approx & \frac{-1}{2(\Delta_3\Phi)^2(\Delta_3 H)}\big[-V(\Phi+\Delta_3\Phi,\Lambda,H-2\Delta_3 H)+4V(\Phi+\Delta_3\Phi,\Lambda,H-\Delta_3 H)\\ & -3V(\Phi+\Delta_3\Phi,\Lambda,H)+2V(\Phi,\Lambda,H-2\Delta_3 H)\\ & -8V(\Phi,\Lambda,H-\Delta_3 H)+6V(\Phi,\Lambda,H)-V(\Phi-\Delta_3\Phi,\Lambda,H-2\Delta_3 H)\\ & +4V(\Phi-\Delta_3\Phi,\Lambda,H-\Delta_3 H)-3V(\Phi-\Delta_3\Phi,\Lambda,H)\big]\end{aligned} \tag{23}$$

(9) else, then based on Eq. (15) in the main text by adopting $u = \Phi$ and $v = H$ it yields:

$$\begin{aligned}\left(\frac{\partial^3 V}{\partial \Phi^2 \partial H}\right)_\Lambda \approx & \frac{1}{2(\Delta_3\Phi)^2(\Delta_3 H)}\big[V(\Phi+\Delta_3\Phi,\Lambda,H+\Delta_3 H)-V(\Phi+\Delta_3\Phi,\Lambda,H-\Delta_3 H)\\ & -2V(\Phi,\Lambda,H+\Delta_3 H)+2V(\Phi,\Lambda,H-\Delta_3 H)\\ & +V(\Phi-\Delta_3\Phi,\Lambda,H+\Delta_3 H)-V(\Phi-\Delta_3\Phi,\Lambda,H-\Delta_3 H)\big]\end{aligned} \tag{24}$$

5 Expressions for the third-order partial derivatives $(\partial^3 V/(\partial\Phi\partial\Lambda^2))_H$

Similarly, the expressions for the $(\partial^3 V/(\partial\Phi\partial\Lambda^2))_H$ can be calculated by:

(1) if $H_\mathrm{B} \leq H \leq H_\mathrm{T}$ **and** either $\Phi_\mathrm{S} < \Phi < \Phi_\mathrm{S} + \Delta_3\Phi$ or $\Phi_\mathrm{N} < \Phi < \Phi_\mathrm{N} + \Delta_3\Phi$ **and** either $\Lambda_\mathrm{W} < \Lambda < \Lambda_\mathrm{W} + \Delta_3\Lambda$ or $\Lambda_\mathrm{E} < \Lambda < \Lambda_\mathrm{E} + \Delta_3\Lambda$, based on Eq. (18) in the main text by adopting $u = \Lambda$ and $v = \Phi$ with two corresponding sign factors of $+1$ and $+1$ it yields:

$$\begin{aligned}\left(\frac{\partial^3 V}{\partial \Phi \partial \Lambda^2}\right)_H \approx & \frac{1}{2(\Delta_3\Phi)(\Delta_3\Lambda)^2}\big[V(\Phi+2\Delta_3\Phi,\Lambda+3\Delta_3\Lambda,H)-4V(\Phi+\Delta_3\Phi,\Lambda+3\Delta_3\Lambda,H)\\ & +3V(\Phi,\Lambda+3\Delta_3\Lambda,H)-4V(\Phi+2\Delta_3\Phi,\Lambda+2\Delta_3\Lambda,H)\\ & +16V(\Phi+\Delta_3\Phi,\Lambda+2\Delta_3\Lambda,H)-12V(\Phi,\Lambda+2\Delta_3\Lambda,H)\\ & +5V(\Phi+2\Delta_3\Phi,\Lambda+\Delta_3\Lambda,H)-20V(\Phi+\Delta_3\Phi,\Lambda+\Delta_3\Lambda,H)\\ & +15V(\Phi,\Lambda+\Delta_3\Lambda,H)-2V(\Phi+2\Delta_3\Phi,\Lambda,H)\\ & +8V(\Phi+\Delta_3\Phi,\Lambda,H)-6V(\Phi,\Lambda,H)\big]\end{aligned} \tag{25}$$

(2) else if $H_{\mathrm{B}} \leq H \leq H_{\mathrm{T}}$ **and** either $\Phi_{\mathrm{S}} - \Delta_3\Phi < \Phi < \Phi_{\mathrm{S}}$ or $\Phi_{\mathrm{N}} - \Delta_3\Phi < \Phi < \Phi_{\mathrm{N}}$ **and** either $\Lambda_{\mathrm{W}} < \Lambda < \Lambda_{\mathrm{W}} + \Delta_3\Lambda$ or $\Lambda_{\mathrm{E}} < \Lambda < \Lambda_{\mathrm{E}} + \Delta_3\Lambda$, based on Eq. (18) in the main text by adopting $u = \Lambda$ and $v = \Phi$ with two corresponding sign factors of $+1$ and -1 it yields:

$$\begin{aligned}\left(\frac{\partial^3 V}{\partial\Phi\partial\Lambda^2}\right)_H \approx & \frac{-1}{2(\Delta_3\Phi)(\Delta_3\Lambda)^2}\big[V(\Phi-2\Delta_3\Phi,\Lambda+3\Delta_3\Lambda,H)-4V(\Phi-\Delta_3\Phi,\Lambda+3\Delta_3\Lambda,H)\\ &+3V(\Phi,\Lambda+3\Delta_3\Lambda,H)-4V(\Phi-2\Delta_3\Phi,\Lambda+2\Delta_3\Lambda,H)\\ &+16V(\Phi-\Delta_3\Phi,\Lambda+2\Delta_3\Lambda,H)-12V(\Phi,\Lambda+2\Delta_3\Lambda,H)\\ &+5V(\Phi-2\Delta_3\Phi,\Lambda+\Delta_3\Lambda,H)-20V(\Phi-\Delta_3\Phi,\Lambda+\Delta_3\Lambda,H)\\ &+15V(\Phi,\Lambda+\Delta_3\Lambda,H)-2V(\Phi-2\Delta_3\Phi,\Lambda,H)\\ &+8V(\Phi-\Delta_3\Phi,\Lambda,H)-6V(\Phi,\Lambda,H)\big]\end{aligned} \tag{26}$$

(3) else if $H_{\mathrm{B}} \leq H \leq H_{\mathrm{T}}$ **and** either $\Phi_{\mathrm{S}} < \Phi < \Phi_{\mathrm{S}} + \Delta_3\Phi$ or $\Phi_{\mathrm{N}} < \Phi < \Phi_{\mathrm{N}} + \Delta_3\Phi$ **and** either $\Lambda_{\mathrm{W}} - \Delta_3\Lambda < \Lambda < \Lambda_{\mathrm{W}}$ or $\Lambda_{\mathrm{E}} - \Delta_3\Lambda < \Lambda < \Lambda_{\mathrm{E}}$, based on Eq. (18) in the main text by adopting $u = \Lambda$ and $v = \Phi$ with two corresponding sign factors of -1 and $+1$ it yields:

$$\begin{aligned}\left(\frac{\partial^3 V}{\partial\Phi\partial\Lambda^2}\right)_H \approx & \frac{1}{2(\Delta_3\Phi)(\Delta_3\Lambda)^2}\big[V(\Phi+2\Delta_3\Phi,\Lambda-3\Delta_3\Lambda,H)-4V(\Phi+\Delta_3\Phi,\Lambda-3\Delta_3\Lambda,H)\\ &+3V(\Phi,\Lambda-3\Delta_3\Lambda,H)-4V(\Phi+2\Delta_3\Phi,\Lambda-2\Delta_3\Lambda,H)\\ &+16V(\Phi+\Delta_3\Phi,\Lambda-2\Delta_3\Lambda,H)-12V(\Phi,\Lambda-2\Delta_3\Lambda,H)\\ &+5V(\Phi+2\Delta_3\Phi,\Lambda-\Delta_3\Lambda,H)-20V(\Phi+\Delta_3\Phi,\Lambda-\Delta_3\Lambda,H)\\ &+15V(\Phi,\Lambda-\Delta_3\Lambda,H)-2V(\Phi+2\Delta_3\Phi,\Lambda,H)\\ &+8V(\Phi+\Delta_3\Phi,\Lambda,H)-6V(\Phi,\Lambda,H)\big]\end{aligned} \tag{27}$$

(4) else if $H_{\mathrm{B}} \leq H \leq H_{\mathrm{T}}$ **and** either $\Phi_{\mathrm{S}} - \Delta_3\Phi < \Phi < \Phi_{\mathrm{S}}$ or $\Phi_{\mathrm{N}} - \Delta_3\Phi < \Phi < \Phi_{\mathrm{N}}$ **and** either $\Lambda_{\mathrm{W}} - \Delta_3\Lambda < \Lambda < \Lambda_{\mathrm{W}}$ or $\Lambda_{\mathrm{E}} - \Delta_3\Lambda < \Lambda < \Lambda_{\mathrm{E}}$, based on Eq. (18) in the main text by adopting $u = \Lambda$ and $v = \Phi$ with two corresponding sign factors of -1 and -1 it yields:

$$\begin{aligned}\left(\frac{\partial^3 V}{\partial\Phi\partial\Lambda^2}\right)_H \approx & \frac{-1}{2(\Delta_3\Phi)(\Delta_3\Lambda)^2}\big[V(\Phi-2\Delta_3\Phi,\Lambda-3\Delta_3\Lambda,H)-4V(\Phi-\Delta_3\Phi,\Lambda-3\Delta_3\Lambda,H)\\ &+3V(\Phi,\Lambda-3\Delta_3\Lambda,H)-4V(\Phi-2\Delta_3\Phi,\Lambda-2\Delta_3\Lambda,H)\\ &+16V(\Phi-\Delta_3\Phi,\Lambda-2\Delta_3\Lambda,H)-12V(\Phi,\Lambda-2\Delta_3\Lambda,H)\\ &+5V(\Phi-2\Delta_3\Phi,\Lambda-\Delta_3\Lambda,H)-20V(\Phi-\Delta_3\Phi,\Lambda-\Delta_3\Lambda,H)\\ &+15V(\Phi,\Lambda-\Delta_3\Lambda,H)-2V(\Phi-2\Delta_3\Phi,\Lambda,H)\\ &+8V(\Phi-\Delta_3\Phi,\Lambda,H)-6V(\Phi,\Lambda,H)\big]\end{aligned} \tag{28}$$

(5) else if $H_{\mathrm{B}} \leq H \leq H_{\mathrm{T}}$ **and** either $\Phi < \Phi_{\mathrm{S}} - \Delta_3\Phi$ or $\Phi_{\mathrm{S}} + \Delta_3\Phi < \Phi < \Phi_{\mathrm{N}} - \Delta_3\Phi$ or $\Phi_{\mathrm{N}} + \Delta_3\Phi < \Phi$ **and** either $\Lambda_{\mathrm{W}} < \Lambda < \Lambda_{\mathrm{W}} + \Delta_3\Lambda$ or $\Lambda_{\mathrm{E}} < \Lambda < \Lambda_{\mathrm{E}} + \Delta_3\Lambda$, then based on Eq. (17) in the main text by adopting $u = \Lambda$ and $v = \Phi$ with the positive sign for

$u = \Lambda$ it yields:

$$\left(\frac{\partial^3 V}{\partial\Phi\partial\Lambda^2}\right)_H \approx \frac{1}{2(\Delta_3\Phi)(\Delta_3\Lambda)^2}\big[-V(\Phi+\Delta_3\Phi,\Lambda+3\Delta_3\Lambda,H)+V(\Phi-\Delta_3\Phi,\Lambda+3\Delta_3\Lambda,H) \\ +4V(\Phi+\Delta_3\Phi,\Lambda+2\Delta_3\Lambda,H)-4V(\Phi-\Delta_3\Phi,\Lambda+2\Delta_3\Lambda,H) \\ -5V(\Phi+\Delta_3\Phi,\Lambda+\Delta_3\Lambda,H)+5V(\Phi-\Delta_3\Phi,\Lambda+\Delta_3\Lambda,H) \\ +2V(\Phi+\Delta_3\Phi,\Lambda,H)-2V(\Phi-\Delta_3\Phi,\Lambda,H)\big] \tag{29}$$

(6) else if $H_\mathrm{B} \le H \le H_\mathrm{T}$ **and** either $\Phi < \Phi_\mathrm{S} - \Delta_3\Phi$ or $\Phi_\mathrm{S}+\Delta_3\Phi < \Phi < \Phi_\mathrm{N}-\Delta_3\Phi$ or $\Phi_\mathrm{N}+\Delta_3\Phi < \Phi$ **and** either $\Lambda_\mathrm{W}-\Delta_3\Lambda < \Lambda < \Lambda_\mathrm{W}$ or $\Lambda_\mathrm{E}-\Delta_3\Lambda < \Lambda < \Lambda_\mathrm{E}$, then based on Eq. (17) in the main text by adopting $u = \Lambda$ and $v = \Phi$ with the negative sign for $u = \Lambda$ it yields:

$$\left(\frac{\partial^3 V}{\partial\Phi\partial\Lambda^2}\right)_H \approx \frac{1}{2(\Delta_3\Phi)(\Delta_3\Lambda)^2}\big[-V(\Phi+\Delta_3\Phi,\Lambda-3\Delta_3\Lambda,H)+V(\Phi-\Delta_3\Phi,\Lambda-3\Delta_3\Lambda,H) \\ +4V(\Phi+\Delta_3\Phi,\Lambda-2\Delta_3\Lambda,H)-4V(\Phi-\Delta_3\Phi,\Lambda-2\Delta_3\Lambda,H) \\ -5V(\Phi+\Delta_3\Phi,\Lambda-\Delta_3\Lambda,H)+5V(\Phi-\Delta_3\Phi,\Lambda-\Delta_3\Lambda,H) \\ +2V(\Phi+\Delta_3\Phi,\Lambda,H)-2V(\Phi-\Delta_3\Phi,\Lambda,H)\big] \tag{30}$$

(7) else if $H_\mathrm{B} \le H \le H_\mathrm{T}$ **and** either $\Phi_\mathrm{S} < \Phi < \Phi_\mathrm{S}+\Delta_3\Phi$ or $\Phi_\mathrm{N} < \Phi < \Phi_\mathrm{N}+\Delta_3\Phi$ **and** either $\Lambda < \Lambda_\mathrm{W}-\Delta_3\Lambda$ or $\Lambda_\mathrm{W}+\Delta_3\Lambda < \Lambda < \Lambda_\mathrm{E}-\Delta_3\Lambda$ or $\Lambda_\mathrm{E}+\Delta_3\Lambda < \Lambda$, then based on Eq. (16) in the main text by adopting $u = \Lambda$ and $v = \Phi$ with the positive sign for $v = \Phi$ it yields:

$$\left(\frac{\partial^3 V}{\partial\Phi\partial\Lambda^2}\right)_H \approx \frac{1}{2(\Delta_3\Phi)(\Delta_3\Lambda)^2}\big[-V(\Phi+2\Delta_3\Phi,\Lambda+\Delta_3\Lambda,H)+4V(\Phi+\Delta_3\Phi,\Lambda+\Delta_3\Lambda,H) \\ -3V(\Phi,\Lambda+\Delta_3\Lambda,H)+2V(\Phi+2\Delta_3\Phi,\Lambda,H) \\ -8V(\Phi+\Delta_3\Phi,\Lambda,H)+6V(\Phi,\Lambda,H)-V(\Phi+2\Delta_3\Phi,\Lambda-\Delta_3\Lambda,H) \\ +4V(\Phi+\Delta_3\Phi,\Lambda-\Delta_3\Lambda,H)-3V(\Phi,\Lambda-\Delta_3\Lambda,H)\big] \tag{31}$$

