Assignment 1

Yun Siyu, 17307110448

2020 年 4 月 30 日

1 Problem 1

1.1 a

Exponential prior with mean 1,which means $\lambda = 1$, so the prior is,

$$p(\lambda) = \begin{cases} e^{-\lambda} & x > 0 \\ 0 & o.w \end{cases}$$

Likelihood,

$$\begin{aligned} L(\lambda|X) &= \Pi_{i=1}^{n} f(x_i|\lambda) \\ &= \Pi_{i=1}^{n} \frac{\lambda^{x_i} e^{-\lambda}}{x_i!} \end{aligned} \tag{1}$$

Posterior distribution,

$$\begin{aligned} p(\lambda|X) &\propto p(\lambda)L(\lambda|X) \\ &\propto \lambda^{\sum_{i=1}^{n} x_i} e^{-(n+1)\lambda} \\ &\propto Gamma(\sum_{i=1}^{n} x_i + 1, n+1) \end{aligned} \tag{2}$$

1.2 b

Using my expert knowledge, the accident rate around this busy intersection may have the prior,

$$\begin{aligned} p(\lambda) &\sim Gamma(2, 1) \\ &\propto \lambda e^{-\lambda} \end{aligned} \tag{3}$$

Likelihood,

$$\begin{aligned} L(\lambda|X) &= \Pi_{i=1}^{n} f(x_i|\lambda) \\ &= \Pi_{i=1}^{n} \frac{\lambda^{x_i} e^{-\lambda}}{x_i!} \end{aligned} \tag{4}$$

Posterior distribution,

$$\begin{aligned} p(\lambda|X) &\propto p(\lambda)L(\lambda|X) \\ &\propto \lambda^{\sum_{i=1}^{n} x_i + 1} e^{-(n+1)\lambda} \\ &\propto Gamma(\sum_{i=1}^{n} x_i + 2, n+1) \end{aligned} \tag{5}$$

1.3 c

According to the standard exponential prior, the posterior distribution is,

$$\begin{aligned} p(\lambda|X) &\propto Gamma(\sum_{i=1}^{n} x_i + 1, n+1) \\ &\propto Gamma(34, 16) \end{aligned} \tag{6}$$

Therefore the posterior median for λ is,

$$\begin{aligned} E(\lambda|X) &= \frac{\alpha}{\beta} \\ &= \frac{17}{8} \end{aligned} \tag{7}$$

a 95% credible interval for λ is,

$$[1.471624, 2.896517]$$

1.4 d

According to the standard exponential prior, the posterior distribution is,

$$\begin{aligned} p(\lambda|X) &\propto Gamma(\sum_{i=1}^{n} x_i + 1, n+1) \\ &\propto Gamma(35, 16) \end{aligned} \tag{8}$$

Therefore the posterior median for λ is,

$$\begin{aligned} E(\lambda|X) &= \frac{\alpha}{\beta} \\ &= \frac{35}{16} \end{aligned} \tag{9}$$

a 95% credible interval for λ is,

$$[1.523674, 2.969475]$$

2 Problem 2

2.1 a

$s_0 = 1$ and $\delta = 200$, the prior,

$$p(\mu|\sigma^2) \propto e^{-\frac{(\mu-220)^2}{2\sigma^2}} \tag{10}$$

$$p(\sigma^2) \propto (\sigma^2)^{-1101} e^{\frac{-250000}{\sigma^2}} \tag{11}$$

The likelihood,

$$\begin{aligned} L(\mu, \sigma|\boldsymbol{x}) &= \Pi_{i=1}^{n} f(x_i|\mu, \sigma) \\ &= \Pi_{i=1}^{n} \frac{1}{\sqrt{2\pi\sigma^2}} e^{\frac{-(x_i-\mu)^2}{2\sigma^2}} \end{aligned} \tag{12}$$

Therefore, the posterior for μ,

$$\begin{aligned} p(\mu|\boldsymbol{x}, \sigma) &\propto L(\mu, \sigma|\boldsymbol{x}) p(\mu|\sigma) p(\sigma) \\ &\propto L(\mu, \sigma|\boldsymbol{x}) p(\mu|\sigma) \\ &\propto e^{-\frac{\sum_{i=1}^{12}(x_i-\mu)^2 + (\mu-220)^2}{2\sigma^2}} \\ &\propto e^{-\frac{(\mu - \frac{\sum_{i=1}^{12} x_i + 220}{13})^2}{2\sigma^2/13}} \\ &\propto N(\frac{\sum_{i=1}^{12} x_i + 220}{13}, \sigma^2/13) \end{aligned} \tag{13}$$

