1) Antiderivates

Definition A function F is called an **antiderivative** of f on an interval I if $F'(x) = f(x)$ for all x in I.

1 Theorem If F is an antiderivative of f on an interval I, then the most general antiderivative of f on I is

$$F(x) + C$$

where C is an arbitrary constant.

2) Indefinite Integral

$$\int f(x)\,dx = F(x) \quad \text{means} \quad F'(x) = f(x)$$

Example : $\int x^4 dx = \frac{x^5}{5} + c$

3) Table of Formulas for Indefinite integral

(1) $\int x^n dx = \frac{x^{n+1}}{n+1} + c$ **for** $n \neq 1$ **.**

1 Table of Indefinite Integrals

$\int cf(x)\,dx = c\int f(x)\,dx$	$\int [f(x) + g(x)]\,dx = \int f(x)\,dx + \int g(x)\,dx$
$\int k\,dx = kx + C$	
$\int x^n\,dx = \frac{x^{n+1}}{n+1} + C \quad (n \neq -1)$	$\int \frac{1}{x}\,dx = \ln\lvert x\rvert + C$
$\int e^x\,dx = e^x + C$	$\int a^x\,dx = \frac{a^x}{\ln a} + C$
$\int \sin x\,dx = -\cos x + C$	$\int \cos x\,dx = \sin x + C$
$\int \sec^2 x\,dx = \tan x + C$	$\int \csc^2 x\,dx = -\cot x + C$
$\int \sec x \tan x\,dx = \sec x + C$	$\int \csc x \cot x\,dx = -\csc x + C$
$\int \frac{1}{x^2+1}\,dx = \tan^{-1}x + C$	$\int \frac{1}{\sqrt{1-x^2}}\,dx = \sin^{-1}x + C$
$\int \sinh x\,dx = \cosh x + C$	$\int \cosh x\,dx = \sinh x + C$

Table 4.9 Indefinite Integrals of Trigonometric Functions

1. $\frac{d}{dx}(\sin ax) = a\cos ax \rightarrow \int \cos ax\,dx = \frac{1}{a}\sin ax + C$
2. $\frac{d}{dx}(\cos ax) = -a\sin ax \rightarrow \int \sin ax\,dx = -\frac{1}{a}\cos ax + C$
3. $\frac{d}{dx}(\tan ax) = a\sec^2 ax \rightarrow \int \sec^2 ax\,dx = \frac{1}{a}\tan ax + C$
4. $\frac{d}{dx}(\cot ax) = -a\csc^2 ax \rightarrow \int \csc^2 ax\,dx = -\frac{1}{a}\cot ax + C$
5. $\frac{d}{dx}(\sec ax) = a\sec ax\tan ax \rightarrow \int \sec ax\tan ax\,dx = \frac{1}{a}\sec ax + C$
6. $\frac{d}{dx}(\csc ax) = -a\csc ax\cot ax \rightarrow \int \csc ax\cot ax\,dx = -\frac{1}{a}\csc ax + C$

Table 4.10 Other Definite Integrals

7. $\frac{d}{dx}(e^{ax}) = ae^{ax} \rightarrow \int e^{ax}\,dx = \frac{1}{a}e^{ax} + C$
8. $\frac{d}{dx}(b^x) = b^x \ln b \rightarrow \int b^x\,dx = \frac{1}{\ln b}b^x + C, b > 0, b \neq 1$
9. $\frac{d}{dx}(\ln\lvert x\rvert) = \frac{1}{x} \rightarrow \int \frac{dx}{x} = \ln\lvert x\rvert + C$
10. $\frac{d}{dx}\left[\sin^{-1}\frac{x}{a}\right] = \frac{1}{\sqrt{a^2-x^2}} \rightarrow \int \frac{dx}{\sqrt{a^2-x^2}} = \sin^{-1}\frac{x}{a} + C$
11. $\frac{d}{dx}\left[\tan^{-1}\frac{x}{a}\right] = \frac{a}{a^2+x^2} \rightarrow \int \frac{dx}{a^2+x^2} = \frac{1}{a}\tan^{-1}\frac{x}{a} + C$
12. $\frac{d}{dx}\left(\sec^{-1}\left\lvert\frac{x}{a}\right\rvert\right) = \frac{a}{x\sqrt{x^2-a^2}} \rightarrow \int \frac{dx}{x\sqrt{x^2-a^2}} = \frac{1}{a}\sec^{-1}\left\lvert\frac{x}{a}\right\rvert + C$

