

Introduction to Smooth Manifolds

Exercise Solutions

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Chapter 1

Smooth Manifolds

1-1 Let X be the set of all points $(x, y) \in \mathbb{R}^2$ such that $y = \pm 1$, and let M be the quotient of X by the equivalence relation generated by $(x, 1) \sim (x, -1)$ for all $x \neq 0$. Show that M is locally Euclidean and second-countable, but not Hausdorff. (This space is called the *line with two origins*.)

Proof Let $\pi: X \rightarrow M$ be the quotient map, where

$$X = \{(x, y) \in \mathbb{R}^2 : y = \pm 1\}$$

and the equivalence relation is generated by $(x, 1) \sim (x, -1)$ for all $x \neq 0$. Let

$$p = \pi(0, 1), \quad q = \pi(0, -1)$$

denote the two distinct equivalence classes in M corresponding to the two origins.

To describe a basis for the topology on M , for any open interval $W \subseteq \mathbb{R}$, define the following sets:

$$\begin{aligned} U_W &= \pi(\{(x, \pm 1) : x \in W\}), & \text{for } 0 \notin W, \\ U_W^+ &= \pi(\{(x, \pm 1) : x \in W \setminus \{0\}\} \cup \{(0, 1)\}), & \text{for } 0 \in W, \\ U_W^- &= \pi(\{(x, \pm 1) : x \in W \setminus \{0\}\} \cup \{(0, -1)\}), & \text{for } 0 \in W. \end{aligned}$$

These sets are open in the quotient topology on M , where:

- U_W is a basic open set in M when W does not contain 0;
- U_W^+ is a neighborhood of the point $p = \pi(0, 1)$;
- U_W^- is a neighborhood of the point $q = \pi(0, -1)$.

We next show that M is second-countable and locally Euclidean but not Hausdorff.

• **Second-countability:**

Define:

$$\mathcal{B} = \{U_W, U_W^\pm : W \text{ is an open interval with rational endpoints.}\}$$

Since there are only countably many open intervals in \mathbb{R} with rational endpoints, the collection \mathcal{B} is countable.

We claim that \mathcal{B} is a basis for the topology on M . Let $U \subseteq M$ be any open set and let $x \in U$. We want to find some $B \in \mathcal{B}$ such that $x \in B \subseteq U$.

There are two cases to consider:

- If $x \in M$ corresponds to a point $\pi(x_0, \pm 1)$ with $x_0 \neq 0$, then $\pi^{-1}(U)$ is an open subset of X containing both $(x_0, 1)$ and $(x_0, -1)$ (since they are identified when $x_0 \neq 0$). Since X inherits the subspace topology from \mathbb{R}^2 , there exists an open interval $W \ni x_0$ with rational endpoints such that $(W \times \{\pm 1\}) \subseteq \pi^{-1}(U)$. Then $U_W = \pi((W \times \{\pm 1\})) \subseteq U$, and $U_W \in \mathcal{B}$ if $0 \notin W$.
- If $x = p = \pi(0, 1)$ or $x = q = \pi(0, -1)$, then $x \in U$ implies $\pi^{-1}(U)$ contains either $(0, 1)$ or $(0, -1)$ respectively. Since $\pi^{-1}(U)$ is open in X , there exists an open interval $W \ni 0$ such that:

- * $(W \setminus \{0\}) \times \{\pm 1\} \subseteq \pi^{-1}(U)$,
- * and either $(0, 1) \in \pi^{-1}(U)$ or $(0, -1) \in \pi^{-1}(U)$.

Hence, either $U_W^+ \subseteq U$ or $U_W^- \subseteq U$, and such sets are in \mathcal{B} because W has rational endpoints.

Therefore, M is second-countable.

• **Local Euclidean property:**

- For $x \notin \{p, q\}$, define a map

$$\varphi: U_{\mathbb{R} \setminus \{0\}} \rightarrow \mathbb{R} \setminus \{0\}, \quad \pi(x, \pm 1) \mapsto x$$

Clearly φ is bijective. Let V be an open subset of $\mathbb{R} \setminus \{0\}$, $\varphi^{-1}(V)$ is open if and only if $\pi^{-1} \circ \varphi^{-1}(V)$ is open in X . Since

$$\pi^{-1} \circ \varphi^{-1}(V) = (V \times \{-1\}) \cup (V \times \{1\})$$

which is open in X , $\varphi^{-1}(V)$ is open in $U_{\mathbb{R} \setminus \{0\}}$. This indicates that φ is continuous. Let $U \subseteq U_{\mathbb{R} \setminus \{0\}}$ be an open subset of M , it means $\pi^{-1}(U)$ is open in X . Since

$$\varphi(U) = \{x: (x, 1) \in \pi^{-1}(U)\} \cup \{x: (x, -1) \in \pi^{-1}(U)\}$$

is open in X , φ yields a homeomorphism. Hence, every point with $x \neq 0$ has a neighborhood homeomorphic to $\mathbb{R} \setminus \{0\}$, which is locally Euclidean.

- For $x = p$, define the map

$$\psi_+: U_{(-1,1)}^+ \rightarrow (-1, 1), \quad \psi_+(\pi(x, \pm 1)) = x, \quad \psi_+(p) = 0.$$

This map is well-defined and bijective. To show that ψ_+ is a homeomorphism, it suffices to verify that both ψ_+ and its inverse are continuous at p and 0, respectively.

For any $\varepsilon \in (0, 1)$, we have

$$\psi_+^{-1}((-\varepsilon, \varepsilon)) = U_{(-\varepsilon, \varepsilon)}^+ \quad \text{and} \quad \psi_+(U_{(-\varepsilon, \varepsilon)}^+) = (-\varepsilon, \varepsilon),$$

which shows that ψ_+ is continuous at p and its inverse is continuous at 0. Therefore, ψ_+ is a homeomorphism.