(8) else if $H_\mathrm{B} \le H \le H_\mathrm{T}$ **and** either $\Phi_\mathrm{S}-\Delta_3\Phi < \Phi < \Phi_\mathrm{S}$ or $\Phi_\mathrm{N}-\Delta_3\Phi < \Phi < \Phi_\mathrm{N}$ **and** either $\Lambda < \Lambda_\mathrm{W}-\Delta_3\Lambda$ or $\Lambda_\mathrm{W}+\Delta_3\Lambda < \Lambda < \Lambda_\mathrm{E}-\Delta_3\Lambda$ or $\Lambda_\mathrm{E}+\Delta_3\Lambda < \Lambda$, then based on Eq. (16) in the main text by adopting $u = \Lambda$ and $v = \Phi$ with the negative sign for $v = \Phi$ it yields:

$$\left(\frac{\partial^3 V}{\partial\Phi\partial\Lambda^2}\right)_H \approx \frac{-1}{2(\Delta_3\Phi)(\Delta_3\Lambda)^2}\big[-V(\Phi-2\Delta_3\Phi,\Lambda+\Delta_3\Lambda,H)+4V(\Phi-\Delta_3\Phi,\Lambda+\Delta_3\Lambda,H) \\ -3V(\Phi,\Lambda+\Delta_3\Lambda,H)+2V(\Phi-2\Delta_3\Phi,\Lambda,H) \\ -8V(\Phi-\Delta_3\Phi,\Lambda,H)+6V(\Phi,\Lambda,H)-V(\Phi-2\Delta_3\Phi,\Lambda-\Delta_3\Lambda,H) \\ +4V(\Phi-\Delta_3\Phi,\Lambda-\Delta_3\Lambda,H)-3V(\Phi,\Lambda-\Delta_3\Lambda,H)\big] \tag{32}$$

(9) else, then based on Eq. (15) in the main text by adopting $u = \Lambda$ and $v = \Phi$ it yields:

$$\left(\frac{\partial^3 V}{\partial \Phi \partial \Lambda^2}\right)_H \approx \frac{1}{2(\Delta_3\Phi)(\Delta_3\Lambda)^2}\big[V(\Phi+\Delta_3\Phi,\Lambda+\Delta_3\Lambda,H) - V(\Phi-\Delta_3\Phi,\Lambda+\Delta_3\Lambda,H) - 2V(\Phi+\Delta_3\Phi,\Lambda,H) + 2V(\Phi-\Delta_3\Phi,\Lambda,H) + V(\Phi+\Delta_3\Phi,\Lambda-\Delta_3\Lambda,H) - V(\Phi-\Delta_3\Phi,\Lambda-\Delta_3\Lambda,H)\big] \tag{33}$$

6 Expressions for the third-order partial derivatives $(\partial^3 V/(\partial \Lambda^2 \partial H))_\Phi$

Analogously, the expressions for the $(\partial^3 V/(\partial \Lambda^2 \partial H))_\Phi$ can be calculated by:

(1) if $\Phi_S \leq \Phi \leq \Phi_N$ **and** either $\Lambda_W < \Lambda < \Lambda_W + \Delta_3\Lambda$ or $\Lambda_E < \Lambda < \Lambda_E + \Delta_3\Lambda$ **and** either $H_B < H < H_B + \Delta_3 H$ or $H_T < H < H_T + \Delta_3 H$, based on Eq. (18) in the main text by adopting $u = \Lambda$ and $v = H$ with two corresponding sign factors of $+1$ and $+1$ it yields:

$$\left(\frac{\partial^3 V}{\partial \Lambda^2 \partial H}\right)_\Phi \approx \frac{1}{2(\Delta_3\Lambda)^2(\Delta_3 H)}\big[V(\Phi,\Lambda+3\Delta_3\Lambda,H+2\Delta_3 H) - 4V(\Phi,\Lambda+3\Delta_3\Lambda,H+\Delta_3 H) + 3V(\Phi,\Lambda+3\Delta_3\Lambda,H) - 4V(\Phi,\Lambda+2\Delta_3\Lambda,H+2\Delta_3 H) + 16V(\Phi,\Lambda+2\Delta_3\Lambda,H+\Delta_3 H) - 12V(\Phi,\Lambda+2\Delta_3\Lambda,H) + 5V(\Phi,\Lambda+\Delta_3\Lambda,H+2\Delta_3 H) - 20V(\Phi,\Lambda+\Delta_3\Lambda,H+\Delta_3 H) + 15V(\Phi,\Lambda+\Delta_3\Lambda,H) - 2V(\Phi,\Lambda,H+2\Delta_3 H) + 8V(\Phi,\Lambda,H+\Delta_3 H) - 6V(\Phi,\Lambda,H)\big] \tag{34}$$

(2) else if $\Phi_S \leq \Phi \leq \Phi_N$ **and** either $\Lambda_W < \Lambda < \Lambda_W + \Delta_3\Lambda$ or $\Lambda_E < \Lambda < \Lambda_E + \Delta_3\Lambda$ **and** either $H_B - \Delta_3 H < H < H_B$ or $H_T - \Delta_3 H < H < H_T$, based on Eq. (18) in the main text by adopting $u = \Lambda$ and $v = H$ with two corresponding sign factors of $+1$ and -1 it yields:

$$\left(\frac{\partial^3 V}{\partial \Lambda^2 \partial H}\right)_\Phi \approx \frac{-1}{2(\Delta_3\Lambda)^2(\Delta_3 H)}\big[V(\Phi,\Lambda+3\Delta_3\Lambda,H-2\Delta_3 H) - 4V(\Phi,\Lambda+3\Delta_3\Lambda,H-\Delta_3 H) + 3V(\Phi,\Lambda+3\Delta_3\Lambda,H) - 4V(\Phi,\Lambda+2\Delta_3\Lambda,H-2\Delta_3 H) + 16V(\Phi,\Lambda+2\Delta_3\Lambda,H-\Delta_3 H) - 12V(\Phi,\Lambda+2\Delta_3\Lambda,H) + 5V(\Phi,\Lambda+\Delta_3\Lambda,H-2\Delta_3 H) - 20V(\Phi,\Lambda+\Delta_3\Lambda,H-\Delta_3 H) + 15V(\Phi,\Lambda+\Delta_3\Lambda,H) - 2V(\Phi,\Lambda,H-2\Delta_3 H) + 8V(\Phi,\Lambda,H-\Delta_3 H) - 6V(\Phi,\Lambda,H)\big] \tag{35}$$

(3) else if $\Phi_S \leq \Phi \leq \Phi_N$ **and** either $\Lambda_W - \Delta_3\Lambda < \Lambda < \Lambda_W$ or $\Lambda_E - \Delta_3\Lambda < \Lambda < \Lambda_E$ **and** either $H_B < H < H_B + \Delta_3 H$ or $H_T < H < H_T + \Delta_3 H$, based on Eq. (18) in the main text by adopting $u = \Lambda$ and $v = H$ with two corresponding sign factors of -1 and $+1$ it

yields:

$$\left(\frac{\partial^3 V}{\partial \Lambda^2 \partial H}\right)_\Phi \approx \frac{1}{2(\Delta_3\Lambda)^2(\Delta_3 H)} \big[V(\Phi, \Lambda - 3\Delta_3\Lambda, H + 2\Delta_3 H) - 4V(\Phi, \Lambda - 3\Delta_3\Lambda, H + \Delta_3 H) \\ + 3V(\Phi, \Lambda - 3\Delta_3\Lambda, H) - 4V(\Phi, \Lambda - 2\Delta_3\Lambda, H + 2\Delta_3 H) \\ + 16V(\Phi, \Lambda - 2\Delta_3\Lambda, H + \Delta_3 H) - 12V(\Phi, \Lambda - 2\Delta_3\Lambda, H) \\ + 5V(\Phi, \Lambda - \Delta_3\Lambda, H + 2\Delta_3 H) - 20V(\Phi, \Lambda - \Delta_3\Lambda, H + \Delta_3 H) \\ + 15V(\Phi, \Lambda - \Delta_3\Lambda, H) - 2V(\Phi, \Lambda, H + 2\Delta_3 H) \\ + 8V(\Phi, \Lambda, H + \Delta_3 H) - 6V(\Phi, \Lambda, H) \big] \tag{36}$$

(4) else if $\Phi_S \leq \Phi \leq \Phi_N$ **and** either $\Lambda_W - \Delta_3\Lambda < \Lambda < \Lambda_W$ or $\Lambda_E - \Delta_3\Lambda < \Lambda < \Lambda_E$ **and** either $H_B - \Delta_3 H < H < H_B$ or $H_T - \Delta_3 H < H < H_T$, based on Eq. (18) in the main text by adopting $u = \Lambda$ and $v = H$ with two corresponding sign factors of -1 and -1 it yields:

$$\left(\frac{\partial^3 V}{\partial \Lambda^2 \partial H}\right)_\Phi \approx \frac{-1}{2(\Delta_3\Lambda)^2(\Delta_3 H)} \big[V(\Phi, \Lambda - 3\Delta_3\Lambda, H - 2\Delta_3 H) - 4V(\Phi, \Lambda - 3\Delta_3\Lambda, H - \Delta_3 H) \\ + 3V(\Phi, \Lambda - 3\Delta_3\Lambda, H) - 4V(\Phi, \Lambda - 2\Delta_3\Lambda, H - 2\Delta_3 H) \\ + 16V(\Phi, \Lambda - 2\Delta_3\Lambda, H - \Delta_3 H) - 12V(\Phi, \Lambda - 2\Delta_3\Lambda, H) \\ + 5V(\Phi, \Lambda - \Delta_3\Lambda, H - 2\Delta_3 H) - 20V(\Phi, \Lambda - \Delta_3\Lambda, H - \Delta_3 H) \\ + 15V(\Phi, \Lambda - \Delta_3\Lambda, H) - 2V(\Phi, \Lambda, H - 2\Delta_3 H) \\ + 8V(\Phi, \Lambda, H - \Delta_3 H) - 6V(\Phi, \Lambda, H) \big] \tag{37}$$

(5) else if $\Phi_S \leq \Phi \leq \Phi_N$ **and** either $\Lambda_W < \Lambda < \Lambda_W + \Delta_3\Lambda$ or $\Lambda_E < \Lambda < \Lambda_E + \Delta_3\Lambda$ **and** either $H < H_B - \Delta_3 H$ or $H_B + \Delta_3 H < H < H_T - \Delta_3 H$ or $H_T + \Delta_3 H < H$, then based on Eq. (17) in the main text by adopting $u = \Lambda$ and $v = H$ with the positive sign for $u = \Lambda$ it yields:

$$\left(\frac{\partial^3 V}{\partial \Lambda^2 \partial H}\right)_\Phi \approx \frac{1}{2(\Delta_3\Lambda)^2(\Delta_3 H)} \big[-V(\Phi, \Lambda + 3\Delta_3\Lambda, H + \Delta_3 H) + V(\Phi, \Lambda + 3\Delta_3\Lambda, H - \Delta_3 H) \\ + 4V(\Phi, \Lambda + 2\Delta_3\Lambda, H + \Delta_3 H) - 4V(\Phi, \Lambda + 2\Delta_3\Lambda, H - \Delta_3 H) \\ - 5V(\Phi, \Lambda + \Delta_3\Lambda, H + \Delta_3 H) + 5V(\Phi, \Lambda + \Delta_3\Lambda, H - \Delta_3 H) \\ + 2V(\Phi, \Lambda, H + \Delta_3 H) - 2V(\Phi, \Lambda, H - \Delta_3 H) \big] \tag{38}$$

(6) else if $\Phi_S \leq \Phi \leq \Phi_N$ **and** either $\Lambda_W - \Delta_3\Lambda < \Lambda < \Lambda_W$ or $\Lambda_E - \Delta_3\Lambda < \Lambda < \Lambda_E$ **and** either $H < H_B - \Delta_3 H$ or $H_B + \Delta_3 H < H < H_T - \Delta_3 H$ or $H_T + \Delta_3 H < H$, then based on Eq. (17) in the main text by adopting $u = \Lambda$ and $v = H$ with the negative sign for $u = \Lambda$ it yields:

$$\left(\frac{\partial^3 V}{\partial \Lambda^2 \partial H}\right)_\Phi \approx \frac{1}{2(\Delta_3\Lambda)^2(\Delta_3 H)} \big[-V(\Phi, \Lambda - 3\Delta_3\Lambda, H + \Delta_3 H) + V(\Phi, \Lambda - 3\Delta_3\Lambda, H - \Delta_3 H) \\ + 4V(\Phi, \Lambda - 2\Delta_3\Lambda, H + \Delta_3 H) - 4V(\Phi, \Lambda - 2\Delta_3\Lambda, H - \Delta_3 H) \\ - 5V(\Phi, \Lambda - \Delta_3\Lambda, H + \Delta_3 H) + 5V(\Phi, \Lambda - \Delta_3\Lambda, H - \Delta_3 H) \\ + 2V(\Phi, \Lambda, H + \Delta_3 H) - 2V(\Phi, \Lambda, H - \Delta_3 H) \big] \tag{39}$$

(7) else if $\Phi_{\mathrm{S}} \leq \Phi \leq \Phi_{\mathrm{N}}$ **and** either $\Lambda < \Lambda_{\mathrm{W}} - \Delta_3\Lambda$ or $\Lambda_{\mathrm{W}} + \Delta_3\Lambda < \Lambda < \Lambda_{\mathrm{E}} - \Delta_3\Lambda$ or $\Lambda_{\mathrm{E}} + \Delta_3\Lambda < \Lambda$ **and** either $H_{\mathrm{B}} < H < H_{\mathrm{B}} + \Delta_3 H$ or $H_{\mathrm{T}} < H < H_{\mathrm{T}} + \Delta_3 H$, then based on Eq. (16) in the main text by adopting $u = \Lambda$ and $v = H$ with the positive sign for $v = H$ it yields:

$$\left(\frac{\partial^3 V}{\partial \Lambda^2 \partial H}\right)_{\Phi} \approx \frac{1}{2(\Delta_3\Lambda)^2(\Delta_3 H)}\big[-V(\Phi,\Lambda+\Delta_3\Lambda,H+2\Delta_3 H)+4V(\Phi,\Lambda+\Delta_3\Lambda,H+\Delta_3 H) \\ -3V(\Phi,\Lambda+\Delta_3\Lambda,H)+2V(\Phi,\Lambda,H+2\Delta_3 H) \\ -8V(\Phi,\Lambda,H+\Delta_3 H)+6V(\Phi,\Lambda,H)-V(\Phi,\Lambda-\Delta_3\Lambda,H+2\Delta_3 H) \\ +4V(\Phi,\Lambda-\Delta_3\Lambda,H+\Delta_3 H)-3V(\Phi,\Lambda-\Delta_3\Lambda,H)\big] \tag{40}$$

(8) else if $\Phi_{\mathrm{S}} \leq \Phi \leq \Phi_{\mathrm{N}}$ **and** either $\Lambda < \Lambda_{\mathrm{W}} - \Delta_3\Lambda$ or $\Lambda_{\mathrm{W}} + \Delta_3\Lambda < \Lambda < \Lambda_{\mathrm{E}} - \Delta_3\Lambda$ or $\Lambda_{\mathrm{E}} + \Delta_3\Lambda < \Lambda$ **and** either either $H_{\mathrm{B}} - \Delta_3 H < H < H_{\mathrm{B}}$ or $H_{\mathrm{T}} - \Delta_3 H < H < H_{\mathrm{T}}$, then based on Eq. (16) in the main text by adopting $u = \Lambda$ and $v = H$ with the negative sign for $v = H$ it yields:

$$\left(\frac{\partial^3 V}{\partial \Lambda^2 \partial H}\right)_{\Phi} \approx \frac{-1}{2(\Delta_3\Lambda)^2(\Delta_3 H)}\big[-V(\Phi,\Lambda+\Delta_3\Lambda,H-2\Delta_3 H)+4V(\Phi,\Lambda+\Delta_3\Lambda,H-\Delta_3 H) \\ -3V(\Phi,\Lambda+\Delta_3\Lambda,H)+2V(\Phi,\Lambda,H-2\Delta_3 H) \\ -8V(\Phi,\Lambda,H-\Delta_3 H)+6V(\Phi,\Lambda,H)-V(\Phi,\Lambda-\Delta_3\Lambda,H-2\Delta_3 H) \\ +4V(\Phi,\Lambda-\Delta_3\Lambda,H-\Delta_3 H)-3V(\Phi,\Lambda-\Delta_3\Lambda,H)\big] \tag{41}$$

