Logic-Based Learning: Tutorial 1

Deductive and Abductive Inference

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The aim of this tutorial is to enable you to practice with logic-based concepts introduced in Unit 2, and with the two key inference methods of deduction, through resolution and SLD resolution, and abduction:

Question 1

Using the methods described in slides 8 and 10, convert the following first-order sentences into clausal representation:

i) $\forall X(literate(X) \rightarrow reads(X) \vee write(X))$

ii) $\forall X(clear(X) \rightarrow (block(X) \wedge \neg\exists Y\, on(Y, X)))$

iii) $\exists Y(g(Y) \wedge \forall Z(r(Z) \rightarrow f(Y, Z)))$

Solution

i) $\forall \mathbf{X}(\mathbf{literate}(\mathbf{X}) \rightarrow \mathbf{reads}(\mathbf{X}) \vee \mathbf{write}(\mathbf{X}))$

$\forall X(\neg literate(X) \vee read(X) \vee write(X))$	remove implications
$\neg literate(X) \vee read(X) \vee write(X)$	remove universal quantifiers

ii) $\forall \mathbf{X}(\mathbf{clear}(\mathbf{X}) \rightarrow (\mathbf{block}(\mathbf{X}) \wedge \neg\exists \mathbf{Yon}(\mathbf{Y}, \mathbf{X})))$

$\forall X(\neg clear(X) \vee (block(X) \wedge \neg\exists Y\, on(Y, X)))$	remove implications
$\forall X(\neg clear(X) \vee (block(X) \wedge \forall Y \neg on(Y, X)))$	push negation next to atoms
$\forall X, Y(\neg clear(X) \vee (block(X) \wedge \neg on(Y, X)))$	move universal quantifiers to the front
$\neg clear(X) \vee (block(X) \wedge \neg on(Y, X))$	remove universal quantifiers
$\{\neg clear(X) \vee block(X), \neg clear(X) \vee \neg on(Y, X)\}$	distribute disjunctions

iii) $\exists \mathbf{Y}(\mathbf{g}(\mathbf{Y}) \wedge \forall \mathbf{Z}(\mathbf{r}(\mathbf{Z}) \rightarrow \mathbf{f}(\mathbf{Y}, \mathbf{Z})))$

$\exists Y(g(Y) \wedge \forall Z(\neg r(Z) \vee f(Y, Z)))$	remove implications
$g(sk1) \wedge \forall Z(\neg r(Z) \vee f(sk1, Z)))$	eliminate existentials using new constants
$\forall Z(g(sk1) \wedge (\neg r(Z) \vee f(sk1, Z)))$	move universal quantifiers to the front
$g(sk1) \wedge (\neg r(Z) \vee f(sk1, Z))$	remove universal quantifiers
$\{g(sk1), \neg r(Z) \vee f(sk1, Z)\}$	

Question 2

Consider the following first-order logic sentence S whose language $\mathcal{L}$ includes only the constants b, c, l:

$$S = \forall X, \forall Y(\neg p(b, Y) \wedge p(c, l) \wedge (p(b, X) \vee \neg p(X, l)))$$

1. Rewrite the sentence S into a set C of clauses.
2. What is the Herbrand domain of C?
3. Write in full *ground*(C) (i.e. the grounding of C).
4. Using the Theorem given in slide 14 state whether C is satisfiable or not and explain why.

Solution

1. $C = \{\neg p(b, Y),\ p(c, l),\ p(b, X) \vee \neg p(X, l)\}$
2. The Herbrand domain of C is $\{b,\ c,\ l\}$
3. The *ground*(C) is:
$$\begin{array}{lll} \{ & \neg p(b, b) & p(b, b) \vee \neg p(b, l) \\ & \neg p(b, c) & p(b, c) \vee \neg p(c, l) \\ & \neg p(b, l) & p(b, l) \vee \neg p(l, l) \\ & p(c, l) & \} \end{array}$$
4. The Theorem in slide 14 states that a set T of clauses is satisfiable if and only if the *ground*(T) is satisfiable. *ground*(T) is satisfiable if there is an assignment of true or false to the atoms in the Herbrand base of *ground*(T) (i.e. an Herbrand interpretation) that makes each ground clause in *ground*(T) true. In *ground*(T) we can see that the subset of ground clauses:
$$\{\neg p(b, c), p(c, l), p(b, c) \vee \neg p(c, l)\}$$
is clearly unsatisfiable. So the initial given sentence S is unsatisfiable.

