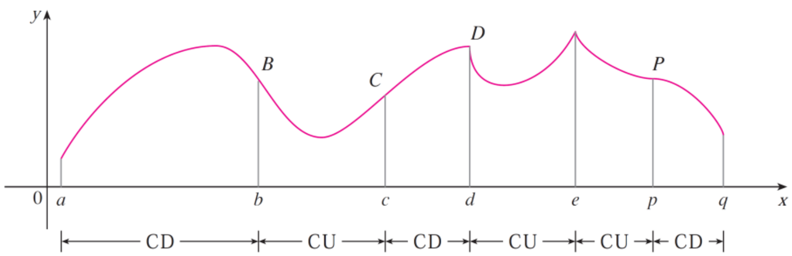
1) Conavity

Definition If the graph of f lies above all of its tangents on an interval I, then it is called **concave upward** on I. If the graph of f lies below all of its tangents on I, it is called **concave downward** on I.



2) Test for Concavity

Concavity Test

(a) If $f''(x) > 0$ for all x in I, then the graph of f is concave upward on I.

(b) If $f''(x) < 0$ for all x in I, then the graph of f is concave downward on I.

3) Point of Inflection

Definition A point P on a curve $y = f(x)$ is called an **inflection point** if f is continuous there and the curve changes from concave upward to concave downward or from concave downward to concave upward at P.

Remark: **The tangent to the curve $y = f(x)$ at the point of infection passs through the curve.**

4) Slant Asymptote

If $y = ax + b$ is a slant asymptote of the curve $y = f(x)$,

Then $a = \lim_{x\to\infty} \frac{f(x)}{x}$ **and** $b = \lim_{x\to\infty} [f(x) - ax]$.

5) Steps for sketching a Curve

Graphing Guidelines for $y = f(x)$

1. **Identify the domain or interval of interest.** On what interval should the function be graphed? It may be the domain of the function or some subset of the domain.
2. **Exploit symmetry.** Take advantage of symmetry. For example, is the function *even* $(f(-x) = f(x))$, *odd* $(f(-x) = -f(x))$, or neither?
3. **Find the first and second derivatives.** They are needed to determine extreme values, concavity, inflection points, and intervals of increase and decrease. Computing derivatives—particularly second derivatives—may not be practical, so some functions may need to be graphed without complete derivative information.
4. **Find critical points and possible inflection points.** Determine points at which $f'(x) = 0$ or f' is undefined. Determine points at which $f''(x) = 0$ or f'' is undefined.
5. **Find intervals on which the function is increasing/decreasing and concave up/down.** The first derivative determines the intervals of increase and decrease. The second derivative determines the intervals on which the function is concave up or concave down.
6. **Identify extreme values and inflection points.** Use either the First or the Second Derivative Test to classify the critical points. Both x- and y-coordinates of maxima, minima, and inflection points are needed for graphing.
7. **Locate vertical/horizontal asymptotes and determine end behavior.** Vertical asymptotes often occur at zeros of denominators. Horizontal asymptotes require examining limits as $x \to \pm\infty$; these limits determine end behavior.
8. **Find the intercepts.** The y-intercept of the graph is found by setting $x = 0$. The x-intercepts are the real zeros (or roots) of a function: those values of x that satisfy $f(x) = 0$.
9. **Choose an appropriate graphing window and make a graph.** Use the results of the above steps to graph the function. If you use graphing software, check for consistency with your analytical work. Is your graph *complete*—that is, does it show all the essential details of the function?

6) Example :

Sketch the graph of the function $f(x) = x^{2/3}(6 - x)^{1/3}$.

