Solutions to Math 1014 (Tutorial 10)

1) $f(x)=\sin x+x-1$

$x_0=1.5$

$x_{n+1}=x_n-\frac{f(x_n)}{f'(x_n)}$

$f'(x)=\cos x+1$

$f'(x_n)=\cos x_n+1$

$\therefore x_{n+1}=x_n-\frac{\sin x_n+x_n-1}{\cos x_n+1}$.

$\therefore$ The root ≈ 0.510973 #

n	x_n
0	1.5
1	0.101436
2	0.501114
3	0.510961
4	0.510973
5	0.510973
6	0.510973
7	0.510973
8	0.510973
9	0.510973
10	0.510973

2) $f(x)=\ln(x+1)-1$

$x_0=1.7$

$x_{n+1}=x_n-\frac{f(x_n)}{f'(x_n)}$

Because $f'(x_n)=\frac{1}{x_n+1}$, we have $x_{n+1}=x_n-(\ln(x_n+1)-1)(x_n+1)=1+2x_n-(x_n+1)\ln(x_n+1)$.

$\therefore$ The root is about 1.71828 #

n	x_n
0	1.7
1	1.71822
2	1.71828
3	1.71828
4	1.71828
5	1.71828
6	1.71828
7	1.71828
8	1.71828
9	1.71828
10	1.71828

3) A preliminary sketch of the two curves seems to indicate that they intersect once, near $x=1.5$. Let $f(x)=x^3-(x^2+1)$. Then $f'(x_n)=3x_n^2-2x_n$. The Newton's method formula becomes

$$x_{n+1}=x_n-\frac{x_n^3-x_n^2-1}{3x_n^2-2x_n}.$$

If we use an initial estimate of $x_0=1.5$, we obtain $x_1=1.46667$, $x_2=1.46557$, $x_3=1.46557$, so there appears to be a point of intersection near $x=1.46557$.

4) A preliminary sketch of the two curves seems to indicate that they intersect twice on the given interval, once just to the right of 0, and once between 2 and 2.5.

Let $f(x)=4\sqrt{x}-(x^2+1)$. Then $f'(x_n)=2/\sqrt{x_n}-2x_n$. The Newton's method formula becomes

$$x_{n+1}=x_n-\frac{4\sqrt{x_n}-(x_n^2+1)}{2/\sqrt{x_n}-2x_n}.$$

If we use an initial estimate of $x_0=.1$, we obtain $x_1=.0583788$, $x_2=.0629053$, $x_3=.0629971$, so there appears to be a point of intersection near $x=.06299$.

If we use an initial estimate of $x_0=2.25$, we obtain $x_1=2.23026$, $x_2=2.23012$, $x_3=2.23012$, so there appears to be a point of intersection near $x=2.23012$.

5) A preliminary sketch of the two curves seems to indicate that they intersect twice on the given interval, once just to the right of 0, and once between 1 and 1.5

Let $f(x)=\ln x-(x^3-2)$. Then $f'(x_n)=1/x_n-3x_n^2$. The Newton's method formula becomes

$$x_{n+1}=x_n-\frac{\ln(x_n)-(x_n^3-2)}{1/x_n-3x_n^2}.$$

If we use an initial estimate of $x_0=.1$, we obtain $x_1=.13045$, $x_2=.13557$, $x_3=.135674$, so there appears to be a point of intersection near $x=.13567$.

If we use an initial estimate of $x_0=1.4$, we obtain $x_1=1.32111$, $x_2=1.31501$, $x_3=1.31498$, and $x_4=1.31498$ so there appears to be a point of intersection near $x=1.31498$.

6) Residual $=f(x_n)$

Because the residuals become small quickly, the convergence of x_n is quite slow. This is related to the extreme flatness of the graph of x^{10} between 0 and 1/2.

n	x_n	Error	Residual
0	.5	.5	.000976563
1	.45	.45	.000340506
2	.405	.405	.000118727
3	.3645	.3645	.0000413976
4	.32805	.32805	.0000144345
5	.295245	.295245	$5.0329x\times10^{-6}$
6	.265721	.265721	1.75489×10^{-6}
7	.239148	.239148	6.11893×10^{-7}
8	.215234	.215234	2.13354×10^{-7}
9	.19371	.19371	7.43919×10^{-8}
10	.174339	.174339	2.59389×10^{-8}

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7)

a. If r is a root of $x^2 - a$, then $r^2 - a = 0$, so $r^2 = a$, and $|r| = \sqrt{a}$, so either $r = \sqrt{a}$ or $r = -\sqrt{a}$. If we also insist that $r > 0$, then $r = \sqrt{a}$.

b. The Newton's method recursion is

$$x_{n+1} = x_n - \frac{x_n^2 - a}{2x_n} = \frac{2x_n^2 - (x_n^2 - a)}{2x_n} = \frac{x_n^2 + a}{2x_n} = \frac{1}{2}\left(x_n + \frac{a}{x_n}\right).$$

c. Because $3^2 = 9 < 13$ and $4^2 = 16 > 13$, a good starting value for $\sqrt{13}$ would be a number between 3 and 4 (but closer to 4), like 3.6.