Question 3

Consider the following set S of first-order formulae:

$$\begin{array}{lll} S = \{ & daughter(rebecca) & \\ & \forall X(daughter(X) \rightarrow (\exists Y(mum(Y, X)) \wedge \exists Z(dad(Z, X))) & \} \end{array}$$

1. Rewrite S into a set C of clauses.
2. Assume an Herbrand domain $D = \{rebecca\}$, construct an Herbrand model for the set C of clauses. Remember, an Herbrand model of a set C of clauses is an Herbrand interpretation that satisfies all the clauses in C.

3. Repeat point 2 above but with Herbrand domain $D = \{rebecca, natascia\}$.
4. State what the minimal Herbrand model is in points 2 and 3 and explain your answer.

Solution

1. $C = \{daughter(rebecca), \neg daughter(X) \vee mum(m(X), X), \neg daughter(X) \vee dad(d(X), X)\}$
2. In C the terms $m(X)$ and $d(X)$ are skolem terms. An Herbrand model M_1 for the set C is given by the set of atoms

   $$M_1 = \{daughter(rebecca), mum(m(rebecca), rebecca), dad(d(rebecca), rebecca)\}$$

3. Assume now the Herbrand domain to be $D = \{rebecca, natascia\}$. The set C has more than one Herbrand model. For instance, the Herbrand model M_1, given in answer 2, is also an Herbrand model with respect to this new Herbrand Domain. The following would be an example of another Herbrand model M_2 for the set C:

   $$\begin{aligned} M_2 = \{&daughter(rebecca), mum(m(rebecca), rebecca), dad(d(rebecca), rebecca),\\ &daughter(natascia), mum(m(natascia), natascia), dad(d(natascia), natascia)\} \end{aligned}$$

4. In answer 2, the Herbrand model M_1 is also a minimal Herbrand model, and it is also a minimal Herbrand model in the case when $D = \{rebecca, natascia\}$. The Herbrand model M_2, given in answer 3, is not a minimal Herbrand model for C.

Question 4

Consider the following set S of sentences. Find the minimal Herbrand model of S, assuming in the second case an Herbrand domain $\{a1, b1\}$:

1. $S = \{a \leftarrow \neg b \wedge c, d \leftarrow \neg c, \neg d\}$
2. $S = \{\forall X, \forall Y (q(X) \leftarrow \neg q(Y))\}$

Solution

1. We first transform S into a set of clauses. We get $S = \{b \vee \neg c \vee a, d \vee c, \neg d\}$. Any minimal Herbrand model has to satisfy the clause $\neg d$, so d must be false in such a model. This implies also that c must be true for the clause $d \vee c$ to be true. For the first sentence, since c is true, if b were true then we would not need a to be true. This would then give the minimal Herbrand model $M_1 = \{c, b\}$. Alternatively, b can be false, which implies that a would have to be true, hence giving the second minimal Herbrand model $M_2 = \{a, c\}$. So, the given set S has two minimal Herbrand models. Note that, the model $\{a, b, c\}$ is also an Herbrand model of S but it is not a minimal model.

2. The sentence in S is equivalent to the clause $q(X) \vee q(Y)$. Given the Herbrand domain $\{a1, b1\}$, we can construct the $ground(S)$ which is given by the set $ground(S) = \{q(a1), q(a1) \vee q(b1), q(b1)\}$ using the commutativity and reflexivity properties of $\vee$. So a minimal Herbrand model must include $q(a1)$ in oder to satisfy the first clause and the atom $q(b1)$ in order to satisfy the third clause. These two atoms are also sufficient to satisfy the clause $q(a1) \vee q(b1)$. So the minimal Herbrand model is $\{q(a1), q(b1)\}$.