The posterior for σ,

$$\begin{aligned}
p(\sigma|\boldsymbol{x},\mu) &\propto L(\mu,\sigma|\boldsymbol{x})p(\mu|\sigma)p(\sigma) \\
&\propto \Pi_{i=1}^{12}\frac{1}{\sqrt{2\pi\sigma^2}}e^{\frac{-(x_i-\mu)^2}{2\sigma^2}} * (\sigma^2)^{-\frac{1}{2}} * e^{-\frac{(\mu-220)^2}{2\sigma^2}} * (\sigma^2)^{-1101}e^{\frac{-250000}{\sigma^2}} \\
&\propto (\sigma^2)^{-1106.5-1}e^{-\frac{\sum_{i=1}^{12}(x_i-\mu)^2/2+(\mu-220)^2/2+250000}{\sigma^2}} \\
&\sim IG(1106.5, \sum_{i=1}^{12}(x_i-\mu)^2/2+(\mu-220)^2/2+250000)
\end{aligned} \tag{14}$$

Therefore, according to the gibbs sampler, a point estimate is,

$$\hat{\mu} = 225.2091$$

And 95% credible interval for the single unknown parameter μ is,

$$[217.0210, 233.4166]$$

2.2 b

The prior,

$$p(\mu) \propto 1 \tag{15}$$

$$p(\sigma) \propto \frac{1}{\sigma} \tag{16}$$

Make $Y = \sigma^2$, then,

$$\begin{aligned}
F(Y) &= P(Y \leq y) \\
&= P(\sigma^2 \leq y) \\
&= P(\sigma \leq \sqrt{y}) \\
&\propto ln(y)
\end{aligned} \tag{17}$$

Therefore,

$$p(\sigma^2) \propto \frac{1}{\sigma^2} \tag{18}$$

The likelihood,

$$\begin{aligned}
L(\mu,\sigma|\boldsymbol{x}) &= \Pi_{i=1}^{n} f(x_i|\mu,\sigma) \\
&= \Pi_{i=1}^{n}\frac{1}{\sqrt{2\pi\sigma^2}}e^{\frac{-(x_i-\mu)^2}{2\sigma^2}}
\end{aligned} \tag{19}$$

The posterior distribution for μ,

$$\begin{aligned}
p(\mu|\boldsymbol{x},\sigma) &\propto L(\mu,\sigma|\boldsymbol{x})p(\mu) \\
&\propto \Pi_{i=1}^{n} f(x_i|\mu,\sigma) \\
&\propto \Pi_{i=1}^{n} e^{\frac{-(x_i-\mu)^2}{2\sigma^2}} \\
&\propto e^{\frac{-(\mu-\frac{1}{12}\sum_{i=1}^{12}x_i)^2}{2\sigma^2/12}} \\
&\sim N(\frac{1}{12}\sum_{i=1}^{12}x_i, \frac{\sigma^2}{12})
\end{aligned} \tag{20}$$

The posterior distribution for σ^2,

$$\begin{aligned}
p(\sigma|\boldsymbol{x},\mu) &\propto L(\mu,\sigma|\boldsymbol{x})p(\sigma) \\
&\propto \Pi_{i=1}^{n} f(x_i|\mu,\sigma)p(\sigma) \\
&\propto (\sigma^2)^{-6-1}e^{\frac{-\sum_{i=1}^{12}(x_i-\mu)^2/2}{\sigma^2}} \\
&\sim IG(6, \frac{\sum_{i=1}^{12}(x_i-\mu)^2}{2})
\end{aligned} \tag{21}$$

Therefore, according to the gibbs sampler, a point estimate is,

$$\hat{\mu} = 225.7792$$

And 95% credible interval for the single unknown parameter μ is,

$$[213.0257, 238.7319]$$

2.3 c

For (a), a point estimate is,

$$\hat{\mu} = 225.2091$$

And 95% credible interval for the single unknown parameter μ is,

$$[217.0210, 233.4166]$$

For (b), a point estimate is,

$$\hat{\mu} = 225.7792$$

And 95% credible interval for the single unknown parameter μ is,

$$[213.0257, 238.7319]$$

We can see that the point estimates are almost equal, so the different choices of priors will not have much impact on the results of point estimates. However, the lengths of the 95% credible interval in (a) and (b) are different. The reason may be that the information of μ learned in (a) is more than that in (b), so the interval length is narrower.