(9) else, then based on Eq. (15) in the main text by adopting $u = \Lambda$ and $v = H$ it yields:

$$\left(\frac{\partial^3 V}{\partial \Lambda^2 \partial H}\right)_{\Phi} \approx \frac{1}{2(\Delta_3\Lambda)^2(\Delta_3 H)}\big[V(\Phi,\Lambda+\Delta_3\Lambda,H+\Delta_3 H)-V(\Phi,\Lambda+\Delta_3\Lambda,H-\Delta_3 H) \\ -2V(\Phi,\Lambda,H+\Delta_3 H)+2V(\Phi,\Lambda,H-\Delta_3 H) \\ +V(\Phi,\Lambda-\Delta_3\Lambda,H+\Delta_3 H)-V(\Phi,\Lambda-\Delta_3\Lambda,H-\Delta_3 H)\big] \tag{42}$$

7 Expressions for the third-order partial derivatives $(\partial^3 V/(\partial\Phi\partial H^2))_{\Lambda}$

Similarly, the expressions for the $(\partial^3 V/(\partial\Phi\partial H^2))_{\Lambda}$ are obtained by:

(1) if $\Lambda_{\mathrm{W}} \leq \Lambda \leq \Lambda_{\mathrm{E}}$ **and** either $\Phi_{\mathrm{S}} < \Phi < \Phi_{\mathrm{S}} + \Delta_3\Phi$ or $\Phi_{\mathrm{N}} < \Phi < \Phi_{\mathrm{N}} + \Delta_3\Phi$ **and** either $H_{\mathrm{B}} < H < H_{\mathrm{B}} + \Delta_3 H$ or $H_{\mathrm{T}} < H < H_{\mathrm{T}} + \Delta_3 H$, based on Eq. (18) in the main text by adopting $u = H$ and $v = \Phi$ with two corresponding sign factors of $+1$ and $+1$ it yields:

$$\left(\frac{\partial^3 V}{\partial \Phi \partial H^2}\right)_{\Lambda} \approx \frac{1}{2(\Delta_3\Phi)(\Delta_3 H)^2}\big[V(\Phi+2\Delta_3\Phi,\Lambda,H+3\Delta_3 H)-4V(\Phi+\Delta_3\Phi,\Lambda,H+3\Delta_3 H) \\ +3V(\Phi,\Lambda,H+3\Delta_3 H)-4V(\Phi+2\Delta_3\Phi,\Lambda,H+2\Delta_3 H) \\ +16V(\Phi+\Delta_3\Phi,\Lambda,H+2\Delta_3 H)-12V(\Phi,\Lambda,H+2\Delta_3 H) \\ +5V(\Phi+2\Delta_3\Phi,\Lambda,H+\Delta_3 H)-20V(\Phi+\Delta_3\Phi,\Lambda,H+\Delta_3 H) \\ +15V(\Phi,\Lambda,H+\Delta_3 H)-2V(\Phi+2\Delta_3\Phi,\Lambda,H) \\ +8V(\Phi+\Delta_3\Phi,\Lambda,H)-6V(\Phi,\Lambda,H)\big] \tag{43}$$

(2) else if $\Lambda_W \leq \Lambda \leq \Lambda_E$ **and** either $\Phi_S - \Delta_3\Phi < \Phi < \Phi_S$ or $\Phi_N - \Delta_3\Phi < \Phi < \Phi_N$ **and** either $H_B < H < H_B + \Delta_3 H$ or $H_T < H < H_T + \Delta_3 H$, based on Eq. (18) in the main text by adopting $u = H$ and $v = \Phi$ with two corresponding sign factors of $+1$ and -1 it yields:

$$\begin{aligned}\left(\frac{\partial^3 V}{\partial\Phi\partial H^2}\right)_\Lambda \approx & \frac{-1}{2(\Delta_3\Phi)(\Delta_3 H)^2}\big[V(\Phi-2\Delta_3\Phi,\Lambda,H+3\Delta_3 H)-4V(\Phi-\Delta_3\Phi,\Lambda,H+3\Delta_3 H)\\ &+3V(\Phi,\Lambda,H+3\Delta_3 H)-4V(\Phi-2\Delta_3\Phi,\Lambda,H+2\Delta_3 H)\\ &+16V(\Phi-\Delta_3\Phi,\Lambda,H+2\Delta_3 H)-12V(\Phi,\Lambda,H+2\Delta_3 H)\\ &+5V(\Phi-2\Delta_3\Phi,\Lambda,H+\Delta_3 H)-20V(\Phi-\Delta_3\Phi,\Lambda,H+\Delta_3 H)\\ &+15V(\Phi,\Lambda,H+\Delta_3 H)-2V(\Phi-2\Delta_3\Phi,\Lambda,H)\\ &+8V(\Phi-\Delta_3\Phi,\Lambda,H)-6V(\Phi,\Lambda,H)\big]\end{aligned} \tag{44}$$

(3) else if $\Lambda_W \leq \Lambda \leq \Lambda_E$ **and** either $\Phi_S < \Phi < \Phi_S + \Delta_3\Phi$ or $\Phi_N < \Phi < \Phi_N + \Delta_3\Phi$ **and** either $H_B - \Delta_3 H < H < H_B$ or $H_T - \Delta_3 H < H < H_T$, based on Eq. (18) in the main text by adopting $u = H$ and $v = \Phi$ with two corresponding sign factors of -1 and $+1$ it yields:

$$\begin{aligned}\left(\frac{\partial^3 V}{\partial\Phi\partial H^2}\right)_\Lambda \approx & \frac{1}{2(\Delta_3\Phi)(\Delta_3 H)^2}\big[V(\Phi+2\Delta_3\Phi,\Lambda,H-3\Delta_3 H)-4V(\Phi+\Delta_3\Phi,\Lambda,H-3\Delta_3 H)\\ &+3V(\Phi,\Lambda,H-3\Delta_3 H)-4V(\Phi+2\Delta_3\Phi,\Lambda,H-2\Delta_3 H)\\ &+16V(\Phi+\Delta_3\Phi,\Lambda,H-2\Delta_3 H)-12V(\Phi,\Lambda,H-2\Delta_3 H)\\ &+5V(\Phi+2\Delta_3\Phi,\Lambda,H-\Delta_3 H)-20V(\Phi+\Delta_3\Phi,\Lambda,H-\Delta_3 H)\\ &+15V(\Phi,\Lambda,H-\Delta_3 H)-2V(\Phi+2\Delta_3\Phi,\Lambda,H)\\ &+8V(\Phi+\Delta_3\Phi,\Lambda,H)-6V(\Phi,\Lambda,H)\big]\end{aligned} \tag{45}$$

(4) else if $\Lambda_W \leq \Lambda \leq \Lambda_E$ **and** either $\Phi_S - \Delta_3\Phi < \Phi < \Phi_S$ or $\Phi_N - \Delta_3\Phi < \Phi < \Phi_N$ **and** either $H_B - \Delta_3 H < H < H_B$ or $H_T - \Delta_3 H < H < H_T$, based on Eq. (18) in the main text by adopting $u = H$ and $v = \Phi$ with two corresponding sign factors of -1 and -1 it yields:

$$\begin{aligned}\left(\frac{\partial^3 V}{\partial\Phi\partial H^2}\right)_\Lambda \approx & \frac{-1}{2(\Delta_3\Phi)(\Delta_3 H)^2}\big[V(\Phi-2\Delta_3\Phi,\Lambda,H-3\Delta_3 H)-4V(\Phi-\Delta_3\Phi,\Lambda,H-3\Delta_3 H)\\ &+3V(\Phi,\Lambda,H-3\Delta_3 H)-4V(\Phi-2\Delta_3\Phi,\Lambda,H-2\Delta_3 H)\\ &+16V(\Phi-\Delta_3\Phi,\Lambda,H-2\Delta_3 H)-12V(\Phi,\Lambda,H-2\Delta_3 H)\\ &+5V(\Phi-2\Delta_3\Phi,\Lambda,H-\Delta_3 H)-20V(\Phi-\Delta_3\Phi,\Lambda,H-\Delta_3 H)\\ &+15V(\Phi,\Lambda,H-\Delta_3 H)-2V(\Phi-2\Delta_3\Phi,\Lambda,H)\\ &+8V(\Phi-\Delta_3\Phi,\Lambda,H)-6V(\Phi,\Lambda,H)\big]\end{aligned} \tag{46}$$

(5) else if $\Lambda_W \leq \Lambda \leq \Lambda_E$ **and** either $\Phi < \Phi_S - \Delta_3\Phi$ or $\Phi_S + \Delta_3\Phi < \Phi < \Phi_N - \Delta_3\Phi$ or $\Phi_N + \Delta_3\Phi < \Phi$ **and** either $H_B < H < H_B + \Delta_3 H$ or $H_T < H < H_T + \Delta_3 H$, then based on Eq. (17) in the main text by adopting $u = H$ and $v = \Phi$ with the positive sign for $u = H$

it yields:

$$\left(\frac{\partial^3 V}{\partial\Phi\partial H^2}\right)_\Lambda \approx \frac{1}{2(\Delta_3\Phi)(\Delta_3 H)^2}\big[-V(\Phi+\Delta_3\Phi,\Lambda,H+3\Delta_3 H)+V(\Phi-\Delta_3\Phi,\Lambda,H+3\Delta_3 H)$$
$$+4V(\Phi+\Delta_3\Phi,\Lambda,H+2\Delta_3 H)-4V(\Phi-\Delta_3\Phi,\Lambda,H+2\Delta_3 H)$$
$$-5V(\Phi+\Delta_3\Phi,\Lambda,H+\Delta_3 H)+5V(\Phi-\Delta_3\Phi,\Lambda,H+\Delta_3 H)$$
$$+2V(\Phi+\Delta_3\Phi,\Lambda,H)-2V(\Phi-\Delta_3\Phi,\Lambda,H)\big] \tag{47}$$

(6) else if $\Lambda_W \leq \Lambda \leq \Lambda_E$ **and** either $\Phi < \Phi_S - \Delta_3\Phi$ or $\Phi_S + \Delta_3\Phi < \Phi < \Phi_N - \Delta_3\Phi$ or $\Phi_N + \Delta_3\Phi < \Phi$ **and** either $H_B - \Delta_3 H < H < H_B$ or $H_T - \Delta_3 H < H < H_T$, then based on Eq. (17) in the main text by adopting $u = H$ and $v = \Phi$ with the negative sign for $u = H$ it yields:

$$\left(\frac{\partial^3 V}{\partial\Phi\partial H^2}\right)_\Lambda \approx \frac{1}{2(\Delta_3\Phi)(\Delta_3 H)^2}\big[-V(\Phi+\Delta_3\Phi,\Lambda,H-3\Delta_3 H)+V(\Phi-\Delta_3\Phi,\Lambda,H-3\Delta_3 H)$$
$$+4V(\Phi+\Delta_3\Phi,\Lambda,H-2\Delta_3 H)-4V(\Phi-\Delta_3\Phi,\Lambda,H-2\Delta_3 H)$$
$$-5V(\Phi+\Delta_3\Phi,\Lambda,H-\Delta_3 H)+5V(\Phi-\Delta_3\Phi,\Lambda,H-\Delta_3 H)$$
$$+2V(\Phi+\Delta_3\Phi,\Lambda,H)-2V(\Phi-\Delta_3\Phi,\Lambda,H)\big] \tag{48}$$

(7) else if $\Lambda_W \leq \Lambda \leq \Lambda_E$ **and** either $\Phi_S < \Phi < \Phi_S + \Delta_3\Phi$ or $\Phi_N < \Phi < \Phi_N + \Delta_3\Phi$ **and** either $H < H_B - \Delta_3 H$ or $H_B + \Delta_3 H < H < H_T - \Delta_3 H$ or $H_T + \Delta_3 H < H$, then based on Eq. (16) in the main text by adopting $u = H$ and $v = \Phi$ with the positive sign for $v = \Phi$ it yields:

$$\left(\frac{\partial^3 V}{\partial\Phi\partial H^2}\right)_\Lambda \approx \frac{1}{2(\Delta_3\Phi)(\Delta_3 H)^2}\big[-V(\Phi+2\Delta_3\Phi,\Lambda,H+\Delta_3 H)+4V(\Phi+\Delta_3\Phi,\Lambda,H+\Delta_3 H)$$
$$-3V(\Phi,\Lambda,H+\Delta_3 H)+2V(\Phi+2\Delta_3\Phi,\Lambda,H)$$
$$-8V(\Phi+\Delta_3\Phi,\Lambda,H)+6V(\Phi,\Lambda,H)-V(\Phi+2\Delta_3\Phi,\Lambda,H-\Delta_3 H)$$
$$+4V(\Phi+\Delta_3\Phi,\Lambda,H-\Delta_3 H)-3V(\Phi,\Lambda,H-\Delta_3 H)\big] \tag{49}$$

(8) else if $\Lambda_W \leq \Lambda \leq \Lambda_E$ **and** either $\Phi_S - \Delta_3\Phi < \Phi < \Phi_S$ or $\Phi_N - \Delta_3\Phi < \Phi < \Phi_N$ **and** either $H < H_B - \Delta_3 H$ or $H_B + \Delta_3 H < H < H_T - \Delta_3 H$ or $H_T + \Delta_3 H < H$, then based on Eq. (16) in the main text by adopting $u = H$ and $v = \Phi$ with the negative sign for $v = \Phi$ it yields:

$$\left(\frac{\partial^3 V}{\partial\Phi\partial H^2}\right)_\Lambda \approx \frac{-1}{2(\Delta_3\Phi)(\Delta_3 H)^2}\big[-V(\Phi-2\Delta_3\Phi,\Lambda,H+\Delta_3 H)+4V(\Phi-\Delta_3\Phi,\Lambda,H+\Delta_3 H)$$
$$-3V(\Phi,\Lambda,H+\Delta_3 H)+2V(\Phi-2\Delta_3\Phi,\Lambda,H)$$
$$-8V(\Phi-\Delta_3\Phi,\Lambda,H)+6V(\Phi,\Lambda,H)-V(\Phi-2\Delta_3\Phi,\Lambda,H-\Delta_3 H)$$
$$+4V(\Phi-\Delta_3\Phi,\Lambda,H-\Delta_3 H)-3V(\Phi,\Lambda,H-\Delta_3 H)\big] \tag{50}$$

(9) else, then based on Eq. (15) in the main text by adopting $u = H$ and $v = \Phi$ it yields:

$$\left(\frac{\partial^3 V}{\partial \Phi \partial H^2}\right)_\Lambda \approx \frac{1}{2(\Delta_3\Phi)(\Delta_3 H)^2}\big[V(\Phi+\Delta_3\Phi,\Lambda,H+\Delta_3 H) - V(\Phi-\Delta_3\Phi,\Lambda,H+\Delta_3 H) \\ -2V(\Phi+\Delta_3\Phi,\Lambda,H) + 2V(\Phi-\Delta_3\Phi,\Lambda,H) \\ +V(\Phi+\Delta_3\Phi,\Lambda,H-\Delta_3 H) - V(\Phi-\Delta_3\Phi,\Lambda,H-\Delta_3 H)\big] \tag{51}$$

8 Expressions for the third-order partial derivatives $\partial^3 V/(\partial\Phi\partial\Lambda\partial H)$

Regarding the non-diagonal components with three variables of the third-order partial derivatives, the different conditional switches of the $\partial^3 V/(\partial\Phi\partial\Lambda\partial H)$ can be obtained by:

(1) if either $\Phi_S < \Phi < \Phi_S + \Delta_3\Phi$ or $\Phi_N < \Phi < \Phi_N + \Delta_3\Phi$ **and** either $\Lambda_W < \Lambda < \Lambda_W + \Delta_3\Lambda$ or $\Lambda_E < \Lambda < \Lambda_E + \Delta_3\Lambda$ **and** either $H_B < H < H_B + \Delta_3 H$ or $H_T < H < H_T + \Delta_3 H$, then based on Eq. (22) in the main text by adopting $u = \Phi$, $v = \Lambda$, and $w = H$ with three corresponding sign factors of $+1$, $+1$, and $+1$ it yields:

$$\frac{\partial^3 V}{\partial\Phi\partial\Lambda\partial H} \approx \frac{1}{8(\Delta_3\Phi)(\Delta_3\Lambda)(\Delta_3 H)}\big[-V(\Phi+2\Delta_3\Phi,\Lambda+2\Delta_3\Lambda,H+2\Delta_3 H) \\ +4V(\Phi+2\Delta_3\Phi,\Lambda+2\Delta_3\Lambda,H+\Delta_3 H) - 3V(\Phi+2\Delta_3\Phi,\Lambda+2\Delta_3\Lambda,H) \\ +4V(\Phi+2\Delta_3\Phi,\Lambda+\Delta_3\Lambda,H+2\Delta_3 H) - 16V(\Phi+2\Delta_3\Phi,\Lambda+\Delta_3\Lambda,H+\Delta_3 H) \\ +12V(\Phi+2\Delta_3\Phi,\Lambda+\Delta_3\Lambda,H) - 3V(\Phi+2\Delta_3\Phi,\Lambda,H+2\Delta_3 H) \\ +12V(\Phi+2\Delta_3\Phi,\Lambda,H+\Delta_3 H) - 9V(\Phi+2\Delta_3\Phi,\Lambda,H) \\ +4V(\Phi+\Delta_3\Phi,\Lambda+2\Delta_3\Lambda,H+2\Delta_3 H) - 16V(\Phi+\Delta_3\Phi,\Lambda+2\Delta_3\Lambda,H+\Delta_3 H) \\ +12V(\Phi+\Delta_3\Phi,\Lambda+2\Delta_3\Lambda,H) - 16V(\Phi+\Delta_3\Phi,\Lambda+\Delta_3\Lambda,H+2\Delta_3 H) \\ +64V(\Phi+\Delta_3\Phi,\Lambda+\Delta_3\Lambda,H+\Delta_3 H) - 48V(\Phi+\Delta_3\Phi,\Lambda+\Delta_3\Lambda,H) \\ +12V(\Phi+\Delta_3\Phi,\Lambda,H+2\Delta_3 H) - 48V(\Phi+\Delta_3\Phi,\Lambda,H+\Delta_3 H) \\ +36V(\Phi+\Delta_3\Phi,\Lambda,H) - 3V(\Phi,\Lambda+2\Delta_3\Lambda,H+2\Delta_3 H) \\ +12V(\Phi,\Lambda+2\Delta_3\Lambda,H+\Delta_3 H) - 9V(\Phi,\Lambda+2\Delta_3\Lambda,H) \\ +12V(\Phi,\Lambda+\Delta_3\Lambda,H+2\Delta_3 H) - 48V(\Phi,\Lambda+\Delta_3\Lambda,H+\Delta_3 H) \\ +36V(\Phi,\Lambda+\Delta_3\Lambda,H) - 9V(\Phi,\Lambda,H+2\Delta_3 H) \\ +36V(\Phi,\Lambda,H+\Delta_3 H) - 27V(\Phi,\Lambda,H)\big] \tag{52}$$

(2) else if either $\Phi_S < \Phi < \Phi_S + \Delta_3\Phi$ or $\Phi_N < \Phi < \Phi_N + \Delta_3\Phi$ **and** either $\Lambda_W < \Lambda < \Lambda_W + \Delta_3\Lambda$ or $\Lambda_E < \Lambda < \Lambda_E + \Delta_3\Lambda$ **and** either $H_B - \Delta_3 H < H < H_B$ or $H_T - \Delta_3 H < H < H_T$, then based on Eq. (22) in the main text by adopting $u = \Phi$, $v = \Lambda$, and $w = H$ with three

corresponding sign factors of $+1$, $+1$, and -1 it yields:

$$\begin{aligned}\frac{\partial^3 V}{\partial\Phi\partial\Lambda\partial H} \approx & \frac{-1}{8(\Delta_3\Phi)(\Delta_3\Lambda)(\Delta_3 H)}\big[-V(\Phi+2\Delta_3\Phi,\Lambda+2\Delta_3\Lambda,H-2\Delta_3 H) \\ & +4V(\Phi+2\Delta_3\Phi,\Lambda+2\Delta_3\Lambda,H-\Delta_3 H)-3V(\Phi+2\Delta_3\Phi,\Lambda+2\Delta_3\Lambda,H) \\ & +4V(\Phi+2\Delta_3\Phi,\Lambda+\Delta_3\Lambda,H-2\Delta_3 H)-16V(\Phi+2\Delta_3\Phi,\Lambda+\Delta_3\Lambda,H-\Delta_3 H) \\ & +12V(\Phi+2\Delta_3\Phi,\Lambda+\Delta_3\Lambda,H)-3V(\Phi+2\Delta_3\Phi,\Lambda,H-2\Delta_3 H) \\ & +12V(\Phi+2\Delta_3\Phi,\Lambda,H-\Delta_3 H)-9V(\Phi+2\Delta_3\Phi,\Lambda,H) \\ & +4V(\Phi+\Delta_3\Phi,\Lambda+2\Delta_3\Lambda,H-2\Delta_3 H)-16V(\Phi+\Delta_3\Phi,\Lambda+2\Delta_3\Lambda,H-\Delta_3 H) \\ & +12V(\Phi+\Delta_3\Phi,\Lambda+2\Delta_3\Lambda,H)-16V(\Phi+\Delta_3\Phi,\Lambda+\Delta_3\Lambda,H-2\Delta_3 H) \\ & +64V(\Phi+\Delta_3\Phi,\Lambda+\Delta_3\Lambda,H-\Delta_3 H)-48V(\Phi+\Delta_3\Phi,\Lambda+\Delta_3\Lambda,H) \\ & +12V(\Phi+\Delta_3\Phi,\Lambda,H-2\Delta_3 H)-48V(\Phi+\Delta_3\Phi,\Lambda,H-\Delta_3 H) \\ & +36V(\Phi+\Delta_3\Phi,\Lambda,H)-3V(\Phi,\Lambda+2\Delta_3\Lambda,H-2\Delta_3 H) \\ & +12V(\Phi,\Lambda+2\Delta_3\Lambda,H-\Delta_3 H)-9V(\Phi,\Lambda+2\Delta_3\Lambda,H) \\ & +12V(\Phi,\Lambda+\Delta_3\Lambda,H-2\Delta_3 H)-48V(\Phi,\Lambda+\Delta_3\Lambda,H-\Delta_3 H) \\ & +36V(\Phi,\Lambda+\Delta_3\Lambda,H)-9V(\Phi,\Lambda,H-2\Delta_3 H) \\ & +36V(\Phi,\Lambda,H-\Delta_3 H)-27V(\Phi,\Lambda,H)\big]\end{aligned} \tag{53}$$

(3) else if either $\Phi_S < \Phi < \Phi_S+\Delta_3\Phi$ or $\Phi_N < \Phi < \Phi_N+\Delta_3\Phi$ **and** either $\Lambda_W-\Delta_3\Lambda < \Lambda < \Lambda_W$ or $\Lambda_E-\Delta_3\Lambda < \Lambda < \Lambda_E$ **and** either $H_B < H < H_B+\Delta_3 H$ or $H_T < H < H_T+\Delta_3 H$, then based on Eq. (22) in the main text by adopting $u=\Phi$, $v=\Lambda$, and $w=H$ with three corresponding sign factors of $+1$, -1, and $+1$ it yields:

$$\begin{aligned}\frac{\partial^3 V}{\partial\Phi\partial\Lambda\partial H} \approx & \frac{-1}{8(\Delta_3\Phi)(\Delta_3\Lambda)(\Delta_3 H)}\big[-V(\Phi+2\Delta_3\Phi,\Lambda-2\Delta_3\Lambda,H+2\Delta_3 H) \\ & +4V(\Phi+2\Delta_3\Phi,\Lambda-2\Delta_3\Lambda,H+\Delta_3 H)-3V(\Phi+2\Delta_3\Phi,\Lambda-2\Delta_3\Lambda,H) \\ & +4V(\Phi+2\Delta_3\Phi,\Lambda-\Delta_3\Lambda,H+2\Delta_3 H)-16V(\Phi+2\Delta_3\Phi,\Lambda-\Delta_3\Lambda,H+\Delta_3 H) \\ & +12V(\Phi+2\Delta_3\Phi,\Lambda-\Delta_3\Lambda,H)-3V(\Phi+2\Delta_3\Phi,\Lambda,H+2\Delta_3 H) \\ & +12V(\Phi+2\Delta_3\Phi,\Lambda,H+\Delta_3 H)-9V(\Phi+2\Delta_3\Phi,\Lambda,H) \\ & +4V(\Phi+\Delta_3\Phi,\Lambda-2\Delta_3\Lambda,H+2\Delta_3 H)-16V(\Phi+\Delta_3\Phi,\Lambda-2\Delta_3\Lambda,H+\Delta_3 H) \\ & +12V(\Phi+\Delta_3\Phi,\Lambda-2\Delta_3\Lambda,H)-16V(\Phi+\Delta_3\Phi,\Lambda-\Delta_3\Lambda,H+2\Delta_3 H) \\ & +64V(\Phi+\Delta_3\Phi,\Lambda-\Delta_3\Lambda,H+\Delta_3 H)-48V(\Phi+\Delta_3\Phi,\Lambda-\Delta_3\Lambda,H) \\ & +12V(\Phi+\Delta_3\Phi,\Lambda,H+2\Delta_3 H)-48V(\Phi+\Delta_3\Phi,\Lambda,H+\Delta_3 H) \\ & +36V(\Phi+\Delta_3\Phi,\Lambda,H)-3V(\Phi,\Lambda-2\Delta_3\Lambda,H+2\Delta_3 H) \\ & +12V(\Phi,\Lambda-2\Delta_3\Lambda,H+\Delta_3 H)-9V(\Phi,\Lambda-2\Delta_3\Lambda,H) \\ & +12V(\Phi,\Lambda-\Delta_3\Lambda,H+2\Delta_3 H)-48V(\Phi,\Lambda-\Delta_3\Lambda,H+\Delta_3 H) \\ & +36V(\Phi,\Lambda-\Delta_3\Lambda,H)-9V(\Phi,\Lambda,H+2\Delta_3 H) \\ & +36V(\Phi,\Lambda,H+\Delta_3 H)-27V(\Phi,\Lambda,H)\big]\end{aligned} \tag{54}$$

(4) else if either $\Phi_S < \Phi < \Phi_S+\Delta_3\Phi$ or $\Phi_N < \Phi < \Phi_N+\Delta_3\Phi$ **and** either $\Lambda_W-\Delta_3\Lambda < \Lambda < \Lambda_W$ or $\Lambda_E-\Delta_3\Lambda < \Lambda < \Lambda_E$ **and** either $H_B-\Delta_3 H < H < H_B$ or $H_T-\Delta_3 H < H < H_T$, then based on Eq. (22) in the main text by adopting $u=\Phi$, $v=\Lambda$, and $w=H$ with three

corresponding sign factors of $+1$, -1, and -1 it yields:

$$\begin{aligned}
\frac{\partial^3 V}{\partial\Phi\partial\Lambda\partial H} \approx & \frac{1}{8(\Delta_3\Phi)(\Delta_3\Lambda)(\Delta_3 H)}\big[-V(\Phi+2\Delta_3\Phi,\Lambda-2\Delta_3\Lambda,H-2\Delta_3 H) \\
& +4V(\Phi+2\Delta_3\Phi,\Lambda-2\Delta_3\Lambda,H-\Delta_3 H)-3V(\Phi+2\Delta_3\Phi,\Lambda-2\Delta_3\Lambda,H) \\
& +4V(\Phi+2\Delta_3\Phi,\Lambda-\Delta_3\Lambda,H-2\Delta_3 H)-16V(\Phi+2\Delta_3\Phi,\Lambda-\Delta_3\Lambda,H-\Delta_3 H) \\
& +12V(\Phi+2\Delta_3\Phi,\Lambda-\Delta_3\Lambda,H)-3V(\Phi+2\Delta_3\Phi,\Lambda,H-2\Delta_3 H) \\
& +12V(\Phi+2\Delta_3\Phi,\Lambda,H-\Delta_3 H)-9V(\Phi+2\Delta_3\Phi,\Lambda,H) \\
& +4V(\Phi+\Delta_3\Phi,\Lambda-2\Delta_3\Lambda,H-2\Delta_3 H)-16V(\Phi+\Delta_3\Phi,\Lambda-2\Delta_3\Lambda,H-\Delta_3 H) \\
& +12V(\Phi+\Delta_3\Phi,\Lambda-2\Delta_3\Lambda,H)-16V(\Phi+\Delta_3\Phi,\Lambda-\Delta_3\Lambda,H-2\Delta_3 H) \\
& +64V(\Phi+\Delta_3\Phi,\Lambda-\Delta_3\Lambda,H-\Delta_3 H)-48V(\Phi+\Delta_3\Phi,\Lambda-\Delta_3\Lambda,H) \\
& +12V(\Phi+\Delta_3\Phi,\Lambda,H-2\Delta_3 H)-48V(\Phi+\Delta_3\Phi,\Lambda,H-\Delta_3 H) \\
& +36V(\Phi+\Delta_3\Phi,\Lambda,H)-3V(\Phi,\Lambda-2\Delta_3\Lambda,H-2\Delta_3 H) \\
& +12V(\Phi,\Lambda-2\Delta_3\Lambda,H-\Delta_3 H)-9V(\Phi,\Lambda-2\Delta_3\Lambda,H) \\
& +12V(\Phi,\Lambda-\Delta_3\Lambda,H-2\Delta_3 H)-48V(\Phi,\Lambda-\Delta_3\Lambda,H-\Delta_3 H) \\
& +36V(\Phi,\Lambda-\Delta_3\Lambda,H)-9V(\Phi,\Lambda,H-2\Delta_3 H) \\
& +36V(\Phi,\Lambda,H-\Delta_3 H)-27V(\Phi,\Lambda,H)\big]
\end{aligned} \tag{55}$$

(5) else if either $\Phi_\mathrm{S}-\Delta_3\Phi<\Phi<\Phi_\mathrm{S}$ or $\Phi_\mathrm{N}-\Delta_3\Phi<\Phi<\Phi_\mathrm{N}$ **and** either $\Lambda_\mathrm{W}<\Lambda<\Lambda_\mathrm{W}+\Delta_3\Lambda$ or $\Lambda_\mathrm{E}<\Lambda<\Lambda_\mathrm{E}+\Delta_3\Lambda$ **and** either $H_\mathrm{B}<H<H_\mathrm{B}+\Delta_3 H$ or $H_\mathrm{T}<H<H_\mathrm{T}+\Delta_3 H$, then based on Eq. (22) in the main text with by adopting $u=\Phi$, $v=\Lambda$, and $w=H$ with three corresponding sign factors of -1, $+1$, and $+1$ it yields:

$$\begin{aligned}
\frac{\partial^3 V}{\partial\Phi\partial\Lambda\partial H} \approx & \frac{-1}{8(\Delta_3\Phi)(\Delta_3\Lambda)(\Delta_3 H)}\big[-V(\Phi-2\Delta_3\Phi,\Lambda+2\Delta_3\Lambda,H+2\Delta_3 H) \\
& +4V(\Phi-2\Delta_3\Phi,\Lambda+2\Delta_3\Lambda,H+\Delta_3 H)-3V(\Phi-2\Delta_3\Phi,\Lambda+2\Delta_3\Lambda,H) \\
& +4V(\Phi-2\Delta_3\Phi,\Lambda+\Delta_3\Lambda,H+2\Delta_3 H)-16V(\Phi-2\Delta_3\Phi,\Lambda+\Delta_3\Lambda,H+\Delta_3 H) \\
& +12V(\Phi-2\Delta_3\Phi,\Lambda+\Delta_3\Lambda,H)-3V(\Phi-2\Delta_3\Phi,\Lambda,H+2\Delta_3 H) \\
& +12V(\Phi-2\Delta_3\Phi,\Lambda,H+\Delta_3 H)-9V(\Phi-2\Delta_3\Phi,\Lambda,H) \\
& +4V(\Phi-\Delta_3\Phi,\Lambda+2\Delta_3\Lambda,H+2\Delta_3 H)-16V(\Phi-\Delta_3\Phi,\Lambda+2\Delta_3\Lambda,H+\Delta_3 H) \\
& +12V(\Phi-\Delta_3\Phi,\Lambda+2\Delta_3\Lambda,H)-16V(\Phi-\Delta_3\Phi,\Lambda+\Delta_3\Lambda,H+2\Delta_3 H) \\
& +64V(\Phi-\Delta_3\Phi,\Lambda+\Delta_3\Lambda,H+\Delta_3 H)-48V(\Phi-\Delta_3\Phi,\Lambda+\Delta_3\Lambda,H) \\
& +12V(\Phi-\Delta_3\Phi,\Lambda,H+2\Delta_3 H)-48V(\Phi-\Delta_3\Phi,\Lambda,H+\Delta_3 H) \\
& +36V(\Phi-\Delta_3\Phi,\Lambda,H)-3V(\Phi,\Lambda+2\Delta_3\Lambda,H+2\Delta_3 H) \\
& +12V(\Phi,\Lambda+2\Delta_3\Lambda,H+\Delta_3 H)-9V(\Phi,\Lambda+2\Delta_3\Lambda,H) \\
& +12V(\Phi,\Lambda+\Delta_3\Lambda,H+2\Delta_3 H)-48V(\Phi,\Lambda+\Delta_3\Lambda,H+\Delta_3 H) \\
& +36V(\Phi,\Lambda+\Delta_3\Lambda,H)-9V(\Phi,\Lambda,H+2\Delta_3 H) \\
& +36V(\Phi,\Lambda,H+\Delta_3 H)-27V(\Phi,\Lambda,H)\big]
\end{aligned} \tag{56}$$