Question 5

For the following definite programs P calculate the Least Herbrand Model $M(P)$:

$$\text{i)}\ P = \begin{cases} t \leftarrow q, r. \\ p \leftarrow q, r. \\ p \leftarrow s. \\ s \leftarrow q. \\ q. \end{cases} \qquad \text{ii)}\ P = \begin{cases} bird(penguin). \\ bird(X) \leftarrow flies(X), animal(X). \\ flies(plane). \\ flies(hawk). \\ flies(sparrow). \\ animal(hawk). \\ animal(sparrow). \end{cases}$$

Solution

1. $\{p, q, s\}$
2. $\{animal(sparrow), animal(hawk), flies(sparrow), flies(hawk),$ $flies(plane), bird(penguin), bird(sparrow), bird(hawk)\}$

Question 6

If possible unify the following pairs and give the substitution ϕ, otherwise explain why they do not unify:

1) $p(X)$ and $r(Y)$
2) $p(X, Y)$ and $p(a, Z)$
3) $p(X, X)$ and $p(a, b)$
4) $p(f(X))$ and $p(g(Y))$
5) $r(f(X), Y, g(Y))$ and $r(f(X), Z, g(X))$
6) $ancestor(X, Y)$ and $ancestor(bill, father(bill))$

Solution

1. $p(X)$ and $r(Y)$
   do not unify because they use different predicates.
2. $p(X, Y)$ and $p(a, Z)$
   unify with $\phi = [X = a, Y = Z]$
3. $p(X, X)$ and $p(a, b)$
   do not unify because a and b are different constants whereas in $p(X, X)$ the arguments are the same for any substitution of X.

4. $p(f(X))$ and $p(g(Y))$
   do not unify as they use different function symbol.

5. $r(f(X), Y, g(Y))$ and $r(f(X), Z, g(X))$
   unify with $\phi = [X = Y = Z]$.

6. $ancestor(X, Y)$ and $ancestor(bill, father(bill))$
   unify with $\phi = [X = bill, Y = father(bill)]$.

Question 7

Consider the following set of a single first-order sentence:

$$S = \{\forall X, \forall Y(subset(X, Y) \leftrightarrow (\forall U(member(U, Y) \leftarrow member(U, X)))\}$$

1. Express in English what the sentence in S represents.

2. Consider a constant e to denote the empty set and define a clause $C4$ that expresses the notion of empty set using the signature of S. Note that the signature of a set of clauses is the set of predicate, functions and constants that appear in the clauses.

3. Use resolution to show that $S \cup \{C4\} \models \forall X(subset(e, X)$

Solution

1. The sentence in S defines the notion of subset relation between any arbitrary sets X and Y.

2. The clause $C4 = \forall U \neg member(U, e)$.

3. First of all we need to transform the sentence into a set of clauses. Note that the $\leftrightarrow$ formula needs to be split into a conjunction of two implication formulae. So the sentence S is transformed into the following set C of clauses:

$$\forall X, \forall Y(\neg subset(X, Y) \vee (\forall U(\neg member(U, X) \vee member(U, Y)))$$
$$\neg subset(X, Y) \vee \neg member(U, X) \vee member(U, Y) \qquad (C1)$$

$$\forall X, \forall Y(\neg\forall U(\neg member(U, X) \vee member(U, Y)) \vee subset(X, Y))$$
$$\forall X, \forall Y(\exists U(member(U, X) \wedge \neg member(U, Y)) \vee subset(X, Y))$$
$$\forall X, \forall Y(member(f(X, Y), X) \wedge \neg member(f(X, Y), Y)) \vee subset(X, Y))$$
$$member(f(X, Y), X) \vee subset(X, Y) \qquad (C2)$$
$$\neg member(f(X, Y), Y) \vee subset(X, Y) \qquad (C3)$$

We need now to show that $\{C1, C2, C3, C4\} \models \forall X(subset(e, X)$. To do so we consider the refutation task

$$\{C1, C2, C3, C4, \neg\forall X(subset(e, X)\} \models []$$

We need first to transfer the negated goal into a clause. $\neg\forall X(subset(e, X)$ becomes $\neg subset(e, s)$ where s is a new skolem constant. So we need to show that

$$\{C1, C2, C3, C4, \neg subset(e, s)\} \models []$$

The resolution proof is given in Figure 1.

