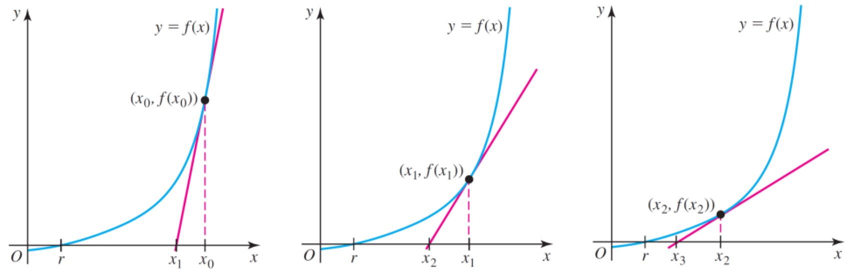
1) Newton's Method of finding Roots



(a) (b) (c)

Solving for x and calling it x_{n+1}, we find that

$$\underbrace{x_{n+1}}_{\text{new approximation}} = \underbrace{x_n}_{\text{current approximation}} - \frac{f(x_n)}{f'(x_n)}, \text{ provided } f'(x_n) \neq 0.$$

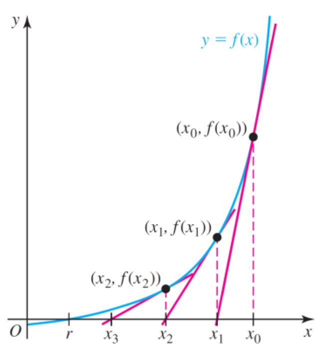
PROCEDURE Newton's Method for Approximating Roots of $f(x) = 0$

1. Choose an initial approximation x_0 as close to a root as possible.
2. For $n = 0, 1, 2, \ldots$

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)},$$

provided $f'(x_n) \neq 0$.

3. End the calculations when a termination condition is met.



EXAMPLE 1 Applying Newton's method Approximate the roots of $f(x) = x^3 - 5x + 1$ using seven steps of Newton's method. Use $x_0 = -3$, $x_0 = 1$, and $x_0 = 4$ as initial approximations (margin figure).

SOLUTION Noting that $f'(x) = 3x^2 - 5$, Newton's method takes the form

$$x_{n+1} = x_n - \frac{\overbrace{x_n^3 - 5x_n + 1}^{f(x_n)}}{\underbrace{3x_n^2 - 5}_{f'(x_n)}} = \frac{2x_n^3 - 1}{3x_n^2 - 5},$$

where $n = 0, 1, 2, \ldots,$ and x_0 is specified. With an initial approximation of $x_0 = -3$, the first approximation is

$$x_1 = \frac{2x_0^3 - 1}{3x_0^2 - 5} = \frac{2(-3)^3 - 1}{3(-3)^2 - 5} = -2.5.$$

The second approximation is

$$x_2 = \frac{2x_1^3 - 1}{3x_1^2 - 5} = \frac{2(-2.5)^3 - 1}{3(-2.5)^2 - 5} \approx -2.345455.$$

Continuing in this fashion, we generate the first seven approximations shown in Table 4.5.

The approximations generated from the initial approximations $x_0 = 1$ and $x_0 = 4$ are also shown in the table.

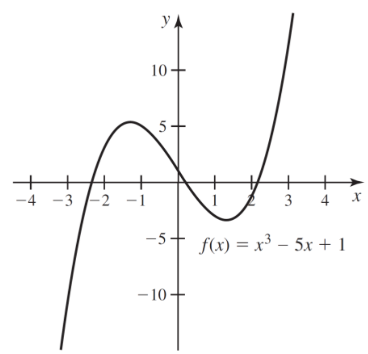


Table 4.5

k	x_k	x_k	x_k
0	−3	1	4
1	−2.500000	−0.500000	2.953488
2	−2.345455	0.294118	2.386813
3	−2.330203	0.200215	2.166534
4	−2.330059	0.201639	2.129453
5	−2.330059	0.201640	2.128420
6	−2.330059	0.201640	2.128419
7	−2.330059	0.201640	2.128419

Notice that with the initial approximation $x_0 = -3$ (second column), the resulting sequence of approximations settles on the value −2.330059 after four iterations, and then there are no further changes in these digits. A similar behavior is seen with the initial approximations $x_0 = 1$ and $x_0 = 4$. Based on this evidence, we conclude that −2.330059, 0.201640, and 2.128419 are approximations to the roots of f with at least six digits (to the right of the decimal point) of accuracy.

Example 2 :

We will illustrate Newton's Method by using it to approximate $\sqrt{2}$.

