Decision Theory*

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1 Fundamentals of Decision Theory

A few years back, Greg Wolfson, one of the author's former students, and his wife were in the Caribbean as Hurricane Andrew approached. Understandably, they were nervous about the possibility of being stuck on the Turks and Caicos Islands during one of the worst storms of the century. Should they stay on the Islands or should they try to make it to Miami on route back home? If they stayed and the hurricane hit the Islands, then they faced having their vacation ruined or worse. If they left, then they gave up the rest of their vacation, incurred additional costs of getting last-minute plane tickets, and ran the risk of being caught in the hurricane while in Miami. Fortunately, Greg had studied *decision theory.* Decision theory helped Greg and his wife to think *systematically* through their decision problem—stay or flee—and reach their best decision. This turned out to be "stay," and a good thing too: While Miami was being battered by Hurricane Andrew, Greg and his wife were on a Hobie Cat, sailing the lovely turquoise waters off the Turks and Caicos Islands—another economics success story!

Decision theory is a set of tools for deciding "which." For example, Greg and his wife were deciding *which* would be the better course of action for them, stay or flee. These tools help by *formalizing* your decision making. They help you recognize the alternatives available to you; they help you see what additional information would be useful in reaching a decision; and they make you aware of the assumptions or conditions that are critical to the decision you make.

A word of caution: As powerful as these tools are, they are *not* a substitute for your own thinking. Rather, they are aids to your thinking. Put another way, they are not magic formulae that can make your decisions for you (which is just as well, since otherwise someone would program a computer with them, which would likely do you out of a job).

2 Some Basic Issues

When you make a decision, you are choosing among *alternatives*. Your objective is to choose the alternative that is *best*, where "best" depends on what your goals are. Indeed, the first rule of decision making is to know what your goals are. For example, if your decision problem is which movie to see at the multiplex, then "best" means "most entertaining" (assuming being entertained is your goal).

The first rule of decision making: *know what your goals (objectives) are.*

Although the first rule of decision making may strike you as obvious, you would be surprised how often people start making decisions without thinking through what their goals are. For instance, obeying the first rule can often be a problem when a committee makes a decision, since the committee members can have different goals. Sometimes the committee members recognize their differences in advance, but sometimes they are unspoken. Occasionally committee members believe they are in agreement with respect to their goals when, in fact, they are not (you've likely had conversations that began "it just didn't occur to me that you wanted ..."). Psychology also plays a role here. You may not, for example, want to admit to yourself what your true goals are—perhaps because they are socially unacceptable—so you convince yourself that your goals are something else. Unfortunately, it is beyond this book to make sure that you obey the first rule. All it can do, as it has just done, is point out that obeying the first rule is not as easy as it may at first seem.

Having identified your goals, you next have to identify your alternatives. For some decision-making problems, your alternatives are obvious. For instance, if you are deciding which movie at the multiplex to see, then your alternatives are the movies playing plus, possibly, not seeing any movie at all. For other problems, however, identifying your alternatives is more difficult. For instance, if you are deciding which personal computer to buy, then it can be quite difficult to identify *all* your alternatives (e.g., you may not know all the companies that make computers or all the optional configurations available). Fortunately, there are ways to overcome, at least partially, such difficulties, as we will see later. We will even study the *decision* of whether you should expend resources expanding your list of alternatives later in this section.

Your choice of alternative will lead to some consequence. Depending on the decision-making problem you face, the consequence of choosing a given alternative will be either known or uncertain. If you are driving in your neighborhood, then you know where you will end up if you turn left at a given intersection. If you are investing in the stock market, then you are uncertain about what returns you will earn. Typically, we will suppose that even if you are uncertain about which particular consequence will occur, you know the set of possible consequences. For instance, although you don't know what your stock price will be a year from now, you do know that it will be some non-negative number. Moreover, you likely know something about which stock prices are more or less likely. For example, you may believe that it is more likely that your stock's price will change by 20% or less than it is to change by 21% or more.

Sometimes, however, you may not know what all the possible consequences are. That is, some possible consequences could be *unforeseen.* To give an example, a vineyard owner was proud of his "green" farming techniques. Unlike many of his fellow vintners, he used pesticides that killed only the "bad" bugs, leaving the "good" bugs—those that ate the bad bugs—alive. A consequence of this, which was unforeseen by the vintner, was that if he successfully killed the bad bugs, then the good bugs would be left with nothing to eat and would starve.

By their very nature, unforeseen consequences are difficult to identify prior to making your decisions. And for the same reason, it is difficult to predict which consequences will be unforeseen by others. As a practical manner, one way to *help* identify unforeseen consequences in your own decision making is to think about what your "un-goals" are; that is, the consequences you would like *not* to happen. For instance, an un-goal of the vintner was to kill the good bugs. Another way to identify unforeseen consequences is to reframe your way of thinking about your goals. For instance, instead of thinking about not killing the good bugs, think instead of helping the good bugs to survive. Reframed in this way, the adverse consequence of killing the good bugs' food supply might be more apparent.

3 Decision Making Under Certainty

To represent, in a schematic way, a decision problem, we draw a *decision tree.* An example of such a tree is shown in Figure 1. It represents the following problem: A firm is considering producing a new product. Prior to producing, the firm can conduct a marketing campaign (e.g., advertise heav-

Decision tree: *A graphical representation of a decision problem as a series of branching alternatives.*

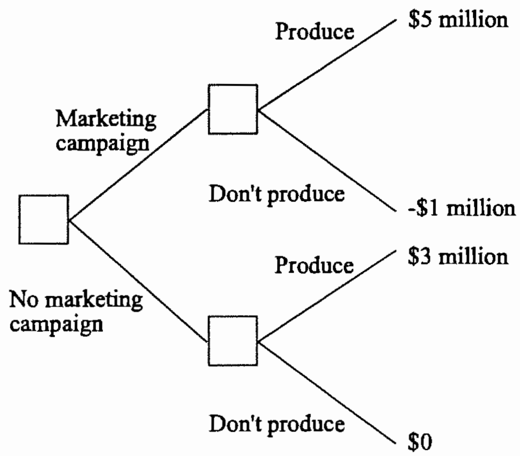

Figure 1: A decision tree for a firm that first must decide whether to have a *marketing campaign* for a new product or to have *no marketing campaign* and, then, must decide whether to *produce* or not to produce the product (i.e., *don't produce*). The squares are decision nodes. From them stem branches. At the end of the tree are the payoffs.

ily) or not. Suppose that a marketing campaign costs \$1 million. Suppose that if the product is marketed and produced, it will generate revenues of \$8 million while costing \$2 million to produce; so profit will be \$5 million ($= \$(8-2-1)$ million). If the product is marketed, but not produced, it will generate no revenue and no production cost; so profit will be -\$1 million (i.e., just the cost of the marketing campaign). If the product is not marketed, but produced, it will generate a revenue of \$4 million but cost \$1 million to produce; so profit will be \$3 million. Finally, if the product is neither marketed, nor produced, then profit will be \$0. Each square in Figure 1 is a *decision node.* A decision node indicates that the decision maker (in this case the firm) has a decision to make. The alternatives available to him at that decision node are represented by the *branches* that stem from the right-side of the decision node. For instance, the alternatives available to the firm at the left-most—first—decision node are "marketing campaign" and "no marketing campaign." Note that the tree is read from left to right; moreover, going from left to right is meant to represent the sequence of decisions. For example, the firm must first decide whether to have a marketing campaign before it decides whether to produce the product. At the end of the tree are the consequences or *payoffs* from the sequence of decisions. For

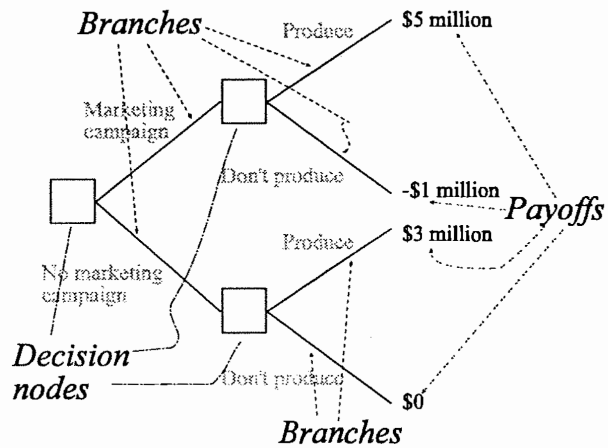

Figure 2: The components of a decision tree labeled.

instance, if the firm chose "no marketing campaign" and, then, "produce," it would make a profit of \$3 million. Payoffs are expressed in terms relevant to the decision maker's goals. Here, the goal is to make money, so they are represented in monetary terms. Figure 2 labels the various parts of decision tree.

