Solutions to Math 1013 (Tutorial 12)

8(a) $\int x^3(x^4+16)^6dx \overset{\text{set}}{=\!=} I$

Let $u = x^4+16$

$du = 4x^3dx$

$\therefore x^3dx = \frac{1}{4}du$

$I = \int u^6\left(\frac{du}{4}\right)$

$= \frac{1}{4}\frac{u^7}{7}+C$

$= \frac{1}{28}(x^4+16)^7+C$ #

8(b) $I = \int \frac{2dx}{x\sqrt{4x^2-1}}$, $x>2$

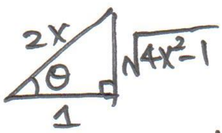
Let $x = \frac{1}{2}\sec\theta$

$\left\{\begin{array}{l}\because 1+\tan^2\theta = \sec^2\theta \\ \Rightarrow \sec^2\theta - 1 = \tan^2\theta\end{array}\right)$

$dx = \frac{1}{2}\sec\theta\tan\theta\, d\theta$

$\therefore I = \int \frac{\frac{2}{2}\sec\theta\tan\theta\, d\theta}{\frac{1}{2}\sec\theta\tan\theta}$

$= 2\int d\theta = 2\theta + C$



$x>2$

$(\because \sec\theta = 2x)$

$\therefore I = 2\sec^{-1}|2x| + C$ #

8(c) $I = \int (x+1)\sqrt{3x+2}\,dx$

Let $u = \sqrt{3x+2}$

$u^2 = 3x+2$

$2udu = 3dx$

$\therefore dx = \frac{2udu}{3}$

$x = \frac{u^2-2}{3}$

$x+1 = \frac{u^2+1}{3}$

$\therefore I = \int\left(\frac{u^2+1}{3}\right)u\left(\frac{2u}{3}\right)du$

$= \frac{2}{9}\int(u^3+u^2)du$

$= \frac{2}{9}\left(\frac{u^4}{4}\right) + \frac{2}{9}\left(\frac{u^3}{3}\right) + C$

$= \frac{1}{18}(3x+2)^2 + \frac{2}{27}(3x+2)^{\frac{3}{2}} + C$ #

8(d) $I = \int_0^{\frac{\pi}{4}} \frac{\sin x}{\cos^2 x}dx$

Let $u = \cos x$

$du = -\sin x\,dx$

$\therefore \sin x\,dx = -du$

when $x = 0$, $u = 1$

$x = \frac{\pi}{4}$, $u = \frac{\sqrt{2}}{2}$

$\therefore I = \int_1^{\frac{\sqrt{2}}{2}} \frac{-du}{u^2}$

$= -\int_1^{\frac{\sqrt{2}}{2}} u^{-2}du$

$= -\left(\frac{u^{-1}}{-1}\right)\Big|_1^{\frac{\sqrt{2}}{2}}$

$= \frac{1}{u}\Big|_1^{\frac{\sqrt{2}}{2}}$

$= \frac{2}{\sqrt{2}} - 1$

$= \frac{2-\sqrt{2}}{\sqrt{2}}$ #

8(e)

