Introduction to Smooth Manifolds Exercise Solutions

Larry

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Chapter 1

Smooth Manifolds

1-1 Let X be the set of all points $(x, y) \in \mathbb{R}^2$ such that $y = \pm 1$, and let M be the quotient of X by the equivalence relation generated by $(x, 1) \sim (x, -1)$ for all $x \neq 0$. Show that M is locally Euclidean and second-countable, but not Hausdorff. (This space is called the *line with two origins*.)

Proof Let $\pi\colon X \to M$ be the quotient map, where

$$X = \{(x, y) \in \mathbb{R}^2 : y = \pm 1\}$$

and the equivalence relation is generated by $(x, 1) \sim (x, -1)$ for all $x \neq 0$. Let

$$p = \pi(0, 1), \quad q = \pi(0, -1)$$

denote the two distinct equivalence classes in M corresponding to the two origins.
To describe a basis for the topology on M, for any open interval $W \subseteq \mathbb{R}$, define the following sets:

$$\begin{aligned} U_W &= \pi\left(\{(x, \pm 1) : x \in W\}\right), && \text{for } 0 \notin W, \\ U_W^+ &= \pi\left(\{(x, \pm 1) : x \in W \setminus \{0\}\} \cup \{(0, 1)\}\right), && \text{for } 0 \in W, \\ U_W^- &= \pi\left(\{(x, \pm 1) : x \in W \setminus \{0\}\} \cup \{(0, -1)\}\right), && \text{for } 0 \in W. \end{aligned}$$

These sets are open in the quotient topology on M, where:

- U_W is a basic open set in M when W does not contain 0;
- U_W^+ is a neighborhood of the point $p = \pi(0, 1)$;
- U_W^- is a neighborhood of the point $q = \pi(0, -1)$.

We next show that M is second-countable and locally Euclidean but not Hausdorff.

- **Second-countability:**
  Define:
  $$\mathcal{B} = \{U_W, U_W^\pm : W \text{is an open interval with rational endpoints.}\}$$
  Since there are only countably many open intervals in $\mathbb{R}$ with rational endpoints, the collection $\mathcal{B}$ is countable.

  We claim that $\mathcal{B}$ is a basis for the topology on M. Let $U \subseteq M$ be any open set and let $x \in U$. We want to find some $B \in \mathcal{B}$ such that $x \in B \subseteq U$.
  There are two cases to consider:

  - If $x \in M$ corresponds to a point $\pi(x_0, \pm 1)$ with $x_0 \neq 0$, then $\pi^{-1}(U)$ is an open subset of X containing both $(x_0, 1)$ and $(x_0, -1)$ (since they are identified when $x_0 \neq 0$). Since X inherits the subspace topology from $\mathbb{R}^2$, there exists an open interval $W \ni x_0$ with rational endpoints such that $(W \times \{\pm 1\}) \subseteq \pi^{-1}(U)$. Then $U_W = \pi((W \times \{\pm 1\})) \subseteq U$, and $U_W \in \mathcal{B}$ if $0 \notin W$.
  - If $x = p = \pi(0, 1)$ or $x = q = \pi(0, -1)$, then $x \in U$ implies $\pi^{-1}(U)$ contains either $(0, 1)$ or $(0, -1)$ respectively. Since $\pi^{-1}(U)$ is open in X, there exists an open interval $W \ni 0$ such that:
    - $(W \setminus \{0\}) \times \{\pm 1\} \subseteq \pi^{-1}(U)$,

- * and either $(0,1) \in \pi^{-1}(U)$ or $(0,-1) \in \pi^{-1}(U)$.

  Hence, either $U_W^+ \subseteq U$ or $U_W^- \subseteq U$, and such sets are in $\mathcal{B}$ because W has rational endpoints.

Therefore, M is second-countable.

- **Local Euclidean property:**
  - For $x \notin \{p, q\}$, define a map

$$\varphi\colon U_{\mathbb{R}\setminus\{0\}} \to \mathbb{R} \setminus \{0\}, \quad \pi(x, \pm 1) \mapsto x$$

Clearly φ is bijective. Let V be an open subset of $\mathbb{R} \setminus \{0\}$, $\varphi^{-1}(V)$ is open if and only if $\pi^{-1} \circ \varphi^{-1}(V)$ is open in X. Since

$$\pi^{-1} \circ \varphi^{-1}(V) = (V \times \{-1\}) \cup (V \times \{1\})$$

which is open in X, $\varphi^{-1}(V)$ is open in $U_{\mathbb{R}\setminus\{0\}}$. This indicates that φ is continuous. Let $U \subseteq U_{\mathbb{R}\setminus\{0\}}$ be an open subset of M, it means $\pi^{-1}(U)$ is open in X. Since

$$\varphi(U) = \{x\colon (x,1) \in \pi^{-1}(U)\} \cup \{x\colon (x,-1) \in \pi^{-1}(U)\}$$

is open in X, φ yields a homeomorphism. Hence, every point with $x \neq 0$ has a neighborhood homeomorphic to $\mathbb{R} \setminus \{0\}$, which is locally Euclidean.

  - For $x = p$, define the map

$$\psi_+\colon U^+_{(-1,1)} \to (-1,1), \quad \psi_+(\pi(x, \pm 1)) = x, \quad \psi_+(p) = 0.$$

This map is well-defined and bijective. To show that ψ_+ is a homeomorphism, it suffices to verify that both ψ_+ and its inverse are continuous at p and 0, respectively.
For any $\varepsilon \in (0,1)$, we have

$$\psi_+^{-1}((-\varepsilon, \varepsilon)) = U^+_{(-\varepsilon,\varepsilon)} \quad \text{and} \quad \psi_+(U^+_{(-\varepsilon,\varepsilon)}) = (-\varepsilon, \varepsilon),$$

which shows that ψ_+ is continuous at p and its inverse is continuous at 0. Therefore, ψ_+ is a homeomorphism.

  - For $x = q$, The proof is identical to Case 2.

- **Not Hausdorff:**
  We show that M is not Hausdorff by exhibiting two points that cannot be separated by disjoint open neighborhoods.

  Consider the two points $p = \pi(0,1)$ and $q = \pi(0,-1)$. Suppose for contradiction that there exist disjoint open sets U and V in M such that $p \in U$ and $q \in V$.

  Since the sets U and V are open neighborhoods of p and q, respectively, there exist basic open sets $U_W^+ \subseteq U$ and $U_{W'}^- \subseteq V$, where W and W' are open intervals containing 0.

Let $W'' = W \cap W'$; then $0 \in W''$, so $W'' \setminus \{0\} \neq \emptyset$. Define:

$$A := \pi\left((W'' \setminus \{0\}) \times \{\pm 1\}\right) = U_{W'' \setminus \{0\}} \subseteq U_W^+ \cap U_{W'}^-.$$

Therefore, U_W^+ and $U_{W'}^-$ are not disjoint; they always intersect in a nonempty open set. This contradicts the assumption that p and q can be separated by disjoint open sets. Hence, M is not Hausdorff.

$\square$

1-2 Show that a disjoint union of uncountably many copies of $\mathbb{R}$ is locally Euclidean and Hausdorff, but not second-countable.

Proof Let $X = \coprod_{\alpha \in A} \mathbb{R}_\alpha$, A is an uncountable index set. $U \subseteq X$ is open if and only if $\forall \alpha \in A$, $\{x \in \mathbb{R} : (\alpha, x) \in U\}$ is open in $\mathbb{R}$.

1. X is locally Euclidean.
   By definition, $\mathbb{R}_\alpha = \{(\alpha, x) : x \in \mathbb{R}\}$. $\forall (\alpha, x) \in \mathbb{R}_\alpha$, since $\mathbb{R}_\alpha$ is an open subset of X(By the definition of topology of X) and $\mathbb{R}_\alpha$ is homeomorphic to $\mathbb{R}$, X is locally Euclidean.

2. X is Hausdorff.
   Let $(\alpha, x), (\beta, y) \in X$. if $\alpha \neq \beta$, clearly we have two disjoint open subset $\mathbb{R}_\alpha$ and $\mathbb{R}_\beta$ such that $(\alpha, x) \in \mathbb{R}_\alpha$ and $(\beta, x) \in \mathbb{R}_\beta$. if $\alpha = \beta$, since $\mathbb{R}$ is Hausdorff, we can find two disjoint open subset $U, V \in \mathbb{R}$ such that $x \in U$ and $y \in V$. $(\alpha, U), (\beta, V)$ are two disjoint open subset of X.

3. X is not second-countable.
   Assume that X is second-countable with its countable basis $\mathcal{B} = \{B_i\}$ and I is an countable index set. Since $\mathbb{R}_\alpha$ is an non-empty open subset of X, we can always find $B_i \in \mathcal{B}$ such that $B_i \subseteq \mathbb{R}_\alpha$. By Axiom of Choice, we can define

   $$f \colon A \to I, \quad \alpha \mapsto i \ \text{ s.t. } \ B_i \subseteq \mathbb{R}_\alpha.$$

   Clearly f is injective. This leads to an contradiction, since A is uncountable but I is countable.

$\square$

1-3 A topology space is said to be σ*-compact* if it can be expressed as a union of countably many compact subspaces. Show that a locally Euclidean Hausdorff space is a topological manifold if and only if it is σ-compact.

Proof

($\Rightarrow$) Every topological manifold admits a countable basis $\mathcal{B} = \{B_i\}$ of precompact coordinate balls(Lemma 1.10). The collection $\{\overline{B_i} \mid B_i \in \mathcal{B}\}$ implies that the manifold is σ-compact.

($\Leftarrow$) Let X be a locally Euclidean Hausdorff space that is σ-compact. By definition, there exists a countable family of compact subsets $\{K_n\}_{n\in\mathbb{N}}$ such that $X = \bigcup_{n\in\mathbb{N}} K_n$. Since X is locally Euclidean, for each K_n, there exists a finite open cover $\{U_{n_i}\}_{i=1}^{k_n}$ where each (U_{n_i}, φ_{n_i}) is a coordinate chart.

