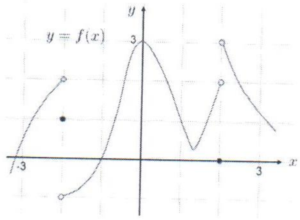
Solutions to Math 1013 (Tutorial 13) (Revision Exercises)

1) Find $\lim\limits_{x\to -2^-} \dfrac{|f(|x|)-2|}{f(x)+1}$ according to the given graph of f below.

$\boxed{\begin{array}{l} \text{as } x\to -2^- \Rightarrow x<-2 \\ \therefore |x|>2 \\ \Rightarrow |x|\to 2^+ \end{array}}$



Solution:

When $x\to -2^-$, $f(x)\to 2$, $|x|\to 2^+$, and $f(|x|)\to 3$, hence

$$\lim_{x\to -2^-} \frac{|f(|x|)-2|}{f(x)+1} = \frac{|3-2|}{2+1} = \frac{1}{3}$$

2) Find the horizontal asymptote of the function $y = \left(\cos\dfrac{1}{2x} - \dfrac{3}{x^2}\right)\left(1 + x\sin\dfrac{1}{x}\right)$.

Solutions:

Then answer is (c). $\lim\limits_{x\to\infty}\left(\cos\dfrac{1}{2x} - \dfrac{3}{x^2}\right)\left(1+x\sin\dfrac{1}{x}\right) = \lim\limits_{x\to\infty}\left(\cos\dfrac{1}{2x}-\dfrac{3}{x^2}\right)\lim\limits_{x\to\infty}\left(1+\dfrac{\sin\frac{1}{x}}{\frac{1}{x}}\right) =$ $1\cdot\left(1+\lim\limits_{x\to\infty}\dfrac{\cos\frac{1}{x}\cdot\frac{-1}{x^2}}{-\frac{1}{x^2}}\right) = 1+1 = 2$ by the L'Hôpital's rule. Note also $\lim\limits_{x\to\infty}\left(\cos\dfrac{1}{2x}-\dfrac{3}{x^2}\right)\left(1+x\sin\dfrac{1}{x}\right) = 2$.

3) If $\dfrac{d}{dx}[f(\frac{1}{3}x^3)] = 2x^5$, what is $f'(x)$?

Solutions:

By the Chain Rule, $\dfrac{d}{dx}[f(\frac{1}{3}x^3)] = f'(\frac{1}{3}x^3)\cdot x^2 = 2x^5$; i.e., $f'(\frac{1}{3}x^3) = 2x^3 =$ $6\cdot(\frac{1}{3}x^3)$, and hence $f'(x) = 6x$.

$\because \dfrac{df(w)}{dw} = f'(w)$

$f'(\frac{x^3}{3}) = \dfrac{d f(\frac{x^3}{3})}{d(\frac{x^3}{3})}$

Hence $\dfrac{d f(\frac{x^3}{3})}{dx} = \dfrac{df(\frac{x^3}{3})}{d(\frac{x^3}{3})}\dfrac{d(\frac{x^3}{3})}{dx}$

$= f'(\frac{x^3}{3})\dfrac{d\frac{x^3}{3}}{dx} = f'(\frac{x^3}{3})(x^2)$

4) Find the limit: $\lim\limits_{x\to 0}(e^x+2x)^{\frac{1}{2x}}$.

Solution: $\lim\limits_{x\to 0}\ln y = \lim\limits_{x\to 0}\dfrac{\ln(e^x+2x)}{2x} = \lim\limits_{x\to 0}\dfrac{\frac{e^x+2}{e^x+2x}}{2} = \dfrac{3}{2}$ by the L'Hôpital Rule.

5) **By L'Hopital Rule,** ($= \frac{0}{0}$ form) As $\int_0^0 g(t)dt = 0$

$$\lim_{x\to 0}\frac{\int_0^{x^2}(t+1)f(t)dt}{3x^2} = \lim_{x\to 0}\frac{(x^2+1)f(x^2)2x}{6x}$$
$$= \lim_{x\to 0}\frac{1}{3}(x^2+1)f(x^2)$$
$$= \frac{1}{3}f(0) = \frac{4}{3}$$

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6) **Sketch the graph of $y = f(x) = 2x^6 - 3x^4$.**