(6) else if either $\Phi_\mathrm{S}-\Delta_3\Phi<\Phi<\Phi_\mathrm{S}$ or $\Phi_\mathrm{N}-\Delta_3\Phi<\Phi<\Phi_\mathrm{N}$ **and** either $\Lambda_\mathrm{W}<\Lambda<\Lambda_\mathrm{W}+\Delta_3\Lambda$ or $\Lambda_\mathrm{E}<\Lambda<\Lambda_\mathrm{E}+\Delta_3\Lambda$ **and** either $H_\mathrm{B}-\Delta_3 H<H<H_\mathrm{B}$ or $H_\mathrm{T}-\Delta_3 H<H<H_\mathrm{T}$, then based on Eq. (22) in the main text by adopting $u=\Phi$, $v=\Lambda$, and $w=H$ with three

corresponding sign factors of -1, $+1$, and -1 it yields:

$$\begin{aligned}
\frac{\partial^3 V}{\partial\Phi\partial\Lambda\partial H} \approx & \frac{1}{8(\Delta_3\Phi)(\Delta_3\Lambda)(\Delta_3 H)}\big[-V(\Phi-2\Delta_3\Phi,\Lambda+2\Delta_3\Lambda,H-2\Delta_3 H) \\
& +4V(\Phi-2\Delta_3\Phi,\Lambda+2\Delta_3\Lambda,H-\Delta_3 H)-3V(\Phi-2\Delta_3\Phi,\Lambda+2\Delta_3\Lambda,H) \\
& +4V(\Phi-2\Delta_3\Phi,\Lambda+\Delta_3\Lambda,H-2\Delta_3 H)-16V(\Phi-2\Delta_3\Phi,\Lambda+\Delta_3\Lambda,H-\Delta_3 H) \\
& +12V(\Phi-2\Delta_3\Phi,\Lambda+\Delta_3\Lambda,H)-3V(\Phi-2\Delta_3\Phi,\Lambda,H-2\Delta_3 H) \\
& +12V(\Phi-2\Delta_3\Phi,\Lambda,H-\Delta_3 H)-9V(\Phi-2\Delta_3\Phi,\Lambda,H) \\
& +4V(\Phi-\Delta_3\Phi,\Lambda+2\Delta_3\Lambda,H-2\Delta_3 H)-16V(\Phi-\Delta_3\Phi,\Lambda+2\Delta_3\Lambda,H-\Delta_3 H) \\
& +12V(\Phi-\Delta_3\Phi,\Lambda+2\Delta_3\Lambda,H)-16V(\Phi-\Delta_3\Phi,\Lambda+\Delta_3\Lambda,H-2\Delta_3 H) \\
& +64V(\Phi-\Delta_3\Phi,\Lambda+\Delta_3\Lambda,H-\Delta_3 H)-48V(\Phi-\Delta_3\Phi,\Lambda+\Delta_3\Lambda,H) \\
& +12V(\Phi-\Delta_3\Phi,\Lambda,H-2\Delta_3 H)-48V(\Phi-\Delta_3\Phi,\Lambda,H-\Delta_3 H) \\
& +36V(\Phi-\Delta_3\Phi,\Lambda,H)-3V(\Phi,\Lambda+2\Delta_3\Lambda,H-2\Delta_3 H) \\
& +12V(\Phi,\Lambda+2\Delta_3\Lambda,H-\Delta_3 H)-9V(\Phi,\Lambda+2\Delta_3\Lambda,H) \\
& +12V(\Phi,\Lambda+\Delta_3\Lambda,H-2\Delta_3 H)-48V(\Phi,\Lambda+\Delta_3\Lambda,H-\Delta_3 H) \\
& +36V(\Phi,\Lambda+\Delta_3\Lambda,H)-9V(\Phi,\Lambda,H-2\Delta_3 H) \\
& +36V(\Phi,\Lambda,H-\Delta_3 H)-27V(\Phi,\Lambda,H)\big]
\end{aligned} \tag{57}$$

(7) else if either $\Phi_S-\Delta_3\Phi<\Phi<\Phi_S$ or $\Phi_N-\Delta_3\Phi<\Phi<\Phi_N$ **and** either $\Lambda_W-\Delta_3\Lambda<\Lambda<\Lambda_W$ or $\Lambda_E-\Delta_3\Lambda<\Lambda<\Lambda_E$ **and** either $H_B<H<H_B+\Delta_3 H$ or $H_T<H<H_T+\Delta_3 H$, then based on Eq. (22) in the main text by adopting $u=\Phi$, $v=\Lambda$, and $w=H$ with three corresponding sign factors of -1, -1, and $+1$ it yields:

$$\begin{aligned}
\frac{\partial^3 V}{\partial\Phi\partial\Lambda\partial H} \approx & \frac{1}{8(\Delta_3\Phi)(\Delta_3\Lambda)(\Delta_3 H)}\big[-V(\Phi-2\Delta_3\Phi,\Lambda-2\Delta_3\Lambda,H+2\Delta_3 H) \\
& +4V(\Phi-2\Delta_3\Phi,\Lambda-2\Delta_3\Lambda,H+\Delta_3 H)-3V(\Phi-2\Delta_3\Phi,\Lambda-2\Delta_3\Lambda,H) \\
& +4V(\Phi-2\Delta_3\Phi,\Lambda-\Delta_3\Lambda,H+2\Delta_3 H)-16V(\Phi-2\Delta_3\Phi,\Lambda-\Delta_3\Lambda,H+\Delta_3 H) \\
& +12V(\Phi-2\Delta_3\Phi,\Lambda-\Delta_3\Lambda,H)-3V(\Phi-2\Delta_3\Phi,\Lambda,H+2\Delta_3 H) \\
& +12V(\Phi-2\Delta_3\Phi,\Lambda,H+\Delta_3 H)-9V(\Phi-2\Delta_3\Phi,\Lambda,H) \\
& +4V(\Phi-\Delta_3\Phi,\Lambda-2\Delta_3\Lambda,H+2\Delta_3 H)-16V(\Phi-\Delta_3\Phi,\Lambda-2\Delta_3\Lambda,H+\Delta_3 H) \\
& +12V(\Phi-\Delta_3\Phi,\Lambda-2\Delta_3\Lambda,H)-16V(\Phi-\Delta_3\Phi,\Lambda-\Delta_3\Lambda,H+2\Delta_3 H) \\
& +64V(\Phi-\Delta_3\Phi,\Lambda-\Delta_3\Lambda,H+\Delta_3 H)-48V(\Phi-\Delta_3\Phi,\Lambda-\Delta_3\Lambda,H) \\
& +12V(\Phi-\Delta_3\Phi,\Lambda,H+2\Delta_3 H)-48V(\Phi-\Delta_3\Phi,\Lambda,H+\Delta_3 H) \\
& +36V(\Phi-\Delta_3\Phi,\Lambda,H)-3V(\Phi,\Lambda-2\Delta_3\Lambda,H+2\Delta_3 H) \\
& +12V(\Phi,\Lambda-2\Delta_3\Lambda,H+\Delta_3 H)-9V(\Phi,\Lambda-2\Delta_3\Lambda,H) \\
& +12V(\Phi,\Lambda-\Delta_3\Lambda,H+2\Delta_3 H)-48V(\Phi,\Lambda-\Delta_3\Lambda,H+\Delta_3 H) \\
& +36V(\Phi,\Lambda-\Delta_3\Lambda,H)-9V(\Phi,\Lambda,H+2\Delta_3 H) \\
& +36V(\Phi,\Lambda,H+\Delta_3 H)-27V(\Phi,\Lambda,H)\big]
\end{aligned} \tag{58}$$

(8) else if either $\Phi_S-\Delta_3\Phi<\Phi<\Phi_S$ or $\Phi_N-\Delta_3\Phi<\Phi<\Phi_N$ **and** either $\Lambda_W-\Delta_3\Lambda<\Lambda<\Lambda_W$ or $\Lambda_E-\Delta_3\Lambda<\Lambda<\Lambda_E$ **and** either $H_B-\Delta_3 H<H<H_B$ or $H_T-\Delta_3 H<H<H_T$, then based on Eq. (22) in the main text by adopting $u=\Phi$, $v=\Lambda$, and $w=H$ with three

corresponding sign factors of -1, -1, and -1 it yields:

$$\begin{aligned}\frac{\partial^3 V}{\partial\Phi\partial\Lambda\partial H} \approx &\frac{-1}{8(\Delta_3\Phi)(\Delta_3\Lambda)(\Delta_3 H)}\big[-V(\Phi-2\Delta_3\Phi,\Lambda-2\Delta_3\Lambda,H-2\Delta_3 H)\\
&+4V(\Phi-2\Delta_3\Phi,\Lambda-2\Delta_3\Lambda,H-\Delta_3 H)-3V(\Phi-2\Delta_3\Phi,\Lambda-2\Delta_3\Lambda,H)\\
&+4V(\Phi-2\Delta_3\Phi,\Lambda-\Delta_3\Lambda,H-2\Delta_3 H)-16V(\Phi-2\Delta_3\Phi,\Lambda-\Delta_3\Lambda,H-\Delta_3 H)\\
&+12V(\Phi-2\Delta_3\Phi,\Lambda-\Delta_3\Lambda,H)-3V(\Phi-2\Delta_3\Phi,\Lambda,H-2\Delta_3 H)\\
&+12V(\Phi-2\Delta_3\Phi,\Lambda,H-\Delta_3 H)-9V(\Phi-2\Delta_3\Phi,\Lambda,H)\\
&+4V(\Phi-\Delta_3\Phi,\Lambda-2\Delta_3\Lambda,H-2\Delta_3 H)-16V(\Phi-\Delta_3\Phi,\Lambda-2\Delta_3\Lambda,H-\Delta_3 H)\\
&+12V(\Phi-\Delta_3\Phi,\Lambda-2\Delta_3\Lambda,H)-16V(\Phi-\Delta_3\Phi,\Lambda-\Delta_3\Lambda,H-2\Delta_3 H)\\
&+64V(\Phi-\Delta_3\Phi,\Lambda-\Delta_3\Lambda,H-\Delta_3 H)-48V(\Phi-\Delta_3\Phi,\Lambda-\Delta_3\Lambda,H)\\
&+12V(\Phi-\Delta_3\Phi,\Lambda,H-2\Delta_3 H)-48V(\Phi-\Delta_3\Phi,\Lambda,H-\Delta_3 H)\\
&+36V(\Phi-\Delta_3\Phi,\Lambda,H)-3V(\Phi,\Lambda-2\Delta_3\Lambda,H-2\Delta_3 H)\\
&+12V(\Phi,\Lambda-2\Delta_3\Lambda,H-\Delta_3 H)-9V(\Phi,\Lambda-2\Delta_3\Lambda,H)\\
&+12V(\Phi,\Lambda-\Delta_3\Lambda,H-2\Delta_3 H)-48V(\Phi,\Lambda-\Delta_3\Lambda,H-\Delta_3 H)\\
&+36V(\Phi,\Lambda-\Delta_3\Lambda,H)-9V(\Phi,\Lambda,H-2\Delta_3 H)\\
&+36V(\Phi,\Lambda,H-\Delta_3 H)-27V(\Phi,\Lambda,H)\big]
\end{aligned} \tag{59}$$

(9) else if either $\Phi < \Phi_S - \Delta_3\Phi$ or $\Phi_S + \Delta_3\Phi < \Phi < \Phi_N - \Delta_3\Phi$ or $\Phi_N + \Delta_3\Phi < \Phi$ **and** either $\Lambda_W < \Lambda < \Lambda_W + \Delta_3\Lambda$ or $\Lambda_E < \Lambda < \Lambda_E + \Delta_3\Lambda$ **and** either $H_B < H < H_B + \Delta_3 H$ or $H_T < H < H_T + \Delta_3 H$, then based on Eq. (21) in the main text by adopting $u = \Phi$, $v = \Lambda$, and $w = H$ with two corresponding sign factors of $+1$ and $+1$ for $v = \Lambda$ and $w = H$ it yields:

$$\begin{aligned}\frac{\partial^3 V}{\partial\Phi\partial\Lambda\partial H} \approx &\frac{1}{8(\Delta_3\Phi)(\Delta_3\Lambda)(\Delta_3 H)}\big[V(\Phi+\Delta_3\Phi,\Lambda+2\Delta_3\Lambda,H+2\Delta_3 H)\\
&-4V(\Phi+\Delta_3\Phi,\Lambda+2\Delta_3\Lambda,H+\Delta_3 H)+3V(\Phi+\Delta_3\Phi,\Lambda+2\Delta_3\Lambda,H)\\
&-4V(\Phi+\Delta_3\Phi,\Lambda+\Delta_3\Lambda,H+2\Delta_3 H)+16V(\Phi+\Delta_3\Phi,\Lambda+\Delta_3\Lambda,H+\Delta_3 H)\\
&-12V(\Phi+\Delta_3\Phi,\Lambda+\Delta_3\Lambda,H)+3V(\Phi+\Delta_3\Phi,\Lambda,H+2\Delta_3 H)\\
&-12V(\Phi+\Delta_3\Phi,\Lambda,H+\Delta_3 H)+9V(\Phi+\Delta_3\Phi,\Lambda,H)\\
&-V(\Phi-\Delta_3\Phi,\Lambda+2\Delta_3\Lambda,H+2\Delta_3 H)+4V(\Phi-\Delta_3\Phi,\Lambda+2\Delta_3\Lambda,H+\Delta_3 H)\\
&-3V(\Phi-\Delta_3\Phi,\Lambda+2\Delta_3\Lambda,H)+4V(\Phi-\Delta_3\Phi,\Lambda+\Delta_3\Lambda,H+2\Delta_3 H)\\
&-16V(\Phi-\Delta_3\Phi,\Lambda+\Delta_3\Lambda,H+\Delta_3 H)+12V(\Phi-\Delta_3\Phi,\Lambda+\Delta_3\Lambda,H)\\
&-3V(\Phi-\Delta_3\Phi,\Lambda,H+2\Delta_3 H)+12V(\Phi-\Delta_3\Phi,\Lambda,H+\Delta_3 H)\\
&-9V(\Phi-\Delta_3\Phi,\Lambda,H)\big]
\end{aligned} \tag{60}$$

(10) else if either $\Phi < \Phi_S - \Delta_3\Phi$ or $\Phi_S + \Delta_3\Phi < \Phi < \Phi_N - \Delta_3\Phi$ or $\Phi_N + \Delta_3\Phi < \Phi$ **and** either $\Lambda_W < \Lambda < \Lambda_W + \Delta_3\Lambda$ or $\Lambda_E < \Lambda < \Lambda_E + \Delta_3\Lambda$ **and** either $H_B - \Delta_3 H < H < H_B$ or $H_T - \Delta_3 H < H < H_T$, then based on Eq. (21) in the main text by adopting $u = \Phi$, $v = \Lambda$, and $w = H$ with two corresponding sign factors of $+1$ and -1 for $v = \Lambda$ and