1. For each U_{n_i}, choose a precompact coordinate ball $B_{n_i} \subseteq U_{n_i}$ (possible by local Euclideanness, see Lemma 1.10). The collection $\{B_{n_i}\}$ is countable and covers X.

2. Each B_{n_i}, being homeomorphic to an open ball in $\mathbb{R}^n$, admits a countable basis. A countable union of countable bases remains countable, thus X is second-countable.

Therefore, X satisfies all axioms of a topological manifold (locally Euclidean + Hausdorff + second-countable).

$\square$

1-4 Let M be a topological manifold, and let $\mathcal{U}$ be an open cover of M.

(a) Assuming that each set in $\mathcal{U}$ intersects only finitely many others, show that $\mathcal{U}$ is locally finite.

(b) Give an example to show that the converse to (a) may be false.

(c) Now assume that the sets in $\mathcal{U}$ are precompact in M, and prove the converse: if $\mathcal{U}$ is locally finite, then each set in $\mathcal{U}$ intersects only finitely many others.

Proof

(a) Omitted as "Easy".

(b) Let $M = \mathbb{R}, \mathcal{U} = \{(n, \infty) : n \in \mathbb{N}\} \cup \{(-\infty, 1)\}$

(c) Assume $\mathcal{U}$ is a locally finite open cover of M, and that each set in $\mathcal{U}$ is precompact. Fix $U \in \mathcal{U}$, and define

$$\mathcal{V} = \{V \in \mathcal{U} \colon V \cap U \neq \emptyset\},$$

the collection of all elements in $\mathcal{U}$ that intersect U.

Since U is precompact, its closure $\overline{U}$ is compact. Because $\mathcal{U}$ is locally finite, for every point $x \in \overline{U}$, there exists an open neighborhood V_x intersects only finitely many elements of $\mathcal{U}$.

Then $\{V_x\}_{x \in \overline{U}}$ is an open cover of $\overline{U}$ by elements of $\mathcal{U}$, so by compactness, there exists a finite subcover:

$$\overline{U} \subseteq \bigcup_{i=1}^{n} V_{x_i}.$$

Now for each $i = 1, \ldots, n$, define

$$\mathcal{V}_i = \{W \in \mathcal{U} \colon W \cap V_{x_i} \neq \emptyset\}.$$

Since each V_{x_i} intersects only finitely many elements of $\mathcal{U}$, each $\mathcal{V}_i$ is finite. Now, take any $V \in \mathcal{V}$. Then $V \cap U \neq \emptyset$, and since $\overline{U} \subseteq \bigcup_{i=1}^{n} V_{x_i}$, there exists some i such that $V \cap V_{x_i} \neq \emptyset$, implying $V \in \mathcal{V}_i$. Thus,

$$\mathcal{V} \subseteq \bigcup_{i=1}^{n} \mathcal{V}_i.$$

As each $\mathcal{V}_i$ is finite and n is finite, it follows that $\mathcal{V}$ is finite.

Therefore, each $U \in \mathcal{U}$ intersects only finitely many other elements of $\mathcal{U}$.

$\square$

1-5 Suppose M is a locally Euclidean Hausdorff space. Show that M is second countable if and only if it is paracompact and has countably many connected components.

Proof

($\Rightarrow$) By Proposition 1.11, second-countable property of topological manifold admits at most countably many connected components. Theorem 1.15 shows that every topological manifold is paracompact.

($\Leftarrow$) Suppose M is paracompact and has countably many connected components. It suffices to show that each connected component is second countable, since a countable union of second countable spaces is second countable.

Let C be a connected component of M. Since M is locally Euclidean, there exists a basis of precompact coordinate charts. Let $\mathcal{U}$ be an open cover of C by such charts. By paracompactness, there exists a locally finite refinement $\mathcal{V}$ of $\mathcal{U}$ consisting of precompact coordinate domains.

To show that C is second countable, we will prove that $\mathcal{V}$ is countable. For this, define an equivalence relation $\sim$ on $\mathcal{V}$: for $U, V \in \mathcal{V}$, declare $U \sim V$ if there exists a finite sequence $U = U_0, U_1, \ldots, U_n = V$ in $\mathcal{V}$ such that $U_i \cap U_{i+1} \neq \emptyset$ for all i. Denote by $[U]$ the equivalence class of U under this relation.

We now show that $[U]$ is an open and closed subset of C:

- $[U]$ *is open:* U is a union of open set by definition.
- $[U]$ *is closed:* Let $x \in C \setminus [U]$. Since $\mathcal{V}$ is an open cover of C, there exists $V \in \mathcal{V}$ such that $x \in V$. If V intersected any element of $[U]$, then V would be connected to U via a finite chain of overlapping sets, and hence $x \in [U]$, contradicting $x \in C \setminus [U]$. Therefore, x has an open neighborhood contained in $C \setminus [U]$.

  Since this holds for arbitrary $x \in C \setminus [U]$, we conclude that $C \setminus [U]$ is open, so $[U]$ is closed.

Since C is connected and $[U]$ is nonempty, open, and closed in C, it must be that $[U] = C$. Hence, every element of $\mathcal{V}$ can be connected to U via a finite chain of overlapping sets.

Now define inductively:

$$\mathcal{V}_1 = \{U\}, \quad \mathcal{V}_{n+1} = \{V \in \mathcal{V} \colon \exists W \in \mathcal{V}_n \text{ with } V \cap W \neq \emptyset\}.$$

Then $\bigcup_{n=1}^{\infty} \mathcal{V}_n = \mathcal{V}$. By Problem 1-4, each $\mathcal{V}_n$ is finite. Thus, $\mathcal{V}$ is a countable collection.

Since $\mathcal{V}$ is a countable open cover of C by coordinate domains, the collection

$$\overline{\mathcal{V}} = \{\overline{V} \colon V \in \mathcal{V}\}$$

covers C with countably many compact subsets, thus C is σ-compact. By Problem 1-3, C is second-countable. Finally, M is a countable disjoint union of its connected components, each of which is second countable, so M is second-countable.

$\square$

1-6 Let M be a nonempty topological manifold of dimension $n \geq 1$. If M has a smooth structure, show that it has uncountably many distinct ones. [Hint: first show that for any $s > 0$, $F_s(x) = |x|^{s-1}x$ defines a homeomorphism from $\mathbb{B}^n$ to itself, which is a diffeomorphism if and only if $s = 1$.]

Proof We proceed in four steps:

1. **Homeomorphism property of F_s:**
For any $s > 0$, the map $F_s \colon \mathbb{B}^n \to \mathbb{B}^n$ defined by $F_s(x) = |x|^{s-1}x$ is a homeomorphism.

   - If $s \geq 1$, F_s is clearly continuous on $\mathbb{B}^n$.
   - If $0 < s < 1$, continuity at $x = 0$ follows from:
   $$\lim_{x \to 0} |F_s(x)| = \lim_{x \to 0} |x|^s = 0 = F_s(0).$$
   - The inverse is $F_{1/s}$, since $F_s \circ F_{1/s} = F_{1/s} \circ F_s = \mathrm{id}_{\mathbb{B}^n}$.

2. **Non-smoothness at origin:**
F_s is a diffeomorphism on $\mathbb{B}^n \setminus \{0\}$ but fails to be smooth at 0 when $s \neq 1$:

   - For $0 < s < 1$, the derivative at 0 does not exist:
   $$\frac{\partial F_s(0)}{\partial x^i} = \lim_{\Delta x^i \to 0} (\Delta x^i)^{s-1}(0, \ldots, 1, \ldots, 0) \quad \text{(diverges)}$$
   - For $s > 1$, the inverse $F_{1/s}$ has $0 < 1/s < 1$ and thus fails to be smooth at 0.

   Hence F_s is a diffeomorphism on $\mathbb{B}^n$ if and only if $s = 1$.

3. **Constructing a modified atlas:**
Fix a point $p \in M$ and choose a smooth chart (U, φ) from the given smooth structure $\mathcal{A}$ on M, such that:
$$\varphi(U) = \mathbb{B}^n \quad \text{and} \quad \varphi(p) = 0.$$
For any $s > 0$, define a new chart (U, φ_s) by:
$$\varphi_s = F_s \circ \varphi,$$
where $F_s(x) = |x|^{s-1}x$ is the homeomorphism from Step 1.

   Construct a new atlas $\mathcal{A}_s$ as follows:
$$\mathcal{A}_s = \{(U, \varphi_s)\} \cup \{(V, \psi) \in \mathcal{A} \colon p \notin V\}.$$
That is, $\mathcal{A}_s$ consists of:

   - The single modified chart (U, φ_s) centered at p,
   - All charts from the original atlas $\mathcal{A}$ that do not contain p.

   $\mathcal{A}_s$ is a smooth atlas:

- The charts in $\mathcal{A}_s$ cover M: every point $q \neq p$ is covered by some chart (V, ψ) in $\mathcal{A}$ with $p \notin V$, and p is covered by (U, φ_s).
- The charts in $\mathcal{A}_s$ are pairwise compatible:
  - For any two charts (V_1, ψ_1) and (V_2, ψ_2) in $\mathcal{A}_s$ not containing p, their transition map $\psi_2 \circ \psi_1^{-1}$ is smooth because $\mathcal{A}$ is a smooth atlas.
  - For (U, φ_s) and any (V, ψ) with $p \notin V$, the transition map on $U \cap V$ is:

    $$\psi \circ \varphi_s^{-1} = \psi \circ \varphi^{-1} \circ F_{1/s}.$$

    This is smooth because $\psi \circ \varphi^{-1}$ is smooth (by compatibility in $\mathcal{A}$) and $F_{1/s}$ is smooth away from 0.

