Math 1013 Tutorial 4 (Limit and Continuity) (T11 & T17)

1) Limit of a function (Definition)

DEFINITION Limit of a Function (Preliminary)

Suppose the function f is defined for all x near a except possibly at a. If $f(x)$ is arbitrarily close to L (as close to L as we like) for all x sufficiently close (but not equal) to a, we write

$$\lim_{x\to a} f(x) = L$$

and say the limit of $f(x)$ as x approaches a equals L.

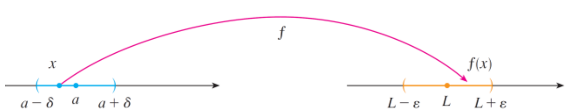
The Precise Definition of Limit (p. 110)

2 Definition Let f be a function defined on some open interval that contains the number a, except possibly at a itself. Then we say that the **limit of $f(x)$ as x approaches a is L**, and we write

$$\lim_{x\to a} f(x) = L$$

if for every number $\varepsilon > 0$ there is a number $\delta > 0$ such that

$$\text{if} \quad 0 < |x-a| < \delta \quad \text{then} \quad |f(x) - L| < \varepsilon$$



DEFINITION One-Sided Limits

1. **Right-sided limit** Suppose f is defined for all x near a with $x > a$. If $f(x)$ is arbitrarily close to L for all x sufficiently close to a with $x > a$, we write

$$\lim_{x\to a^+} f(x) = L$$

and say the limit of $f(x)$ as x approaches a from the right equals L.

2. **Left-sided limit** Suppose f is defined for all x near a with $x < a$. If $f(x)$ is arbitrarily close to L for all x sufficiently close to a with $x < a$, we write

$$\lim_{x\to a^-} f(x) = L$$

and say the limit of $f(x)$ as x approaches a from the left equals L.

2)

(a) If Limit exists, the limit is unique.

(b) If Limit exists, left hand Limit = Right hand Limit

$$\lim_{x\to a} f(x) = L \quad \text{if and only if} \quad \lim_{x\to a^-} f(x) = L \quad \text{and} \quad \lim_{x\to a^+} f(x) = L$$

Example 1 :

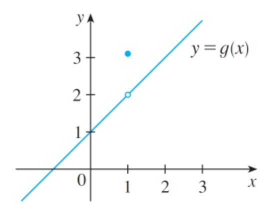
Find $\lim_{x\to 1} g(x)$ where

$$g(x) = \begin{cases} x+1 & \text{if } x \neq 1 \\ \pi & \text{if } x = 1 \end{cases}$$

Solution :

Here g is defined at $x = 1$ and $g(1) = \pi$, but the value of a limit as x approaches 1 does not depend on the value of the function at 1. Since $g(x) = x + 1$ for $x \neq 1$, we have

$$\lim_{x\to 1} g(x) = \lim_{x\to 1} (x+1) = 2$$



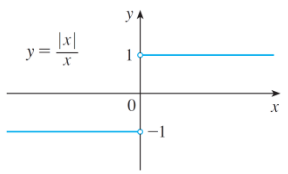
Example 2 : Prove that $\lim_{x\to 0} \frac{|x|}{x}$ does not exist.

Solution :

$$\lim_{x\to 0^+} \frac{|x|}{x} = \lim_{x\to 0^+} \frac{x}{x} = \lim_{x\to 0^+} 1 = 1$$

$$\lim_{x\to 0^-} \frac{|x|}{x} = \lim_{x\to 0^-} \frac{-x}{x} = \lim_{x\to 0^-} (-1) = -1$$

Since the right- and left-hand limits are different, it follows from Theorem 1 that $\lim_{x\to 0} |x|/x$ does not exist. The graph of the function $f(x) = |x|/x$ is shown in Figure 4 and supports the one-sided limits that we found.