Solution :

If we let $f(x) := x^2 - 2$ for $x \in \mathbb{R}$, then we seek the positive root of the equation $f(x) = 0$. Since $f'(x) = 2x$, the iteration formula is

$$\begin{aligned} x_{n+1} &= x_n - \frac{f(x_n)}{f'(x_n)} \\ &= x_n - \frac{x_n^2 - 2}{2x_n} = \frac{1}{2}\left(x_n + \frac{2}{x_n}\right). \end{aligned}$$

If we take $x_1 := 1$ as our initial estimate, we obtain the successive values $x_2 = 3/2 = 1.5$, $x_3 = 17/12 = 1.416\,666\ldots$, $x_4 = 577/408 = 1.414\,215\ldots$, and $x_5 = 665\,857/470\,832 = 1.414\,213\,562\,374\ldots$, which is correct to eleven places. □

2) The Difficulty of Newton's Method

Pitfalls of Newton's Method

Given a good initial approximation, Newton's method usually converges to a root. And when it converges, it usually does so quickly. However, when Newton's method fails, it does so in curious and spectacular ways. The formula for Newton's method suggests one way in which the method could encounter difficulties: The term $f'(x_n)$ appears in a denominator, so if at any step $f'(x_n) = 0$, then the method breaks down. Furthermore, if $f'(x_n)$ is close to zero at any step, then the method may be slow to converge or may fail to converge. The following example shows three ways in which Newton's method may go awry.

EXAMPLE 4 **Difficulties with Newton's method** Find the root of $f(x) = \dfrac{8x^2}{3x^2 + 1}$ using Newton's method with initial approximations of $x_0 = 1$, $x_0 = 0.15$, and $x_0 = 1.1$.

SOLUTION Notice that f has the single root $x = 0$. So the point of the example is not to find the root, but to investigate the performance of Newton's method. Computing f' and doing a few steps of algebra show that the formula for Newton's method is

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} = \frac{x_n}{2}\left(1 - 3x_n^2\right).$$

The results of five iterations of Newton's method are displayed in Table 4.8, and they tell three different stories.

Table 4.8

k	x_k	x_k	x_k
0	1	0.15	1.1
1	−1	0.0699375	−1.4465
2	1	0.0344556	3.81665
3	−1	0.0171665	−81.4865
4	1	0.00857564	8.11572×10^5
5	−1	0.00428687	-8.01692×10^{17}

The approximations generated using $x_0 = 1$ (second column) get stuck in a cycle that alternates between $+1$ and -1. The geometry underlying this rare occurrence is illustrated in Figure 4.83.

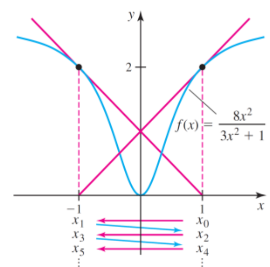


FIGURE 4.83

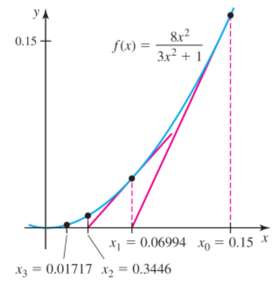


FIGURE 4.84

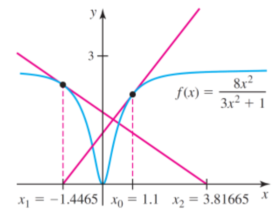


FIGURE 4.85

The approximations generated using $x_0 = 0.15$ (third column) actually converge to the root 0, but they converge slowly (Figure 4.84). Notice that the error is reduced by a factor of approximately 2 with each step. Newton's method usually has a faster rate of error reduction. The slow convergence is due to the fact that both f and f' have zeros at 0. As mentioned earlier, if the approximations x_n approach a zero of f', the rate of convergence is often compromised.

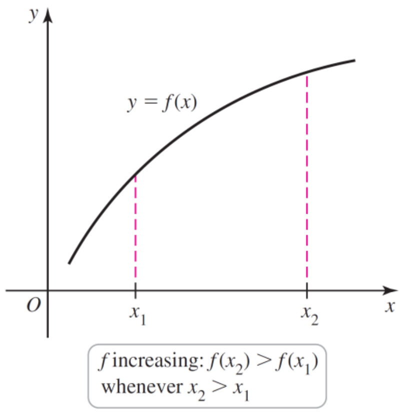
The approximations generated using $x_0 = 1.1$ (fourth column) increase in magnitude quickly and do not converge to a finite value, even though this initial approximation seems reasonable. The geometry of this case is shown in Figure 4.85.

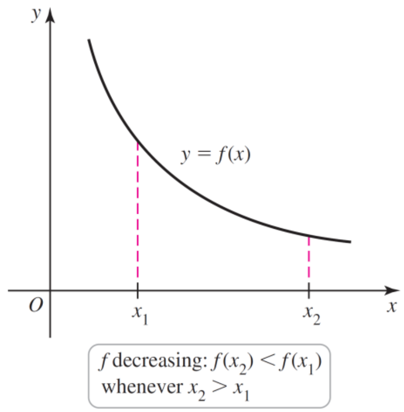
The three cases in this example illustrate the most common ways that Newton's method may fail to converge at its usual rate: The approximations may cycle or wander, they may converge slowly, or they may diverge (often at a rapid rate).