4. **Distinct smooth structures:**
   We show that the smooth structures induced by $\mathcal{A}_s$ and $\mathcal{A}_{s'}$ are distinct unless $s = s'$.

   - Suppose $s \neq s'$. Consider the transition map between (U, φ_s) and $(U, \varphi_{s'})$:

     $$\varphi_{s'} \circ \varphi_s^{-1} = F_{s'} \circ F_{1/s} = F_{s'/s}.$$

     This is a diffeomorphism on $\mathbb{B}^n \setminus \{0\}$ but fails to be smooth at 0 unless $s'/s = 1$ (i.e., $s = s'$), as shown in Step 2.
   - Thus, $\mathcal{A}_s$ and $\mathcal{A}_{s'}$ are not smoothly compatible unless $s = s'$.

   Since there are uncountably many choices for $s > 0$, this yields uncountably many distinct smooth structures on M.

$\square$

1-7 Let N denote the north pole $(0, \dots, 0, 1) \in \mathbb{S}^n \subseteq \mathbb{R}^{n+1}$, and let S denote the south pole $(0, \dots, 0, -1)$. Define the stereographic projection $\sigma \colon \mathbb{S}^n \setminus \{N\} \to \mathbb{R}^n$ by

$$\sigma(x^1, \dots, x^{n+1}) = \frac{(x^1, \dots, x^n)}{1 - x^{n+1}}.$$

Let $\tilde{\sigma}(x) = -\sigma(-x)$ for $x \in \mathbb{S}^n \setminus \{S\}$.

(a) For any $x \in \mathbb{S}^n \setminus \{N\}$, show that $\sigma(x) = u$, where $(u, 0)$ is the point where the line through N and x intersects the hyperplane $x^{n+1} = 0$. Similarly, show that $\tilde{\sigma}(x)$ is the point where the line through S and x intersects the same hyperplane. (Thus $\tilde{\sigma}$ is called stereographic projection from the south pole.)

(b) Show that σ is bijective, with inverse given by

$$\sigma^{-1}(u^1, \dots, u^n) = \frac{(2u^1, \dots, 2u^n, |u|^2 - 1)}{|u|^2 + 1}.$$

(c) Compute the transition map $\tilde{\sigma} \circ \sigma^{-1}$ and verify that the atlas $\{(\mathbb{S}^n \setminus \{N\}, \sigma), (\mathbb{S}^n \setminus \{S\}, \tilde{\sigma})\}$ defines a smooth structure on $\mathbb{S}^n$. These are called stereographic coordinates.

(d) Show that this smooth structure agrees with the one defined in Example 1.31.

Proof

(a) Since N, x, and $\sigma(x)$ are collinear, there exists $\lambda \in \mathbb{R}$ such that

$$x = \lambda N + (1 - \lambda)\sigma(x).$$

Solving for λ and $\sigma(x)$ gives:

$$\lambda = x^{n+1},$$

$$\sigma(x) = \frac{(x^1, \dots, x^n)}{1 - x^{n+1}}.$$

The symmetry $\tilde{\sigma}(-x) = -\sigma(x)$ implies $\tilde{\sigma}(x) = -\sigma(-x)$.

(b) Verify that $\sigma \circ \sigma^{-1} = \mathrm{id}_{\mathbb{R}^n}$ and $\sigma^{-1} \circ \sigma = \mathrm{id}_{\mathbb{S}^n \setminus \{N\}}$

- For $\sigma \circ \sigma^{-1}$, let $(u^1, \dots, u^n) \in \mathbb{R}^n$, we have

$$\begin{aligned}
\sigma \circ \sigma^{-1}(u^1, \dots, u^n) &= \sigma\left(\frac{(2u^1, \dots, 2u^n, |u|^2 - 1)}{|u|^2 + 1}\right) \\
&= \frac{\left(\frac{2u^1}{|u|^2+1}, \dots, \frac{2u^n}{|u|^2+1}\right)}{1 - \frac{|u|^2-1}{|u|^2+1}} \\
&= (u^1, \dots, u^n).
\end{aligned}$$

- For $\sigma^{-1} \circ \sigma$, let $(x^1, \ldots, x^{n+1}) \in \mathbb{S}^n \setminus \{N\}$, which means $x^{n+1} \neq 1$ and

$$|\sigma(x)|^2 = \frac{(x^1)^2 + \cdots + (x^n)^2}{(1 - x^{n+1})^2} = \frac{1 + x^{n+1}}{1 - x^{n+1}},$$

$$\begin{aligned} \sigma^{-1} \circ \sigma(x^1, \ldots, x^{n+1}) &= \sigma^{-1}\left(\frac{(x^1, \ldots, x^n)}{1 - x^{n+1}}\right) \\ &= \frac{\left(\frac{2x^1}{1-x^{n+1}}, \ldots, \frac{2x^n}{1-x^{n+1}}, \frac{|x|^2-1}{1-x^{n+1}}\right)}{\frac{|x|^2+1}{1-x^{n+1}}} \\ &= (x^1, \ldots, x^{n+1}). \end{aligned}$$

(c) It's sufficient to proof that $\tilde{\sigma} \circ \sigma^{-1}$ and $\sigma \circ \tilde{\sigma}^{-1}$ are smooth on $\mathbb{R}^n \setminus \{0\}$. Let $u = (u^1, \ldots, u^n) \in \mathbb{R}^n \setminus \{0\}$, it can be easily verified $\tilde{\sigma}^{-1}(u) = \sigma^{-1}(u)$.

$$\tilde{\sigma} \circ \sigma^{-1}(u) = \sigma \circ \tilde{\sigma}^{-1}(u) = \frac{u}{|u|^2},$$

both are smooth on $\mathbb{R}^n \setminus \{0\}$.

(d) We only proof that $\sigma \circ \pi_i^{-1}$ and $\pi_i \circ \sigma^{-1}$ are smooth for $i = 1, \ldots, n+1$, $\tilde{\sigma}$ is completely the same.

- For $i = n + 1$,
  - For transition map $\pi_{n+1} \circ \sigma^{-1}$:

$$\sigma(U_{n+1}^+ \setminus \{N\}) = \sigma\{x^{n+1} \in (0, 1)\} = \{|u| > 1 \colon u \in \mathbb{R}^n\}.$$

$$\begin{aligned} \pi_{n+1} \circ \sigma^{-1}(u) &= \pi_{n+1}\left(\frac{(2u^1, \ldots, 2u^n, |u|^2 - 1)}{1 + |u|^2}\right) \\ &= \frac{2u}{1 + |u|^2}. \end{aligned}$$

  - For transition map $\sigma \circ \pi_i^{-1}$:

$$\pi_{n+1}(U_{n+1}^+ \setminus \{N\}) = \pi_{n+1}\{x^{n+1} \in (0, 1)\} = \mathbb{B}^n \setminus \{0\}.$$

$$\begin{aligned} \sigma \circ \pi_{n+1}^{-1}(u) &= \sigma(u^1, \ldots, u^n, \sqrt{1 - |u|^2}) \\ &= \frac{u}{1 - \sqrt{1 - |u|^2}}. \end{aligned}$$

  Both of them are smooth on their domains.

- For $i = 1, \ldots, n$,

– For transition map $\pi_i \circ \sigma^{-1}$:

$$\sigma(U_i^+ \setminus \{N\}) = \sigma(U_i^+) = \{u \in \mathbb{R}^n \colon u^i > 0\}.$$

$$\begin{aligned}
\pi_i \circ \sigma^{-1}(u) &= \pi_i\left(\frac{(2u^1, \dots, 2u^n, |u|^2 - 1)}{1 + |u|^2}\right) \\
&= \frac{(2u^1, \dots, \widehat{2u^i}, \dots, |u|^2 - 1)}{|u|^2 + 1}
\end{aligned}$$

– For transition map $\sigma \circ \pi_i^{-1}$:

$$\pi_i(U_i^+ \setminus \{N\}) = \pi_i(U_i^+) = \mathbb{B}^n \cap \{u \in \mathbb{R}^n \colon u^i > 0\}.$$

$$\begin{aligned}
\sigma \circ \pi_i^{-1}(u) &= \sigma(u^1, \dots, \sqrt{1 - |u|^2}, u^i, \dots, u^n) \\
&= \frac{(u^1, \dots, \sqrt{1 - |u|^2}, \dots, u^{n-1})}{1 - u^n}
\end{aligned}$$

All transition maps are smooth on their domains, confirming compatibility.

$\square$

1-8 By identifying $\mathbb{R}^2$ with $\mathbb{C}$, we can think of the unit circle $\mathbb{S}^1$ as a subset of the complex plane. An angle function on a subset $U \subseteq \mathbb{S}^1$ is a continuous function $\theta\colon U \to \mathbb{R}$ such that $\mathrm{e}^{\mathrm{i}\theta(z)} = z$ for all $z \in U$. Show that there exists an angle function on an open subset $U \subseteq \mathbb{S}^1$ if and only if $U \neq \mathbb{S}^1$. For any such angle function, show that (U, θ) is a smooth coordinate chart for $\mathbb{S}^1$ with its standard smooth structure.

Proof We prove the existence of an angle function θ on an open subset $U \subseteq \mathbb{S}^1$ for two cases: $U = \mathbb{S}^1$ and $U \subsetneq \mathbb{S}^1$.

- **Nonexistence for $U = \mathbb{S}^1$:**
  Assume such $\theta\colon \mathbb{S}^1 \to \mathbb{R}$ exists. Define the exponential map:
  $$f\colon \mathbb{R} \to \mathbb{S}^1, \quad t \mapsto \mathrm{e}^{\mathrm{i}t}.$$
  By definition, θ satisfies $f \circ \theta(z) = z$ for all $z \in \mathbb{S}^1$, implying f is injective. However, f is periodic ($f(t + 2\pi) = f(t)$), contradicting injectivity. Thus, θ cannot exist globally.

