homework1

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1 实验要求

使用 C/C++ 实现两种解线性方程组的算法：

1. 列主元 Guass 消元法
2. Gauss-Seidel 迭代法

2 算法原理

2.1 待解线性方程组

考虑两点边值问题

$$\begin{cases} \varepsilon \frac{d^2y}{dx^2} + \frac{dy}{dx} = a, \quad 0 < a < 1 \\ y(0) = 0, \quad y(1) = 1 \end{cases}$$

容易知道它们的精确解为

$$y = \frac{1-a}{1-e^{-\frac{1}{\varepsilon}}}(1 - e^{-\frac{x}{e}}) + ax$$

为了把微分方程离散，把 $[0,1]$ 区间 n 等分，令 $h = \frac{1}{n}$，

$$x_i = ih, \quad i = 1, 2, ..., n-1$$

得到差分方程

$$\varepsilon\frac{y_{i-1}-2y_i+y_{i+1}}{h^2}+\frac{y_{i+1}}{h}=a$$

简化为

$$(\varepsilon+h)y_{i+1}-(2\varepsilon+h)y_i+\varepsilon y_{i-1}=ah^2$$

从而离散后得到的微分方程组的系数矩阵为

$$A=\begin{bmatrix} -(2\varepsilon+h) & \varepsilon+h & & & & \\ \varepsilon & -(2\varepsilon+h) & \varepsilon+h & & & \\ & \varepsilon & -(2\varepsilon+h) & \ddots & & \\ & & \ddots & \ddots & \varepsilon+h & \\ & & & \varepsilon & -(2\varepsilon+h) & \end{bmatrix}$$

2.2 列主元 Guass 消元法

列主元 Guass 消元法是对顺序 Gauss 消元法的一种改进，顺序 Gauss 消元可行的充分必要条件是矩阵 A 的各阶顺序主子式不为 0，但是只要 $\det A\neq 0$，方程组 $Ax=b$ 就有解，故顺序 Gauss 消元本身具有局限性。列主元 Guass 消元法对每个 $\{k,k=1,2,...,n-1\}$ 在消元前，选出 $\{|a_{kk}^{(k-1)}|,|a_{k+1,k}^{(k-1)}|,...,|a_{n,k}^{(k-1)}|\}$ 中绝对值最大的元素 $a_{mk}^{(k-1)}$，对 k 行和 m 行进行交换后，再做消元运算，由于 $\det A\neq 0$，可证 $\{a_{kk}^{(k-1)},a_{k+1,k}^{(k-1)},...,a_{n,k}^{(k-1)}\}$ 中至少有一个元素不为 0，因此，列主行消元总是可行的。

2.3 Gauss-Seidel 迭代法

Gauss-Seidel 迭代法与 jacobi 迭代步骤大致相同，其迭代的核心部分是计算迭代式的过程中，要及时将 $x_i^{(k+1)}$ 放到 $x_i^{(k)}$ 的位置上。

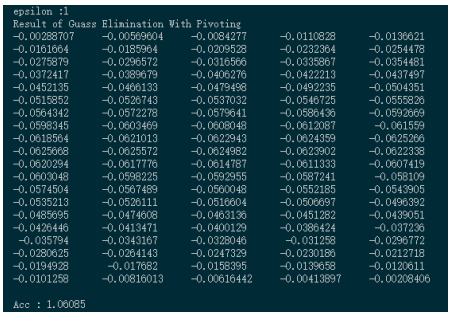
$$\begin{cases} x_1^{k+1}=x_1^k+x_2^k+...+x_n^k+b_1 \\ x_2^{k+1}=x_1^{k+1}+x_2^k+...+x_n^k+b_2 \\ \vdots \\ x_n^{k+1}=x_1^{k+1}+x_2^{k+1}+...+x_n^k+b_n \end{cases}$$