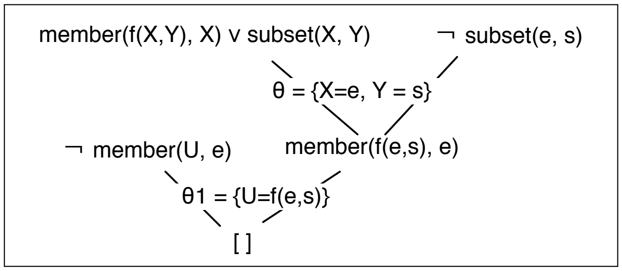

Figure 1: Resolution tree for Question 7

Question 8

Consider the following set C of definite clauses. Draw the SLD-tree of the goal $\leftarrow p(X)$ using the top to bottom left-to-right ordering and state what the computed answer substitutions are.

$p(X) \leftarrow q(X, Y), r(Y)$
$p(X) \leftarrow q(X, X)$
$q(X, X) \leftarrow s(X)$
$r(b)$
$s(a)$
$s(b)$

Solution The SLD proof tree constructed using the top-down and left-to-right proof strategy is given in Figure 2. The substitutions in the three success derivations are $\theta_1 = [X = Y = b]$ and $\theta_2 = [X = a]$, and $\theta_3 = [X = b]$.

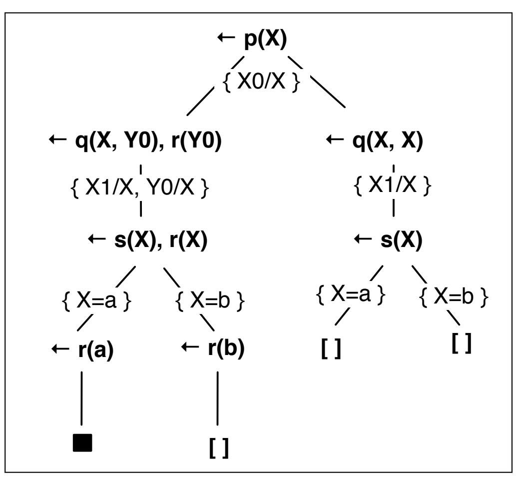

Figure 2: SLD proof tree for Question 8

Question 9

Consider the following set C of normal clauses. Draw the SLDNF derivation of the goal $\leftarrow subset(s1, s2)$ using the top to bottom left-to-right ordering.

$subset(X, Y) \leftarrow not\ \ exception(X, Y)$
$exception(X, Y) \leftarrow member(U, X), not\ \ member(U, Y)$
$member(a, s1)$
$member(a, s2)$
$member(b, s2)$

Solution

The SLDNF proof tree is given in Figure 3.

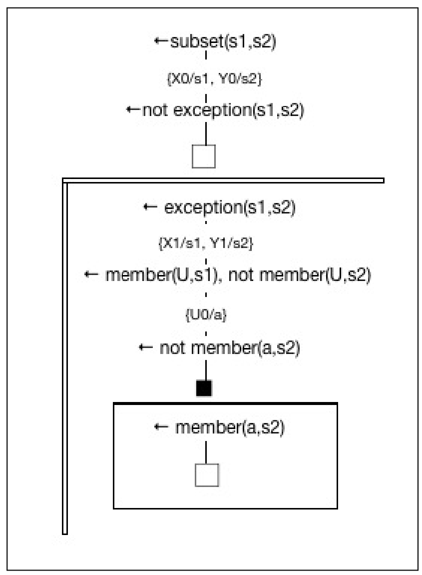

Figure 3: SLDNF proof tree for Question 9

Question 10

Consider the following abductive model $\langle KB, Ab, IC \rangle$, where KB, Ab and IC are defined as follows:

$$KB = \begin{bmatrix} carDoesNotStart(X) \leftarrow batteryFlat(X) \\ carDoesNotStart(X) \leftarrow hasNoFuel(X) \\ lightsGoOn(mycar) \\ fuelIndicatorEmpty(mycar) \end{bmatrix} \quad Ab = \begin{bmatrix} batteryFlat(mycar) \\ batteryFlat(yourcar) \\ hasNoFuel(mycar) \\ hasNoFuel(yourcar) \\ brokenIndicator(mycar) \\ brokenIndicator(yourcar) \end{bmatrix}$$

$$IC = \begin{bmatrix} \leftarrow batteryFlat(X), \\ \quad lightsGoOn(X) \\ \leftarrow hasNoFuel(X), \\ \quad not\ \ fuelIndicatorEmpty(X), \\ \quad not\ \ brokenIndicator(X) \end{bmatrix}$$

State all the possible abductive solutions (if any) for each of the following observations and draw the abductive proof derivation for each of them.

1. $O = carDoesNotStart(mycar)$
2. $O = carDoesNotStart(yourcar)$

Solution

1. $carDoesNotStart(mycar)$
   In this case there is only one possible solution $\Delta = \{hasNoFuel(mycar)\}$ as shown in the abductive derivation below (Figure 4). Note that in this case the abductive proof procedure fails to find a possible explanation when the first rule that defines $carDoesNotStart(mycar)$ is used, but succeeds on the second alternative derivation that makes use of the second rule definition of $carDoesNotStart(mycar)$.

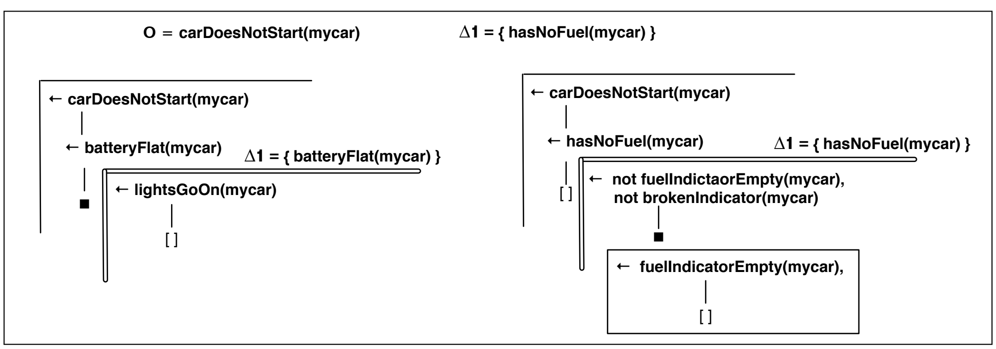

Figure 4: Abductive derivation for observation $carDoesNotStart(mycar)$

2. $carDoesNotStart(yourcar)$
   In this case there are two possible alternative explanations: $\Delta = \{batteryFlat(yourcar)\}$ and $\Delta = \{hasNoFuel(yourcar), fuelIndicatorEmpty(yourcar)\}$ as shown in the two alternative abductive derivations depicted in Figure 5.

