1) Well Defined Function

A function is well defined if it satisfies the following two conditions: $f: A \to B$

(i) **(Existence of Image)**

For all $x \in A$, there exists a $y \in B$,

Such that $f(x) = y$.

$(\forall x \in A, \exists y \in B, \ni f(x) = y.)$

(ii) **(Uniqueness of Image)**

(Accept One to one or Many to one only)

For any $x_1 \in A$, $y_1 and\ y_2 \in B$, such that

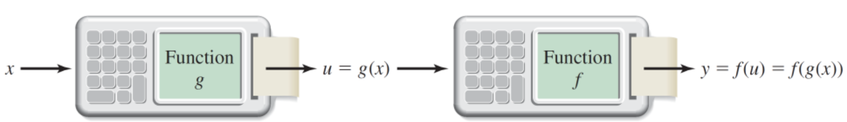
$f(x_1) = y_1$ and $f(x_1) = y_2$,

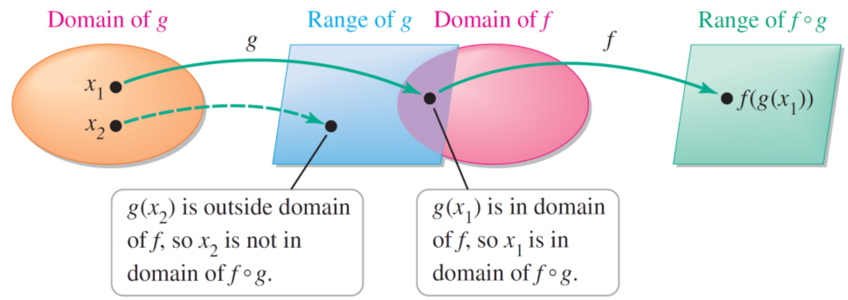
Then $y_1 = y_2$.

2) Composition Function

DEFINITION Composite Functions

Given two functions f and g, the composite function $f \circ g$ is defined by $(f \circ g)(x) = f(g(x))$. It is evaluated in two steps: $y = f(u)$, where $u = g(x)$. The domain of $f \circ g$ consists of all x in the domain of g such that $u = g(x)$ is in the domain of f (Figure 1.8).





EXAMPLE 5 Working with composite functions Identify possible choices for the inner and outer functions in the following composite functions. Give the domain of the composite function.

a. $h(x) = \sqrt{9x - x^2}$ **b.** $h(x) = \dfrac{2}{(x^2 - 1)^3}$

SOLUTION

a. An obvious outer function is $f(x) = \sqrt{x}$, which works on the inner function $g(x) = 9x - x^2$. Therefore, h can be expressed as $h = f \circ g$ or $h(x) = f(g(x))$. The domain of $f \circ g$ consists of all values of x such that $9x - x^2 \geq 0$. Solving this inequality gives $\{x: 0 \leq x \leq 9\}$ as the domain of $f \circ g$.

b. A good choice for an outer function is $f(x) = 2/x^3 = 2x^{-3}$, which works on the inner function $g(x) = x^2 - 1$. Therefore, h can be expressed as $h = f \circ g$ or $h(x) = f(g(x))$. The domain of $f \circ g$ consists of all values of $g(x)$ such that $g(x) \neq 0$, which is $\{x: x \neq \pm 1\}$.

EXAMPLE 6 More composite functions Given $f(x) = \sqrt[3]{x}$ and $g(x) = x^2 - x - 6$, find **(a)** $g \circ f$ and **(b)** $g \circ g$, and their domains.

SOLUTION

a. We have

$$(g \circ f)(x) = g(f(x)) = (\underbrace{\sqrt[3]{x}}_{f(x)})^2 - \underbrace{\sqrt[3]{x}}_{f(x)} - 6 = x^{2/3} - x^{1/3} - 6.$$

Because the domains of f and g are $(-\infty, \infty)$, the domain of $f \circ g$ is also $(-\infty, \infty)$.

b. In this case, we have the composition of two polynomials:

$$\begin{aligned}(g \circ g)(x) &= g(g(x)) \\ &= g(x^2 - x - 6) \\ &= \underbrace{(x^2 - x - 6)}_{g(x)}{}^2 - \underbrace{(x^2 - x - 6)}_{g(x)} - 6 \\ &= x^4 - 2x^3 - 12x^2 + 13x + 36.\end{aligned}$$

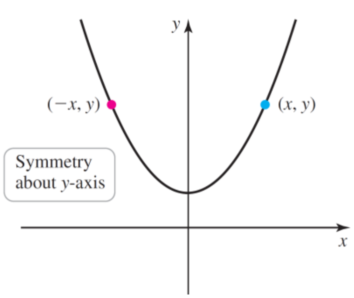
The domain of the composition of two polynomials is $(-\infty, \infty)$.

