Introduction to Smooth Manifolds
Exercise Solutions

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GitHub Repository: Solutions for *Introduction to Smooth Manifolds*

Contents

Chapter 1

Smooth Manifolds

1-1 Let X be the set of all points $(x, y) \in \mathbb{R}^2$ such that $y = \pm 1$, and let M be the quotient of X by the equivalence relation generated by $(x, 1) \sim (x, -1)$ for all $x \neq 0$. Show that M is locally Euclidean and second-countable, but not Hausdorff. (This space is called the *line with two origins.*)

Proof Let $\pi\colon X \to M$ be the quotient map, where

$$X = \{(x, y) \in \mathbb{R}^2 : y = \pm 1\}$$

and the equivalence relation is generated by $(x, 1) \sim (x, -1)$ for all $x \neq 0$. Let

$$p = \pi(0, 1), \quad q = \pi(0, -1)$$

denote the two distinct equivalence classes in M corresponding to the two origins.

To describe a basis for the topology on M, for any open interval $W \subseteq \mathbb{R}$, define the following sets:

$$\begin{aligned} U_W &= \pi\left(\{(x, \pm 1) : x \in W\}\right), && \text{for } 0 \notin W, \\ U_W^+ &= \pi\left(\{(x, \pm 1) : x \in W \setminus \{0\}\} \cup \{(0, 1)\}\right), && \text{for } 0 \in W, \\ U_W^- &= \pi\left(\{(x, \pm 1) : x \in W \setminus \{0\}\} \cup \{(0, -1)\}\right), && \text{for } 0 \in W. \end{aligned}$$

These sets are open in the quotient topology on M, where:

- U_W is a basic open set in M when W does not contain 0;
- U_W^+ is a neighborhood of the point $p = \pi(0, 1)$;
- U_W^- is a neighborhood of the point $q = \pi(0, -1)$.

We next show that M is second-countable and locally Euclidean but not Hausdorff.

- **Second-countability:**
  Define:
  $$\mathcal{B} = \{U_W, U_W^{\pm} \colon W \text{is an open interval with rational endpoints.}\}$$
  Since there are only countably many open intervals in $\mathbb{R}$ with rational endpoints, the collection $\mathcal{B}$ is countable.

  We claim that $\mathcal{B}$ is a basis for the topology on M. Let $U \subseteq M$ be any open set and let $x \in U$. We want to find some $B \in \mathcal{B}$ such that $x \in B \subseteq U$.
  There are two cases to consider:

  - If $x \in M$ corresponds to a point $\pi(x_0, \pm 1)$ with $x_0 \neq 0$, then $\pi^{-1}(U)$ is an open subset of X containing both $(x_0, 1)$ and $(x_0, -1)$ (since they are identified when $x_0 \neq 0$). Since X inherits the subspace topology from $\mathbb{R}^2$, there exists an open interval $W \ni x_0$ with rational endpoints such that $(W \times \{\pm 1\}) \subseteq \pi^{-1}(U)$. Then $U_W = \pi((W \times \{\pm 1\})) \subseteq U$, and $U_W \in \mathcal{B}$ if $0 \notin W$.
  - If $x = p = \pi(0, 1)$ or $x = q = \pi(0, -1)$, then $x \in U$ implies $\pi^{-1}(U)$ contains either $(0, 1)$ or $(0, -1)$ respectively. Since $\pi^{-1}(U)$ is open in X, there exists an open interval $W \ni 0$ such that:

    * $(W \setminus \{0\}) \times \{\pm 1\} \subseteq \pi^{-1}(U)$,
    * and either $(0,1) \in \pi^{-1}(U)$ or $(0,-1) \in \pi^{-1}(U)$.

  Hence, either $U_W^+ \subseteq U$ or $U_W^- \subseteq U$, and such sets are in $\mathcal{B}$ because W has rational endpoints.

  Therefore, M is second-countable.

- **Local Euclidean property:**
  - For $x \notin \{p, q\}$, define a map

    $$\varphi\colon U_{\mathbb{R}\setminus\{0\}} \to \mathbb{R} \setminus \{0\}, \quad \pi(x, \pm 1) \mapsto x$$

    Clearly φ is bijective. Let V be an open subset of $\mathbb{R} \setminus \{0\}$, $\varphi^{-1}(V)$ is open if and only if $\pi^{-1} \circ \varphi^{-1}(V)$ is open in X. Since

    $$\pi^{-1} \circ \varphi^{-1}(V) = (V \times \{-1\}) \cup (V \times \{1\})$$

    which is open in X, $\varphi^{-1}(V)$ is open in $U_{\mathbb{R}\setminus\{0\}}$. This indicates that φ is continuous. Let $U \subseteq U_{\mathbb{R}\setminus\{0\}}$ be an open subset of M, it means $\pi^{-1}(U)$ is open in X. Since

    $$\varphi(U) = \{x\colon (x,1) \in \pi^{-1}(U)\} \cup \{x\colon (x,-1) \in \pi^{-1}(U)\}$$

    is open in X, φ yields a homeomorphism. Hence, every point with $x \neq 0$ has a neighborhood homeomorphic to $\mathbb{R} \setminus \{0\}$, which is locally Euclidean.
  - For $x = p$, define the map

    $$\psi_+\colon U^+_{(-1,1)} \to (-1,1), \quad \psi_+(\pi(x, \pm 1)) = x, \quad \psi_+(p) = 0.$$

    This map is well-defined and bijective. To show that ψ_+ is a homeomorphism, it suffices to verify that both ψ_+ and its inverse are continuous at p and 0, respectively.
    For any $\varepsilon \in (0,1)$, we have

    $$\psi_+^{-1}((-\varepsilon, \varepsilon)) = U^+_{(-\varepsilon,\varepsilon)} \quad \text{and} \quad \psi_+(U^+_{(-\varepsilon,\varepsilon)}) = (-\varepsilon, \varepsilon),$$

    which shows that ψ_+ is continuous at p and its inverse is continuous at 0. Therefore, ψ_+ is a homeomorphism.
  - For $x = q$, The proof is identical to Case 2.

- **Not Hausdorff:**
  We show that M is not Hausdorff by exhibiting two points that cannot be separated by disjoint open neighborhoods.

  Consider the two points $p = \pi(0,1)$ and $q = \pi(0,-1)$. Suppose for contradiction that there exist disjoint open sets U and V in M such that $p \in U$ and $q \in V$.

  Since the sets U and V are open neighborhoods of p and q, respectively, there exist basic open sets $U_W^+ \subseteq U$ and $U_{W'}^- \subseteq V$, where W and W' are open intervals containing 0.

Let $W'' = W \cap W'$; then $0 \in W''$, so $W'' \setminus \{0\} \neq \emptyset$. Define:

$$A := \pi\left((W'' \setminus \{0\}) \times \{\pm 1\}\right) = U_{W'' \setminus \{0\}} \subseteq U_W^+ \cap U_{W'}^-.$$

Therefore, U_W^+ and $U_{W'}^-$ are not disjoint; they always intersect in a nonempty open set. This contradicts the assumption that p and q can be separated by disjoint open sets. Hence, M is not Hausdorff.

$\square$

1-2 Show that a disjoint union of uncountably many copies of $\mathbb{R}$ is locally Euclidean and Hausdorff, but not second-countable.

Proof Let $X = \coprod_{\alpha \in A} \mathbb{R}_\alpha$, A is an uncountable index set. $U \subseteq X$ is open if and only if $\forall \alpha \in A$, $\{x \in \mathbb{R} : (\alpha, x) \in U\}$ is open in $\mathbb{R}$.

1. X is locally Euclidean.
   By definition, $\mathbb{R}_\alpha = \{(\alpha, x) : x \in \mathbb{R}\}$. $\forall (\alpha, x) \in \mathbb{R}_\alpha$, since $\mathbb{R}_\alpha$ is an open subset of X(By the definition of topology of X) and $\mathbb{R}_\alpha$ is homeomorphic to $\mathbb{R}$, X is locally Euclidean.

2. X is Hausdorff.
   Let $(\alpha, x), (\beta, y) \in X$. if $\alpha \neq \beta$, clearly we have two disjoint open subset $\mathbb{R}_\alpha$ and $\mathbb{R}_\beta$ such that $(\alpha, x) \in \mathbb{R}_\alpha$ and $(\beta, x) \in \mathbb{R}_\beta$. if $\alpha = \beta$, since $\mathbb{R}$ is Hausdorff, we can find two disjoint open subset $U, V \in \mathbb{R}$ such that $x \in U$ and $y \in V$. $(\alpha, U), (\beta, V)$ are two disjoint open subset of X.

3. X is not second-countable.
   Assume that X is second-countable with its countable basis $\mathcal{B} = \{B_i\}$ and I is an countable index set. Since $\mathbb{R}_\alpha$ is an non-empty open subset of X, we can always find $B_i \in \mathcal{B}$ such that $B_i \subseteq \mathbb{R}_\alpha$. By Axiom of Choice, we can define

$$f \colon A \to I, \quad \alpha \mapsto i \ \text{ s.t. } \ B_i \subseteq \mathbb{R}_\alpha.$$

   Clearly f is injective. This leads to an contradiction, since A is uncountable but I is countable.

$\square$

1-3 A topology space is said to be *σ-compact* if it can be expressed as a union of countably many compact subspaces. Show that a locally Euclidean Hausdorff space is a topological manifold if and only if it is σ-compact.

Proof

($\Rightarrow$) Every topological manifold admits a countable basis $\mathcal{B} = \{B_i\}$ of precompact coordinate balls(Lemma 1.10). The collection $\{\overline{B_i} \mid B_i \in \mathcal{B}\}$ implies that the manifold is σ-compact.

($\Leftarrow$) Let X be a locally Euclidean Hausdorff space that is σ-compact. By definition, there exists a countable family of compact subsets $\{K_n\}_{n\in\mathbb{N}}$ such that $X = \bigcup_{n\in\mathbb{N}} K_n$. Since X is locally Euclidean, for each K_n, there exists a finite open cover $\{U_{n_i}\}_{i=1}^{k_n}$ where each (U_{n_i}, φ_{n_i}) is a coordinate chart.

1. For each U_{n_i}, choose a precompact coordinate ball $B_{n_i} \subseteq U_{n_i}$ (possible by local Euclideanness, see Lemma 1.10). The collection $\{B_{n_i}\}$ is countable and covers X.
2. Each B_{n_i}, being homeomorphic to an open ball in $\mathbb{R}^n$, admits a countable basis. A countable union of countable bases remains countable, thus X is second-countable.

Therefore, X satisfies all axioms of a topological manifold (locally Euclidean + Hausdorff + second-countable).

$\square$

1-4 Let M be a topological manifold, and let $\mathcal{U}$ be an open cover of M.

(a) Assuming that each set in $\mathcal{U}$ intersects only finitely many others, show that $\mathcal{U}$ is locally finite.

(b) Give an example to show that the converse to (a) may be false.

(c) Now assume that the sets in $\mathcal{U}$ are precompact in M, and prove the converse: if $\mathcal{U}$ is locally finite, then each set in $\mathcal{U}$ intersects only finitely many others.

Proof

(a) Omitted as "Easy".

(b) Let $M = \mathbb{R}, \mathcal{U} = \{(n, \infty) : n \in \mathbb{N}\} \cup \{(-\infty, 1)\}$

(c) Assume $\mathcal{U}$ is a locally finite open cover of M, and that each set in $\mathcal{U}$ is precompact. Fix $U \in \mathcal{U}$, and define

$$\mathcal{V} = \{V \in \mathcal{U} \colon V \cap U \neq \emptyset\},$$

the collection of all elements in $\mathcal{U}$ that intersect U.

Since U is precompact, its closure $\overline{U}$ is compact. Because $\mathcal{U}$ is locally finite, for every point $x \in \overline{U}$, there exists an open neighborhood V_x intersects only finitely many elements of $\mathcal{U}$.

Then $\{V_x\}_{x \in \overline{U}}$ is an open cover of $\overline{U}$ by elements of $\mathcal{U}$, so by compactness, there exists a finite subcover:

$$\overline{U} \subseteq \bigcup_{i=1}^{n} V_{x_i}.$$

Now for each $i = 1, \ldots, n$, define

$$\mathcal{V}_i = \{W \in \mathcal{U} \colon W \cap V_{x_i} \neq \emptyset\}.$$

Since each V_{x_i} intersects only finitely many elements of $\mathcal{U}$, each $\mathcal{V}_i$ is finite. Now, take any $V \in \mathcal{V}$. Then $V \cap U \neq \emptyset$, and since $\overline{U} \subseteq \bigcup_{i=1}^{n} V_{x_i}$, there exists some i such that $V \cap V_{x_i} \neq \emptyset$, implying $V \in \mathcal{V}_i$. Thus,

$$\mathcal{V} \subseteq \bigcup_{i=1}^{n} \mathcal{V}_i.$$

As each $\mathcal{V}_i$ is finite and n is finite, it follows that $\mathcal{V}$ is finite.

Therefore, each $U \in \mathcal{U}$ intersects only finitely many other elements of $\mathcal{U}$.

□

1-5 Suppose M is a locally Euclidean Hausdorff space. Show that M is second countable if and only if it is paracompact and has countably many connected components.

Proof

($\Rightarrow$) By Proposition 1.11, second-countable property of topological manifold admits at most countably many connected components. Theorem 1.15 shows that every topological manifold is paracompact.

($\Leftarrow$) Suppose M is paracompact and has countably many connected components. It suffices to show that each connected component is second countable, since a countable union of second countable spaces is second countable.

Let C be a connected component of M. Since M is locally Euclidean, there exists a basis of precompact coordinate charts. Let $\mathcal{U}$ be an open cover of C by such charts. By paracompactness, there exists a locally finite refinement $\mathcal{V}$ of $\mathcal{U}$ consisting of precompact coordinate domains.

To show that C is second countable, we will prove that $\mathcal{V}$ is countable. For this, define an equivalence relation $\sim$ on $\mathcal{V}$: for $U, V \in \mathcal{V}$, declare $U \sim V$ if there exists a finite sequence $U = U_0, U_1, \ldots, U_n = V$ in $\mathcal{V}$ such that $U_i \cap U_{i+1} \neq \emptyset$ for all i. Denote by $[U]$ the equivalence class of U under this relation.

We now show that $[U]$ is an open and closed subset of C:

- $[U]$ *is open:* U is a union of open set by definition.
- $[U]$ *is closed:* Let $x \in C \setminus [U]$. Since $\mathcal{V}$ is an open cover of C, there exists $V \in \mathcal{V}$ such that $x \in V$. If V intersected any element of $[U]$, then V would be connected to U via a finite chain of overlapping sets, and hence $x \in [U]$, contradicting $x \in C \setminus [U]$. Therefore, x has an open neighborhood contained in $C \setminus [U]$.
  Since this holds for arbitrary $x \in C \setminus [U]$, we conclude that $C \setminus [U]$ is open, so $[U]$ is closed.

Since C is connected and $[U]$ is nonempty, open, and closed in C, it must be that $[U] = C$. Hence, every element of $\mathcal{V}$ can be connected to U via a finite chain of overlapping sets.