$I = \int_{-1}^{2} x^2e^{x^3+1}dx$

Set $u = x^3+1$

$du = 3x^2dx$

$\therefore x^2dx = \frac{1}{3}du$

when $x = -1$, $u = 0$

$x = 2$, $u = 9$

$\therefore I = \int_0^9 e^u du$

$= e^u\Big|_0^9$

$= e^9 - e^0 = e^9 - 1$ #

8(f) $I = \int \frac{(\sqrt{x}+1)^4}{2\sqrt{x}}dx$

Let $u = \sqrt{x}+1$

$du = \frac{dx}{2\sqrt{x}}$

$\therefore I = \int u^4 du$

$= \frac{u^5}{5} + C$

$= \frac{(\sqrt{x}+1)^5}{5} + C$

8(h) $\int_0^{\frac{\pi}{6}} \frac{\sin 2y\, dy}{\sin^2 y + 2} = I$

As $\sin 2y = 2\sin y\cos y$

Let $u = \sin^2 y + 2$

$du = 2\sin y\cos y\, dy$

when $y = 0$, $u = 2$

$y = \frac{\pi}{6}$, $u = 2 + \frac{1}{4} = \frac{9}{4}$

$\therefore I = \int_2^{\frac{9}{4}} \frac{du}{u}$

$= \ln u\Big|_2^{\frac{9}{4}} = \ln\left(\frac{9}{4}\right) - \ln 2$

$= \ln\left(\frac{9}{2\times 4}\right) = \ln\left|\frac{9}{8}\right|$ #

8(i) $\int \frac{dx}{\sqrt{1+\sqrt{1+x}}} = I$

Let $u = \sqrt{1+x}$

$u^2 = 1+x$

$2udu = dx$

$I = 2\int \frac{udu}{\sqrt{1+u}}$

let $w = \sqrt{1+u}$

$w^2 = 1+u$

$\therefore 2wdw = du$

$\Rightarrow I = 4\int \frac{(w^2-1)(wdw)}{w}$

$= 4\int(w^2-1)dw$

$= 4\left(\frac{w^3}{3}\right) - 4w + C$

$= 4\,\frac{(1+u)^{\frac{3}{2}}}{3} - 4\sqrt{1+u} + C$

$= \frac{4}{3}(1+\sqrt{1+x})^{\frac{3}{2}} - 4\sqrt{1+\sqrt{1+x}} + C$ #

Solutions to Math 1013 (Tutorial 12) Further Exercises

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11)

a. $A(-2) = \int_{-2}^{-2} f(t)\,dt = 0.$

b. $F(8) = \int_4^8 f(t)\,dt = -9.$

c. $A(4) = \int_{-2}^{4} f(t)\,dt = 8 + 17 = 25.$

d. $F(4) = \int_4^4 f(t)\,dt = 0.$

e. $A(8) = \int_{-2}^{8} f(t)\,dt = 25 - 9 = 16.$

12)

a. $A(2) = \int_0^{-2} f(t)\,dt = 8.$

b. $F(5) = \int_2^5 f(t)\,dt = -5.$

c. $A(0) = \int_0^0 f(t)\,dt = 0.$

d. $F(8) = \int_2^8 f(t)\,dt = -16.$

e. $A(8) = \int_0^8 f(t)\,dt = 8 - 16 = -8.$

f. $A(5) = \int_0^5 f(t)\,dt = 8 - 5 = 3.$

g. $F(2) = \int_2^2 f(t)\,dt = 0.$

23 $\displaystyle\int_0^1 (x^2 - 2x + 3)\,dx = \left(\frac{x^3}{3} - x^2 + 3x\right)\Big|_0^1 = \frac{1}{3} - 1 + 3 - (0 - 0 + 0) = \frac{7}{3}$. It does appear that the area is between 2 and 3.

24 $\displaystyle\int_{-\pi/4}^{7\pi/4} (\sin x + \cos x)\,dx = (-\cos x + \sin x)\Big|_{-\pi/4}^{7\pi/4} = -\sqrt{2}/2 + -\sqrt{2}/2 - (-\sqrt{2}/2 + -\sqrt{2}/2) = 0$. It does appear that the area above the axis is equal to the area below, so the net area is 0.

8 Let $\Delta x = \frac{1-0}{n} = \frac{1}{n}$. Let $x_k = 0 + k(\Delta x) = \frac{k}{n}$. Then $f(x_k) = \frac{4k}{n} - 2$. Thus,

$$\lim_{n\to\infty} \sum_{k=1}^{n} f(x_k)\Delta x = \lim_{n\to\infty} \sum_{k=1}^{n} \left(\frac{4k}{n} - 2\right)\left(\frac{1}{n}\right) = \lim_{n\to\infty} \left(\frac{4}{n^2}\sum_{k=1}^{n} k - \frac{2}{n}\sum_{k=1}^{n} 1\right) =$$

$$\lim_{n\to\infty} \left(\frac{4}{n^2}\left(\frac{n^2+n}{2}\right) - 2\right) = 2 - 2 = 0.$$

9 Let $\Delta x = \frac{2-0}{n} = \frac{2}{n}$. Let $x_k = 0 + k(\Delta x) = \frac{2k}{n}$. Then $f(x_k) = \frac{4k^2}{n^2} - 4$. Thus,

$$\lim_{n\to\infty} \sum_{k=1}^{n} f(x_k)\Delta x = \lim_{n\to\infty} \sum_{k=1}^{n} \left(\frac{4k^2}{n^2} - 4\right)\left(\frac{2}{n}\right) = \lim_{n\to\infty} \left(\frac{8}{n^3}\sum_{k=1}^{n} k^2 - \frac{8}{n}\sum_{k=1}^{n} 1\right) =$$

$$\lim_{n\to\infty} \left(\frac{8}{n^3}\left(\frac{n(n+1)(2n+1)}{6}\right) - 8\right) = \frac{8}{3} - 8 = \frac{-16}{3}.$$