3) Important Theorems of Limit of a Function

Theorem 1 :

If $f(x) \leqslant g(x)$ when x is near a (except possibly at a) and the limits of f and g both exist as x approaches a, then

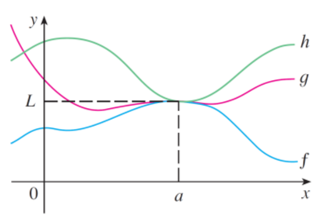
$$\lim_{x \to a} f(x) \leqslant \lim_{x \to a} g(x)$$

Theorem 2 : The Sandwich Theorem (The Squeeze Theorem)

3 The Squeeze Theorem If $f(x) \leqslant g(x) \leqslant h(x)$ when x is near a (except possibly at a) and

$$\lim_{x \to a} f(x) = \lim_{x \to a} h(x) = L$$

then
$$\lim_{x \to a} g(x) = L$$



Example 3 : Show that $\lim_{x \to 0} x^2 \sin\frac{1}{x} = 0$.

Solution:

First note that we **cannot** use

$$\lim_{x \to 0} x^2 \sin\frac{1}{x} = \lim_{x \to 0} x^2 \cdot \lim_{x \to 0} \sin\frac{1}{x}$$

because $\lim_{x \to 0} \sin(1/x)$ does not exist (see Example 4 in Section 2.2).

Instead we apply the Squeeze Theorem, and so we need to find a function f smaller than $g(x) = x^2 \sin(1/x)$ and a function h bigger than g such that both $f(x)$ and $h(x)$ approach 0. To do this we use our knowledge of the sine function. Because the sine of any number lies between -1 and 1, we can write

$$\boxed{4} \qquad -1 \leqslant \sin\frac{1}{x} \leqslant 1$$

Any inequality remains true when multiplied by a positive number. We know that $x^2 \geqslant 0$ for all x and so, multiplying each side of the inequalities in $\boxed{4}$ by x^2, we get

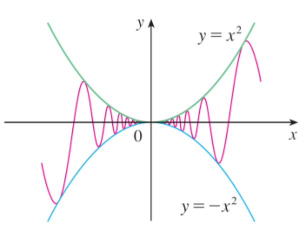
$$-x^2 \leqslant x^2 \sin\frac{1}{x} \leqslant x^2$$

as illustrated by Figure 8. We know that

$$\lim_{x \to 0} x^2 = 0 \qquad \text{and} \qquad \lim_{x \to 0} (-x^2) = 0$$

Taking $f(x) = -x^2$, $g(x) = x^2 \sin(1/x)$, and $h(x) = x^2$ in the Squeeze Theorem, we obtain

$$\lim_{x \to 0} x^2 \sin\frac{1}{x} = 0$$



$y = x^2 \sin(1/x)$

4) Laws of Limit

Assume $\lim_{x \to a} f(x)$ and $\lim_{x \to a} g(x)$ exist. The following properties hold, where c is a real number, and $m > 0$ and $n > 0$ are integers.

1. **Sum** $\lim_{x \to a} [f(x) + g(x)] = \lim_{x \to a} f(x) + \lim_{x \to a} g(x)$
2. **Difference** $\lim_{x \to a} [f(x) - g(x)] = \lim_{x \to a} f(x) - \lim_{x \to a} g(x)$
3. **Constant multiple** $\lim_{x \to a} [cf(x)] = c \lim_{x \to a} f(x)$
4. **Product** $\lim_{x \to a} [f(x)g(x)] = \left[\lim_{x \to a} f(x)\right]\left[\lim_{x \to a} g(x)\right]$
5. **Quotient** $\lim_{x \to a} \left[\frac{f(x)}{g(x)}\right] = \frac{\lim_{x \to a} f(x)}{\lim_{x \to a} g(x)}$, provided $\lim_{x \to a} g(x) \neq 0$
6. **Power** $\lim_{x \to a} [f(x)]^n = \left[\lim_{x \to a} f(x)\right]^n$
7. **Fractional power** $\lim_{x \to a} [f(x)]^{n/m} = \left[\lim_{x \to a} f(x)\right]^{n/m}$, provided $f(x) \geq 0$, for x near a, if m is even and n/m is reduced to lowest terms