3) Increasing and Decreasing Function

DEFINITION Increasing and Decreasing Functions

Suppose a function f is defined on an interval I. We say that f is **increasing** on I if $f(x_2) > f(x_1)$ whenever x_1 and x_2 are in I and $x_2 > x_1$. We say that f is **decreasing** on I if $f(x_2) < f(x_1)$ whenever x_1 and x_2 are in I and $x_2 > x_1$.

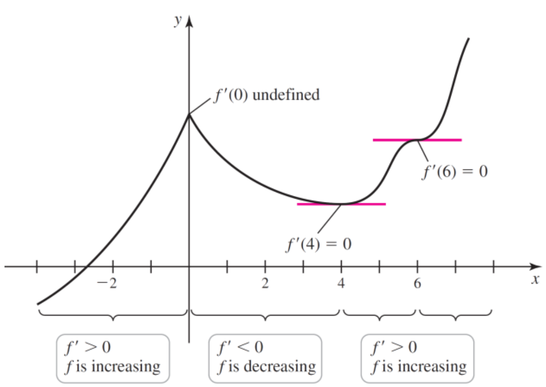




4) Test for Intervals of Increase and Decrease

THEOREM 4.3 Test for Intervals of Increase and Decrease

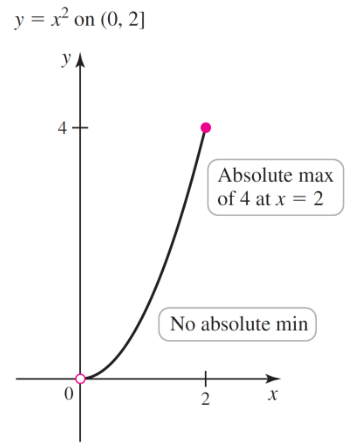
Suppose f is continuous on an interval I and differentiable at all interior points of I. If $f'(x) > 0$ at all interior points of I, then f is increasing on I. If $f'(x) < 0$ at all interior points of I, then f is decreasing on I.

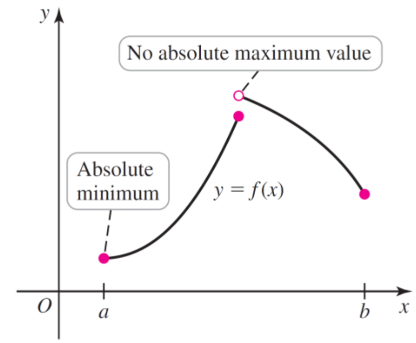


5) Absolute Maximum and Absolute Minimum

DEFINITION Absolute Maximum and Minimum

Let f be defined on an interval I containing c. If $f(c) \geq f(x)$ for every x in I, then f has an **absolute maximum** value of $f(c)$ on I at c. If $f(c) \leq f(x)$ for every x in I, then f has an **absolute minimum** value of $f(c)$ on I at c.

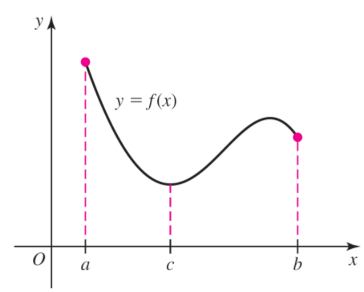




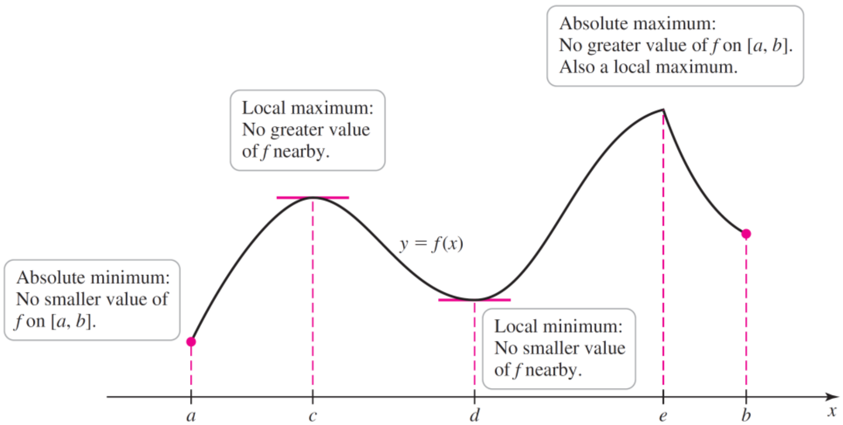
6) Extreme Value Theorem

THEOREM 4.1 Extreme Value Theorem

A function that is continuous on a closed interval $[a, b]$ has an absolute maximum value and an absolute minimum value on that interval.



