Solutions Math 1013 (Tutorial 11)

(6) P.5

$f(x) = \dfrac{x|x|(x+7)}{x-1}$ for all $x \neq 1$

(a) For $x > 0$,

$$f(x) = \frac{x^2(x+7)}{x-1} = \frac{x^3+7x^2}{x-1}$$

$$f'(x) = \frac{(3x^2+14x)(x-1)-(x^3+7x^2)(1)}{(x-1)^2}$$

$$= \frac{2x^3+4x^2-14x}{(x-1)^2}$$

$$f''(x) = \frac{(6x^2+8x-14)(x-1)^2-(2x^3+4x^2-14x)2(x-1)}{(x-1)^4}$$

$$= \frac{2x^3-6x^2+6x+14}{(x-1)^3}$$

For $x < 0$,

$$f(x) = \frac{-x^2(x+7)}{x-1} = \frac{-x^3-7x^2}{x-1}$$

$$f'(x) = \frac{(-3x^2-14x)(x-1)-(-x^3-7x^2)(1)}{(x-1)^2}$$

$$= \frac{-2x^3-4x^2+14x}{(x-1)^2}$$

$$f''(x) = \frac{-2x^3+6x^2-6x-14}{(x-1)^3}$$

$$= \frac{|x|}{x}\left[\frac{2x^3-6x^2+6x+14}{(x-1)^3}\right] \quad \text{for } x \neq 0, 1$$

(b) $f'(x) = 0,\ 2x^3+4x^2-14x = 0$

$x = 0$ or $x^2+2x-7 = 0$

$$x = \frac{-2 \pm \sqrt{4+28}}{2} = -1 \pm 2\sqrt{2}$$

At $x = 0$ rejected for $f''(x)$ definition.

At $x = -1-2\sqrt{2}$, $x = -1+2\sqrt{2}$ by sign test

For $x = -1-2\sqrt{2}$, $f''(x) < 0$

$x = -1+2\sqrt{2}$, $f''(x) > 0$

For $x = -1-2\sqrt{2}$,

$$y = \frac{-(-1-2\sqrt{2})^3-7(-1-2\sqrt{2})^2}{(-1-2\sqrt{2}-1)}$$

$$= \frac{(1+2\sqrt{2})^2(2\sqrt{2}-6)}{-2(1+\sqrt{2})}$$

$$= \frac{(9+4\sqrt{2})(2\sqrt{2}-6)(\sqrt{2}-1)}{-2(2-1)}$$

$$= \frac{(-38-6\sqrt{2})(\sqrt{2}-1)}{-2} = -13+16\sqrt{2}$$

For $x = -1+2\sqrt{2}$,

$$y = \frac{(-1+2\sqrt{2})^3+7(-1+2\sqrt{2})^2}{(-1+2\sqrt{2}-1)} = 13+16\sqrt{2}$$

So $(-1-2\sqrt{2}, 16\sqrt{2}-13)$ maximum point

$(-1+2\sqrt{2}, 16\sqrt{2}+13)$ minimum point

For $-1+2\sqrt{2} > x > 0$, $f'(x) < 0$.

For $-1-2\sqrt{2} < x < 0$, $f'(x) < 0$.

$\therefore$ $(0, 0)$ is neither max nor min.

(c) $x = 1$ vertical asymptote

(d) For $f''(x) = 0$ for $x \neq 0$ case

$$2x^3-6x^2+6x+14 = 0$$

$$2(x+1)(x^2-4x+7) = 0$$

$x = -1$ and $x^2-4x+7 > 0$ for all x.

$f''(x)$ changes sign as x increases through -1.

$\therefore$ $(-1, 3)$ is a point of inflexion.