Having represented a decision problem by a tree, the next step is to solve it. Solving a tree means determining which decisions will best accomplish the decision maker's goals. If, as in Figure 1, the goal is to make money, then this means determining the decisions that will lead to the most money. Trees are solved by working *backwards*: Start at the *right-most* decision nodes and select the branches that give the decision maker the largest payoffs. For the tree in Figure 1, this means choosing "produce" at the top right-most decision node—since a \$5 million gain is better than a \$1 million loss—and choosing "produce" at the bottom right-most decision node—since a \$3 million gain is better than \$0. Next move *left* to the preceding decision nodes. Again, select the branches that give the decision maker the largest payoffs *taking into account, if necessary, the future decisions that will be made.* In Figure 1, this means choosing "marketing campaign" at the first node because "market" ultimately leads to a payoff of \$5 million, while "no marketing campaign" ultimately leads to a payoff of \$3 million. Were there any decision nodes to the left of the "marketing campaign/no marketing campaign" node (i.e., were there decisions to be taken prior to the decision of whether to market), then you would choose the alternatives at those

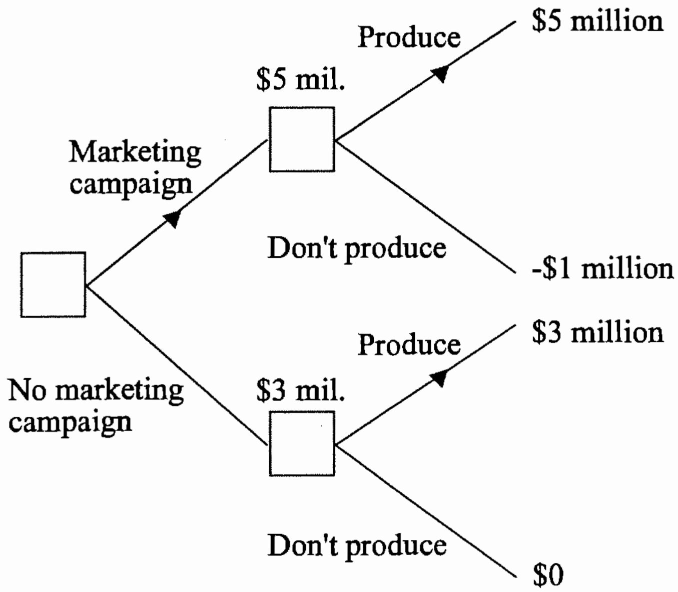

Figure 3: The correct branches to follow are noted with arrows (▲). The values of the intermediate decision nodes are also noted.

nodes taking into account your decision to have a marketing campaign at the "marketing campaign/no marketing campaign" node.

The tree in Figure 1 is straightforward, so solving it is fairly straightforward as well. For more complicated trees, however, it is important to keep track of where you are as you work backwards. Two devices for keeping track are *arrowing* the correct decision and *valuing* the intermediate decision nodes (i.e., the decision nodes other than the first). Arrowing means putting a little arrow (or other mark) on the correct decision. When you're done arrowing, the arrows will "guide" you through the tree. For instance, in Figure 1, you would put an arrow on the "marketing campaign" branch, because that is the correct alternative to choose. Valuing a decision node means writing the payoff from making the correct decision at that decision node. Figure 3 shows Figure 1 with arrows and values.

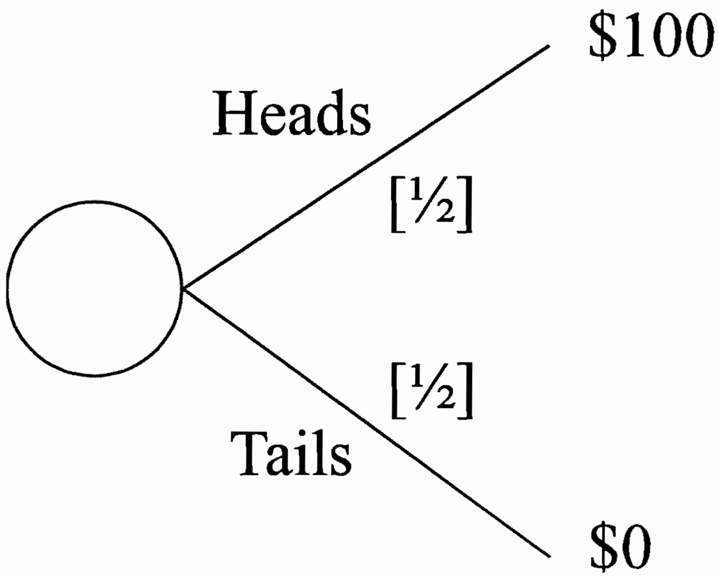

Figure 4: A depiction of a random event in which the decision maker receives \$100 if a coin lands heads up and \$0 if it lands tails up. The circle node is a *chance* node. The numbers in brackets (e.g., $[\frac{1}{2}]$) are the probabilities associated with the possible outcomes.

4 Decision Making Under Uncertainty (Part I)

Many decisions are made in situations of uncertainty. To represent such decision problems, we use a second kind of node: a *chance* node. A chance node, drawn as a circle, indicates that what follows is uncertain. Each branch stemming from the chance node shows a possible outcome of the random process that the chance node represents. For instance, Figure 4 represents the "decision" tree associated with flipping a fair coin. The possible outcomes, heads and tails, are indicated. In addition, the probabilities of the two outcomes are shown in brackets next to the branches. Note that which outcome occurs after a chance node is determined by the random process in question and *not* by the decision maker. For instance, "nature" or "chance," not the decision maker, decides whether the coin lands heads up or tails up. At the end of the tree are the payoffs: \$100 to the decision maker if the coin lands heads up and \$0 to him if the coin lands tails up.

Recall that the *expected value* of a gamble, denoted EV, is given by the formula

$$EV = p_1 \cdot x_1 + \ldots + p_N \cdot x_N,$$

where p_n is the probability of the nth outcome and x_n is the payoff should the nth outcome occur. For example, the expected value of the gamble shown in Figure 4 is

$$\$50 = \frac{1}{2} \cdot \$100 + \frac{1}{2} \cdot \$0.$$

It might strike you as odd that \$50 is called the "expected" value of the gamble in Figure 4: How could \$50 be "expected" if the only two possible values are \$0 and \$100? There are two justifications for the term "expected." First, suppose that we repeated the gamble many times, say 1000 times. Your average winnings—that is, the total of your winnings for the 1000 repetitions divided by 1000—would very likely be very close to \$50.[1] In other words, we would *expect* your average winnings to be \$50. As a second justification, suppose that you ran a life insurance company. Then, for each insured, you are essentially gambling on when he or she will die (i.e., x would be age at death). The expected value would, thus, refer to the expected age of death of an insured. If you insured a large enough population, then the average age at death among your insureds would be close to the expected age of death of a single insured.