Solution :

Calculation of the first two derivatives gives

$$f'(x) = \frac{4 - x}{x^{1/3}(6 - x)^{2/3}} \qquad f''(x) = \frac{-8}{x^{4/3}(6 - x)^{5/3}}$$

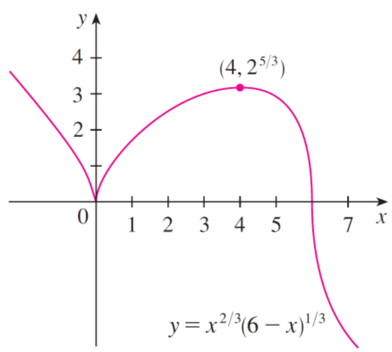
Since $f'(x) = 0$ when $x = 4$ and $f'(x)$ does not exist when $x = 0$ or $x = 6$, the critical numbers are 0, 4, and 6.

Interval	$4 - x$	$x^{1/3}$	$(6 - x)^{2/3}$	$f'(x)$	f
$x < 0$	$+$	$-$	$+$	$-$	decreasing on $(-\infty, 0)$
$0 < x < 4$	$+$	$+$	$+$	$+$	increasing on $(0, 4)$
$4 < x < 6$	$-$	$+$	$+$	$-$	decreasing on $(4, 6)$
$x > 6$	$-$	$+$	$+$	$-$	decreasing on $(6, \infty)$

To find the local extreme values we use the First Derivative Test. Since f' changes from negative to positive at 0, $f(0) = 0$ is a local minimum. Since f' changes from positive to negative at 4, $f(4) = 2^{5/3}$ is a local maximum. The sign of f' does not change at 6, so there is no minimum or maximum there. (The Second Derivative Test could be used at 4 but not at 0 or 6 since f'' does not exist at either of these numbers.)

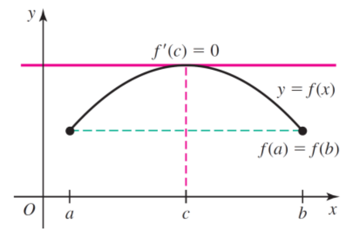
Looking at the expression for $f''(x)$ and noting that $x^{4/3} \geqslant 0$ for all x, we have $f''(x) < 0$ for $x < 0$ and for $0 < x < 6$ and $f''(x) > 0$ for $x > 6$. So f is concave downward on $(-\infty, 0)$ and $(0, 6)$ and concave upward on $(6, \infty)$, and the only inflection point is $(6, 0)$. The graph is sketched in Figure 12. Note that the curve has vertical tangents at $(0, 0)$ and $(6, 0)$ because $|f'(x)| \to \infty$ as $x \to 0$ and as $x \to 6$.

$$y = (x^2)^{1/3} \cdot \frac{6 - x}{|6 - x|}|6 - x|^{1/3}$$



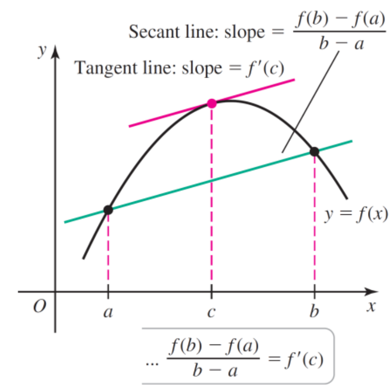
7) Mean Value Theorem

THEOREM 4.8 Rolle's Theorem
Let f be continuous on a closed interval $[a, b]$ and differentiable on (a, b) with $f(a) = f(b)$. There is at least one point c in (a, b) such that $f'(c) = 0$.



THEOREM 4.9 Mean Value Theorem
If f is continuous on the closed interval $[a, b]$ and differentiable on (a, b), then there is at least one point c in (a, b) such that

$$\frac{f(b) - f(a)}{b - a} = f'(c).$$



THEOREM 4.10 Zero Derivative Implies Constant Function
If f is differentiable and $f'(x) = 0$ at all points of an interval I, then f is a constant function on I.