4) The Rule of substitution

4 The Substitution Rule If $u = g(x)$ is a differentiable function whose range is an interval I and f is continuous on I, then

$$\int f(g(x))\,g'(x)\,dx = \int f(u)\,du$$

Example : $\int tanx\ dx = \int \frac{sinx\ dx}{cosx} = \int -\frac{dcosx}{cosx} = -ln|cosx| + c$

5) Reimann Sum

> **DEFINITION Riemann Sum**
>
> Suppose f is defined on a closed interval $[a, b]$, which is divided into n subintervals of equal length Δx. If x_k^* is any point in the kth subinterval $[x_{k-1}, x_k]$, for $k = 1, 2, \ldots, n$, then
>
> $$f(x_1^*)\Delta x + f(x_2^*)\Delta x + \cdots + f(x_n^*)\Delta x$$
>
> is called a **Riemann sum** for f on $[a, b]$. This sum is
>
> - a **left Riemann sum** if x_k^* is the left endpoint of $[x_{k-1}, x_k]$ (Figure 5.9);
> - a **right Riemann sum** if x_k^* is the right endpoint of $[x_{k-1}, x_k]$ (Figure 5.10); and
> - a **midpoint Riemann sum** if x_k^* is the midpoint of $[x_{k-1}, x_k]$ (Figure 5.11), for $k = 1, 2, \ldots, n$.

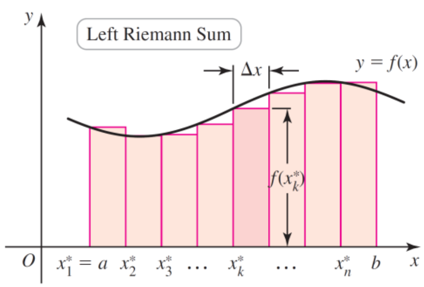


FIGURE 5.9

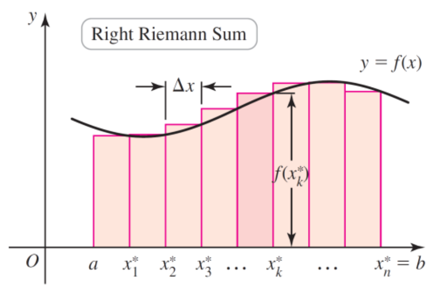


FIGURE 5.10

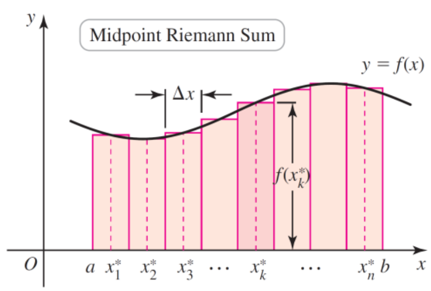


FIGURE 5.11

6) Special Sums

Let n be a positive integer.

$$\sum_{k=1}^{n} c = cn \qquad \sum_{k=1}^{n} k = \frac{n(n+1)}{2}$$

$$\sum_{k=1}^{n} k^2 = \frac{n(n+1)(2n+1)}{6} \qquad \sum_{k=1}^{n} k^3 = \frac{n^2(n+1)^2}{4}$$

7) Definite Integral

> **2 Definition of a Definite Integral** If f is a function defined for $a \leqslant x \leqslant b$, we divide the interval $[a, b]$ into n subintervals of equal width $\Delta x = (b - a)/n$. We let $x_0\,(= a), x_1, x_2, \ldots, x_n\,(= b)$ be the endpoints of these subintervals and we let $x_1^*, x_2^*, \ldots, x_n^*$ be any **sample points** in these subintervals, so x_i^* lies in the ith subinterval $[x_{i-1}, x_i]$. Then the **definite integral of f from a to b** is
>
> $$\int_a^b f(x)\,dx = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i^*)\,\Delta x$$
>
> provided that this limit exists and gives the same value for all possible choices of sample points. If it does exist, we say that f is **integrable** on $[a, b]$.