Now define inductively:

$$\mathcal{V}_1 = \{U\}, \quad \mathcal{V}_{n+1} = \{V \in \mathcal{V} \colon \exists W \in \mathcal{V}_n \text{ with } V \cap W \neq \emptyset\}.$$

Then $\bigcup_{n=1}^{\infty} \mathcal{V}_n = \mathcal{V}$. By Problem 1-4, each $\mathcal{V}_n$ is finite. Thus, $\mathcal{V}$ is a countable collection.

Since $\mathcal{V}$ is a countable open cover of C by coordinate domains, the collection

$$\overline{\mathcal{V}} = \{\overline{V} \colon V \in \mathcal{V}\}$$

covers C with countably many compact subsets, thus C is σ-compact. By Problem 1-3, C is second-countable. Finally, M is a countable disjoint union of its connected components, each of which is second countable, so M is second-countable.

$\square$

1-6 Let M be a nonempty topological manifold of dimension $n \geq 1$. If M has a smooth structure, show that it has uncountably many distinct ones. [Hint: first show that for any $s > 0$, $F_s(x) = |x|^{s-1}x$ defines a homeomorphism from $\mathbb{B}^n$ to itself, which is a diffeomorphism if and only if $s = 1$.]

Proof We proceed in four steps:

1. **Homeomorphism property of F_s:**
   For any $s > 0$, the map $F_s \colon \mathbb{B}^n \to \mathbb{B}^n$ defined by $F_s(x) = |x|^{s-1}x$ is a homeomorphism.

   - If $s \geq 1$, F_s is clearly continuous on $\mathbb{B}^n$.
   - If $0 < s < 1$, continuity at $x = 0$ follows from:

   $$\lim_{x \to 0} |F_s(x)| = \lim_{x \to 0} |x|^s = 0 = F_s(0).$$

   - The inverse is $F_{1/s}$, since $F_s \circ F_{1/s} = F_{1/s} \circ F_s = \mathrm{id}_{\mathbb{B}^n}$.

2. **Non-smoothness at origin:**
   F_s is a diffeomorphism on $\mathbb{B}^n \setminus \{0\}$ but fails to be smooth at 0 when $s \neq 1$:

   - For $0 < s < 1$, the derivative at 0 does not exist:

   $$\frac{\partial F_s(0)}{\partial x^i} = \lim_{\Delta x^i \to 0} (\Delta x^i)^{s-1}(0, \ldots, 1, \ldots, 0) \quad \text{(diverges)}$$

   - For $s > 1$, the inverse $F_{1/s}$ has $0 < 1/s < 1$ and thus fails to be smooth at 0.

   Hence F_s is a diffeomorphism on $\mathbb{B}^n$ if and only if $s = 1$.

3. **Constructing a modified atlas:**
   Fix a point $p \in M$ and choose a smooth chart (U, φ) from the given smooth structure $\mathcal{A}$ on M, such that:

   $$\varphi(U) = \mathbb{B}^n \quad \text{and} \quad \varphi(p) = 0.$$

   For any $s > 0$, define a new chart (U, φ_s) by:

   $$\varphi_s = F_s \circ \varphi,$$

   where $F_s(x) = |x|^{s-1}x$ is the homeomorphism from Step 1.

   Construct a new atlas $\mathcal{A}_s$ as follows:

   $$\mathcal{A}_s = \{(U, \varphi_s)\} \cup \{(V, \psi) \in \mathcal{A} \colon p \notin V\}.$$

   That is, $\mathcal{A}_s$ consists of:

   - The single modified chart (U, φ_s) centered at p,
   - All charts from the original atlas $\mathcal{A}$ that do not contain p.

   $\mathcal{A}_s$ is a smooth atlas:

- The charts in $\mathcal{A}_s$ cover M: every point $q \neq p$ is covered by some chart (V, ψ) in $\mathcal{A}$ with $p \notin V$, and p is covered by (U, φ_s).
- The charts in $\mathcal{A}_s$ are pairwise compatible:
  - For any two charts (V_1, ψ_1) and (V_2, ψ_2) in $\mathcal{A}_s$ not containing p, their transition map $\psi_2 \circ \psi_1^{-1}$ is smooth because $\mathcal{A}$ is a smooth atlas.
  - For (U, φ_s) and any (V, ψ) with $p \notin V$, the transition map on $U \cap V$ is:

    $$\psi \circ \varphi_s^{-1} = \psi \circ \varphi^{-1} \circ F_{1/s}.$$

    This is smooth because $\psi \circ \varphi^{-1}$ is smooth (by compatibility in $\mathcal{A}$) and $F_{1/s}$ is smooth away from 0.

4. **Distinct smooth structures:**
   We show that the smooth structures induced by $\mathcal{A}_s$ and $\mathcal{A}_{s'}$ are distinct unless $s = s'$.

   - Suppose $s \neq s'$. Consider the transition map between (U, φ_s) and $(U, \varphi_{s'})$:

     $$\varphi_{s'} \circ \varphi_s^{-1} = F_{s'} \circ F_{1/s} = F_{s'/s}.$$

     This is a diffeomorphism on $\mathbb{B}^n \setminus \{0\}$ but fails to be smooth at 0 unless $s'/s = 1$ (i.e., $s = s'$), as shown in Step 2.
   - Thus, $\mathcal{A}_s$ and $\mathcal{A}_{s'}$ are not smoothly compatible unless $s = s'$.

   Since there are uncountably many choices for $s > 0$, this yields uncountably many distinct smooth structures on M.

□

1-7 Let N denote the north pole $(0,\dots,0,1) \in \mathbb{S}^n \subseteq \mathbb{R}^{n+1}$, and let S denote the south pole $(0,\dots,0,-1)$. Define the stereographic projection $\sigma\colon \mathbb{S}^n \setminus \{N\} \to \mathbb{R}^n$ by

$$\sigma(x^1,\dots,x^{n+1}) = \frac{(x^1,\dots,x^n)}{1-x^{n+1}}.$$

Let $\tilde{\sigma}(x) = -\sigma(-x)$ for $x \in \mathbb{S}^n \setminus \{S\}$.

(a) For any $x \in \mathbb{S}^n \setminus \{N\}$, show that $\sigma(x) = u$, where $(u,0)$ is the point where the line through N and x intersects the hyperplane $x^{n+1} = 0$. Similarly, show that $\tilde{\sigma}(x)$ is the point where the line through S and x intersects the same hyperplane. (Thus $\tilde{\sigma}$ is called stereographic projection from the south pole.)

(b) Show that σ is bijective, with inverse given by

$$\sigma^{-1}(u^1,\dots,u^n) = \frac{(2u^1,\dots,2u^n,|u|^2-1)}{|u|^2+1}.$$

(c) Compute the transition map $\tilde{\sigma}\circ\sigma^{-1}$ and verify that the atlas $\{(\mathbb{S}^n \setminus \{N\},\sigma),(\mathbb{S}^n \setminus \{S\},\tilde{\sigma})\}$ defines a smooth structure on $\mathbb{S}^n$. These are called stereographic coordinates.

(d) Show that this smooth structure agrees with the one defined in Example 1.31.

Proof

(a) Since N, x, and $\sigma(x)$ are collinear, there exists $\lambda \in \mathbb{R}$ such that

$$x = \lambda N + (1-\lambda)\sigma(x).$$

Solving for λ and $\sigma(x)$ gives:

$$\lambda = x^{n+1},$$

$$\sigma(x) = \frac{(x^1,\dots,x^n)}{1-x^{n+1}}.$$

The symmetry $\tilde{\sigma}(-x) = -\sigma(x)$ implies $\tilde{\sigma}(x) = -\sigma(-x)$.

(b) Verify that $\sigma\circ\sigma^{-1} = \mathrm{id}_{\mathbb{R}^n}$ and $\sigma^{-1}\circ\sigma = \mathrm{id}_{\mathbb{S}^n\setminus\{N\}}$

- For $\sigma\circ\sigma^{-1}$, let $(u^1,\dots,u^n) \in \mathbb{R}^n$, we have

$$\begin{aligned}\sigma\circ\sigma^{-1}(u^1,\dots,u^n) &= \sigma\left(\frac{(2u^1,\dots,2u^n,|u|^2-1)}{|u|^2+1}\right)\\ &= \frac{\left(\frac{2u^1}{|u|^2+1},\dots,\frac{2u^n}{|u|^2+1}\right)}{1-\frac{|u|^2-1}{|u|^2+1}}\\ &= (u^1,\dots,u^n).\end{aligned}$$

- For $\sigma^{-1} \circ \sigma$, let $(x^1, \dots, x^{n+1}) \in \mathbb{S}^n \setminus \{N\}$, which means $x^{n+1} \neq 1$ and

$$|\sigma(x)|^2 = \frac{(x^1)^2 + \cdots + (x^n)^2}{(1 - x^{n+1})^2} = \frac{1 + x^{n+1}}{1 - x^{n+1}},$$

$$\begin{aligned} \sigma^{-1} \circ \sigma(x^1, \dots, x^{n+1}) &= \sigma^{-1}\left(\frac{(x^1, \dots, x^n)}{1 - x^{n+1}}\right) \\ &= \frac{\left(\frac{2x^1}{1-x^{n+1}}, \dots, \frac{2x^n}{1-x^{n+1}}, \frac{|x|^2-1}{1-x^{n+1}}\right)}{\frac{|x|^2+1}{1-x^{n+1}}} \\ &= (x^1, \dots, x^{n+1}). \end{aligned}$$

(c) It's sufficient to proof that $\tilde{\sigma} \circ \sigma^{-1}$ and $\sigma \circ \tilde{\sigma}^{-1}$ are smooth on $\mathbb{R}^n \setminus \{0\}$. Let $u = (u^1, \dots, u^n) \in \mathbb{R}^n \setminus \{0\}$, it can be easily verified $\tilde{\sigma}^{-1}(u) = \sigma^{-1}(u)$.

$$\tilde{\sigma} \circ \sigma^{-1}(u) = \sigma \circ \tilde{\sigma}^{-1}(u) = \frac{u}{|u|^2},$$

both are smooth on $\mathbb{R}^n \setminus \{0\}$.

(d) We only proof that $\sigma \circ \pi_i^{-1}$ and $\pi_i \circ \sigma^{-1}$ are smooth for $i = 1, \dots, n+1$, $\tilde{\sigma}$ is completely the same.

- For $i = n + 1$,
  - For transition map $\pi_{n+1} \circ \sigma^{-1}$:

$$\sigma(U_{n+1}^+ \setminus \{N\}) = \sigma\{x^{n+1} \in (0,1)\} = \{|u| > 1 \colon u \in \mathbb{R}^n\}.$$

$$\begin{aligned} \pi_{n+1} \circ \sigma^{-1}(u) &= \pi_{n+1}\left(\frac{(2u^1, \dots, 2u^n, |u|^2 - 1)}{1 + |u|^2}\right) \\ &= \frac{2u}{1 + |u|^2}. \end{aligned}$$

  - For transition map $\sigma \circ \pi_i^{-1}$:

$$\pi_{n+1}(U_{n+1}^+ \setminus \{N\}) = \pi_{n+1}\{x^{n+1} \in (0,1)\} = \mathbb{B}^n \setminus \{0\}.$$

$$\begin{aligned} \sigma \circ \pi_{n+1}^{-1}(u) &= \sigma(u^1, \dots, u^n, \sqrt{1 - |u|^2}) \\ &= \frac{u}{1 - \sqrt{1 - |u|^2}}. \end{aligned}$$

  Both of them are smooth on their domains.

- For $i = 1, \dots, n$,

- For transition map $\pi_i \circ \sigma^{-1}$:

$$\sigma(U_i^+ \setminus \{N\}) = \sigma(U_i^+) = \{u \in \mathbb{R}^n \colon u^i > 0\}.$$

$$\begin{aligned} \pi_i \circ \sigma^{-1}(u) &= \pi_i \left(\frac{(2u^1, \ldots, 2u^n, |u|^2 - 1)}{1 + |u|^2} \right) \\ &= \frac{(2u^1, \ldots, \widehat{2u^i}, \ldots, |u|^2 - 1)}{|u|^2 + 1} \end{aligned}$$

- For transition map $\sigma \circ \pi_i^{-1}$:

$$\pi_i(U_i^+ \setminus \{N\}) = \pi_i(U_i^+) = \mathbb{B}^n \cap \{u \in \mathbb{R}^n \colon u^i > 0\}.$$

$$\begin{aligned} \sigma \circ \pi_i^{-1}(u) &= \sigma(u^1, \ldots, \sqrt{1 - |u|^2}, u^i, \ldots, u^n) \\ &= \frac{(u^1, \ldots, \sqrt{1 - |u|^2}, \ldots, u^{n-1})}{1 - u^n} \end{aligned}$$

All transition maps are smooth on their domains, confirming compatibility.

$\square$

1-8 By identifying $\mathbb{R}^2$ with $\mathbb{C}$, we can think of the unit circle $\mathbb{S}^1$ as a subset of the complex plane. An angle function on a subset $U \subseteq \mathbb{S}^1$ is a continuous function $\theta\colon U \to \mathbb{R}$ such that $\mathrm{e}^{\mathrm{i}\theta(z)} = z$ for all $z \in U$. Show that there exists an angle function on an open subset $U \subseteq \mathbb{S}^1$ if and only if $U \neq \mathbb{S}^1$. For any such angle function, show that (U, θ) is a smooth coordinate chart for $\mathbb{S}^1$ with its standard smooth structure.

Proof We prove the existence of an angle function θ on an open subset $U \subseteq \mathbb{S}^1$ for two cases: $U = \mathbb{S}^1$ and $U \subsetneq \mathbb{S}^1$.

- **Nonexistence for $U = \mathbb{S}^1$:**
  Assume such $\theta\colon \mathbb{S}^1 \to \mathbb{R}$ exists. Define the exponential map:
  $$f\colon \mathbb{R} \to \mathbb{S}^1, \quad t \mapsto \mathrm{e}^{\mathrm{i}t}.$$
  By definition, θ satisfies $f \circ \theta(z) = z$ for all $z \in \mathbb{S}^1$, implying f is injective. However, f is periodic ($f(t + 2\pi) = f(t)$), contradicting injectivity. Thus, θ cannot exist globally.

- **Existence for $U \subsetneq \mathbb{S}^1$:**
  Without loss of generality, assume $U = \mathbb{S}^1 \setminus \{p\}$ where $p = (1, 0)$. Restrict f to $(0, 2\pi)$:
  $$g \coloneqq f|_{(0,2\pi)}\colon (0, 2\pi) \to U, \quad t \mapsto \mathrm{e}^{\mathrm{i}t}.$$
  - *Bijectivity:* g is bijective by construction, with each $z \in U$ uniquely corresponding to $t \in (0, 2\pi)$.
  - *Smoothness:* The Jacobian of g at t is:
    $$J(g) = \begin{pmatrix} -\sin t \\ \cos t \end{pmatrix},$$
    which has rank 1 everywhere. By the Constant Rank Theorem, g is a diffeomorphism. Its inverse $\varphi \coloneqq g^{-1}$ defines a local angle function on U.