Proof: Suppose $f'(x) = 0$ on $[a, b]$, where a and b are distinct points of I. By the Mean Value Theorem, there exists a point c in (a, b) such that

$$\frac{f(b) - f(a)}{b - a} = \underbrace{f'(c) = 0.}_{\substack{f'(x) = 0 \text{ for} \\ \text{all } x \text{ in } I}}$$

Multiplying both sides of this equation by $b - a \neq 0$, it follows that $f(b) = f(a)$, and this is true for every pair of points a and b in I. If $f(b) = f(a)$ for every pair of points in an interval, then f is a constant function on that interval.

Example :

(a) Prove that $\dfrac{b-a}{1+b^2} < \tan^{-1} b - \tan^{-1} a < \dfrac{b-a}{1+a^2}$ if $a < b$.

(b) Show that $\dfrac{\pi}{4} + \dfrac{3}{25} < \tan^{-1} \dfrac{4}{3} < \dfrac{\pi}{4} + \dfrac{1}{6}$.

Solution :

(a) Let $f(x) = \tan^{-1} x$. Since $f'(x) = 1/(1+x^2)$ and $f'(\xi) = 1/(1+\xi^2)$,

We have by the mean value theorem,

$$\frac{\tan^{-1} b - \tan^{-1} a}{b-a} = \frac{1}{1+\xi^2} \qquad a < \xi < b$$

Since $\xi > a$, $1/(1+\xi^2) < 1/(1+a^2)$. Since $\xi < b$, $1/(1+\xi^2) > 1/(1+b^2)$. Then

$$\frac{1}{1+b^2} < \frac{\tan^{-1} b - \tan^{-1} a}{b-a} < \frac{1}{1+a^2}$$

and the required result follows on multiplying by $b - a$.

(b) Let $b = 4/3$ and $a = 1$ in the result of part (a). Then since $\tan^{-1} 1 = \pi/4$, we have

$$\frac{3}{25} < \tan^{-1}\frac{4}{3} - \tan^{-1} 1 < \frac{1}{6} \qquad \text{or} \qquad \frac{\pi}{4} + \frac{3}{25} < \tan^{-1}\frac{4}{3} < \frac{\pi}{4} + \frac{1}{6}$$

8) Exercises:

1) **Sketch the graph of** $y = \sqrt{\dfrac{x^3}{x-a}}$, **for** $a > 0$.

2) **Sketch the graph of** $y = xe^{\frac{2}{x}} + 1$.

3) **Let** $f(x) = \dfrac{x^2+3}{\sqrt{x^2+1}}$ **be a real valued function defined on the real line.**

(a) **Show that** $\lim\limits_{x\to\infty} [f(x) - x] = 0$.

(b) **Find the local maximum and minimum of** $f(x)$.

(c) **Make use of (a) and (b) to sketch the graph** $y = f(x)$.

4) **Let** $f(x) = \dfrac{(x-1)^3}{(x+1)^2}$ **be a real valued function defined on the real line except** $x = -1$.

(a) **Prove that the straight lines** $x = -1$ **and** $y = x - 5$ **are asymptotes of the graph** $y = f(x)$.

(b) **Find, if any,**

(i) **the local maximum and minimum of** $f(x)$.

(ii) **the intersection of the graph** $y = f(x)$ **and its asymptotes.**

(c) **Sketch the graph** $y = f(x)$.

(d) **Sketch the graph** $y = |f(x)|$.

5) Let $f(x) = \sqrt[3]{x^2 - x^3}$.

(a) Show that $f'(0)$ and $f'(1)$ do not exist.

(b) Find the range of values of x such that

(i) $f'(x) > 0$, (ii) $f'(x) < 0$,

(iii) $f''(x) > 0$, (iv) $f''(x) < 0$.

(c) Find the maximum, minimum and inflexional points of the graph $y = f(x)$.

(d) Find the asymptote(s) of the graph $y = f(x)$.

(e) Sketch the graph of $y = f(x)$.

6) Given a function $f(x) = \frac{x|x|(x+7)}{x-1}$ for all $x \neq 1$.