- For $x = q$, The proof is identical to Case 2.

• **Not Hausdorff:**

We show that M is not Hausdorff by exhibiting two points that cannot be separated by disjoint open neighborhoods.

Consider the two points $p = \pi(0, 1)$ and $q = \pi(0, -1)$. Suppose for contradiction that there exist disjoint open sets U and V in M such that $p \in U$ and $q \in V$.

Since the sets U and V are open neighborhoods of p and q , respectively, there exist basic open sets $U_W^+ \subseteq U$ and $U_{W'}^- \subseteq V$, where W and W' are open intervals containing 0.

Let $W'' = W \cap W'$; then $0 \in W''$, so $W'' \setminus \{0\} \neq \emptyset$. Define:

$$A := \pi((W'' \setminus \{0\}) \times \{\pm 1\}) = U_{W'' \setminus \{0\}} \subseteq U_W^+ \cap U_{W'}^-.$$

Therefore, U_W^+ and $U_{W'}^-$ are not disjoint; they always intersect in a nonempty open set. This contradicts the assumption that p and q can be separated by disjoint open sets. Hence, M is not Hausdorff.

□

1-2 Show that a disjoint union of uncountably many copies of \mathbb{R} is locally Euclidean and Hausdorff, but not second-countable.

Proof Let $X = \coprod_{\alpha \in A} \mathbb{R}_\alpha$, A is an uncountable index set. $U \subseteq X$ is open if and only if $\forall \alpha \in A$, $\{x \in \mathbb{R} : (\alpha, x) \in U\}$ is open in \mathbb{R} .

1. X is locally Euclidean.

By definition, $\mathbb{R}_\alpha = \{(\alpha, x) : x \in \mathbb{R}\}$. $\forall (\alpha, x) \in \mathbb{R}_\alpha$, since \mathbb{R}_α is an open subset of X (By the definition of topology of X) and \mathbb{R}_α is homeomorphic to \mathbb{R} , X is locally Euclidean.

2. X is Hausdorff.

Let $(\alpha, x), (\beta, y) \in X$. if $\alpha \neq \beta$, clearly we have two disjoint open subset \mathbb{R}_α and \mathbb{R}_β such that $(\alpha, x) \in \mathbb{R}_\alpha$ and $(\beta, y) \in \mathbb{R}_\beta$. if $\alpha = \beta$, since \mathbb{R} is Hausdorff, we can find two disjoint open subset $U, V \subseteq \mathbb{R}$ such that $x \in U$ and $y \in V$. $(\alpha, U), (\beta, V)$ are two disjoint open subset of X .

3. X is not second-countable.

Assume that X is second-countable with its countable basis $\mathcal{B} = \{B_i\}$ and I is an countable index set. Since \mathbb{R}_α is a non-empty open subset of X , we can always find $B_i \in \mathcal{B}$ such that $B_i \subseteq \mathbb{R}_\alpha$. By Axiom of Choice, we can define

$$f: A \rightarrow I, \quad \alpha \mapsto i \text{ s.t. } B_i \subseteq \mathbb{R}_\alpha.$$

Clearly f is injective. This leads to a contradiction, since A is uncountable and I is countable.

□

1-3 A topology space is said to be σ -compact if it can be expressed as a union of countably many compact subspaces. Show that a locally Euclidean Hausdorff space is a topological manifold if and only if it is σ -compact.

Proof

(\Rightarrow) Every topological manifold admits a countable basis $\mathcal{B} = \{B_i\}$ of precompact coordinate balls (Lemma 1.10). The collection $\{\overline{B_i} \mid B_i \in \mathcal{B}\}$ implies that the manifold is σ -compact.

(\Leftarrow) Let X be a locally Euclidean Hausdorff space that is σ -compact. By definition, there exists a countable family of compact subsets $\{K_n\}_{n \in \mathbb{N}}$ such that $X = \bigcup_{n \in \mathbb{N}} K_n$. Since X is locally Euclidean, for each K_n , there exists a finite open cover $\{U_{n_i}\}_{i=1}^{k_n}$ where each (U_{n_i}, φ_{n_i}) is a coordinate chart.

1. For each U_{n_i} , choose a precompact coordinate ball $B_{n_i} \subseteq U_{n_i}$ (possible by local Euclideanness, see Lemma 1.10). The collection $\{B_{n_i}\}$ is countable and covers X .
2. Each B_{n_i} , being homeomorphic to an open ball in \mathbb{R}^n , admits a countable basis. A countable union of countable bases remains countable, thus X is second-countable.

Therefore, X satisfies all axioms of a topological manifold (locally Euclidean + Hausdorff + second-countable).

□

1-4 Let M be a topological manifold, and let \mathcal{U} be an open cover of M .

- (a) Assuming that each set in \mathcal{U} intersects only finitely many others, show that \mathcal{U} is locally finite.
- (b) Give an example to show that the converse to (a) may be false.
- (c) Now assume that the sets in \mathcal{U} are precompact in M , and prove the converse: if \mathcal{U} is locally finite, then each set in \mathcal{U} intersects only finitely many others.

Proof

- (a) Omitted as "Easy".
- (b) Let $M = \mathbb{R}$, $\mathcal{U} = \{(n, \infty) : n \in \mathbb{N}\} \cup \{(-\infty, 1)\}$
- (c) Assume \mathcal{U} is a locally finite open cover of M , and that each set in \mathcal{U} is precompact. Fix $U \in \mathcal{U}$, and define

$$\mathcal{V} = \{V \in \mathcal{U} : V \cap U \neq \emptyset\},$$

the collection of all elements in \mathcal{U} that intersect U .

