# Homework5

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# 1 实验要求

1. 实现三次样条插值算法
2. 计算三次样条插值函数

# 2 算法原理

## 2.1 三次样条插值

三次样条插值函数用 $S(x)$ 表示，其在区间 $[x_i, x_{i+1}]$ 上基本的表达式为

$$S(x) = ax^3 + bx^2 + c * x^3 + d$$

从此定义可知要求出 $S(x)$ ，在每个小区间 $[x_i, x_{i+1}]$ 上要确定 4 个待定系数，而共有 n 个小区间，故应该确定 4n 个待定系数。

三次样条方程满足以下条件：

- 在每个分段小区间 $[x_i, x_{i+1}]$ $S(x) = S_i(x)$ 都是一个三次方程
- 满足插值条件，即 $S(x_i) = y_i \quad (i = 0, 1, ..., n)$
- 曲线光滑，即 $S(x)$ $S$'$(x)$ $S$''$(x)$ 连续。

这个三次方程可以构造成：

$$y = a_i + b_i x + c_i x^2 + d_i x^3$$

我们称这个方程为三次样条函数 $S_i(x)$ 。从 $S_i(x)$ 可以看出每个小区间有四个未知数 $a_i, b_i, c_i, d_i$ ，有 n 个小区间，则有 4n 个未知数，要解出这些未知数，则我们需要 4n 个方程来求解。

## 2.2 大 M 法计算三次样条插值函数 (自然边界条件)

给定插值点 $\{(x_i, f(x_i)), i = 0, 1 \ldots, n\}$, 记 $S''(x_i) = M_i, h_i = x_{i+1} - x_i$, 则

$$\begin{aligned}
S(x) =& \frac{(x_{i+1} - x)^3 M_i + (x - x_i)^3 M_{i+1}}{6h_i} + \frac{(x_{i+1} - x) y_i + (x - x_i) y_{i+1}}{h_i} \\
&- \frac{h_i}{6} \left[(x_{i+1} - x) M_i + (x - x_i) M_{i+1}\right] \\
S(x) =& \frac{M_{i+1} - M_i}{6h_i} x^3 + \frac{x_{i+1} M_i - x_i M_{i+1}}{2h_i} x^2 \\
&+ \frac{3\left(x_i^2 M_{i+1} - x_{i+1}^2 M_i\right) + 6\left(y_{i+1} - y_i\right) - h_i^2\left(M_{i+1} - M_i\right)}{6h_i} x \\
&+ \frac{x_{i+1}^3 M_i - X_i^3 M_{i+1} + 6\left(x_{i+1} y_i - x_i y_{i+1}\right) - h_i^2\left(x_{i+1} M_i - x_i M_{i+1}\right)}{6h_i}, \quad x \in [x_i, x_{i+1}]
\end{aligned}$$

其中 $M$ 满足

$$\mu_i M_{i-1} + 2M_i + \lambda_i M_{i+1} = d_i, \quad i = 1, 2, \ldots, n-1$$

当

$$\begin{aligned}
\lambda_i &= \frac{h_i}{h_i + h_{i-1}} \quad \mu_i = 1 - \lambda_i \\
d_i &= \frac{6}{h_i + h_{i-1}} \left(\frac{y_{i+1} - y_i}{h_i} - \frac{y_i - y_{i-1}}{h_{i-1}}\right) = 6f\left[x_{i-1}, x_i, x_{i+1}\right]
\end{aligned}$$

在自然边界条件下 $(M_0 = M_n = 0)$, 方程组为

$$\begin{bmatrix}
2 & \lambda_1 & & & \\
\mu_2 & 2 & \lambda_2 & & \\
& \ddots & \ddots & \ddots & \\
& & \mu_{n-2} & 2 & \lambda_{n-2} \\
& & & \mu_{n-1} & 2
\end{bmatrix}
\begin{bmatrix}
M_1 \\ M2 \\ \vdots \\ M_{n-2} \\ M_{n-1}
\end{bmatrix}
=
\begin{bmatrix}
d_1 - \mu_1 M_0 \\ d_2 \\ \vdots \\ d_{n-2} \\ d_{n-1} - \lambda_{n-1} M_n
\end{bmatrix}$$