Expected value is useful for decision theory because many decision makers are *expected-value maximizers*:

Definition 1 *A decision maker is an* expected-value maximizer *if he chooses among his alternatives the one that yields the greatest expected value.*

To better understand an expected-value maximizer's behavior, consider the decision problem shown in Figure 5: If a firm launches a new product, then there are three possible outcomes:

1. Highly successful, which means profit of \$7 million, and which occurs with a 55% probability;

2. Moderately successful, which means profit of \$3 million, and which occurs with a 40% probability; and

[1] For instance, there is a 94% probability that your average winnings would fall between \$47 and \$53 and a 99% probability that your average winnings would fall between \$46 and \$54. If you have had an advanced course in probability, you may recognize that this is nothing more than the law of large numbers.

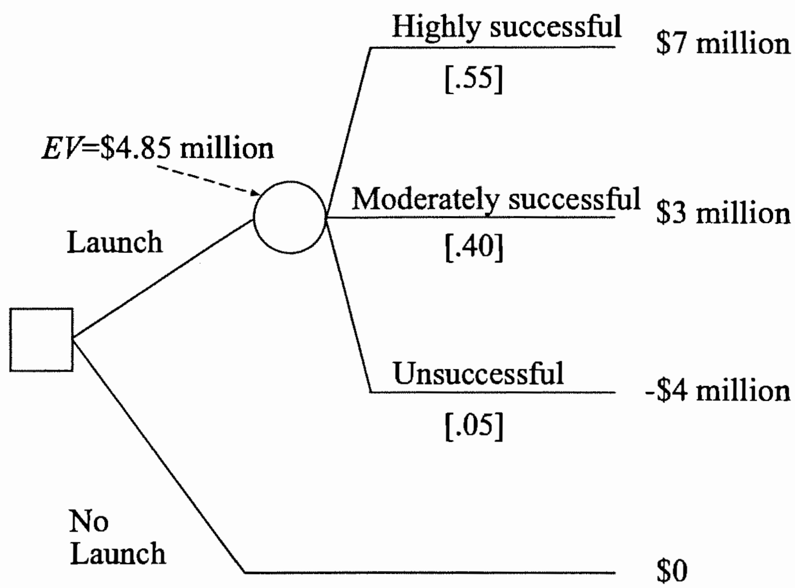

Figure 5: A firm faces the decision of launching a new product or not launching it. There are three possible outcomes if it launches: highly successful, moderately successful, and unsuccessful. The payoff from each is shown at the end of the tree and the probability of each is indicated in brackets. Since the *expected value* of launching exceeds the value of not launching, the firm would choose to launch if it is an expected-value maximizer.

3. Unsuccessful, which means a loss of \$4 million, and which occurs with a 5% probability.

If the firm chooses not to launch, then it earns nothing. The expected value of launching is

$$\$4.85 \text{ million} = .55 \times \$7 \text{ million} + .4 \times \$3 \text{ million} + .05 \times (-\$4 \text{ million})\,.$$

Since \$4.85 million is greater than \$0, the firm's payoff if it doesn't launch, the firm would choose to launch if it is an expected-value maximizer.

Note that we solve the tree in Figure 5 in the same way we solved the others we encountered—in fact, the way we solve all trees—by working backwards: We started at the right-most node, in this case a chance node, calculated the value of that node (i.e., its expected value), and then moved left to the preceding node (i.e., launch/no launch).

As a last example, suppose that prior to making the launch/no launch decision, the firm could commission a survey that would tell it how successful the product will be. Of course, whether to conduct the survey is itself a decision, so it must be added to the tree. Figure 6 revises the tree in Figure 5 to reflect these changes.

Assume the firm is an expected-value maximizer. As always, we solve the tree by working backwards. Since the bottom of the tree is the same as Figure 5, we know from our analysis of that tree that the firm would choose to launch. Its expected payoff is \$4.85 million. At the top of the tree, the right-most nodes are all decision nodes. Note that they *follow* the chance node because if the firm does a survey it will know how successful its new product will be *at the time* it decides whether to launch. At the top two decision nodes, the firm would launch—positive amounts of money beat nothing, while at the bottom of the three decision nodes it would not launch—\$0 beats suffering a loss. Note that the values of these three decision nodes have been appropriately labeled. When the firm decides to take a survey, it doesn't know what it will learn, so what it will learn is uncertain. This is reflected by the chance node that precedes the top right-most decision nodes. The expected value at that node is

$$\$5.05 \text{ million } = .55 \times \$7 \text{ million} + .4 \times \$3 \text{ million } + .05 \times \$0.$$

Comparing \$5.05 million to \$4.85 million, it follows that the firm would prefer to undertake the survey than make a decision without it.

This last example also illustrates how we can calculate the value of this survey: The value of the survey is the difference in the expected payoff with

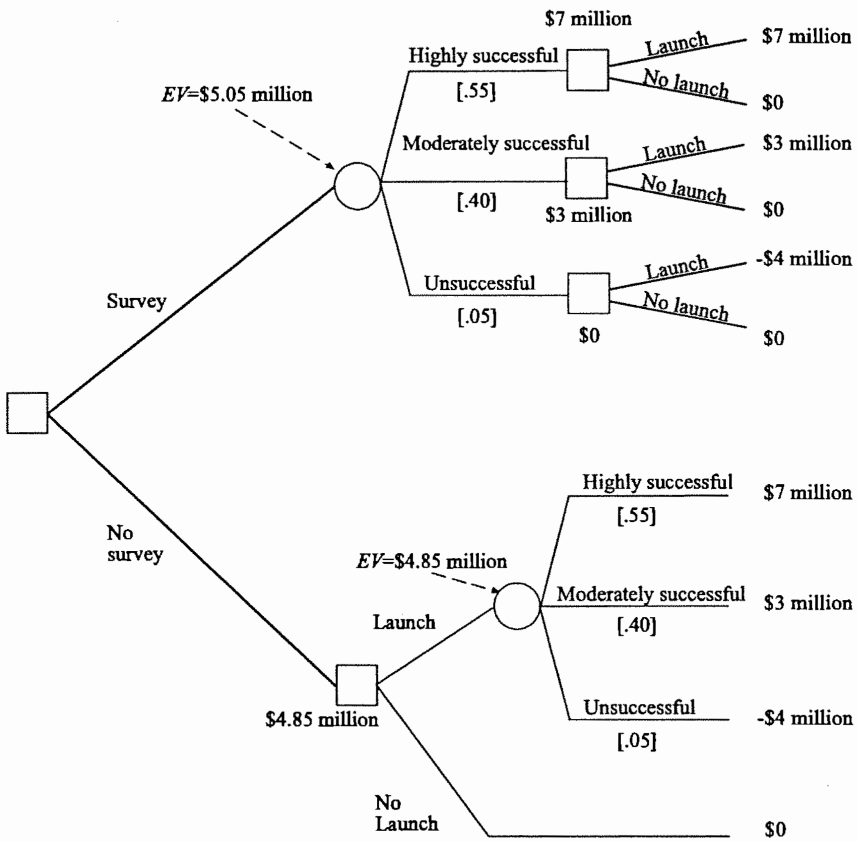

Figure 6: Now, the firm can conduct a survey, if it wishes, prior to making its launch decision.

the survey, \$5.05 million, and the expected payoff without the survey, \$4.85 million; that is, the survey is worth \$200,000 (\$0.2 million). The firm would pay up to \$200,000 to have this survey conducted. We will return to the value of information later in Section 8.

To summarize:

Solving trees for expected-value maximizers:

1. For each of the right-most nodes proceed as follows:

    (a) If the node is a decision node, determine the best alternative to take. The payoff from this alternative is the *value of this decision node.* Arrow the best alternative.