Since U is precompact, its closure \overline{U} is compact. Because \mathcal{U} is locally finite, for every point $x \in \overline{U}$, there exists an open neighborhood V_x intersects only finitely many elements of \mathcal{U} .

Then $\{V_x\}_{x \in \overline{U}}$ is an open cover of \overline{U} by elements of \mathcal{U} , so by compactness, there exists a finite subcover:

$$\overline{U} \subseteq \bigcup_{i=1}^n V_{x_i}.$$

Now for each $i = 1, \dots, n$, define

$$\mathcal{V}_i = \{W \in \mathcal{U} : W \cap V_{x_i} \neq \emptyset\}.$$

Since each V_{x_i} intersects only finitely many elements of \mathcal{U} , each \mathcal{V}_i is finite. Now, take any $V \in \mathcal{V}$. Then $V \cap U \neq \emptyset$, and since $\overline{U} \subseteq \bigcup_{i=1}^n V_{x_i}$, there exists some i such that $V \cap V_{x_i} \neq \emptyset$, implying $V \in \mathcal{V}_i$. Thus,

$$\mathcal{V} \subseteq \bigcup_{i=1}^n \mathcal{V}_i.$$

As each \mathcal{V}_i is finite and n is finite, it follows that \mathcal{V} is finite.

Therefore, each $U \in \mathcal{U}$ intersects only finitely many other elements of \mathcal{U} .

□

1-5 Suppose M is a locally Euclidean Hausdorff space. Show that M is second countable if and only if it is paracompact and has countably many connected components.

Proof

- (\Rightarrow) By Proposition 1.11, second-countable property of topological manifold admits at most countably many connected components. Theorem 1.15 shows that every topological manifold is paracompact.
- (\Leftarrow) Suppose M is paracompact and has countably many connected components. It suffices to show that each connected component is second countable, since a countable union of second countable spaces is second countable.

Let C be a connected component of M . Since M is locally Euclidean, there exists a basis of precompact coordinate charts. Let \mathcal{U} be an open cover of C by such charts. By paracompactness, there exists a locally finite refinement \mathcal{V} of \mathcal{U} consisting of precompact coordinate domains.

To show that C is second countable, we will prove that \mathcal{V} is countable. For this, define an equivalence relation \sim on \mathcal{V} : for $U, V \in \mathcal{V}$, declare $U \sim V$ if there exists a finite sequence $U = U_0, U_1, \dots, U_n = V$ in \mathcal{V} such that $U_i \cap U_{i+1} \neq \emptyset$ for all i . Denote by $[U]$ the equivalence class of U under this relation.

We now show that $[U]$ is an open and closed subset of C :

- $[U]$ is open: U is a union of open set by definition.
- $[U]$ is closed: Let $x \in C \setminus [U]$. Since \mathcal{V} is an open cover of C , there exists $V \in \mathcal{V}$ such that $x \in V$. If V intersected any element of $[U]$, then V would be connected to U via a finite chain of overlapping sets, and hence $x \in [U]$, contradicting $x \in C \setminus [U]$. Therefore, x has an open neighborhood contained in $C \setminus [U]$.

Since this holds for arbitrary $x \in C \setminus [U]$, we conclude that $C \setminus [U]$ is open, so $[U]$ is closed.

Since C is connected and $[U]$ is nonempty, open, and closed in C , it must be that $[U] = C$. Hence, every element of \mathcal{V} can be connected to U via a finite chain of overlapping sets.

Now define inductively:

$$\mathcal{V}_1 = \{U\}, \quad \mathcal{V}_{n+1} = \{V \in \mathcal{V} : \exists W \in \mathcal{V}_n \text{ with } V \cap W \neq \emptyset\}.$$

Then $\bigcup_{n=1}^{\infty} \mathcal{V}_n = \mathcal{V}$. By Problem 1-4, each \mathcal{V}_n is finite. Thus, \mathcal{V} is a countable collection.

Since \mathcal{V} is a countable open cover of C by coordinate domains, the collection

$$\overline{\mathcal{V}} = \{\overline{V} : V \in \mathcal{V}\}$$

covers C with countably many compact subsets, thus C is σ -compact. By Problem 1-3, C is second-countable. Finally, M is a countable disjoint union of its connected components, each of which is second countable, so M is second-countable.

□

1-6 Let M be a nonempty topological manifold of dimension $n \geq 1$. If M has a smooth structure, show that it has uncountably many distinct ones. [Hint: first show that for any $s > 0$, $F_s(x) = |x|^{s-1}x$ defines a homeomorphism from \mathbb{B}^n to itself, which is a diffeomorphism if and only if $s = 1$.]

Proof We proceed in four steps:

1. Homeomorphism property of F_s :

For any $s > 0$, the map $F_s: \mathbb{B}^n \rightarrow \mathbb{B}^n$ defined by $F_s(x) = |x|^{s-1}x$ is a homeomorphism.

- If $s \geq 1$, F_s is clearly continuous on \mathbb{B}^n .
- If $0 < s < 1$, continuity at $x = 0$ follows from:

$$\lim_{x \rightarrow 0} |F_s(x)| = \lim_{x \rightarrow 0} |x|^s = 0 = F_s(0).$$

- The inverse is $F_{1/s}$, since $F_s \circ F_{1/s} = F_{1/s} \circ F_s = \text{id}_{\mathbb{B}^n}$.