7) Local Maximum and Local Minimum



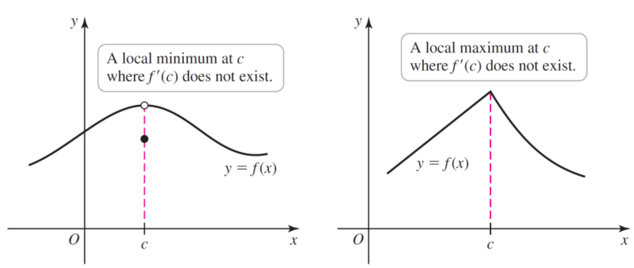
DEFINITION Local Maximum and Minimum Values

Suppose I is an interval on which f is defined and c is an interior point of I. If $f(c) \geq f(x)$ for all x in some open interval containing c, then $f(c)$ is a **local maximum** value of f. If $f(c) \leq f(x)$ for all x in some open interval containing c, then $f(c)$ is a **local minimum** value of f.

8) Critical Point

DEFINITION Critical Point

An interior point c of the domain of f at which $f'(c) = 0$ or $f'(c)$ fails to exist is called a **critical point** of f.



(a) (b)

9) Local Extreme Point Theorem

THEOREM 4.2 Local Extreme Point Theorem

If f has a local maximum or minimum value at c and $f'(c)$ exists, then $f'(c) = 0$.

10) Use the 1st Derivative of $f(x)$ to find the Extreme Values

FIRST DERIVATIVE TEST FOR ABSOLUTE EXTREME VALUES Suppose that c is a critical number of a continuous function f defined on an interval.

(a) If $f'(x) > 0$ for all $x < c$ and $f'(x) < 0$ for all $x > c$, then $f(c)$ is the absolute maximum value of f.

(b) If $f'(x) < 0$ for all $x < c$ and $f'(x) > 0$ for all $x > c$, then $f(c)$ is the absolute minimum value of f.

11) The 2nd Derivative Test for Extremums

THE SECOND DERIVATIVE TEST Suppose f'' is continuous near c.

(a) If $f'(c) = 0$ and $f''(c) > 0$, then f has a local minimum at c.

(b) If $f'(c) = 0$ and $f''(c) < 0$, then f has a local maximum at c.

(a) **For the curve $y = f(x)$. If both $f(x)$ and $f'(x)$ are differentiable at $x = x_0$, and $f'(x_0) = 0$, $f''(x_0) < 0$, then $(x_0, f(x_0))$ is a local maximum point**

(b) **For the curve $y = f(x)$. If both $f(x)$ and $f'(x)$ are differentiable at $x = x_0$, and $f'(x_0) = 0$, $f''(x_0) > 0$, then $(x_0, f(x_0))$ is a local minimum point**

12) Examples

EXAMPLE 4 **Absolute extreme values** Find the absolute maximum and minimum values of the following functions.

a. $f(x) = x^4 - 2x^3$ on the interval $[-2, 2]$

b. $g(x) = x^{2/3}(2 - x)$ on the interval $[-1, 2]$

SOLUTION

a. Because f is a polynomial, its derivative exists everywhere. So, if f has critical points, they are points at which $f'(x) = 0$. Computing f' and setting it equal to zero, we have

$$f'(x) = 4x^3 - 6x^2 = 2x^2(2x - 3) = 0.$$

Solving this equation gives the critical points $x = 0$ and $x = \frac{3}{2}$, both of which lie in the interval $[-2, 2]$; these points and the endpoints are *candidates* for the location of absolute extrema. Evaluating f at each of these points, we have

$$f(-2) = 32, f(0) = 0, f\left(\tfrac{3}{2}\right) = -\tfrac{27}{16}, \text{ and } f(2) = 0.$$

The largest of these function values is $f(-2) = 32$, which is the absolute maximum of f on $[-2, 2]$. The smallest of these values is $f\left(\frac{3}{2}\right) = -\frac{27}{16}$, which is the absolute minimum of f on $[-2, 2]$. The graph of f (Figure 4.11) shows that the critical point $x = 0$ corresponds to neither a local maximum nor a local minimum.