    (b) If the node is a chance node, calculate the expected value. This expected value is the *value of this chance node.*

2. For the nodes one to the left proceed as follows:

    (a) If the node is a decision node, determine the best alternative to take, using, as needed, the values of future nodes (nodes to the right) as payoffs. The payoff from this alternative is the value of this decision node. Arrow the best alternative.

    (b) If the node is a chance node, calculated the expected value, using, as needed, the values of future nodes (nodes to the right) as payoffs. The expected value is the value of this chance node.

3. Repeat Step 2 as needed, until the left-most node is reached. Following the arrows from *left* to *right* gives the sequence of appropriate decisions to take.

5 Decision Making Under Uncertainty (Part II)

Consider the following situation. You own a "lottery" ticket with the following properties. Tomorrow, a fair coin will be flipped. If it lands heads up, you will receive \$1 million. If it lands tails up, you will receive \$0. The ticket is transferable; that is, you can give or sell it to another person, in which case this other person is entitled to the winnings from the lottery ticket (if any). Between today and tomorrow, people may approach you about buying your lottery ticket. What is the *smallest* price that you would accept in exchange for your ticket?

A possible answer is \$500,000, since that it is the expected value of this lottery ticket.[2] Many people, however, would be willing to accept less than \$500,000. You, for instance, might be willing to sell the ticket for as little as \$400,000 under the view that a certain \$400,000 was worth the same as a half chance at \$1 million. Moreover, even if you are unwilling to sell your ticket for \$400,000, many people would be.

Selling your ticket for less than \$500,000 is, however, *in*consistent with being an expected-value maximizer, since you wouldn't be choosing the alternative that yielded you the greatest expected value. So if you would, in fact, accept \$400,000, then you are not an expected-value maximizer. Moreover, even if *you* are (i.e., *you* would not sell the ticket for less than \$500,000), others clearly aren't. Consequently, we need some way to model decision makers who are not expected-value maximizers.

One reason that someone is not an expected-value maximizer is that he is concerned with the riskiness of the gambles he faces. For instance, a difference between an *expected* value of \$500,000 and a *certain* value of \$400,000 is that there is considerable risk with the former—although you might win \$1 million, you might also end up with nothing—but no risk with the latter. Most people don't like risk, and are willing, in fact, to pay to avoid it. Such people are called *risk averse.* By accepting less than \$500,000 for the lottery ticket, you are effectively paying to avoid risk; that is, you are behaving in a risk-averse fashion. When you buy insurance, thereby reducing or eliminating your risk, you are behaving in a risk-averse fashion. When you forego some amount of expected wealth by saving some of your wealth in low-risk assets, such as Federally insured deposit accounts, rather than investing them all in high-risk stocks with higher expected returns, you are behaving in a risk-averse fashion. In short, there are many instances of risk-averse behavior in day-to-day life.

To define risk aversion more formally, we begin with the concept of a *certainty-equivalent value:*

Definition 2 ***Certainty-equivalent value:*** *The minimum payment a decision maker would accept, if paid with certainty, rather than face a gamble. The certainty-equivalent value is often abbreviated as CE.*

For example, if \$400,000 is the smallest amount that you would be willing to accept in exchange for the lottery ticket, then your certainty-equivalent value for that gamble is \$400,000 (i.e., $CE = \$400,000$). We can now define risk aversion formally:

[2] $\$500,000 = \frac{1}{2} \times \$1,000,000 + \frac{1}{2} \times \$0$.

Definition 3 *A decision maker is* risk averse *if his certainty-equivalent value for any given gamble is less than the expected value of that gamble. Notationally, we can say an individual is risk averse if $CE \leq EV$ for all gambles and $CE < EV$ for some gambles.*

For instance, if you would be willing to accept less than \$500,000 for your lottery ticket, then your certainty-equivalent value is less than the expected value (i.e., \$500,000), which would be consistent with your being risk averse.

In contrast, an expected-value maximizer is *risk neutral*—his decisions are unaffected by risk. Formally,

Definition 4 *A decision maker is* risk neutral *if the certainty-equivalent value for any given gamble is equal to the expected value of that gamble. Notationally, we can say an individual is risk neutral if $CE = EV$ for all gambles.*

In rare instances, a decision maker *likes* risk; that is, he would be willing to pay to take on more risk (or, equivalently, require compensation to part with risk). For instance, if someone's certainty-equivalent value for the lottery ticket were \$600,000 (i.e., he had to be compensated for giving up the risk represented by the lottery ticket), then he would be called *risk loving*:

Definition 5 *A decision maker is* risk loving *if the certainty-equivalent value for any given gamble is greater than the expected value of that gamble. Notationally, we can say an individual is risk loving if $CE \geq EV$ for all gambles and $CE > EV$ for some gambles.*

It is worth emphasizing that risk loving behavior is fairly rare.[3]

At this point you might ask: When is it appropriate (i.e., reasonably accurate) to assume a decision maker is risk neutral and when is it appropriate to assume he is risk averse? Some answers:

Small stakes versus large stakes: If the amounts of money involved in the gamble are small *relative* to the decision maker's

[3] Although it is true that a number of people like to pay to gamble on occasion (e.g., they visit casinos or play state lotteries), their more typical behavior can be described as risk neutral or risk averse (e.g., even people who visit casinos typically purchase life insurance). Moreover, it is not clear that people gamble because they love the risk *per se*; they may like the excitement of the casino, or like to dream about what they would do if they won the lottery, or, possibly, they are like the author's mother, who has on occasion purchased a lottery ticket because she worries that she would have a hard time living with herself if she failed to buy a ticket for a drawing in which "her numbers" won.

wealth or income, then his behavior will tend to be approximately risk neutral. For example, for gambles involving sums less than \$10, most people's behavior is approximately risk neutral. On the other hand, if the amounts of money involved are large *relative* to the decision maker's wealth or income, then his behavior will tend to be risk averse. For example, for gambles involving sums of more than \$10,000, most people's behavior exhibits risk aversion.

Small risks versus large risks: If the possible payoffs (or at least the most likely to be realized payoffs) are close to the expected value, then the risk is small and the decision maker's behavior will be approximately risk neutral. For instance, if the gamble is heads you win \$500,001, but tails you win \$499,999, then your behavior will be close to risk neutral since both payoffs are close to the expected value (i.e., \$500,000). On the other hand, if the possible payoffs are far from the expected value, then the risk is greater and the decision maker's behavior will tend to be risk averse. For instance, we saw that we should expect risk-averse behavior when the gamble was heads you win \$1 million, but tails you win \$0.[4]

Diversification: So far the question of whether someone takes a gamble has been presented as an all-or-nothing proposition. In many instances, however, a person purchases a portion of a gamble. For example, investing in General Motors (or any other company) is a gamble, but you don't have to buy all of General Motors to participate in that gamble. Moreover, at the same time you buy stock in General Motors, you can purchase other securities, giving you a *portfolio* of investments. If you choose your portfolio wisely, you can *diversify* away much of the risk that is unique to a given company. That is, the risk that is unique to a given company in your portfolio no longer concerns you—you are *risk neutral* with respect to it. Consequently, you would like your firm to act as an expected-value maximizer. In summary, diversified decision makers are risk neutral (or approximately so), while undiversified decision makers are more likely to be risk averse.

[4] For a more rigorous (and much more mathematical) treatment of large versus small risk, see Chapter 2 of Chi-fu Huang and Robert H. Litzenberger's *Foundations for Financial Economics*, Amsterdam: North-Holland, 1988.