2. Non-smoothness at origin:

F_s is a diffeomorphism on $\mathbb{B}^n \setminus \{0\}$ but fails to be smooth at 0 when $s \neq 1$:

- For $0 < s < 1$, the derivative at 0 does not exist:

$$\frac{\partial F_s(0)}{\partial x^i} = \lim_{\Delta x^i \rightarrow 0} (\Delta x^i)^{s-1} (0, \dots, 1, \dots, 0) \quad (\text{diverges})$$

- For $s > 1$, the inverse $F_{1/s}$ has $0 < 1/s < 1$ and thus fails to be smooth at 0.

Hence F_s is a diffeomorphism on \mathbb{B}^n if and only if $s = 1$.

3. Constructing a modified atlas:

Fix a point $p \in M$ and choose a smooth chart (U, φ) from the given smooth structure \mathcal{A} on M , such that:

$$\varphi(U) = \mathbb{B}^n \quad \text{and} \quad \varphi(p) = 0.$$

For any $s > 0$, define a new chart (U, φ_s) by:

$$\varphi_s = F_s \circ \varphi,$$

where $F_s(x) = |x|^{s-1}x$ is the homeomorphism from Step 1.

Construct a new atlas \mathcal{A}_s as follows:

$$\mathcal{A}_s = \{(U, \varphi_s)\} \cup \{(V, \psi) \in \mathcal{A}: p \notin V\}.$$

That is, \mathcal{A}_s consists of:

- The single modified chart (U, φ_s) centered at p ,
- All charts from the original atlas \mathcal{A} that do not contain p .

\mathcal{A}_s is a smooth atlas:

- The charts in \mathcal{A}_s cover M : every point $q \neq p$ is covered by some chart (V, ψ) in \mathcal{A} with $p \notin V$, and p is covered by (U, φ_s) .
- The charts in \mathcal{A}_s are pairwise compatible:
 - For any two charts (V_1, ψ_1) and (V_2, ψ_2) in \mathcal{A}_s not containing p , their transition map $\psi_2 \circ \psi_1^{-1}$ is smooth because \mathcal{A} is a smooth atlas.
 - For (U, φ_s) and any (V, ψ) with $p \notin V$, the transition map on $U \cap V$ is:

$$\psi \circ \varphi_s^{-1} = \psi \circ \varphi^{-1} \circ F_{1/s}.$$

This is smooth because $\psi \circ \varphi^{-1}$ is smooth (by compatibility in \mathcal{A}) and $F_{1/s}$ is smooth away from 0.

4. Distinct smooth structures:

We show that the smooth structures induced by \mathcal{A}_s and $\mathcal{A}_{s'}$ are distinct unless $s = s'$.

- Suppose $s \neq s'$. Consider the transition map between (U, φ_s) and $(U, \varphi_{s'})$:

$$\varphi_{s'} \circ \varphi_s^{-1} = F_{s'} \circ F_{1/s} = F_{s'/s}.$$

This is a diffeomorphism on $\mathbb{B}^n \setminus \{0\}$ but fails to be smooth at 0 unless $s'/s = 1$ (i.e., $s = s'$), as shown in Step 2.

- Thus, \mathcal{A}_s and $\mathcal{A}_{s'}$ are not smoothly compatible unless $s = s'$.

Since there are uncountably many choices for $s > 0$, this yields uncountably many distinct smooth structures on M .

□

1-7 Let N denote the north pole $(0, \dots, 0, 1) \in \mathbb{S}^n \subseteq \mathbb{R}^{n+1}$, and let S denote the south pole $(0, \dots, 0, -1)$. Define the stereographic projection $\sigma: \mathbb{S}^n \setminus \{N\} \rightarrow \mathbb{R}^n$ by

$$\sigma(x^1, \dots, x^{n+1}) = \frac{(x^1, \dots, x^n)}{1 - x^{n+1}}.$$

Let $\tilde{\sigma}(x) = -\sigma(-x)$ for $x \in \mathbb{S}^n \setminus \{S\}$.

- (a) For any $x \in \mathbb{S}^n \setminus \{N\}$, show that $\sigma(x) = u$, where $(u, 0)$ is the point where the line through N and x intersects the hyperplane $x^{n+1} = 0$. Similarly, show that $\tilde{\sigma}(x)$ is the point where the line through S and x intersects the same hyperplane. (Thus $\tilde{\sigma}$ is called stereographic projection from the south pole.)
- (b) Show that σ is bijective, with inverse given by

$$\sigma^{-1}(u^1, \dots, u^n) = \frac{(2u^1, \dots, 2u^n, |u|^2 - 1)}{|u|^2 + 1}.$$

- (c) Compute the transition map $\tilde{\sigma} \circ \sigma^{-1}$ and verify that the atlas $\{(\mathbb{S}^n \setminus \{N\}, \sigma), (\mathbb{S}^n \setminus \{S\}, \tilde{\sigma})\}$ defines a smooth structure on \mathbb{S}^n . These are called stereographic coordinates.
- (d) Show that this smooth structure agrees with the one defined in Example 1.31.

Proof

- (a) Since N , x , and $\sigma(x)$ are collinear, there exists $\lambda \in \mathbb{R}$ such that

$$x = \lambda N + (1 - \lambda)\sigma(x).$$

Solving for λ and $\sigma(x)$ gives:

$$\begin{aligned} \lambda &= x^{n+1}, \\ \sigma(x) &= \frac{(x^1, \dots, x^n)}{1 - x^{n+1}}. \end{aligned}$$

The symmetry $\tilde{\sigma}(-x) = -\sigma(x)$ implies $\tilde{\sigma}(x) = -\sigma(-x)$.