- **Existence for $U \subsetneq \mathbb{S}^1$:**
  Without loss of generality, assume $U = \mathbb{S}^1 \setminus \{p\}$ where $p = (1, 0)$. Restrict f to $(0, 2\pi)$:
  $$g \coloneqq f|_{(0,2\pi)}\colon (0, 2\pi) \to U, \quad t \mapsto \mathrm{e}^{\mathrm{i}t}.$$
  - *Bijectivity:* g is bijective by construction, with each $z \in U$ uniquely corresponding to $t \in (0, 2\pi)$.
  - *Smoothness:* The Jacobian of g at t is:
    $$J(g) = \begin{pmatrix} -\sin t \\ \cos t \end{pmatrix},$$
    which has rank 1 everywhere. By the Constant Rank Theorem, g is a diffeomorphism. Its inverse $\varphi \coloneqq g^{-1}$ defines a local angle function on U.

- **Smooth Atlas Construction:**
  Let $V = \mathbb{S}^1 \setminus \{q\}$ where $q = (-1, 0)$, and define:
  $$\psi\colon V \to (-\pi, \pi), \quad \mathrm{e}^{\mathrm{i}t} \mapsto t.$$
  The transition maps between charts (U, φ) and (V, ψ) are:
  $$\psi \circ \varphi^{-1}(t) = \begin{cases} t & t \in (0, \pi), \\ t - 2\pi & t \in (\pi, 2\pi), \end{cases}$$
  $$\varphi \circ \psi^{-1}(t) = \begin{cases} t & t \in (0, \pi), \\ t + 2\pi & t \in (-\pi, 0). \end{cases}$$
  Both are smooth on their domains, confirming $\mathcal{A} = \{(U, \varphi), (V, \psi)\}$ is a smooth atlas for $\mathbb{S}^1$.

$\square$

1-9 Complex projective n-space, denoted by $\mathbb{CP}^n$, is the set of all 1-dimensional complex-linear subspaces of $\mathbb{C}^{n+1}$, with the quotient topology inherited from the natural projection $\pi\colon \mathbb{C}^{n+1} \setminus \{0\} \to \mathbb{CP}^n$. Show that $\mathbb{CP}^n$ is a compact $2n$-dimensional topological manifold, and show how to give it a smooth structure analogous to the one we constructed for $\mathbb{RP}^n$. (We use the correspondence

$$(x^1 + \mathrm{i}y^1, \ldots, x^{n+1} + \mathrm{i}y^{n+1}) \leftrightarrow (x^1, y^1, \ldots, x^{n+1}, y^{n+1})$$

to identify $\mathbb{C}^{n+1}$ with $\mathbb{R}^{2n+2}$.)

Proof The construction of smooth structure are exactly the same as in Example 1.5. Here we only prove $\mathbb{CP}^n$ is Hausdorff and second-countable.

- the quotient map π is an open map
  Let U be an open subset of $\mathbb{C}^{n+1} \setminus \{0\}$, to prove the quotient map π is an open map, it only suffices to prove that $\pi^{-1} \circ \pi(U)$ is open in $\mathbb{C}^{n+1} \setminus \{0\}$. Since

  $$\pi^{-1} \circ \pi(U) = \bigcup_{t \in \mathbb{C}^\times} tU$$

  for any fixed $t \in \mathbb{C}^\times$, tU is an open subset, we show that their union $\pi^{-1} \circ \pi(U)$ must be open.

- **Hausdorff property**
  Let $[z] = [z_0, \ldots, z_n]$ and $[w] = [w_0, \ldots, w_n]$ be two distinct points in $\mathbb{CP}^n$. Then, z and w are not proportional, i.e., there is no $\lambda \in \mathbb{C}^\times$ such that $w = \lambda z$.

  Define the function $f\colon \mathbb{C}^{n+1} \times \mathbb{C}^{n+1} \to \mathbb{C}$ by

  $$f(z, w) = \sum_{i<j} |z_i w_j - z_j w_i|^2.$$

  This function is zero if and only if z and w are proportional. Since $[z] \neq [w]$, we have $f(z, w) > 0$.

  By continuity of f, there exist open neighborhoods $A \subset \mathbb{C}^{n+1} \setminus \{0\}$ of z and $B \subset \mathbb{C}^{n+1} \setminus \{0\}$ of w such that $f(a, b) > 0$ for all $a \in A$ and $b \in B$.

  Let $U = \pi(A)$ and $V = \pi(B)$. Since π is an open map, U and V are open in $\mathbb{CP}^n$. Moreover, U and V are disjoint, because if $[a] = [b]$ for some $a \in A$ and $b \in B$, then $f(a, b) = 0$, which contradicts the construction of A and B.

  Hence, $\mathbb{CP}^n$ is Hausdorff.

- **Second-countable**
  Since $\mathbb{C}^{n+1} \setminus \{0\}$ is second-countable, and π is a continuous open map, the quotient space $\mathbb{CP}^n$ is also second-countable.

- **The compactness of $\mathbb{CP}^n$**
  The compactness of $\mathbb{CP}^n$ follows from the fact that it is the continuous image of the unit sphere $\mathbb{S}^{2n+1} \subseteq \mathbb{C}^{n+1}$ under π.

$\square$

1-10 Let k and n be integers satisfying $0 < k < n$, and let $P, Q \subseteq \mathbb{R}^n$ be the linear subspaces spanned by $(e_1, \ldots, e_k)$ and $(e_{k+1}, \ldots, e_n)$, respectively, where e_i is the ith standard basis vector for $\mathbb{R}^n$. For any k-dimensional subspace $\subseteq \mathbb{R}^n$ that has trivial intersection with Q, show that the coordinate representation $\varphi(S)$ constructed in Example 1.36 is the unique $(n-k) \times k$ matrix B such that S is spanned by the columns of the matrix $\left(\begin{smallmatrix} I_k \\ B \end{smallmatrix}\right)$, where I_k denotes the $k \times k$ identity matrix.

Proof We prove the existence and uniqueness of the coordinate representation $\varphi(S) = B$ for a k-dimensional subspace $S \subseteq \mathbb{R}^n$ with $S \cap Q = \{0\}$.

- **Existence of the matrix representation:**
  Consider the projection map $\pi_P \colon S \to P$. We claim π_P is an isomorphism:

  - *Injectivity*: Suppose $\pi_P(s) = 0$ for some $s \in S$. Then s has the form uniquely:
  $$s = \pi_P(s) + \pi_Q(s) = \pi_Q(s) \in Q.$$
  Since $S \cap Q = \{0\}$ by hypothesis, we must have $s = 0$.
  - *Surjectivity*: As $\dim S = \dim P = k$ and π_P is injective, it is automatically surjective by the rank-nullity theorem.

  Thus π_P is a vector space isomorphism between S and P. Choose
  $$\{\pi_P^{-1}(e_1), \ldots, \pi_P^{-1}(e_k)\}$$
  for the basis of S. Since $\{e_1, e_n\}$ is a basis of V, we have
  $$\pi_P^{-1}(e_i) = e_i + \sum_{j=k+1}^{n} b_{ij} e_j$$
  Thus S can be spanned by the columns of the matrix
  $$\begin{pmatrix} I_k \\ B \end{pmatrix}$$
  under the basis $\{e_1, \ldots, e_n\}$ where $B = (b_{ij})$.

- **Uniqueness of the matrix B:**
  Suppose there exist two $(n-k) \times k$ matrices B and B' such that:
  $$\operatorname{span}\left(\begin{pmatrix} I_k \\ B \end{pmatrix}\right) = \operatorname{span}\left(\begin{pmatrix} I_k \\ B' \end{pmatrix}\right) = S.$$
  Then there exists an invertible matrix $C \in \mathbb{R}^{k \times k}$ such that:
  $$\begin{pmatrix} I_k \\ B' \end{pmatrix} = \begin{pmatrix} I_k \\ B \end{pmatrix} C.$$
  This matrix equation implies:
  $$\begin{aligned} &I_k = I_k C \quad \Rightarrow \quad C = I_k, \\ &B' = BC = B. \end{aligned}$$
  Therefore, B is uniquely determined by S.

$\square$

1-11 Let $M = \overline{\mathbb{B}^n}$, the closed unit ball in $\mathbb{R}^n$. Show that M is a topological manifold with boundary in which each point in $\mathbb{S}^{n-1}$ is a boundary point and each point in $\mathbb{B}^n$ is an interior point. Show how to give it a smooth structure such that every smooth interior chart is a smooth chart for the standard smooth structure on $\mathbb{B}^n$. [Hint: consider the map $\pi \circ \sigma^{-1} \colon \mathbb{R}^n \to \mathbb{R}^n$, where $\sigma \colon \mathbb{S}^n \to \mathbb{R}^n$ is the stereographic projection (Problem 1-7) and π is a projection from $\mathbb{R}^{n+1}$ to $\mathbb{R}^n$ that omits some coordinate other than the last.]

Proof We establish that $\overline{\mathbb{B}}^n$ is a smooth manifold with boundary, where $\mathbb{S}^{n-1}$ constitutes the boundary and $\mathbb{B}^n$ the interior, by constructing an explicit smooth structure. (This proof proceeds independently of the hint.)

- **Topological manifold structure:**
  - For $x \in \mathbb{B}^n$: The identity chart $(\mathbb{B}^n, \mathrm{id}_{\mathbb{B}^n})$ suffices.
  - For $x \in \mathbb{S}^{n-1}$: We define charts via coordinate projection:

$$\begin{aligned} U_i^+ &= \{x \in \mathbb{R}^n \mid x_i > 0\}, \\ V_i^+ &= U_i^+ \cap \overline{\mathbb{B}}^n, \\ \varphi_i &= \pi_i \circ \pi_{n+1}^{-1} : V_i^+ \to \mathbb{H}^n \cap \mathbb{B}^n, \\ \varphi_i(x^1, \dots, x^n) &= \pi_i(x^1, \dots, x^n, \sqrt{1 - |x|^2}) = (x^1, \dots, \widehat{x^i}, \dots, \sqrt{1 - |x|^2}) \end{aligned}$$

    where $\pi_i \colon \mathbb{S}^n \to \mathbb{R}^n$ omits the i-th coordinate. The collection

$$\{(V_i^\pm, \varphi_i)\}$$

    forms boundary charts since π_i and π_{n+1} are both homeomorphic on V_i^+.

