Introduction to Smooth Manifolds Exercise Solutions

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Chapter 1

Smooth Manifolds

1-1 Let X be the set of all points $(x,y) \in \mathbb{R}^2$ such that $y = \pm 1$, and let M be the quotient of X by the equivalence relation generated by $(x,1) \sim (x,-1)$ for all $x \neq 0$. Show that M is locally Euclidean and second-countable, but not Hausdorff. (This space is called the *line with two origins*.)

Proof Let $\pi\colon X \to M$ be the quotient map, where

$$X = \{(x,y) \in \mathbb{R}^2 : y = \pm 1\}$$

and the equivalence relation is generated by $(x,1) \sim (x,-1)$ for all $x \neq 0$. Let

$$p = \pi(0,1), \quad q = \pi(0,-1)$$

denote the two distinct equivalence classes in M corresponding to the two origins.
To describe a basis for the topology on M, for any open interval $W \subset \mathbb{R}$, define the following sets:

$$\begin{aligned} U_W &= \pi\left(\{(x,\pm 1) : x \in W\}\right), && \text{for } 0 \notin W, \\ U_W^+ &= \pi\left(\{(x,\pm 1) : x \in W \setminus \{0\}\} \cup \{(0,1)\}\right), && \text{for } 0 \in W, \\ U_W^- &= \pi\left(\{(x,\pm 1) : x \in W \setminus \{0\}\} \cup \{(0,-1)\}\right), && \text{for } 0 \in W. \end{aligned}$$

These sets are open in the quotient topology on M, where:

- U_W is a basic open set in M when W does not contain 0;
- U_W^+ is a neighborhood of the point $p = \pi(0,1)$;
- U_W^- is a neighborhood of the point $q = \pi(0,-1)$.

We next show that M is second-countable and locally Euclidean but not Hausdorff.

- **Second-countability:**

- **Local Euclidean property:**

- **Not Hausdorff:**

□

1-2 Show that a disjoint union of uncountably many copies of $\mathbb{R}$ is locally Euclidean and Hausdorff, but not second-countable.

Proof Let $X = \coprod_{\alpha \in A} \mathbb{R}_\alpha$, A is an uncountable index set. $U \subseteq X$ is open if and only if $\forall \alpha \in A$, $\{x \in \mathbb{R} : (\alpha, x) \in U\}$ is open in $\mathbb{R}$.

1. X is locally Euclidean.
   By definiton, $\mathbb{R}_\alpha = \{(\alpha, x) : x \in \mathbb{R}\}$. $\forall (\alpha, x) \in \mathbb{R}_\alpha$, since $\mathbb{R}_\alpha$ is an open subset of X(By the definiton of topology of X) and $\mathbb{R}_\alpha$ is homeomorphic to $\mathbb{R}$, X is locally Euclidean.

2. X is Hausdorff.
   Let $(\alpha, x), (\beta, y) \in X$. if $\alpha \neq \beta$, clearly we have two disjoint open subset $\mathbb{R}_\alpha$ and $\mathbb{R}_\beta$ such that $(\alpha, x) \in \mathbb{R}_\alpha$ and $(\beta, x) \in \mathbb{R}_\beta$. if $\alpha = \beta$, since $\mathbb{R}$ is Hausdorff, we can find two disjoint open subset $U, V \in \mathbb{R}$ such that $x \in U$ and $y \in V$. $(\alpha, U), (\beta, V)$ are two disjoint open subset of X.

3. X is not second-countable.
   Assume that X is second-countable with its countable basis $\mathcal{B} = \{B_i\}$ and I is an countable index set. Since $\mathbb{R}_\alpha$ is an non-empty open subset of X, we can always find $B_i \in \mathcal{B}$ such that $B_i \subseteq \mathbb{R}_\alpha$. By Axiom of Choice, we can define

$$f \colon A \to I, \quad \alpha \mapsto i \ \text{ s.t. } \ B_i \subseteq \mathbb{R}_\alpha.$$

   Clearly f is injective. This leads to an contradiction, since A is uncountable and I is countable.

$\square$

1-3 A topology space is said to be σ-*compact* if it can be expressed as a union of countably many compact subspaces. Show that a locally Euclidean Hausdorff space is a topological manifold if and only if it is σ-compact.