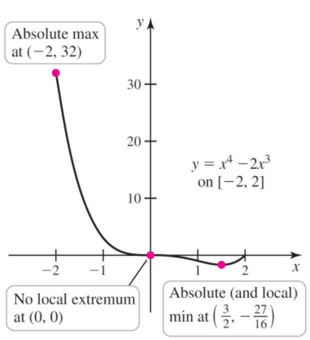


FIGURE 4.11

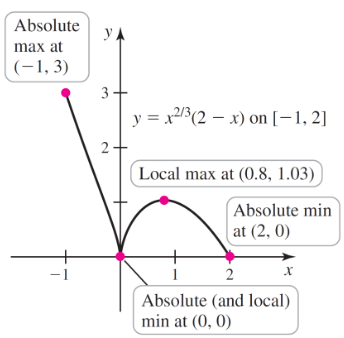


FIGURE 4.12

b. Differentiating $g(x) = x^{2/3}(2 - x) = 2x^{2/3} - x^{5/3}$, we have

$$g'(x) = \frac{4}{3}x^{-1/3} - \frac{5}{3}x^{2/3} = \frac{4 - 5x}{3x^{1/3}}.$$

Because $g'(0)$ is undefined and 0 is in the domain of g, $x = 0$ is a critical point. In addition, $g'(x) = 0$ when $4 - 5x = 0$, so $x = \frac{4}{5}$ is also a critical point. These two critical points and the endpoints are *candidates* for the location of absolute extrema. The next step is to evaluate f at the critical points and endpoints:

$$g(-1) = 3, \quad g(0) = 0, \quad g(4/5) \approx 1.03, \quad \text{and } g(2) = 0.$$

The largest of these function values is $g(-1) = 3$, which is the absolute maximum value of g on $[-1, 2]$. The least of these values is 0, which occurs twice. Therefore, g has its absolute minimum value on $[-1, 2]$ at the critical point $x = 0$ and the endpoint $x = 2$ (Figure 4.12). *Related Exercises 37–50*◄

EXAMPLE 2 **Intervals of increase and decrease** Find the intervals on which the following functions are increasing and decreasing.

a. $f(x) = xe^{-x}$ **b.** $f(x) = 2x^3 + 3x^2 + 1$

SOLUTION

a. By the Product Rule, $f'(x) = e^{-x} + x(-e^{-x}) = (1 - x)e^{-x}$. Solving $f'(x) = 0$ and noting that $e^{-x} \neq 0$ for all x, the sole critical point is $x = 1$. Therefore, if f' changes sign, then it does so at $x = 1$ and nowhere else. By evaluating f' at selected points in $(-\infty, 1)$ and $(1, \infty)$, we can determine the sign of f' on the entire interval:

- At $x = 0$, $f'(0) = 1 > 0$. So, $f' > 0$ on $(-\infty, 1)$, which means that f is increasing on $(-\infty, 1)$. (In fact, f is increasing on $(-\infty, 1]$.)
- At $x = 2$, $f'(2) = -e^{-2} < 0$. So $f' < 0$ on $(1, \infty)$, which means that f is decreasing on $(1, \infty)$. (In fact, f is decreasing on $[1, \infty)$.)

Note also that the graph has a horizontal tangent line at $x = 1$. We verify these conclusions by plotting f and f' (Figure 4.17).

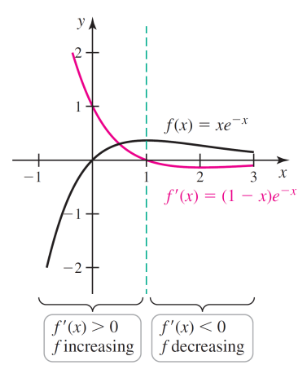


FIGURE 4.17

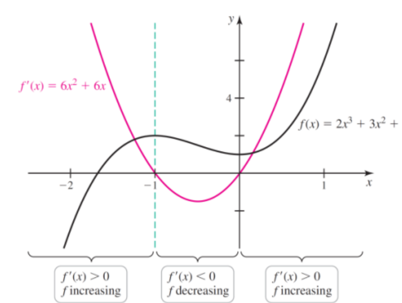


FIGURE 4.18

b. In this case, $f'(x) = 6x^2 + 6x = 6x(x+1)$. To find the intervals of increase, we first solve $6x(x+1) = 0$ and determine that the critical points are $x = 0$ and $x = -1$. If f' changes sign, then it does so at these points and nowhere else; that is, f' has the same sign throughout each of the intervals $(-\infty, -1)$, $(-1, 0)$, and $(0, \infty)$. Evaluating f' at selected points of each interval determines the sign of f' on that interval.

- At $x = -2$, $f'(-2) = 12 > 0$, so $f' > 0$ and f is increasing on $(-\infty, -1)$.
- At $x = -\frac{1}{2}$, $f'\left(-\frac{1}{2}\right) = -\frac{3}{2} < 0$, so $f' < 0$ and f is decreasing on $(-1, 0)$.
- At $x = 1$, $f'(1) = 12 > 0$, so $f' > 0$ and f is increasing on $(0, \infty)$.