- **Smooth Atlas Construction:**
  Let $V = \mathbb{S}^1 \setminus \{q\}$ where $q = (-1, 0)$, and define:
  $$\psi\colon V \to (-\pi, \pi), \quad \mathrm{e}^{\mathrm{i}t} \mapsto t.$$
  The transition maps between charts (U, φ) and (V, ψ) are:
  $$\psi \circ \varphi^{-1}(t) = \begin{cases} t & t \in (0, \pi), \\ t - 2\pi & t \in (\pi, 2\pi), \end{cases}$$
  $$\varphi \circ \psi^{-1}(t) = \begin{cases} t & t \in (0, \pi), \\ t + 2\pi & t \in (-\pi, 0). \end{cases}$$
  Both are smooth on their domains, confirming $\mathcal{A} = \{(U, \varphi), (V, \psi)\}$ is a smooth atlas for $\mathbb{S}^1$.

$\square$

1-9 Complex projective n-space, denoted by $\mathbb{CP}^n$, is the set of all 1-dimensional complex-linear subspaces of $\mathbb{C}^{n+1}$, with the quotient topology inherited from the natural projection $\pi\colon \mathbb{C}^{n+1} \setminus \{0\} \to \mathbb{CP}^n$. Show that $\mathbb{CP}^n$ is a compact $2n$-dimensional topological manifold, and show how to give it a smooth structure analogous to the one we constructed for $\mathbb{RP}^n$. (We use the correspondence

$$(x^1 + \mathrm{i}y^1, \dots, x^{n+1} + \mathrm{i}y^{n+1}) \leftrightarrow (x^1, y^1, \dots, x^{n+1}, y^{n+1})$$

to identify $\mathbb{C}^{n+1}$ with $\mathbb{R}^{2n+2}$.)

Proof The construction of smooth structure are exactly the same as in Example 1.5. Here we only prove $\mathbb{CP}^n$ is Hausdorff and second-countable.

- the quotient map π is an open map
  Let U be an open subset of $\mathbb{C}^{n+1} \setminus \{0\}$, to prove the quotient map π is an open map, it only suffices to prove that $\pi^{-1} \circ \pi(U)$ is open in $\mathbb{C}^{n+1} \setminus \{0\}$. Since
  $$\pi^{-1} \circ \pi(U) = \bigcup_{t \in \mathbb{C}^\times} tU$$
  for any fixed $t \in \mathbb{C}^\times$, tU is an open subset, we show that their union $\pi^{-1} \circ \pi(U)$ must be open.

- **Hausdorff property**
  Let $[z] = [z_0, \dots, z_n]$ and $[w] = [w_0, \dots, w_n]$ be two distinct points in $\mathbb{CP}^n$. Then, z and w are not proportional, i.e., there is no $\lambda \in \mathbb{C}^\times$ such that $w = \lambda z$.

  Define the function $f\colon \mathbb{C}^{n+1} \times \mathbb{C}^{n+1} \to \mathbb{C}$ by
  $$f(z, w) = \sum_{i<j} |z_i w_j - z_j w_i|^2.$$
  This function is zero if and only if z and w are proportional. Since $[z] \neq [w]$, we have $f(z, w) > 0$.

  By continuity of f, there exist open neighborhoods $A \subset \mathbb{C}^{n+1} \setminus \{0\}$ of z and $B \subset \mathbb{C}^{n+1} \setminus \{0\}$ of w such that $f(a, b) > 0$ for all $a \in A$ and $b \in B$.

  Let $U = \pi(A)$ and $V = \pi(B)$. Since π is an open map, U and V are open in $\mathbb{CP}^n$. Moreover, U and V are disjoint, because if $[a] = [b]$ for some $a \in A$ and $b \in B$, then $f(a, b) = 0$, which contradicts the construction of A and B.

  Hence, $\mathbb{CP}^n$ is Hausdorff.

- **Second-countable**
  Since $\mathbb{C}^{n+1} \setminus \{0\}$ is second-countable, and π is a continuous open map, the quotient space $\mathbb{CP}^n$ is also second-countable.

- **The compactness of** $\mathbb{CP}^n$
  The compactness of $\mathbb{CP}^n$ follows from the fact that it is the continuous image of the unit sphere $\mathbb{S}^{2n+1} \subseteq \mathbb{C}^{n+1}$ under π.

$\square$

1-10 Let k and n be integers satisfying $0 < k < n$, and let $P, Q \subseteq \mathbb{R}^n$ be the linear subspaces spanned by $(e_1, \dots, e_k)$ and $(e_{k+1}, \dots, e_n)$, respectively, where e_i is the ith standard basis vector for $\mathbb{R}^n$. For any k-dimensional subspace $\subseteq \mathbb{R}^n$ that has trivial intersection with Q, show that the coordinate representation $\varphi(S)$ constructed in Example 1.36 is the unique $(n-k) \times k$ matrix B such that S is spanned by the columns of the matrix $\left(\begin{smallmatrix} I_k \\ B \end{smallmatrix}\right)$, where I_k denotes the $k \times k$ identity matrix.

Proof We prove the existence and uniqueness of the coordinate representation $\varphi(S) = B$ for a k-dimensional subspace $S \subseteq \mathbb{R}^n$ with $S \cap Q = \{0\}$.

- **Existence of the matrix representation:**
  Consider the projection map $\pi_P \colon S \to P$. We claim π_P is an isomorphism:
  - *Injectivity*: Suppose $\pi_P(s) = 0$ for some $s \in S$. Then s has the form uniquely:
  $$s = \pi_P(s) + \pi_Q(s) = \pi_Q(s) \in Q.$$
  Since $S \cap Q = \{0\}$ by hypothesis, we must have $s = 0$.
  - *Surjectivity*: As $\dim S = \dim P = k$ and π_P is injective, it is automatically surjective by the rank-nullity theorem.

  Thus π_P is a vector space isomorphism between S and P. Choose
  $$\{\pi_P^{-1}(e_1), \dots, \pi_P^{-1}(e_k)\}$$
  for the basis of S. Since $\{e_1, e_n\}$ is a basis of V, we have
  $$\pi_P^{-1}(e_i) = e_i + \sum_{j=k+1}^{n} b_{ij} e_j$$
  Thus S can be spanned by the columns of the matrix
  $$\begin{pmatrix} I_k \\ B \end{pmatrix}$$
  under the basis $\{e_1, \dots, e_n\}$ where $B = (b_{ij})$.

- **Uniqueness of the matrix B:**
  Suppose there exist two $(n-k) \times k$ matrices B and B' such that:
  $$\operatorname{span}\left(\begin{pmatrix} I_k \\ B \end{pmatrix}\right) = \operatorname{span}\left(\begin{pmatrix} I_k \\ B' \end{pmatrix}\right) = S.$$
  Then there exists an invertible matrix $C \in \mathbb{R}^{k \times k}$ such that:
  $$\begin{pmatrix} I_k \\ B' \end{pmatrix} = \begin{pmatrix} I_k \\ B \end{pmatrix} C.$$
  This matrix equation implies:
  $$\begin{aligned} I_k &= I_k C \quad \Rightarrow \quad C = I_k, \\ B' &= BC = B. \end{aligned}$$
  Therefore, B is uniquely determined by S.

$\square$

1-11 Let $M = \overline{\mathbb{B}^n}$, the closed unit ball in $\mathbb{R}^n$. Show that M is a topological manifold with boundary in which each point in $\mathbb{S}^{n-1}$ is a boundary point and each point in $\mathbb{B}^n$ is an interior point. Show how to give it a smooth structure such that every smooth interior chart is a smooth chart for the standard smooth structure on $\mathbb{B}^n$. [Hint: consider the map $\pi \circ \sigma^{-1} \colon \mathbb{R}^n \to \mathbb{R}^n$, where $\sigma \colon \mathbb{S}^n \to \mathbb{R}^n$ is the stereographic projection (Problem 1-7) and π is a projection from $\mathbb{R}^{n+1}$ to $\mathbb{R}^n$ that omits some coordinate other than the last.]

Proof We establish that $\overline{\mathbb{B}}^n$ is a smooth manifold with boundary, where $\mathbb{S}^{n-1}$ constitutes the boundary and $\mathbb{B}^n$ the interior, by constructing an explicit smooth structure. (This proof proceeds independently of the hint.)

- **Topological manifold structure:**
  - For $x \in \mathbb{B}^n$: The identity chart $(\mathbb{B}^n, \mathrm{id}_{\mathbb{B}^n})$ suffices.
  - For $x \in \mathbb{S}^{n-1}$: We define charts via coordinate projection:

$$\begin{aligned} U_i^+ &= \{x \in \mathbb{R}^n \mid x_i > 0\}, \\ V_i^+ &= U_i^+ \cap \overline{\mathbb{B}}^n, \\ \varphi_i &= \pi_i \circ \pi_{n+1}^{-1} : V_i^+ \to \mathbb{H}^n \cap \mathbb{B}^n, \\ \varphi_i(x^1, \ldots, x^n) &= \pi_i(x^1, \ldots, x^n, \sqrt{1 - |x|^2}) = (x^1, \ldots, \widehat{x^i}, \ldots, \sqrt{1 - |x|^2}) \end{aligned}$$

    where $\pi_i \colon \mathbb{S}^n \to \mathbb{R}^n$ omits the i-th coordinate. The collection

$$\{(V_i^\pm, \varphi_i)\}$$

    forms boundary charts since π_i and π_{n+1} are both homeomorphic on V_i^+.

- **Smooth structure:**
  - The charts $\{(V_i^\pm, \varphi_i)\}$ are compatible with each other, since the standard smooth structure of $\mathbb{S}^n$ ensures transition maps

$$\varphi_j \circ \varphi_i^{-1} = \pi_j \circ \pi_{n+1}^{-1} \circ \pi_{n+1} \circ \pi_i^{-1} = \pi_j \circ \pi_i^{-1}$$

    are diffeomorphisms on their domains $\varphi_i(V_i^+ \cap V_j^+)$.
  - Boundary charts and interior chart are compatible, since the Jacobian of transition map

$$|J(\varphi_i \circ \mathrm{id}_{\mathbb{B}^n}^{-1})| = (-1)^{n-1} \frac{x^i}{\sqrt{1 - |x|^2}} \neq 0$$

    on its domain $\mathbb{B}^n \cap V_i^\pm$. Thus the smooth atlas

$$\mathcal{A} = \{(V_i^\pm, \varphi_i)\} \cup (\mathbb{B}^n, \mathrm{id}_{\mathbb{B}^n})$$

    yields a smooth structure of $\overline{\mathbb{B}}^n$.

- **Boundary and interior identification:**

- For $x \in \mathbb{S}^{n-1}$, some some boundary chart $(V_i^{\pm}, \varphi_i)$ satisfies

$$\varphi_i(x) = (x^1, \ldots, \widehat{x^i}, \ldots, 0) \in \partial\mathbb{H}^n,$$

  confirming $\mathbb{S}^{n-1} \subseteq \partial\overline{\mathbb{B}}^n$ via Theorem 1.46 (Boundary Invariance).
- For $x \in \mathbb{B}^n$, the identity chart maps x to $\mathbb{B}^n \subseteq \mathbb{R}^n$, proving $\mathbb{B}^n \subseteq \operatorname{Int}(\overline{\mathbb{B}}^n)$.
- Since $\overline{\mathbb{B}}^n = \mathbb{B}^n \cup \mathbb{S}^{n-1}$, we conclude:

$$\partial\overline{\mathbb{B}}^n = \mathbb{S}^{n-1}, \quad \operatorname{Int}(\overline{\mathbb{B}}^n) = \mathbb{B}^n.$$

□

1-12 Prove Proposition 1.45 (a product of smooth manifolds together with one smooth manifold with boundary is a smooth manifold with boundary).

Proof

- **Model Space Identification**: First observe that $\mathbb{R}^m \times \mathbb{H}^n \cong \mathbb{H}^{m+n}$ via the diffeomorphism:

$$\varphi \colon \mathbb{R}^m \times \mathbb{H}^n \to \mathbb{H}^{m+n}$$
$$(x^1, \ldots, x^m, y^1, \ldots, y^n) \mapsto (x^1, \ldots, x^m, y^1, \ldots, y^n)$$

  This preserves boundaries since $\varphi(\mathbb{R}^m \times \partial\mathbb{H}^n) = \partial\mathbb{H}^{m+n}$.

- **Chart Construction**: Let $M = M_1 \times \cdots \times M_k$ (dim $m = \sum m_i$) and N (dim n) with $\partial N \neq \emptyset$.
  - **Interior Charts**: For $(p, q) \in M \times \operatorname{Int}(N)$:
    (a) Take smooth charts (U_i, φ_i) about $p_i \in M_i$ with $\varphi_i \colon U_i \to \mathbb{R}^{m_i}$
    (b) Take interior chart (V, ψ) about $q \in N$ with $\psi \colon V \to \mathbb{R}^n$
    (c) The product chart is:

$$\left(\prod_{i=1}^{k} U_i \times V, (\varphi_1, \ldots, \varphi_k, \psi) \right)$$

    mapping to $\mathbb{R}^m \times \mathbb{R}^n \subseteq \mathbb{H}^{m+n}$
  - **Boundary Charts**: For $(p, q) \in M \times \partial N$:
    (a) Take smooth charts (U_i, φ_i) as above
    (b) Take boundary chart (V, ψ) with $\psi \colon V \to \mathbb{H}^n$ and $\psi(q) \in \partial\mathbb{H}^n$
    (c) The product chart is:

$$\left(\prod_{i=1}^{k} U_i \times V, (\varphi_1, \ldots, \varphi_k, \psi) \right)$$

    mapping to $\mathbb{R}^m \times \mathbb{H}^n \cong \mathbb{H}^{m+n}$ with boundary points precisely when $q \in \partial N$

- **Chart Compatibility**:
  - For two interior charts, the transition map is:

$$(\varphi_1', \ldots, \varphi_k', \psi') \circ (\varphi_1, \ldots, \varphi_k, \psi)^{-1} = (\varphi_1' \circ \varphi_1^{-1}, \ldots, \varphi_k' \circ \varphi_k^{-1}, \psi' \circ \psi^{-1})$$

    which is smooth since each component is smooth.
  - For boundary charts, the same holds because $\psi' \circ \psi^{-1}$ is smooth as a map between subsets of $\mathbb{H}^n$.

  - For mixed cases (one interior, one boundary chart), the transition maps are smooth by the boundary compatibility of N's charts.

- **Boundary Characterization**:

  - If (p, q) is mapped to $\partial\mathbb{H}^{m+n}$ in some chart, then by Theorem 1.46 it holds in all charts, this occurs precisely when $q \in \partial N$, proving:

$$\partial(M \times N) = M \times \partial N$$

  - The interior is correspondingly $M \times \mathrm{Int}(N)$

Thus $M \times N$ is a smooth manifold with boundary as claimed.

$\square$

Chapter 2

Smooth Maps

2-1 Define $f : \mathbb{R} \to \mathbb{R}$ by

$$f(x) = \begin{cases} 1, & x \geq 0, \\ 0, & x < 0. \end{cases}$$

Show that for every $x \in \mathbb{R}$, there exist smooth coordinate charts (U, φ) containing x, and (V, ψ) containing $f(x)$, such that the map $\psi \circ f \circ \varphi^{-1}$ is smooth as a function from $\varphi(U \cap f^{-1}(V))$ to $\mathbb{R}$. However, f is not smooth in the sense we have defined in this chapter.