- (b) Verify that $\sigma \circ \sigma^{-1} = \text{id}_{\mathbb{R}^n}$ and $\sigma^{-1} \circ \sigma = \text{id}_{\mathbb{S}^n \setminus \{N\}}$

- For $\sigma \circ \sigma^{-1}$, let $(u^1, \dots, u^n) \in \mathbb{R}^n$, we have

$$\begin{aligned} \sigma \circ \sigma^{-1}(u^1, \dots, u^n) &= \sigma \left(\frac{(2u^1, \dots, 2u^n, |u|^2 - 1)}{|u|^2 + 1} \right) \\ &= \frac{\left(\frac{2u^1}{|u|^2 + 1}, \dots, \frac{2u^n}{|u|^2 + 1} \right)}{1 - \frac{|u|^2 - 1}{|u|^2 + 1}} \\ &= (u^1, \dots, u^n). \end{aligned}$$

- For $\sigma^{-1} \circ \sigma$, let $(x^1, \dots, x^{n+1}) \in \mathbb{S}^n \setminus \{N\}$, which means $x^{n+1} \neq 1$ and

$$|\sigma(x)|^2 = \frac{(x^1)^2 + \dots + (x^n)^2}{(1 - x^{n+1})^2} = \frac{1 + x^{n+1}}{1 - x^{n+1}},$$

$$\begin{aligned} \sigma^{-1} \circ \sigma(x^1, \dots, x^{n+1}) &= \sigma^{-1} \left(\frac{(x^1, \dots, x^n)}{1 - x^{n+1}} \right) \\ &= \frac{\left(\frac{2x^1}{1 - x^{n+1}}, \dots, \frac{2x^n}{1 - x^{n+1}}, \frac{|x|^2 - 1}{1 - x^{n+1}} \right)}{\frac{|x|^2 + 1}{1 - x^{n+1}}} \\ &= (x^1, \dots, x^{n+1}). \end{aligned}$$

- (c) It's sufficient to proof that $\tilde{\sigma} \circ \sigma^{-1}$ and $\sigma \circ \tilde{\sigma}^{-1}$ are smooth on $\mathbb{R}^n \setminus \{0\}$. Let $u = (u^1, \dots, u^n) \in \mathbb{R}^n \setminus \{0\}$, it can be easily verified $\tilde{\sigma}^{-1}(u) = \sigma^{-1}(u)$.

$$\tilde{\sigma} \circ \sigma^{-1}(u) = \sigma \circ \tilde{\sigma}^{-1}(u) = \frac{u}{|u|^2},$$

both are smooth on $\mathbb{R}^n \setminus \{0\}$.

- (d) We only proof that $\sigma \circ \pi_i^{-1}$ and $\pi_i \circ \sigma^{-1}$ are smooth for $i = 1, \dots, n+1$, $\tilde{\sigma}$ is completely the same.

- For $i = n+1$,
 - For transition map $\pi_{n+1} \circ \sigma^{-1}$:

$$\sigma(U_{n+1}^+ \setminus \{N\}) = \sigma\{x^{n+1} \in (0, 1)\} = \{|u| > 1 : u \in \mathbb{R}^n\}.$$

$$\begin{aligned} \pi_{n+1} \circ \sigma^{-1}(u) &= \pi_{n+1} \left(\frac{(2u^1, \dots, 2u^n, |u|^2 - 1)}{1 + |u|^2} \right) \\ &= \frac{2u}{1 + |u|^2}. \end{aligned}$$

- For transition map $\sigma \circ \pi_i^{-1}$:

$$\pi_{n+1}(U_{n+1}^+ \setminus \{N\}) = \pi_{n+1}\{x^{n+1} \in (0, 1)\} = \mathbb{B}^n \setminus \{0\}.$$

$$\begin{aligned} \sigma \circ \pi_{n+1}^{-1}(u) &= \sigma(u^1, \dots, u^n, \sqrt{1 - |u|^2}) \\ &= \frac{u}{1 - \sqrt{1 - |u|^2}}. \end{aligned}$$

Both of them are smooth on their domains.

- For $i = 1, \dots, n$,

– For transition map $\pi_i \circ \sigma^{-1}$:

$$\sigma(U_i^+ \setminus \{N\}) = \sigma(U_i^+) = \{u \in \mathbb{R}^n : u^i > 0\}.$$

$$\begin{aligned} \pi_i \circ \sigma^{-1}(u) &= \pi_i \left(\frac{(2u^1, \dots, 2u^n, |u|^2 - 1)}{1 + |u|^2} \right) \\ &= \frac{(2u^1, \dots, \widehat{2u^i}, \dots, |u|^2 - 1)}{|u|^2 + 1} \end{aligned}$$

– For transition map $\sigma \circ \pi_i^{-1}$:

$$\pi_i(U_i^+ \setminus \{N\}) = \pi_i(U_i^+) = \mathbb{B}^n \cap \{u \in \mathbb{R}^n : u^i > 0\}.$$

$$\begin{aligned} \sigma \circ \pi_i^{-1}(u) &= \sigma(u^1, \dots, \sqrt{1 - |u|^2}, u^i, \dots, u^n) \\ &= \frac{(u^1, \dots, \sqrt{1 - |u|^2}, \dots, u^{n-1})}{1 - u^n} \end{aligned}$$

All transition maps are smooth on their domains, confirming compatibility.

□

1-8 By identifying \mathbb{R}^2 with \mathbb{C} , we can think of the unit circle \mathbb{S}^1 as a subset of the complex plane. An angle function on a subset $U \subseteq \mathbb{S}^1$ is a continuous function $\theta: U \rightarrow \mathbb{R}$ such that $e^{i\theta(z)} = z$ for all $z \in U$. Show that there exists an angle function on an open subset $U \subseteq \mathbb{S}^1$ if and only if $U \neq \mathbb{S}^1$. For any such angle function, show that (U, θ) is a smooth coordinate chart for \mathbb{S}^1 with its standard smooth structure.

