Assignment 1

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1 Problem 1

1.1 MLE

According to the MLE

$$\begin{aligned} L(\theta) &= \Pi_{i=1}^{10} f(x_i|\theta) \\ &= (\frac{2\theta}{3})^2 \cdot (\frac{\theta}{3})^3 \cdot (\frac{2(1-\theta)}{3})^3 \cdot (\frac{1-\theta}{3})^3 \end{aligned} \tag{1}$$

So we can get

$$\begin{aligned} l(\theta) &= logL(\theta) \\ &= 2ln\frac{2\theta}{3} + 3ln\frac{\theta}{3} + 3ln\frac{2(1-\theta)}{3} + 2ln\frac{1-\theta}{3} \end{aligned} \tag{2}$$

Therefore, the score function is,

$$\begin{aligned} U(\theta) &= \frac{dl(\theta)}{d\theta} \\ &= \frac{5}{\theta} - \frac{5}{1-\theta} \end{aligned} \tag{3}$$

We know that,

$$\frac{dU(\theta)}{d\theta} = -\frac{5}{\theta^2} - \frac{5}{(1-\theta)^2} < 0 \tag{4}$$

So we can make $U(\theta) = 0$. Thus maximum likelihood estimator of θ is,

$$\hat{\theta}_{MLE} = \frac{1}{2}$$

1.2 MME

According to the MME

$$\frac{1}{10}\sum_{i=1}^{10} x_i = 1.5 \tag{5}$$

$$\begin{aligned} E(X) &= \sum_{i=0}^{3} iP(X=i) \\ &= \frac{7-6\theta}{3} \end{aligned} \tag{6}$$

We know that,

$$\frac{1}{10}\sum_{i=1}^{10} x_i = E(X) \tag{7}$$

So we can calculate method of moments estimator of θ,

$$\hat{\theta}_{MME} = \frac{5}{12}$$

2 Problem 2

2.1 MLE

According to the MLE

$$
\begin{aligned}
L(\sigma) &= \Pi_{i=1}^{n} f(X_i|\sigma) \\
&= \Pi_{i=1}^{n} \frac{1}{2\sigma} e^{-\frac{|X_i|}{\sigma}} \\
&= \frac{1}{(2\sigma)^n} e^{-\frac{\sum_{i=1}^{n}|X_i|}{\sigma}}
\end{aligned} \tag{8}
$$

So we can get

$$
\begin{aligned}
l(\sigma) &= logL(\sigma) \\
&= -nln2\sigma - \frac{\sum_{i=1}^{n}|X_i|}{\sigma}
\end{aligned} \tag{9}
$$

Therefore, the score function is,

$$
\begin{aligned}
U(\theta) &= \frac{dl(\theta)}{d\theta} \\
&= -\frac{n}{\sigma} + \frac{\sum_{i=1}^{n}|X_i|}{\sigma^2}
\end{aligned} \tag{10}
$$

So we can make $U(\theta) = 0$. Thus maximum likelihood estimator of θ is,

$$
\hat{\sigma}_{MLE} = \frac{1}{n}\sum_{i=1}^{n}|X_i|
$$

2.2 MME

We can calculate that,

$$
\begin{aligned}
E(X) &= \int_{-\infty}^{\infty} \frac{1}{2\sigma} x e^{-\frac{|x|}{\sigma}}\, dx \\
&= 0
\end{aligned} \tag{11}
$$

So we need to use secondary moment,

$$
\begin{aligned}
E(X^2) &= \int_{-\infty}^{\infty} \frac{1}{2\sigma} x^2 e^{-\frac{|x|}{\sigma}}\, dx \\
&= 2\sigma^2
\end{aligned} \tag{12}
$$

We know that,

$$
\frac{1}{n}\sum_{i=1}^{n} X_i^2 = E(X) \tag{13}
$$

So we can calculate method of moments estimator of θ,

$$
\hat{\sigma}_{MME} = \sqrt{\frac{1}{2n}\sum_{i=1}^{n} X_i^2}
$$

3 Problem 3

3.1 MLE

According to the MLE

$$
\begin{aligned}
L(\theta) &= \Pi_{i=1}^{n} f(x_i|\theta) \\
&= (\frac{1}{\theta})^n
\end{aligned} \tag{14}
$$