2.4 d

From the wikipedia, we know that the conjugate prior for μ is normal distribution, so we can get the prior,

$$p(\mu) = \frac{1}{\sqrt{50\pi}} e^{-\frac{(\mu-220)^2}{50}}$$

The likelihood,

$$\begin{aligned} L(\mu|\boldsymbol{x}) &= \Pi_{i=1}^{n} f(x_i|\mu) \\ &= \Pi_{i=1}^{n} \frac{1}{\sqrt{456\pi}} e^{\frac{-(x_i-\mu)^2}{456}} \end{aligned} \tag{22}$$

Therefore, the posterior,

$$\begin{aligned} p(\mu|x) &\propto e^{-\frac{\sum_{i=1}^{12}(x_i-\mu)^2}{456} - \frac{(u-220)^2}{50}} \\ &\propto N(223.22, 10.8) \end{aligned} \tag{23}$$

Therefore, a point estimate is,

$$\hat{\mu} = 223.22$$

-4

And 95% credible interval for the single unknown parameter μ is,

$$[202.0524, 244.3876]$$

qnorm第二个参数是标准差不是方差

2.5 e

It can be seen that the lengths of point estimate and 95% credible interval in (d) are different from those in (a) and (b). The possible reason is that the prior distribution of μ in (d) follows the normal distribution and is different from (a) and (b), and (d) gives the value of σ^2 but not in (a) and (b).

3 Problem 3

3.1 a

the likelihood,

$$\begin{aligned} L(\theta|\boldsymbol{x}) &= \Pi_{i=1}^{n} f(x_i|\theta) \\ &= \Pi_{i=1}^{n} \frac{1}{\sqrt{2\pi\theta}} e^{\frac{-x_i^2}{2\theta}} \end{aligned} \tag{24}$$

3.2 b

Based on a single observation x,

$$p(x|\theta) = \frac{1}{\sqrt{2\pi\theta}} e^{\frac{-x^2}{2\theta}} \tag{25}$$

Then,

$$logp(x|\theta) = -\frac{x_i^2}{2\theta} - \frac{1}{2} ln2\pi - \frac{1}{2} ln\theta$$

$$\begin{aligned} I(\theta) &= -E(\frac{dlog^2 p(x|\theta)}{d\theta^2}) \\ &= -E(\frac{1}{2\theta^2} - \frac{x_i^2}{\theta^3}) \\ &= -\frac{1}{2\theta^2} + E(\frac{x_i^2}{\theta^3}) \\ &= -\frac{1}{2\theta^2} + \frac{E(x_i^2)}{\theta^3} \\ &= -\frac{1}{2\theta^2} + \frac{0+\theta}{\theta^3} \\ &= -\frac{1}{2\theta^2} + \frac{1}{\theta^2} \\ &= \frac{1}{2\theta^2} \end{aligned} \tag{26}$$

Jeffrey's prior

$$\begin{aligned} p(\theta) &\propto I(\theta)^{\frac{1}{2}} \\ &\propto \frac{\sqrt{2}}{2\theta} \end{aligned} \tag{27}$$

3.3 c

Jeffrey's prior

$$p(\theta) \propto \frac{\sqrt{2}}{2\theta} \tag{28}$$

the likelihood,

$$\begin{aligned} L(\theta|\boldsymbol{x}) &= \Pi_{i=1}^{6} f(x_i|\theta) \\ &= \Pi_{i=1}^{6} \frac{1}{\sqrt{2\pi\theta}} e^{\frac{-x_i^2}{2\theta}} \end{aligned} \tag{29}$$

Posterior distribution,

$$\begin{aligned} p(\theta|X) &\propto p(\theta) L(\theta|X) \\ &\propto \theta^{-4} e^{\sum_{i=1}^{6} \frac{-x_i^2}{2\theta}} \\ &\propto \theta^{-4} e^{-\frac{8.84995}{\theta}} \\ &\sim IG(3, 8.84995) \end{aligned} \tag{30}$$

Therefore, the posterior distribution is Inverse Gamma Dist with parameter,

$$\alpha = 3, \quad \beta = 8.84995$$

4 Problem 4

4.1 a

For the normal prior for θ, the parameters of the prior are,

$$\mu = 400, \quad \sigma = 87.86242$$

The medical expert's best guess of the most likely mean survival time is 400 days, so μ is most likely 400. The expert also believes there is a $\frac{2}{3}$ chance that the mean survival time is between 315 and 485 days. Therefore,

$$\begin{aligned} \phi(\frac{\theta - \mu}{\sigma}) &= \phi(\frac{85}{\sigma}) \\ &= \frac{5}{6} \end{aligned} \tag{31}$$

Therefore, according to the R code, $\sigma = 87.86242$

4.2 b

For the gamma prior for θ, the parameters of the prior are,

$$\alpha = 400, \quad \beta = 87.86242$$

The medical expert's best guess of the most likely mean survival time is 400 days, so,

$$\begin{aligned} E(\theta) &= \frac{\alpha}{\beta} \\ &= 400 \end{aligned} \tag{32}$$

The expert also believes there is a $\frac{2}{3}$ chance that the mean survival time is between 315 and 485 days. And we know that $Var(\theta) = \frac{\alpha}{\beta^2}$ Therefore,

$$P(315 \leq \theta \leq 485) = \frac{2}{3} \tag{33}$$

According to the R code,

$$\alpha = 20.42$$

$$\beta = 0.05105$$