Proof f is not smooth because f is not continuous. Let $U = (-1, 1)$, $V = (1/2, 3/2)$, $\varphi = \psi = \mathbf{id}$. Then $\varphi(U \cap f^{-1}(V)) = [0, 1)$, $\psi(V) = \{1\}$. $\psi \circ f \circ \varphi^{-1}$ is smooth from $\varphi(U \cap f^{-1}(V))$ to $\psi(V)$ because it is a constant map.

$\square$

2-2 Prove Proposition 2.12(smoothness of maps into product manifolds).

Proof Let $p \in N$ be arbitrary. Choose charts

$$\phi : U \subseteq N \to \mathbb{R}^n, \quad \psi_i : V_i \subseteq M_i \to \mathbb{R}^{m_i}, \quad \text{for } i = 1, \dots, k,$$

such that $F(p) \in V_1 \times \cdots \times V_k$, and $F(U) \subseteq V_1 \times \cdots \times V_k$.

Define $\psi = \psi_1 \times \cdots \times \psi_k : V_1 \times \cdots \times V_k \to \mathbb{R}^{m_1 + \cdots + m_k}$, which is a smooth chart on the product manifold $M_1 \times \cdots \times M_k$.

Then the local expression of F in coordinates is:

$$\psi \circ F \circ \phi^{-1} : \phi(U) \subseteq \mathbb{R}^n \to \mathbb{R}^{m_1 + \cdots + m_k}.$$

This map can be written as

$$\psi \circ F \circ \phi^{-1}(x) = \big(\psi_1 \circ F_1 \circ \phi^{-1}(x), \dots, \psi_k \circ F_k \circ \phi^{-1}(x)\big).$$

So in coordinates, the map $\psi \circ F \circ \phi^{-1}$ is smooth if and only if each component $\psi_i \circ F_i \circ \phi^{-1}$ is smooth. Hence, F is smooth if and only if each F_i is smooth.

$\square$

2-3 For each of the following maps between spheres, compute sufficiently many coordinate representations to prove that it is smooth.

(a) $p_n \colon \mathbb{S}^1 \to \mathbb{S}^1$ is the nth power map for $n \in \mathbb{Z}$, given in complex notation by

$$p_n(z) = z^n.$$

(b) $\alpha \colon \mathbb{S}^n \to \mathbb{S}^n$ is the antipodal map given by

$$\alpha(x) = -x.$$

(c) $F \colon \mathbb{S}^3 \to \mathbb{S}^2$ is given by

$$F(w, z) = (\bar{z}w + \bar{w}z,\ i\bar{w}z - i\bar{z}w,\ |z|^2 - |w|^2),$$

where we think of $\mathbb{S}^3$ as the subset

$$\{(w, z) \in \mathbb{C}^2 \mid |w|^2 + |z|^2 = 1\}.$$

Proof

(a) First, p_n is continuous:

$$|z_1 - z_2| = |\mathrm{e}^{in\theta_1} - \mathrm{e}^{in\theta_2}| \leq n|\theta_1 - \theta_2|.$$

Now we prove that p_n is smooth. $\forall z \in \mathbb{S}^1$, there exists an open subset U that contains z and diffeomorphic to an open interval I, the diffeomorphism denotes

$$\varphi \colon U \to I \quad \mathrm{e}^{\mathrm{i}\theta} \mapsto \theta.$$

Similarly we can find an open subset V of $\mathbb{S}^1$ that contains $p_n(z) = z^n$ and diffeomorphic to an open interval J, the diffeomorphism denotes ψ. Since p_n is continuous, we may shrink U small enough that $p_n(U) \subseteq V$. Thus the coordinate representation of p_n is

$$\psi \circ p_n \circ \varphi^{-1}(\theta) = n\theta + 2k(\theta)\pi.$$

Since $k(\theta)$ must be integers and $\psi \circ p_n \circ \varphi^{-1}$ is a continuous map on an interval I, $k(\theta)$ must be constant thus p_n is smooth.

(b) For any point $x \in \mathbb{S}^n$, it is contained in a smooth chart (U_i^+, φ_i^+) such that

$$\varphi_i^+(x^1, \ldots, x^{n+1}) = (x^1, \ldots, \widehat{x^i}, \ldots, x^{n+1}).$$

Then another smooth chart (U_i^-, φ_i^-) must contains $\alpha(x)$. $\alpha^{-1}(U_i^-) \cap U_i^+ = U_i^+$ and $\varphi_i^+(\alpha^{-1}(U_i^-) \cap U_i^+ = U_i^+) = \mathbb{B}^n$. The coordinate representation of α is

$$\varphi_i^- \circ \alpha \circ (\varphi_i^+)^{-1}(u^1, \ldots, u^n) = -(u^1, \ldots, u^n),$$

which is clearly smooth.

(c) Let $U_1 = \mathbb{S}^3 \setminus \{N\}$ and $V_1 = \mathbb{S}^2 \setminus \{N\}$, and let φ and ψ be the corresponding coordinate charts. The coordinate expression of F is computed as

$$\psi \circ F \circ \varphi^{-1}(u^1, u^2, u^3) = \left(\frac{2u^1u^3 + u^2(|u|^2 - 1)}{2(u^1)^2 + 2(u^2)^2}, \frac{u^1(|u|^2 - 1) - 2u^2u^3}{2(u^1)^2 + 2(u^2)^2} \right),$$

which is smooth on its domain $\varphi(U_1 \cap F^{-1}(V_1))$.

The computation using other coordinate charts proceeds similarly and yields smooth coordinate expressions as well. Hence, the map F is smooth on all of $\mathbb{S}^3$.

$\square$

2-4 Show that the inclusion map $\overline{\mathbb{B}}^n \to \mathbb{R}^n$ is smooth when $\overline{\mathbb{B}}^n$ is regarded as a smooth manifold with boundary.

Proof We only prove that the inclusion map is smooth at boundary points. Use the smooth structure defined in Problem 1-11, let $p \in \partial\overline{\mathbb{B}}^n$ and choose a boundary chart (V_i, φ_i) contains p. Coordinate expression of the inclusion map ι

$$\iota \circ \varphi_i^{-1} \colon \mathbb{H}^n \cap \mathbb{B}^n \to \mathbb{R}^n \quad (u^1, \ldots, u^n) \mapsto (u^1, \ldots, u^{i-1}, \sqrt{1 - |u|^2}, u^i, \ldots, u^{n-1})$$

can be easily extended to a smooth map on $\mathbb{B}^n$, thus the inclusion map ι is smooth.

$\square$

2-5 Let $\mathbb{R}$ be the real line with its standard smooth structure, and let $\tilde{\mathbb{R}}$ denote the same underlying topological manifold equipped with the smooth structure defined in Example 1.23. Let $f\colon \mathbb{R} \to \mathbb{R}$ be a function that is smooth in the usual sense.

(a) Show that f is also smooth as a map from $\mathbb{R}$ to $\tilde{\mathbb{R}}$.

(b) Show that f is smooth as a map from $\tilde{\mathbb{R}}$ to $\mathbb{R}$ if and only if $f^{(n)}(0) = 0$ whenever n is not an integral multiple of 3.

Proof

(a) Denote by $\tilde{f}$ the map from $\mathbb{R}$ to $\tilde{\mathbb{R}}$. Since $\tilde{\mathbb{R}}$ has a globally defined smooth chart $(\mathbb{R}, \psi)$, we consider the composition $\psi \circ \tilde{f} = \psi \circ f$. Both ψ and f are smooth maps from $\mathbb{R}$ to $\mathbb{R}$ in the standard sense, hence their composition is smooth. Therefore, $\tilde{f}$ is smooth.

(b) Suppose first that $f \circ \psi^{-1}$ is smooth. We aim to show that $f^{(n)}(0) = 0$ whenever n is not an integral multiple of 3, using the Faà di Bruno formula:

$$\frac{\mathrm{d}^n}{\mathrm{d}x^n} F(G(x)) = \sum \frac{n!}{m_1!1!^{m_1}\, m_2!2!^{m_2} \cdots m_n!n!^{m_n}} \cdot F^{(m_1+\cdots+m_n)}(G(x)) \cdot \prod_{j=1}^{n} \left(G^{(j)}(x)\right)^{m_j},$$

where the sum ranges over all nonnegative integers $m_1, \ldots, m_n$ such that

$$1 \cdot m_1 + 2 \cdot m_2 + \cdots + n \cdot m_n = n.$$

Since $f = f \circ \psi^{-1} \circ \psi$, we set $F = f \circ \psi^{-1}$ and $G = \psi$. Note that $G^{(j)}(0) \neq 0$ if and only if $j = 3$. For $n = 3k + 1$ or $n = 3k + 2$, any choice of $(m_1, \ldots, m_n)$ satisfying the above condition must include some $m_j \neq 0$ with $j \neq 3$. Therefore, every term in the sum evaluates to zero at $x = 0$, which implies that $f^{(n)}(0) = 0$ whenever n is not divisible by 3.

Suppose $f^{(n)}(0) = 0$ whenever n is not an integral multiple of 3. We now show that $f \circ \psi^{-1}$ is smooth. Since f is smooth, by Taylor's theorem we have

$$f(x) = \sum_{k=0}^{n} \frac{f^{(3k)}(0)}{(3k)!} x^{3k} + x^{3n+1} g(x),$$

where $g(x)$ is smooth. Substituting x with $x^{1/3}$ gives

$$f \circ \psi^{-1}(x) = f(x^{1/3}) = \sum_{k=0}^{n} \frac{f^{(3k)}(0)}{(3k)!} x^{k} + x^{n+\frac{1}{3}} g(x^{1/3}).$$

It suffices to show that for any $n \in \mathbb{N}$, the function $x^{n+\frac{1}{3}} g(x^{1/3})$ lies in $C^n(\mathbb{R})$. We prove this by induction.

For the base case $n = 0$, the function $x^{1/3} g(x^{1/3})$ is continuous, since both $x^{1/3}$ and $g(x^{1/3})$ are continuous.

Now suppose the statement holds for $n = k$, i.e., if $g \in C^k(\mathbb{R})$, then $x^{k+\frac{1}{3}}g(x^{1/3}) \in C^k(\mathbb{R})$. We aim to show the case for $n = k+1$. Note that $g \in C^{k+1}(\mathbb{R})$ implies $g(x) \in C^k(\mathbb{R})$ and $xg'(x) \in C^k(\mathbb{R})$ as well.

By the chain rule, we compute the derivative:

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(x^{k+\frac{1}{3}}g(x^{\frac{1}{3}})\right) = \left(k+\frac{1}{3}\right)x^{k-\frac{2}{3}}g(x^{\frac{1}{3}}) + \tfrac{1}{3}x^{k-\frac{1}{3}}g'(x^{\frac{1}{3}}).$$

By the inductive hypothesis, both terms on the right-hand side belong to $C^k(\mathbb{R})$. Hence, the derivative lies in $C^k(\mathbb{R})$, which implies $x^{k+\frac{1}{3}}g(x^{1/3}) \in C^{k+1}(\mathbb{R})$.

This completes the induction, and thus $f \circ \psi^{-1}$ is smooth.

$\square$

2-6 Let $P\colon \mathbb{R}^{n+1} \setminus \{0\} \to \mathbb{R}^{k+1} \setminus \{0\}$ be a smooth function, and suppose that for some $d \in \mathbb{Z}$, $P(\lambda x) = \lambda^d P(x) \quad \text{for all } \lambda \in \mathbb{R} \setminus \{0\},\ x \in \mathbb{R}^{n+1} \setminus \{0\}$.(Such a function is said to be *homogeneous of degree d*.) Show that the map $\tilde{P}\colon \mathbb{RP}^n \to \mathbb{RP}^k$ defined by $\tilde{P}([x]) = [P(x)]$ is well-defined and smooth.

Proof To show that $\tilde{P}$ is well-defined, suppose $x, y \in \mathbb{R}^{n+1} \setminus \{0\}$ and $[x] = [y]$ in $\mathbb{RP}^n$. Then there exists $\lambda \in \mathbb{R} \setminus \{0\}$ such that $x = \lambda y$. Using the homogeneity of P, we compute:

$$\tilde{P}([x]) = [P(x)] = [P(\lambda y)] = [\lambda^d P(y)] = [P(y)] = \tilde{P}([y]).$$

Thus, $\tilde{P}$ is well-defined.

We now show that $\tilde{P}$ is continuous. Consider the commutative diagram:

$$\begin{tikzcd}
\mathbb{R}^{n+1} \setminus \{0\} \arrow[r, "P"] \arrow[d, "\pi"'] & \mathbb{R}^{k+1} \setminus \{0\} \arrow[d, "\pi"] \\
\mathbb{RP}^n \arrow[r, "\tilde{P}"'] & \mathbb{RP}^k
\end{tikzcd}$$

where π denotes the canonical projection $\pi(x) = [x]$. Since both P and π are continuous, and $\pi \circ P = \tilde{P} \circ \pi$, it follows from the universal property of quotient maps that $\tilde{P}$ is continuous.

To show that $\tilde{P}$ is smooth, we examine it in local coordinates. Let $[x] \in \mathbb{RP}^n$, and choose a standard coordinate chart (U_i, φ_i) around $[x]$, where

$$U_i = \{[x^1 : \cdots : x^{n+1}] \in \mathbb{RP}^n \mid x^i \neq 0\}, \quad \varphi_i([x]) = \left(\frac{x^1}{x^i}, \ldots, \widehat{\frac{x^i}{x^i}}, \ldots, \frac{x^{n+1}}{x^i} \right).$$

Similarly, let (U_j, φ_j) be a coordinate chart on $\mathbb{RP}^k$ containing $\tilde{P}([x])$. Then on the domain $\varphi_i(U_i \cap \tilde{P}^{-1}(U_j))$, the coordinate representation of $\tilde{P}$ is given by:

$$\begin{aligned}
\varphi_j \circ \tilde{P} \circ \varphi_i^{-1}(u^1, \ldots, u^n) &= \varphi_j \circ \tilde{P}([u^1, \ldots, u^{i-1}, 1, u^i, \ldots, u^n]) \\
&= \varphi_j \circ [P(u)] \\
&= \varphi_j \circ [P^1(u), \ldots, P^{k+1}(u)] \\
&= \frac{1}{P^j(u)} \left(P^1(u), \ldots, \widehat{P^j(u)}, \ldots, P^{k+1}(u) \right).
\end{aligned}$$

On this chart, $P^j(u) \neq 0$ by construction, and each $P^l(u)$ is a smooth function of u. Therefore, the expression above is smooth, which proves that $\tilde{P}$ is smooth.

$\square$

2-7 Let M be a nonempty smooth n-manifold with or without boundary, and suppose $n \geq 1$. Show that the vector space $C^\infty(M)$ of smooth real-valued functions on M is infinite-dimensional.[Hint: Show that if $f_1, \ldots, f_k$ are elements of $C^\infty(M)$ with nonempty disjoint supports, then they are linearly independent.]

Proof Suppose $f_1, \ldots, f_k \in C^\infty(M)$ are smooth functions with nonempty, pairwise disjoint supports. We claim that these functions are linearly independent.

Consider a linear combination $f = a_1 f_1 + \cdots + a_k f_k$ that is identically zero on M. Fix $i \in \{1, \ldots, k\}$, and choose a point $x \in \operatorname{supp}(f_i)$, which is nonempty by assumption. Since the supports of the f_j are disjoint, we have $f_j(x) = 0$ for all $j \neq i$. Then

$$0 = f(x) = a_i f_i(x).$$

Because $f_i(x) \neq 0$, it follows that $a_i = 0$. Since this holds for each i, all coefficients $a_1, \ldots, a_k$ must be zero, and hence $f_1, \ldots, f_k$ are linearly independent.