For $0 < x < 1$,

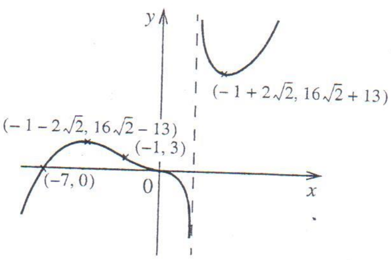
$$f''(x) = \frac{2(x+1)(x^2-4x+7)}{(x-1)^3} < 0$$

For $-1 < x < 0$,

$$f''(x) = \frac{-2(x+1)(x^2-4x+7)}{(x-1)^3} > 0$$

$\therefore$ $(0, 0)$ is another point of inflexion.

(e) The sketch of graph is



Solutions Math 1013 (Tutorial 11)

7) $f(x) = \begin{cases} x^2(x+1)^{\frac{2}{3}} & \text{if } x \geq 0 \\ -x^2(x+1)^{\frac{2}{3}} & \text{if } x < 0 \end{cases}$

(a) $f'(x) = 2x(x+1)^{\frac{2}{3}} + \frac{2x^2}{3}(x+1)^{\frac{-1}{3}}$ for $x > 0$

$= \frac{2x}{(x+1)^{1/3}}[x + 1 + \frac{x}{3}]$

$= \frac{2x(4x+3)}{3(x+1)^{1/3}}$ for $x > 0$

$f'(x) = \frac{-2x(4x+3)}{3(x+1)^{1/3}}$ for $x < 0$ and $x \neq -1$

$f''(x)$

$= \frac{2}{3} \cdot \frac{(x+1)^{\frac{1}{3}}(8x+3) - (4x^2+3x) \cdot \frac{1}{3}(x+1)^{-\frac{2}{3}}}{(x+1)^{2/3}}$ for $x > 0$

$= \frac{2}{3(x+1)^{4/3}}\left[(x+1)(8x+3) - \frac{4x^2+3x}{3}\right]$

$= \frac{2}{9(x+1)^{4/3}}(20x^2 + 30x + 9)$ for $x > 0$

$f''(x) = \frac{-2}{9(x+1)^{4/3}}(20x^2 + 30x + 9)$

for $x < 0$ and $x \neq -1$

(b) $f'_+(0) = \lim_{x \to 0^+} \frac{f(x) - f(0)}{x - 0} = \lim_{x \to 0^+} x(x+1)^{\frac{2}{3}} = 0$

$f'_-(0) = \lim_{x \to 0^-} \frac{f(x) - f(0)}{x - 0} = \lim_{x \to 0^-} \left[-x(x+1)^{\frac{2}{3}}\right] = 0$

$\therefore \quad f'_+(0) = f'_-(0) = 0$

$\therefore \quad f'(0) = 0$

$f''_+(0) = \lim_{x \to 0^+} \frac{f'(x) - f'(0)}{x - 0} = \lim_{x \to 0^+} \frac{2(4x+3)}{3(x+1)^{1/3}} = 2$

$f''_-(0) = \lim_{x \to 0^-} \frac{f'(x) - f'(0)}{x - 0} = \lim_{x \to 0^-} \frac{-2(4x+3)}{3(x+1)^{1/3}} = -2$

$f''_+(0) \neq f''_-(0)$

$\therefore$ $f''(0)$ does not exist.

$f'_+(-1) = \lim_{x \to -1^+} \frac{f(x) - f(-1)}{x + 1} = \lim_{x \to -1^+} \frac{-x^2}{(x+1)^{1/3}} = -\infty$

$\therefore$ $f'(-1)$ does not exist.

(c)

x	$x < -1$	$-1 < x < \frac{-3}{4}$	$\frac{-3}{4} < x < 0$	$x > 0$
$f'(x)$	+	−	+	+

$\therefore$ $f(x)$ attains maximum at $x = -1$, minimum at $x = \frac{-3}{4}$.