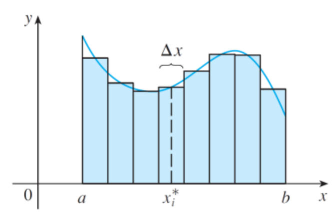


FIGURE 1
If $f(x) \geqslant 0$, the Riemann sum $\Sigma f(x_i^*)\,\Delta x$ is the sum of areas of rectangles.

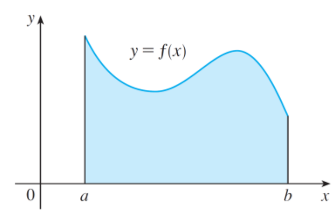


FIGURE 2
If $f(x) \geqslant 0$, the integral $\int_a^b f(x)\,dx$ is the area under the curve $y = f(x)$ from a to b.

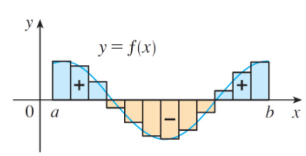


FIGURE 3
$\Sigma f(x_i^*)\,\Delta x$ is an approximation to the net area.

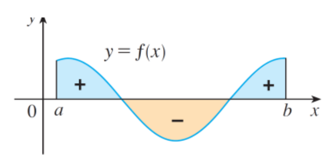


FIGURE 4
$\int_a^b f(x)\,dx$ is the net area.

8) Integrable Function

THEOREM 5.2 Integrable Functions
If f is continuous on $[a, b]$ or bounded on $[a, b]$ with a finite number of discontinuities, then f is integrable on $[a, b]$.

9) Definite Integral and Reimann Sum

4 Theorem If f is integrable on $[a, b]$, then

$$\int_a^b f(x)\,dx = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i)\,\Delta x$$

where $\Delta x = \dfrac{b-a}{n}$ and $x_i = a + i\,\Delta x$

Such that $\int_a^b f(x)dx = \lim\limits_{n\to\infty} \sum_{k=1}^{n} f\left(a + k\left(\frac{b-a}{n}\right)\right)\left(\frac{b-a}{n}\right)$

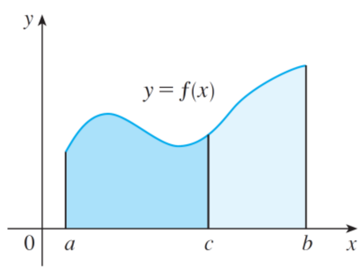
$$= \lim_{n\to\infty} \sum_{k=0}^{n-1} f\left(a + k\left(\frac{b-a}{n}\right)\right)\left(\frac{b-a}{n}\right).$$

10) Properties of Integrals

(a) $\int_a^b f(x)dx = -\int_b^a f(x)dx$

(b) $\int_a^a f(x)dx = 0$

(c) $\int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx$



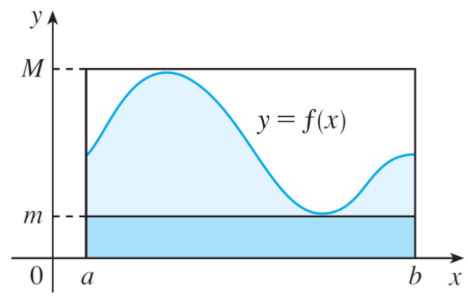
Properties of the Integral

1. $\int_a^b c\,dx = c(b-a)$, where c is any constant
2. $\int_a^b [f(x) + g(x)]\,dx = \int_a^b f(x)\,dx + \int_a^b g(x)\,dx$
3. $\int_a^b cf(x)\,dx = c\int_a^b f(x)\,dx$, where c is any constant
4. $\int_a^b [f(x) - g(x)]\,dx = \int_a^b f(x)\,dx - \int_a^b g(x)\,dx$