Proof

($\Rightarrow$) Every topological manifold admits a countable basis $\mathcal{B} = \{B_i\}$ of precompact coordinate balls(Lemma 1.10). The collection $\{\overline{B_i} \mid B_i \in \mathcal{B}\}$ implies that the manifold is σ-compact.

($\Leftarrow$) Let X be a locally Euclidean Hausdorff space that is σ-compact. By definition, there exists a countable family of compact subsets $\{K_n\}_{n\in\mathbb{N}}$ such that $X = \bigcup_{n\in\mathbb{N}} K_n$. Since X is locally Euclidean, for each K_n, there exists a finite open cover $\{U_{n_i}\}_{i=1}^{k_n}$ where each (U_{n_i}, φ_{n_i}) is a coordinate chart.

1. For each U_{n_i}, choose a precompact coordinate ball $B_{n_i} \subseteq U_{n_i}$ (possible by local Euclideanness, see Lemma 1.10). The collection $\{B_{n_i}\}$ is countable and covers X.
2. Each B_{n_i}, being homeomorphic to an open ball in $\mathbb{R}^n$, admits a countable basis. A countable union of countable bases remains countable, thus X is second-countable.

Therefore, X satisfies all axioms of a topological manifold (locally Euclidean + Hausdorff + second-countable).

$\square$

1-4 Let M be a topological manifold, and let $\mathcal{U}$ be an open cover of M.

(a) Assuming that each set in $\mathcal{U}$ intersects only finitely many others, show that $\mathcal{U}$ is locally finite.

(b) Give an example to show that the converse to (a) may be false.

(c) Now assume that the sets in $\mathcal{U}$ are precompact in M, and prove the converse: if $\mathcal{U}$ is locally finite, then each set in $\mathcal{U}$ intersects only finitely many others.

Proof

(a) Easy.

(b) Let $M = \mathbb{R}$, $\mathcal{U} = \{(n, \infty) : n \in \mathbb{N}\} \cup \{(-\infty, 1)\}$

$\square$

1-5 Suppose M is a locally Euclidean Hausdorff space. Show that M is second countable if and only if it is paracompact and has countably many connected components. [Hint: assuming M is paracompact, show that each component of M has a locally finite cover by precompact coordinate domains, and extract from this a countable subcover.]

1-6 Let M be a nonempty topological manifold of dimension $n \geq 1$. If M has a smooth structure, show that it has uncountably many distinct ones. [Hint: first show that for any $s > 0$, $F_s(x) = |x|^{s-1}x$ defines a homeomorphism from $\mathbb{B}^n$ to itself, which is a diffeomorphism if and only if $s = 1$.]

Proof We proceed in four steps:

1. **Homeomorphism property of F_s:**
   For any $s > 0$, the map $F_s \colon \mathbb{B}^n \to \mathbb{B}^n$ defined by $F_s(x) = |x|^{s-1}x$ is a homeomorphism.

   - If $s \geq 1$, F_s is clearly continuous on $\mathbb{B}^n$.
   - If $0 < s < 1$, continuity at $x = 0$ follows from:

   $$\lim_{x\to 0} |F_s(x)| = \lim_{x\to 0} |x|^s = 0 = F_s(0).$$

   - The inverse is $F_{1/s}$, since $F_s \circ F_{1/s} = F_{1/s} \circ F_s = \mathrm{id}_{\mathbb{B}^n}$.

2. **Non-smoothness at origin:**
   F_s is a diffeomorphism on $\mathbb{B}^n \setminus \{0\}$ but fails to be smooth at 0 when $s \neq 1$:

   - For $0 < s < 1$, the derivative at 0 does not exist:

   $$\frac{\partial F_s(0)}{\partial x^i} = \lim_{\Delta x^i \to 0} (\Delta x^i)^{s-1}(0, \dots, 1, \dots, 0) \quad \text{(diverges)}$$

   - For $s > 1$, the inverse $F_{1/s}$ has $0 < 1/s < 1$ and thus fails to be smooth at 0.

   Hence F_s is a diffeomorphism on $\mathbb{B}^n$ if and only if $s = 1$.