Proof We prove the existence of an angle function θ on an open subset $U \subseteq \mathbb{S}^1$ for two cases: $U = \mathbb{S}^1$ and $U \subsetneq \mathbb{S}^1$.

- **Nonexistence for $U = \mathbb{S}^1$:**

Assume such $\theta: \mathbb{S}^1 \rightarrow \mathbb{R}$ exists. Define the exponential map:

$$f: \mathbb{R} \rightarrow \mathbb{S}^1, \quad t \mapsto e^{it}.$$

By definition, θ satisfies $f \circ \theta(z) = z$ for all $z \in \mathbb{S}^1$, implying f is injective. However, f is periodic ($f(t + 2\pi) = f(t)$), contradicting injectivity. Thus, θ cannot exist globally.

- **Existence for $U \subsetneq \mathbb{S}^1$:**

Without loss of generality, assume $U = \mathbb{S}^1 \setminus \{p\}$ where $p = (1, 0)$. Restrict f to $(0, 2\pi)$:

$$g := f|_{(0, 2\pi)}: (0, 2\pi) \rightarrow U, \quad t \mapsto e^{it}.$$

- *Bijectivity*: g is bijective by construction, with each $z \in U$ uniquely corresponding to $t \in (0, 2\pi)$.
- *Smoothness*: The Jacobian of g at t is:

$$J(g) = \begin{pmatrix} -\sin t \\ \cos t \end{pmatrix},$$

which has rank 1 everywhere. By the Constant Rank Theorem, g is a diffeomorphism. Its inverse $\varphi := g^{-1}$ defines a local angle function on U .

- **Smooth Atlas Construction:**

Let $V = \mathbb{S}^1 \setminus \{q\}$ where $q = (-1, 0)$, and define:

$$\psi: V \rightarrow (-\pi, \pi), \quad e^{it} \mapsto t.$$

The transition maps between charts (U, φ) and (V, ψ) are:

$$\begin{aligned} \psi \circ \varphi^{-1}(t) &= \begin{cases} t & t \in (0, \pi), \\ t - 2\pi & t \in (\pi, 2\pi), \end{cases} \\ \varphi \circ \psi^{-1}(t) &= \begin{cases} t & t \in (0, \pi), \\ t + 2\pi & t \in (-\pi, 0). \end{cases} \end{aligned}$$

Both are smooth on their domains, confirming $\mathcal{A} = \{(U, \varphi), (V, \psi)\}$ is a smooth atlas for \mathbb{S}^1 .

□

1-9 Complex projective n -space, denoted by \mathbb{CP}^n , is the set of all 1-dimensional complex-linear subspaces of \mathbb{C}^{n+1} , with the quotient topology inherited from the natural projection $\pi: \mathbb{C}^{n+1} \setminus \{0\} \rightarrow \mathbb{CP}^n$. Show that \mathbb{CP}^n is a compact $2n$ -dimensional topological manifold, and show how to give it a smooth structure analogous to the one we constructed for \mathbb{RP}^n . (We use the correspondence

$$(x^1 + iy^1, \dots, x^{n+1} + iy^{n+1}) \leftrightarrow (x^1, y^1, \dots, x^{n+1}, y^{n+1})$$

to identify \mathbb{C}^{n+1} with \mathbb{R}^{2n+2} .)

Proof The construction of smooth structure are exactly the same as in Example 1.5. Here we only prove \mathbb{CP}^n is Hausdorff and second-countable.

- the quotient map π is an open map

Let U be an open subset of $\mathbb{C}^{n+1} \setminus \{0\}$, to prove the quotient map π is an open map, it only suffices to prove that $\pi^{-1} \circ \pi(U)$ is open in $\mathbb{C}^{n+1} \setminus \{0\}$. Since

$$\pi^{-1} \circ \pi(U) = \bigcup_{t \in \mathbb{C}^\times} tU$$

for any fixed $t \in \mathbb{C}^\times$, tU is an open subset, we show that their union $\pi^{-1} \circ \pi(U)$ must be open.

- **Hausdorff property**

Let $[z] = [z_0, \dots, z_n]$ and $[w] = [w_0, \dots, w_n]$ be two distinct points in \mathbb{CP}^n . Then, z and w are not proportional, i.e., there is no $\lambda \in \mathbb{C}^\times$ such that $w = \lambda z$.

Define the function $f: \mathbb{C}^{n+1} \times \mathbb{C}^{n+1} \rightarrow \mathbb{C}$ by

$$f(z, w) = \sum_{i < j} |z_i w_j - z_j w_i|^2.$$

This function is zero if and only if z and w are proportional. Since $[z] \neq [w]$, we have $f(z, w) > 0$.

By continuity of f , there exist open neighborhoods $A \subset \mathbb{C}^{n+1} \setminus \{0\}$ of z and $B \subset \mathbb{C}^{n+1} \setminus \{0\}$ of w such that $f(a, b) > 0$ for all $a \in A$ and $b \in B$.

Let $U = \pi(A)$ and $V = \pi(B)$. Since π is an open map, U and V are open in \mathbb{CP}^n . Moreover, U and V are disjoint, because if $[a] = [b]$ for some $a \in A$ and $b \in B$, then $f(a, b) = 0$, which contradicts the construction of A and B .

Hence, \mathbb{CP}^n is Hausdorff.

- **Second-countable**

Since $\mathbb{C}^{n+1} \setminus \{0\}$ is second-countable, and π is a continuous open map, the quotient space \mathbb{CP}^n is also second-countable.

- **The compactness of \mathbb{CP}^n**

The compactness of \mathbb{CP}^n follows from the fact that it is the continuous image of the unit sphere $\mathbb{S}^{2n+1} \subseteq \mathbb{C}^{n+1}$ under π .