- **Smooth structure:**
  - The charts $\{(V_i^\pm, \varphi_i)\}$ are compatible with each other, since the standard smooth structure of $\mathbb{S}^n$ ensures transition maps

$$\varphi_j \circ \varphi_i^{-1} = \pi_j \circ \pi_{n+1}^{-1} \circ \pi_{n+1} \circ \pi_i^{-1} = \pi_j \circ \pi_i^{-1}$$

    are diffeomorphisms on their domains $\varphi_i(V_i^+ \cap V_j^+)$.
  - Boundary charts and interior chart are compatible, since the Jacobian of transition map

$$|J(\varphi_i \circ \mathrm{id}_{\mathbb{B}^n}^{-1})| = (-1)^{n-1} \frac{x^i}{\sqrt{1 - |x|^2}} \neq 0$$

    on its domain $\mathbb{B}^n \cap V_i^\pm$. Thus the smooth atlas

$$\mathcal{A} = \{(V_i^\pm, \varphi_i)\} \cup (\mathbb{B}^n, \mathrm{id}_{\mathbb{B}^n})$$

    yields a smooth structure of $\overline{\mathbb{B}}^n$.

- **Boundary and interior identification:**

- For $x \in \mathbb{S}^{n-1}$, some some boundary chart $(V_i^{\pm}, \varphi_i)$ satisfies

$$\varphi_i(x) = (x^1, \ldots, \widehat{x^i}, \ldots, 0) \in \partial\mathbb{H}^n,$$

  confirming $\mathbb{S}^{n-1} \subseteq \partial\overline{\mathbb{B}}^n$ via Theorem 1.46 (Boundary Invariance).
- For $x \in \mathbb{B}^n$, the identity chart maps x to $\mathbb{B}^n \subseteq \mathbb{R}^n$, proving $\mathbb{B}^n \subseteq \mathrm{Int}(\overline{\mathbb{B}}^n)$.
- Since $\overline{\mathbb{B}}^n = \mathbb{B}^n \cup \mathbb{S}^{n-1}$, we conclude:

$$\partial\overline{\mathbb{B}}^n = \mathbb{S}^{n-1}, \quad \mathrm{Int}(\overline{\mathbb{B}}^n) = \mathbb{B}^n.$$

□

1-12 Prove Proposition 1.45 (a product of smooth manifolds together with one smooth manifold with boundary is a smooth manifold with boundary).

Proof

- **Model Space Identification**: First observe that $\mathbb{R}^m \times \mathbb{H}^n \cong \mathbb{H}^{m+n}$ via the diffeomorphism:

$$\varphi\colon \mathbb{R}^m \times \mathbb{H}^n \to \mathbb{H}^{m+n}$$
$$(x^1, \ldots, x^m, y^1, \ldots, y^n) \mapsto (x^1, \ldots, x^m, y^1, \ldots, y^n)$$

  This preserves boundaries since $\varphi(\mathbb{R}^m \times \partial\mathbb{H}^n) = \partial\mathbb{H}^{m+n}$.

- **Chart Construction**: Let $M = M_1 \times \cdots \times M_k$ (dim $m = \sum m_i$) and N (dim n) with $\partial N \neq \emptyset$.

  - **Interior Charts**: For $(p, q) \in M \times \mathrm{Int}(N)$:

    (a) Take smooth charts (U_i, φ_i) about $p_i \in M_i$ with $\varphi_i\colon U_i \to \mathbb{R}^{m_i}$

    (b) Take interior chart (V, ψ) about $q \in N$ with $\psi\colon V \to \mathbb{R}^n$

    (c) The product chart is:

$$\left(\prod_{i=1}^{k} U_i \times V, (\varphi_1, \ldots, \varphi_k, \psi)\right)$$

    mapping to $\mathbb{R}^m \times \mathbb{R}^n \subseteq \mathbb{H}^{m+n}$

  - **Boundary Charts**: For $(p, q) \in M \times \partial N$:

    (a) Take smooth charts (U_i, φ_i) as above

    (b) Take boundary chart (V, ψ) with $\psi\colon V \to \mathbb{H}^n$ and $\psi(q) \in \partial\mathbb{H}^n$

    (c) The product chart is:

$$\left(\prod_{i=1}^{k} U_i \times V, (\varphi_1, \ldots, \varphi_k, \psi)\right)$$

    mapping to $\mathbb{R}^m \times \mathbb{H}^n \cong \mathbb{H}^{m+n}$ with boundary points precisely when $q \in \partial N$

- **Chart Compatibility**:

  - For two interior charts, the transition map is:

$$(\varphi_1', \ldots, \varphi_k', \psi') \circ (\varphi_1, \ldots, \varphi_k, \psi)^{-1} = (\varphi_1' \circ \varphi_1^{-1}, \ldots, \varphi_k' \circ \varphi_k^{-1}, \psi' \circ \psi^{-1})$$

    which is smooth since each component is smooth.

  - For boundary charts, the same holds because $\psi' \circ \psi^{-1}$ is smooth as a map between subsets of $\mathbb{H}^n$.

- For mixed cases (one interior, one boundary chart), the transition maps are smooth by the boundary compatibility of N's charts.

- **Boundary Characterization**:
  - If (p, q) is mapped to $\partial\mathbb{H}^{m+n}$ in some chart, then by Theorem 1.46 it holds in all charts, this occurs precisely when $q \in \partial N$, proving:

$$\partial(M \times N) = M \times \partial N$$

  - The interior is correspondingly $M \times \mathrm{Int}(N)$

Thus $M \times N$ is a smooth manifold with boundary as claimed.

$\square$

Chapter 2

Smooth Maps

2-1 Define $f : \mathbb{R} \to \mathbb{R}$ by

$$f(x) = \begin{cases} 1, & x \geq 0, \\ 0, & x < 0. \end{cases}$$

Show that for every $x \in \mathbb{R}$, there exist smooth coordinate charts (U, φ) containing x, and (V, ψ) containing $f(x)$, such that the map $\psi \circ f \circ \varphi^{-1}$ is smooth as a function from $\varphi(U \cap f^{-1}(V))$ to $\mathbb{R}$. However, f is not smooth in the sense we have defined in this chapter.

Proof f is not smooth because f is not continuous. Let $U = (-1, 1)$, $V = (1/2, 3/2)$, $\varphi = \psi = \mathbf{id}$. Then $\varphi(U \cap f^{-1}(V)) = [0, 1)$, $\psi(V) = \{1\}$. $\psi \circ f \circ \varphi^{-1}$ is smooth from $\varphi(U \cap f^{-1}(V))$ to $\psi(V)$ because it is a constant map.

$\square$

2-2 Prove Proposition 2.12(smoothness of maps into product manifolds).

Proof Let $p \in N$ be arbitrary. Choose charts

$$\phi : U \subseteq N \to \mathbb{R}^n, \quad \psi_i : V_i \subseteq M_i \to \mathbb{R}^{m_i}, \quad \text{for } i = 1, \ldots, k,$$

such that $F(p) \in V_1 \times \cdots \times V_k$, and $F(U) \subseteq V_1 \times \cdots \times V_k$.
Define $\psi = \psi_1 \times \cdots \times \psi_k : V_1 \times \cdots \times V_k \to \mathbb{R}^{m_1 + \cdots + m_k}$, which is a smooth chart on the product manifold $M_1 \times \cdots \times M_k$.
Then the local expression of F in coordinates is:

$$\psi \circ F \circ \phi^{-1} : \phi(U) \subseteq \mathbb{R}^n \to \mathbb{R}^{m_1 + \cdots + m_k}.$$

This map can be written as

$$\psi \circ F \circ \phi^{-1}(x) = \big(\psi_1 \circ F_1 \circ \phi^{-1}(x), \ldots, \psi_k \circ F_k \circ \phi^{-1}(x)\big).$$

So in coordinates, the map $\psi \circ F \circ \phi^{-1}$ is smooth if and only if each component $\psi_i \circ F_i \circ \phi^{-1}$ is smooth. Hence, F is smooth if and only if each F_i is smooth.

$\square$

2-3 For each of the following maps between spheres, compute sufficiently many coordinate representations to prove that it is smooth.

(a) $p_n\colon \mathbb{S}^1 \to \mathbb{S}^1$ is the nth power map for $n \in \mathbb{Z}$, given in complex notation by

$$p_n(z) = z^n.$$

(b) $\alpha\colon \mathbb{S}^n \to \mathbb{S}^n$ is the antipodal map given by

$$\alpha(x) = -x.$$

(c) $F\colon \mathbb{S}^3 \to \mathbb{S}^2$ is given by

$$F(w,z) = (\bar{z}w + \bar{w}z,\ i\bar{w}z - i\bar{z}w,\ |z|^2 - |w|^2),$$

where we think of $\mathbb{S}^3$ as the subset

$$\{(w,z) \in \mathbb{C}^2 \mid |w|^2 + |z|^2 = 1\}.$$

Proof

(a) First, p_n is continuous:

$$|z_1 - z_2| = |\mathrm{e}^{\mathrm{i}n\theta_1} - \mathrm{e}^{\mathrm{i}n\theta_2}| \le n|\theta_1 - \theta_2|.$$

Now we prove that p_n is smooth. $\forall z \in \mathbb{S}^1$, there exists an open subset U that contains z and diffeomorphic to an open interval I, the diffeomorphism denotes

$$\varphi\colon U \to I \quad \mathrm{e}^{\mathrm{i}\theta} \mapsto \theta.$$

Similarly we can find an open subset V of $\mathbb{S}^1$ that contains $p_n(z) = z^n$ and diffeomorphic to an open interval J, the diffeomorphism denotes ψ. Since p_n is continuous, we may shrink U small enough that $p_n(U) \subseteq V$. Thus the coordinate representation of p_n is

$$\psi \circ p_n \circ \varphi^{-1}(\theta) = n\theta + 2k(\theta)\pi.$$

Since $k(\theta)$ must be integers and $\psi \circ p_n \circ \varphi^{-1}$ is a continuous map on an interval I, $k(\theta)$ must be constant thus p_n is smooth.

