Math 1013 T8 (Tutorial 1) (Chapter 1 : Function)

Function

1) Domain, Codomain and Range

A function defined as $f: A \to B$

A is called the Domain of f,

B is called the Codomain of f,

$f(A)$ is called the Range of f.

2) Well Defined Function

A function is well defined if it satisfies the following two conditions: $\boldsymbol{f: A \to B}$

(i) (Existence of Image)

For all $\boldsymbol{x \in A}$, there exists a $\boldsymbol{y \in B}$,

Such that $\boldsymbol{f(x) = y}$.

($\boldsymbol{\forall x \in A, \exists y \in B, \ni f(x) = y}$.)

(ii) (Uniqueness of Image)

(Accept One to one or Many to one only)

For any $\boldsymbol{\in A}$, $\boldsymbol{b_1 and\ b_2 \in B}$, such that

$\boldsymbol{f(a) = b_1}$ and $\boldsymbol{f(a) = b_2}$,

Then $\boldsymbol{b_1 = b_2}$.

Example 1:

If $f(x) = \sqrt{x}$ and $g(x) = \sqrt{2 - x}$, find each function and its domain.

(a) $f \circ g$ (b) $g \circ f$ (c) $f \circ f$ (d) $g \circ g$

Solution:

(a) $$(f \circ g)(x) = f(g(x)) = f(\sqrt{2 - x}) = \sqrt{\sqrt{2 - x}} = \sqrt[4]{2 - x}$$

The domain of $f \circ g$ is $\{x \mid 2 - x \geqslant 0\} = \{x \mid x \leqslant 2\} = (-\infty, 2]$.

(b) $$(g \circ f)(x) = g(f(x)) = g(\sqrt{x}) = \sqrt{2 - \sqrt{x}}$$

For $\sqrt{x}$ to be defined we must have $x \geqslant 0$. For $\sqrt{2 - \sqrt{x}}$ to be defined we must h $2 - \sqrt{x} \geqslant 0$, that is, $\sqrt{x} \leqslant 2$, or $x \leqslant 4$. Thus we have $0 \leqslant x \leqslant 4$, so the domai $g \circ f$ is the closed interval $[0, 4]$.

(c) $$(f \circ f)(x) = f(f(x)) = f(\sqrt{x}) = \sqrt{\sqrt{x}} = \sqrt[4]{x}$$

The domain of $f \circ f$ is $[0, \infty)$.

(d) $$(g \circ g)(x) = g(g(x)) = g(\sqrt{2 - x}) = \sqrt{2 - \sqrt{2 - x}}$$

This expression is defined when both $2 - x \geqslant 0$ and $2 - \sqrt{2 - x} \geqslant 0$. The first inequality means $x \leqslant 2$, and the second is equivalent to $\sqrt{2 - x} \leqslant 2$, or $2 - x \leqslant 4$, $x \geqslant -2$. Thus $-2 \leqslant x \leqslant 2$, so the domain of $g \circ g$ is the closed interval $[-2, 2]$.

Example 2 :

Given $F(x) = \cos^2(x + 9)$, find functions f, g, and h such that $F = f \circ g$

SOLUTION Since $F(x) = [\cos(x + 9)]^2$, the formula for F says: First add 9, then take cosine of the result, and finally square. So we let

$$h(x) = x + 9 \qquad g(x) = \cos x \qquad f(x) = x^2$$

Then

$$(f \circ g \circ h)(x) = f(g(h(x))) = f(g(x + 9)) = f(\cos(x + 9))$$
$$= [\cos(x + 9)]^2 = F(x)$$

3) Injective

A function is injective (one to one) if

For any $y \in B$, such that

$f(x_1) = y$ and $f(x_2) = y$,

Then $x_1 = x_2$.

4) Surjective

A function is Surjective (onto) if

(Codomain = Range)

For any $y \in B$, there is a $x \in A$,

Such that $f(x) = y$.

5) Bijective

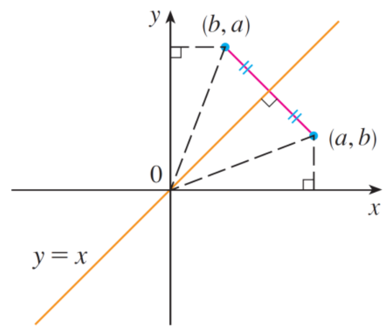
If a function f is both injective and surjective, the function is said to be Bijective.

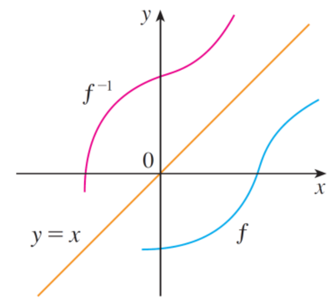
6) Inverse Function

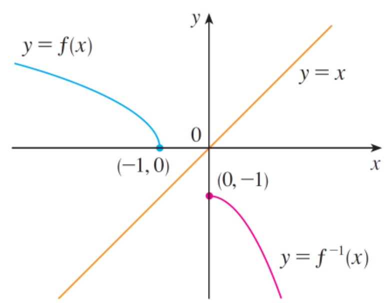
(a) If a function $f(x)$ is bijective, the function $f(x)$ has an inverse function $f^{-1}(x)$.

i.e. It's inverse function is well defined.