(a) Find $f''(x)$ if $x \neq 0$ and $x \neq 1$.

(b) Show that the maximum point of the graph $y = f(x)$ is given by $(-1 - 2\sqrt{2}\ ,\ 16\sqrt{2} - 13)$ and that the minimum point is by $(-1 + 2\sqrt{2}\ ,\ 16\sqrt{2} + 13)$.

(c) Find the vertical asymptote of the graph $y = f(x)$.

(d) Show that (-l , 3) and (0 , 0) are the points of inflexion.

(e) Sketch the graph of $y = f(x)$.

7) Let $f(x) = \begin{cases} x^2(x+1)^{\frac{2}{3}} & if \quad x \geq 0 \\ -x^2(x+1)^{\frac{2}{3}} & if \quad x < 0 \end{cases}$.

(a) For $x \neq -0$ and -1 , find $f'(x)$ and $f''(x)$.

(b) Show that $f'(0) = 0$ but both $f''(0)$ and $f'(-1)$ do not exist .

(c) Show that the graph of $y = f(x)$ has extreme points at $x = -1$ and $x = \frac{-3}{4}$ and has inflexional points at $x = 0$ and $x = \frac{-15 \pm 3\sqrt{5}}{20}$.

(d) Sketch the graph of $y = f(x)$.

8) Let $f(x) = x^{\frac{2}{3}}(2x - 1)$.

(a) Show that $f'(0)$ does not exist .

(b) Find $f'(x)$ and $f''(x)$ for $x \neq 0$.

(c) Find the maximum, minimum and inflexional points of the graph $y = f(x)$.

(d) Show that the graph of $y = f(x)$ has no asymptote .

(e) Sketch the graph of $y = f(x)$.

For Questions 9 -10

a. *Determine whether the Mean Value Theorem applies to the following functions on the given interval $[a, b]$.*
b. *If so, find or approximate the point(s) that are guaranteed to exist by the Mean Value Theorem.*
c. *Make a sketch of the function and the line that passes through $(a, f(a))$ and $(b, f(b))$. Mark the points P (if they exist) at which the slope of the function equals the slope of the secant line. Then sketch the tangent line at P.*

9) $f(x) = e^x$; [0 , ln4]

10) $f(x) = x + \frac{1}{x}$; [1, 3]

11) $f(x) = \frac{x}{x+2}$; [-1, 2]

(Q 12 -16) Application of Mean Value Theorem

12)

100-m speed The Jamaican sprinter Usain Bolt set a world record of 9.58 s in the 100-m dash in the summer of 2009. Did his speed ever exceed 37 km/hr during the race? Explain.

13)

Mean Value Theorem and the police A state patrol officer saw a car start from rest at a highway on-ramp. She radioed ahead to a patrol officer 30 mi along the highway. When the car reached the location of the second officer 28 min later, it was clocked going 60 mi/hr. The driver of the car was given a ticket for exceeding the 60-mi/hr speed limit. Why can the officer conclude that the driver exceeded the speed limit?

14)

Mean Value Theorem and the police again Compare carefully to Exercise 31. A state patrol officer saw a car start from rest at a highway on-ramp. She radioed ahead to another officer 30 mi along the highway. When the car reached the location of the second officer 30 min later, it was clocked going 60 mi/hr. Can the patrol officer conclude that the driver exceeded the speed limit?

15)

Running pace Explain why if a runner completes a 6.2-mi (10-km) race in 32 min, then he must have been running at exactly 11 mi/hr at least twice in the race. Assume the runner's speed at the finish line is zero.

16)

Mean Value Theorem for quadratic functions Consider the quadratic function $f(x) = Ax^2 + Bx + C$, where A, B, and C are real numbers with $A \neq 0$. Show that when the Mean Value Theorem is applied to f on the interval $[a, b]$, the number c guaranteed by the theorem is the midpoint of the interval.