## 2.3 追赶法求解三对角矩阵

追赶法是一种快速有效的方法，用于求解三对角矩阵线性方程组。它本质上是 LU 分解。

追赶法求解三对角矩阵的步骤为：

**Algorithm 1:** Alg

**Input:** *vector* $a$ $b$ $c$ $d$;

**Output:** *vector* $d$;

$b[1] \leftarrow b[1], y[1] \leftarrow d[1]$

**for** $i$ = *2 to* $n$ **do**

  $a[i] \leftarrow \frac{a[i]}{\beta[i-1]}$;

  $b[i] = b[i] - a[i]c[i-1]$;

  $d[i] = d[i] - a[i]d[i-1]$;

$d[n] = d[n]/b[n]$

**for** $i$ = $n$ *- 1 to 1* **do**

  $d[i] = (d[i] - c[i]d[i+1])/b[i]$;

**return** $d$

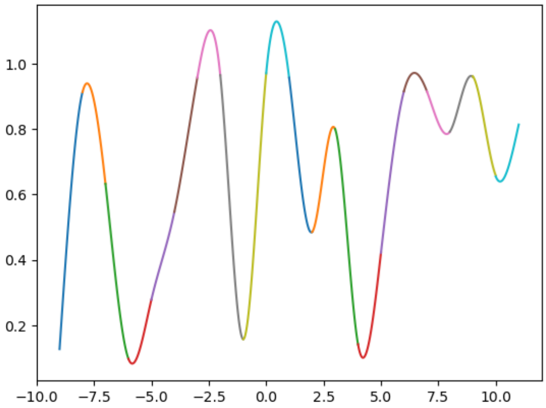
# 3 实验结果

```
[-9,-8]  : (-0.254241)*x^3 + (-6.86449)*x^2 + (-60.7398)*x + (-175.849)
[-8,-7]  : (0.203803)*x^3 + (4.12854)*x^2 + (27.2045)*x + (58.6696)
[-7,-6]  : (0.25253)*x^3 + (5.15181)*x^2 + (34.3673)*x + (75.3829)
[-6,-5]  : (-0.244121)*x^3 + (-3.7879)*x^2 + (-19.2709)*x + (-31.8937)
[-5,-4]  : (0.0954546)*x^3 + (1.30573)*x^2 + (6.19725)*x + (10.5533)
[-4,-3]  : (-0.0828972)*x^3 + (-0.834491)*x^2 + (-2.36364)*x + (-0.861222)
[-3,-2]  : (-0.309266)*x^3 + (-2.87181)*x^2 + (-8.47559)*x + (-6.97317)
[-2,-1]  : (0.90846)*x^3 + (4.43455)*x^2 + (6.13712)*x + (2.76864)
[-1,0]  : (-0.889575)*x^3 + (-0.959557)*x^2 + (0.743018)*x + (0.9706)
[0,1]  : (0.203139)*x^3 + (-0.959557)*x^2 + (0.743018)*x + (0.9706)
[1,2]  : (0.445017)*x^3 + (-1.68519)*x^2 + (1.46865)*x + (0.728722)
[2,3]  : (-0.738108)*x^3 + (5.41356)*x^2 + (-12.7289)*x + (10.1937)
[3,4]  : (0.747415)*x^3 + (-7.95615)*x^2 + (27.3803)*x + (-29.9154)
[4,5]  : (-0.339953)*x^3 + (5.09227)*x^2 + (-24.8134)*x + (39.6762)
[5,6]  : (-0.111902)*x^3 + (1.67151)*x^2 + (-7.7096)*x + (11.1698)
[6,7]  : (0.0796627)*x^3 + (-1.77666)*x^2 + (12.9794)*x + (-30.2082)
[7,8]  : (0.163651)*x^3 + (-3.54043)*x^2 + (25.3258)*x + (-59.0164)
[8,9]  : (-0.319969)*x^3 + (8.06646)*x^2 + (-67.5293)*x + (188.597)
[9,10]  : (0.354323)*x^3 + (-10.1394)*x^2 + (96.3237)*x + (-302.962)
[10,11]  : (-0.163425)*x^3 + (5.39301)*x^2 + (-59.0007)*x + (214.786)
```