2.4 对实验结果的修正

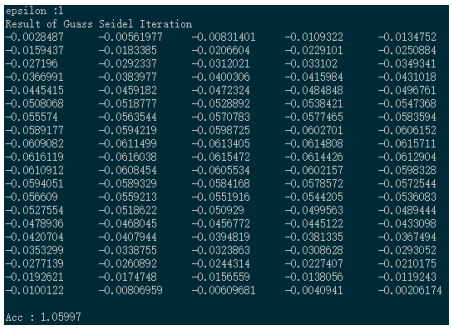
事实上，对实验给定的待解方程组，部分初始值是已知的，若在计算过程中不对初值条件进行修正，将会产生较大误差，故根据已知条件 $y(0)=0, y(1)=1$ 对初始条件进行修正 $b[99]=ah^2-\varepsilon-h$。

3 实验结果

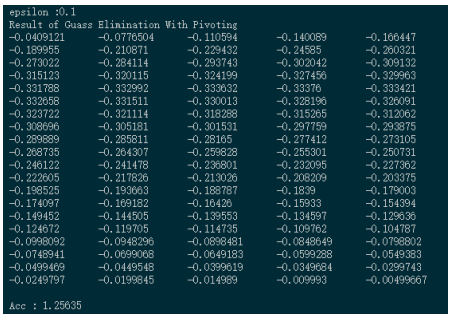
未修正前，$\varepsilon=1, 0.1, 0.01, 0.0001$ 的结果分别为


epsilon :1
Result of Guass Elimination With Pivoting
-0.00288707 -0.00569604 -0.0084277 -0.0110828 -0.0136621
-0.0161664 -0.0185964 -0.0209528 -0.0232364 -0.0254478
-0.0275879 -0.0296572 -0.0316566 -0.0335867 -0.0354481
-0.0372417 -0.0389679 -0.0406276 -0.0422213 -0.0437497
-0.0452135 -0.0466133 -0.0479498 -0.0492235 -0.0504351
-0.0515852 -0.0526743 -0.0537032 -0.0546725 -0.0555826
-0.0564342 -0.0572278 -0.0579641 -0.0586436 -0.0592669
-0.0598345 -0.0603469 -0.0608048 -0.0612087 -0.061559
-0.0618564 -0.0621013 -0.0622943 -0.0624359 -0.0625266
-0.0625668 -0.0625572 -0.0624982 -0.0623902 -0.0622338
-0.0620294 -0.0617776 -0.0614787 -0.0611333 -0.0607419
-0.0603048 -0.0598225 -0.0592955 -0.0587241 -0.058109
-0.0574504 -0.0567489 -0.0560048 -0.0552185 -0.0543905
-0.0535213 -0.0526111 -0.0516604 -0.0506697 -0.0496392
-0.0485695 -0.0474608 -0.0463136 -0.0451282 -0.0439051
-0.0426446 -0.0413471 -0.0400129 -0.0386424 -0.037236
-0.035794 -0.0343167 -0.0328046 -0.031258 -0.0296772
-0.0280635 -0.0264143 -0.0247329 -0.0230186 -0.0212718
-0.0194928 -0.017682 -0.0158395 -0.0139658 -0.0120611
-0.0101258 -0.00816013 -0.00616442 -0.00413897 -0.00208406
Acc : 1.06085

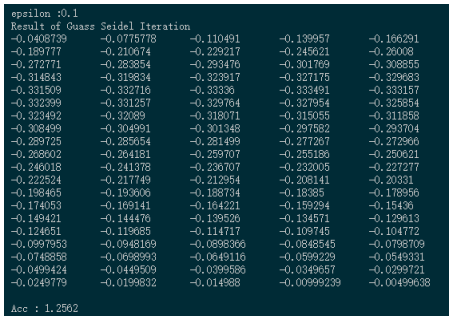
(a)


