1) Solve the equation $|2x-1|-|x+5|=3$.

Key:

$$|2x-1| = \begin{cases} 2x-1 & \text{if } x \geq \frac{1}{2} \\ 1-2x & \text{if } x < \frac{1}{2} \end{cases} \quad \text{and}$$

$$|x+5| = \begin{cases} x+5 & \text{if } x \geq -5 \\ -x-5 & \text{if } x < -5 \end{cases}$$

Therefore, we consider the three cases

$x < -5$, $-5 \leq x < \frac{1}{2}$, and $x \geq \frac{1}{2}$.

If $x < -5$, we must have $1-2x-(-x-5)=3 \quad \Leftrightarrow \quad x=3$,

which is false, since we are considering $x < -5$.

If $-5 \leq x < \frac{1}{2}$, we must have $1-2x-(x+5)=3$

$\Leftrightarrow \quad x = -\frac{7}{3}$.

If $x \geq \frac{1}{2}$, we must have $2x-1-(x+5)=3 \quad \Leftrightarrow \quad x=9$.

So the two solutions of the equation are $x=-\frac{7}{3}$ and $x=9$.

2) Solve the inequality $|x-1|-|x-3| \geq 5$.

Key: we first divide the real line into 3 part :

$x < 1$, $1 \leq x < 3$ *and* $x \geq 3$.

Case 1 : $x > 1$, for $|x-1|-|x-3| \geq 5$

$\Rightarrow -(x-1)-\{-(x-3)\} \geq 5$

$\Rightarrow 1-x+x-3 \geq 5 \quad \Rightarrow 4 \geq 5$ **(impossible)**

Case 2 : $1 \leq x < 3$,

for $|x-1|-|x-3| \geq 5$

$\Rightarrow (x-1)-\{-(x-3)\} \geq 5$

$\Rightarrow x-1+x-3 \geq 5 \Rightarrow x \geq \frac{9}{2} > 3$ **(impossible)**

Case 3 : $x \geq 3$, for $|x-1|-|x-3| \geq 5$

$\Rightarrow (x-1)-(x-3) \geq 5$

$\Rightarrow 1-x-x+3 \geq 5 \Rightarrow x \leq \frac{-1}{2}$ **(impossible)**

Therefore there is no solution for the inequality.

3) Sketch the graph of the function $f(x)=\left|x^2-4|x|+3\right|$.

Key:

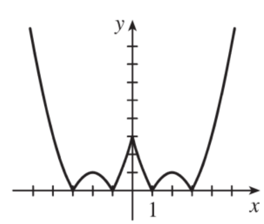
$f(x)=\left|x^2-4|x|+3\right|$. If $x \geq 0$, then $f(x)=\left|x^2-4x+3\right|=|(x-1)(x-3)|$.

Case (i): If $0 < x \leq 1$, then $f(x)=x^2-4x+3$.

Case (ii): If $1 < x \leq 3$, then $f(x)=-(x^2-4x+3)=-x^2+4x-3$.

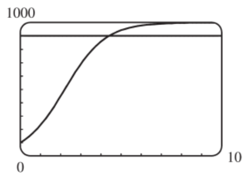
Case (iii): If $x > 3$, then $f(x)=x^2-4x+3$.

This enables us to sketch the graph for $x \geq 0$. Then we use the fact that f is an even function to reflect this part of the graph about the y-axis to obtain the entire graph. Or, we could consider also the cases $x < -3$, $-3 \leq x < -1$, and $-1 \leq x < 0$.



Key:

(a)



The population would reach 900 in about 4.4 years.

(b)

$$P = \frac{100{,}000}{100 + 900e^{-t}} \Rightarrow \quad 100P + 900Pe^{-t} = 100{,}000$$

$$\Rightarrow \quad 900Pe^{-t} = 100{,}000 - 100P \quad \Rightarrow$$

$$e^{-t} = \frac{100{,}000 - 100P}{900P} \Rightarrow \quad -t = \ln\left(\frac{1000 - P}{9P}\right)$$

$$\Rightarrow \quad t = -\ln\left(\frac{1000 - P}{9P}\right), \text{ or } \ln\left(\frac{9P}{1000 - P}\right);$$

This is the time required for the population to reach a given number P.