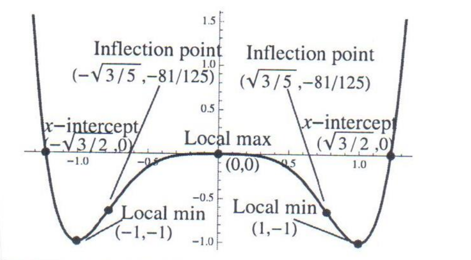
The domain of f is $(-\infty, \infty)$, and there is even symmetry, because $f(-x) = f(x)$. $f'(x) = 12x^5 - 12x^3 = 12x^3(x^2-1)$. This is 0 when $x = \pm 1$. $f''(x) = 60x^4 - 36x^2 = 12x^2(5x^2-3)$, which is 0 when $x = \pm\sqrt{3/5}$. Note that $f'(-2) < 0$, $f'(-.5) > 0$, $f'(.5) < 0$, and $f'(2) > 0$. Thus f is decreasing on $(-\infty, -1)$ and on $(0, 1)$, while it is increasing on $(-1, 0)$ and on $(1, \infty)$.

There is a local maximum of 0 at $x = 0$ and local minimums of -1 at $x = \pm 1$. Note also that $f''(x) > 0$ for $x < -\sqrt{3/5}$ and for $x > \sqrt{3/5}$, and $f''(x) < 0$ for $-1 < x < 1$, so there are inflection points at $x = \pm\sqrt{3/5}$, and f is concave up on $(-\infty, -\sqrt{3/5})$ and on $(\sqrt{3/5}, \infty)$, and is concave down on $(-\sqrt{3/5}, \sqrt{3/5})$.

X	$x<-\sqrt{\frac{3}{2}}$	$-\sqrt{\frac{3}{2}}$	$-\sqrt{\frac{3}{2}}<x<-1$	-1	$-1<x<-\sqrt{\frac{3}{5}}$	$-\sqrt{\frac{3}{5}}$	$-\sqrt{\frac{3}{5}}<x<0$	0	$0<x<\sqrt{\frac{3}{5}}$	$\sqrt{\frac{3}{5}}$	$\sqrt{\frac{3}{5}}<x<1$
$f(x)$		0		-1		$\frac{-81}{125}$		0		$\frac{-81}{125}$	
$f'(x)$	$-$	$-$	$-$	0	+	+	+	0	$-$	$-$	$-$
$f''(x)$	+	+	+	+	+	0	$-$	$-$	$-$	0	+
	↘	x-intercept $(-\sqrt{\frac{3}{2}}, 0)$	↘	min. point $(-1,-1)$	↗	point of inflection $(-\sqrt{\frac{3}{5}}, \frac{-81}{125})$	↗	max. point $(0,0)$	↘	Point of inflection $(\sqrt{\frac{3}{5}}, \frac{81}{25})$	↘

X	$\sqrt{\frac{3}{5}}<x<1$	1	$1<x<\sqrt{\frac{3}{2}}$	$\sqrt{\frac{3}{2}}$	$x>\sqrt{\frac{3}{2}}$
$f(x)$		-1		0	
$f'(x)$	$-$	0	+	+	+
$f''(x)$	+	+	+	+	+
	↘	min. point $(1,-1)$	↗	x-intercept $(\sqrt{\frac{3}{2}}, 0)$	↗

Sketch the graph:



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7) $y = f(x) = \dfrac{x^2+x}{2-x^2}$

The derivatives of f are $f'(x) = \frac{x^2+8x+4}{(x^2-4)^2}$, $f''(x) = -2 \cdot \frac{x^3+12x^2+12x+16}{(x^2-4)^3}$. Solving $f'(x) = 0$ gives critical points $x = -4 \pm 2\sqrt{3} \approx -7.464, -0.536$, and solving $f''(x) = 0$ numerically gives a possible inflection point at $x \approx -11.045$. Also note that f' and f'' are undefined at $x = \pm 2$; f has vertical asymptotes at these points. Testing the sign of $f'(x)$ shows that f is decreasing on the intervals $(-7.464, -2)$ and $(-2, -0.536)$ and increasing on $(-\infty, -7.464)$, $(-0.536, 2)$ and $(2, \infty)$. The First Derivative Test shows that a local minimum occurs at $x \approx -0.536$ and a local maximum occurs at $x \approx -7.464$.