The graph has a horizontal tangent line at $x = -1$ and $x = 0$. Figure 4.18 shows the graph of f superimposed on the graph of f', confirming our conclusions.

13) 7 Inderminate forms :

(a) $\frac{0}{0}, \frac{\infty}{\infty}, \infty - \infty, 0 \cdot \infty, 1^{\infty}, 0^0, \infty^0$

(b) **For the forms $1^{\infty}, 0^0, \infty^0$, Taking the log to both sides.**

14) L'hôpital Rule

L'HOSPITAL'S RULE Suppose f and g are differentiable and $g'(x) \neq 0$ on an open interval I that contains a (except possibly at a). Suppose that

$$\lim_{x \to a} f(x) = 0 \quad \text{and} \quad \lim_{x \to a} g(x) = 0$$

or that

$$\lim_{x \to a} f(x) = \pm\infty \quad \text{and} \quad \lim_{x \to a} g(x) = \pm\infty$$

(In other words, we have an indeterminate form of type $\frac{0}{0}$ or ∞/∞.) Then

$$\lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{f'(x)}{g'(x)}$$

if the limit on the right side exists (or is ∞ or $-\infty$).

15) Examples

Example 1 : Calculate $\displaystyle\lim_{x \to \infty} \frac{\ln x}{\sqrt[3]{x}}$.

SOLUTION Since $\ln x \to \infty$ and $\sqrt[3]{x} \to \infty$ as $x \to \infty$, l'Hospital's Rule applies:

$$\lim_{x \to \infty} \frac{\ln x}{\sqrt[3]{x}} = \lim_{x \to \infty} \frac{1/x}{\frac{1}{3}x^{-2/3}}$$

Notice that the limit on the right side is now indeterminate of type $\frac{0}{0}$. But instead of applying l'Hospital's Rule a second time as we did in Example 2, we simplify the expression and see that a second application is unnecessary:

$$\lim_{x \to \infty} \frac{\ln x}{\sqrt[3]{x}} = \lim_{x \to \infty} \frac{1/x}{\frac{1}{3}x^{-2/3}} = \lim_{x \to \infty} \frac{3}{\sqrt[3]{x}} = 0$$

Example 2 : Calculate $\displaystyle\lim_{x \to 0^+} (1 + \sin 4x)^{\cot x}$.

SOLUTION First notice that as $x \to 0^+$, we have $1 + \sin 4x \to 1$ and $\cot x \to \infty$, so the given limit is indeterminate. Let

$$y = (1 + \sin 4x)^{\cot x}$$

Then $$\ln y = \ln[(1 + \sin 4x)^{\cot x}] = \cot x \ln(1 + \sin 4x)$$

so l'Hospital's Rule gives

$$\lim_{x \to 0^+} \ln y = \lim_{x \to 0^+} \frac{\ln(1 + \sin 4x)}{\tan x} = \lim_{x \to 0^+} \frac{\dfrac{4 \cos 4x}{1 + \sin 4x}}{\sec^2 x} = 4$$

So far we have computed the limit of $\ln y$, but what we want is the limit of y. To find this we use the fact that $y = e^{\ln y}$:

$$\lim_{x \to 0^+} (1 + \sin 4x)^{\cot x} = \lim_{x \to 0^+} y = \lim_{x \to 0^+} e^{\ln y} = e^4$$

Example 3 : Evaluate $\lim_{x\to\infty}\left(1+\frac{1}{x}\right)^x$.

This limit has the form 1^∞. Noting that $(1 + 1/x)^x = e^{x\ln(1+1/x)}$, the first step is to evaluate

$$L = \lim_{x\to\infty} x\ln\left(1 + \frac{1}{x}\right),$$

which has the form $0 \cdot \infty$. Proceeding as in part (a), we have

$$L = \lim_{x\to\infty} x\ln\left(1+\frac{1}{x}\right) = \lim_{x\to\infty}\frac{\ln(1+1/x)}{1/x} \qquad x = \frac{1}{1/x}$$

$$= \lim_{x\to\infty}\frac{\frac{1}{1+1/x}\cdot\left(-\frac{1}{x^2}\right)}{\left(-\frac{1}{x^2}\right)} \qquad \text{L'Hôpital's Rule for } 0/0 \text{ form}$$

$$= \lim_{x\to\infty}\frac{1}{1+1/x} = 1. \qquad \text{Simplify and evaluate.}$$

The second step is to exponentiate:

$$\lim_{x\to\infty}\left(1+\frac{1}{x}\right)^x = e^L = e^1 = e.$$

16) Exercises :

Q1 - 2) Finding roots with Newton's method

Use a calculator or program to compute the first 10 iterations of Newton's method when they are applied to the following functions with the given initial approximation. Make a table similar to show the steps.