Comparison Properties of the Integral

6. If $f(x) \geq 0$ for $a \leq x \leq b$, then $\int_a^b f(x)\,dx \geq 0$.
7. If $f(x) \geq g(x)$ for $a \leq x \leq b$, then $\int_a^b f(x)\,dx \geq \int_a^b g(x)\,dx$.
8. If $m \leq f(x) \leq M$ for $a \leq x \leq b$, then

$$m(b-a) \leq \int_a^b f(x)\,dx \leq M(b-a)$$

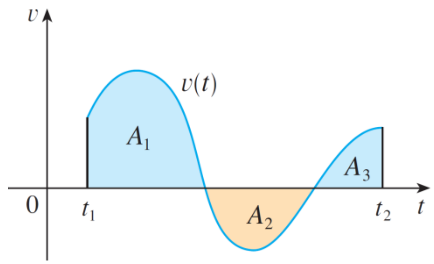


11) Fundamental Theorem of Calculus

The Fundamental Theorem of Calculus Suppose f is continuous on $[a, b]$.

1. If $g(x) = \int_a^x f(t)\,dt$, then $g'(x) = f(x)$.
2. $\int_a^b f(x)\,dx = F(b) - F(a)$, where F is any antiderivative of f, that is, $F' = f$.

12) Displacement and Distance



$$\text{displacement} = \int_{t_1}^{t_2} v(t)\,dt = A_1 - A_2 + A_3$$

$$\text{distance} = \int_{t_1}^{t_2} |v(t)|\,dt = A_1 + A_2 + A_3$$

13) The Substitution Rule for Definite Integral

6 The Substitution Rule for Definite Integrals If g' is continuous on $[a, b]$ and f is continuous on the range of $u = g(x)$, then

$$\int_a^b f(g(x))\,g'(x)\,dx = \int_{g(a)}^{g(b)} f(u)\,du$$

14) Example 1 : Evaluate $\int_0^4 \sqrt{2x+1}\,dx$.

SOLUTION Using the substitution from Solution 1 of Example 2, we have $u = 2x + 1$ and $dx = \frac{1}{2}\,du$. To find the new limits of integration we note that

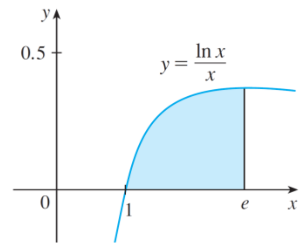
when $x = 0$, $u = 2(0) + 1 = 1$ and when $x = 4$, $u = 2(4) + 1 = 9$

Therefore
$$\int_0^4 \sqrt{2x+1}\,dx = \int_1^9 \tfrac{1}{2}\sqrt{u}\,du$$
$$= \tfrac{1}{2}\cdot\tfrac{2}{3}u^{3/2}\Big]_1^9$$
$$= \tfrac{1}{3}(9^{3/2} - 1^{3/2}) = \tfrac{26}{3}$$

Example 2: Calculate $\int_1^e \frac{\ln x}{x}\,dx$.

SOLUTION We let $u = \ln x$ because its differential $du = dx/x$ occurs in the integral. When $x = 1$, $u = \ln 1 = 0$; when $x = e$, $u = \ln e = 1$. Thus

$$\int_1^e \frac{\ln x}{x}\,dx = \int_0^1 u\,du = \frac{u^2}{2}\Big]_0^1 = \frac{1}{2}$$



15) Integration of Symmetric Functions

THEOREM 5.4 Integrals of Even and Odd Functions

Let a be a positive real number and let f be an integrable function on the interval $[-a, a]$.