Testing the sign of $f''(x)$ shows that f is concave down on the intervals $(-11.045, -2)$ and $(2, \infty)$ and concave up on the intervals $(-\infty, -11.045)$ and $(-2, 2)$. Therefore an inflection point occurs at $x \approx -11.045$. The graph has x-intercepts at $x = -1, 0$. Observe that $\lim\limits_{x\to 2^-} \dfrac{x^2+x}{4-x^2} = \infty$, $\lim\limits_{x\to 2^+} \dfrac{x^2+x}{4-x^2} = -\infty$; therefore f has no absolute min or max. We also observe that $\lim\limits_{x\to\pm\infty} f(x) = -1$.

① Vertical asympotetes, $x = -2$, $x = 2$

② max. point $(-4-2\sqrt{3}, -0.93) \approx (-7.46, -0.93)$

min. point $(-4+2\sqrt{3}, -0.67) \approx (-0.54, -0.67)$

③ points of inflection: $(-11.04, -0.94)$,

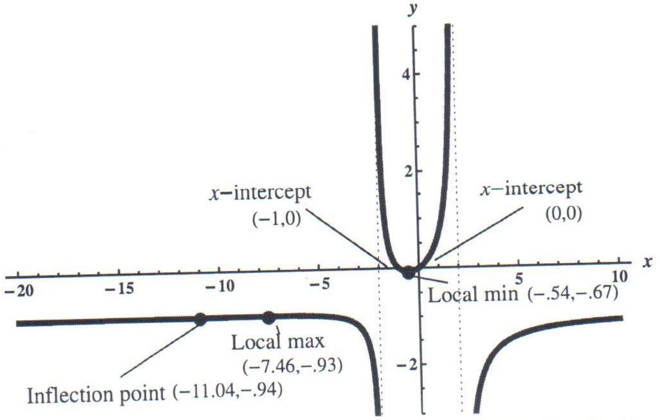
④ x-intercept : $(-1, 0)$, $(0, 0)$

⑤ horizontal asymptote $y = -1$

⑥ Concave downward $x \in (-11.045, -2)$ and $(2, \infty)$

Concave upward $x \in (-\infty, -11.045)$ and $(-2, 2)$

Sketch the graph of $y = \dfrac{x^2+x}{2-x}$.

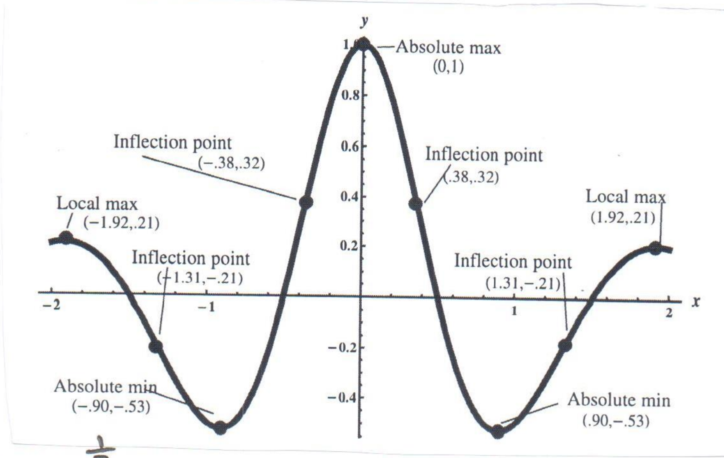


Solutions to Math 1013 (tutorial 13)

(8) $y = \dfrac{\cos \pi x}{1+x^2}$ for $x \in [-2, 2]$

Observe that f is an even function, so its graph is symmetric in the y-axis. The derivatives of f are $f'(x) = -\frac{\pi(1+x^2)\sin \pi x + 2x\cos \pi x}{(1+x^2)^2}$, $f''(x) = \frac{4\pi(x^3+x)\sin \pi x + (-\pi^2 x^4 + (6-2\pi^2)x^2 - \pi^2 - 2)\cos \pi x}{(1+x^2)^3}$. Note that $x = 0$ is a critical point; solving $f'(x) = 0$ numerically gives additional critical points $x = \pm 0.90, \pm 1.92$, and solving $f''(x) = 0$ gives possible inflection points at $x \approx \pm 0.38, \pm 1.31$. Testing the sign of f' between critical points shows that f is decreasing on the intervals $(-1.92, -0.90)$, $(0, 0.90)$ and $(1.92, 2)$ and increasing on $(-2, -1.92)$, $(-0.90, 0)$ and $(0.90, 1.92)$. The First Derivative Test shows that local minima occur at $x \approx \pm 0.90$ and local maxima occur at $x = 0$ and $x \approx \pm 1.92$. Comparing the values of f at the critical points and endpoints shows that the absolute maximum occurs at $x = 0$ and the absolute minimum at $x \approx \pm 0.90$.