5.1 An Example of Diversification

To clarify the issue of diversification, consider the following example. There are two companies in which you can invest. One sells ice cream. The other sells umbrellas. Ice cream sales are greater on sunny days than on rainy days, while umbrella sales are greater on rainy days than on sunny days. Suppose that, on average, one out of four days is rainy; that is, the probability of rain is $\frac{1}{4}$. On a rainy day, the umbrella company makes a profit of \$100 and the ice cream company makes a profit of \$0. On a sunny day, the umbrella company makes a profit of \$0 and the ice cream factory makes a profit of \$200. Suppose you invest in the umbrella company only; specifically, suppose you own all of it. Then you face a gamble: on rainy days you receive \$100 and on sunny days you receive nothing. Your expected value is

$$\$25 = \frac{1}{4} \times \$100 + \frac{3}{4} \times \$0.$$

Suppose, in contrast, that you sell three quarters of your holdings in the umbrella company and use some of the proceeds to buy one eighth of the ice cream factory. Now on rainy days you receive \$25 from the umbrella company (since you can claim one quarter of the \$100 profit), but nothing from the ice cream company (since there are no profits). On sunny days you receive \$25 from the ice cream company (since you can claim one eighth of the \$200 profit), but nothing from the umbrella company (since there are no profits). That is, rain or shine, you receive \$25—your risk has disappeared! Your expected value, however, has remained the same (i.e., \$25). This is the magic of diversification.

Moreover, once you can diversify, you want your companies to make expected-value-maximizing decisions. Suppose, for instance, that the umbrella company could change its strategy so that it made a profit of \$150 on rainy days, but lost \$10 on sunny days. This would increase its daily *expected* profit by \$5 (the new EV calculation is

$$\frac{1}{4} \times \$150 + \frac{3}{4} \times (-\$10) = \$30).$$

It would also, arguably, increase the riskiness of its profits by changing its strategy in this way. Suppose, for convenience, that 100% of a company trades on the stock exchange for 100 times its expected daily earnings.[5] The entire ice cream company would, then, be worth \$15,000 ($= 100 \times \left(\frac{1}{4} \times \$0 + \frac{3}{4} \times \$200\right)$) and the entire umbrella company would, then,

[5] The price-to-earnings ratio is 100 here, but the value of the price-to-earnings ratio

be worth \$3000. To return to your position of complete diversification and earning \$25 a day, you would have to reduce your position in the umbrella company to hold one sixth of the company and you would have to increase your holdings of the ice cream company to $\frac{2}{15}$th of the company:

$$\begin{aligned} \text{Earnings on a rainy day} &: \frac{1}{6} \times \$150 + \frac{2}{15} \times \$0 = \$25; \text{ and} \\ \text{Earnings on a sunny day} &: \frac{1}{6} \times (-\$10) + \frac{2}{15} \times \$200 = \$25. \end{aligned}$$

Going from holding one fourth of the umbrella company to owning one sixth of the umbrella company means selling $\frac{1}{12}$th of the umbrella company,[6] which would yield you \$250 ($= \frac{1}{12} \times \$3000$). Going from holding one eighth of the ice cream company to owning 2/15ths means buying an additional $\frac{1}{120}$th of the ice cream company,[7] which would cost you \$125 ($= \frac{1}{120} \times \$15,000$). Your profit from these stock market trades would be \$125. Moreover, you would still receive a riskless \$25 per day. So because you can diversify, you benefit by having your umbrella company do something that increases its expected value, *even if it is riskier.*

6 Modeling Risk-Averse Decision Makers

In this section, we will explore one way to model risk-averse decision makers. Specifically, we will assume a decision maker is motivated not by the money he may receive, but rather by the happiness that the money represents. In the parlance of economics, the decision maker is motivated by the *utility* the money represents and not the money itself. "Utility" is the word economists use when sensible people would use the word "happiness"; that is, "utility" and "happiness" are synonyms. [A lamentable characteristic pertaining to economists is a penchant for eschewing the quotidian term or phrase when an unorthodox and obfuscating term or phrase can be substituted.]

Utility: *a synonym for happiness.*

does not matter for the conclusions reached here. If the ratio were r, then decreasing your holdings of the umbrella company to $\frac{1}{6}$th of the company and increasing your holdings of the ice cream company to $\frac{2}{15}$th would yield a trading profit of

$$\frac{30r}{12} - \frac{150r}{120} = \frac{5}{4}r > 0.$$

Now you might wonder whether it is appropriate to use the same price-to-earnings ratio for both firms. In this case it is, at least if you believe that the stock price is driven by fundamentals (that is, future profits).

[6] Since $\frac{1}{4} - \frac{1}{6} = \frac{3}{12} - \frac{2}{12} = \frac{1}{12}$.

[7] Since $\frac{2}{15} - \frac{1}{8} = \frac{16}{120} - \frac{15}{120} = \frac{1}{120}$.

maximized by approximately 30% coverage (i.e., $\alpha = .3$).[15] This is a general proposition:

Proposition 11 *As a rule, a risk-averse individual will purchase less than full insurance (under insure) if the insurance that is offered to him is actuarially unfair.*

8 Information

Often when we make a decision under uncertainty we would like more information about the uncertainty we face. You, for example, might read the prospectus for a security that you are considering purchasing, or you may ask a friend or a stock broker for advice about that security. We have, in fact, already seen a situation of information gathering. Recall Figure 6 from page 11. In that tree, a firm was deciding whether to launch a new product or not. Prior to making this decision, the firm could, if it wished conduct a survey that would *perfectly* reveal how successful the product would be. Without conducting a survey, the firm would have to make its launch decision "in the dark." This is an example of a firm deciding whether to acquire additional information before making a decision. Note that whether to acquire additional information is, itself, a decision; one that we will study in this section.

Information is divided into two classes: perfect and imperfect. *Perfect information,* like that in Figure 6, is information that completely reveals in advance the outcome of some future uncertain event. In Figure 6, for instance, the survey completely revealed how successful the launch would be.

Perfect information: *information that completely reveals what will happen; that is, information that once learned means there is no longer uncertainty about a given event.*

In other circumstances, we might expect to receive *imperfect information.* Imperfect information helps us to have a better idea of what will happen in the future, but it does not completely reveal what will happen. For instance, consider Figure 14. A firm uses sheet metal in making its product (e.g., car parts). It is concerned with whether a recent shipment of sheet metal is up to its standards. It can use the sheet metal and do an entire production run, return the sheet metal to the supplier, or do a test run and then decide whether to do a production run or return the sheet metal. Unfortunately, a test run is not definitive as to whether the metal is up to the firm's standards, although it gives some indications: "likely okay"

[15] A more accurate approximation of the α that maximizes the individual's expected utility is $\alpha = .30077$.