To construct infinitely many such functions, observe that every smooth manifold is locally Euclidean. Therefore, for any $n \geq 1$, we can choose countably many pairwise disjoint open subsets $U_1, U_2, \cdots \subset M$, each diffeomorphic to an open ball in $\mathbb{R}^n$. Within each U_i, we can find a smooth bump function $f_i \in C^\infty(M)$ with compact support contained in U_i.

These bump functions $f_1, f_2, \ldots$ are smooth, have disjoint (and nonempty) supports, and hence are linearly independent by the argument above. Therefore, $C^\infty(M)$ contains an infinite linearly independent set and is thus infinite-dimensional.

□

2-8 Define $F\colon \mathbb{R}^n \to \mathbb{RP}^n$ by $F(x^1,\dots,x^n) = [x^1,\dots,x^n,1]$. Show that F is a diffeomorphism onto a dense open subset of $\mathbb{RP}^n$. Do the same for $G\colon \mathbb{C}^n \to \mathbb{CP}^n$ defined by $G(z^1,\dots,z^n) = [z^1,\dots,z^n,1]$(see Problem 1-9)

Proof The map $F\colon \mathbb{R}^n \to \mathbb{RP}^n$ given by

$$F(x^1,\dots,x^n) = [x^1,\dots,x^n,1]$$

is a diffeomorphism onto its image. In fact, this image is precisely the standard coordinate chart $U_{n+1} \subseteq \mathbb{RP}^n$, defined by

$$U_{n+1} = \left\{[x^1,\cdots,x^{n+1}] \in \mathbb{RP}^n \colon x^{n+1} \neq 0\right\}.$$

The coordinate chart map $\varphi_{n+1}\colon U_{n+1} \to \mathbb{R}^n$ is defined by

$$\varphi_{n+1}([x^1 : \cdots : x^{n+1}]) = \left(\frac{x^1}{x^{n+1}},\dots,\frac{x^n}{x^{n+1}}\right).$$

One can easily check that F is the inverse of φ_{n+1}, so F is a diffeomorphism from $\mathbb{R}^n$ onto U_{n+1}.

We now show that U_{n+1} is a dense open subset of $\mathbb{RP}^n$. By definition, U_{n+1} is open in $\mathbb{RP}^n$, so it remains to show that it is dense. That is, for any non-empty open subset $V \subseteq \mathbb{RP}^n$, we must show that $V \cap U_{n+1} \neq \emptyset$.

Consider the canonical projection map $\pi\colon \mathbb{R}^{n+1} \setminus \{0\} \to \mathbb{RP}^n$, which is surjective and continuous. Let us define the subset

$$\tilde{U}_{n+1} = \left\{x \in \mathbb{R}^{n+1} \setminus \{0\} \colon x^{n+1} \neq 0\right\}.$$

Note that

$$\pi^{-1}(U_{n+1}) = \tilde{U}_{n+1}.$$

For any open set $V \subseteq \mathbb{RP}^n$, we consider the preimage

$$\pi^{-1}(V \cap U_{n+1}) = \pi^{-1}(V) \cap \pi^{-1}(U_{n+1}) = \pi^{-1}(V) \cap \tilde{U}_{n+1}.$$

Since $\pi^{-1}(V)$ is open in $\mathbb{R}^{n+1} \setminus \{0\}$, and $\tilde{U}_{n+1}$ is dense there, their intersection is non-empty. Hence,

$$V \cap U_{n+1} \neq \emptyset,$$

showing that U_{n+1} is dense in $\mathbb{RP}^n$.

The proof for the complex case is entirely analogous.

$\square$

2-9 Given a polynomial p in one variable with complex coefficients, not identically zero, show that there is a unique smooth map $\tilde{p}\colon \mathbb{CP}^1 \to \mathbb{CP}^1$ that makes the following diagram commute, where $\mathbb{CP}^1$ is 1-dimensional complex projective space and $G\colon \mathbb{C} \to \mathbb{CP}^1$ is the map of Problem 2-8:

$$\begin{tikzcd} \mathbb{C} \arrow[r, "G"] \arrow[d, "p"'] & \mathbb{CP}^1 \arrow[d, "\tilde{p}"] \\ \mathbb{C} \arrow[r, "G"'] & \mathbb{CP}^1 \end{tikzcd}$$

Proof Let $p(z) = a_0 + a_1 z + \cdots + a_d z^d$ be a nonzero complex polynomial. Define a map $\tilde{p}\colon \mathbb{CP}^1 \to \mathbb{CP}^1$ by

$$\tilde{p}([z,w]) = [a_0 w^d + a_1 z w^{d-1} + \cdots + a_d z^d, w^d] = [p_h(z,w), w^d],$$

where $p_h(z,w)$ is the homogenization of p, so that $p_h(z,1) = p(z)$.

This map is well-defined and smooth on $\mathbb{CP}^1$ because of Problem 2-6.

For any $z \in \mathbb{C}$, we have $G(z) = [z,1]$, so

$$\tilde{p}(G(z)) = \tilde{p}([z,1]) = [p(z),1] = G(p(z)).$$

Therefore, $\tilde{p} \circ G = G \circ p$, and the diagram commutes.

Uniqueness follows from the fact that $G(\mathbb{C}) = \{[z,1] \mid z \in \mathbb{C}\}$ is an open dense subset of $\mathbb{CP}^1$, and any smooth map $\tilde{p}$ agreeing with $G \circ p$ on this subset must coincide with the above construction everywhere on $\mathbb{CP}^1$.

$\square$

2-10 For any topological space M, let $C(M)$ denote the algebra of continuous functions $f\colon M \to \mathbb{R}$. Given a continuous map $F\colon M \to N$, define $F^*\colon C(N) \to C(M)$ by $F^*(f) = f \circ F$.

(a) Show that F^* is a linear map.

(b) Suppose M and N are smooth manifolds. Show that $F\colon M \to N$ is smooth if and only if $F^*(C^\infty(N)) \subseteq C^\infty(M)$.

(c) Suppose $F\colon M \to N$ is a homeomorphism between smooth manifolds. Show that it is a diffeomorphism if and only if F^* restricts to an isomorphism from $C^\infty(N)$ to $C^\infty(M)$.

Proof

(a) It's trivial.

(b) Suppose first that $F\colon M \to N$ is smooth. Then for any $f \in C^\infty(N)$, the composition $f \circ F \in C^\infty(M)$, so $F^*(f) = f \circ F$ is smooth. Hence, $F^*(C^\infty(N)) \subseteq C^\infty(M)$.

Conversely, suppose that $F^*(C^\infty(N)) \subseteq C^\infty(M)$. Let $p \in M$ be arbitrary, and let $q = F(p)$. Choose a smooth coordinate chart (V, ψ) around q, where $\psi = (y^1, \ldots, y^n)\colon V \to \mathbb{R}^n$. For each component function $y^i\colon V \to \mathbb{R}$, choose a smooth bump function ρ supported in V, such that $\rho \equiv 1$ on a smaller neighborhood $\tilde{V} \subseteq V$ of q.

Define the function $\tilde{y}^i = \rho y^i$. Then $\tilde{y}^i$ extends to a smooth function on all of N, and agrees with y^i on $\tilde{V}$. By assumption, $\tilde{y}^i \circ F \in C^\infty(M)$. Since F is continuous, there exists a neighborhood $U \subseteq M$ of p such that $F(U) \subseteq \tilde{V}$. On U, we have

$$y^i \circ F = \tilde{y}^i \circ F,$$

so $y^i \circ F$ is smooth on U. This shows that each component function of $\psi \circ F$ is smooth in a neighborhood of p, so F is smooth at p. Since p was arbitrary, it follows that F is smooth.

(c) Suppose F is a diffeomorphism, let $G = F^{-1}$ and define $G^*\colon C^\infty(M) \to C^\infty(N)$ by $G^*(g) = g \circ G$. By (a), G^* is a linear map and it is easy to verify that G^* is the inverse of F^*, thus F^* is a isomorphism.

Suppose F^* is an isomorphism between $C^\infty(N)$ and $C^\infty(M)$, since F is a homeomorphism, by (b), it suffices to show that $G^*(C^\infty(M)) \subseteq C^\infty(N)$. Since F^* is an isomorphism ,for any $g \in C^\infty(M)$, there exists $f \in C^\infty(N)$ such that $g = F^*(f)$. Thus

$$G^*(g) = G^*(F^*(f)) = G^*(f \circ F) = f \circ F \circ G = f \in C^\infty(N)$$

and F is a diffeomorphism.

$\square$

2-11 Suppose V is a real vector space of dimension $n \geq 1$. Define the projectivization of V, denoted by $\mathbb{P}(V)$, to be the set of 1-dimensional linear subspaces of V, with the quotient topology induced by the map $\pi\colon V \setminus \{0\} \to \mathbb{P}(V)$ that sends x to its span. (Thus $\mathbb{P}(\mathbb{R}^n) = \mathbb{RP}^{n-1}$.) Show that $\mathbb{P}(V)$ is a topological $(n-1)$-manifold, and has a unique smooth structure with the property that for each basis $(E_1, \dots, E_n)$ for V, the map $E\colon \mathbb{RP}^{n-1} \to \mathbb{P}(V)$ defined by $E[v^1, \dots, v^n] = [v^i E_i]$ (where brackets denote equivalence classes) is a diffeomorphism.

Proof

- **$\mathbb{P}(V)$ is a topological $(n-1)$-manifold.**
  Fix a basis $(E_1, \dots, E_n)$ of V. This determines a linear isomorphism $\varphi_B\colon \mathbb{R}^n \to V$ given by $\varphi_B(v^1, \dots, v^n) = \sum v^i E_i$. This isomorphism equips V with a topology via pullback from $\mathbb{R}^n$, and restricts to a homeomorphism $\mathbb{R}^n \setminus \{0\} \to V \setminus \{0\}$.

  Consider the standard projection maps $\pi_{\mathbb{R}^n}\colon \mathbb{R}^n \setminus \{0\} \to \mathbb{RP}^{n-1}$ and $\pi_V\colon V \setminus \{0\} \to \mathbb{P}(V)$. These induce a map

  $$\widetilde{\varphi}_B\colon \mathbb{RP}^{n-1} \to \mathbb{P}(V), \quad [x] \mapsto [\varphi_B(x)].$$

  This map is well-defined because scalar multiplication is preserved under φ_B, and it is bijective because φ_B is an isomorphism. By the universal property of quotient maps, $\widetilde{\varphi}_B$ is continuous, and so is its inverse. Thus, $\widetilde{\varphi}_B$ is a homeomorphism. Since $\mathbb{RP}^{n-1}$ is a topological $(n-1)$-manifold, so is $\mathbb{P}(V)$.

- **Existence of a smooth structure.**
  We define a smooth structure on $\mathbb{P}(V)$ by pulling back the standard smooth structure from $\mathbb{RP}^{n-1}$ via $\widetilde{\varphi}_B$. That is, a chart (U, ψ) on $\mathbb{P}(V)$ is declared smooth if and only if $(\widetilde{\varphi}_B^{-1}(U), \psi \circ \widetilde{\varphi}_B)$ is a smooth chart on $\mathbb{RP}^{n-1}$. By construction, $\widetilde{\varphi}_B$ is a diffeomorphism.

  Now let $(F_1, \dots, F_n)$ be another basis of V, and define

  $$F\colon \mathbb{RP}^{n-1} \to \mathbb{P}(V), \quad [v^1, \dots, v^n] \mapsto [v^i F_i].$$

  Since both $(E_1, \dots, E_n)$ and $(F_1, \dots, F_n)$ are bases of V, there exists an invertible matrix $A \in GL(n, \mathbb{R})$ such that $F_i = A_i^j E_j$. Define a map $P\colon \mathbb{RP}^{n-1} \to \mathbb{RP}^{n-1}$ by $P([v]) = [Av]$. This is a diffeomorphism by Problem 2-6.

  Observe that the map F can be written as the composition

  $$[v] \overset{P}{\longmapsto} [Av] \overset{\widetilde{\varphi}_B}{\longmapsto} [A_i^j v^i E_j] = [v^i F_i],$$

  i.e., $F = \widetilde{\varphi}_B \circ P$. Since both $\widetilde{\varphi}_B$ and P are diffeomorphisms, it follows that F is a diffeomorphism.

- **Uniqueness of the smooth structure.**
  Let $\mathcal{S}$ denote the smooth structure constructed above. Suppose there is another smooth structure $\mathcal{S}'$ on $\mathbb{P}(V)$ such that for every basis B of V, the map $\widetilde{\psi}_B\colon \mathbb{RP}^{n-1} \to \mathbb{P}(V)$ is a diffeomorphism with respect to $\mathcal{S}'$. Then the identity map

  $$\mathrm{id} = \widetilde{\varphi}_B \circ \widetilde{\psi}_B^{-1}\colon (\mathbb{P}(V), \mathcal{S}') \to (\mathbb{P}(V), \mathcal{S})$$

is a diffeomorphism. Therefore, $\mathcal{S}' = \mathcal{S}$. This shows that the smooth structure on $\mathbb{P}(V)$ is uniquely determined by the property that for every basis B, the map $\widetilde{\psi}_B$ is a diffeomorphism.

$\square$

2-12 State and prove an analogue of Problem 2-11 for complex vector spaces.

Proof The proof is analogous to the real case in Problem 2-11

□

2-13 Suppose M is a topological space with the property that for every indexed open cover $\mathcal{X}$ of M, there exists a patition of unity subordinate to $\mathcal{X}$. Show that M is paracompact.

Proof Let $\mathcal{X} = \{X_\alpha\}_{\alpha\in A}$ be an arbitrary open cover of M, and let $\{\psi_\alpha\}_{\alpha\in A}$ be a partition of unity subordinate to $\mathcal{X}$. For each $\alpha \in A$, define the open set

$$U_\alpha = \{p \in M \colon \psi_\alpha(p) > 0\}.$$

Then $\{U_\alpha\}_{\alpha\in A}$ is an open cover of M, since each ψ_α is nonnegative, for any $p \in M$, we have $\sum_\alpha \psi_\alpha(p) = 1$, there exists some α such that $\psi_\alpha(p) > 0$, i.e., $p \in U_\alpha$. Furthermore, since $U_\alpha \subseteq \operatorname{supp}(\psi_\alpha) \subseteq X_\alpha$, it follows that $\{U_\alpha\}_{\alpha\in A}$ is a refinement of $\mathcal{X}$. Finally, the collection $\{\operatorname{supp}(\psi_\alpha)\}_{\alpha\in A}$ is locally finite by the definition of a partition of unity. Since $U_\alpha \subseteq \operatorname{supp}(\psi_\alpha)$, the subcollection $\{U_\alpha\}_{\alpha\in A}$ is also locally finite. Therefore, $\{U_\alpha\}_{\alpha\in A}$ is a locally finite open refinement of $\mathcal{X}$. Since $\mathcal{X}$ was arbitrary, this proves that M is paracompact.