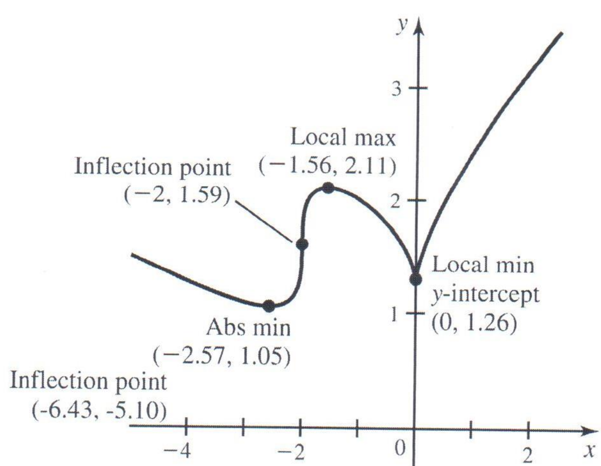
Testing the sign of $f''(x)$ shows that f is concave down on the intervals $(-2, -1.31)$, $(-0.38, 0.38)$ and $(1.31, 2)$ and concave up on the intervals $(-1.31, -0.38)$ and $(0.38, 1.31)$. Therefore inflection points occur at $x = \pm 0.38, \pm 1.31$. The x-intercepts of the graph occur when $\cos \pi x = 0$, which gives $x = \pm 1/2, \pm 3/2$.



(9) $y = x^{\frac{2}{3}} + (x+2)^{\frac{1}{3}}$

The derivatives of f are $f'(x) = \frac{2}{3}x^{-1/3} + \frac{1}{3}(x+2)^{-2/3}$, $f''(x) = -\frac{2}{9}\left(x^{-4/3} + (x+2)^{-5/3}\right)$. Solving $f'(x) = 0$ gives critical points $x \approx -2.57, -1.56$; we also have critical points at $x = -2, 0$ because $f'(x)$ is undefined at these points. Solving $f''(x) = 0$ numerically gives a possible inflection point at $x \approx -6.43$. We also have possible inflection points at $x = -2, 0$ because $f''(x)$ is undefined at these points. Testing the sign of $f'(x)$ shows that f is decreasing on the intervals $(-\infty, -2.57)$ and $(-1.56, 0)$ and increasing on $(-2.57, -1.56)$ and $(0, \infty)$. The First Derivative Test shows that local mins occur at $x \approx -2.57$ and $x = 0$ and a local max occurs at $x \approx -1.56$.

Testing the sign of $f''(x)$ shows that f is concave down on the intervals $(-\infty, -6.43)$, $(-2, 0)$ and $(0, \infty)$ and concave up on the interval $(-6.43, -2)$. Therefore inflection points occur at $x \approx -6.43$ and $x = -2$. Because $\lim_{x \to \pm\infty} f(x) = \infty$, f has no absolute maximum. The absolute minimum occurs at $x \approx -2.57$.



Solutions to Math 1013 (Tutorial 13)

10) $y = \dfrac{-x\sqrt{x^2-4}}{x-2}$

① The domain of f is $(-\infty,-2]\cup(2,\infty)$ and there is no symmetry.

$\because\ x^2-4=(x+2)(x-2)>0$ and $x\neq 2$

② There is a vertical asymptote at $x=2$ because $\lim_{x\to 2^+}\frac{-x\sqrt{x^2-4}}{x-2}=-\infty$.

③ There are no horizontal asymptotes.