(b) For any point $x \in \mathbb{S}^n$, it is contained in a smooth chart (U_i^+, φ_i^+) such that

$$\varphi_i^+(x^1, \dots, x^{n+1}) = (x^1, \dots, \widehat{x^i}, \dots, x^{n+1}).$$

Then another smooth chart (U_i^-, φ_i^-) must contains $\alpha(x)$. $\alpha^{-1}(U_i^-) \cap U_i^+ = U_i^+$ and $\varphi_i^+(\alpha^{-1}(U_i^-) \cap U_i^+ = U_i^+) = \mathbb{B}^n$. The coordinate representation of α is

$$\varphi_i^- \circ \alpha \circ (\varphi_i^+)^{-1}(u^1, \dots, u^n) = -(u^1, \dots, u^n),$$

which is clearly smooth.

(c) Let $U_1 = \mathbb{S}^3 \setminus \{N\}$ and $V_1 = \mathbb{S}^2 \setminus \{N\}$, and let φ and ψ be the corresponding coordinate charts. The coordinate expression of F is computed as

$$\psi \circ F \circ \varphi^{-1}(u^1, u^2, u^3) = \left(\frac{2u^1u^3 + u^2(|u|^2 - 1)}{2(u^1)^2 + 2(u^2)^2}, \frac{u^1(|u|^2 - 1) - 2u^2u^3}{2(u^1)^2 + 2(u^2)^2} \right),$$

which is smooth on its domain $\varphi(U_1 \cap F^{-1}(V_1))$.

The computation using other coordinate charts proceeds similarly and yields smooth coordinate expressions as well. Hence, the map F is smooth on all of $\mathbb{S}^3$.

$\square$

2-4 Show that the inclusion map $\overline{\mathbb{B}}^n \to \mathbb{R}^n$ is smooth when $\overline{\mathbb{B}}^n$ is regarded as a smooth manifold with boundary.

Proof We only prove that the inclusion map is smooth at boundary points. Use the smooth structure defined in Problem 1-11, let $p \in \partial\overline{\mathbb{B}}^n$ and choose a boundary chart (V_i, φ_i) contains p. Coordinate expression of the inclusion map ι

$$\iota \circ \varphi_i^{-1} \colon \mathbb{H}^n \cap \mathbb{B}^n \to \mathbb{R}^n \quad (u^1, \dots, u^n) \mapsto (u^1, \dots, u^{i-1}, \sqrt{1 - |u|^2}, u^i, \dots, u^{n-1})$$

can be easily extended to a smooth map on $\mathbb{B}^n$, thus the inclusion map ι is smooth.

$\square$

2-5 Let $\mathbb{R}$ be the real line with its standard smooth structure, and let $\tilde{\mathbb{R}}$ denote the same underlying topological manifold equipped with the smooth structure defined in Example 1.23. Let $f\colon \mathbb{R} \to \mathbb{R}$ be a function that is smooth in the usual sense.

(a) Show that f is also smooth as a map from $\mathbb{R}$ to $\tilde{\mathbb{R}}$.

(b) Show that f is smooth as a map from $\tilde{\mathbb{R}}$ to $\mathbb{R}$ if and only if $f^{(n)}(0) = 0$ whenever n is not an integral multiple of 3.

Proof

(a) Denote by $\tilde{f}$ the map from $\mathbb{R}$ to $\tilde{\mathbb{R}}$. Since $\tilde{\mathbb{R}}$ has a globally defined smooth chart $(\mathbb{R}, \psi)$, we consider the composition $\psi \circ \tilde{f} = \psi \circ f$. Both ψ and f are smooth maps from $\mathbb{R}$ to $\mathbb{R}$ in the standard sense, hence their composition is smooth. Therefore, $\tilde{f}$ is smooth.

(b) Suppose first that $f \circ \psi^{-1}$ is smooth. We aim to show that $f^{(n)}(0) = 0$ whenever n is not an integral multiple of 3, using the Faà di Bruno formula:

$$\frac{\mathrm{d}^n}{\mathrm{d}x^n} F(G(x)) = \sum \frac{n!}{m_1!1!^{m_1}\, m_2!2!^{m_2} \cdots m_n!n!^{m_n}} \cdot F^{(m_1+\cdots+m_n)}(G(x)) \cdot \prod_{j=1}^{n} \left(G^{(j)}(x)\right)^{m_j},$$

where the sum ranges over all nonnegative integers $m_1, \ldots, m_n$ such that

$$1 \cdot m_1 + 2 \cdot m_2 + \cdots + n \cdot m_n = n.$$

Since $f = f \circ \psi^{-1} \circ \psi$, we set $F = f \circ \psi^{-1}$ and $G = \psi$. Note that $G^{(j)}(0) \neq 0$ if and only if $j = 3$. For $n = 3k+1$ or $n = 3k+2$, any choice of $(m_1, \ldots, m_n)$ satisfying the above condition must include some $m_j \neq 0$ with $j \neq 3$. Therefore, every term in the sum evaluates to zero at $x = 0$, which implies that $f^{(n)}(0) = 0$ whenever n is not divisible by 3.

Suppose $f^{(n)}(0) = 0$ whenever n is not an integral multiple of 3. We now show that $f \circ \psi^{-1}$ is smooth. Since f is smooth, by Taylor's theorem we have

$$f(x) = \sum_{k=0}^{n} \frac{f^{(3k)}(0)}{(3k)!} x^{3k} + x^{3n+1} g(x),$$

where $g(x)$ is smooth. Substituting x with $x^{1/3}$ gives

$$f \circ \psi^{-1}(x) = f(x^{1/3}) = \sum_{k=0}^{n} \frac{f^{(3k)}(0)}{(3k)!} x^{k} + x^{n+\frac{1}{3}} g(x^{1/3}).$$

It suffices to show that for any $n \in \mathbb{N}$, the function $x^{n+\frac{1}{3}} g(x^{1/3})$ lies in $C^n(\mathbb{R})$. We prove this by induction.

For the base case $n = 0$, the function $x^{1/3} g(x^{1/3})$ is continuous, since both $x^{1/3}$ and $g(x^{1/3})$ are continuous.

Now suppose the statement holds for $n = k$, i.e., if $g \in C^k(\mathbb{R})$, then $x^{k+\frac{1}{3}} g(x^{1/3}) \in C^k(\mathbb{R})$. We aim to show the case for $n = k+1$. Note that $g \in C^{k+1}(\mathbb{R})$ implies $g(x) \in C^k(\mathbb{R})$ and $x g'(x) \in C^k(\mathbb{R})$ as well.

By the chain rule, we compute the derivative:

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(x^{k+\frac{1}{3}}g(x^{\frac{1}{3}})\right) = \left(k+\frac{1}{3}\right)x^{k-\frac{2}{3}}g(x^{\frac{1}{3}}) + \tfrac{1}{3}x^{k-\frac{1}{3}}g'(x^{\frac{1}{3}}).$$

By the inductive hypothesis, both terms on the right-hand side belong to $C^k(\mathbb{R})$. Hence, the derivative lies in $C^k(\mathbb{R})$, which implies $x^{k+\frac{1}{3}}g(x^{1/3}) \in C^{k+1}(\mathbb{R})$.

This completes the induction, and thus $f \circ \psi^{-1}$ is smooth.

$\square$

2-6 Let $P\colon \mathbb{R}^{n+1}\setminus\{0\} \to \mathbb{R}^{k+1}\setminus\{0\}$ be a smooth function, and suppose that for some $d \in \mathbb{Z}$, $P(\lambda x) = \lambda^d P(x) \quad$ for all $\lambda \in \mathbb{R}\setminus\{0\},\ x \in \mathbb{R}^{n+1}\setminus\{0\}$.(Such a function is said to be *homogeneous of degree d*.) Show that the map $\tilde{P}\colon \mathbb{RP}^n \to \mathbb{RP}^k$ defined by $\tilde{P}([x]) = [P(x)]$ is well-defined and smooth.

