Additional Note for The Derivative of an Inverse Function

1) Derivative of an Inverse Function

Theorem 1

Let f be differentiable and have an inverse on an interval I. If x_0 is a point of I at which $f'(x_0) \neq 0$, then f^{-1} is differentiable at $y_0 = f(x_0)$ and

$$(f^{-1})'(y_0) = \frac{1}{f'(x_0)}, \text{ where } \quad y_0 = f(x_0).$$

Proof: Before doing a short calculation, we note two facts:

- At a point x_0 where f is differentiable, $y_0 = f(x_0)$ and $x_0 = f^{-1}(y_0)$.
- As a differentiable function, f is continuous at x_0 (Theorem 3.1), which implies that f^{-1} is continuous at y_0 (Theorem 2.13). Therefore, as $y \to y_0$, $x \to x_0$.

Using the definition of the derivative, we have

$$\begin{aligned}
(f^{-1})'(y_0) &= \lim_{y \to y_0} \frac{f^{-1}(y) - f^{-1}(y_0)}{y - y_0} && \text{Definition of derivative of } f^{-1} \\
&= \lim_{x \to x_0} \frac{x - x_0}{f(x) - f(x_0)} && y = f(x) \text{ and } x = f^{-1}(y);\ x \to x_0 \text{ as } y \to y_0 \\
&= \lim_{x \to x_0} \frac{1}{\dfrac{f(x) - f(x_0)}{x - x_0}} && \frac{a}{b} = \frac{1}{b/a} \\
&= \frac{1}{f'(x_0)}. && \text{Definition of derivative of } f
\end{aligned}$$

We have shown that $(f^{-1})'(y_0)$ exists (f^{-1} is differentiable at y_0) and it equals the reciprocal of $f'(x_0)$. ◄

2) Example

EXAMPLE 5 **Derivative of an inverse function** The function $f(x) = \sqrt{x} + x^2 + 1$ is one-to-one, for $x \geq 0$, and has an inverse on that interval. Find the slope of the curve $y = f^{-1}(x)$ at the point $(3, 1)$.

SOLUTION The point $(1, 3)$ is on the graph of f; therefore, $(3, 1)$ is on the graph of f^{-1}. In this case, the slope of the curve $y = f^{-1}(x)$ at the point $(3, 1)$ is the reciprocal of the slope of the curve $y = f(x)$ at $(1, 3)$ (Figure 3.66). Note that $f'(x) = \frac{1}{2\sqrt{x}} + 2x$, which means that $f'(1) = \frac{1}{2} + 2 = \frac{5}{2}$. Therefore,

$$(f^{-1})'(3) = \frac{1}{f'(1)} = \frac{1}{5/2} = \frac{2}{5}.$$

Observe that it is not necessary to find a formula for f^{-1} in order to evaluate its derivative at a point. *Related Exercises 40–50* ◄

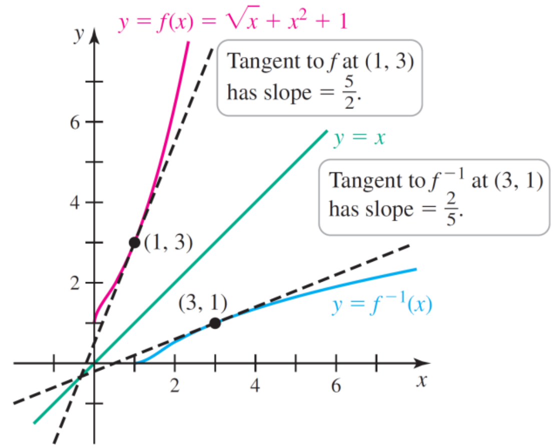

FIGURE 3.66

Additional Note for The Derivative of an Inverse Function

3) Exercises

37–42. Derivatives of inverse functions at a point *Find the derivative of the inverse of the following functions at the specified point on the graph of the inverse function. You do not need to find f^{-1}.*

37. $f(x) = 3x + 4;\ (16, 4)$

38. $f(x) = \frac{1}{2}x + 8;\ (10, 4)$

39. $f(x) = -5x + 4;\ (-1, 1)$

40. $f(x) = x^2 + 1$, for $x \geq 0;\ (5, 2)$

41. $f(x) = \tan x;\ (1, \pi/4)$

42. $f(x) = x^2 - 2x - 3$, for $x \leq 1;\ (12, -3)$

43–46. Slopes of tangent lines *Given the function f, find the slope of the line tangent to the graph of f^{-1} at the specified point on the graph of f^{-1}.*

43. $f(x) = \sqrt{x};\ (2, 4)$

44. $f(x) = x^3;\ (8, 2)$

45. $f(x) = (x + 2)^2;\ (36, 4)$

46. $f(x) = -x^2 + 8;\ (7, 1)$

47–50. Derivatives and inverse functions

47. Find $(f^{-1})'(3)$ if $f(x) = x^3 + x + 1$.

48. Find the slope of the curve $y = f^{-1}(x)$ at $(4, 7)$ if the slope of the curve $y = f(x)$ at $(7, 4)$ is $\frac{2}{3}$.

49. Suppose the slope of the curve $y = f^{-1}(x)$ at $(4, 7)$ is $\frac{4}{5}$. Find $f'(7)$.

50. Suppose the slope of the curve $y = f(x)$ at $(4, 7)$ is $\frac{1}{5}$. Find $(f^{-1})'(7)$.

4) Answers to Exercises

37 $f(4) = 16$ so $(f^{-1})'(16) = \frac{1}{f'(4)} = \frac{1}{3}$.

38 $f(4) = 10$ so $(f^{-1})'(10) = \frac{1}{f'(4)} = \frac{1}{1/2} = 2$.

39 $f(1) = -1$ so $(f^{-1})'(-1) = \frac{1}{f'(1)} = \frac{1}{-5} = \frac{-1}{5}$.

40 $f(2) = 5$ so $(f^{-1})'(5) = \frac{1}{f'(2)} = \frac{1}{4}$.

41 $f\left(\frac{\pi}{4}\right) = 1$ so $(f^{-1})'(1) = \frac{1}{f'\left(\frac{\pi}{4}\right)} = \frac{1}{\sec^2\left(\frac{\pi}{4}\right)} = \frac{1}{2}$.

42 $f(-3) = 12$ so $(f^{-1})'(12) = \frac{1}{f'(-3)} = \frac{1}{-8} = \frac{-1}{8}$.

43 $f(4) = 2$ so $(f^{-1})'(2) = \frac{1}{f'(4)} = \frac{1}{(1/2\sqrt{4})} = 4$.

44 $f(2) = 8$ and $(f^{-1})'(8) = \frac{1}{f'(2)} = \frac{1}{3 \cdot 2^2} = \frac{1}{12}$.

45 $f(4) = 36$ and $(f^{-1})'(36) = \frac{1}{f'(4)} = \frac{1}{2(4+2)} = \frac{1}{12}$.

46 $f(1) = 7$ and $(f^{-1})'(7) = \frac{1}{f'(1)} = \frac{1}{-2 \cdot 1} = \frac{-1}{2}$.

47 Note that $f(1) = 3$. So $(f^{-1})'(3) = \frac{1}{f'(1)} = \frac{1}{4}$.

48 $(f^{-1})'(4) = \frac{1}{f'(7)} = \frac{1}{2/3} = \frac{3}{2}$.

49 $(f^{-1})'(4) = \frac{1}{f'(7)} = \frac{4}{5}$, so $f'(7) = \frac{5}{4}$.

50 $(f^{-1})'(7) = \frac{1}{f'(4)} = \frac{1}{1/5} = 5$.