epsilon :1
Result of Guass Seidel Iteration
-0.0028487 -0.00561977 -0.00831401 -0.0109322 -0.0134752
-0.0159437 -0.0183385 -0.0206604 -0.0229101 -0.0250884
-0.027196 -0.0292337 -0.0312021 -0.033102 -0.0349341
-0.0366991 -0.0383977 -0.0400306 -0.0415984 -0.0431018
-0.0445415 -0.0459182 -0.0472324 -0.0484848 -0.0496761
-0.0508068 -0.0518777 -0.0528892 -0.0538421 -0.0547368
-0.055574 -0.0563544 -0.0570783 -0.0577465 -0.0583594
-0.0589177 -0.0594219 -0.0598725 -0.0602701 -0.0606152
-0.0609082 -0.0611499 -0.0613405 -0.0614808 -0.0615711
-0.0616119 -0.0616038 -0.0615472 -0.0614426 -0.0612904
-0.0610912 -0.0608454 -0.0605534 -0.0602157 -0.0598328
-0.0594051 -0.0589329 -0.0584168 -0.0578572 -0.0572544
-0.056609 -0.0559213 -0.0551916 -0.0544205 -0.0536083
-0.0527554 -0.0518622 -0.050929 -0.0499563 -0.0489444
-0.0478936 -0.0468045 -0.0456772 -0.0445122 -0.0433098
-0.0420704 -0.0407944 -0.0394819 -0.0381335 -0.0367494
-0.0353299 -0.0338755 -0.0323863 -0.0308628 -0.0293052
-0.0277139 -0.0260892 -0.0244314 -0.0227407 -0.0210175
-0.0192621 -0.0174748 -0.0156559 -0.0138056 -0.0119243
-0.0100122 -0.00806959 -0.00609681 -0.0040941 -0.00206174
Acc : 1.05997

(b)


epsilon :0.1
Result of Guass Elimination With Pivoting
-0.0409121 -0.0776504 -0.110594 -0.140089 -0.166447
-0.189955 -0.210871 -0.229432 -0.24585 -0.260321
-0.273022 -0.284114 -0.293743 -0.302042 -0.309132
-0.315123 -0.320115 -0.324199 -0.327456 -0.329963
-0.331788 -0.332992 -0.333632 -0.33376 -0.333421
-0.332658 -0.331511 -0.330013 -0.328196 -0.326091
-0.323722 -0.321114 -0.318288 -0.315265 -0.312062
-0.308696 -0.305181 -0.301531 -0.297759 -0.293875
-0.289889 -0.285811 -0.28165 -0.277412 -0.273105
-0.268735 -0.264307 -0.259828 -0.255301 -0.250731
-0.246122 -0.241478 -0.236801 -0.232095 -0.227362
-0.222605 -0.217826 -0.213026 -0.208209 -0.203375
-0.198525 -0.193663 -0.188787 -0.1839 -0.179003
-0.174097 -0.169182 -0.16426 -0.15933 -0.154394
-0.149452 -0.144505 -0.139553 -0.134597 -0.129636
-0.124672 -0.119705 -0.114735 -0.109762 -0.104787
-0.0998092 -0.0948296 -0.0898481 -0.0848649 -0.0798802
-0.0748941 -0.0699068 -0.0649183 -0.0599288 -0.0549383
-0.0499469 -0.0449548 -0.0399619 -0.0349684 -0.0299743
-0.0249797 -0.0199845 -0.014989 -0.009993 -0.00499667
Acc : 1.25635

(c)


epsilon :0.1
Result of Guass Seidel Iteration
-0.0408739 -0.0775778 -0.110491 -0.139957 -0.166291
-0.189777 -0.210674 -0.229217 -0.245621 -0.26008
-0.272771 -0.283854 -0.293476 -0.301769 -0.308855
-0.314843 -0.319834 -0.323917 -0.327175 -0.329683
-0.331509 -0.332716 -0.33336 -0.333491 -0.333157
-0.332399 -0.331257 -0.329764 -0.327954 -0.325854
-0.323492 -0.32089 -0.318071 -0.315055 -0.311858
-0.308499 -0.304991 -0.301348 -0.297582 -0.293704
-0.289725 -0.285654 -0.281499 -0.277267 -0.272966
-0.268602 -0.264181 -0.259707 -0.255186 -0.250621
-0.246018 -0.241378 -0.236707 -0.232005 -0.227277
-0.222524 -0.217749 -0.212954 -0.208141 -0.20331
-0.198465 -0.193606 -0.188734 -0.18385 -0.178956
-0.174053 -0.169141 -0.164221 -0.159294 -0.15436
-0.149421 -0.144476 -0.139526 -0.134571 -0.129613
-0.124651 -0.119685 -0.114717 -0.109745 -0.104772
-0.0997953 -0.0948169 -0.0898366 -0.0848545 -0.0798709
-0.0748858 -0.0698993 -0.0649116 -0.0599229 -0.0549331
-0.0499424 -0.0449509 -0.0399586 -0.0349657 -0.0299721
-0.0249779 -0.0199832 -0.014988 -0.00999239 -0.00499638
Acc : 1.2562