Proof To show that $\tilde{P}$ is well-defined, suppose $x, y \in \mathbb{R}^{n+1}\setminus\{0\}$ and $[x] = [y]$ in $\mathbb{RP}^n$. Then there exists $\lambda \in \mathbb{R}\setminus\{0\}$ such that $x = \lambda y$. Using the homogeneity of P, we compute:

$$\tilde{P}([x]) = [P(x)] = [P(\lambda y)] = [\lambda^d P(y)] = [P(y)] = \tilde{P}([y]).$$

Thus, $\tilde{P}$ is well-defined.
We now show that $\tilde{P}$ is continuous. Consider the commutative diagram:

$$\begin{array}{ccc} \mathbb{R}^{n+1}\setminus\{0\} & \xrightarrow{P} & \mathbb{R}^{k+1}\setminus\{0\} \\ {\scriptstyle\pi}\big\downarrow & & \big\downarrow{\scriptstyle\pi} \\ \mathbb{RP}^n & \xrightarrow[\tilde{P}]{} & \mathbb{RP}^k \end{array}$$

where π denotes the canonical projection $\pi(x) = [x]$. Since both P and π are continuous, and $\pi \circ P = \tilde{P} \circ \pi$, it follows from the universal property of quotient maps that $\tilde{P}$ is continuous.
To show that $\tilde{P}$ is smooth, we examine it in local coordinates. Let $[x] \in \mathbb{RP}^n$, and choose a standard coordinate chart (U_i, φ_i) around $[x]$, where

$$U_i = \{[x^1 : \cdots : x^{n+1}] \in \mathbb{RP}^n \mid x^i \neq 0\}, \quad \varphi_i([x]) = \left(\frac{x^1}{x^i}, \ldots, \frac{\widehat{x^i}}{x^i}, \ldots, \frac{x^{n+1}}{x^i}\right).$$

Similarly, let (U_j, φ_j) be a coordinate chart on $\mathbb{RP}^k$ containing $\tilde{P}([x])$. Then on the domain $\varphi_i(U_i \cap \tilde{P}^{-1}(U_j))$, the coordinate representation of $\tilde{P}$ is given by:

$$\begin{aligned} \varphi_j \circ \tilde{P} \circ \varphi_i^{-1}(u^1, \ldots, u^n) &= \varphi_j \circ \tilde{P}([u^1, \ldots, u^{i-1}, 1, u^i, \ldots, u^n]) \\ &= \varphi_j \circ [P(u)] \\ &= \varphi_j \circ [P^1(u), \ldots, P^{k+1}(u)] \\ &= \frac{1}{P^j(u)}\left(P^1(u), \ldots, \widehat{P^j(u)}, \ldots, P^{k+1}(u)\right). \end{aligned}$$

On this chart, $P^j(u) \neq 0$ by construction, and each $P^l(u)$ is a smooth function of u. Therefore, the expression above is smooth, which proves that $\tilde{P}$ is smooth.

$\square$

2-7 Let M be a nonempty smooth n-manifold with or without boundary, and suppose $n \geq 1$. Show that the vector space $C^\infty(M)$ of smooth real-valued functions on M is infinite-dimensional.[Hint: Show that if $f_1, \ldots, f_k$ are elements of $C^\infty(M)$ with nonempty disjoint supports, then they are linearly independent.]

Proof Suppose $f_1, \ldots, f_k \in C^\infty(M)$ are smooth functions with nonempty, pairwise disjoint supports. We claim that these functions are linearly independent.
Consider a linear combination $f = a_1 f_1 + \cdots + a_k f_k$ that is identically zero on M. Fix $i \in \{1, \ldots, k\}$, and choose a point $x \in \operatorname{supp}(f_i)$, which is nonempty by assumption. Since the supports of the f_j are disjoint, we have $f_j(x) = 0$ for all $j \neq i$. Then

$$0 = f(x) = a_i f_i(x).$$

Because $f_i(x) \neq 0$, it follows that $a_i = 0$. Since this holds for each i, all coefficients $a_1, \ldots, a_k$ must be zero, and hence $f_1, \ldots, f_k$ are linearly independent.
To construct infinitely many such functions, observe that every smooth manifold is locally Euclidean. Therefore, for any $n \geq 1$, we can choose countably many pairwise disjoint open subsets $U_1, U_2, \cdots \subset M$, each diffeomorphic to an open ball in $\mathbb{R}^n$. Within each U_i, we can find a smooth bump function $f_i \in C^\infty(M)$ with compact support contained in U_i.
These bump functions $f_1, f_2, \ldots$ are smooth, have disjoint (and nonempty) supports, and hence are linearly independent by the argument above. Therefore, $C^\infty(M)$ contains an infinite linearly independent set and is thus infinite-dimensional.

$\square$

2-8 Define $F\colon \mathbb{R}^n \to \mathbb{RP}^n$ by $F(x^1,\dots,x^n) = [x^1,\dots,x^n,1]$. Show that F is a diffeomorphism onto a dense open subset of $\mathbb{RP}^n$. Do the same for $G\colon \mathbb{C}^n \to \mathbb{CP}^n$ defined by $G(z^1,\dots,z^n) = [z^1,\dots,z^n,1]$(see Problem 1-9)

Proof The map $F\colon \mathbb{R}^n \to \mathbb{RP}^n$ given by

$$F(x^1,\dots,x^n) = [x^1,\dots,x^n,1]$$

is a diffeomorphism onto its image. In fact, this image is precisely the standard coordinate chart $U_{n+1} \subseteq \mathbb{RP}^n$, defined by

$$U_{n+1} = \left\{[x^1,\cdots,x^{n+1}] \in \mathbb{RP}^n \colon x^{n+1} \neq 0\right\}.$$

The coordinate chart map $\varphi_{n+1}\colon U_{n+1} \to \mathbb{R}^n$ is defined by

$$\varphi_{n+1}([x^1 : \cdots : x^{n+1}]) = \left(\frac{x^1}{x^{n+1}},\dots,\frac{x^n}{x^{n+1}}\right).$$

One can easily check that F is the inverse of φ_{n+1}, so F is a diffeomorphism from $\mathbb{R}^n$ onto U_{n+1}.
We now show that U_{n+1} is a dense open subset of $\mathbb{RP}^n$. By definition, U_{n+1} is open in $\mathbb{RP}^n$, so it remains to show that it is dense. That is, for any non-empty open subset $V \subseteq \mathbb{RP}^n$, we must show that $V \cap U_{n+1} \neq \emptyset$.
Consider the canonical projection map $\pi\colon \mathbb{R}^{n+1}\setminus\{0\} \to \mathbb{RP}^n$, which is surjective and continuous. Let us define the subset

$$\tilde{U}_{n+1} = \left\{x \in \mathbb{R}^{n+1}\setminus\{0\} \colon x^{n+1} \neq 0\right\}.$$

Note that

$$\pi^{-1}(U_{n+1}) = \tilde{U}_{n+1}.$$

For any open set $V \subseteq \mathbb{RP}^n$, we consider the preimage

$$\pi^{-1}(V \cap U_{n+1}) = \pi^{-1}(V) \cap \pi^{-1}(U_{n+1}) = \pi^{-1}(V) \cap \tilde{U}_{n+1}.$$

Since $\pi^{-1}(V)$ is open in $\mathbb{R}^{n+1}\setminus\{0\}$, and $\tilde{U}_{n+1}$ is dense there, their intersection is non-empty. Hence,

$$V \cap U_{n+1} \neq \emptyset,$$

showing that U_{n+1} is dense in $\mathbb{RP}^n$.
The proof for the complex case is entirely analogous.

$\square$

2-9 Given a polynomial p in one variable with complex coefficients, not identically zero, show that there is a unique smooth map $\tilde{p}\colon \mathbb{CP}^1 \to \mathbb{CP}^1$ that makes the following diagram commute, where $\mathbb{CP}^1$ is 1-dimensional complex projective space and $G\colon \mathbb{C} \to \mathbb{CP}^1$ is the map of Problem 2-8:

$$\begin{tikzcd}
\mathbb{C} \arrow[r, "G"] \arrow[d, "p"'] & \mathbb{CP}^1 \arrow[d, "\tilde{p}"] \\
\mathbb{C} \arrow[r, "G"'] & \mathbb{CP}^1
\end{tikzcd}$$

Proof Let $p(z) = a_0 + a_1 z + \cdots + a_d z^d$ be a nonzero complex polynomial. Define a map $\tilde{p}\colon \mathbb{CP}^1 \to \mathbb{CP}^1$ by

$$\tilde{p}([z, w]) = [a_0 w^d + a_1 z w^{d-1} + \cdots + a_d z^d, w^d] = [p_h(z, w), w^d],$$

where $p_h(z, w)$ is the homogenization of p, so that $p_h(z, 1) = p(z)$.
This map is well-defined and smooth on $\mathbb{CP}^1$ because of Problem 2-6.
For any $z \in \mathbb{C}$, we have $G(z) = [z, 1]$, so

$$\tilde{p}(G(z)) = \tilde{p}([z, 1]) = [p(z), 1] = G(p(z)).$$

Therefore, $\tilde{p} \circ G = G \circ p$, and the diagram commutes.
Uniqueness follows from the fact that $G(\mathbb{C}) = \{[z, 1] \mid z \in \mathbb{C}\}$ is an open dense subset of $\mathbb{CP}^1$, and any smooth map $\tilde{p}$ agreeing with $G \circ p$ on this subset must coincide with the above construction everywhere on $\mathbb{CP}^1$.

$\square$

2-10 For any topological space M, let $C(M)$ denote the algebra of continuous functions $f: M \to \mathbb{R}$. Given a continuous map $F: M \to N$, define $F^*: C(N) \to C(M)$ by $F^*(f) = f \circ F$.

(a) Show that F^* is a linear map.

(b) Suppose M and N are smooth manifolds. Show that $F: M \to N$ is smooth if and only if $F^*(C^\infty(N)) \subseteq C^\infty(M)$.

(c) Suppose $F: M \to N$ is a homeomorphism between smooth manifolds. Show that it is a diffeomorphism if and only if F^* restricts to an isomorphism from $C^\infty(N)$ to $C^\infty(M)$.

Proof

(a) It's trivial.

(b) Suppose first that $F: M \to N$ is smooth. Then for any $f \in C^\infty(N)$, the composition $f \circ F \in C^\infty(M)$, so $F^*(f) = f \circ F$ is smooth. Hence, $F^*(C^\infty(N)) \subseteq C^\infty(M)$.

Conversely, suppose that $F^*(C^\infty(N)) \subseteq C^\infty(M)$. Let $p \in M$ be arbitrary, and let $q = F(p)$. Choose a smooth coordinate chart (V, ψ) around q, where $\psi = (y^1, \ldots, y^n): V \to \mathbb{R}^n$. For each component function $y^i: V \to \mathbb{R}$, choose a smooth bump function ρ supported in V, such that $\rho \equiv 1$ on a smaller neighborhood $\tilde{V} \subseteq V$ of q.