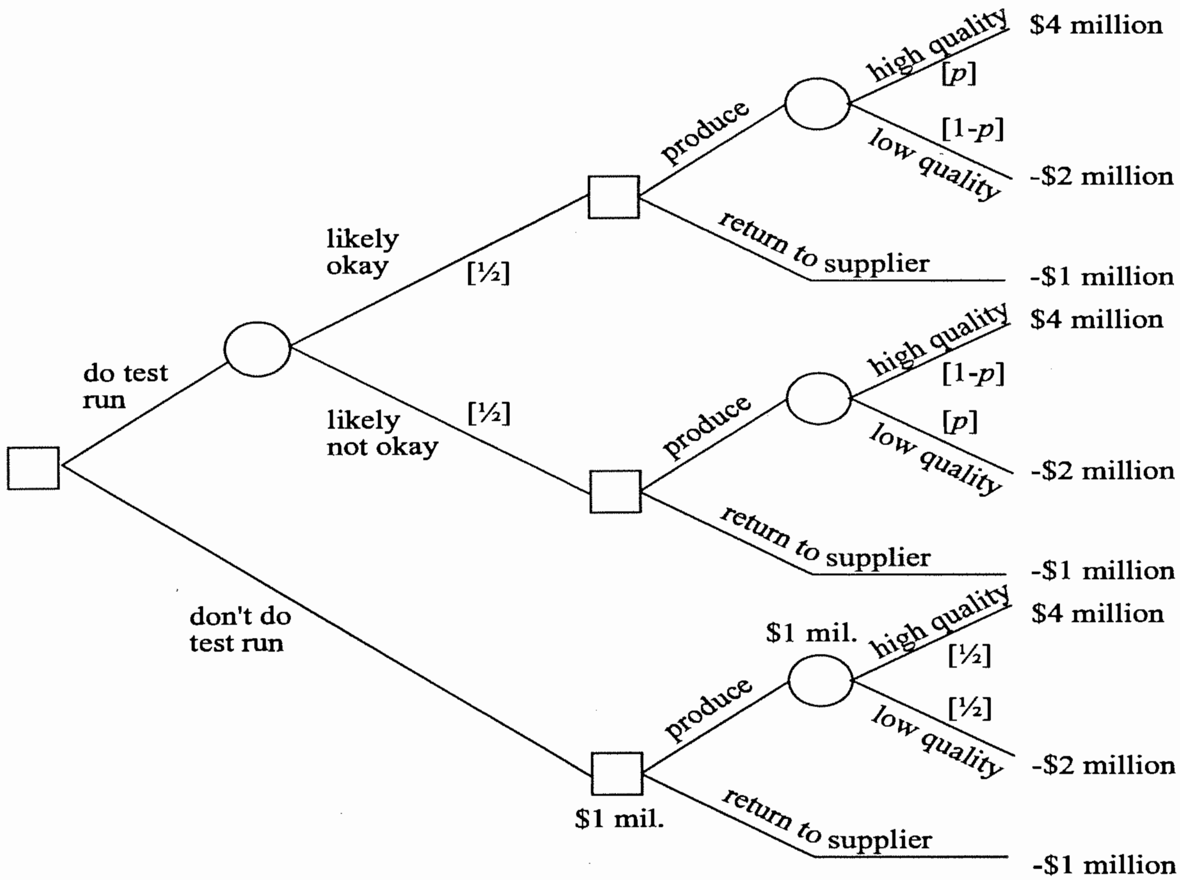

Figure 14: An example of *imperfect* information: By doing a test run, the firm gets information about how likely it is that the sheet metal is high quality versus low quality.

and "likely not okay." If the test run comes back "likely okay," then the probability that the metal is high quality is p and, hence the probability that the metal is low quality is $1-p$. If the test run comes back "likely not okay," then the probability that the metal is high quality is $1-p$ and the probability that it is low quality is p. Absent a test run, the probability that the metal is high quality is $\frac{1}{2}$ and, hence, the probability that it is low quality is $\frac{1}{2}$. Since we want the test run to be informative, assume that $p > \frac{1}{2}$; that is, if the test run comes back "likely okay," then the firm's updated probability that the metal is high quality is greater than if it had received no information. Similarly, if the test run comes back "likely not okay," then the firm's update probability that the metal is high quality is lower than if it had received no information.

Imperfect information: *information that* improves *a decision maker's ability to predict the outcome of a future event, but which does not completely reveal what will happen.*

The variable p is, therefore, a measure of how informative the test run is. The closer p is to $\frac{1}{2}$, the less informative the test run is. Indeed, were $p = \frac{1}{2}$, then there would be no information in a test run, since the outcome of a test run would not change the probability that metal is high quality. Conversely, the further p is from $\frac{1}{2}$ (equivalently, the closer it is to 1), the more informative the test run is. Indeed, were $p = 1$, then we would have perfect information like we had in Figure 6.

In Figure 14, the probability that a test run comes back with "likely okay" is $\frac{1}{2}$. Even if we hadn't written that into the tree, we would have know that this probability was $\frac{1}{2}$. How? Prior to doing the test run, we have no information. Hence, information-wise, we are in the same situation as the bottom of the tree. That is, we must believe that there is a $\frac{1}{2}$ probability that the metal is high quality. We know that if we do the test run and it comes back "likely okay," then the probability that the metal is high quality is p; in other words, *conditional* on the result "likely okay," the probability of high quality metal is p. We can write, therefore,

$$\Pr\{\text{high quality} \mid \text{"likely okay"}\} = p,$$

where the notation $\Pr\{A|B\}$ means the probability of A conditional on B. Similarly, we have

$$\begin{aligned}\Pr\{\text{low quality} \mid \text{"likely okay"}\} &= 1-p;\\ \Pr\{\text{high quality} \mid \text{"likely not okay"}\} &= 1-p;\text{ and}\\ \Pr\{\text{low quality} \mid \text{"likely not okay"}\} &= p.\end{aligned}$$

The law of total probability tells us that

$$\begin{aligned}\Pr\{\text{high quality}\} = {} & \Pr\{\text{high quality} \mid \text{"likely okay"}\} \times \Pr\{\text{"likely okay"}\}\\ & + \Pr\{\text{high quality} \mid \text{"likely not okay"}\} \times \Pr\{\text{"likely not okay"}\}\end{aligned}$$

Let q be the probability that a test run comes back "likely okay." Then substituting into the last equation, we get

$$\frac{1}{2} = p \times q + (1-p) \times (1-q)\,.$$

We can solve this last equation for q:

$$\begin{aligned}\frac{1}{2} &= 2 \times p \times q - q - p + 1 \\ &= q \times 2 \times \left(p - \frac{1}{2}\right) - p + 1.\end{aligned}$$

Adding a $p - 1$ to both sides yields:

$$p - \frac{1}{2} = q \times 2 \times \left(p - \frac{1}{2}\right);$$

hence $\frac{1}{2} = q = \Pr\{\text{"likely okay"}\}$, as drawn in the tree. What's the moral? All trees have to be consistent in this way or, to put it another way, we are not free to pick whatever probabilities we want for the results of the test run.

We solve the tree in Figure 14 like any other tree—starting at the right and moving left. The top right-most chance node yields an expected value of

$$EV_{\text{top}} = p \times \$4 \text{ million } + (1-p) \times (-\$2) \text{ million}.$$

Since $p > \frac{1}{2}$, $EV_{\text{top}} > \$1$ million. Consequently, at the top right decision node the decision would be to produce, and the value of this decision node would be EV_{top}.

Going down to the bottom right-most chance node, we see it has an expected value of

$$\frac{1}{2} \times \$4 \text{ million } + \frac{1}{2} \times (-\$2) \text{ million } = \$1 \text{ million}$$

(note this indicated on the tree). Since +\$1 million is better than -\$1 million, the correct decision at the preceding decision node is to produce; hence, the value of this decision node is \$1 million.

Now consider the middle right-most chance node. It has an expected value of

$$\begin{aligned}EV_{\text{middle}} &= (1-p) \times \$4 \text{ million } + p \times (-\$2) \text{ million} \\ &= \$4 \text{ million } - p \times \$6 \text{ million}.\end{aligned}$$

The decision to take at the preceding decision node depends on the value of p: If $EV_{\text{middle}} \geq -\$1$ million, then the firm should produce. If $EV_{\text{middle}} < -\$1$ million, then the firm should not produce. When is $EV_{\text{middle}} \geq -\$1$ million? Answer: when

$$\$4 \text{ million } - p \times \$6 \text{ million} \geq -\$1 \text{ million};$$

or, dividing through by $-\$1$ million (remember this reverses the inequality sign), when

$$6 \times p - 4 \leq 1;$$

or, adding 4 to both sides and, then, dividing both sides by 6, when

$$p \leq \frac{5}{6}.$$

So when $p \leq \frac{5}{6}$—it is not too likely that the metal is low quality conditional on the test run returning "likely not okay"—then the firm should produce even if the result of the test run is not promising. On the other hand, if $p > \frac{5}{6}$—it is very likely that the metal is low quality conditional on the test run returning "likely not okay"—then the firm should not produce if the result of the test run is not promising.