3. **Constructing a modified atlas:**
   Fix a point $p \in M$ and choose a smooth chart (U, φ) from the given smooth structure $\mathcal{A}$ on M, such that:

   $$\varphi(U) = \mathbb{B}^n \quad \text{and} \quad \varphi(p) = 0.$$

   For any $s > 0$, define a new chart (U, φ_s) by:

   $$\varphi_s = F_s \circ \varphi,$$

   where $F_s(x) = |x|^{s-1}x$ is the homeomorphism from Step 1.

   Construct a new atlas $\mathcal{A}_s$ as follows:

   $$\mathcal{A}_s = \{(U, \varphi_s)\} \cup \{(V, \psi) \in \mathcal{A} \colon p \notin V\}.$$

   That is, $\mathcal{A}_s$ consists of:

   - The single modified chart (U, φ_s) centered at p,
   - All charts from the original atlas $\mathcal{A}$ that do not contain p.

   $\mathcal{A}_s$ is a smooth atlas:

- The charts in $\mathcal{A}_s$ cover M: every point $q \neq p$ is covered by some chart (V, ψ) in $\mathcal{A}$ with $p \notin V$, and p is covered by (U, φ_s).
- The charts in $\mathcal{A}_s$ are pairwise compatible:
  - For any two charts (V_1, ψ_1) and (V_2, ψ_2) in $\mathcal{A}_s$ not containing p, their transition map $\psi_2 \circ \psi_1^{-1}$ is smooth because $\mathcal{A}$ is a smooth atlas.
  - For (U, φ_s) and any (V, ψ) with $p \notin V$, the transition map on $U \cap V$ is:

$$\psi \circ \varphi_s^{-1} = \psi \circ \varphi^{-1} \circ F_{1/s}.$$

    This is smooth because $\psi \circ \varphi^{-1}$ is smooth (by compatibility in $\mathcal{A}$) and $F_{1/s}$ is smooth away from 0.

4. **Distinct smooth structures:**
   We show that the smooth structures induced by $\mathcal{A}_s$ and $\mathcal{A}_{s'}$ are distinct unless $s = s'$.

   - Suppose $s \neq s'$. Consider the transition map between (U, φ_s) and $(U, \varphi_{s'})$:

$$\varphi_{s'} \circ \varphi_s^{-1} = F_{s'} \circ F_{1/s} = F_{s'/s}.$$

     This is a diffeomorphism on $\mathbb{B}^n \setminus \{0\}$ but fails to be smooth at 0 unless $s'/s = 1$ (i.e., $s = s'$), as shown in Step 2.
   - Thus, $\mathcal{A}_s$ and $\mathcal{A}_{s'}$ are not smoothly compatible unless $s = s'$.

   Since there are uncountably many choices for $s > 0$, this yields uncountably many distinct smooth structures on M.

□

1-7 Let N denote the north pole $(0,\dots,0,1) \in \mathbb{S}^n \subset \mathbb{R}^{n+1}$, and let S denote the south pole $(0,\dots,0,-1)$. Define the stereographic projection $\sigma\colon \mathbb{S}^n \setminus \{N\} \to \mathbb{R}^n$ by

$$\sigma(x^1,\dots,x^{n+1}) = \frac{(x^1,\dots,x^n)}{1-x^{n+1}}.$$

Let $\tilde{\sigma}(x) = -\sigma(-x)$ for $x \in \mathbb{S}^n \setminus \{S\}$.

(a) For any $x \in \mathbb{S}^n \setminus \{N\}$, show that $\sigma(x) = u$, where $(u,0)$ is the point where the line through N and x intersects the hyperplane $x^{n+1} = 0$. Similarly, show that $\tilde{\sigma}(x)$ is the point where the line through S and x intersects the same hyperplane. (Thus $\tilde{\sigma}$ is called stereographic projection from the south pole.)

(b) Show that σ is bijective, with inverse given by

$$\sigma^{-1}(u^1,\dots,u^n) = \frac{(2u^1,\dots,2u^n,|u|^2-1)}{|u|^2+1}.$$

(c) Compute the transition map $\tilde{\sigma} \circ \sigma^{-1}$ and verify that the atlas $\{(\mathbb{S}^n \setminus \{N\},\sigma),(\mathbb{S}^n \setminus \{S\},\tilde{\sigma})\}$ defines a smooth structure on $\mathbb{S}^n$. These are called stereographic coordinates.

(d) Show that this smooth structure agrees with the one defined in Example 1.31.