10 Let $\Delta x = \frac{2-1}{n} = \frac{1}{n}$. Let $x_k = 1 + k(\Delta x) = 1 + \frac{k}{n} = \frac{n+k}{n}$. Then $f(x_k) = 3\left(\frac{(n+k)^2}{n^2}\right) + \frac{n+k}{n}$. Thus,

$$\lim_{n\to\infty} \sum_{k=1}^{n} f(x_k)\Delta x = \lim_{n\to\infty} \sum_{k=1}^{n} \left(3\left(\frac{(n+k)^2}{n^2}\right) + \frac{n+k}{n}\right)\left(\frac{1}{n}\right) = \lim_{n\to\infty} \sum_{k=1}^{n} \left(\frac{3}{n} + \frac{6k}{n^2} + \frac{3k^2}{n^3} + \frac{1}{n} + \frac{k}{n^2}\right) =$$

$$\lim_{n\to\infty} \left(\frac{4}{n}\sum_{k=1}^{n} 1 + \frac{7}{n^2}\sum_{k=1}^{n} k + \frac{3}{n^3}\sum_{k=1}^{n} k^2\right) = \lim_{n\to\infty} \left(4 + \frac{7}{2}\cdot\frac{n^2+n}{n^2} + \frac{3}{6}\cdot\frac{n(n+1)(2n+1)}{n^3}\right) = 4 + \frac{7}{2} + 1 = 8.5$$

11 Let $\Delta x = \frac{4-0}{n} = \frac{4}{n}$. Let $x_k = 1 + k(\Delta x) = 0 + \frac{4k}{n}$. Then $f(x_k) = \frac{64k^3}{n^3} - \frac{4k}{n}$. Thus,

$$\lim_{n\to\infty} \sum_{k=1}^{n} f(x_k)\Delta x = \lim_{n\to\infty} \sum_{k=1}^{n} \left(\frac{64k^3}{n^3} - \frac{4k}{n}\right)\cdot\frac{4}{n} = \lim_{n\to\infty} \left(\frac{256}{n^4}\sum_{k=1}^{n} k^3 - \frac{16}{n^2}\sum_{k=1}^{n} k\right) =$$

$$\lim_{n\to\infty} \left(\frac{256}{n^4}\cdot\frac{n^2(n+1)^2}{4} - \frac{16}{n^2}\cdot\frac{n(n+1)}{2}\right) = 64 - 8 = 56.$$

15 $\displaystyle\int_{-2}^{2} (3x^4 - 2x + 1)\,dx = \left(\frac{3x^5}{5} - x^2 + x\right)\Big|_{-2}^{2} = \frac{96}{5} - 4 + 2 - \left(\frac{-96}{5} - 4 - 2\right) = \frac{192}{5} + 4 = \frac{212}{5}.$

16 $\displaystyle\int \cos(3x)\,dx = \frac{\sin 3x}{3} + C.$

Solutions to Math 1013 (Tutorial 12) Further Exercises

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17 $\int_0^2 (x+1)^3\,dx = \left(\frac{(x+1)^4}{4}\right)\Big|_0^2 = \frac{81}{4} - \frac{1}{4} = 20.$

18 $\int_0^1 (4x^{21} - 2x^{16} + 1)\,dx = \left(\frac{4x^{22}}{22} - \frac{2x^{17}}{17} + x\right)\Big|_0^1 = \frac{2}{11} - \frac{2}{17} + 1 = \frac{199}{187}.$

19 $\int (9x^8 - 7x^6)\,dx = x^9 - x^7 + C.$

20 $\int_{-2}^2 e^{4x+8}\,dx = \left(\frac{1}{4}\cdot e^{4x+8}\right)\Big|_{-2}^2 = \frac{1}{4}(e^{16} - 1).$

21 $\int_0^1 (x + \sqrt{x})\,dx = \left(\frac{x^2}{2} + \frac{2x^{3/2}}{3}\right)\Big|_0^1 = \frac{1}{2} + \frac{2}{3} = \frac{7}{6}.$

22 Let $u = x^3 + 27$, and note that $du = 3x^2\,dx$. Substituting yields $\frac{1}{3}\int \frac{1}{u}\,du = \frac{1}{3}\ln|u| + C = \frac{\ln|x^3+27|}{3} + C.$