Working back to the first chance node, we find that the expected value of that node is

$$\frac{1}{2} \times EV_{\text{top}} + \frac{1}{2} \times \begin{cases} EV_{\text{middle}} & \text{if } p \leq \frac{5}{6} \\ -\$1 \text{ million} & \text{if } p > \frac{5}{6} \end{cases}.$$

Finally, we can decide what to do at the first decision node; that is, we can determine whether it is worthwhile to do a test run or not. Suppose first, that a test run is not that informative, so $p \leq \frac{5}{6}$. The value of doing the test run is, then,

$$\begin{aligned} \frac{1}{2} \times EV_{\text{top}} + \frac{1}{2} \times EV_{\text{middle}} &= \frac{1}{2} \times (p \times \$4 \text{ million } + (1-p) \times (-\$2) \text{ million}) \\ &\quad + \frac{1}{2} \times ((1-p) \times \$4 \text{ million } + p \times (-\$2) \text{ million}) \\ &= \frac{1}{2} \times \$4 \text{ million } + \frac{1}{2} \times (-\$2) \text{ million} \\ &= \$1 \text{ million.} \end{aligned}$$

In which case, there is *no* additional value to doing a test run.

Suppose, in contrast, that a test run is very information, so $p > \frac{5}{6}$. The value of doing the test run is, then,

$$\begin{aligned}
\frac{1}{2} \times EV_{\text{top}} + \frac{1}{2} \times (-\$1) \text{ million} &= \frac{1}{2} \times (p \times \$4 \text{ million } + (1-p) \times (-\$2) \text{ million}) \\
&\quad + \frac{1}{2} \times (-\$1) \text{ million} \\
&= p \times \$3 \text{ million } - \$1.5 \text{ million} \\
&> \$1 \text{ million (if } p > \frac{5}{6}).
\end{aligned}$$

In this case, there is an additional value to doing a test run.

Why were the two cases so different? That is, why was there additional value to doing a test run when $p > \frac{5}{6}$ but not otherwise? The answer has to do with the difference in how the firm responds to "likely not okay" in the two cases. When $p \leq \frac{5}{6}$, the firm produces even if the result from the test run is "likely not okay." This is also the action it takes absent any information (i.e., at the bottom decision node) and if the result is "likely okay." That is, when $p \leq \frac{5}{6}$, the information learned from the test run has *no* potential to affect the firm's action—when $p \leq \frac{5}{6}$, the firm always produces. In contrast, when $p > \frac{5}{6}$, then the firm does not produce if the result from the test run is "likely not okay." That is, when $p > \frac{5}{6}$, the information learned from the test run *has* the potential to affect the firm's action—the firm takes *different* actions depending on the result of the test run. This reflects a general proposition about the value of information:

Proposition 12 (The fundamental rule of information) *Information has value* only if *it has the potential to affect a decision maker's choice of action.*

When $p \leq \frac{5}{6}$, the information has *no* potential to affect the firm's decision, but when $p > \frac{5}{6}$, the information *has* the potential to affect the firm's decision. This is why the information is valueless when $p \leq \frac{5}{6}$, but valuable when $p > \frac{5}{6}$.

How valuable is the information when it is valuable? To find out, we subtract the expected value of not doing the test run from the expected value of doing the test run:

$$\begin{aligned}
&\underbrace{p \times \$3 \text{ million } - \$1.5 \text{ million}}_{\text{expected value if do test run}} \quad - \quad \underbrace{\$1 \text{ million}}_{\text{expected value if don't do test run}} \\
&= \quad p \times \$3 \text{ million } - \$2.5 \text{ million}
\end{aligned}$$

(for $p > \frac{5}{6}$). Figure 15 plots the value of information for p between $\frac{1}{2}$ and 1.

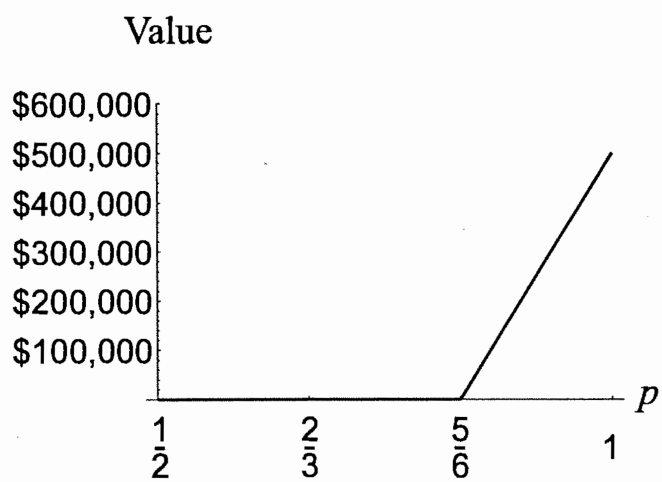

Figure 15: A plot of the value of information. Information is valueless for $p \leq \frac{5}{6}$ because it, then, has no potential to affect the firm's choice of action. It has positive value when $p > \frac{5}{6}$ because it, then, has a potential to affect the firm's choice of action. Note that the value of information is increasing between $p = \frac{5}{6}$ and $p = 1$, reaching a maximum value at $p = 1$; that is, reaching a maximum value at perfect information.

Note that the value is zero for p between $\frac{1}{2}$ and $\frac{5}{6}$ and is increasing (upward sloping) for p between $\frac{5}{6}$ and 1. The information is most valuable when $p = 1$, which makes sense: We would expected *perfect* information (which is what $p = 1$ represents) to be more valuable than *imperfect* information. This, too, is a general proposition:

Proposition 13 *Perfect information is always at least as valuable as imperfect information.*

Figure 16 repeats Figure 14, except now the payoff if the metal is returned to the supplier is left as a variable, z, and the value of p is fixed at $\frac{5}{6}$. What we want to do now is comparative statics with respect to z. In particular, we want to see how the value of information changes as z changes. Note that the expected values have been written in for each of the right-most chance nodes. From this information, we see that we want to divide our analysis into four regions:

1. $z \leq -\$1$ million;
2. $-\$1$ million $< z \leq \$1$ million;
3. $\$1$ million $< z \leq \$3$ million; and
4. $\$3$ million $< z$.

In region 1, the decision to make at each of the three right-most decision nodes is "produce." Moreover, this is the decision regardless of the information learned. Since the information, therefore, has no potential to change the firm's action, the information must be worthless (this is just the Fundamental Rule of Information).

In region 2, the firm produces at the top and bottom decision node, but returns the metal to the supplier at the middle node. The information has, therefore, the potential to change the firm's action, so we know it has value. Specifically,

$$\begin{aligned}\text{Value of information in region 2} &= \frac{1}{2} \times \$3 \text{ mil.} + \frac{1}{2} \times z - \$1 \text{ mil.} \\ &= \frac{1}{2} \times z + \$500,000.\end{aligned}$$

Note that, in region 2, the value of information is increasing in z.