Proof

(a) Since N, x, and $\sigma(x)$ are collinear, there exists $\lambda \in \mathbb{R}$ such that

$$x = \lambda N + (1-\lambda)\sigma(x).$$

Solving for λ and $\sigma(x)$ gives:

$$\lambda = x^{n+1},$$

$$\sigma(x) = \frac{(x^1,\dots,x^n)}{1-x^{n+1}}.$$

The symmetry $\tilde{\sigma}(-x) = -\sigma(x)$ implies $\tilde{\sigma}(x) = -\sigma(-x)$.

(b) Verify that $\sigma \circ \sigma^{-1} = \mathrm{id}_{\mathbb{R}^n}$ and $\sigma^{-1} \circ \sigma = \mathrm{id}_{\mathbb{S}^n \setminus \{N\}}$

- For $\sigma \circ \sigma^{-1}$, let $(u^1,\dots,u^n) \in \mathbb{R}^n$, we have

$$\begin{aligned}
\sigma \circ \sigma^{-1}(u^1,\dots,u^n) &= \sigma\left(\frac{(2u^1,\dots,2u^n,|u|^2-1)}{|u|^2+1}\right) \\
&= \frac{\left(\frac{2u^1}{|u|^2+1},\dots,\frac{2u^n}{|u|^2+1}\right)}{1-\frac{|u|^2-1}{|u|^2+1}} \\
&= (u^1,\dots,u^n).
\end{aligned}$$

- For $\sigma^{-1} \circ \sigma$, let $(x^1, \ldots, x^{n+1}) \in \mathbb{S}^n \setminus \{N\}$, which means $x^{n+1} \neq 1$ and

$$|\sigma(x)|^2 = \frac{(x^1)^2 + \cdots + (x^n)^2}{(1 - x^{n+1})^2} = \frac{1 + x^{n+1}}{1 - x^{n+1}},$$

$$\begin{aligned} \sigma^{-1} \circ \sigma(x^1, \ldots, x^{n+1}) &= \sigma^{-1}\left(\frac{(x^1, \ldots, x^n)}{1 - x^{n+1}}\right) \\ &= \frac{\left(\frac{2x^1}{1-x^{n+1}}, \ldots, \frac{2x^n}{1-x^{n+1}}, \frac{|x|^2-1}{1-x^{n+1}}\right)}{\frac{|x|^2+1}{1-x^{n+1}}} \\ &= (x^1, \ldots, x^{n+1}). \end{aligned}$$

(c) It's sufficient to proof that $\tilde{\sigma} \circ \sigma^{-1}$ and $\sigma \circ \tilde{\sigma}^{-1}$ are smooth on $\mathbb{R}^n \setminus \{0\}$. Let $u = (u^1, \ldots, u^n) \in \mathbb{R}^n \setminus \{0\}$, it can be easily verified $\tilde{\sigma}^{-1}(u) = \sigma^{-1}(u)$.

$$\tilde{\sigma} \circ \sigma^{-1}(u) = \sigma \circ \tilde{\sigma}^{-1}(u) = \frac{u}{|u|^2},$$

both are smooth on $\mathbb{R}^n \setminus \{0\}$.

(d) We only proof that $\sigma \circ \pi_i^{-1}$ and $\pi_i \circ \sigma^{-1}$ are smooth for $i = 1, \ldots, n+1$, $\tilde{\sigma}$ is completely the same.

- For $i = n + 1$,
  - For transition map $\pi_{n+1} \circ \sigma^{-1}$:

$$\sigma(U_{n+1}^+ \setminus \{N\}) = \sigma\{x^{n+1} \in (0, 1)\} = \{|u| > 1 \colon u \in \mathbb{R}^n\}.$$

$$\begin{aligned} \pi_{n+1} \circ \sigma^{-1}(u) &= \pi_{n+1}\left(\frac{(2u^1, \ldots, 2u^n, |u|^2 - 1)}{1 + |u|^2}\right) \\ &= \frac{2u}{1 + |u|^2}. \end{aligned}$$

  - For transition map $\sigma \circ \pi_i^{-1}$:

$$\pi_{n+1}(U_{n+1}^+ \setminus \{N\}) = \pi_{n+1}\{x^{n+1} \in (0, 1)\} = \mathbb{B}^n \setminus \{0\}.$$

$$\begin{aligned} \sigma \circ \pi_{n+1}^{-1}(u) &= \sigma(u^1, \ldots, u^n, \sqrt{1 - |u|^2}) \\ &= \frac{u}{1 - \sqrt{1 - |u|^2}}. \end{aligned}$$

  Both of them are smooth on their domains.