23 $\frac{1}{2}\int_0^1 \frac{dx}{\sqrt{1-(x/2)^2}} = \left(\sin^{-1}\left(\frac{x}{2}\right)\right)\Big|_0^1 = \frac{\pi}{6}.$

24 Let $u = 3y^3+1$, and note that $du = 9y^2\,dy$. Substituting yields $\frac{1}{9}\int u^4\,du = \frac{u^5}{45} + C = \frac{(3y^3+1)^5}{45} + C.$

25 Let $u = 25 - x^2$, and note that $du = -2x\,dx$. Substituting yields $\frac{-1}{2}\int_{25}^{16} u^{-1/2}\,du = -\sqrt{u}\big|_{25}^{16} = 5 - 4 = 1.$

26 Let $u = \cos x^2$ and note that $du = -\sin x^2 \cdot 2x\,dx$. Substituting yields $\frac{-1}{2}\int u^8\,du = \frac{-u^9}{18} + C = \frac{-\cos^9 x^2}{18} + C.$

27 $\int \sin^2(5\theta)\,d\theta = \int \frac{1-\cos(10\theta)}{2}\,d\theta = \frac{\theta}{2} - \frac{\sin(10\theta)}{20} + C.$

28 $\int_0^\pi (1 - \cos^2(3\theta))\,d\theta = \int_0^\pi \sin^2(3\theta)\,d\theta = \int_0^\pi \frac{1-\cos(6\theta)}{2}\,d\theta = \left(\frac{\theta}{2} - \frac{\sin(6\theta)}{12}\right)\Big|_0^\pi = \frac{\pi}{2}.$

29 Let $u = x^3 + 3x^2 - 6x$, and note that $du = 3x^2 + 6x - 6\,dx = 3(x^2+2x-2)\,dx$. Substituting yields $\frac{1}{3}\int \frac{1}{u}\,du = \frac{1}{3}\ln|u| + C = \frac{\ln|x^3+3x^2-6x|}{3} + C.$

30 Let $u = e^x$ so that $du = e^x\,dx$. Substituting yields $\int_1^2 \frac{1}{1+u^2}\,du = \left(\tan^{-1}(u)\right)\big|_1^2 = \tan^{-1}(2) - \frac{\pi}{4}.$

38)

a. $\int_a^c f(x)\,dx = 20 - 12 = 8.$

b. $\int_b^d f(x)\,dx = 15 - 12 = 3.$

c. $2\int_c^b f(x)\,dx = -2\int_b^x f(x)\,dx = -2(-12) = 24.$

d. $4\int_a^d f(x)\,dx = 80 - 48 + 60 = 92.$

e. $3\int_a^b f(x)\,dx = 3(20) = 60.$

f. $2\int_b^d f(x)\,dx = 2(15 - 12) = 6.$

Solutions to Math 1013 (Tutorial 12) Further Exercises

56 Let $u = 1 + \cos^2 x$. Then $du = -2\sin x \cos x\, dx$. Substituting yields $-\int \frac{1}{u}\, du = -\log|u| + C = -\log|1 + \cos^2 x| + C$.

57 Let $u = \frac{1}{x}$. Then $du = \frac{-1}{x^2}\, dx$. Substituting yields $-\int \sin u\, du = \cos u + C = \cos\left(\frac{1}{x}\right) + C$.

58 Let $u = \tan^{-1}(x)$. Then $du = \frac{1}{1+x^2}\, dx$. Substituting yields $\int u^5\, du = \frac{u^6}{6} + C = \frac{(\tan^{-1}(x))^6}{6} + C$.

59 Let $u = \tan^{-1}(x)$. Then $du = \frac{1}{1+x^2}\, dx$. Substituting yields $\int \frac{1}{u}\, du = \ln|u| + C = \ln|\tan^{-1}(x)| + C$.

60 Let $u = \sin^{-1}(x)$. Then $du = \frac{1}{\sqrt{1-x^2}}\, dx$. Substituting yields $\int u\, du = \frac{u^2}{2} + C = \frac{(\sin^{-1}(x))^2}{2} + C$.

61 Let $u = e^x + e^{-x}$. Then $du = (e^x - e^{-x})\, dx$. Substituting yields $\int \frac{1}{u}\, du = \ln|u| + C = \ln|e^x + e^{-x}| + C$.