$w = H$ it yields:

$$\frac{\partial^3 V}{\partial\Phi\partial\Lambda\partial H} \approx \frac{-1}{8(\Delta_3\Phi)(\Delta_3\Lambda)(\Delta_3 H)}\big[V(\Phi+\Delta_3\Phi,\Lambda+2\Delta_3\Lambda,H-2\Delta_3 H)$$
$$-4V(\Phi+\Delta_3\Phi,\Lambda+2\Delta_3\Lambda,H-\Delta_3 H)+3V(\Phi+\Delta_3\Phi,\Lambda+2\Delta_3\Lambda,H)$$
$$-4V(\Phi+\Delta_3\Phi,\Lambda+\Delta_3\Lambda,H-2\Delta_3 H)+16V(\Phi+\Delta_3\Phi,\Lambda+\Delta_3\Lambda,H-\Delta_3 H)$$
$$-12V(\Phi+\Delta_3\Phi,\Lambda+\Delta_3\Lambda,H)+3V(\Phi+\Delta_3\Phi,\Lambda,H-2\Delta_3 H)$$
$$-12V(\Phi+\Delta_3\Phi,\Lambda,H-\Delta_3 H)+9V(\Phi+\Delta_3\Phi,\Lambda,H)$$
$$-V(\Phi-\Delta_3\Phi,\Lambda+2\Delta_3\Lambda,H-2\Delta_3 H)+4V(\Phi-\Delta_3\Phi,\Lambda+2\Delta_3\Lambda,H-\Delta_3 H)$$
$$-3V(\Phi-\Delta_3\Phi,\Lambda+2\Delta_3\Lambda,H)+4V(\Phi-\Delta_3\Phi,\Lambda+\Delta_3\Lambda,H-2\Delta_3 H)$$
$$-16V(\Phi-\Delta_3\Phi,\Lambda+\Delta_3\Lambda,H-\Delta_3 H)+12V(\Phi-\Delta_3\Phi,\Lambda+\Delta_3\Lambda,H)$$
$$-3V(\Phi-\Delta_3\Phi,\Lambda,H-2\Delta_3 H)+12V(\Phi-\Delta_3\Phi,\Lambda,H-\Delta_3 H)$$
$$-9V(\Phi-\Delta_3\Phi,\Lambda,H)\big] \tag{61}$$

(11) else if either $\Phi < \Phi_S - \Delta_3\Phi$ or $\Phi_S + \Delta_3\Phi < \Phi < \Phi_N - \Delta_3\Phi$ or $\Phi_N + \Delta_3\Phi < \Phi$ **and** either $\Lambda_W - \Delta_3\Lambda < \Lambda < \Lambda_W$ or $\Lambda_E - \Delta_3\Lambda < \Lambda < \Lambda_E$ **and** either $H_B < H < H_B + \Delta_3 H$ or $H_T < H < H_T + \Delta_3 H$, then based on Eq. (21) in the main text by adopting $u = \Phi$, $v = \Lambda$, and $w = H$ with two corresponding sign factors of -1 and $+1$ for $v = \Lambda$ and $w = H$ it yields:

$$\frac{\partial^3 V}{\partial\Phi\partial\Lambda\partial H} \approx \frac{-1}{8(\Delta_3\Phi)(\Delta_3\Lambda)(\Delta_3 H)}\big[V(\Phi+\Delta_3\Phi,\Lambda-2\Delta_3\Lambda,H+2\Delta_3 H)$$
$$-4V(\Phi+\Delta_3\Phi,\Lambda-2\Delta_3\Lambda,H+\Delta_3 H)+3V(\Phi+\Delta_3\Phi,\Lambda-2\Delta_3\Lambda,H)$$
$$-4V(\Phi+\Delta_3\Phi,\Lambda-\Delta_3\Lambda,H+2\Delta_3 H)+16V(\Phi+\Delta_3\Phi,\Lambda-\Delta_3\Lambda,H+\Delta_3 H)$$
$$-12V(\Phi+\Delta_3\Phi,\Lambda-\Delta_3\Lambda,H)+3V(\Phi+\Delta_3\Phi,\Lambda,H+2\Delta_3 H)$$
$$-12V(\Phi+\Delta_3\Phi,\Lambda,H+\Delta_3 H)+9V(\Phi+\Delta_3\Phi,\Lambda,H)$$
$$-V(\Phi-\Delta_3\Phi,\Lambda-2\Delta_3\Lambda,H+2\Delta_3 H)+4V(\Phi-\Delta_3\Phi,\Lambda-2\Delta_3\Lambda,H+\Delta_3 H)$$
$$-3V(\Phi-\Delta_3\Phi,\Lambda-2\Delta_3\Lambda,H)+4V(\Phi-\Delta_3\Phi,\Lambda-\Delta_3\Lambda,H+2\Delta_3 H)$$
$$-16V(\Phi-\Delta_3\Phi,\Lambda-\Delta_3\Lambda,H+\Delta_3 H)+12V(\Phi-\Delta_3\Phi,\Lambda-\Delta_3\Lambda,H)$$
$$-3V(\Phi-\Delta_3\Phi,\Lambda,H+2\Delta_3 H)+12V(\Phi-\Delta_3\Phi,\Lambda,H+\Delta_3 H)$$
$$-9V(\Phi-\Delta_3\Phi,\Lambda,H)\big] \tag{62}$$

(12) else if either $\Phi < \Phi_S - \Delta_3\Phi$ or $\Phi_S + \Delta_3\Phi < \Phi < \Phi_N - \Delta_3\Phi$ or $\Phi_N + \Delta_3\Phi < \Phi$ **and** either $\Lambda_W - \Delta_3\Lambda < \Lambda < \Lambda_W$ or $\Lambda_E - \Delta_3\Lambda < \Lambda < \Lambda_E$ **and** either $H_B - \Delta_3 H < H < H_B$ or $H_T - \Delta_3 H < H < H_T$, then based on Eq. (21) in the main text by adopting $u = \Phi$, $v = \Lambda$, and $w = H$ with two corresponding sign factors of -1 and -1 for $v = \Lambda$ and

$w = H$ it yields:

$$\begin{aligned}
\frac{\partial^3 V}{\partial\Phi\partial\Lambda\partial H} \approx & \frac{1}{8(\Delta_3\Phi)(\Delta_3\Lambda)(\Delta_3 H)} \big[V(\Phi+\Delta_3\Phi,\Lambda-2\Delta_3\Lambda,H-2\Delta_3 H) \\
& -4V(\Phi+\Delta_3\Phi,\Lambda-2\Delta_3\Lambda,H-\Delta_3 H)+3V(\Phi+\Delta_3\Phi,\Lambda-2\Delta_3\Lambda,H) \\
& -4V(\Phi+\Delta_3\Phi,\Lambda-\Delta_3\Lambda,H-2\Delta_3 H)+16V(\Phi+\Delta_3\Phi,\Lambda-\Delta_3\Lambda,H-\Delta_3 H) \\
& -12V(\Phi+\Delta_3\Phi,\Lambda-\Delta_3\Lambda,H)+3V(\Phi+\Delta_3\Phi,\Lambda,H-2\Delta_3 H) \\
& -12V(\Phi+\Delta_3\Phi,\Lambda,H-\Delta_3 H)+9V(\Phi+\Delta_3\Phi,\Lambda,H) \\
& -V(\Phi-\Delta_3\Phi,\Lambda-2\Delta_3\Lambda,H-2\Delta_3 H)+4V(\Phi-\Delta_3\Phi,\Lambda-2\Delta_3\Lambda,H-\Delta_3 H) \\
& -3V(\Phi-\Delta_3\Phi,\Lambda-2\Delta_3\Lambda,H)+4V(\Phi-\Delta_3\Phi,\Lambda-\Delta_3\Lambda,H-2\Delta_3 H) \\
& -16V(\Phi-\Delta_3\Phi,\Lambda-\Delta_3\Lambda,H-\Delta_3 H)+12V(\Phi-\Delta_3\Phi,\Lambda-\Delta_3\Lambda,H) \\
& -3V(\Phi-\Delta_3\Phi,\Lambda,H-2\Delta_3 H)+12V(\Phi-\Delta_3\Phi,\Lambda,H-\Delta_3 H) \\
& -9V(\Phi-\Delta_3\Phi,\Lambda,H)\big]
\end{aligned} \tag{63}$$

(13) else if either $\Phi_S < \Phi < \Phi_S + \Delta_3\Phi$ or $\Phi_N < \Phi < \Phi_N + \Delta_3\Phi$ **and** either $\Lambda < \Lambda_W - \Delta_3\Lambda$ or $\Lambda_W + \Delta_3\Lambda < \Lambda < \Lambda_E - \Delta_3\Lambda$ or $\Lambda_E + \Delta_3\Lambda < \Lambda$ **and** either $H_B < H < H_B + \Delta_3 H$ or $H_T < H < H_T + \Delta_3 H$, then based on Eq. (21) in the main text by adopting $u = \Lambda$, $v = \Phi$, and $w = H$ with two corresponding sign factors of $+1$ and $+1$ for $v = \Phi$ and $w = H$ it yields:

$$\begin{aligned}
\frac{\partial^3 V}{\partial\Phi\partial\Lambda\partial H} \approx & \frac{1}{8(\Delta_3\Phi)(\Delta_3\Lambda)(\Delta_3 H)} \big[V(\Phi+2\Delta_3\Phi,\Lambda+\Delta_3\Lambda,H+2\Delta_3 H) \\
& -4V(\Phi+2\Delta_3\Phi,\Lambda+\Delta_3\Lambda,H+\Delta_3 H)+3V(\Phi+2\Delta_3\Phi,\Lambda+\Delta_3\Lambda,H) \\
& -4V(\Phi+\Delta_3\Phi,\Lambda+\Delta_3\Lambda,H+2\Delta_3 H)+16V(\Phi+\Delta_3\Phi,\Lambda+\Delta_3\Lambda,H+\Delta_3 H) \\
& -12V(\Phi+\Delta_3\Phi,\Lambda+\Delta_3\Lambda,H)+3V(\Phi,\Lambda+\Delta_3\Lambda,H+2\Delta_3 H) \\
& -12V(\Phi,\Lambda+\Delta_3\Lambda,H+\Delta_3 H)+9V(\Phi,\Lambda+\Delta_3\Lambda,H) \\
& -V(\Phi+2\Delta_3\Phi,\Lambda-\Delta_3\Lambda,H+2\Delta_3 H)+4V(\Phi+2\Delta_3\Phi,\Lambda-\Delta_3\Lambda,H+\Delta_3 H) \\
& -3V(\Phi+2\Delta_3\Phi,\Lambda-\Delta_3\Lambda,H)+4V(\Phi+\Delta_3\Phi,\Lambda-\Delta_3\Lambda,H+2\Delta_3 H) \\
& -16V(\Phi+\Delta_3\Phi,\Lambda-\Delta_3\Lambda,H+\Delta_3 H)+12V(\Phi+\Delta_3\Phi,\Lambda-\Delta_3\Lambda,H) \\
& -3V(\Phi,\Lambda-\Delta_3\Lambda,H+2\Delta_3 H)+12V(\Phi,\Lambda-\Delta_3\Lambda,H+\Delta_3 H) \\
& -9V(\Phi,\Lambda-\Delta_3\Lambda,H)\big]
\end{aligned} \tag{64}$$

(14) else if either $\Phi_S < \Phi < \Phi_S + \Delta_3\Phi$ or $\Phi_N < \Phi < \Phi_N + \Delta_3\Phi$ **and** either $\Lambda < \Lambda_W - \Delta_3\Lambda$ or $\Lambda_W + \Delta_3\Lambda < \Lambda < \Lambda_E - \Delta_3\Lambda$ or $\Lambda_E + \Delta_3\Lambda < \Lambda$ **and** either $H_B - \Delta_3 H < H < H_B$ or $H_T - \Delta_3 H < H < H_T$, then based on Eq. (21) in the main text by adopting $u = \Lambda$, $v = \Phi$, and $w = H$ with two corresponding sign factors of $+1$ and -1 for $v = \Phi$ and $w = H$ it

yields:

$$
\begin{aligned}
\frac{\partial^3 V}{\partial\Phi\partial\Lambda\partial H} \approx & \frac{-1}{8(\Delta_3\Phi)(\Delta_3\Lambda)(\Delta_3 H)}\big[V(\Phi+2\Delta_3\Phi,\Lambda+\Delta_3\Lambda,H-2\Delta_3 H) \\
& -4V(\Phi+2\Delta_3\Phi,\Lambda+\Delta_3\Lambda,H-\Delta_3 H)+3V(\Phi+2\Delta_3\Phi,\Lambda+\Delta_3\Lambda,H) \\
& -4V(\Phi+\Delta_3\Phi,\Lambda+\Delta_3\Lambda,H-2\Delta_3 H)+16V(\Phi+\Delta_3\Phi,\Lambda+\Delta_3\Lambda,H-\Delta_3 H) \\
& -12V(\Phi+\Delta_3\Phi,\Lambda+\Delta_3\Lambda,H)+3V(\Phi,\Lambda+\Delta_3\Lambda,H-2\Delta_3 H) \\
& -12V(\Phi,\Lambda+\Delta_3\Lambda,H-\Delta_3 H)+9V(\Phi,\Lambda+\Delta_3\Lambda,H) \\
& -V(\Phi+2\Delta_3\Phi,\Lambda-\Delta_3\Lambda,H-2\Delta_3 H)+4V(\Phi+2\Delta_3\Phi,\Lambda-\Delta_3\Lambda,H-\Delta_3 H) \\
& -3V(\Phi+2\Delta_3\Phi,\Lambda-\Delta_3\Lambda,H)+4V(\Phi+\Delta_3\Phi,\Lambda-\Delta_3\Lambda,H-2\Delta_3 H) \\
& -16V(\Phi+\Delta_3\Phi,\Lambda-\Delta_3\Lambda,H-\Delta_3 H)+12V(\Phi+\Delta_3\Phi,\Lambda-\Delta_3\Lambda,H) \\
& -3V(\Phi,\Lambda-\Delta_3\Lambda,H-2\Delta_3 H)+12V(\Phi,\Lambda-\Delta_3\Lambda,H-\Delta_3 H) \\
& -9V(\Phi,\Lambda-\Delta_3\Lambda,H)\big]
\end{aligned}
\tag{65}
$$

(15) else if either $\Phi_S-\Delta_3\Phi<\Phi<\Phi_S$ or $\Phi_N-\Delta_3\Phi<\Phi<\Phi_N$ **and** either $\Lambda<\Lambda_W-\Delta_3\Lambda$ or $\Lambda_W+\Delta_3\Lambda<\Lambda<\Lambda_E-\Delta_3\Lambda$ or $\Lambda_E+\Delta_3\Lambda<\Lambda$ **and** either $H_B<H<H_B+\Delta_3 H$ or $H_T<H<H_T+\Delta_3 H$, then based on Eq. (21) in the main text by adopting $u=\Lambda$, $v=\Phi$, and $w=H$ with two corresponding sign factors of -1 and $+1$ for $v=\Phi$ and $w=H$ it yields:

$$
\begin{aligned}
\frac{\partial^3 V}{\partial\Phi\partial\Lambda\partial H} \approx & \frac{-1}{8(\Delta_3\Phi)(\Delta_3\Lambda)(\Delta_3 H)}\big[V(\Phi-2\Delta_3\Phi,\Lambda+\Delta_3\Lambda,H+2\Delta_3 H) \\
& -4V(\Phi-2\Delta_3\Phi,\Lambda+\Delta_3\Lambda,H+\Delta_3 H)+3V(\Phi-2\Delta_3\Phi,\Lambda+\Delta_3\Lambda,H) \\
& -4V(\Phi-\Delta_3\Phi,\Lambda+\Delta_3\Lambda,H+2\Delta_3 H)+16V(\Phi-\Delta_3\Phi,\Lambda+\Delta_3\Lambda,H+\Delta_3 H) \\
& -12V(\Phi-\Delta_3\Phi,\Lambda+\Delta_3\Lambda,H)+3V(\Phi,\Lambda+\Delta_3\Lambda,H+2\Delta_3 H) \\
& -12V(\Phi,\Lambda+\Delta_3\Lambda,H+\Delta_3 H)+9V(\Phi,\Lambda+\Delta_3\Lambda,H) \\
& -V(\Phi-2\Delta_3\Phi,\Lambda-\Delta_3\Lambda,H+2\Delta_3 H)+4V(\Phi-2\Delta_3\Phi,\Lambda-\Delta_3\Lambda,H+\Delta_3 H) \\
& -3V(\Phi-2\Delta_3\Phi,\Lambda-\Delta_3\Lambda,H)+4V(\Phi-\Delta_3\Phi,\Lambda-\Delta_3\Lambda,H+2\Delta_3 H) \\
& -16V(\Phi-\Delta_3\Phi,\Lambda-\Delta_3\Lambda,H+\Delta_3 H)+12V(\Phi-\Delta_3\Phi,\Lambda-\Delta_3\Lambda,H) \\
& -3V(\Phi,\Lambda-\Delta_3\Lambda,H+2\Delta_3 H)+12V(\Phi,\Lambda-\Delta_3\Lambda,H+\Delta_3 H) \\
& -9V(\Phi,\Lambda-\Delta_3\Lambda,H)\big]
\end{aligned}
\tag{66}
$$