④ $f'(x)=\frac{(x-2)(-x^2(x^2-4)^{-1/2}+(x^2-4)^{1/2}(-1))-((-x)(x^2-4)^{1/2})}{(x-2)^2}$. This can be written as $\frac{-x^2+2x+4}{(x-2)\sqrt{x^2-4}}$,

$f'(x)=0 \Rightarrow x=1+\sqrt{5}\approx 3.236$

⑤ $f'(x)>0$ on $(-\infty,-2)$, and on $(2,1+\sqrt{5})$, so f is increasing on those intervals. $f'(x)<0$ on $(1+\sqrt{5},\infty)$, so f is decreasing on that interval. There is a local maximum at $x=1+\sqrt{5}$.

max. point $=(3.236,-6.660)$

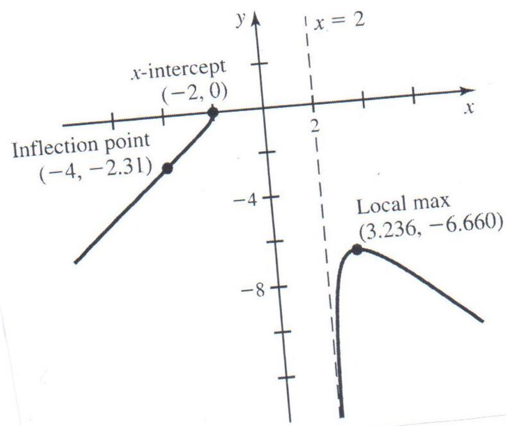
⑥ $f''(x)$ simplifies to be $\frac{-4(x+4)}{(x-2)^2(x+2)\sqrt{x^2-4}}$, which is 0 at $x=-4$

$f''(x)>0$ on $(-4,-2)$, so f is concave up on that interval. $f''(x)<0$ on $(-\infty,-4)$ and on $(1+\sqrt{5},\infty)$, so f is concave down on those intervals. There is a point of inflection at $x=-4$.

point of inflection $=(-4,-2.31)$

⑦ x-intercept $\rightarrow (-2,0)$

⑧ Sketch the graph of $y=\dfrac{-x\sqrt{x^2-4}}{x-2}$



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11. a. Let x be the distance from the point on the shoreline nearest to the boat to the point where the woman lands on shore; then the remaining distance she must travel on shore is $6-x$. By the Pythagorean theorem, the distance the woman must row is $\sqrt{x^2+16}$. So the time for the rowing leg is $\frac{\text{distance}}{\text{rate}} = \frac{\sqrt{x^2+16}}{2}$ and the time for the walking leg is $\frac{\text{distance}}{\text{rate}} = \frac{6-x}{3}$. The total travel time for the trip is the objective function $T(x) = \frac{\sqrt{x^2+16}}{2} + \frac{6-x}{3}$. We wish to minimize this function for $0 \le x \le 6$. The critical points of the objective function satisfy $T'(x) = \frac{x}{2\sqrt{x^2+16}} - \frac{1}{3} = 0$ which when simplified gives $5x^2 = 64$ and so $x = 8/\sqrt{5}$ is the only critical point in $(0,6)$. From the First Derivative Test we see that T has a local minimum at this point, so $x = 8/\sqrt{5}$ must give the minimum value of T on $[0,6]$.

b. Let $v > 0$ be the woman's rowing speed. Then the total travel time is now given by $T(x) = \frac{\sqrt{x^2+16}}{v} + \frac{6-x}{3}$. The derivative of the objective function is $T'(x) = \frac{x}{v\sqrt{x^2+16}} - \frac{1}{3}$. If we try to solve the equation $T'(x) = 0$ as in part (a) above, we see that there is at most one solution $x > 0$. Therefore there can be at most one critical point of T in the interval $(0,6)$. Observe also that $T'(0) = -1/3 < 0$ so the absolute minimum of T on $[0,6]$ cannot occur at $x = 0$. So one of two things must happen: there is a unique critical point for T in $(0,6)$ which is the absolute minimum for T on $[0,6]$, and then $T'(6) > 0$; or, T is decreasing on $[0,6]$, and then $T'(6) \le 0$ (the quickest way to the restaurant is to row directly in this case). The condition $T'(6) \le 0$ is equivalent to $\frac{6}{\sqrt{6^2+16}} \le \frac{v}{3}$ which gives $v \ge 9/\sqrt{13}$ mi/hr.