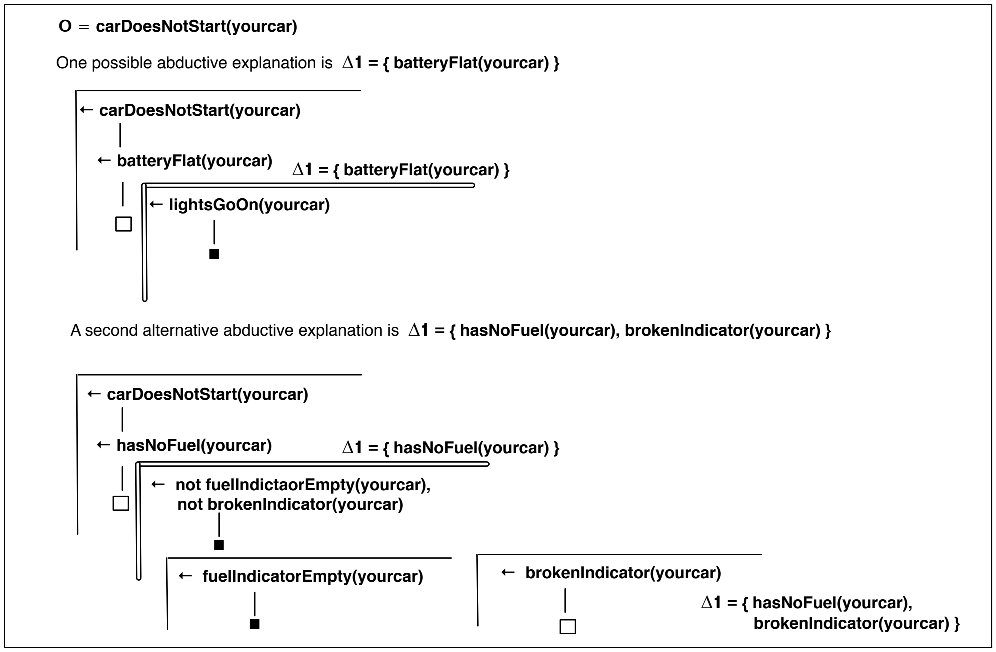

Figure 5: Two alternative abductive derivations for observation $carDoesNotStart(yourcar)$

Question 11

Consider the following abductive model $\langle KB, Ab, IC\rangle$, where KB, Ab and IC are defined as follows. Remember that Ab includes the complement of the atoms defined below as well as the complement of ground instances of defined predicates.

$$KB = \begin{bmatrix} headache(X) \leftarrow jaundice(X) \\ headache(X) \leftarrow migraine(X) \\ sickness(X) \leftarrow stomachBug(X) \end{bmatrix} \qquad Ab = \begin{bmatrix} jaundice(bob) \\ migraine(bob) \\ yellowEyes(bob) \end{bmatrix}$$

$$IC = \begin{bmatrix} \leftarrow migraine(X), jaundice(X) \\ \leftarrow jaundice(X), not\ \ yellowEyes(X) \\ \leftarrow jaundice(X), not\ \ sickness(X) \end{bmatrix}$$

Show whether there is an abductive solution for the observation $headache(bob)$ by drawing its abductive proof derivation.

Solution

Consider the observation $headache(bob)$. The abductive derivation that uses the first definition of the concept $headache(X)$ fails because one of the constraints is violated since there is no possible way of proving that b is sick. This is illustrated in Figure 6.

If the second definition of $headache(bob)$ is used instead, the observation $headache(bob)$ has a possible explanation $\Delta = \{migraine(bob), notjaundice(bob)\}$ as shown in the abductive derivation depicted in Figure 7.