□

1-10 Let k and n be integers satisfying $0 < k < n$, and let $P, Q \subseteq \mathbb{R}^n$ be the linear subspaces spanned by (e_1, \dots, e_k) and (e_{k+1}, \dots, e_n) , respectively, where e_i is the i th standard basis vector for \mathbb{R}^n . For any k -dimensional subspace $S \subseteq \mathbb{R}^n$ that has trivial intersection with Q , show that the coordinate representation $\varphi(S)$ constructed in Example 1.36 is the unique $(n-k) \times k$ matrix B such that S is spanned by the columns of the matrix $\begin{pmatrix} I_k \\ B \end{pmatrix}$, where I_k denotes the $k \times k$ identity matrix.

Proof We prove the existence and uniqueness of the coordinate representation $\varphi(S) = B$ for a k -dimensional subspace $S \subseteq \mathbb{R}^n$ with $S \cap Q = \{0\}$.

- **Existence of the matrix representation:**

Consider the projection map $\pi_P: S \rightarrow P$. We claim π_P is an isomorphism:

- *Injectivity:* Suppose $\pi_P(s) = 0$ for some $s \in S$. Then s has the form uniquely:

$$s = \pi_P(s) + \pi_Q(s) = \pi_Q(s) \in Q.$$

Since $S \cap Q = \{0\}$ by hypothesis, we must have $s = 0$.

- *Surjectivity:* As $\dim S = \dim P = k$ and π_P is injective, it is automatically surjective by the rank-nullity theorem.

Thus π_P is a vector space isomorphism between S and P . Choose

$$\{\pi_P^{-1}(e_1), \dots, \pi_P^{-1}(e_k)\}$$

for the basis of S . Since $\{e_1, e_n\}$ is a basis of V , we have

$$\pi_P^{-1}(e_i) = e_i + \sum_{j=k+1}^n b_{ij}e_j$$

Thus S can be spanned by the columns of the matrix

$$\begin{pmatrix} I_k \\ B \end{pmatrix}$$

under the basis $\{e_1, \dots, e_n\}$ where $B = (b_{ij})$.

- **Uniqueness of the matrix B :**

Suppose there exist two $(n-k) \times k$ matrices B and B' such that:

$$\text{span} \left(\begin{pmatrix} I_k \\ B \end{pmatrix} \right) = \text{span} \left(\begin{pmatrix} I_k \\ B' \end{pmatrix} \right) = S.$$

Then there exists an invertible matrix $C \in \mathbb{R}^{k \times k}$ such that:

$$\begin{pmatrix} I_k \\ B' \end{pmatrix} = \begin{pmatrix} I_k \\ B \end{pmatrix} C.$$

This matrix equation implies:

$$\begin{aligned} I_k &= I_k C &\Rightarrow & C = I_k, \\ B' &= BC = B. \end{aligned}$$

Therefore, B is uniquely determined by S .

□

1-11 Let $M = \overline{\mathbb{B}^n}$, the closed unit ball in \mathbb{R}^n . Show that M is a topological manifold with boundary in which each point in \mathbb{S}^{n-1} is a boundary point and each point in \mathbb{B}^n is an interior point. Show how to give it a smooth structure such that every smooth interior chart is a smooth chart for the standard smooth structure on \mathbb{B}^n . [Hint: consider the map $\pi \circ \sigma^{-1}: \mathbb{R}^n \rightarrow \mathbb{R}^n$, where $\sigma: \mathbb{S}^n \rightarrow \mathbb{R}^n$ is the stereographic projection (Problem 1-7) and π is a projection from \mathbb{R}^{n+1} to \mathbb{R}^n that omits some coordinate other than the last.]

Proof We establish that $\overline{\mathbb{B}^n}$ is a smooth manifold with boundary, where \mathbb{S}^{n-1} constitutes the boundary and \mathbb{B}^n the interior, by constructing an explicit smooth structure. (This proof proceeds independently of the hint.)

• **Topological manifold structure:**

- For $x \in \mathbb{B}^n$: The identity chart $(\mathbb{B}^n, \text{id}_{\mathbb{B}^n})$ suffices.
- For $x \in \mathbb{S}^{n-1}$: We define charts via coordinate projection:

$$\begin{aligned} U_i^+ &= \{x \in \mathbb{R}^n \mid x_i > 0\}, \\ V_i^+ &= U_i^+ \cap \overline{\mathbb{B}^n}, \\ \varphi_i &= \pi_i \circ \pi_{n+1}^{-1}: V_i^+ \rightarrow \mathbb{H}^n \cap \mathbb{B}^n, \\ \varphi_i(x^1, \dots, x^n) &= \pi_i(x^1, \dots, x^n, \sqrt{1 - |x|^2}) = (x^1, \dots, \widehat{x^i}, \dots, \sqrt{1 - |x|^2}) \end{aligned}$$

where $\pi_i: \mathbb{S}^n \rightarrow \mathbb{R}^n$ omits the i -th coordinate. The collection

$$\{(V_i^\pm, \varphi_i)\}$$

forms boundary charts since π_i and π_{n+1} are both homeomorphic on V_i^+ .

• **Smooth structure:**

- The charts $\{(V_i^\pm, \varphi_i)\}$ are compatible with each other, since the standard smooth structure of \mathbb{S}^n ensures transition maps

$$\varphi_j \circ \varphi_i^{-1} = \pi_j \circ \pi_{n+1}^{-1} \circ \pi_{n+1} \circ \pi_i^{-1} = \pi_j \circ \pi_i^{-1}$$

are diffeomorphisms on their domains $\varphi_i(V_i^+ \cap V_j^+)$.