- If f is even, $\int_{-a}^{a} f(x)\,dx = 2\int_0^a f(x)\,dx$.
- If f is odd, $\int_{-a}^{a} f(x)\,dx = 0$.

16) Exercises :

1) **Evaluate the following integrals :**

(a) $\int (3x^5 - 5x^9)\,dx$ **(b)** $\int \left(4\sqrt{x} - \frac{4}{\sqrt{x}}\right) dx$

(c) $\int \frac{1}{x\sqrt{x^2 - 25}}\,dx$ **(d)** $\int \frac{12t^8 - t}{t^3}\,dt$

(e) $\int (\sin 2y + \cos 3y)\,dy$ **(f)** $\int (\sec^2\theta + \sec\theta\tan\theta)\,d\theta$

(g) $\int \sec 4\theta \tan 4\theta\,d\theta$ **(h)** $\int e^{x+2}\,dx$

(i) $\int \sqrt{x}\,(2x^6 - 4\sqrt[3]{x})\,dx$ **(j)** $\int \frac{2 + x^2}{1 + x^2}\,dx$

2) **Velocity to position**

Given the following velocity functions of an object moving along a line, find the position function with the given initial position. Then graph both the velocity and position functions.

(a) $v(t) = 6t^2 + 4t - 10;\ s(0) = 0$

(b) $v(t) = 2\cos t;\ s(0) = 0$

3) Evaluate $\lim_{n\to\infty}\left\{\frac{1}{n+1} + \frac{1}{n+2} + \cdots + \frac{1}{n+n}\right\}$.

4) Prove that $\lim_{n\to\infty} \frac{1}{n}\left\{\sin\frac{t}{n} + \sin\frac{2t}{n} + \cdots + \sin\frac{(n-1)t}{n}\right\} = \frac{1-\cos t}{t}$.

5)

Sum to integral Evaluate the following limit by identifying the integral that it represents:

$$\lim_{n\to\infty} \sum_{k=1}^{n} \left[\left(\frac{4k}{n}\right)^8 + 1\right]\left(\frac{4}{n}\right).$$

6) **Fundamental Theorem of Calculus :**

a. $\frac{d}{dx}\int_1^x \sin^2 t\,dt$ **b.** $\frac{d}{dx}\int_x^5 \sqrt{t^2 + 1}\,dt$ **c.** $\frac{d}{dx}\int_0^{x^2} \cos t^2\,dt$

(d) $\frac{d}{dx}\int_{-x}^{x} \sqrt{1 + t^2}\,dt$ **(e)** $\frac{d}{dx}\int_{e^x}^{e^{2x}} \ln t^2\,dt$

(f) $\frac{d}{dx}\int_{x^2}^{10} \frac{dz}{z^2 + 1}$ **(g)** $\frac{d}{dx}\int_0^{\cos x} (t^4 + 6)\,dt$

7) **Even and Odd Functions**

(a) $\int_{-\pi/4}^{\pi/4} \tan x\,dx$ **(b)** $\int_{-2}^{2} \frac{x^3 - 4x}{x^2 + 1}\,dx$

8) **Integration by Substitutions**

(a) $\int x^3(x^4 + 16)^6\,dx$ **(b)** $\int \frac{2}{x\sqrt{4x^2 - 1}}\,dx,\ x > 2$

(c) $\int (x + 1)\sqrt{3x + 2}\,dx$ **(d)** $\int_0^{\pi/4} \frac{\sin x}{\cos^2 x}\,dx$

(e) $\int_{-1}^{2} x^2 e^{x^3+1}\,dx$ **(f)** $\int \frac{(\sqrt{x} + 1)^4}{2\sqrt{x}}\,dx$

(g) $\displaystyle\int_0^{\pi/6} \frac{\sin 2y}{\sin^2 y + 2}\,dy$ (*Hint:* $\sin 2y = 2\sin y \cos y$.)