1) $f(x) = \sin x + x - 1;\ x_0 = 1.5$

2) $f(x) = \ln(x+1) - 1;\ x_0 = 1.7$

Q3 - 5) Finding intersection points

Use Newton's method to approximate all the intersection points of the following pairs of curves. Some preliminary graphing or analysis may help in choosing good initial approximations.

3) $y = x^3$ and $y = x^2 + 1$

4) $y = 4\sqrt{x}$ and $y = x^2 + 1$

5) $y = \ln x$ and $y = x^3 - 2$

6)	**Residuals and errors** Approximate the root of $f(x) = x^{10}$ at $x = 0$ using Newton's method with an initial approximation of $x_0 = 0.5$. Make a table showing the first 10 approximations, the error in these approximations (which is $\lvert x_n - 0\rvert = \lvert x_n\rvert$), and the residual of these approximations (which is $f(x_n)$). Comment on the relative size of the errors and the residuals, and give an explanation.

7) **Approximating square roots** Let $a > 0$ be given and suppose we want to approximate $\sqrt{a}$ using Newton's method.

a. Explain why the square root problem is equivalent to finding the positive root of $f(x) = x^2 - a$.

b. Show that Newton's method applied to this function takes the form (sometimes called the Babylonian method)

$$x_{n+1} = \frac{1}{2}\left(x_n + \frac{a}{x_n}\right), \text{ for } n = 0, 1, 2, \ldots.$$

c. How would you choose initial approximations to approximate $\sqrt{13}$ and $\sqrt{73}$?

d. Approximate $\sqrt{13}$ and $\sqrt{73}$ with at least 10 significant digits.

Fixed points

An important question about many functions concerns the existence and location of **fixed points**. **A fixed point of $f(x)$** is a value of x that satisfies the equation f (x) = x; it corresponds to a point at which the graph of $y = f(x)$ intersects the line y = x.

8) **Fixed points of quadratics and quartics** Let $f(x) = ax(1 - x)$, where a is a real number and $0 \le x \le 1$. Recall that the fixed point of a function is a value of x such that $f(x) = x$ (Exercises 28–31).

a. Without using a calculator, find the values of a, with $0 < a \le 4$, such that f has a fixed point. Give the fixed point in terms of a.

b. Consider the polynomial $g(x) = f(f(x))$. Write g in terms of a and powers of x. What is its degree?

c. Graph g for $a = 2, 3$, and 4.

d. Find the number and location of the fixed points of g for $a = 2, 3$, and 4 on the interval $0 \le x \le 1$.

9) Prove that the function

$$f(x) = x^{101} + x^{51} + x + 1$$

has neither a local maximum nor a local minimum.

10) **Find the absolute maximum and absolute minimum values of f on the given interval.**

(a) $f(x) = 3x^2 - 12x + 5, \quad [0, 3]$

(b) $f(x) = \dfrac{x}{x^2 + 1}, \quad [0, 2]$

(c) $f(x) = xe^{-x^2/8}, \quad [-1, 4]$

(d) $f(x) = \ln(x^2 + x + 1), \quad [-1, 1]$

11) **Evaluate**

(a) $$\lim_{x \to 1} \frac{1 - x + \ln x}{1 + \cos \pi x}$$

(b) $$\lim_{x \to \infty} x^3 e^{-x^2}$$

(c) $$\lim_{x \to \infty} \left(1 + \frac{3}{x} + \frac{5}{x^2}\right)^x$$

12)

If f' is continuous, $f(2) = 0$, and $f'(2) = 7$, evaluate

$$\lim_{x \to 0} \frac{f(2 + 3x) + f(2 + 5x)}{x}$$

Questions 13 – 23 : Optimization Problems

13)

A model used for the yield Y of an agricultural crop as a function of the nitrogen level N in the soil (measured in appropriate units) is

$$Y = \frac{kN}{1 + N^2}$$

where k is a positive constant. What nitrogen level gives the best yield?

14)

A farmer wants to fence an area of 1.5 million square feet in a rectangular field and then divide it in half with a fence parallel to one of the sides of the rectangle. How can he do this so as to minimize the cost of the fence?

15)

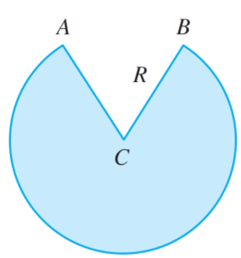
Find the point on the line $y = 4x + 7$ that is closest to the origin.

16)

A right circular cylinder is inscribed in a sphere of radius r. Find the largest possible volume of such a cylinder.

17)

A cone-shaped drinking cup is made from a circular piece of paper of radius R by cutting out a sector and joining the edges CA and CB. Find the maximum capacity of such a cup.



18)

A cone with height h is inscribed in a larger cone with height H so that its vertex is at the center of the base of the larger cone. Show that the inner cone has maximum volume when $h = \frac{1}{3}H$.