- For $i = 1, \ldots, n$,

- For transition map $\pi_i \circ \sigma^{-1}$:

$$\sigma(U_i^+ \setminus \{N\}) = \sigma(U_i^+) = \{u \in \mathbb{R}^n \colon u^i > 0\}.$$

$$\begin{aligned} \pi_i \circ \sigma^{-1}(u) &= \pi_i\left(\frac{(2u^1, \ldots, 2u^n, |u|^2 - 1)}{1 + |u|^2}\right) \\ &= \frac{(2u^1, \ldots, \widehat{2u^i}, \ldots, |u|^2 - 1)}{|u|^2 + 1} \end{aligned}$$

- For transition map $\sigma \circ \pi_i^{-1}$:

$$\pi_i(U_i^+ \setminus \{N\}) = \pi_i(U_i^+) = \mathbb{B}^n \cap \{u \in \mathbb{R}^n \colon u^i > 0\}.$$

$$\begin{aligned} \sigma \circ \pi_i^{-1}(u) &= \sigma(u^1, \ldots, \sqrt{1 - |u|^2}, u^i, \ldots, u^n) \\ &= \frac{(u^1, \ldots, \sqrt{1 - |u|^2}, \ldots, u^{n-1})}{1 - u^n} \end{aligned}$$

All transition maps are smooth on their domains, confirming compatibility.

$\square$

1-8 By identifying $\mathbb{R}^2$ with $\mathbb{C}$, we can think of the unit circle $\mathbb{S}^1$ as a subset of the complex plane. An angle function on a subset $U \subset \mathbb{S}^1$ is a continuous function $\theta\colon U \to \mathbb{R}$ such that $\mathrm{e}^{\mathrm{i}\theta(z)} = z$ for all $z \in U$. Show that there exists an angle function on an open subset $U \subset \mathbb{S}^1$ if and only if $U \neq \mathbb{S}^1$. For any such angle function, show that (U, θ) is a smooth coordinate chart for $\mathbb{S}^1$ with its standard smooth structure.

Proof We prove the existence of an angle function θ on an open subset $U \subset \mathbb{S}^1$ for two cases: $U = \mathbb{S}^1$ and $U \subsetneq \mathbb{S}^1$.

- **Nonexistence for $U = \mathbb{S}^1$:**
  Assume such $\theta\colon \mathbb{S}^1 \to \mathbb{R}$ exists. Define the exponential map:
  $$f\colon \mathbb{R} \to \mathbb{S}^1, \quad t \mapsto \mathrm{e}^{\mathrm{i}t}.$$
  By definition, θ satisfies $f \circ \theta(z) = z$ for all $z \in \mathbb{S}^1$, implying f is injective. However, f is periodic ($f(t + 2\pi) = f(t)$), contradicting injectivity. Thus, θ cannot exist globally.

- **Existence for $U \subsetneq \mathbb{S}^1$:**
  Without loss of generality, assume $U = \mathbb{S}^1 \setminus \{p\}$ where $p = (1, 0)$. Restrict f to $(0, 2\pi)$:
  $$g \coloneqq f|_{(0,2\pi)}\colon (0, 2\pi) \to U, \quad t \mapsto \mathrm{e}^{\mathrm{i}t}.$$
  - *Bijectivity:* g is bijective by construction, with each $z \in U$ uniquely corresponding to $t \in (0, 2\pi)$.
  - *Smoothness:* The Jacobian of g at t is:
    $$J(g) = \begin{pmatrix} -\sin t \\ \cos t \end{pmatrix},$$
    which has rank 1 everywhere. By the Constant Rank Theorem, g is a diffeomorphism. Its inverse $\varphi \coloneqq g^{-1}$ defines a local angle function on U.