12. Let L be the ladder length and x be the distance between the foot of the ladder and the fence. The Pythagorean theorem gives the relationship $L^2 = (x+5)^2 + b^2$, where b is the height of the top of the ladder. We see that $b/(x+5) = 8/x$ by similar triangles, which gives $b = 8(x+5)/x$. Substituting in the expression for L^2 above gives $L^2 = (x+5)^2 + 64\frac{(x+5)^2}{x^2} = (x+5)^2\left(1 + \frac{64}{x^2}\right)$. It suffices to minimize L^2 instead of L. However in this case x and b must satisfy $x, b \le 20$. Solving $20 = 8(x+5)/x$ for x gives $x = 10/3$, so the condition $b \le 20$ corresponds to $x \ge 10/3$, and we see that we must minimize L^2 for $10/3 \le x \le 20$. We have $\frac{d}{dx}L^2 = (x+5)^2\left(-\frac{128}{x^3}\right) + 2(x+5)\left(1+\frac{64}{x^2}\right) = \frac{2(x+5)(x^3-320)}{x^3}$. Because $x > 0$, the only critical point is $x = \sqrt[3]{320} \approx 6.84$. By the First Derivative Test, this critical point corresponds to a local minimum, and by Theorem 4.5, this solitary local minimum is also the absolute minimum on the interval $[10/3, 20]$. Substituting $x \approx 6.84$ in the expression for L^2 we find the length of the shortest ladder $L \approx 18.22$ ft.

13. Let x and y be the dimensions of the flower garden; the area of the flower garden is 30, so we have the constraint $xy = 30$ which gives $y = 30/x$. The dimensions of the garden and borders are $x+4$ and $y+2$, so the objective function to be minimized for $x > 0$ is $A = (x+4)(y+2) = (x+4)\left(\frac{30}{x}+2\right) = 2x + \frac{120}{x} + 38$. The critical points of $A(x)$ satisfy $A'(x) = 2 - \frac{120}{x^2} = 0$, which has unique solution $x = \sqrt{60} = 2\sqrt{15}$. By the First (or Second) Derivative test, this critical point gives a local minimum, which by Theorem 4.5 must be the absolute minimum of A over $(0,\infty)$. The corresponding value of y is $30/2\sqrt{15} = \sqrt{15}$, so the dimensions are $\sqrt{15}$ by $2\sqrt{15}$ m.

14. Let x be the distance from the point on shore nearest the island to the point where the underwater cable meets the shore, and let y be the be the length of the underwater cable. By the Pythagorean theorem, $y = \sqrt{x^2+3.5^2}$. The objective function to be minimized is the cost given by $C(x) = 2400\sqrt{x^2+3.5^2} + 1200 \cdot (8-x) = 2400\sqrt{x^2+3.5^2} - 1200x + 9600$. We wish to minimize this function for $0 \le x \le 8$. The critical points of $C(x)$ satisfy $C'(x) = \frac{2400x}{\sqrt{x^2+3.5^2}} - 1200 = 1200\left(\frac{2x}{\sqrt{x^2+3.5^2}} - 1\right) = 0$, which we solve to obtain $x = 7\sqrt{3}/6$. By the First Derivative Test, this critical point corresponds to a local minimum, and by Theorem 4.5, this solitary local minimum is also the absolute minimum on the interval $[0,8]$. Therefore the optimal point on shore has distance $x = 7\sqrt{3}/6$ mi from the point on shore nearest the island, in the direction of the power station.

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15.

a. The dimensions of the box are $3-2x, 4-2x$ and x, so the volume is given by $V(x) = x(3-2x)(4-2x) = 4x^3 - 14x^2 + 12x$. The dimensions cannot be negative, so we must have $0 \le x \le 3/2$. The critical points of $V(x)$ satisfy $V'(x) = 12x^2 - 28x + 12 = 4(3x^2 - 7x + 3) = 0$. This quadratic equation has roots $x = (7-\sqrt{13})/6 \approx 0.57$ and $x = (7+\sqrt{13})/6 \approx 1.77$, so the only critical point in $(0, 3/2)$ is $x = (7-\sqrt{13})/6 \approx 0.57$. We have $V(0) = V(3/2) = 0$, so the maximum volume is $V(0.57) \approx 3.03$ ft^3.

b. In this case the dimensions of the box are $l-2x$, $l-2x$ and x, so the volume is given by $V(x) = x(l-2x)^2 = 4x^3 - 4lx^2 + l^2x$. The dimensions cannot be negative, so we must have $0 \le x \le l/2$. The critical points of $V(x)$ satisfy $V'(x) = 12x^2 - 8lx + l^2 = (6x-l)(2x-l) = 0$. This quadratic equation has roots $x = l/6$ and $l/2$, so the only critical point in $(0, l/2)$ is $x = l/6$. We have $V(0) = V(l/2) = 0$, so the maximum volume is $V(l/6) = 2l^3/27$.