Because $8^2 = 64 < 73$ and $9^2 = 81 > 73$, a good starting value for $\sqrt{73}$ would be a number between 8 and 9, like 8.5.

d. The first chart is for $\sqrt{13}$ and the second is for $\sqrt{73}$.

n	x_n
0	3.6
1	3.605555555556
2	3.60555127547
3	3.6055127546
4	3.6055127546
5	3.6055127546
6	3.6055127546
7	3.6055127546
8	3.6055127546
9	3.6055127546
10	3.6055127546

n	x_n
0	8.5
1	8.54411764706
2	8.54400374608
3	8.54400374532
4	8.54400374532
5	8.54400374532
6	8.54400374532
7	8.54400374532
8	8.54400374532
9	8.54400374532
10	8.54400374532

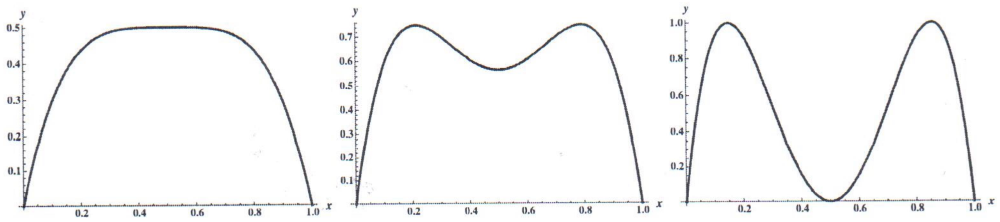
Solutions to Math 1014 (Tutorial 10)

8)

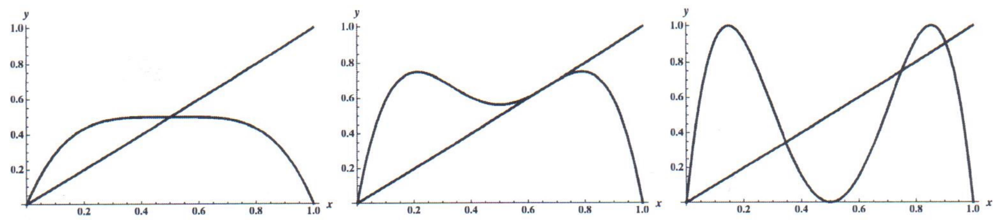
a. We are seeking solutions of $f(x) = ax(1-x) = x$. This can be written as $ax^2 + x(1-a) = 0$, or $x(ax + (1-a)) = 0$. The solutions of this equation are $x = 0$ and $x = \frac{a-1}{a}$. If $0 < a < 1$, this does not give a value of x in the range $(0, 1)$. If $1 \leq a \leq 4$, we do get a fixed point $x = \frac{a-1}{a}$.

b. $g(x) = f(f(x)) = f(ax(1-x)) = a(ax(1-x))(1-ax(1-x)) = (a^2x - a^2x^2)(1 - ax + ax^2) = a^2x - a^2x^2 - a^3x^2 + a^3x^3 + a^3x^3 - a^3x^4 = a^2x - a^2x^2 - a^3x^2 + 2a^3x^3 - a^3x^4$. This is a fourth degree polynomial.

c. From left to right, with $a = 2$, then $a = 3$, then $a = 4$:



d. The graphs of $y = g(x)$ together with $y = x$ for $a = 2$, then $a = 3$, then $a = 4$.



When $a = 2$, we have $g(x) = -8x^4 + 16x^3 - 12x^2 + 4x$, so we are looking for a root of $g(x) - x = -8x^4 + 16x^3 - 12x^2 + 4x - x = -8x^4 + 16x^3 - 12x^2 + 3x = x(-8x^3 + 16x^2 - 12x + 3)$. Clearly $x = 0$ is one root, and the diagram indicates that $g(x) = x$ near $x = .5$. A quick check shows that $x = .5$ is a root of $g(x) - x$, so $.5$ is a fixed point of g.

When $a = 3$, we have $g(x) = -27x^4 + 54x^3 - 36x^2 + 9x$, so we are looking for a root of $g(x) - x = -27x^4 + 54x^3 - 36x^2 + 9x - x = -27x^4 + 54x^3 - 36x^2 + 8x = x(-27x^3 + 54x^2 - 36x + 8)$. Clearly $x = 0$ is one root, and the diagram indicates that $g(x) = x$ near $x = .6$. Applying Newton's method to $g(x) - x$ with an initial estimate of $.6$ yields a root of approximately $.\overline{6} = 2/3$. A quick check shows that $2/3$ is a fixed point of g.

When $a = 4$, we have $g(x) = -64x^4 + 128x^3 - 80x^2 + 16x$, so we are looking for a root of $g(x) - x = -64x^4 + 128x^3 - 80x^2 + 16x - x = -64x^4 + 128x^3 - 80x^2 + 15x = x(-64x^3 + 128x^2 - 80x + 15)$. Clearly $x = 0$ is one root, and the diagram indicates that $g(x) = x$ near $x = .3$ and $x = .75$ and $x = .9$. Checking the value of $.75 = 3/4$, we confirm that $g(3/4) = 3/4$. Applying Newton's method to $g(x) - x$ with an initial estimate of $.3$ yields a root of approximately $.345492$, and applying it with an initial estimate of $.9$ yields a root of approximately $.904508$.

Thus the fixed points of g with $a = 4$ are 0, $.345492$, $.75$, and $.904508$.