(c) $P = 900 \quad \Rightarrow \quad t = \ln\left(\frac{9 \cdot 900}{1000 - 900}\right) = \ln 81 \approx 4.4$ years, as in part (a).

5) Under ideal conditions a certain bacteria population is known to double every three hours. Suppose that there are initially 100 bacteria.

(a) What is the size of the population after 15 hours?

(b) What is the size of the population after t hours?

(c) Estimate the size of the population after 20 hours.

(d) Graph the population function and estimate the time for the population to reach 50,000.

Key:

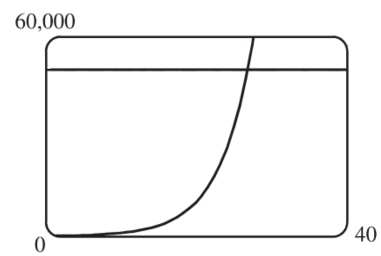
(a) Fifteen hours represents 5 doubling periods (one doubling period is three hours).

$$100 \cdot 2^5 = 3200$$

(b) In t hours, there will be $t/3$ doubling periods. The initial population is 100, so the population y at time t is $y = 100 \cdot 2^{t/3}$.

(c) $t = 20 \quad \Rightarrow \quad y = 100 \cdot 2^{20/3} \approx 10{,}159$

(d) We graph $y_1 = 100 \cdot 2^{x/3}$ and $y_2 = 50{,}000$. The two curves intersect at $x \approx 26.9$, so the population reaches 50,000 in about 26.9 hours.



6) Human Population

Use a graphing calculator with exponential regression capability to model the population of the world with the data from 1950 to 2000 in Table 1 below. Use the exponential

model to estimate the population in 1993 and to predict the population in the year 2010.

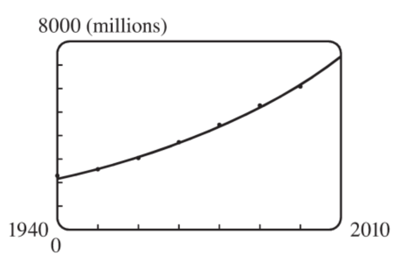
TABLE I

Year	Population (millions)
1900	1650
1910	1750
1920	1860
1930	2070
1940	2300
1950	2560
1960	3040
1970	3710
1980	4450
1990	5280
2000	6080

Table 1 shows data for the population of the world in the 20th century

Key:

An exponential model is $y = ab^t$, where $a = 3.154832569 \times 10^{-12}$ and $b = 1.017764706$. This model gives $y(1993) \approx 5498$ million and $y(2010) \approx 7417$ million.



7) (a) The domain is the whole real line.

(b) $2x^2 - 8 \geq 0 \Rightarrow (x+2)(x-2) \geq 0$

The domain = $(-\infty, -2] \cup [2, \infty)$.

(c) The domain is $R\backslash\{3\}$.

(d) $x \neq 1$ *and* $3x^2 - 12 \geq 0, \Rightarrow (x+2)(x-2) \geq 0$,

The domain is $(-\infty, -2] \cup [2, \infty)$.

(e) The domain is the whole real line.

8) $f(x) = x + \frac{1}{x}$ **and** $g(x) = \frac{x+1}{x+2}$

Find the functions (a) fog, **(b)** gof ,

(c) fof , **and (d)** gog **and their domains.**

Key:

$$f(x) = x + \frac{1}{x},\ D = \{x \mid x \neq 0\};\quad g(x) = \frac{x+1}{x+2},\ D = \{x \mid x \neq -2\}$$

(a) $$(f \circ g)(x) = f(g(x)) = f\left(\frac{x+1}{x+2}\right) = \frac{x+1}{x+2} + \frac{1}{\frac{x+1}{x+2}} = \frac{x+1}{x+2} + \frac{x+2}{x+1}$$

$$= \frac{(x+1)(x+1) + (x+2)(x+2)}{(x+2)(x+1)} = \frac{(x^2+2x+1) + (x^2+4x+4)}{(x+2)(x+1)} = \frac{2x^2+6x+5}{(x+2)(x+1)}$$

Since $g(x)$ is not defined for $x = -2$ and $f(g(x))$ is not defined for $x = -2$ and $x = -1$, the domain of $(f \circ g)(x)$ is $D = \{x \mid x \neq -2, -1\}$.