5) Continuity (Definition)

A Function $f(x)$ is continuous at a number a if

$$\lim_{x \to a} f(x) = f(a)$$

Such that $\lim_{x \to a^-} f(x) = \lim_{x \to a^+} f(x) = f(a)$

6) Continuous Function can pass the Limit

If $f(x)$ **is continuous at a number a,**

$$\lim_{x \to a} f(x) = f(\lim_{x \to a} x)$$

7) Examples of Continuous Functions

The following types of functions are continuous at every number in their domains:

polynomials rational functions root functions

trigonometric functions inverse trigonometric functions

exponential functions logarithmic functions

8) Pass Limit of Continuous Composite Functions

$\boxed{8}$ **Theorem** If f is continuous at b and $\lim_{x \to a} g(x) = b$, then $\lim_{x \to a} f(g(x)) = f(b)$. In other words,

$$\lim_{x \to a} f(g(x)) = f\left(\lim_{x \to a} g(x)\right)$$

9) Limit of a function $f(x)$ for x tends to infinity

(1) Definition

$\boxed{7}$ **Definition** Let f be a function defined on some interval (a, ∞). Then

$$\lim_{x \to \infty} f(x) = L$$

means that for every $\varepsilon > 0$ there is a corresponding number N such that

$$\text{if} \quad x > N \quad \text{then} \quad |f(x) - L| < \varepsilon$$

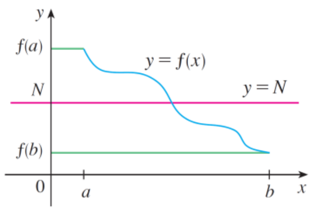
(2) Theorems

(a) $\lim_{x \to \infty} \frac{1}{x} = 0$

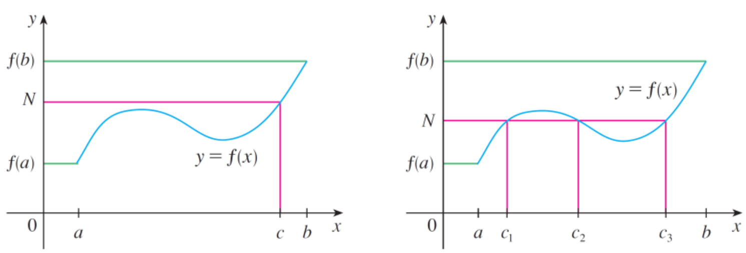
(b) $\lim_{x \to -\infty} f(x) = \lim_{t \to \infty} f(-t)$

10) The Intermediate Value Theorem

10 The Intermediate Value Theorem Suppose that f is continuous on the closed interval $[a, b]$ and let N be any number between $f(a)$ and $f(b)$, where $f(a) \neq f(b)$. Then there exists a number c in (a, b) such that $f(c) = N$.



The Intermediate Value Theorem states that a continuous function takes on every intermediate value between the function values $f(a)$ and $f(b)$. It is illustrated by Figure 8. Note that the value N can be taken on once [as in part (a)] or more than once [as in part (b)].



(a) (b)

11) Exercises :

Section 2.2 (no. 29 - 37) #1-9

29–37 Determine the infinite limit.

29. $\displaystyle \lim_{x \to -3^+} \frac{x+2}{x+3}$

30. $\displaystyle \lim_{x \to -3^-} \frac{x+2}{x+3}$

31. $\displaystyle \lim_{x \to 1} \frac{2-x}{(x-1)^2}$

32. $\displaystyle \lim_{x \to 5^-} \frac{e^x}{(x-5)^3}$

33. $\displaystyle \lim_{x \to 3^+} \ln(x^2-9)$

34. $\displaystyle \lim_{x \to \pi^-} \cot x$

35. $\displaystyle \lim_{x \to 2\pi^-} x \csc x$

36. $\displaystyle \lim_{x \to 2^-} \frac{x^2-2x}{x^2-4x+4}$

37. $\displaystyle \lim_{x \to 2^+} \frac{x^2-2x-8}{x^2-5x+6}$

Section 2.3 (no. 7-9, 41-46, 57- 58, 62) #10-18

7-9 Evaluate the Limits :