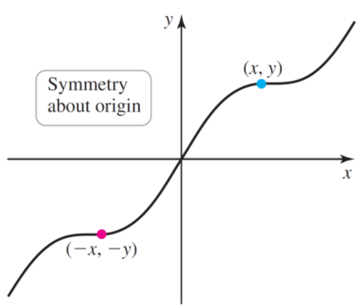
3) Even function and Odd function

DEFINITION Symmetry in Functions

An **even function** f has the property that $f(-x) = f(x)$, for all x in the domain. The graph of an even function is symmetric about the y-axis. Polynomials consisting of only even powers of the variable (of the form x^{2n}, where n is a nonnegative integer) are even functions.

An **odd function** f has the property that $f(-x) = -f(x)$, for all x in the domain. The graph of an odd function is symmetric about the origin. Polynomials consisting of only odd powers of the variable (of the form x^{2n+1}, where n is a nonnegative integer) are odd functions.





4) Inverse Functions:

> **2 Definition** Let f be a one-to-one function with domain A and range B. Then its **inverse function** f^{-1} has domain B and range A and is defined by
>
> $$f^{-1}(y) = x \iff f(x) = y$$
>
> for any y in B.

(a) Existence of inverse Functions

Let f be a one-to-one function on a domain D with a range R. Then f has a unique inverse f^{-1} with domain R and range D such that

$$f^{-1}(f(x)) = x \quad \text{and} \quad f(f^{-1}(y)) = y,$$

where x is in D and y is in R.

Such that $f(x) and\ f^{-1}(x)$ are both bijective functions.

(b) The relation between the domain and range of $f(x) and\ f^{-1}(x)$

> domain of f^{-1} = range of f
>
> range of f^{-1} = domain of f

5) Simplification of inverse Trigonometric Functions

EXAMPLE 13 Simplify the expression $\cos(\tan^{-1}x)$.

SOLUTION 1 Let $y = \tan^{-1}x$. Then $\tan y = x$ and $-\pi/2 < y < \pi/2$. We want to find $\cos y$ but, since $\tan y$ is known, it is easier to find $\sec y$ first:

$$\sec^2 y = 1 + \tan^2 y = 1 + x^2$$

$$\sec y = \sqrt{1 + x^2} \qquad (\text{since } \sec y > 0 \text{ for } -\pi/2 < y < \pi/2)$$

Thus

$$\cos(\tan^{-1}x) = \cos y = \frac{1}{\sec y} = \frac{1}{\sqrt{1 + x^2}}$$

6) Solving inequality with Absolute value sign

EXAMPLE 2 Solve the inequality $|x-3|+|x+2|<11$.

SOLUTION Recall the definition of absolute value:

$$|x| = \begin{cases} x & \text{if } x \geqslant 0 \\ -x & \text{if } x < 0 \end{cases}$$

It follows that

$$|x-3| = \begin{cases} x-3 & \text{if } x-3 \geqslant 0 \\ -(x-3) & \text{if } x-3 < 0 \end{cases}$$

$$= \begin{cases} x-3 & \text{if } x \geqslant 3 \\ -x+3 & \text{if } x < 3 \end{cases}$$

Similarly

$$|x+2| = \begin{cases} x+2 & \text{if } x+2 \geqslant 0 \\ -(x+2) & \text{if } x+2 < 0 \end{cases}$$

$$= \begin{cases} x+2 & \text{if } x \geqslant -2 \\ -x-2 & \text{if } x < -2 \end{cases}$$