(b) $$(g \circ f)(x) = g(f(x)) = g\left(x + \frac{1}{x}\right) = \frac{\left(x + \frac{1}{x}\right) + 1}{\left(x + \frac{1}{x}\right) + 2}$$

$$= \frac{\frac{x^2+1+x}{x}}{\frac{x^2+1+2x}{x}} = \frac{x^2+x+1}{x^2+2x+1} = \frac{x^2+x+1}{(x+1)^2}$$

Since $f(x)$ is not defined for $x=0$ and $g(f(x))$ is not defined for $x=-1$, the domain of $(g\circ f)(x)$ is $D=\{x \mid x \neq -1, 0\}$.

(c) $(f\circ f)(x) = f(f(x)) = f\left(x+\frac{1}{x}\right) = \left(x+\frac{1}{x}\right) + \frac{1}{x+\frac{1}{x}}$

$$= x + \frac{1}{x} + \frac{1}{\frac{x^2+1}{x}} = x + \frac{1}{x} + \frac{x}{x^2+1}$$

$$= \frac{x(x)(x^2+1)+1\,(x^2+1)+x(x)}{x(x^2+1)} = \frac{x^4+x^2+x^2+1+x^2}{x(x^2+1)}$$

$$= \frac{x^4+3x^2+1}{x(x^2+1)}, \quad D=\{x \mid x\neq 0\}$$

(d) $(g\circ g)(x) = g(g(x)) = g\left(\frac{x+1}{x+2}\right) = \dfrac{\frac{x+1}{x+2}+1}{\frac{x+1}{x+2}+2}$

$$= \frac{\frac{x+1+1(x+2)}{x+2}}{\frac{x+1+2(x+2)}{x+2}} = \frac{x+1+x+2}{x+1+2x+4} = \frac{2x+3}{3x+5}$$

Since $g(x)$ is not defined for $x=-2$ and $g(g(x))$ is not defined for $x=-\frac{5}{3}$, the domain of $(g\circ g)(x)$ is $D=\left\{x \mid x\neq -2, -\frac{5}{3}\right\}$.

9) Prove that $cos(\sin^{-1}x) = \sqrt{1-x^2}$.

Key : Let $\theta = \sin^{-1}x$, $\sin\theta = \sin(\sin^{-1}x) = x$,

$cos\theta = \sqrt{1-x^2}$.

$\therefore \quad cos(\sin^{-1}x) = \sqrt{1-x^2}$.

10) Simplify the following expressions :

(a) $tan(\sin^{-1}x)$ (b) $sin(\tan^{-1}x)$

(c) $cos(2\tan^{-1}x)$

Key : (a) Let $\theta = \sin^{-1}x$, $\sin\theta = \sin(\sin^{-1}x) = x$,

$tan\theta = \frac{x}{\sqrt{1-x^2}}$.

$\therefore\ tan(\sin^{-1}x) = tan\theta = \frac{x}{\sqrt{1-x^2}}$.

(b) Let $= \tan^{-1}x$, $\tan\theta = x$, $\sin\theta = \frac{x}{\sqrt{1+x^2}}$.

$\therefore sin(\tan^{-1}x) = \frac{x}{\sqrt{1+x^2}}$.

(c) Let $= \tan^{-1}x$, , $\tan\theta = x$, $\sin\theta = \frac{x}{\sqrt{1+x^2}}$

$\therefore cos(2\tan^{-1}x) = \cos 2\theta = 1 - 2sin^2\theta$

$= 1 - 2\frac{x^2}{1+x^2} = \frac{1-x^2}{1+x^2}$.

11) Find the domains and the ranges of $y = f(x) = \frac{x+1}{2x+1}$ and it's inverse function $y = f^{-1}(x)$.

Key: The domain of the function $y = f(x) = \frac{x+1}{2x+1}$ is $R\backslash\left\{\frac{-1}{2}\right\}$.

Interchange x and y, we have :

$$x = \frac{y+1}{2y+1} \Rightarrow y = \frac{1-x}{2x-1} = f^{-1}(x)$$

Thus, the domain of $f^{-1}(x)$ is $R\backslash\left\{\frac{1}{2}\right\}$.