Define the function $\tilde{y}^i = \rho y^i$. Then $\tilde{y}^i$ extends to a smooth function on all of N, and agrees with y^i on $\tilde{V}$. By assumption, $\tilde{y}^i \circ F \in C^\infty(M)$. Since F is continuous, there exists a neighborhood $U \subseteq M$ of p such that $F(U) \subseteq \tilde{V}$. On U, we have

$$y^i \circ F = \tilde{y}^i \circ F,$$

so $y^i \circ F$ is smooth on U. This shows that each component function of $\psi \circ F$ is smooth in a neighborhood of p, so F is smooth at p. Since p was arbitrary, it follows that F is smooth.

(c) Suppose F is a diffeomorphism, let $G = F^{-1}$ and define $G^*: C^\infty(M) \to C^\infty(N)$ by $G^*(g) = g \circ G$. By (a), G^* is a linear map and it is easy to verify that G^* is the inverse of F^*, thus F^* is a isomorphism.

Suppose F^* is an isomorphism between $C^\infty(N)$ and $C^\infty(M)$, since F is a homeomorphism, by (b), it suffices to show that $G^*(C^\infty(M)) \subseteq C^\infty(N)$. Since F^* is an isomorphism ,for any $g \in C^\infty(M)$, there exists $f \in C^\infty(N)$ such that $g = F^*(f)$. Thus

$$G^*(g) = G^*(F^*(f)) = G^*(f \circ F) = f \circ F \circ G = f \in C^\infty(N)$$

and F is a diffeomorphism.

$\square$

2-11 Suppose V is a real vector space of dimension $n \geq 1$. Define the projectivization of V, denoted by $\mathbb{P}(V)$, to be the set of 1-dimensional linear subspaces of V, with the quotient topology induced by the map $\pi\colon V \setminus \{0\} \to \mathbb{P}(V)$ that sends x to its span. (Thus $\mathbb{P}(\mathbb{R}^n) = \mathbb{RP}^{n-1}$.) Show that $\mathbb{P}(V)$ is a topological $(n-1)$-manifold, and has a unique smooth structure with the property that for each basis $(E_1, \dots, E_n)$ for V, the map $E\colon \mathbb{RP}^{n-1} \to \mathbb{P}(V)$ defined by $E[v^1, \dots, v^n] = [v^i E_i]$ (where brackets denote equivalence classes) is a diffeomorphism.

Proof

- **$\mathbb{P}(V)$ is a topological $(n-1)$-manifold.**
  Fix a basis $(E_1, \dots, E_n)$ of V. This determines a linear isomorphism $\varphi_B\colon \mathbb{R}^n \to V$ given by $\varphi_B(v^1, \dots, v^n) = \sum v^i E_i$. This isomorphism equips V with a topology via pullback from $\mathbb{R}^n$, and restricts to a homeomorphism $\mathbb{R}^n \setminus \{0\} \to V \setminus \{0\}$.

  Consider the standard projection maps $\pi_{\mathbb{R}^n}\colon \mathbb{R}^n \setminus \{0\} \to \mathbb{RP}^{n-1}$ and $\pi_V\colon V \setminus \{0\} \to \mathbb{P}(V)$. These induce a map

  $$\widetilde{\varphi}_B\colon \mathbb{RP}^{n-1} \to \mathbb{P}(V), \quad [x] \mapsto [\varphi_B(x)].$$

  This map is well-defined because scalar multiplication is preserved under φ_B, and it is bijective because φ_B is an isomorphism. By the universal property of quotient maps, $\widetilde{\varphi}_B$ is continuous, and so is its inverse. Thus, $\widetilde{\varphi}_B$ is a homeomorphism. Since $\mathbb{RP}^{n-1}$ is a topological $(n-1)$-manifold, so is $\mathbb{P}(V)$.

- **Existence of a smooth structure.**
  We define a smooth structure on $\mathbb{P}(V)$ by pulling back the standard smooth structure from $\mathbb{RP}^{n-1}$ via $\widetilde{\varphi}_B$. That is, a chart (U, ψ) on $\mathbb{P}(V)$ is declared smooth if and only if $(\widetilde{\varphi}_B^{-1}(U), \psi \circ \widetilde{\varphi}_B)$ is a smooth chart on $\mathbb{RP}^{n-1}$. By construction, $\widetilde{\varphi}_B$ is a diffeomorphism.

  Now let $(F_1, \dots, F_n)$ be another basis of V, and define

  $$F\colon \mathbb{RP}^{n-1} \to \mathbb{P}(V), \quad [v^1, \dots, v^n] \mapsto [v^i F_i].$$

  Since both $(E_1, \dots, E_n)$ and $(F_1, \dots, F_n)$ are bases of V, there exists an invertible matrix $A \in GL(n, \mathbb{R})$ such that $F_i = A_i^j E_j$. Define a map $P\colon \mathbb{RP}^{n-1} \to \mathbb{RP}^{n-1}$ by $P([v]) = [Av]$. This is a diffeomorphism by Problem 2-6.

  Observe that the map F can be written as the composition

  $$[v] \overset{P}{\longmapsto} [Av] \overset{\widetilde{\varphi}_B}{\longmapsto} [A_i^j v^i E_j] = [v^i F_i],$$

  i.e., $F = \widetilde{\varphi}_B \circ P$. Since both $\widetilde{\varphi}_B$ and P are diffeomorphisms, it follows that F is a diffeomorphism.

- **Uniqueness of the smooth structure.**
  Let $\mathcal{S}$ denote the smooth structure constructed above. Suppose there is another smooth structure $\mathcal{S}'$ on $\mathbb{P}(V)$ such that for every basis B of V, the map $\widetilde{\psi}_B\colon \mathbb{RP}^{n-1} \to \mathbb{P}(V)$ is a diffeomorphism with respect to $\mathcal{S}'$. Then the identity map

  $$\mathrm{id} = \widetilde{\varphi}_B \circ \widetilde{\psi}_B^{-1}\colon (\mathbb{P}(V), \mathcal{S}') \to (\mathbb{P}(V), \mathcal{S})$$

is a diffeomorphism. Therefore, $\mathcal{S}' = \mathcal{S}$. This shows that the smooth structure on $\mathbb{P}(V)$ is uniquely determined by the property that for every basis B, the map $\widetilde{\psi}_B$ is a diffeomorphism.

$\square$

2-12 State and prove an analogue of Problem 2-11 for complex vector spaces.

Proof The proof is analogous to the real case in Problem 2-11

$\square$

2-13 Suppose M is a topological space with the property that for every indexed open cover $\mathcal{X}$ of M, there exists a patition of unity subordinate to $\mathcal{X}$. Show that M is paracompact.

Proof Let $\mathcal{X} = \{X_\alpha\}_{\alpha \in A}$ be an arbitrary open cover of M, and let $\{\psi_\alpha\}_{\alpha \in A}$ be a partition of unity subordinate to $\mathcal{X}$. For each $\alpha \in A$, define the open set

$$U_\alpha = \{p \in M \colon \psi_\alpha(p) > 0\}.$$

Then $\{U_\alpha\}_{\alpha \in A}$ is an open cover of M, since each ψ_α is nonnegative, for any $p \in M$, we have $\sum_\alpha \psi_\alpha(p) = 1$, there exists some α such that $\psi_\alpha(p) > 0$, i.e., $p \in U_\alpha$. Furthermore, since $U_\alpha \subseteq \operatorname{supp}(\psi_\alpha) \subseteq X_\alpha$, it follows that $\{U_\alpha\}_{\alpha \in A}$ is a refinement of $\mathcal{X}$. Finally, the collection $\{\operatorname{supp}(\psi_\alpha)\}_{\alpha \in A}$ is locally finite by the definition of a partition of unity. Since $U_\alpha \subseteq \operatorname{supp}(\psi_\alpha)$, the subcollection $\{U_\alpha\}_{\alpha \in A}$ is also locally finite. Therefore, $\{U_\alpha\}_{\alpha \in A}$ is a locally finite open refinement of $\mathcal{X}$. Since $\mathcal{X}$ was arbitrary, this proves that M is paracompact.

$\square$

2-14 Suppose A and B are disjoint closed subsets of a smooth manifold M. Show that there exists $f \in C^\infty(M)$ such that $0 \geq f(x) \geq 1$ for all $x \in M$, $f^{-1}(0) = A$, and $f^{-1}(1) = B$.

Proof By Theorem 2.29, there exist smooth, nonnegative functions $f_A, f_B \in C^\infty(M)$ such that $f_A^{-1}(0) = A$ and $f_B^{-1}(0) = B$.
Define a smooth function $f \colon M \to \mathbb{R}$ by

$$f = \frac{f_A}{f_A + f_B}.$$

Since both f_A and f_B are smooth and nonnegative, and their sum is strictly positive everywhere (because A and B are disjoint), the function f is well-defined and smooth on all of M.
Now, consider the behavior of f on different subsets of M:

- If $x \in A$, then $f_A(x) = 0$ and $f_B(x) > 0$, so $f(x) = 0$.
- If $x \in B$, then $f_B(x) = 0$ and $f_A(x) > 0$, so $f(x) = 1$.
- If $x \in M \setminus (A \cup B)$, then both $f_A(x)$ and $f_B(x)$ are strictly positive, so $f(x) \in (0, 1)$.

Therefore, $f \in C^\infty(M)$ satisfies $0 \leq f(x) \leq 1$ for all $x \in M$, with $f^{-1}(0) = A$ and $f^{-1}(1) = B$, as required.

$\square$

Chapter 3

Tangent Vectors

3-1 Suppose M and N are smooth manifolds with or without boundary, and $F\colon M \to N$ is a smooth map. Show that $dF_p\colon T_pM \to T_{F(p)}N$ is the zero map for each $p \in M$ if and only if F is constant on each component of M.

3-2 Prove Proposition 3.14(the tangent space to a product manifold).

3-3 Prove that if M and N are smooth manifolds, then $T(M \times N)$ is diffeomorphism to $TM \times TN$.

3-4 Show that $T\mathbb{S}^1$ is diffeomorphic to $\mathbb{S}^1 \times \mathbb{R}$.