These expressions show that we must consider three cases:

$$x < -2 \qquad -2 \leqslant x < 3 \qquad x \geqslant 3$$

CASE I If $x < -2$, we have

$$|x-3|+|x+2| < 11$$
$$-x+3-x-2 < 11$$
$$-2x < 10$$
$$x > -5$$

CASE II If $-2 \leqslant x < 3$, the given inequality becomes

$$-x+3+x+2 < 11$$
$$5 < 11 \quad \text{(always true)}$$

CASE III If $x \geqslant 3$, the inequality becomes

$$x-3+x+2 < 11$$
$$2x < 12$$
$$x < 6$$

Combining cases I, II, and III, we see that the inequality is satisfied when $-5 < x < 6$. So the solution is the interval $(-5, 6)$. □

7) Logarithmic Function

$$\boxed{\log_a x = y \iff a^y = x}$$

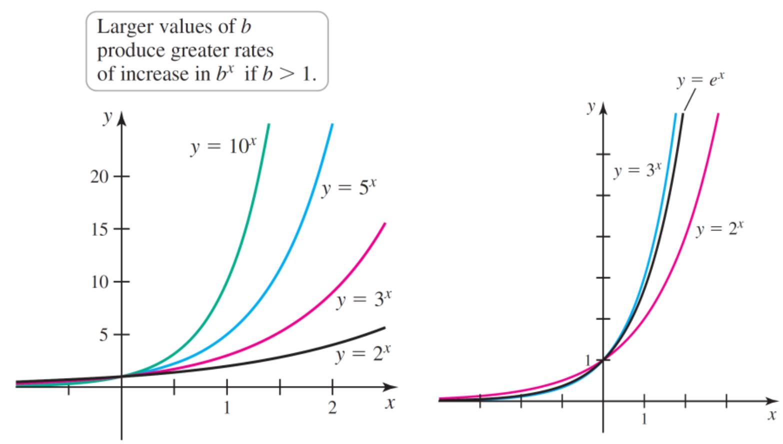
$$\boxed{\begin{array}{ll} \log_a(a^x) = x & \text{for every } x \in \mathbb{R} \\ a^{\log_a x} = x & \text{for every } x > 0 \end{array}}$$

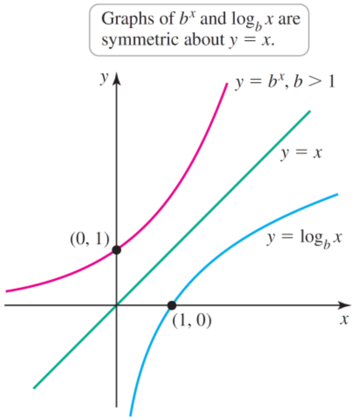
8) Graphs

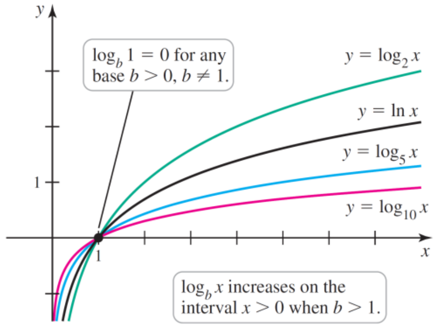
(a) Graphs of Exponential Functions and Logarithmic Function

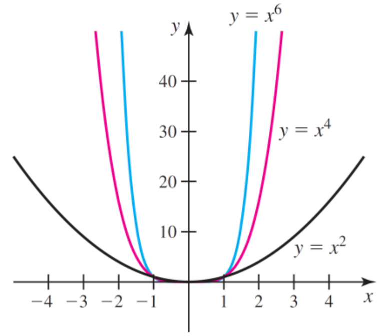
DEFINITION The Natural Exponential Function

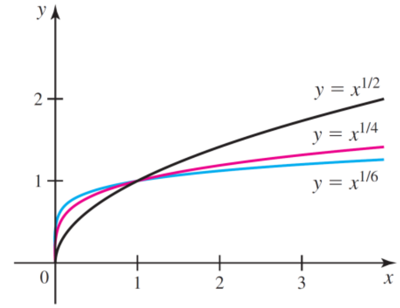
The **natural exponential function** is $f(x) = e^x$, which has the base $e = 2.718281828459\ldots$.