(16) else if either $\Phi_S-\Delta_3\Phi<\Phi<\Phi_S$ or $\Phi_N-\Delta_3\Phi<\Phi<\Phi_N$ **and** either $\Lambda<\Lambda_W-\Delta_3\Lambda$ or $\Lambda_W+\Delta_3\Lambda<\Lambda<\Lambda_E-\Delta_3\Lambda$ or $\Lambda_E+\Delta_3\Lambda<\Lambda$ **and** either $H_B-\Delta_3 H<H<H_B$ or $H_T-\Delta_3 H<H<H_T$, then based on Eq. (21) in the main text by adopting $u=\Lambda$, $v=\Phi$, and $w=H$ with two corresponding sign factors of -1 and -1 for $v=\Phi$ and $w=H$ it

yields:

$$
\begin{aligned}
\frac{\partial^3 V}{\partial\Phi\partial\Lambda\partial H} \approx & \frac{1}{8(\Delta_3\Phi)(\Delta_3\Lambda)(\Delta_3 H)}\big[V(\Phi-2\Delta_3\Phi,\Lambda+\Delta_3\Lambda,H-2\Delta_3 H)\\
&-4V(\Phi-2\Delta_3\Phi,\Lambda+\Delta_3\Lambda,H-\Delta_3 H)+3V(\Phi-2\Delta_3\Phi,\Lambda+\Delta_3\Lambda,H)\\
&-4V(\Phi-\Delta_3\Phi,\Lambda+\Delta_3\Lambda,H-2\Delta_3 H)+16V(\Phi-\Delta_3\Phi,\Lambda+\Delta_3\Lambda,H-\Delta_3 H)\\
&-12V(\Phi-\Delta_3\Phi,\Lambda+\Delta_3\Lambda,H)+3V(\Phi,\Lambda+\Delta_3\Lambda,H-2\Delta_3 H)\\
&-12V(\Phi,\Lambda+\Delta_3\Lambda,H-\Delta_3 H)+9V(\Phi,\Lambda+\Delta_3\Lambda,H)\\
&-V(\Phi-2\Delta_3\Phi,\Lambda-\Delta_3\Lambda,H-2\Delta_3 H)+4V(\Phi-2\Delta_3\Phi,\Lambda-\Delta_3\Lambda,H-\Delta_3 H)\\
&-3V(\Phi-2\Delta_3\Phi,\Lambda-\Delta_3\Lambda,H)+4V(\Phi-\Delta_3\Phi,\Lambda-\Delta_3\Lambda,H-2\Delta_3 H)\\
&-16V(\Phi-\Delta_3\Phi,\Lambda-\Delta_3\Lambda,H-\Delta_3 H)+12V(\Phi-\Delta_3\Phi,\Lambda-\Delta_3\Lambda,H)\\
&-3V(\Phi,\Lambda-\Delta_3\Lambda,H-2\Delta_3 H)+12V(\Phi,\Lambda-\Delta_3\Lambda,H-\Delta_3 H)\\
&-9V(\Phi,\Lambda-\Delta_3\Lambda,H)\big]
\end{aligned} \tag{67}
$$

(17) else if either $\Phi_S < \Phi < \Phi_S + \Delta_3\Phi$ or $\Phi_N < \Phi < \Phi_N + \Delta_3\Phi$ **and** either $\Lambda_W < \Lambda < \Lambda_W + \Delta_3\Lambda$ or $\Lambda_E < \Lambda < \Lambda_E + \Delta_3\Lambda$ **and** either $H < H_B - \Delta_3 H$ or $H_B + \Delta_3 H < H < H_T - \Delta_3 H$ or $H_T + \Delta_3 H < H$, then based on Eq. (21) in the main text by adopting $u = H$, $v = \Phi$, and $w = \Lambda$ with two corresponding sign factors of $+1$ and $+1$ for $v = \Phi$ and $w = \Lambda$ it yields:

$$
\begin{aligned}
\frac{\partial^3 V}{\partial\Phi\partial\Lambda\partial H} \approx & \frac{1}{8(\Delta_3\Phi)(\Delta_3\Lambda)(\Delta_3 H)}\big[V(\Phi+2\Delta_3\Phi,\Lambda+2\Delta_3\Lambda,H+\Delta_3 H)\\
&-4V(\Phi+2\Delta_3\Phi,\Lambda+\Delta_3\Lambda,H+\Delta_3 H)+3V(\Phi+2\Delta_3\Phi,\Lambda,H+\Delta_3 H)\\
&-4V(\Phi+\Delta_3\Phi,\Lambda+2\Delta_3\Lambda,H+\Delta_3 H)+16V(\Phi+\Delta_3\Phi,\Lambda+\Delta_3\Lambda,H+\Delta_3 H)\\
&-12V(\Phi+\Delta_3\Phi,\Lambda,H+\Delta_3 H)+3V(\Phi,\Lambda+2\Delta_3\Lambda,H+\Delta_3 H)\\
&-12V(\Phi,\Lambda+\Delta_3\Lambda,H+\Delta_3 H)+9V(\Phi,\Lambda,H+\Delta_3 H)\\
&-V(\Phi+2\Delta_3\Phi,\Lambda+2\Delta_3\Lambda,H-\Delta_3 H)+4V(\Phi+2\Delta_3\Phi,\Lambda+\Delta_3\Lambda,H-\Delta_3 H)\\
&-3V(\Phi+2\Delta_3\Phi,\Lambda,H-\Delta_3 H)+4V(\Phi+\Delta_3\Phi,\Lambda+2\Delta_3\Lambda,H-\Delta_3 H)\\
&-16V(\Phi+\Delta_3\Phi,\Lambda+\Delta_3\Lambda,H-\Delta_3 H)+12V(\Phi+\Delta_3\Phi,\Lambda,H-\Delta_3 H)\\
&-3V(\Phi,\Lambda+2\Delta_3\Lambda,H-\Delta_3 H)+12V(\Phi,\Lambda+\Delta_3\Lambda,H-\Delta_3 H)\\
&-9V(\Phi,\Lambda,H-\Delta_3 H)\big]
\end{aligned} \tag{68}
$$

(18) else if either $\Phi_S < \Phi < \Phi_S + \Delta_3\Phi$ or $\Phi_N < \Phi < \Phi_N + \Delta_3\Phi$ **and** either $\Lambda_W - \Delta_3\Lambda < \Lambda < \Lambda_W$ or $\Lambda_E - \Delta_3\Lambda < \Lambda < \Lambda_E$ **and** either $H < H_B - \Delta_3 H$ or $H_B + \Delta_3 H < H < H_T - \Delta_3 H$ or $H_T + \Delta_3 H < H$, then based on Eq. (21) in the main text by adopting $u = H$, $v = \Phi$, and $w = \Lambda$ with two corresponding sign factors of $+1$ and -1 for $v = \Phi$ and $w = \Lambda$ it

yields:

$$\frac{\partial^3 V}{\partial\Phi\partial\Lambda\partial H} \approx \frac{-1}{8(\Delta_3\Phi)(\Delta_3\Lambda)(\Delta_3 H)} \big[V(\Phi+2\Delta_3\Phi,\Lambda-2\Delta_3\Lambda,H+\Delta_3 H)$$
$$-4V(\Phi+2\Delta_3\Phi,\Lambda-\Delta_3\Lambda,H+\Delta_3 H)+3V(\Phi+2\Delta_3\Phi,\Lambda,H+\Delta_3 H)$$
$$-4V(\Phi+\Delta_3\Phi,\Lambda-2\Delta_3\Lambda,H+\Delta_3 H)+16V(\Phi+\Delta_3\Phi,\Lambda-\Delta_3\Lambda,H+\Delta_3 H)$$
$$-12V(\Phi+\Delta_3\Phi,\Lambda,H+\Delta_3 H)+3V(\Phi,\Lambda-2\Delta_3\Lambda,H+\Delta_3 H)$$
$$-12V(\Phi,\Lambda-\Delta_3\Lambda,H+\Delta_3 H)+9V(\Phi,\Lambda,H+\Delta_3 H)$$
$$-V(\Phi+2\Delta_3\Phi,\Lambda-2\Delta_3\Lambda,H-\Delta_3 H)+4V(\Phi+2\Delta_3\Phi,\Lambda-\Delta_3\Lambda,H-\Delta_3 H)$$
$$-3V(\Phi+2\Delta_3\Phi,\Lambda,H-\Delta_3 H)+4V(\Phi+\Delta_3\Phi,\Lambda-2\Delta_3\Lambda,H-\Delta_3 H)$$
$$-16V(\Phi+\Delta_3\Phi,\Lambda-\Delta_3\Lambda,H-\Delta_3 H)+12V(\Phi+\Delta_3\Phi,\Lambda,H-\Delta_3 H)$$
$$-3V(\Phi,\Lambda-2\Delta_3\Lambda,H-\Delta_3 H)+12V(\Phi,\Lambda-\Delta_3\Lambda,H-\Delta_3 H)$$
$$-9V(\Phi,\Lambda,H-\Delta_3 H)\big] \tag{69}$$

(19) else if either $\Phi_S-\Delta_3\Phi<\Phi<\Phi_S$ or $\Phi_N-\Delta_3\Phi<\Phi<\Phi_N$ **and** either $\Lambda_W<\Lambda<\Lambda_W+\Delta_3\Lambda$ or $\Lambda_E<\Lambda<\Lambda_E+\Delta_3\Lambda$ **and** either $H<H_B-\Delta_3 H$ or $H_B+\Delta_3 H<H<H_T-\Delta_3 H$ or $H_T+\Delta_3 H<H$, then based on Eq. (21) in the main text by adopting $u=H$, $v=\Phi$, and $w=\Lambda$ with two corresponding sign factors of -1 and $+1$ for $v=\Phi$ and $w=\Lambda$ it yields:

$$\frac{\partial^3 V}{\partial\Phi\partial\Lambda\partial H} \approx \frac{-1}{8(\Delta_3\Phi)(\Delta_3\Lambda)(\Delta_3 H)} \big[V(\Phi-2\Delta_3\Phi,\Lambda+2\Delta_3\Lambda,H+\Delta_3 H)$$
$$-4V(\Phi-2\Delta_3\Phi,\Lambda+\Delta_3\Lambda,H+\Delta_3 H)+3V(\Phi-2\Delta_3\Phi,\Lambda,H+\Delta_3 H)$$
$$-4V(\Phi-\Delta_3\Phi,\Lambda+2\Delta_3\Lambda,H+\Delta_3 H)+16V(\Phi-\Delta_3\Phi,\Lambda+\Delta_3\Lambda,H+\Delta_3 H)$$
$$-12V(\Phi-\Delta_3\Phi,\Lambda,H+\Delta_3 H)+3V(\Phi,\Lambda+2\Delta_3\Lambda,H+\Delta_3 H)$$
$$-12V(\Phi,\Lambda+\Delta_3\Lambda,H+\Delta_3 H)+9V(\Phi,\Lambda,H+\Delta_3 H)$$
$$-V(\Phi-2\Delta_3\Phi,\Lambda+2\Delta_3\Lambda,H-\Delta_3 H)+4V(\Phi-2\Delta_3\Phi,\Lambda+\Delta_3\Lambda,H-\Delta_3 H)$$
$$-3V(\Phi-2\Delta_3\Phi,\Lambda,H-\Delta_3 H)+4V(\Phi-\Delta_3\Phi,\Lambda+2\Delta_3\Lambda,H-\Delta_3 H)$$
$$-16V(\Phi-\Delta_3\Phi,\Lambda+\Delta_3\Lambda,H-\Delta_3 H)+12V(\Phi-\Delta_3\Phi,\Lambda,H-\Delta_3 H)$$
$$-3V(\Phi,\Lambda+2\Delta_3\Lambda,H-\Delta_3 H)+12V(\Phi,\Lambda+\Delta_3\Lambda,H-\Delta_3 H)$$
$$-9V(\Phi,\Lambda,H-\Delta_3 H)\big] \tag{70}$$

(20) else if either $\Phi_S-\Delta_3\Phi<\Phi<\Phi_S$ or $\Phi_N-\Delta_3\Phi<\Phi<\Phi_N$ **and** either $\Lambda_W-\Delta_3\Lambda<\Lambda<\Lambda_W$ or $\Lambda_E-\Delta_3\Lambda<\Lambda<\Lambda_E$ **and** either $H<H_B-\Delta_3 H$ or $H_B+\Delta_3 H<H<H_T-\Delta_3 H$ or $H_T+\Delta_3 H<H$, then based on Eq. (21) in the main text by adopting $u=H$, $v=\Phi$, and $w=\Lambda$ with two corresponding sign factors of -1 and -1 for $v=\Phi$ and $w=\Lambda$ it

yields:

$$\frac{\partial^3 V}{\partial\Phi\partial\Lambda\partial H} \approx \frac{1}{8(\Delta_3\Phi)(\Delta_3\Lambda)(\Delta_3 H)}\big[V(\Phi-2\Delta_3\Phi,\Lambda-2\Delta_3\Lambda,H+\Delta_3 H)$$
$$-4V(\Phi-2\Delta_3\Phi,\Lambda-\Delta_3\Lambda,H+\Delta_3 H)+3V(\Phi-2\Delta_3\Phi,\Lambda,H+\Delta_3 H)$$
$$-4V(\Phi-\Delta_3\Phi,\Lambda-2\Delta_3\Lambda,H+\Delta_3 H)+16V(\Phi-\Delta_3\Phi,\Lambda-\Delta_3\Lambda,H+\Delta_3 H)$$
$$-12V(\Phi-\Delta_3\Phi,\Lambda,H+\Delta_3 H)+3V(\Phi,\Lambda-2\Delta_3\Lambda,H+\Delta_3 H)$$
$$-12V(\Phi,\Lambda-\Delta_3\Lambda,H+\Delta_3 H)+9V(\Phi,\Lambda,H+\Delta_3 H)$$
$$-V(\Phi-2\Delta_3\Phi,\Lambda-2\Delta_3\Lambda,H-\Delta_3 H)+4V(\Phi-2\Delta_3\Phi,\Lambda-\Delta_3\Lambda,H-\Delta_3 H)$$
$$-3V(\Phi-2\Delta_3\Phi,\Lambda,H-\Delta_3 H)+4V(\Phi-\Delta_3\Phi,\Lambda-2\Delta_3\Lambda,H-\Delta_3 H)$$
$$-16V(\Phi-\Delta_3\Phi,\Lambda-\Delta_3\Lambda,H-\Delta_3 H)+12V(\Phi-\Delta_3\Phi,\Lambda,H-\Delta_3 H)$$
$$-3V(\Phi,\Lambda-2\Delta_3\Lambda,H-\Delta_3 H)+12V(\Phi,\Lambda-\Delta_3\Lambda,H-\Delta_3 H)$$
$$-9V(\Phi,\Lambda,H-\Delta_3 H)\big] \tag{71}$$