(d)

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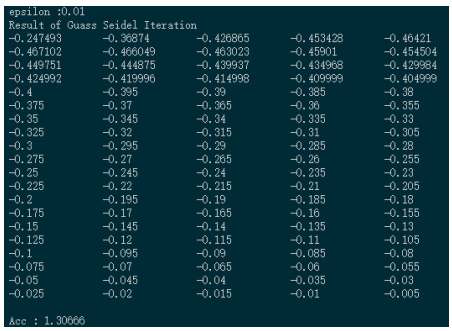
epsilon :0.01
Result of Guass Elimination With Pivoting
-0.2475         -0.36875        -0.426875       -0.453438       -0.464219
-0.467109       -0.466055       -0.463027       -0.459014       -0.454507
-0.449753       -0.444877       -0.439938       -0.434969       -0.429985
-0.424992       -0.419996       -0.414998       -0.409999       -0.405
-0.4            -0.395          -0.39           -0.385          -0.38
-0.375          -0.37           -0.365          -0.36           -0.355
-0.35           -0.345          -0.34           -0.335          -0.33
-0.325          -0.32           -0.315          -0.31           -0.305
-0.3            -0.295          -0.29           -0.285          -0.28
-0.275          -0.27           -0.265          -0.26           -0.255
-0.25           -0.245          -0.24           -0.235          -0.23
-0.225          -0.22           -0.215          -0.21           -0.205
-0.2            -0.195          -0.19           -0.185          -0.18
-0.175          -0.17           -0.165          -0.16           -0.155
-0.15           -0.145          -0.14           -0.135          -0.13
-0.125          -0.12           -0.115          -0.11           -0.105
-0.1            -0.095          -0.09           -0.085          -0.08
-0.075          -0.07           -0.065          -0.06           -0.055
-0.05           -0.045          -0.04           -0.035          -0.03
-0.025          -0.02           -0.015          -0.01           -0.005

Acc : 1.30666
```

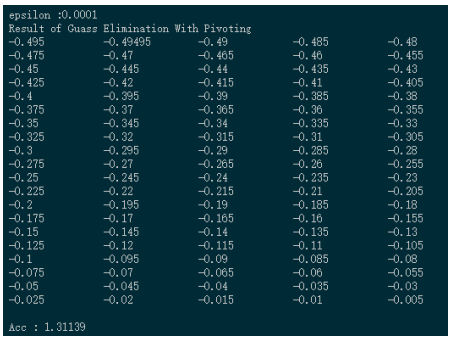
(e)



```
epsilon :0.01
Result of Guass Seidel Iteration
-0.247493       -0.36874        -0.426865       -0.453428       -0.46421
-0.467102       -0.466049       -0.463023       -0.45901        -0.454504
-0.449751       -0.444875       -0.439937       -0.434968       -0.429984
-0.424992       -0.419996       -0.414998       -0.409999       -0.404999
-0.4            -0.395          -0.39           -0.385          -0.38
-0.375          -0.37           -0.365          -0.36           -0.355
-0.35           -0.345          -0.34           -0.335          -0.33
-0.325          -0.32           -0.315          -0.31           -0.305
-0.3            -0.295          -0.29           -0.285          -0.28
-0.275          -0.27           -0.265          -0.26           -0.255
-0.25           -0.245          -0.24           -0.235          -0.23
-0.225          -0.22           -0.215          -0.21           -0.205
-0.2            -0.195          -0.19           -0.185          -0.18
-0.175          -0.17           -0.165          -0.16           -0.155
-0.15           -0.145          -0.14           -0.135          -0.13
-0.125          -0.12           -0.115          -0.11           -0.105
-0.1            -0.095          -0.09           -0.085          -0.08
-0.075          -0.07           -0.065          -0.06           -0.055
-0.05           -0.045          -0.04           -0.035          -0.03
-0.025          -0.02           -0.015          -0.01           -0.005