□

2-14 Suppose A and B are disjoint closed subsets of a smooth manifold M. Show that there exists $f \in C^\infty(M)$ such that $0 \geq f(x) \geq 1$ for all $x \in M$, $f^{-1}(0) = A$, and $f^{-1}(1) = B$.

Proof By Theorem 2.29, there exist smooth, nonnegative functions $f_A, f_B \in C^\infty(M)$ such that $f_A^{-1}(0) = A$ and $f_B^{-1}(0) = B$.

Define a smooth function $f \colon M \to \mathbb{R}$ by

$$f = \frac{f_A}{f_A + f_B}.$$

Since both f_A and f_B are smooth and nonnegative, and their sum is strictly positive everywhere (because A and B are disjoint), the function f is well-defined and smooth on all of M.

Now, consider the behavior of f on different subsets of M:

- If $x \in A$, then $f_A(x) = 0$ and $f_B(x) > 0$, so $f(x) = 0$.
- If $x \in B$, then $f_B(x) = 0$ and $f_A(x) > 0$, so $f(x) = 1$.
- If $x \in M \setminus (A \cup B)$, then both $f_A(x)$ and $f_B(x)$ are strictly positive, so $f(x) \in (0, 1)$.

Therefore, $f \in C^\infty(M)$ satisfies $0 \leq f(x) \leq 1$ for all $x \in M$, with $f^{-1}(0) = A$ and $f^{-1}(1) = B$, as required.

$\square$

Chapter 3

Tangent Vectors

3-1 Suppose M and N are smooth manifolds with or without boundary, and $F\colon M \to N$ is a smooth map. Show that $\mathrm{d}F_p\colon T_pM \to T_{F(p)}N$ is the zero map for each $p \in M$ if and only if F is constant on each component of M.

Proof

(⇒) Suppose that $F\colon M \to N$ is constant on each connected component of M. Fix any point $p \in M$, and let $X \in T_pM$ be a tangent vector. Since F is constant in a neighborhood of p, for any smooth function $f \in C^\infty(N)$, the composition $f \circ F$ is locally constant near p. Therefore,

$$\mathrm{d}F_p(X)(f) = X(f \circ F) = 0.$$

This holds for all $f \in C^\infty(N)$, so $\mathrm{d}F_p(X) = 0$. Hence, $\mathrm{d}F_p = 0$ at every $p \in M$.

(⇐) Now suppose that $\mathrm{d}F_p = 0$ for all $p \in M$. We want to show that F is constant on each connected component of M.

Fix a point $p \in M$. Since $\mathrm{d}F_p = 0$, the differential in local coordinates is also zero. Choose smooth charts (U, φ) around $p \in M$ and (V, ψ) around $F(p) \in N$, such that $F(U) \subseteq V$, and

$$\varphi(U) = \begin{cases} B^n \subset \mathbb{R}^n, & \text{if } p \text{ is an interior point,} \\ B^n \cap \mathbb{H}^n, & \text{if } p \text{ is a boundary point,} \end{cases}$$

where B^n is an open ball centered at the origin in $\mathbb{R}^n$. Let $\widehat{F} = \psi \circ F \circ \varphi^{-1}\colon \varphi(U) \to \psi(V)$ denote the coordinate expression of F.

Then $\widehat{F}$ is a smooth map between open subsets of Euclidean space, and since $\mathrm{d}F_p = 0$, we have that the Jacobian matrix $D\widehat{F}$ is zero at all points in $\varphi(U)$. Therefore, each component function $\widehat{F}^j$ has vanishing partial derivatives on $\varphi(U)$, i.e.,

$$\frac{\partial \widehat{F}^j}{\partial x^i} = 0 \quad \text{for all } i, j.$$

It follows that each $\widehat{F}^j$ is constant on $\varphi(U)$, so $\widehat{F}$ is constant on $\varphi(U)$, and hence F is constant on U.

Therefore, F is *locally constant* on M. But any locally constant function on a connected topological space is constant. Hence, F is constant on each connected component of M.

□

3-2 Prove Proposition 3.14(the tangent space to a product manifold).

Proof It is clear that α is a linear map, and both the domain and codomain have the same dimension:

$$\dim T_p(M_1 \times \cdots \times M_k) = \sum_{i=1}^{k} \dim T_{p_i} M_i.$$

Therefore, it suffices to show that α is surjective.

Let $v_i \in T_{p_i} M_i$ for each $i = 1, \ldots, k$. For each v_i, there exists a smooth curve $c_i \colon (-\varepsilon_i, \varepsilon_i) \to M_i$ such that $c_i(0) = p_i$ and $c_i'(0) = v_i$. Let $\varepsilon = \min\{\varepsilon_1, \ldots, \varepsilon_k\}$, and define a smooth curve

$$c \colon (-\varepsilon, \varepsilon) \to M_1 \times \cdots \times M_k, \quad t \mapsto (c_1(t), \ldots, c_k(t)).$$

Then $c(0) = p$, and define $v := c'(0) \in T_p(M_1 \times \cdots \times M_k)$. By the definition of α, we have

$$\alpha(v) = \Big(\mathrm{d}(\pi_1)_p \,(c'(0)), \ldots, \mathrm{d}(\pi_k)_p \,(c'(0)) \Big).$$

Since $\pi_i \circ c = c_i$, we compute:

$$\mathrm{d}(\pi_i)_p(c'(0)) = \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} (\pi_i \circ c)(t) = c_i'(0) = v_i.$$

Therefore, $\alpha(v) = (v_1, \ldots, v_k)$, so α is surjective. As a linear map between vector spaces of the same finite dimension, surjectivity implies bijectivity. Hence, α is an isomorphism.

$\square$

3-3 Prove that if M and N are smooth manifolds, then $T(M \times N)$ is diffeomorphism to $TM \times TN$.

Proof We define a map

$$\Phi\colon T(M \times N) \to TM \times TN$$

by sending a tangent vector at a point $(p, q) \in M \times N$,

$$\Phi(v) = (\pi_{M \times N}(v), \mathrm{d}\pi_M(v)) \times (\pi_{M \times N}(v), \mathrm{d}\pi_N(v)),$$

where $\pi_M : M \times N \to M$, $\pi_N : M \times N \to N$ are the natural projections, and $\mathrm{d}\pi_M$, $\mathrm{d}\pi_N$ are the differentials.

More concretely, under the standard identification of tangent spaces of a product manifold, for any $(p, q) \in M \times N$,

$$T_{(p,q)}(M \times N) \cong T_pM \oplus T_qN.$$

So any vector $v \in T_{(p,q)}(M \times N)$ can be written as $v = (v_M, v_N)$, where $v_M \in T_pM$, $v_N \in T_qN$. Then we define

$$\Phi((p, q), v) = ((p, v_M), (q, v_N)) \in TM \times TN.$$

This map is clearly bijective: given any $(p, v_M) \in TM$ and $(q, v_N) \in TN$, we can construct $((p, q), (v_M, v_N)) \in T(M \times N)$, which is the inverse of Φ.

To show that Φ is a diffeomorphism, we check smoothness in local coordinates. Let (U, φ) be a coordinate chart on M, and (V, ψ) a chart on N. Then $(U \times V, \varphi \times \psi)$ is a chart on $M \times N$, and the corresponding tangent bundle charts are

$$T(U) \cong \varphi(U) \times \mathbb{R}^{\dim M}, \quad T(V) \cong \psi(V) \times \mathbb{R}^{\dim N}, \quad T(U \times V) \cong \varphi(U) \times \psi(V) \times \mathbb{R}^{\dim M + \dim N}.$$

In these coordinates, Φ acts as the identity map:

$$\Phi(x, y, v, w) = ((x, v), (y, w)),$$

where $x = \varphi(p)$, $y = \psi(q)$, $v \in \mathbb{R}^{\dim M}$, and $w \in \mathbb{R}^{\dim N}$. This is clearly a diffeomorphism in Euclidean space.

Therefore, Φ is a diffeomorphism globally, and we conclude that

$$T(M \times N) \cong TM \times TN$$

as smooth manifolds.

$\square$

3-4 Show that $T\mathbb{S}^1$ is diffeomorphic to $\mathbb{S}^1 \times \mathbb{R}$.

Proof We prove that $T\mathbb{S}^1$ is diffeomorphic to $\mathbb{S}^1 \times \mathbb{R}$ by constructing a global trivialization using coordinate charts.

Consider the standard embedding $\mathbb{S}^1 \subset \mathbb{C}$, where each point can be written as e^{it} for some $t \in \mathbb{R}$. Define two coordinate charts:

$$U_1 = \mathbb{S}^1 \setminus \{-1\}, \quad \varphi_1(e^{it}) = t \in (-\pi, \pi),$$
$$U_2 = \mathbb{S}^1 \setminus \{1\}, \quad \varphi_2(e^{it}) = t \in (0, 2\pi).$$

For any $(p, v) \in T\mathbb{S}^1$, we define a map

$$\Phi : T\mathbb{S}^1 \to \mathbb{S}^1 \times \mathbb{R}, \quad \Phi(p, v) = (p, w),$$

where w is the coordinate representation of v in any chart where $p \in U_i$

$$\mathrm{d}(\varphi_i)_p\,(v) = w\,\frac{\mathrm{d}}{\mathrm{d}t}\bigg|_{\varphi_i(p)},$$

then we define $\Phi(p, v) = (p, w)$.

It remains to show that this is well-defined, i.e., the scalar w does not depend on the choice of coordinate chart. Suppose $p \in U_1 \cap U_2$. Observe that $w = v\varphi_i$ and $\varphi_1 - \varphi_2$ is locally constant, it implies $v(\varphi_1 - \varphi_2) = 0$. Hence, the value of w is the same in both charts. So Φ is well-defined globally.

Now, we show that Φ is a diffeomorphism. The map is clearly bijective: given any $(p, w) \in \mathbb{S}^1 \times \mathbb{R}$, define the inverse $\Phi^{-1}(p, w) = (p, v)$,

$$v = \mathrm{d}\varphi_i^{-1}\left(w\,\frac{\mathrm{d}}{\mathrm{d}t}\bigg|_{\varphi_i(p)}\right),$$

Since this again does not depend on the chart i, the inverse is well-defined.

In local coordinates, Φ is just the identity map $(t, w) \mapsto (t, w)$ on $\varphi_i(U_i) \times \mathbb{R}$, so it is smooth, and so is its inverse.

Therefore, $\Phi : T\mathbb{S}^1 \to \mathbb{S}^1 \times \mathbb{R}$ is a diffeomorphism.

$\square$

3-5 Let $\mathbb{S}^1 \subseteq \mathbb{R}^2$ be the unit circle, and let $K \subseteq \mathbb{R}^2$ be the boundary of the square of side 2 centered at the origin: $K = \{(x, y) \colon \max(|x|, |y|) = 1\}$. Show that there is a homeomorphism $F \colon \mathbb{R}^2 \to \mathbb{R}^2$ such that $F(\mathbb{S}^1) = K$, but there is no diffeomorphism with the same property. [Hint: let γ be a smooth curve whose image lies in $\mathbb{S}^1$, and consider the action of $\mathrm{d}F(\gamma'(t))$ on the coordinate function x and y.]

Proof Define $F \colon \mathbb{R}^2 \to \mathbb{R}^2$ by

$$F(v) = \begin{cases} \dfrac{\|v\|_2}{\|v\|_\infty} v & \text{if } v \neq (0,0), \\ (0,0) & \text{if } v = (0,0). \end{cases}$$

For any $v \in \mathbb{S}^1$, we have $\|v\|_2 = 1$, so

$$F(v) = \frac{1}{\|v\|_\infty} v \quad \Rightarrow \quad \|F(v)\|_\infty = \frac{\|v\|_\infty}{\|v\|_\infty} = 1.$$

Hence F maps $\mathbb{S}^1$ onto K.

For $v \neq 0$, F is clearly continuous since it is composed of continuous functions on its domain. To verify continuity at $v = 0$, observe that

$$\|F(v) - F(0)\|_\infty = \|F(v)\|_\infty = \|v\|_2 \leq \sqrt{2}\|v\|_\infty \to 0 \quad \text{as } v \to 0.$$

Therefore, F is continuous on all of $\mathbb{R}^2$.

The inverse of F can be defined explicitly as

$$F^{-1}(w) = \begin{cases} \dfrac{\|w\|_\infty}{\|w\|_2} w & \text{if } w \neq (0,0), \\ (0,0) & \text{if } w = (0,0). \end{cases}$$

A similar argument shows that F^{-1} is also continuous. Therefore, F is a homeomorphism.

Now we show that there is no diffeomorphism $F \colon \mathbb{R}^2 \to \mathbb{R}^2$ such that $F(\mathbb{S}^1) = K$. Suppose for contradiction that such a diffeomorphism exists.

Let $\gamma \colon (-\varepsilon, \varepsilon) \to \mathbb{S}^1$ be a smooth curve with $\gamma(0) = F^{-1}(1,1)$ and $\gamma'(0) \neq 0$. Define $\eta(t) = F(\gamma(t)) = (x(t), y(t))$. Then η is a smooth curve in K with $\eta(0) = (1,1)$.

Since $(1,1)$ is a corner point of K, we must have:

$$x(t) \leq 1, \quad y(t) \leq 1 \quad \text{with equalities at } t = 0.$$

Thus, $x(t)$ and $y(t)$ each attain a local maximum at $t = 0$. By Fermat's Theorem, which states that the derivative of a smooth function must vanish at a local extremum, we get:

$$x'(0) = y'(0) = 0.$$

However, by the chain rule,

$$\eta'(0) = \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} F(\gamma(t)) = \mathrm{d}F_{\gamma(0)}\left(\gamma'(0)\right).$$

Since F is a diffeomorphism, its differential $\mathrm{d}F$ is an isomorphism at every point. Therefore, if $\gamma'(0) \neq 0$, then $\mathrm{d}F_{\gamma(0)}(\gamma'(0)) \neq 0$, implying that $\eta'(0) \neq 0$.

This contradicts the fact that $x'(0) = y'(0) = 0$, which implies $\eta'(0) = 0$. Hence, such a diffeomorphism F cannot exist.

$\square$

3-6 Consider $\mathbb{S}^3$ as the unit sphere in $\mathbb{C}^2$ under the usual indentification $\mathbb{C}^2 \leftrightarrow \mathbb{R}^4$. For each $z = (z^1, z^2) \in \mathbb{S}^3$, define a curve $\gamma_z \colon \mathbb{R} \to \mathbb{S}^3$ by $\gamma_z(t) = (\mathrm{e}^{\mathrm{i}t} z^1, \mathrm{e}^{\mathrm{i}t} z^2)$. Show that γ_z is a smooth curve whose velocity is never zero.

Proof Under the standard identification $\mathbb{C}^2 \cong \mathbb{R}^4$, the curve $\gamma_z(t)$ can be expressed as

$$\gamma_z(t) = (a\cos t - b\sin t,\ a\sin t + b\cos t,\ c\cos t - d\sin t,\ c\sin t + d\cos t),$$

where $z_1 = a + b\mathrm{i}$, $z_2 = c + d\mathrm{i}$, and the coefficients satisfy $a^2 + b^2 + c^2 + d^2 = 1$. Clearly, γ_z is a smooth map from $\mathbb{R}$ to $\mathbb{R}^4$. Since $\mathbb{S}^3 \subset \mathbb{R}^4$ is a smooth embedded submanifold, it follows by Corollary 5.30 that γ_z is also a smooth map from $\mathbb{R}$ to $\mathbb{S}^3$.