- **Smooth Atlas Construction:**
  Let $V = \mathbb{S}^1 \setminus \{q\}$ where $q = (-1, 0)$, and define:
  $$\psi\colon V \to (-\pi, \pi), \quad \mathrm{e}^{\mathrm{i}t} \mapsto t.$$
  The transition maps between charts (U, φ) and (V, ψ) are:
  $$\psi \circ \varphi^{-1}(t) = \begin{cases} t & t \in (0, \pi), \\ t - 2\pi & t \in (\pi, 2\pi), \end{cases}$$
  $$\varphi \circ \psi^{-1}(t) = \begin{cases} t & t \in (0, \pi), \\ t + 2\pi & t \in (-\pi, 0). \end{cases}$$
  Both are smooth on their domains, confirming $\mathcal{A} = \{(U, \varphi), (V, \psi)\}$ is a smooth atlas for $\mathbb{S}^1$.

□

1-9 Complex projective n-space, denoted by $\mathbb{CP}^n$, is the set of all 1-dimensional complex-linear subspaces of $\mathbb{C}^{n+1}$, with the quotient topology inherited from the natural projection $\pi\colon \mathbb{C}^{n+1} \setminus \{0\} \to \mathbb{CP}^n$. Show that $\mathbb{CP}^n$ is a compact $2n$-dimensional topological manifold, and show how to give it a smooth structure analogous to the one we constructed for $\mathbb{RP}^n$. (We use the correspondence

$$(x^1 + \mathrm{i}y^1, \dots, x^{n+1} + \mathrm{i}y^{n+1}) \leftrightarrow (x^1, y^1, \dots, x^{n+1}, y^{n+1})$$

to identify $\mathbb{C}^{n+1}$ with $\mathbb{R}^{2n+2}$.)

1-10 Let k and n be integers satisfying $0 < k < n$, and let $P, Q \subseteq \mathbb{R}^n$ be the linear subspaces spanned by $(e_1, \ldots, e_k)$ and $(e_{k+1}, \ldots, e_n)$, respectively, where e_i is the ith standard basis vector for $\mathbb{R}^n$. For any k-dimensional subspace $S \subseteq \mathbb{R}^n$ that has trivial intersection with Q, show that the coordinate representation $\varphi(S)$ constructed in Example 1.36 is the unique $(n-k) \times k$ matrix B such that S is spanned by the columns of the matrix $\left(\begin{smallmatrix} I_k \\ B \end{smallmatrix}\right)$, where I_k denotes the $k \times k$ identity matrix.

Proof We prove the existence and uniqueness of the coordinate representation $\varphi(S) = B$ for a k-dimensional subspace $S \subseteq \mathbb{R}^n$ with $S \cap Q = \{0\}$.

- **Existence of the matrix representation:**
  Consider the projection map $\pi_P \colon S \to P$. We claim π_P is an isomorphism:

  - *Injectivity*: Suppose $\pi_P(s) = 0$ for some $s \in S$. Then s has the form uniquely:

    $$s = \pi_P(s) + \pi_Q(s) = \pi_Q(s) \in Q.$$

    Since $S \cap Q = \{0\}$ by hypothesis, we must have $s = 0$.
  - *Surjectivity*: As $\dim S = \dim P = k$ and π_P is injective, it is automatically surjective by the rank-nullity theorem.

  Thus π_P is a vector space isomorphism between S and P. Choose

  $$\{\pi_P^{-1}(e_1), \ldots, \pi_P^{-1}(e_k)\}$$

  for the basis of S. Since $\{e_1, e_n\}$ is a basis of V, we have

  $$\pi_P^{-1}(e_i) = e_i + \sum_{j=k+1}^{n} b_{ij} e_j$$

  Thus S can be spanned by the columns of the matrix

  $$\begin{pmatrix} I_k \\ B \end{pmatrix}$$

  under the basis $\{e_1, \ldots, e_n\}$ where $B = (b_{ij})$.

- **Uniqueness of the matrix B:**
  Suppose there exist two $(n-k) \times k$ matrices B and B' such that:

  $$\operatorname{span}\left(\begin{pmatrix} I_k \\ B \end{pmatrix}\right) = \operatorname{span}\left(\begin{pmatrix} I_k \\ B' \end{pmatrix}\right) = S.$$

  Then there exists an invertible matrix $C \in \mathbb{R}^{k \times k}$ such that:

  $$\begin{pmatrix} I_k \\ B' \end{pmatrix} = \begin{pmatrix} I_k \\ B \end{pmatrix} C.$$

  This matrix equation implies:

  $$\begin{aligned} I_k &= I_k C \quad \Rightarrow \quad C = I_k, \\ B' &= BC = B. \end{aligned}$$

  Therefore, B is uniquely determined by S.