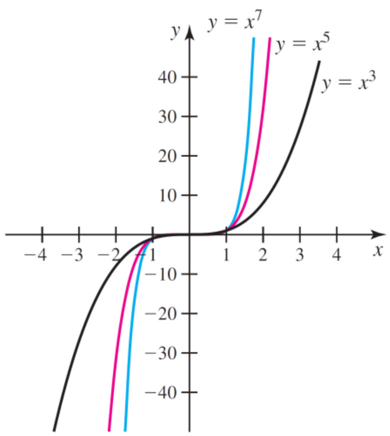


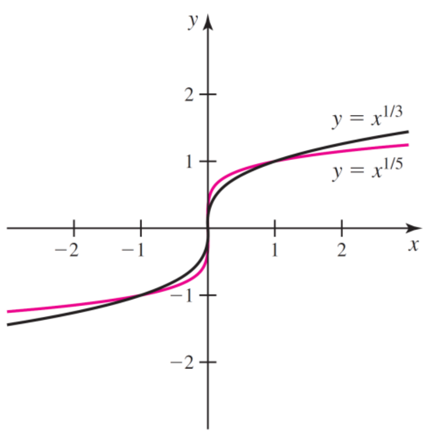






(b) Graphs of Power Functions and Root Functions





9) Change of Base Formula

For any positive number a $(a \neq 1)$, we have

$$\log_a x = \frac{\ln x}{\ln a}$$

Proof :

Let $y = log_a x$ Then $a^y = x$

Taking natural logarithms of the both sides of this equation,

We get $ylna = lnx$.

Therefore $y = \frac{lna}{lnx}$.

10) Exercises:

1) Solve the equation $|2x-1|-|x+5|=3$.

2) Solve the inequality $|x-1|-|x-3|\geq 5$.

3) Sketch the graph of the function $f(x)=\left|x^2-4|x|+3\right|$.

4) The population of a certain species in a limited environment with initial population 100 and carrying capacity 1000 is

$$P(t)=\frac{100{,}000}{100+900e^{-t}}$$

Where t is measured in years.

(a) Graph this function and estimate how long it takes for the population to reach 900.

(b) Find the inverse of this function and explain its meaning.

(c) Use the inverse function to find the time required for the population to reach 900. Compare with the result of part (a).

5) **Bacteria Population**

Under ideal conditions a certain bacteria population is known to double every three hours. Suppose that there are initially 100 bacteria.

(a) What is the size of the population after 15 hours?

(b) What is the size of the population after t hours?

(c) Estimate the size of the population after 20 hours.

(d) Graph the population function and estimate the time for the population to reach 50,000.

6) **Human Population**

Use a graphing calculator with exponential regression capability to model the population of the world with the data from 1950 to 2000 in Table 1 below. Use the exponential model to estimate the population in 1993 and to predict the population in the year 2010.

TABLE 1

Year	Population (millions)
1900	1650
1910	1750
1920	1860
1930	2070
1940	2300
1950	2560
1960	3040
1970	3710
1980	4450
1990	5280
2000	6080

Table 1 shows data for the population of the world in the 20th century

7) Give the domains of the following functions :

(a) $f(x) = x^3 - 2x^2 + 6$

(b) $f(x) = \sqrt[3]{2x^2 - 8}$

(c) $g(x) = \left|\frac{x^2-4}{x+3}\right|$

(d) $f(x) = \frac{\sqrt{3x^2-12}}{x+1}$

(e) $f(x) = \begin{cases} \frac{|x-1|}{x-1} & if \quad x \neq 1 \\ 0 & if \quad x = 1 \end{cases}$

8) $f(x) = x + \frac{1}{x}$ and $g(x) = \frac{x+1}{x+2}$.

Find the following functions :

(a) fog, (b) gof , (c) fof and (d) gog

and their domains.

9) Prove that $cos(\sin^{-1} x) = \sqrt{1 - x^2}$.

10) Simplify the following expressions :

(a) $tan(\sin^{-1} x)$ (b) $sin(\tan^{-1} x)$

(c) $cos(2\tan^{-1} x)$

11) Find the domains and the ranges of $y = f(x) = \frac{x+1}{2x+1}$ and it's inverse function $y = f^{-1}(x)$.