$f''(x) = 0 \Rightarrow 20x^2 + 30x + 9 = 0$

$\Rightarrow x = \frac{-30 \pm \sqrt{900 - 720}}{40} = \frac{-15 \pm 3\sqrt{5}}{20}$

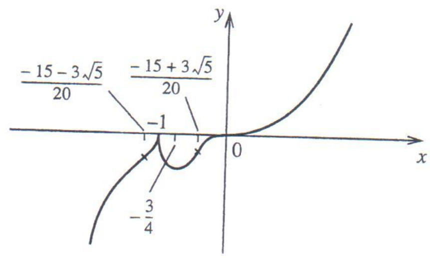
Also by (b), $f''(0)$ does not exist.

x	$x < \frac{-15-3\sqrt{5}}{20}$	$\frac{-15-3\sqrt{5}}{20} < x < -1$
$f''(x)$	−	+

x	$-1 < x < \frac{-15+3\sqrt{5}}{20}$	$\frac{-15+3\sqrt{5}}{20} < x < 0$	$x > 0$
$f''(x)$	+	−	+

The graph of $f(x)$ has inflexional points at $x = 0$, $x = \frac{-15 \pm 3\sqrt{5}}{20}$.

(d)



Solutions Math 1013 (Tutorial 11)

R7

(8) $f(x) = x^{\frac{2}{3}}(2x-1)$

(a) $f'_+(0) = \lim_{x \to 0^+} \dfrac{x^{2/3}(2x-1)-0}{x-0}$

$= \lim_{x \to 0^+} x^{\frac{-1}{3}}(2x-1) = -\infty$

$\therefore$ $f'(0)$ does not exist.

(b) $f'(x) = 2x^{\frac{2}{3}} + \frac{2}{3}x^{\frac{-1}{3}}(2x-1) = \dfrac{2}{x^{1/3}}[x + \frac{1}{3}(2x-1)]$

$= \dfrac{2(5x-1)}{3x^{1/3}}$ for $x \neq 0$

$f''(x) = \dfrac{2}{3} \cdot \dfrac{5x^{1/3} - \frac{1}{3}x^{-2/3}(5x-1)}{x^{2/3}}$

$= \dfrac{2(10x+1)}{9x^{4/3}}$ for $x \neq 0$

(c) $f'(x) = 0 \Rightarrow x = \frac{1}{5}$, $f''(\frac{1}{5}) > 0$

$\therefore$ $f(x)$ attains a minimum at $x = \frac{1}{5}$ $\left(\frac{1}{5}, \frac{-3}{5^{5/3}}\right)$

$\because$ $f'(0)$ does not exist.

$\therefore$ Using sign test, we get

x	$x < 0$	$0 < x < \frac{1}{5}$
$f'(x)$	+	−

$\therefore$ $f(x)$ attains maximum at $x = 0$

$(0, 0)$ is an inflexional point

x	$x < \frac{-1}{10}$	$\frac{-1}{10} < x < 0$	$x > 0$
$f''(x)$	−	+	+

$\therefore$ $\left(\frac{-1}{10}, -\frac{6}{5 \times 10^{2/3}}\right)$ is point of inflexion.

(d) $\lim_{x \to \infty} x^{\frac{2}{3}}(2x-1) = \infty$

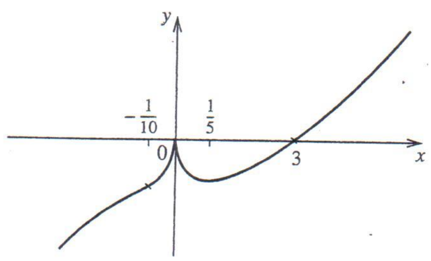
$\therefore$ no horizontal asymptote.

Obviously there is no vertical asymptote

$\lim_{x \to \infty} \dfrac{x^{2/3}(2x-1)}{x} = \infty$

$\therefore$ there is no oblique asymptote.