□

1-11 Let $M = \overline{\mathbb{B}^n}$, the closed unit ball in $\mathbb{R}^n$. Show that M is a topological manifold with boundary in which each point in $\mathbb{S}^{n-1}$ is a boundary point and each point in $\mathbb{B}^n$ is an interior point. Show how to give it a smooth structure such that every smooth interior chart is a smooth chart for the standard smooth structure on $\mathbb{B}^n$. [Hint: consider the map $\pi \circ \sigma^{-1} \colon \mathbb{R}^n \to \mathbb{R}^n$, where $\sigma \colon \mathbb{S}^n \to \mathbb{R}^n$ is the stereographic projection (Problem 1-7) and π is a projection from $\mathbb{R}^{n+1}$ to $\mathbb{R}^n$ that omits some coordinate other than the last.]

Proof We establish that $\overline{\mathbb{B}}^n$ is a smooth manifold with boundary, where $\mathbb{S}^{n-1}$ constitutes the boundary and $\mathbb{B}^n$ the interior, by constructing an explicit smooth structure. (This proof proceeds independently of the hint.)

- **Topological manifold structure:**
  - For $x \in \mathbb{B}^n$: The identity chart $(\mathbb{B}^n, \mathrm{id}_{\mathbb{B}^n})$ suffices.
  - For $x \in \mathbb{S}^{n-1}$: We define charts via coordinate projection:

$$\begin{aligned} U_i^+ &= \{x \in \mathbb{R}^n \mid x_i > 0\}, \\ V_i^+ &= U_i^+ \cap \overline{\mathbb{B}}^n, \\ \varphi_i &= \pi_i \circ \pi_{n+1}^{-1} : V_i^+ \to \mathbb{H}^n \cap \mathbb{B}^n, \\ \varphi_i(x^1, \ldots, x^n) &= \pi_i(x^1, \ldots, x^n, \sqrt{1 - |x|^2}) = (x^1, \ldots, \widehat{x^i}, \ldots, \sqrt{1 - |x|^2}) \end{aligned}$$

    where $\pi_i \colon \mathbb{S}^n \to \mathbb{R}^n$ omits the i-th coordinate. The collection

$$\{(V_i^\pm, \varphi_i)\}$$

    forms boundary charts since π_i and π_{n+1} are both homeomorphic on V_i^+.

- **Smooth structure:**
  - The charts $\{(V_i^\pm, \varphi_i)\}$ are compatible with each other, since the standard smooth structure of $\mathbb{S}^n$ ensures transition maps

$$\varphi_j \circ \varphi_i^{-1} = \pi_j \circ \pi_{n+1}^{-1} \circ \pi_{n+1} \circ \pi_i^{-1} = \pi_j \circ \pi_i^{-1}$$

    are diffeomorphisms on their domains $\varphi_i(V_i^+ \cap V_j^+)$.
  - Boundary charts and interior chart are compatible, since the Jacobian of transition map

$$|J(\varphi_i \circ \mathrm{id}_{\mathbb{B}^n}^{-1})| = (-1)^{n-1} \frac{x^i}{\sqrt{1 - |x|^2}} \neq 0$$

    on its domain $\mathbb{B}^n \cap V_i^\pm$. Thus the smooth atlas

$$\mathcal{A} = \{(V_i^\pm, \varphi_i)\} \cup (\mathbb{B}^n, \mathrm{id}_{\mathbb{B}^n})$$

    yields a smooth structure of $\overline{\mathbb{B}}^n$.

- **Boundary and interior identification:**

- For $x \in \mathbb{S}^{n-1}$, some some boundary chart $(V_i^{\pm}, \varphi_i)$ satisfies

$$\varphi_i(x) = (x^1, \ldots, \widehat{x^i}, \ldots, 0) \in \partial\mathbb{H}^n,$$

  confirming $\mathbb{S}^{n-1} \subset \partial\overline{\mathbb{B}}^n$ via Theorem 1.46 (Boundary Invariance).
- For $x \in \mathbb{B}^n$, the identity chart maps x to $\mathbb{B}^n \subset \mathbb{R}^n$, proving $\mathbb{B}^n \subset \operatorname{Int}(\overline{\mathbb{B}}^n)$.
- Since $\overline{\mathbb{B}}^n = \mathbb{B}^n \cup \mathbb{S}^{n-1}$, we conclude:

$$\partial\overline{\mathbb{B}}^n = \mathbb{S}^{n-1}, \quad \operatorname{Int}(\overline{\mathbb{B}}^n) = \mathbb{B}^n.$$

□

1-12 Prove Proposition 1.45 (a product of smooth manifolds together with one smooth manifold with boundary is a smooth manifold with boundary).