(b) The graph of $y = f^{-1}(x)$ is obtained by reflecting the graph of $y = f(x)$ about the line $x = y$.







7) Even Function and Odd function:

(a) Even Function if $f(-x) = f(x)$

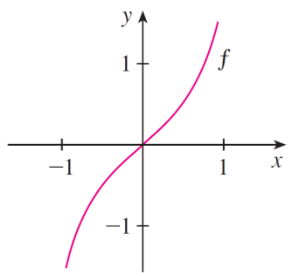
For instance, the function $f(x) = x^2$ is even because

$$f(-x) = (-x)^2 = x^2 = f(x)$$

(b) Odd function if $f(x) = -f(x)$

For example, the function $f(x) = x^3$ is odd because

$$f(-x) = (-x)^3 = -x^3 = -f(x)$$



$$f(x) = x^5 + x$$

8) Increasing and Decreasing function

A function f is called **increasing** on an interval I if

$$f(x_1) < f(x_2) \qquad \text{whenever } x_1 < x_2 \text{ in } I$$

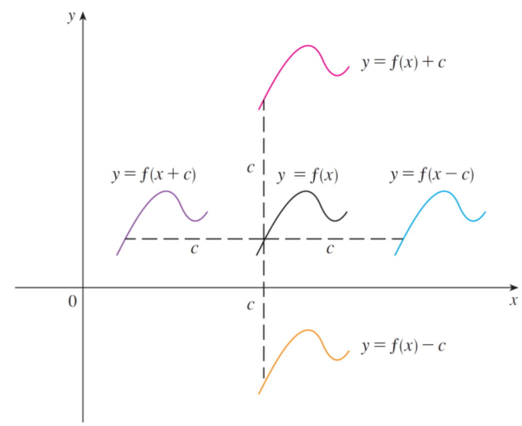
It is called **decreasing** on I if

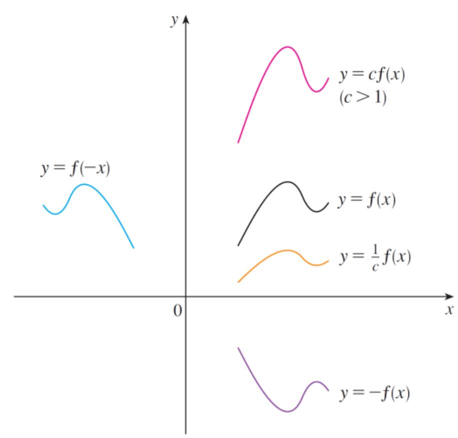
$$f(x_1) > f(x_2) \qquad \text{whenever } x_1 < x_2 \text{ in } I$$

9) Translation, Rotation, Stretching and Symmetry.

VERTICAL AND HORIZONTAL STRETCHING AND REFLECTING Suppose $c > 1$. To obtain the graph of

$y = cf(x)$, stretch the graph of $y = f(x)$ vertically by a factor of c
$y = (1/c)f(x)$, compress the graph of $y = f(x)$ vertically by a factor of c
$y = f(cx)$, compress the graph of $y = f(x)$ horizontally by a factor of c
$y = f(x/c)$, stretch the graph of $y = f(x)$ horizontally by a factor of c
$y = -f(x)$, reflect the graph of $y = f(x)$ about the x-axis
$y = f(-x)$, reflect the graph of $y = f(x)$ about the y-axis





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10) Excercises :

1) Find the domain and range the function (in which the function is well defined)

$$h(x) = \sqrt{4 - x^2}.$$

And sketch the graph of the function

2) Find the domain and range of the function

$$g(x) = sin^{-1}(3x + 1)$$

so that the function is well defined.

3) Find the domain and sketch the graph of the following functions :

(a) $g(x) = \sqrt{x - 5}$

(b) $G(x) = \frac{3x+|x|}{x}$

(c) $g(x) = \frac{|x|}{x^2}$

4) Determine whether $f(x)$ is even, odd, or neither. If you have a graphing calculator, use it to check your answer visually.

(a) $f(x) = \frac{x}{x^2+1}$

(b) $f(x) = 1 + 3x^2 - x^4$

5) Find the inverse function of

(a) $y = f(x) = \frac{x+1}{2x+1}$

(b) $y = f(x) = \sqrt[4]{x^5 + 56}$

6) Prove that the function $f(x) = \sqrt{x^3 + x^2 + x + 1}$

is injective (one to one function).

7) Find the domain and range of $f(x)$ and $f^{-1}(x)$ for the following functions:

(a) $f(x) = \sqrt{3 - e^{2x}}$

(b) $f(x) = \ln(2 + lnx)$.