(e)



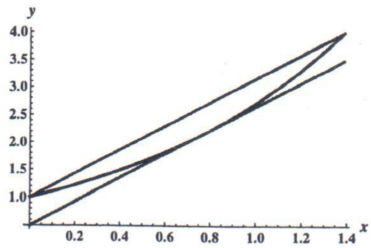
Solutions to Math 1013 (Tutorial 11)

P.8

9) a. The function f is differentiable on $[0, \ln 4]$ so the mean value theorem applies.

b. The average rate of change of f on $[0, \ln 4]$ is $\frac{f(\ln 4)-f(0)}{\ln 4-0} = \frac{4-1}{\ln 4} = \frac{3}{\ln 4}$. We wish to find a point c in $(0, \ln 4)$ such that $f'(c) = 3/\ln 4$, or equivalently $e^c = 3/\ln 4$ which gives $c = \ln\left(\frac{3}{\ln 4}\right)$.

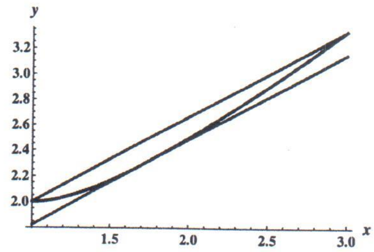
c.



10) a. The function f is differentiable on $[1, 3]$ so the mean value theorem applies.

b. The average rate of change of f on $[1, 3]$ is $\frac{f(3)-f(1)}{3-1} = \frac{\frac{10}{3}-2}{2} = \frac{2}{3}$. We wish to find a point c in $(1, 3)$ such that $f'(c) = 2/3$, or equivalently $1 - \frac{1}{c^2} = \frac{2}{3}$, so $c = \sqrt{3}$.

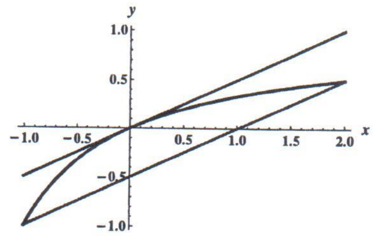
c.



11) a. The function f is differentiable on $[-1, 2]$ so the mean value theorem applies.

b. The average rate of change of f on $[-1, 2]$ is $\frac{f(2)-f(-1)}{2-(-1)} = \frac{\frac{1}{2}-(-1)}{3} = \frac{1}{2}$. We wish to find a point c in $(-1, 2)$ such that $f'(c) = 1/2$, or equivalently $\frac{2}{(c+2)^2} = \frac{1}{2}$, so $c = 0$.

c.



12) Bolt's average speed during the race was $\frac{100}{9.58}$ m/s $= \frac{100}{9.58} \cdot \frac{3600}{1000}$ km/hr ≈ 37.58 km/hr, so by the mean value theorem he must have exceeded 37 km/hr during the race.

13) The average speed of the car over the 28 minute period ($= 28/60$ hr) is $\frac{30-0}{28/60} \approx 64$mi/hr, so the officer can conclude by the mean value theorem that at some point the car exceeded the speed limit.

14) The average speed of the car over the 30 minute period ($= 1/2$ hr) is exactly 60 mi/hr. But because the car started from rest, the average speed for the first few seconds of the trip is less than 60 mi/hr, and therefore the average speed for the remainder of the trip must exceed 60 mi/hr, and the officer can conclude that the driver exceeded the speed limit.

15) The runner's average speed is $6.2/(32/60) \approx 11.6$ mi /hr. By the mean value theorem, the runner's speed was 11.6 mi/hr at least once. By the intermediate value theorem, all speeds between 0 and 11.6 mi/hr were reached. Because the initial and final speed was 0 mi/hr, the speed of 11 mi/hr was reached at least twice.

16) Observe that

$$\frac{f(b)-f(a)}{b-a} = \frac{A(b^2-a^2)+B(b-a)}{b-a} = A(a+b)+B$$

and $f'(c) = 2Ac + B$, so the point c that satisfies the conclusion of the mean value theorem is $c = (a+b)/2$.