Proof

- **Model Space Identification**: First observe that $\mathbb{R}^m \times \mathbb{H}^n \cong \mathbb{H}^{m+n}$ via the diffeomorphism:

$$\varphi\colon \mathbb{R}^m \times \mathbb{H}^n \to \mathbb{H}^{m+n}$$
$$(x^1, \ldots, x^m, y^1, \ldots, y^n) \mapsto (x^1, \ldots, x^m, y^1, \ldots, y^n)$$

  This preserves boundaries since $\varphi(\mathbb{R}^m \times \partial\mathbb{H}^n) = \partial\mathbb{H}^{m+n}$.

- **Chart Construction**: Let $M = M_1 \times \cdots \times M_k$ (dim $m = \sum m_i$) and N (dim n) with $\partial N \neq \emptyset$.

  - **Interior Charts**: For $(p, q) \in M \times \operatorname{Int}(N)$:
    (a) Take smooth charts (U_i, φ_i) about $p_i \in M_i$ with $\varphi_i\colon U_i \to \mathbb{R}^{m_i}$
    (b) Take interior chart (V, ψ) about $q \in N$ with $\psi\colon V \to \mathbb{R}^n$
    (c) The product chart is:

$$\left(\prod_{i=1}^{k} U_i \times V, (\varphi_1, \ldots, \varphi_k, \psi)\right)$$

    mapping to $\mathbb{R}^m \times \mathbb{R}^n \subset \mathbb{H}^{m+n}$

  - **Boundary Charts**: For $(p, q) \in M \times \partial N$:
    (a) Take smooth charts (U_i, φ_i) as above
    (b) Take boundary chart (V, ψ) with $\psi\colon V \to \mathbb{H}^n$ and $\psi(q) \in \partial\mathbb{H}^n$
    (c) The product chart is:

$$\left(\prod_{i=1}^{k} U_i \times V, (\varphi_1, \ldots, \varphi_k, \psi)\right)$$

    mapping to $\mathbb{R}^m \times \mathbb{H}^n \cong \mathbb{H}^{m+n}$ with boundary points precisely when $q \in \partial N$

- **Chart Compatibility**:

  - For two interior charts, the transition map is:

$$(\varphi_1', \ldots, \varphi_k', \psi') \circ (\varphi_1, \ldots, \varphi_k, \psi)^{-1} = (\varphi_1' \circ \varphi_1^{-1}, \ldots, \varphi_k' \circ \varphi_k^{-1}, \psi' \circ \psi^{-1})$$

    which is smooth since each component is smooth.

  - For boundary charts, the same holds because $\psi' \circ \psi^{-1}$ is smooth as a map between subsets of $\mathbb{H}^n$.

  - For mixed cases (one interior, one boundary chart), the transition maps are smooth by the boundary compatibility of N's charts.

- **Boundary Characterization**:
  - If (p, q) is mapped to $\partial\mathbb{H}^{m+n}$ in some chart, then by Theorem 1.46 it holds in all charts, this occurs precisely when $q \in \partial N$, proving:

$$\partial(M \times N) = M \times \partial N$$

  - The interior is correspondingly $M \times \text{Int}(N)$

Thus $M \times N$ is a smooth manifold with boundary as claimed. $\square$

Chapter 2

Smooth Maps

2-1 Define $f : \mathbb{R} \to \mathbb{R}$ by

$$f(x) = \begin{cases} 1, & x \geq 0, \\ 0, & x < 0. \end{cases}$$

Show that for every $x \in \mathbb{R}$, there exist smooth coordinate charts (U, φ) containing x, and (V, ψ) containing $f(x)$, such that the map

$$\psi \circ f \circ \varphi^{-1}$$

is smooth as a function from $\varphi(U \cap f^{-1}(V))$ to $\mathbb{R}$. However, f is not smooth in the sense we have defined in this chapter.