- Boundary charts and interior chart are compatible, since the Jacobian of transition map

$$|J(\varphi_i \circ \text{id}_{\mathbb{B}^n}^{-1})| = (-1)^{n-1} \frac{x^i}{\sqrt{1 - |x|^2}} \neq 0$$

on its domain $\mathbb{B}^n \cap V_i^\pm$. Thus the smooth atlas

$$\mathcal{A} = \{(V_i^\pm, \varphi_i)\} \cup (\mathbb{B}^n, \text{id}_{\mathbb{B}^n})$$

yields a smooth structure of $\overline{\mathbb{B}^n}$.

• **Boundary and interior identification:**

- For $x \in \mathbb{S}^{n-1}$, some boundary chart (V_i^\pm, φ_i) satisfies

$$\varphi_i(x) = (x^1, \dots, \widehat{x^i}, \dots, 0) \in \partial\mathbb{H}^n,$$

confirming $\mathbb{S}^{n-1} \subseteq \partial\overline{\mathbb{B}}^n$ via Theorem 1.46 (Boundary Invariance).

- For $x \in \mathbb{B}^n$, the identity chart maps x to $\mathbb{B}^n \subseteq \mathbb{R}^n$, proving $\mathbb{B}^n \subseteq \text{Int}(\overline{\mathbb{B}}^n)$.
- Since $\overline{\mathbb{B}}^n = \mathbb{B}^n \cup \mathbb{S}^{n-1}$, we conclude:

$$\partial\overline{\mathbb{B}}^n = \mathbb{S}^{n-1}, \quad \text{Int}(\overline{\mathbb{B}}^n) = \mathbb{B}^n.$$

□

1-12 Prove Proposition 1.45 (a product of smooth manifolds together with one smooth manifold with boundary is a smooth manifold with boundary).

Proof

- **Model Space Identification:** First observe that $\mathbb{R}^m \times \mathbb{H}^n \cong \mathbb{H}^{m+n}$ via the diffeomorphism:

$$\begin{aligned} \varphi: \mathbb{R}^m \times \mathbb{H}^n &\rightarrow \mathbb{H}^{m+n} \\ (x^1, \dots, x^m, y^1, \dots, y^n) &\mapsto (x^1, \dots, x^m, y^1, \dots, y^n) \end{aligned}$$

This preserves boundaries since $\varphi(\mathbb{R}^m \times \partial\mathbb{H}^n) = \partial\mathbb{H}^{m+n}$.

- **Chart Construction:** Let $M = M_1 \times \dots \times M_k$ ($\dim m = \sum m_i$) and N ($\dim n$) with $\partial N \neq \emptyset$.

- **Interior Charts:** For $(p, q) \in M \times \text{Int}(N)$:

- (a) Take smooth charts (U_i, φ_i) about $p_i \in M_i$ with $\varphi_i: U_i \rightarrow \mathbb{R}^{m_i}$
- (b) Take interior chart (V, ψ) about $q \in N$ with $\psi: V \rightarrow \mathbb{R}^n$
- (c) The product chart is:

$$\left(\prod_{i=1}^k U_i \times V, (\varphi_1, \dots, \varphi_k, \psi) \right)$$

mapping to $\mathbb{R}^m \times \mathbb{R}^n \subseteq \mathbb{H}^{m+n}$

- **Boundary Charts:** For $(p, q) \in M \times \partial N$:

- (a) Take smooth charts (U_i, φ_i) as above
- (b) Take boundary chart (V, ψ) with $\psi: V \rightarrow \mathbb{H}^n$ and $\psi(q) \in \partial\mathbb{H}^n$
- (c) The product chart is:

$$\left(\prod_{i=1}^k U_i \times V, (\varphi_1, \dots, \varphi_k, \psi) \right)$$

mapping to $\mathbb{R}^m \times \mathbb{H}^n \cong \mathbb{H}^{m+n}$ with boundary points precisely when $q \in \partial N$

- **Chart Compatibility:**

- For two interior charts, the transition map is:

$$(\varphi'_1, \dots, \varphi'_k, \psi') \circ (\varphi_1, \dots, \varphi_k, \psi)^{-1} = (\varphi'_1 \circ \varphi_1^{-1}, \dots, \varphi'_k \circ \varphi_k^{-1}, \psi' \circ \psi^{-1})$$

which is smooth since each component is smooth.

- For boundary charts, the same holds because $\psi' \circ \psi^{-1}$ is smooth as a map between subsets of \mathbb{H}^n .

- For mixed cases (one interior, one boundary chart), the transition maps are smooth by the boundary compatibility of N 's charts.

- **Boundary Characterization:**

- If (p, q) is mapped to $\partial\mathbb{H}^{m+n}$ in some chart, then by Theorem 1.46 it holds in all charts, this occurs precisely when $q \in \partial N$, proving:

$$\partial(M \times N) = M \times \partial N$$

- The interior is correspondingly $M \times \text{Int}(N)$

Thus $M \times N$ is a smooth manifold with boundary as claimed. □

Chapter 2

Smooth Maps

2-1 Define $f : \mathbb{R} \rightarrow \mathbb{R}$ by

$$f(x) = \begin{cases} 1, & x \geq 0, \\ 0, & x < 0. \end{cases}$$

Show that for every $x \in \mathbb{R}$, there exist smooth coordinate charts (U, φ) containing x , and (V, ψ) containing $f(x)$, such that the map

$$\psi \circ f \circ \varphi^{-1}$$

is smooth as a function from $\varphi(U \cap f^{-1}(V))$ to \mathbb{R} . However, f is not smooth in the sense we have defined in this chapter.