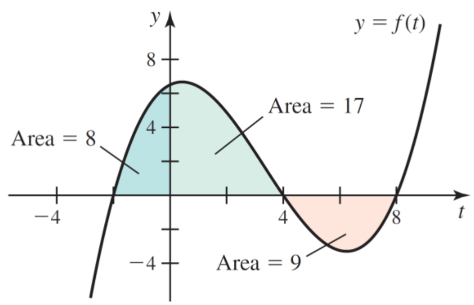
h) $\displaystyle\int_1^{e^2} \frac{\ln x}{x}\,dx$

i) $\displaystyle\int \frac{dx}{\sqrt{1+\sqrt{1+x}}}$ (*Hint:* Begin with $u = \sqrt{1+x}$.)

Further Exercises

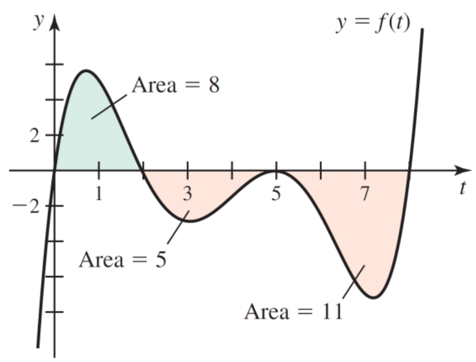
11) **Area functions** The graph of f is shown in the figure. Let $A(x) = \int_{-2}^{x} f(t)\,dt$ and $F(x) = \int_4^x f(t)\,dt$ be two area functions for f. Evaluate the following area functions.

a. $A(-2)$ **b.** $F(8)$ **c.** $A(4)$ **d.** $F(4)$ **e.** $A(8)$



12) **Area functions** The graph of f is shown in the figure. Let $A(x) = \int_0^x f(t)\,dt$ and $F(x) = \int_2^x f(t)\,dt$ be two area functions for f. Evaluate the following area functions.

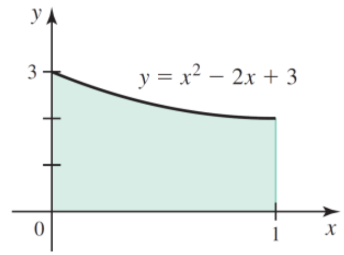
a. $A(2)$ **b.** $F(5)$ **c.** $A(0)$ **d.** $F(8)$ **e.** $A(8)$
f. $A(5)$ **g.** $F(2)$

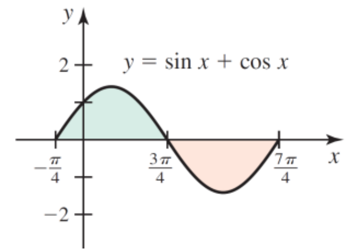


23–24. Definite integrals *Evaluate the following integrals using the Fundamental Theorem of Calculus. Discuss whether your result is consistent with the figure.*

23. $\displaystyle\int_0^1 (x^2 - 2x + 3)\,dx$

24. $\displaystyle\int_{-\pi/4}^{7\pi/4} (\sin x + \cos x)\,dx$





8–11. Limit definition of the definite integral *Use the limit definitior of the definite integral with right Riemann sums and a regular partion* $\left(\int_a^b f(x)\,dx = \lim_{n\to\infty} \sum_{k=1}^{n} f(x_k^*)\Delta x\right)$ *to evaluate the following definite integrals. Use the Fundamental Theorem of Calculus to check your answer.*