Acc : 1.30666
```

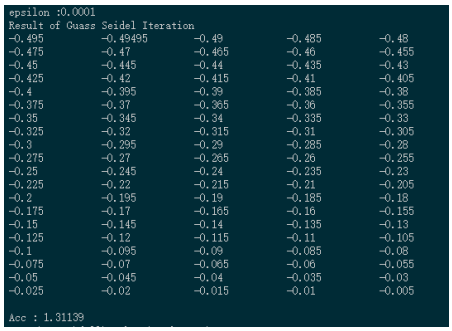
(f)



```
epsilon :0.0001
Result of Guass Elimination With Pivoting
-0.495          -0.49495        -0.49           -0.485          -0.48
-0.475          -0.47           -0.465          -0.46           -0.455
-0.45           -0.445          -0.44           -0.435          -0.43
-0.425          -0.42           -0.415          -0.41           -0.405
-0.4            -0.395          -0.39           -0.385          -0.38
-0.375          -0.37           -0.365          -0.36           -0.355
-0.35           -0.345          -0.34           -0.335          -0.33
-0.325          -0.32           -0.315          -0.31           -0.305
-0.3            -0.295          -0.29           -0.285          -0.28
-0.275          -0.27           -0.265          -0.26           -0.255
-0.25           -0.245          -0.24           -0.235          -0.23
-0.225          -0.22           -0.215          -0.21           -0.205
-0.2            -0.195          -0.19           -0.185          -0.18
-0.175          -0.17           -0.165          -0.16           -0.155
-0.15           -0.145          -0.14           -0.135          -0.13
-0.125          -0.12           -0.115          -0.11           -0.105
-0.1            -0.095          -0.09           -0.085          -0.08
-0.075          -0.07           -0.065          -0.06           -0.055
-0.05           -0.045          -0.04           -0.035          -0.03
-0.025          -0.02           -0.015          -0.01           -0.005

Acc : 1.31139
```

(g)

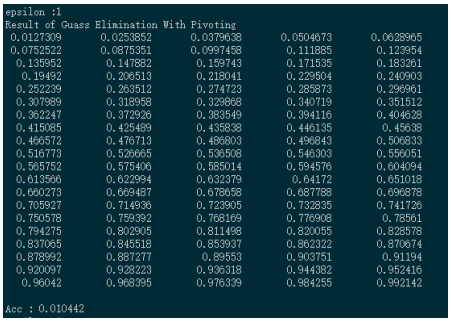


```
epsilon :0.0001
Result of Guass Seidel Iteration
-0.495          -0.49495        -0.49           -0.485          -0.48
-0.475          -0.47           -0.465          -0.46           -0.455
-0.45           -0.445          -0.44           -0.435          -0.43
-0.425          -0.42           -0.415          -0.41           -0.405
-0.4            -0.395          -0.39           -0.385          -0.38
-0.375          -0.37           -0.365          -0.36           -0.355
-0.35           -0.345          -0.34           -0.335          -0.33
-0.325          -0.32           -0.315          -0.31           -0.305
-0.3            -0.295          -0.29           -0.285          -0.28
-0.275          -0.27           -0.265          -0.26           -0.255
-0.25           -0.245          -0.24           -0.235          -0.23
-0.225          -0.22           -0.215          -0.21           -0.205
-0.2            -0.195          -0.19           -0.185          -0.18
-0.175          -0.17           -0.165          -0.16           -0.155
-0.15           -0.145          -0.14           -0.135          -0.13
-0.125          -0.12           -0.115          -0.11           -0.105
-0.1            -0.095          -0.09           -0.085          -0.08
-0.075          -0.07           -0.065          -0.06           -0.055
-0.05           -0.045          -0.04           -0.035          -0.03
-0.025          -0.02           -0.015          -0.01           -0.005