7. $\displaystyle \lim_{x \to 8} \left(1+\sqrt[3]{x}\right)\left(2-6x^2+x^3\right)$

8. $\displaystyle \lim_{t \to 2} \left(\frac{t^2-2}{t^3-3t+5}\right)^2$

9. $\displaystyle \lim_{x \to 2} \sqrt{\frac{2x^2+1}{3x-2}}$

41–46 Find the limit, if it exists. If the limit does not exist, explain why.

41. $\displaystyle \lim_{x \to 3} \left(2x+|x-3|\right)$

42. $\displaystyle \lim_{x \to -6} \frac{2x+12}{|x+6|}$

43. $\displaystyle \lim_{x \to 0.5^-} \frac{2x-1}{|2x^3-x^2|}$

44. $\displaystyle \lim_{x \to -2} \frac{2-|x|}{2+x}$

45. $\displaystyle \lim_{x \to 0^-} \left(\frac{1}{x}-\frac{1}{|x|}\right)$

46. $\displaystyle \lim_{x \to 0^+} \left(\frac{1}{x}-\frac{1}{|x|}\right)$

Section 2.3 (no. 57- 58, 62) #19-21

57. If $\lim_{x\to 1} \frac{f(x) - 8}{x - 1} = 10$, find $\lim_{x\to 1} f(x)$.

58. If $\lim_{x\to 0} \frac{f(x)}{x^2} = 5$, find the following limits.

(a) $\lim_{x\to 0} f(x)$ (b) $\lim_{x\to 0} \frac{f(x)}{x}$

62. Evaluate $\lim_{x\to 2} \frac{\sqrt{6 - x} - 2}{\sqrt{3 - x} - 1}$.

Section 2.5 (no. 12 -14) #22-24

12–14 Use the definition of continuity and the properties of limits to show that the function is continuous at the given number a.

12. $f(x) = 3x^4 - 5x + \sqrt[3]{x^2 + 4}, \quad a = 2$

13. $f(x) = (x + 2x^3)^4, \quad a = -1$

14. $h(t) = \frac{2t - 3t^2}{1 + t^3}, \quad a = 1$

Section 2.5 (no. 23 - 24) #25-26

23–24 How would you "remove the discontinuity" of f? In other words, how would you define $f(2)$ in order to make f continuous at 2?

23. $f(x) = \frac{x^2 - x - 2}{x - 2}$ **24.** $f(x) = \frac{x^3 - 8}{x^2 - 4}$

Section 2.5 (no. 51 - 54) #27-30

51–54 Use the Intermediate Value Theorem to show that there is a root of the given equation in the specified interval.

51. $x^4 + x - 3 = 0, \quad (1, 2)$ **52.** $\sqrt[3]{x} = 1 - x, \quad (0, 1)$

53. $e^x = 3 - 2x, \quad (0, 1)$ **54.** $\sin x = x^2 - x, \quad (1, 2)$

(B) Evaluate the Limits for x tends to infinity :

(31) $\lim_{x\to -\infty} \frac{4x^3 + 6x^2 - 2}{2x^3 - 4x + 5}$

(32) $\lim_{x\to \infty} \frac{x^2}{\sqrt{x^4 + 1}}$

(33) $\lim_{x\to -\infty} \left(x + \sqrt{x^2 + 2x}\right)$

(34) Find $\lim_{x\to -\infty} f(x)$ for

$$f(x) = \sqrt{3x^2 + 8x + 6} - \sqrt{3x^2 + 3x + 1}$$

(35) $\lim_{x\to -\infty} \frac{10x^3 - 3x^2 + 8}{\sqrt{25x^6 + x^4 + 2}}$

(C) 1 More Exercise :

(36) $\lim_{x\to 4} \frac{3(x - 4)\sqrt{x + 5}}{3 - \sqrt{x + 5}}$