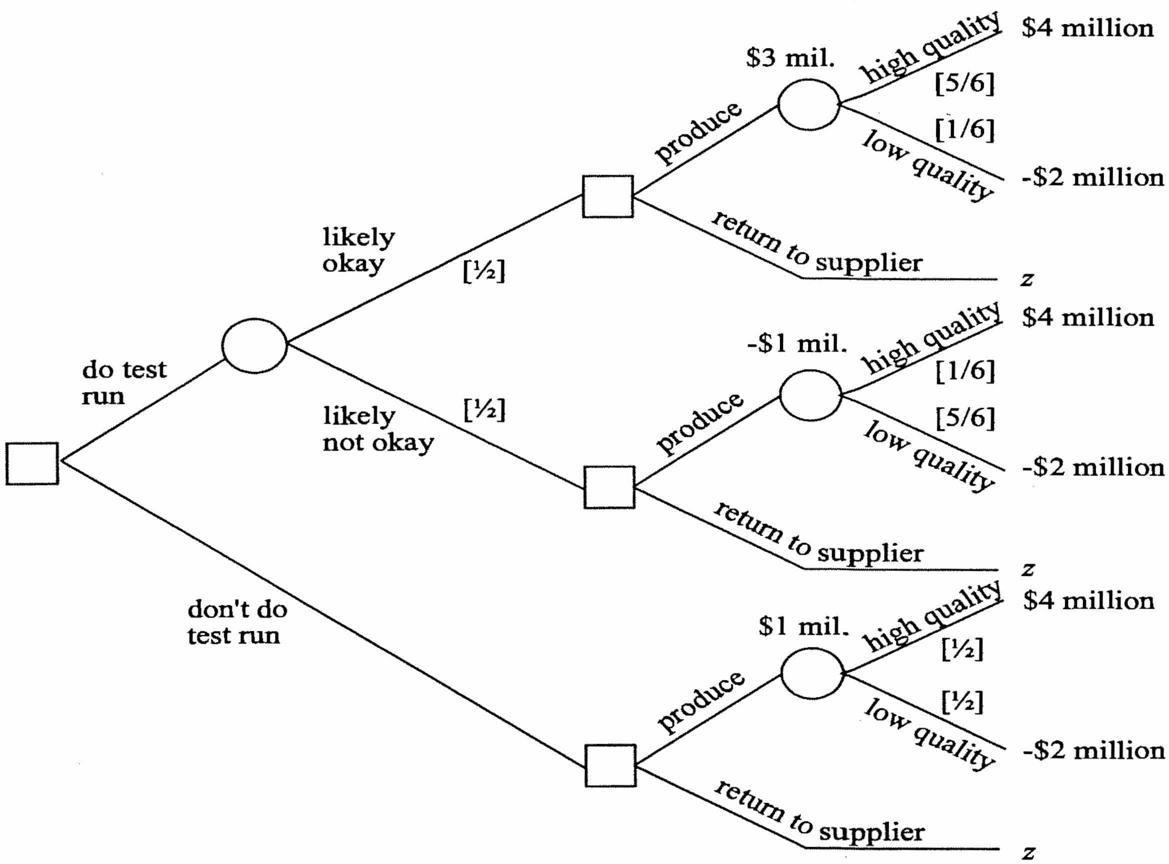

Figure 16: More on the value of information

In region 3, the firm produces only at the top decision node, but returns the metal to the supplier at the bottom two nodes. Again, the information has the potential to change the firm's action, so we know it has value. Specifically,

$$\begin{aligned}\text{Value of information in region 3} &= \frac{1}{2} \times \$3 \text{ mil.} + \frac{1}{2} \times z - z \\ &= \$1.5 \text{ mil.} - \frac{1}{2} \times z.\end{aligned}$$

Note that, in region 3, the value of information is *decreasing* in z.

Finally, in region 4, the firm always does best to return the metal to the supplier. Moreover, this is the decision regardless of the information learned. Since the information has, therefore, no potential to change the firm's action, the information must be worthless.

Figure 17 plots the value of the information as a function of z. From the figure, it is clear that the information is most valuable when $z = \$1$ million. What's significant about \$1 million? It's the value of z that makes the firm indifferent between producing and returning the metal when it has *no* information. This is not a fluke, but an illustration of a general proposition:

Proposition 14 *The value of information is maximized when the decision maker would,* absent the information, *view alternative actions as equally attractive.*

9 Summary

We began this article by considering decision making under certainty. Such decision problems could be represented by *decision trees* consisting of decision nodes (squares), branches for the alternatives, and the payoffs. These trees, *like all trees,* were solved by working from right to left (although the tree is read left to right).

Next we added the possibility of uncertainty. We indicated uncertainty in our trees by using chance nodes (circles). The branches stemming from a chance node are the possible outcomes of the random event that the chance node represents. We practiced solving trees with uncertainty under the assumption that the decision makers were *expected-value maximizers.*

A problem with the assumption that decision makers are expected-value maximizers is that expected-value maximizers are unconcerned by risk (they are *risk neutral*), whereas many real decision makers are concerned by risk. Specifically, they dislike risk and are willing to pay to avoid it. These decision

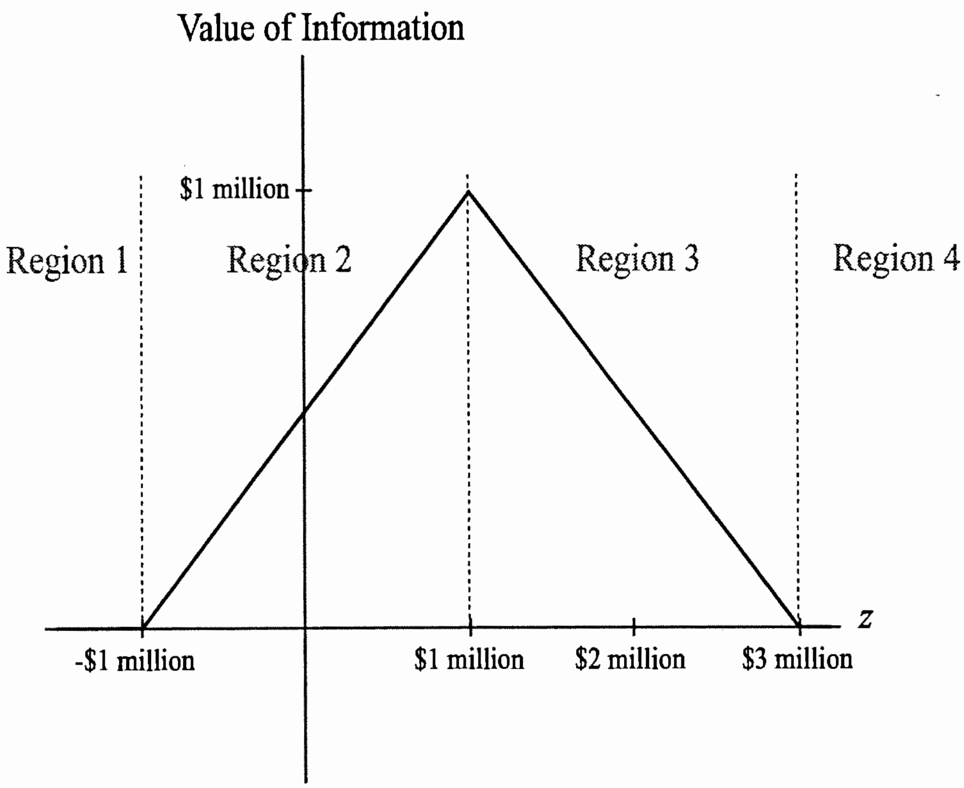

Figure 17: A plot of the value of information against the value of returning the metal to the supplier, z.

makers we called *risk averse.* The defining characteristic of risk-averse decision makers is that the minimum amount they would accept with certainty rather than face a gamble, *the certainty-equivalent value of the gamble*, is less than the expected value of the gamble.

One way to model risk-averse decision makers is to assume that they have a utility or happiness score for every amount of money. This association between money and utility is called a decision maker's *utility function.* Rather than maximize their expected values, risk-averse decision makers maximize their *expected utility*; that is, they are *expected-utility maximizers.*

We ended this article by considering an application of expected-utility maximization to the problem of insurance and by analyzing the value of information in decision-making problems.