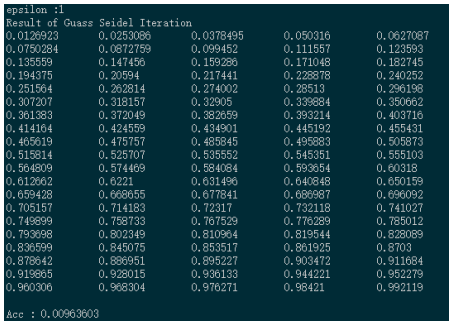
Acc : 1.31139
```

(h)

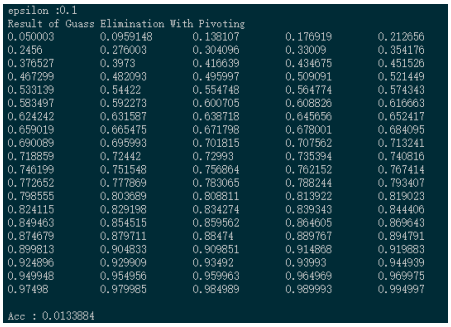
修正后，$\varepsilon = 1, 0.1, 0.01, 0.0001$ 的结果分别为



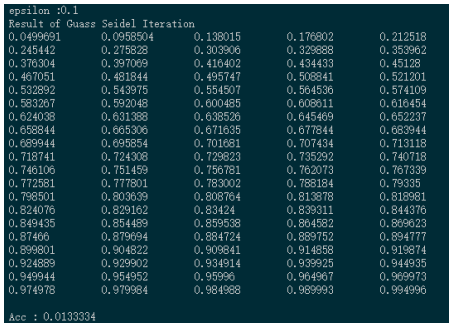
(i)



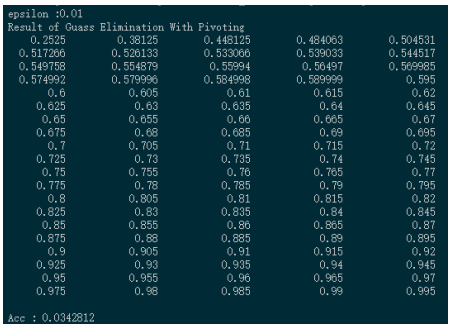
(j)



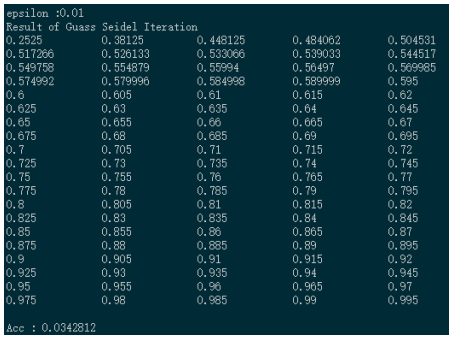
(k)



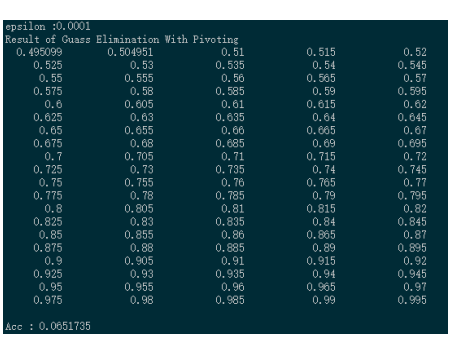
(l)



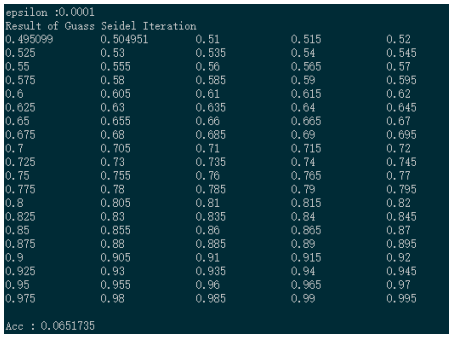
(m)



(n)



(o)



(p)

4 实验分析

在执行解线性方程组的算法之前，首先考虑该方程组是否有已知的初始值，若有则需要做相应的修正，否则将产生较大的误差。