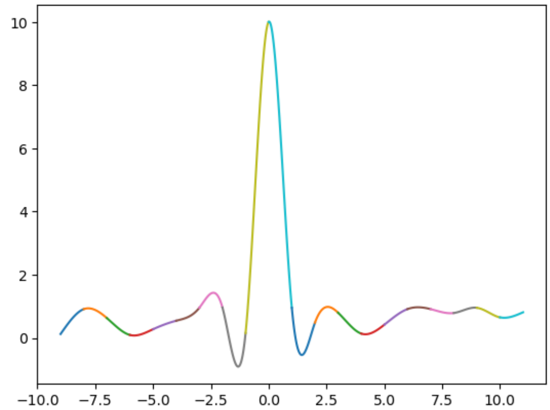
result_1 输出

```
[-9,-8]  : (-0.254626)*x^3 + (-6.87491)*x^2 + (-60.8332)*x + (-176.126)
[-8,-7]  : (0.205731)*x^3 + (4.17367)*x^2 + (27.5555)*x + (59.5768)
[-7,-6]  : (0.245201)*x^3 + (5.00253)*x^2 + (33.3575)*x + (73.1148)
[-6,-5]  : (-0.216734)*x^3 + (-3.3123)*x^2 + (-16.5315)*x + (-26.6632)
[-5,-4]  : (-0.00676361)*x^3 + (-0.162739)*x^2 + (-0.783675)*x + (-0.41684)
[-4,-3]  : (0.298589)*x^3 + (3.50149)*x^2 + (13.8732)*x + (19.1257)
[-3,-2]  : (-1.73299)*x^3 + (-14.7827)*x^2 + (-40.9794)*x + (-35.7269)
[-2,-1]  : (6.22188)*x^3 + (32.9465)*x^2 + (54.479)*x + (27.912)
[-1,0]  : (-11.6901)*x^3 + (-20.7895)*x^2 + (0.743018)*x + (10)
[0,1]  : (11.0037)*x^3 + (-20.7895)*x^2 + (0.743018)*x + (10)
[1,2]  : (-4.8684)*x^3 + (26.8267)*x^2 + (-46.8732)*x + (25.8721)
[2,3]  : (0.685617)*x^3 + (-6.49736)*x^2 + (19.775)*x + (-18.5601)
[3,4]  : (0.365929)*x^3 + (-3.62017)*x^2 + (11.1434)*x + (-9.92847)
[4,5]  : (-0.237734)*x^3 + (3.62379)*x^2 + (-17.8325)*x + (28.706)
[5,6]  : (-0.139292)*x^3 + (2.14716)*x^2 + (-10.4493)*x + (16.4007)
[6,7]  : (0.0870017)*x^3 + (-1.92613)*x^2 + (13.9904)*x + (-32.4787)
[7,8]  : (0.161685)*x^3 + (-3.49448)*x^2 + (24.9689)*x + (-58.0952)
[8,9]  : (-0.319443)*x^3 + (8.05258)*x^2 + (-67.4076)*x + (188.242)
[9,10]  : (0.354185)*x^3 + (-10.1354)*x^2 + (96.2838)*x + (-302.832)
[10,11]  : (-0.163397)*x^3 + (5.3921)*x^2 + (-58.9907)*x + (214.75)
```

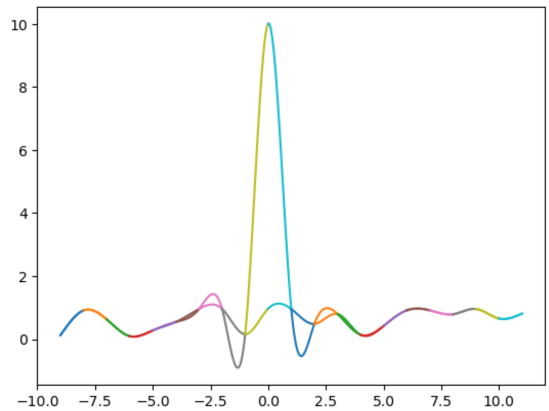
result_2 输出




(a) $result\_1$




(b) $result\_2$




(c) 二者对比

# 4 实验分析

将第 10 个点的位置改变后，有 6 个区间的函数曲线发生变化。