It is obviously that $L(\theta)$ is a monotonically decreasing function. Therefore, if we want to maximize $L(\theta)$, we need to minimize θ. We know that θ is a parameter from a uniform distribution and X_i, ..., X_n form a random sample from the distribution, so we can get,

$$
\theta \geq max\{X_1, ..., X_n\}
$$

Thus maximum likelihood estimator of θ is,

$$\hat{\theta}_{MLE} = max\{X_1, ..., X_n\}$$

3.2 MME

According to the MME

$$\begin{aligned} E(X) &= \int_0^\theta \frac{x}{\theta}\, dx \\ &= \frac{\theta}{2} \end{aligned} \tag{15}$$

We know that,

$$\frac{1}{n}\sum_{i=1}^{n} X_i = E(X) \tag{16}$$

So we can calculate method of moments estimator of θ,

$$\hat{\theta}_{MME} = \frac{2}{n}\sum_{i=1}^{n} X_i$$

4 Problem 4

4.1 Central Limit Theorem

We know that, X_n can be decomposed into n independent random variables that obey the Bernoulli distribution with the parameter p,

$$X_n = x_1 + ... + x_n, x_i \sim b(1,p)$$

Therefore, our target can be changed as,

$$\frac{X_n}{n} = \frac{1}{n}\sum_{i=1}^{n} x_i = \overline{x}$$

According to the Bernoulli distribution

$$E(x_i) = p$$

$$Var(x_i) = p(1-p)$$

So, we can get,

$$E(\overline{x}) = p$$

$$Var(\overline{x}) = \frac{p(1-p)}{n}$$

According to the Central Limit Theorem,

$$\begin{aligned} \frac{\overline{x} - p}{\sqrt{\frac{p(1-p)}{n}}} &\sim N(0,1) \\ \overline{x} &\sim N(p, \frac{p(1-p)}{n}) \end{aligned} \tag{17}$$

Therefore,

$$\frac{X_n}{n} \sim N(p, \frac{p(1-p)}{n})$$

4.2 Delta Method

According to the Delta Method,

$$\mu = p$$

$$\begin{aligned} g(\mu) &= g(p) \\ &= p(1-p) \end{aligned} \tag{18}$$

$$\begin{aligned} g(\overline{x}) &= g(\frac{X_n}{n}) \\ &= \delta_n \end{aligned} \tag{19}$$

We know that $p \neq \frac{1}{2}$, so,

$$g^{'}(p) = 1 - 2p \neq 0$$

Therefore, according to the usual first order Delta method states,

$$\delta_n \sim N(p(1-p), \frac{p(1-p)(1-2p)^2}{n})$$

5 Problem 5

According to the Bernoulli distribution

$$E(x_i) = \mu$$

$$Var(x_i) = \sigma^2$$

So, we can get,

$$E(\overline{X}_n) = \mu$$

$$Var(\overline{X}_n) = \frac{\sigma^2}{n}$$

According to the Central Limit Theorem,

$$\sqrt{n}(\overline{X}_n - \mu) \sim N(0, \sigma^2) \tag{20}$$

According to the Delta Method,

$$g(\mu) = \mu^2$$

$$g(\overline{X}_n) = \overline{X}_n^2$$

$$g^{'}(\mu) = 2\mu$$

if $\mu \neq 0$, according to the usual first order Delta method states,

$$\overline{X}_n^2 \sim N(\mu^2, \frac{4\sigma^2\mu^2}{n})$$

if $\mu = 0$, the second derivative,

$$g^{''} = 2 \neq 0$$

Therefore, According to the usual second order Delta method states,

$$n(\overline{X}_n^2 - \mu^2) \sim \sigma^2\chi_1^2$$

$$\overline{X}_n^2 \sim \frac{\sigma^2}{n}\chi_1^2 + \mu^2$$

6 Problem 6

According to the problem 4,

$$\frac{\hat{p}-p}{\sqrt{\frac{p(1-p)}{n}}} \sim N(0,1)$$

Using the asymptotic normal approximation,

$$\frac{\hat{p}-p}{\sqrt{\frac{\hat{p}(1-\hat{p})}{n}}} \sim N(0,1)$$

Therefore, 100(1-α)% confidence interval for p,

$$[\hat{p}-z_{1-\frac{\alpha}{2}}\sqrt{\frac{\hat{p}(1-\hat{p})}{n}}, \hat{p}+z_{1-\frac{\alpha}{2}}\sqrt{\frac{\hat{p}(1-\hat{p})}{n}}]$$

We know that, $\hat{p}=0.2$, $\alpha=0.05$, so the 95% confidence interval for p,

$$[0.103, 0.297]$$