c. In this case the dimensions of the box are $l-2x$, $L-2x$ and x, so the volume is given by $V(x) = x(l-2x)(L-2x) = 4x^3 - 2(l+L)x^2 + lLx$. The dimensions cannot be negative, so we must have $0 \le x \le l/2$ (because we are letting $L \to \infty$, we may assume that $l \le L$). The critical points of $V(x)$ satisfy $V'(x) = 12x^2 - 4(l+L)x + lL = 0$, and this quadratic equation has roots $x = \frac{L+l\pm\sqrt{L^2-lL+l^2}}{6}$. Now $V(x)$ is a cubic polynomial with roots $x = 0, l/2, L/2$, and so has exactly one critical point between 0 and $l/2$, which gives the maximum of $V(x)$ on the interval $[0, l/2]$. This critical point is given by the smaller root of the quadratic above: $x = \frac{L+l-\sqrt{L^2-lL+l^2}}{6} = \frac{L+l-\sqrt{L^2-lL+l^2}}{6}\frac{(L+l+\sqrt{L^2-lL+l^2})}{(L+l+\sqrt{L^2-lL+l^2})} = \frac{(L+l)^2-(L^2-lL+l^2)}{6(L+l+\sqrt{L^2-lL+l^2})} = \frac{3lL}{6(L+l+\sqrt{L^2-lL+l^2})} = \frac{l}{2\left(1+\frac{l}{L}+\sqrt{1-\frac{l}{L}+\frac{l^2}{L^2}}\right)}$ (for the last step, divide all terms by L). As $L \to \infty$ with l fixed, $l/L \to 0$ so the size x of the corner squares that maximizes the volume has limit $l/4$ as $L \to \infty$.

16.

The cross-section is a trapezoid with height $3\sin\theta$; the larger of the parallel sides has length $3 + 2 \cdot 3\cos\theta = 3 + 6\cos\theta$ and the smaller parallel side has length 3. The area of this trapezoid is given by

$$A(\theta) = \frac{1}{2}(3 + (3 + 6\cos\theta)) \cdot 3\sin\theta = 9(1+\cos\theta)\sin\theta = 9\left(\sin\theta + \frac{\sin 2\theta}{2}\right),$$

using the identity $\sin 2\theta = 2\sin\theta\cos\theta$. We wish to maximize this function for $0 \le \theta \le \pi/2$. The critical points of $A(\theta)$ satisfy $\cos\theta + \cos 2\theta = \cos\theta + 2\cos^2\theta - 1 = 0$, using the identity $\cos 2\theta = 2\cos^2\theta - 1$. Therefore $x = \cos\theta$ satisfies the quadratic equation $2x^2 + x - 1 = 0$, which has roots $x = 1/2$ and -1. So the only critical point in $(0, \pi/2)$ is $\theta = \cos^{-1}(1/2) = \pi/3$, which by the First (or Second) Derivative Test and Theorem 4.5 gives the maximum area.

17.

Let r and h be the radius and height of both the cylinder and cones. The surface area of each cone is $\pi r\sqrt{r^2+h^2}$ and the surface area of the cylinder is $2\pi rh$, so we have the constraint $2\pi r\sqrt{r^2+h^2} + 2\pi rh = A$, which we rewrite as $h + \sqrt{r^2+h^2} = \frac{A}{2\pi r}$. Square to obtain $h^2 + 2h\sqrt{r^2+h^2} + r^2 + h^2 = \left(\frac{A}{2\pi r}\right)^2$, and substitute $\sqrt{r^2+h^2} = A/(2\pi r) - h$ in this equation to obtain $h^2 + 2h\left(\frac{A}{2\pi r} - h\right) + r^2 + h^2 = \left(\frac{A}{2\pi r}\right)^2$. Solving for h yields $h = \frac{\pi r}{A}\left(\frac{A^2}{4\pi^2 r^2} - r^2\right) = \frac{A}{4\pi r} - \frac{\pi r^3}{A}$.