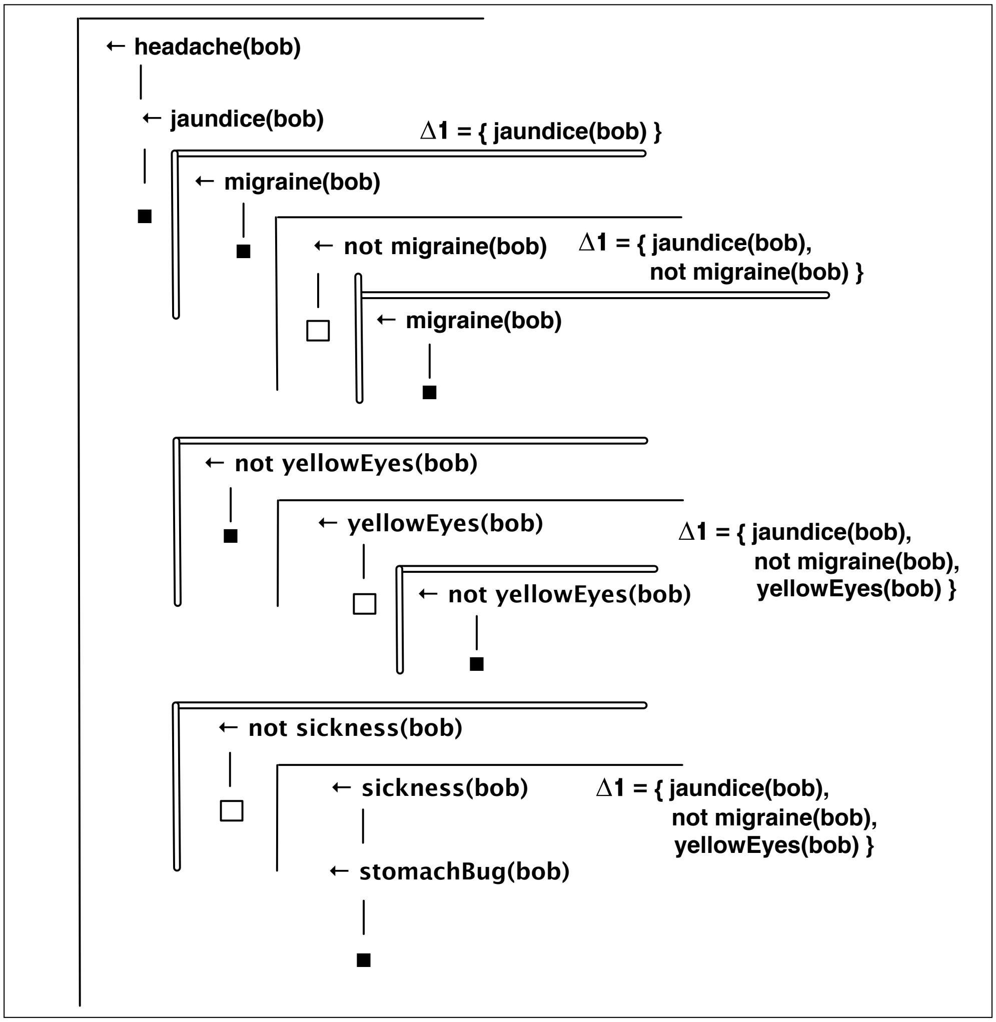

Figure 6: Failure of abductive derivation for observation $headache(bob)$

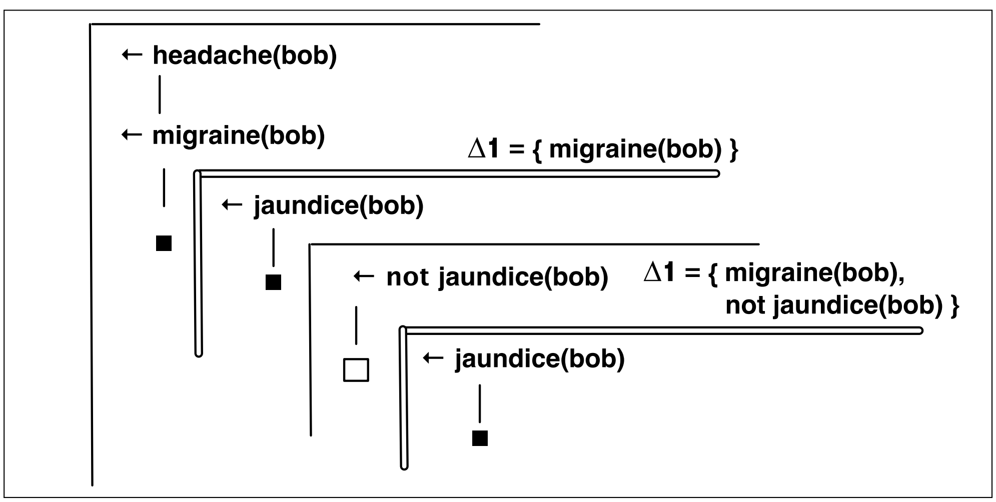

Figure 7: Successful abductive derivation for observation $headache(bob)$