Identifying $T_p\mathbb{S}^3$ with a subspace of $T_p\mathbb{R}^4$, we compute the velocity vector:

$$\gamma_z'(t) = (-a\sin t - b\cos t,\ a\cos t - b\sin t, -c\sin t - d\cos t,\ c\cos t - d\sin t).$$

Its squared norm is

$$|\gamma_z'(t)|^2 = a^2 + b^2 + c^2 + d^2 = 1.$$

Therefore, the velocity of γ_z is never zero.

$\square$

3-7 Let M be a smooth manifold with or without boundary and p be a point of M. Let $C_p^\infty(M)$ denote the algebra of germs of smooth real-valued functions at p, and let $\mathcal{D}_pM$ denote the vector space of derivations of $C_p^\infty(M)$. Define a map $\Phi\colon \mathcal{D}_pM \to T_pM$ by $(\Phi v)f = v([f]_p)$. Show that Φ is an isomorphism.

Proof It is straightforward to verify that Φ is a linear map. To prove that Φ is an isomorphism, we show that it is both injective and surjective.

- **Injectivity**: Suppose $\Phi(v) = 0$ for some $v \in \mathcal{D}_pM$. Then for any germ $[f]_p \in C_p^\infty(M)$, choose a representative $\tilde{f} \in C^\infty(U)$ defined on some neighborhood U of p such that $\tilde{f} \in [f]_p$. By the definition of Φ, we have

  $$v([f]_p) = (\Phi(v))(\tilde{f}) = 0.$$

  Since this holds for all $[f]_p \in C_p^\infty(M)$, it follows that $v = 0$. Thus, Φ is injective.

- **Surjectivity**: Let $w \in T_pM$ be an arbitrary tangent vector at p. Define a map $v : C_p^\infty(M) \to \mathbb{R}$ by

  $$v([f]_p) = w(\tilde{f}),$$

  where $\tilde{f}$ is any representative of the germ $[f]_p$. By Proposition 3.8 (which states that the action of a tangent vector at a point depends only on the germ of a function at that point), this definition is independent of the choice of representative, so v is well-defined. It is easy to verify that v is $\mathbb{R}$-linear and satisfies the Leibniz rule:

  $$v([fg]_p) = w(\widetilde{fg}) = w(\tilde{f}\tilde{g}) = \tilde{f}(p)\,w(\tilde{g}) + \tilde{g}(p)\,w(\tilde{f}) = \tilde{f}(p)v([g]_p) + \tilde{g}(p)v([f]_p),$$

  hence $v \in \mathcal{D}_pM$. Then for any $f \in C^\infty(M)$, we have

  $$\Phi(v)(f) = v([f]_p) = w(f),$$

  so $\Phi(v) = w$, and thus Φ is surjective.

Since Φ is both injective and surjective, it is an isomorphism.

$\square$

3-8 Let M be a smooth manifold with or without boundary and $p \in M$. Let $\mathcal{V}_p M$ denote the set of equivalence classes of smooth curves starting at p under the relation $\gamma_1 \sim \gamma_2$ if $(f \circ \gamma_1)'(0) = (f \circ \gamma_2)'(0)$ for every smooth real-valued function f defined in a neighborhood of p. Show that the map $\Psi \colon \mathcal{V}_p M \to T_p M$ defined by $\Psi[\gamma] = \gamma'(0)$ is well defined and bijective.

Proof We first show that the map

$$\Psi \colon \mathcal{V}_p M \to T_p M, \quad [\gamma] \mapsto \gamma'(0)$$

is well defined. Suppose $\gamma_1 \sim \gamma_2$, i.e., for all $f \in C^\infty(M)$ defined in a neighborhood of p, we have

$$(f \circ \gamma_1)'(0) = (f \circ \gamma_2)'(0).$$

Then, for any $f \in C^\infty(M)$,

$$\gamma_1'(0)(f) = \frac{\mathrm{d}}{\mathrm{d}t}(f \circ \gamma_1)(t)\bigg|_{t=0} = \frac{\mathrm{d}}{\mathrm{d}t}(f \circ \gamma_2)(t)\bigg|_{t=0} = \gamma_2'(0)(f).$$

Thus, $\gamma_1'(0) = \gamma_2'(0) \in T_p M$, and so Ψ is well defined.

To prove that Ψ is bijective, we construct its inverse. Given any $v \in T_p M$, by Proposition 3.23, there exists a smooth curve $c \colon (-\varepsilon, \varepsilon) \to M$ such that $c(0) = p$ and $c'(0) = v$. Define

$$\Psi^{-1}(v) = [c],$$

the equivalence class of such a curve.

We now check that Ψ^{-1} is well defined. Suppose $c_1(0) = c_2(0) = p$ and $c_1'(0) = c_2'(0) = v$. Then for any $f \in C^\infty(M)$,

$$(f \circ c_1)'(0) = c_1'(0)(f) = c_2'(0)(f) = (f \circ c_2)'(0),$$

so $c_1 \sim c_2$, hence $[c_1] = [c_2]$.

It is straightforward to verify that Ψ^{-1} is the inverse of Ψ. Indeed, for any $[\gamma] \in \mathcal{V}_p M$, we have

$$\Psi^{-1}(\Psi([\gamma])) = \Psi^{-1}(\gamma'(0)) = [\gamma],$$

and for any $v \in T_p M$, we have

$$\Psi(\Psi^{-1}(v)) = \Psi([c]) = c'(0) = v.$$

Therefore, Ψ is a bijection.

$\square$

Chapter 4

Submersions, Immersions, and Embeddings

4-1 Use the inclusion map $\mathbb{H}^n \to \mathbb{R}^n$ to show that Theorem 4.5 does not extend to the case in which M is a manifold with boundary.

Proof Let p be the origin, $\mathrm{d}F_p$ is clearly an isomorphism. For any neighborhood U of p in $\mathbb{H}^n$, we have $F(U) = U$. If F were a local diffeomorphism at p in the sense of Theorem 4.5, there would exist a neighborhood $U \subseteq \mathbb{H}^n$ of p such that $F(U) = U$ is an open subset of $\mathbb{R}^n$. But if U were open in $\mathbb{R}^n$, it would contain some open ball B centered at p. This is impossible because $U \subseteq \mathbb{H}^n$ and $\mathbb{H}^n$ contains no open ball in $\mathbb{R}^n$ centered at a boundary point. Hence F is not a local diffeomorphism at p, showing that Theorem 4.5 does not extend to manifolds with boundary.

$\square$

4-2 Suppose M is a smooth manifold(without boundary), N is a smooth manifold with boundary, and $F\colon M \to N$ is smooth. Show that if $p \in M$ is a point such that $\mathrm{d}F_p$ is nonsingular, then $F(p) \in \operatorname{Int} N$.

Proof First we prove the statement in Euclidean space. Suppose U is an open subset of $\mathbb{R}^n$ and $F\colon U \to \mathbb{H}^n$ is a smooth map. Let $\iota\colon \mathbb{H}^n \hookrightarrow \mathbb{R}^n$ be the inclusion map, and define $\widehat{F} = \iota \circ F$. Clearly $\widehat{F}$ is smooth. Since $\mathrm{d}\iota$ is nonsingular everywhere, we have

$$\mathrm{d}\widehat{F}_p = \mathrm{d}\iota_{F(p)} \circ \mathrm{d}F_p\,,$$

which is also nonsingular. By the inverse function theorem, there exist neighborhoods $U_0 \subseteq U$ of p and $V_0 \subseteq \mathbb{R}^n$ of $\widehat{F}(p)$ such that $\widehat{F}|_{U_0}\colon U_0 \to V_0$ is a diffeomorphism. Since V_0 is an open subset of $\mathbb{R}^n$ and $\widehat{F}(U_0) \subseteq \mathbb{H}^n$, it follows that $V_0 \cap \mathbb{H}^n$ is an open subset of $\mathbb{H}^n$ containing $\widehat{F}(p)$. In particular, $\widehat{F}(p) \in \operatorname{Int} \mathbb{H}^n$.

Now we return to the manifold setting. Choose charts

$$\varphi\colon U \to \mathbb{R}^n \quad \text{around } p \in M, \qquad \psi\colon V \to \mathbb{H}^n \quad \text{around } F(p) \in N.$$

Then the local representative

$$\psi \circ F \circ \varphi^{-1}\colon \varphi(U) \to \mathbb{H}^n$$

is smooth. By the chain rule, its differential at $\varphi(p)$ is nonsingular, so the Euclidean case applies. Hence $\psi(F(p)) \in \operatorname{Int} \mathbb{H}^n$, which means $F(p) \in \operatorname{Int} N$.

$\square$

4-3 Formulate and prove a version of the rank theorem for a map of constant rank whose domain is a smooth manifold with boundary. [Hint: after extending F arbitrarily as we did in the proof of Theorem 4.15, follow through the proof of the rank theorem until the point at which the constant-rank hypothesis is used, and then explain how to modify the extended map so that it has constant rank.]

4-4 Let $\gamma\colon \mathbb{R} \to \mathbb{T}^2$ be the curve of Example 4.20. Show that the image set $\gamma(\mathbb{R})$ is dense in $\mathbb{T}^2$.

Proof Let α be an irrational number. We first show that the set

$$\alpha\mathbb{Z} + \mathbb{Z} = \{m + n\alpha : m, n \in \mathbb{Z}\}$$

is dense in $\mathbb{R}$.

Indeed, let $x \in \mathbb{R}$ and $\varepsilon > 0$ be arbitrary. Choose $N \in \mathbb{N}$ such that $1/N < \varepsilon$. By **Dirichlet's Approximation Lemma**, there exist integers m, n with $n > 0$ such that

$$|n\alpha - m| < \frac{1}{N} < \varepsilon.$$

Set $\delta = n\alpha - m$, so that $|\delta| < \varepsilon$. Consider the set

$$B := \{k\delta : k \in \mathbb{Z}\}.$$

By the division algorithm, there exists $k \in \mathbb{Z}$ such that

$$|x - k\delta| \le \frac{|\delta|}{2} < \varepsilon.$$

Since $k\delta = kn\alpha - km \in \alpha\mathbb{Z} + \mathbb{Z}$, we have found integers $n' = kn$, $m' = -km$ such that

$$|x - (n'\alpha + m')| < \varepsilon.$$

This proves that $\alpha\mathbb{Z} + \mathbb{Z}$ is dense in $\mathbb{R}$.

Now, take an arbitrary point

$$p = \left(e^{2\pi i x},\, e^{2\pi i y}\right) \in \mathbb{T}^2$$

and $\varepsilon > 0$. Since $\alpha\mathbb{Z} + \mathbb{Z}$ is dense in $\mathbb{R}$, there exist integers m, n such that

$$|(\alpha x - y) + \alpha m - n| < \frac{\varepsilon}{2\pi}.$$

Let $t = x + m$. Then

$$\begin{aligned}
\|\gamma(t) - p\|_1 &= \left|e^{2\pi i x} - e^{2\pi i t}\right| \;+\; \left|e^{2\pi i y} - e^{2\pi i \alpha t}\right| \\
&= \left|e^{2\pi i y} - e^{2\pi i \alpha(x+m)}\right| \\
&\le 2\pi \left|\alpha(x+m) - y - n\right| \\
&= 2\pi \left|(\alpha x - y) + \alpha m - n\right| \\
&< \varepsilon.
\end{aligned}$$

Thus, for any point $p \in \mathbb{T}^2$ and any $\varepsilon > 0$, there exists $t \in \mathbb{R}$ such that $\|\gamma(t) - p\|_1 < \varepsilon$. Therefore,

$$\overline{\gamma(\mathbb{R})} = \mathbb{T}^2,$$

i.e., $\gamma(\mathbb{R})$ is dense in $\mathbb{T}^2$.

$\square$

4-5 Let $\mathbb{CP}^n$ denote the n-dimensional complex projective space, as defined in Problem 1-9.

(a) Show that the quotient map $\pi\colon \mathbb{C}^{n+1} \setminus \{0\} \to \mathbb{CP}^n$ is a surjective smooth submersion.

(b) Show that $\mathbb{CP}^1$ is diffeomorphic to $\mathbb{S}^2$.

Proof

(a) It is clear that π is a smooth surjective map. We now show that it is a submersion. Let $p = (z^1, \ldots, z^{n+1}) \in \mathbb{C}^{n+1} \setminus \{0\}$. Without loss of generality, assume $z^{n+1} \neq 0$. Then $p \in \widehat{U}_{n+1}$ and $\pi(p) \in U_{n+1}$, where we work in the corresponding local chart.

In these coordinates, the Jacobian matrix of π at p is

$$\begin{pmatrix} \frac{1}{z^{n+1}} & 0 & \cdots & 0 & -\frac{z^1}{(z^{n+1})^2} \\ 0 & \frac{1}{z^{n+1}} & \cdots & 0 & -\frac{z^2}{(z^{n+1})^2} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & \frac{1}{z^{n+1}} & -\frac{z^n}{(z^{n+1})^2} \end{pmatrix}.$$

This matrix is clearly surjective, hence π is a submersion.

(b) Define a map $F\colon \mathbb{CP}^1 \to \mathbb{S}^2$ by

$$F([z,w]) = \frac{(z\bar{w} + w\bar{z},\, iw\bar{z} - iz\bar{w},\, |z|^2 - |w|^2)}{|z|^2 + |w|^2}.$$

This map is bijective, since we can explicitly write its inverse:

$$F^{-1}(x,y,z) = \begin{cases} [x+iy,\, 1-z], & (x,y,z) \neq (0,0,1), \\ [1,0], & (x,y,z) = (0,0,1). \end{cases}$$

The map F is manifestly smooth as a map from $\mathbb{CP}^1$ into $\mathbb{R}^3$, and by Corollary 5.30 it is smooth onto $\mathbb{S}^2$. We now verify that F^{-1} is smooth as well.

Consider the stereographic projection $\sigma\colon \mathbb{S}^2 \setminus \{(0,0,1)\} \to \mathbb{R}^2$ and the standard smooth chart (U_2, φ_2) on $\mathbb{CP}^1$, where

$$U_2 = \{[z,w] \in \mathbb{CP}^1 : w \neq 0\}.$$

Then the coordinate expression from $\mathbb{R}^2$ into U_2 is

$$\begin{aligned} \varphi \circ F^{-1} \circ \sigma^{-1}(u,v) &= \varphi \circ F^{-1}\left(\frac{(2u,\, 2v,\, u^2+v^2-1)}{u^2+v^2+1}\right) \\ &= \varphi([u+iv, 1]) \\ &= u + iv. \end{aligned}$$

The same holds for stereographic projection from the south pole. Therefore F^{-1} is smooth.

□

4-6 Let M be a nonempty smooth compact manifold. Show that there is no smooth submersion $F\colon M \to \mathbb{R}^k$ for any $k > 0$.

Proof Suppose, for the sake of contradiction, that there exists a smooth submersion $F\colon M \to \mathbb{R}^k$. By Theorem 4.28, F is an open map. Hence $F(M)$ is an open subset of $\mathbb{R}^k$.