8. $\displaystyle\int_0^1 (4x - 2)\,dx$

9. $\displaystyle\int_0^2 (x^2 - 4)\,dx$

10. $\displaystyle\int_1^2 (3x^2 + x)\,dx$

11. $\displaystyle\int_0^4 (x^3 - x)\,dx$

15–30. Evaluating integrals *Evaluate the following integrals.*

15. $\int_{-2}^{2} (3x^4 - 2x + 1)\, dx$

16. $\int \cos 3x\, dx$

17. $\int_{0}^{2} (x + 1)^3\, dx$

18. $\int_{0}^{1} (4x^{21} - 2x^{16} + 1)\, dx$

19. $\int (9x^8 - 7x^6)\, dx$

20. $\int_{-2}^{2} e^{4x+8}\, dx$

21. $\int_{0}^{1} \sqrt{x}(\sqrt{x} + 1)\, dx$

22. $\int \frac{x^2}{x^3 + 27}\, dx$

23. $\int_{0}^{1} \frac{dx}{\sqrt{4 - x^2}}$

24. $\int y^2(3y^3 + 1)^4\, dy$

25. $\int_{0}^{3} \frac{x}{\sqrt{25 - x^2}}\, dx$

26. $\int x \sin x^2 \cos^8 x^2\, dx$

27. $\int \sin^2 5\theta\, d\theta$

28. $\int_{0}^{\pi} (1 - \cos^2 3\theta)\, d\theta$

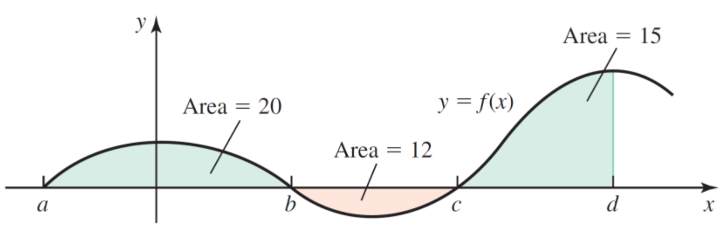
29. $\int \frac{x^2 + 2x - 2}{x^3 + 3x^2 - 6x}\, dx$

30. $\int_{0}^{\ln 2} \frac{e^x}{1 + e^{2x}}\, dx$

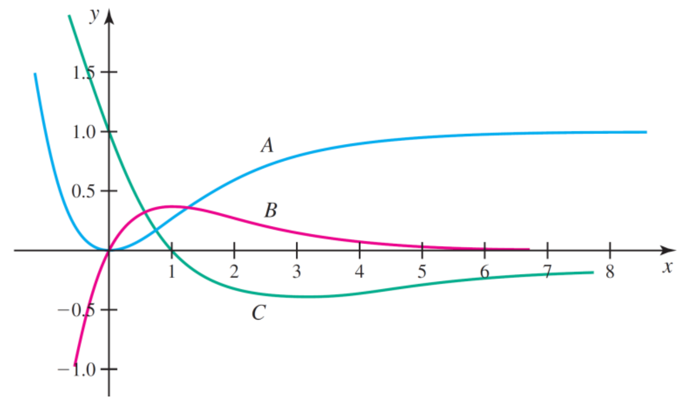
38. Properties of integrals The figure shows the areas of regions bounded by the graph of f and the x-axis. Evaluate the following integrals.

a. $\int_{a}^{c} f(x)\, dx$ **b.** $\int_{b}^{d} f(x)\, dx$ **c.** $\int_{c}^{b} 2\, f(x)\, dx$

d. $\int_{a}^{d} 4\, f(x)\, dx$ **e.** $\int_{a}^{b} 3\, f(x)\, dx$ **f.** $\int_{b}^{d} 2\, f(x)\, dx$



52. Identifying functions Match the graphs A, B, and C in the figure with the functions $f(x)$, $f'(x)$, and $\int_0^x f(t)\, dt$.



56–61. Additional integrals *Evaluate the following integrals.*

56. $\int \frac{\sin 2x}{1 + \cos^2 x}\, dx$ (*Hint*: $\sin 2x = 2 \sin x \cos x$.)

57. $\int \frac{1}{x^2} \sin \frac{1}{x}\, dx$

58. $\int \frac{(\tan^{-1} x)^5}{1 + x^2}\, dx$

59. $\int \frac{dx}{(\tan^{-1} x)(1 + x^2)}$

60. $\int \frac{\sin^{-1} x}{\sqrt{1 - x^2}}\, dx$

61. $\int \frac{e^x - e^{-x}}{e^x + e^{-x}}\, dx$