(21) else if either $\Phi < \Phi_S - \Delta_3\Phi$ or $\Phi_S + \Delta_3\Phi < \Phi < \Phi_N - \Delta_3\Phi$ or $\Phi_N + \Delta_3\Phi < \Phi$ **and** either $\Lambda < \Lambda_W - \Delta_3\Lambda$ or $\Lambda_W + \Delta_3\Lambda < \Lambda < \Lambda_E - \Delta_3\Lambda$ or $\Lambda_E + \Delta_3\Lambda < \Lambda$ **and** either $H_B < H < H_B + \Delta_3 H$ or $H_T < H < H_T + \Delta_3 H$, then based on Eq. (20) in the main text by adopting $u = \Phi$, $v = \Lambda$, and $w = H$ with the positive sign for $w = H$ it yields:

$$\frac{\partial^3 V}{\partial\Phi\partial\Lambda\partial H} \approx \frac{1}{8(\Delta_3\Phi)(\Delta_3\Lambda)(\Delta_3 H)}\big[-V(\Phi+\Delta_3\Phi,\Lambda+\Delta_3\Lambda,H+2\Delta_3 H)$$
$$+4V(\Phi+\Delta_3\Phi,\Lambda+\Delta_3\Lambda,H+\Delta_3 H)-3V(\Phi+\Delta_3\Phi,\Lambda+\Delta_3\Lambda,H)$$
$$+V(\Phi+\Delta_3\Phi,\Lambda-\Delta_3\Lambda,H+2\Delta_3 H)-4V(\Phi+\Delta_3\Phi,\Lambda-\Delta_3\Lambda,H+\Delta_3 H)$$
$$+3V(\Phi+\Delta_3\Phi,\Lambda-\Delta_3\Lambda,H)+V(\Phi-\Delta_3\Phi,\Lambda+\Delta_3\Lambda,H+2\Delta_3 H)$$
$$-4V(\Phi-\Delta_3\Phi,\Lambda+\Delta_3\Lambda,H+\Delta_3 H)+3V(\Phi-\Delta_3\Phi,\Lambda+\Delta_3\Lambda,H)$$
$$-V(\Phi-\Delta_3\Phi,\Lambda-\Delta_3\Lambda,H+2\Delta_3 H)+4V(\Phi-\Delta_3\Phi,\Lambda-\Delta_3\Lambda,H+\Delta_3 H)$$
$$-3V(\Phi-\Delta_3\Phi,\Lambda-\Delta_3\Lambda,H)\big] \tag{72}$$

(22) else if either $\Phi < \Phi_S - \Delta_3\Phi$ or $\Phi_S + \Delta_3\Phi < \Phi < \Phi_N - \Delta_3\Phi$ or $\Phi_N + \Delta_3\Phi < \Phi$ **and** either $\Lambda < \Lambda_W - \Delta_3\Lambda$ or $\Lambda_W + \Delta_3\Lambda < \Lambda < \Lambda_E - \Delta_3\Lambda$ or $\Lambda_E + \Delta_3\Lambda < \Lambda$ **and** either $H_B - \Delta_3 H < H < H_B$ or $H_T - \Delta_3 H < H < H_T$, then based on Eq. (20) in the main text by adopting $u = \Phi$, $v = \Lambda$, and $w = H$ with the negative sign for $w = H$ it yields:

$$\frac{\partial^3 V}{\partial\Phi\partial\Lambda\partial H} \approx \frac{-1}{8(\Delta_3\Phi)(\Delta_3\Lambda)(\Delta_3 H)}\big[-V(\Phi+\Delta_3\Phi,\Lambda+\Delta_3\Lambda,H-2\Delta_3 H)$$
$$+4V(\Phi+\Delta_3\Phi,\Lambda+\Delta_3\Lambda,H-\Delta_3 H)-3V(\Phi+\Delta_3\Phi,\Lambda+\Delta_3\Lambda,H)$$
$$+V(\Phi+\Delta_3\Phi,\Lambda-\Delta_3\Lambda,H-2\Delta_3 H)-4V(\Phi+\Delta_3\Phi,\Lambda-\Delta_3\Lambda,H-\Delta_3 H)$$
$$+3V(\Phi+\Delta_3\Phi,\Lambda-\Delta_3\Lambda,H)+V(\Phi-\Delta_3\Phi,\Lambda+\Delta_3\Lambda,H-2\Delta_3 H)$$
$$-4V(\Phi-\Delta_3\Phi,\Lambda+\Delta_3\Lambda,H-\Delta_3 H)+3V(\Phi-\Delta_3\Phi,\Lambda+\Delta_3\Lambda,H)$$
$$-V(\Phi-\Delta_3\Phi,\Lambda-\Delta_3\Lambda,H-2\Delta_3 H)+4V(\Phi-\Delta_3\Phi,\Lambda-\Delta_3\Lambda,H-\Delta_3 H)$$
$$-3V(\Phi-\Delta_3\Phi,\Lambda-\Delta_3\Lambda,H)\big] \tag{73}$$

(23) else if either $\Phi < \Phi_S - \Delta_3\Phi$ or $\Phi_S + \Delta_3\Phi < \Phi < \Phi_N - \Delta_3\Phi$ or $\Phi_N + \Delta_3\Phi < \Phi$ **and** either $\Lambda_W < \Lambda < \Lambda_W + \Delta_3\Lambda$ or $\Lambda_E < \Lambda < \Lambda_E + \Delta_3\Lambda$ **and** either $H < H_B - \Delta_3 H$ or

$H_\text{B}+\Delta_3 H < H < H_\text{T}-\Delta_3 H$ or $H_\text{T}+\Delta_3 H < H$, then based on Eq. (20) in the main text by adopting $u=\Phi$, $v=H$, and $w=\Lambda$ with the positive sign for $w=\Lambda$ it yields:

$$\begin{aligned}\frac{\partial^3 V}{\partial\Phi\partial\Lambda\partial H} \approx \frac{1}{8(\Delta_3\Phi)(\Delta_3\Lambda)(\Delta_3 H)}\big[&-V(\Phi+\Delta_3\Phi,\Lambda+2\Delta_3\Lambda,H+\Delta_3 H)\\
&+4V(\Phi+\Delta_3\Phi,\Lambda+\Delta_3\Lambda,H+\Delta_3 H)-3V(\Phi+\Delta_3\Phi,\Lambda,H+\Delta_3 H)\\
&+V(\Phi+\Delta_3\Phi,\Lambda+2\Delta_3\Lambda,H-\Delta_3 H)-4V(\Phi+\Delta_3\Phi,\Lambda+\Delta_3\Lambda,H-\Delta_3 H)\\
&+3V(\Phi+\Delta_3\Phi,\Lambda,H-\Delta_3 H)+V(\Phi-\Delta_3\Phi,\Lambda+2\Delta_3\Lambda,H+\Delta_3 H)\\
&-4V(\Phi-\Delta_3\Phi,\Lambda+\Delta_3\Lambda,H+\Delta_3 H)+3V(\Phi-\Delta_3\Phi,\Lambda,H+\Delta_3 H)\\
&-V(\Phi-\Delta_3\Phi,\Lambda+2\Delta_3\Lambda,H-\Delta_3 H)+4V(\Phi-\Delta_3\Phi,\Lambda+\Delta_3\Lambda,H-\Delta_3 H)\\
&-3V(\Phi-\Delta_3\Phi,\Lambda,H-\Delta_3 H)\big]\end{aligned} \tag{74}$$

(24) else if either $\Phi < \Phi_\text{S}-\Delta_3\Phi$ or $\Phi_\text{S}+\Delta_3\Phi < \Phi < \Phi_\text{N}-\Delta_3\Phi$ or $\Phi_\text{N}+\Delta_3\Phi < \Phi$ **and** either $\Lambda_\text{W}-\Delta_3\Lambda < \Lambda < \Lambda_\text{W}$ or $\Lambda_\text{E}-\Delta_3\Lambda < \Lambda < \Lambda_\text{E}$ **and** either $H < H_\text{B}-\Delta_3 H$ or $H_\text{B}+\Delta_3 H < H < H_\text{T}-\Delta_3 H$ or $H_\text{T}+\Delta_3 H < H$, then based on Eq. (20) in the main text by adopting $u=\Phi$, $v=H$, and $w=\Lambda$ with the negative sign for $w=\Lambda$ it yields:

$$\begin{aligned}\frac{\partial^3 V}{\partial\Phi\partial\Lambda\partial H} \approx \frac{-1}{8(\Delta_3\Phi)(\Delta_3\Lambda)(\Delta_3 H)}\big[&-V(\Phi+\Delta_3\Phi,\Lambda-2\Delta_3\Lambda,H+\Delta_3 H)\\
&+4V(\Phi+\Delta_3\Phi,\Lambda-\Delta_3\Lambda,H+\Delta_3 H)-3V(\Phi+\Delta_3\Phi,\Lambda,H+\Delta_3 H)\\
&+V(\Phi+\Delta_3\Phi,\Lambda-2\Delta_3\Lambda,H-\Delta_3 H)-4V(\Phi+\Delta_3\Phi,\Lambda-\Delta_3\Lambda,H-\Delta_3 H)\\
&+3V(\Phi+\Delta_3\Phi,\Lambda,H-\Delta_3 H)+V(\Phi-\Delta_3\Phi,\Lambda-2\Delta_3\Lambda,H+\Delta_3 H)\\
&-4V(\Phi-\Delta_3\Phi,\Lambda-\Delta_3\Lambda,H+\Delta_3 H)+3V(\Phi-\Delta_3\Phi,\Lambda,H+\Delta_3 H)\\
&-V(\Phi-\Delta_3\Phi,\Lambda-2\Delta_3\Lambda,H-\Delta_3 H)+4V(\Phi-\Delta_3\Phi,\Lambda-\Delta_3\Lambda,H-\Delta_3 H)\\
&-3V(\Phi-\Delta_3\Phi,\Lambda,H-\Delta_3 H)\big]\end{aligned} \tag{75}$$

(25) else if either $\Phi_\text{S} < \Phi < \Phi_\text{S}+\Delta_3\Phi$ or $\Phi_\text{N} < \Phi < \Phi_\text{N}+\Delta_3\Phi$ **and** either $\Lambda < \Lambda_\text{W}-\Delta_3\Lambda$ or $\Lambda_\text{W}+\Delta_3\Lambda < \Lambda < \Lambda_\text{E}-\Delta_3\Lambda$ or $\Lambda_\text{E}+\Delta_3\Lambda < \Lambda$ **and** either $H < H_\text{B}-\Delta_3 H$ or $H_\text{B}+\Delta_3 H < H < H_\text{T}-\Delta_3 H$ or $H_\text{T}+\Delta_3 H < H$, then based on Eq. (20) in the main text by adopting $u=\Lambda$, $v=H$, and $w=\Phi$ with the positive sign for $w=\Phi$ it yields:

$$\begin{aligned}\frac{\partial^3 V}{\partial\Phi\partial\Lambda\partial H} \approx \frac{1}{8(\Delta_3\Phi)(\Delta_3\Lambda)(\Delta_3 H)}\big[&-V(\Phi+2\Delta_3\Phi,\Lambda+\Delta_3\Lambda,H+\Delta_3 H)\\
&+4V(\Phi+\Delta_3\Phi,\Lambda+\Delta_3\Lambda,H+\Delta_3 H)-3V(\Phi,\Lambda+\Delta_3\Lambda,H+\Delta_3 H)\\
&+V(\Phi+2\Delta_3\Phi,\Lambda+\Delta_3\Lambda,H-\Delta_3 H)-4V(\Phi+\Delta_3\Phi,\Lambda+\Delta_3\Lambda,H-\Delta_3 H)\\
&+3V(\Phi,\Lambda+\Delta_3\Lambda,H-\Delta_3 H)+V(\Phi+2\Delta_3\Phi,\Lambda-\Delta_3\Lambda,H+\Delta_3 H)\\
&-4V(\Phi+\Delta_3\Phi,\Lambda-\Delta_3\Lambda,H+\Delta_3 H)+3V(\Phi,\Lambda-\Delta_3\Lambda,H+\Delta_3 H)\\
&-V(\Phi+2\Delta_3\Phi,\Lambda-\Delta_3\Lambda,H-\Delta_3 H)+4V(\Phi+\Delta_3\Phi,\Lambda-\Delta_3\Lambda,H-\Delta_3 H)\\
&-3V(\Phi,\Lambda-\Delta_3\Lambda,H-\Delta_3 H)\big]\end{aligned} \tag{76}$$

(26) else if either $\Phi_\text{S}-\Delta_3\Phi < \Phi < \Phi_\text{S}$ or $\Phi_\text{N}-\Delta_3\Phi < \Phi < \Phi_\text{N}$ **and** either $\Lambda < \Lambda_\text{W}-\Delta_3\Lambda$ or $\Lambda_\text{W}+\Delta_3\Lambda < \Lambda < \Lambda_\text{E}-\Delta_3\Lambda$ or $\Lambda_\text{E}+\Delta_3\Lambda < \Lambda$ **and** either $H < H_\text{B}-\Delta_3 H$ or $H_\text{B}+\Delta_3 H < H < H_\text{T}-\Delta_3 H$ or $H_\text{T}+\Delta_3 H < H$, then based on Eq. (20) in the main text by adopting

$u=\Lambda$, $v=H$, and $w=\Phi$ with the negative sign for $w=\Phi$ it yields:

$$\begin{aligned}\frac{\partial^3 V}{\partial\Phi\partial\Lambda\partial H} \approx \frac{-1}{8(\Delta_3\Phi)(\Delta_3\Lambda)(\Delta_3 H)}\big[&-V(\Phi-2\Delta_3\Phi,\Lambda+\Delta_3\Lambda,H+\Delta_3 H)\\
&+4V(\Phi-\Delta_3\Phi,\Lambda+\Delta_3\Lambda,H+\Delta_3 H)-3V(\Phi,\Lambda+\Delta_3\Lambda,H+\Delta_3 H)\\
&+V(\Phi-2\Delta_3\Phi,\Lambda+\Delta_3\Lambda,H-\Delta_3 H)-4V(\Phi-\Delta_3\Phi,\Lambda+\Delta_3\Lambda,H-\Delta_3 H)\\
&+3V(\Phi,\Lambda+\Delta_3\Lambda,H-\Delta_3 H)+V(\Phi-2\Delta_3\Phi,\Lambda-\Delta_3\Lambda,H+\Delta_3 H)\\
&-4V(\Phi-\Delta_3\Phi,\Lambda-\Delta_3\Lambda,H+\Delta_3 H)+3V(\Phi,\Lambda-\Delta_3\Lambda,H+\Delta_3 H)\\
&-V(\Phi-2\Delta_3\Phi,\Lambda-\Delta_3\Lambda,H-\Delta_3 H)+4V(\Phi-\Delta_3\Phi,\Lambda-\Delta_3\Lambda,H-\Delta_3 H)\\
&-3V(\Phi,\Lambda-\Delta_3\Lambda,H-\Delta_3 H)\big]\end{aligned} \tag{77}$$

(27) else, based on Eq. (19) in the main text by adopting $u=\Phi$, $v=\Lambda$, and $w=H$ it yields:

$$\begin{aligned}\frac{\partial^3 V}{\partial\Phi\partial\Lambda\partial H} \approx \frac{1}{8(\Delta_3\Phi)(\Delta_3\Lambda)(\Delta_3 H)}\big[&V(\Phi+\Delta_3\Phi,\Lambda+\Delta_3\Lambda,H+\Delta_3 H)\\
&-V(\Phi+\Delta_3\Phi,\Lambda+\Delta_3\Lambda,H-\Delta_3 H)-V(\Phi+\Delta_3\Phi,\Lambda-\Delta_3\Lambda,H+\Delta_3 H)\\
&+V(\Phi+\Delta_3\Phi,\Lambda-\Delta_3\Lambda,H-\Delta_3 H)-V(\Phi-\Delta_3\Phi,\Lambda+\Delta_3\Lambda,H+\Delta_3 H)\\
&+V(\Phi-\Delta_3\Phi,\Lambda+\Delta_3\Lambda,H-\Delta_3 H)+V(\Phi-\Delta_3\Phi,\Lambda-\Delta_3\Lambda,H+\Delta_3 H)\\
&-V(\Phi-\Delta_3\Phi,\Lambda-\Delta_3\Lambda,H-\Delta_3 H)\big]\end{aligned} \tag{78}$$