On the other hand, since M is compact, its image $F(M)$ is also compact, and therefore closed and bounded in $\mathbb{R}^k$. Thus $F(M)$ is both open and closed in $\mathbb{R}^k$.

Because $\mathbb{R}^k$ is connected when $k > 0$, it follows that $F(M) = \mathbb{R}^k$. But this is impossible, since $\mathbb{R}^k$ is unbounded whereas $F(M)$, being compact, is bounded. This contradiction shows that no such smooth submersion can exist.

$\square$

4-7 Suppose M and N are smooth manifolds, and $\pi\colon M \to N$ is a surjective smooth submersion. Show that there is no other smooth manifold structure on N that satisfies the conclusion of Theorem 4.29; in other words,assuming that $\tilde{N}$ represents the same set as N with a possibly different topology and smooth structure, and that for every smooth manifold P with or without boundary, a map $F\colon \tilde{N} \to P$ is smooth if and only if $F \circ \pi$ is smooth, show that Id_N is a diffeomorphism between N and $\tilde{N}$. [Remark: this shows that the property described in Theorem 4.29 is "characteristic" in the same sense as that in which Theorem A.27(a) is characteristic of the quotient topology.]

Proof Denote by $\mathrm{Id}_{\tilde{N}}\colon \tilde{N} \to N$ and $\mathrm{Id}_N\colon N \to \tilde{N}$ the two set-theoretic identity maps, and write Id for the identity on N or on $\tilde{N}$ as appropriate.

We first show that Id_N is smooth. Consider the commutative diagram

$$\begin{tikzcd} M \arrow[d, "\pi"'] \arrow[dr, "\pi"] & \\ \tilde{N} \arrow[r, "\mathrm{Id}"'] & \tilde{N} \end{tikzcd}$$

Since Id is smooth, the universal property of $\tilde{N}$ ensures that $\pi\colon M \to \tilde{N}$ is smooth. Now consider

$$\begin{tikzcd} M \arrow[d, "\pi"'] \arrow[dr, "\pi"] & \\ N \arrow[r, "\mathrm{Id}_N"'] & \tilde{N} \end{tikzcd}$$

which commutes. By the defining property of the smooth structure on $\tilde{N}$, this implies that Id_N is smooth.

Next we check that $\mathrm{Id}_{\tilde{N}}$ is smooth. Consider the diagram

$$\begin{tikzcd} M \arrow[d, "\pi"'] \arrow[dr, "\pi"] & \\ \tilde{N} \arrow[r, "\mathrm{Id}_{\tilde{N}}"'] & N \end{tikzcd}$$

which also commutes. By the same reasoning, $\mathrm{Id}_{\tilde{N}}$ is smooth.

Thus Id_N and $\mathrm{Id}_{\tilde{N}}$ are smooth inverses of one another, so they are diffeomorphisms. Therefore the smooth structures on N and $\tilde{N}$ agree.

$\square$

4-8 This problem shows that the converse of Theorem 4.9 is false. Let $\pi\colon \mathbb{R}^2 \to \mathbb{R}$ be defined by $\pi(x,y) = xy$. Show that π is surjective ans smooth, and for each smooth manifold P, a map $F\colon \mathbb{R} \to P$ is smooth if and only if $F \circ \pi$ is smooth; but π is not a smooth submersion.

Proof It is clear that π is surjective, since for any $x \in \mathbb{R}$ we have $\pi(x,1) = x$. Moreover, π is smooth as a polynomial map. However, π is not a submersion, because its differential vanishes at the origin:

$$d\pi_{(0,0)}(v_1, v_2) = 0 \quad \text{for all } (v_1, v_2) \in \mathbb{R}^2.$$

Now suppose $F \circ \pi$ is smooth for some map $F\colon \mathbb{R} \to P$, where P is a smooth manifold. Consider the smooth curve

$$\gamma\colon \mathbb{R} \to \mathbb{R}^2, \quad t \mapsto (1, t).$$

Then for all $t \in \mathbb{R}$ we have

$$F(t) = F \circ \pi(1, t) = (F \circ \pi \circ \gamma)(t).$$

Since both π and γ are smooth, their composition $\pi \circ \gamma$ is smooth. Hence $F \circ \pi \circ \gamma$ is smooth, and therefore F itself is smooth.

This shows that F is smooth if and only if $F \circ \pi$ is smooth.

$\square$

4-9 Let M be a connected smooth manifold, and let $\pi\colon E \to M$ be a topological covering map. Complete the proof of Proposition 4.40 by showing that there is only one smooth structure on E such that π is a smooth covering map. [Hint: use the existence of smooth local sections.]

Proof Let E_1 and E_2 denote the same topological space E equipped with two possibly different smooth structures such that $\pi\colon E_i \to M$ is a smooth covering map for $i = 1, 2$. To show that the smooth structures coincide, it suffices to prove that the identity map

$$\mathrm{Id}\colon E_1 \to E_2$$

is a diffeomorphism.

Since Id is a bijection, we need only check that it is locally smooth with smooth inverse. Fix $p \in E$ and write $q = \pi(p) \in M$. Because $\pi\colon E_1 \to M$ is a smooth covering map, there exists a neighborhood $U \subseteq M$ of q and an open neighborhood $V \subseteq E$ of p such that

$$\pi|_V\colon V \to U$$

is a diffeomorphism. The same holds for $\pi\colon E_2 \to M$, and since the underlying topology of E is the same, we may take the same $V \subseteq E$ and $U \subseteq M$ for both structures.

Now observe that on V,

$$\mathrm{Id}_V = \big((\pi|_V)^{-1}\big)_{E_2} \circ \pi|_{V,E_1},$$

that is, the identity on V can be written as the composition of two diffeomorphisms. Hence Id_V is a diffeomorphism. Since p was arbitrary, Id is locally a diffeomorphism everywhere, and therefore a global diffeomorphism.

Thus E_1 and E_2 have the same smooth structure, proving uniqueness.

$\square$

4-10 Show that the map $q\colon \mathbb{S}^n \to \mathbb{RP}^n$ defined in Example 2.13(f) is a smooth covering map.

Proof Let (U_i, φ_i) be the standard coordinate charts on $\mathbb{RP}^n$, where

$$U_i = \{[x^1, \dots, x^{n+1}] \in \mathbb{RP}^n : x^i \neq 0\}, \qquad \varphi_i([x^1, \dots, x^{n+1}]) = \left(\frac{x^1}{x^i}, \dots, \widehat{\frac{x^i}{x^i}}, \dots, \frac{x^{n+1}}{x^i}\right).$$

By the definition of q, the preimage of U_i is the disjoint union

$$q^{-1}(U_i) = V_i^+ \sqcup V_i^-,$$

where

$$V_i^+ = \{(x^1, \dots, x^{n+1}) \in \mathbb{S}^n : x^i > 0\}, \qquad V_i^- = \{(x^1, \dots, x^{n+1}) \in \mathbb{S}^n : x^i < 0\}.$$

We claim that the restrictions $q|_{V_i^+}$ and $q|_{V_i^-}$ are diffeomorphisms onto U_i. Consider $q|_{V_i^+}$. Define a chart on V_i^+ by

$$\psi_i\colon V_i^+ \to \mathbb{R}^n, \qquad \psi_i(x^1, \dots, x^{n+1}) = (x^1, \dots, \widehat{x^i}, \dots, x^{n+1}).$$

Its inverse is

$$\psi_i^{-1}(u^1, \dots, u^n) = (u^1, \dots, u^{i-1}, \sqrt{1 - |u|^2}, u^i, \dots, u^n),$$

where $|u|^2 = (u^1)^2 + \cdots + (u^n)^2$. Then

$$\begin{aligned}(\varphi_i \circ q \circ \psi_i^{-1})(u^1, \dots, u^n) &= \varphi_i\big([\, u^1, \dots, u^{i-1}, \sqrt{1 - |u|^2}, u^i, \dots, u^n \,]\big)\\ &= \frac{1}{\sqrt{1 - |u|^2}}\,(u^1, \dots, u^n).\end{aligned}$$

This map is clearly a diffeomorphism from $\psi_i(V_i^+)$ onto $\varphi_i(U_i)$. The same argument applies to $q|_{V_i^-}$, replacing $\sqrt{1 - |u|^2}$ by $-\sqrt{1 - |u|^2}$.

Since $\{U_i\}$ is an open cover of $\mathbb{RP}^n$, the local triviality condition is satisfied, and hence q is a smooth covering map.

$\square$

4-11 Show that a topological covering map is proper if and only if its fibers are finite, and therefore the converse of Proposition 4.46 is false.

Proof Suppose first that the covering map π is proper. For any $p \in M$, the set $\{p\}$ is compact, hence $\pi^{-1}(\{p\}) = \pi^{-1}(p)$ is compact. Choose an evenly covered neighborhood U of p, so

$$\pi^{-1}(U) = \bigsqcup_{\alpha \in A} V_\alpha$$

with each V_α mapped homeomorphically onto U. In particular, each V_α contains exactly one point q with $\pi(q) = p$. Then $\{V_\alpha\}_{\alpha \in A}$ is an open cover of the fiber $\pi^{-1}(p)$ by pairwise disjoint singletons. If the fiber were infinite, no finite subfamily of these sets could cover it, contradicting the compactness of $\pi^{-1}(p)$. Hence $\pi^{-1}(p)$ is finite.

Conversely, assume all fibers are finite. Let $K \subseteq M$ be compact, and let $\{U_\alpha\}$ be an open cover of $\pi^{-1}(K)$. For each $p \in K$, write the finite fiber as

$$\pi^{-1}(p) = \{q_1, \ldots, q_{n(p)}\}.$$

Choose an evenly covered neighborhood W_p of p such that

$$\pi^{-1}(W_p) = \bigsqcup_{i=1}^{n(p)} V_{p,i},$$

with each $V_{p,i}$ mapped homeomorphically onto W_p and containing q_i. For each i, pick $U_{p,i} \in \{U_\alpha\}$ with $q_i \in U_{p,i}$, and shrink W_p if necessary so that $V_{p,i} \subseteq U_{p,i}$ for all i.

Then $\{W_p : p \in K\}$ covers K, so by compactness there exist $p_1, \ldots, p_m$ with $K \subseteq \bigcup_{j=1}^m W_{p_j}$. Consequently,

$$\pi^{-1}(K) \subseteq \bigcup_{j=1}^m \pi^{-1}(W_{p_j}) = \bigcup_{j=1}^m \bigcup_{i=1}^{n(p_j)} V_{p_j,i} \subseteq \bigcup_{j=1}^m \bigcup_{i=1}^{n(p_j)} U_{p_j,i},$$

and the right-hand side is a finite subfamily of $\{U_\alpha\}$ covering $\pi^{-1}(K)$. Hence $\pi^{-1}(K)$ is compact, so π is proper.

Therefore, a covering map is proper if and only if all its fibers are finite.

$\square$

4-12 Using the covering map $\varepsilon^2 \colon \mathbb{R}^2 \to \mathbb{T}^2$ (see Example 4.35), show that the immersion $X \colon \mathbb{R}^2 \to \mathbb{R}^3$ defined in Example 4.2(d) descends to a smooth embedding of $\mathbb{T}^2$ into $\mathbb{R}^3$. Specifically, show that X passes to the quotient to define a smooth map $\widetilde{X} \colon \mathbb{T}^2 \to \mathbb{R}^3$, and then show that $\widetilde{X}$ is a smooth embedding whose image is the given surface of revolution.

Proof It's clear that X is constant on each fiber of ε^2, by Theorem 4.30 there exists a unique smooth map $\widetilde{X}$ s.t. $\widetilde{X} \circ \varepsilon^2 = X$.

First we prove that $\widetilde{X}$ is injective. Suppose $\widetilde{X}(q_1) = \widetilde{X}(q_2)$, since ε^2 is surjective, there exist $p_1, p_2 \in \mathbb{R}^2$ s.t. $\varepsilon^2(p_i) = q_i$ for $i = 1, 2$. Thus $X(p_1) = X(p_2)$, which implies $p_1 = p_2 + (n_1, n_2)$, $(n_1, n_2) \in \mathbb{Z}^2$. Since $\varepsilon^2(p_1) = \varepsilon^2(p_2)$ we have $q_1 = q_2$, thus $\widetilde{X}$ is injective.

Since X is an immersion and ε^2 is a submersion, $\mathrm{d}X = \mathrm{d}\widetilde{X} \circ \mathrm{d}\varepsilon^2$ implies $\mathrm{d}\widetilde{X}$ is injective, thus $\widetilde{X}$ is a injective smooth immersion.

By Proposition 4.22(c), $\widetilde{X}$ yields an embedding since $\mathbb{T}^2$ is compact.

$\square$

4-13 Define a map $F\colon \mathbb{S}^2 \to \mathbb{R}^4$ by $F(x,y,z) = (x^2 - y^2, xy, xz, yz)$. Using the smooth covering map of Example 2.13(f) and Problem 4-10, show that F descends to a smooth embedding of $\mathbb{RP}^2$ into $\mathbb{R}^4$.

Proof Let $q\colon \mathbb{S}^2 \to \mathbb{RP}^2$ be the smooth covering map from Problem 4-10, which is a surjective smooth submersion. Since F takes the same value on both points of each fiber of q, we have

$$q^{-1}([x,y,z]) = \left\{ \pm\frac{(x,y,z)}{\sqrt{x^2+y^2+z^2}} \right\}, \quad F\left(\frac{(x,y,z)}{\sqrt{x^2+y^2+z^2}}\right) = F\left(-\frac{(x,y,z)}{\sqrt{x^2+y^2+z^2}}\right).$$

By Theorem 4.30, there exists a unique smooth map

$$\widetilde{F}\colon \mathbb{RP}^2 \to \mathbb{R}^4$$

such that $F = \widetilde{F} \circ q$.

To show that $\widetilde{F}$ is a smooth embedding, it suffices to prove that it is an injective immersion. Suppose

$$F(x,y,z) = (a,b,c,d) = (x^2 - y^2,\ xy,\ yz,\ xz).$$

Since $(x^2+y^2)^2 = (x^2-y^2)^2 + 4x^2y^2 = a^2 + 4b^2$, we obtain

$$x^2 + y^2 = \sqrt{a^2+4b^2}, \quad x^2 = \frac{\sqrt{a^2+4b^2} \pm a}{2}, \quad z^2 = 1 - \sqrt{a^2+4b^2}.$$

Thus the triple (x^2, y^2, z^2) is uniquely determined by (a,b,c,d). To determine the signs of x, y, z, note that

$$\begin{cases} \operatorname{sgn}(x)\operatorname{sgn}(y) = \operatorname{sgn}(b), \\ \operatorname{sgn}(y)\operatorname{sgn}(z) = \operatorname{sgn}(c), \\ \operatorname{sgn}(x)\operatorname{sgn}(z) = \operatorname{sgn}(d). \end{cases}$$

This system has exactly two solutions, which differ by an overall sign change. Hence F identifies only antipodal points, and so $\widetilde{F}$ is injective on $\mathbb{RP}^2$.

That $\widetilde{F}$ is an immersion follows by an argument similar to that in Problem 4-12. Since $\mathbb{RP}^2$ is compact, an injective immersion into $\mathbb{R}^4$ is a smooth embedding. Therefore, $\widetilde{F}$ is a smooth embedding of $\mathbb{RP}^2$ into $\mathbb{R}^4$.

$\square$