19)

A piece of wire 10 m long is cut into two pieces. One piece is bent into a square and the other is bent into an equilateral triangle. How should the wire be cut so that the total area enclosed is (a) a maximum? (b) A minimum?

20)

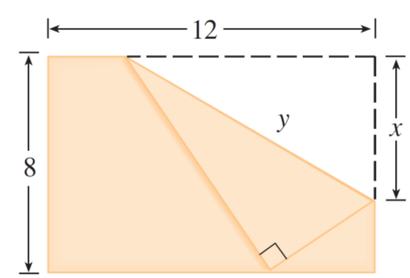
A baseball team plays in a stadium that holds 55,000 spectators. With ticket prices at \$10, the average attendance had been 27,000. When ticket prices were lowered to \$8, the average attendance rose to 33,000.

(a) Find the demand function, assuming that it is linear.

(b) How should ticket prices be set to maximize revenue?

21)

The upper right-hand corner of a piece of paper, 12 in. by 8 in., as in the figure, is folded over to the bottom edge. How would you fold it so as to minimize the length of the fold? In other words, how would you choose x to minimize y?

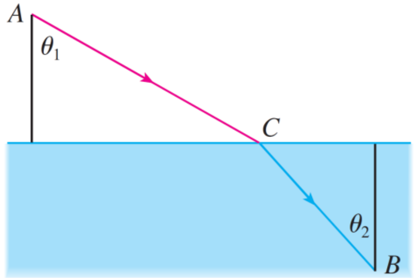


22)

Let v_1 be the velocity of light in air and v_2 the velocity of light in water. According to Fermat's Principle, a ray of light will travel from a point A in the air to a point B in the water by a path ACB that minimizes the time taken. Show that

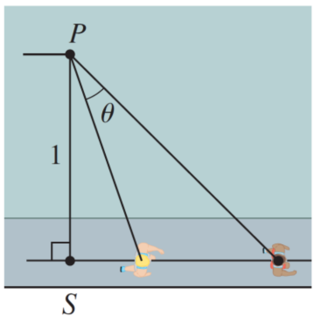
$$\frac{\sin\theta_1}{\sin\theta_2} = \frac{v_1}{v_2}$$

where θ_1 (the angle of incidence) and θ_2 (the angle of refraction) are as shown. This equation is known as Snell's Law.



23)

An observer stands at a point P, one unit away from a track. Two runners start at the point S in the figure and run along the track. One runner runs three times as fast as the other. Find the maximum value of the observer's angle of sight θ between the runners. [*Hint:* Maximize $\tan\theta$.]



(Q24 - 29) = (45-50)

45–50. $0 \cdot \infty$ form *Evaluate the following limits.*

45. $\lim_{x\to 0} x\csc x$

46. $\lim_{x\to 1^-} (1-x)\tan\left(\frac{\pi x}{2}\right)$

47. $\lim_{x\to 0} (\csc 6x \sin 7x)$

48. $\lim_{x\to\infty} (\csc(1/x)(e^{1/x}-1))$

49. $\lim_{x\to\pi/2^-} \left(\frac{\pi}{2}-x\right)\sec x$

50. $\lim_{x\to 0^+} (\sin x)\sqrt{\frac{1-x}{x}}$

(Q30 – 33) = (51-54)

51–54. $\infty - \infty$ form *Evaluate the following limits.*

51. $\lim_{x\to 0^+} \left(\cot x - \frac{1}{x}\right)$

52. $\lim_{x\to\infty} (x - \sqrt{x^2+1})$

53. $\lim_{\theta\to\pi/2^-} (\tan\theta - \sec\theta)$

54. $\lim_{x\to\infty} (x - \sqrt{x^2+4x})$

(Q34 – 42) = (55-63)

55–68. 1^∞, 0^0, ∞^0 forms *Evaluate the following limits or explain why they do not exist. Check your results by graphing.*

55. $\lim_{x\to 0^+} x^{2x}$

56. $\lim_{x\to 0} (1+4x)^{3/x}$

57. $\lim_{\theta\to\pi/2^-} (\tan\theta)^{\cos\theta}$

58. $\lim_{\theta\to 0^+} (\sin\theta)^{\tan\theta}$

59. $\lim_{x\to 0^+} (1+x)^{\cot x}$

60. $\lim_{x\to\infty} \left(1+\frac{1}{x}\right)^{\ln x}$

61. $\lim_{x\to\infty} \left(1+\frac{a}{x}\right)^{x}$, for a constant a

62. $\lim_{x\to 0} (e^{5x}+x)^{1/x}$

63. $\lim_{x\to 0} (e^{ax}+x)^{1/x}$, for a constant a

64. $\lim_{x\to 0} (2^{ax}+x)^{1/x}$, for a constant a