We must have $h \ge 0$, which is equivalent to the condition $r \le \sqrt{A}/\sqrt{2\pi}$. So the possible r under consideration satisfy $0 \le r \le \sqrt{A}/\sqrt{2\pi}$. The objective function to be maximized is the combined volume of the cylinder and cones, which is given by

$$V = \pi r^2 h + 2 \cdot \frac{\pi}{3}r^2 h = \frac{5\pi}{3}r^2 h = \frac{5\pi}{3}r^2\left(\frac{A}{4\pi r} - \frac{\pi r^3}{A}\right) = \frac{5A}{12}r - \frac{5\pi^2}{3A}r^5.$$

The critical points of $V(r)$ satisfy $V'(r) = \frac{5A}{12} - \frac{25\pi^2}{3A}r^4 = 0$, which has unique positive solution $r = \sqrt{A}/(\sqrt[4]{20}\sqrt{\pi})$. To find the corresponding value of h, observe that $\pi r^3/A = A/(20\pi r)$, so $h = \frac{A}{4\pi r} - \frac{\pi r^3}{A} = \frac{A}{4\pi r} - \frac{A}{20\pi r} = \frac{A}{5\pi r}$ which gives $h = \sqrt{A}\sqrt[4]{20}/(5\sqrt{\pi})$. Note that $V(r) = 0$ at the endpoints of the interval $[0, \sqrt{A}/\sqrt{2\pi}]$, so the maximum volume must occur at the values of r and h given above.

18.

The viewing angle θ is given by $\theta = \cot^{-1}\left(\frac{x}{10}\right) - \cot^{-1}\left(\frac{x}{3}\right)$, and we wish to maximize this function for $x > 0$. The critical points satisfy $\theta'(x) = -\frac{1}{1+\left(\frac{x}{10}\right)^2} \cdot \frac{1}{10} - (-)\frac{1}{1+\left(\frac{x}{3}\right)^2} \cdot \frac{1}{3} = \frac{3}{x^2+3^2} - \frac{10}{x^2+10^2} = 0$ which simplifies to $3(x^2+100) = 10(x^2+9)$ or $x^2 = 30$. Therefore $x = \sqrt{30} \approx 5.5$ ft is the only critical point in $(0, \infty)$. By the First (or Second) Derivative Test, this critical point corresponds to a local maximum, and by Theorem 4.5, this solitary local maximum must be the absolute maximum on the interval $(0, \infty)$.

Solutions to Math 1013 (tutorial 13)

19. We have $x = 100\tan\theta$, so the rate at which the beam sweeps along the highway is

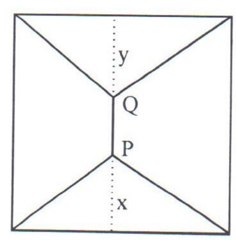
$$\frac{dx}{dt} = 100\sec^2\theta\frac{d\theta}{dt} = 100\sec^2\theta\cdot\frac{\pi}{6} = \frac{50\pi}{3}\sec^2\theta.$$

The beam meets the highway provided that the angle θ satisfies $-\pi/2 < \theta < \pi/2$. The function $\sec^2\theta$ is unbounded on this interval, and so has no maximum. The minimum value occurs at $\theta = 0$, because everywhere else $\sec^2\theta > 1$. Therefore the minimum rate is $50\pi/3 \approx 52.36$ m/s, and there is no maximum rate.

20. a. Let r and h be the radius and height of the inscribed cylinder. The region that lies above the cylinder inside the cone is a cone with radius r and height $H-h$; by similar triangles we have $\frac{H-h}{r} = \frac{H}{R}$ so $h = \frac{H}{R}(R-r)$. The volume of the cylinder is $V = \pi r^2 h = \frac{\pi H}{R}(Rr^2 - r^3)$, which we must maximize over $0 \le r \le R$. The critical points of $V(r)$ satisfy $V'(r) = \frac{\pi H}{R}(2Rr - 3r^2) = 0$, which has unique solution $r = 2R/3$ in $(0, R)$. Because $V(r) = 0$ at the endpoints $r = 0$ and $r = R$, the cylinder with maximum volume has radius $r = 2R/3$, height $h = H/3$ and volume $V = \pi r^2 h = \frac{4\pi}{27}R^2 H = \frac{4}{9}\cdot\frac{\pi}{3}R^2 H$, i.e. 4/9 the volume of the cone.

b. The lateral surface area of the cylinder is $A = 2\pi r h = 2\pi r\cdot\frac{H}{R}(R-r) = \frac{2\pi H}{R}r(R-r)$. This function takes its maximum over $0 \le r \le R$ at $r = R/2$, so the cylinder with maximum lateral surface area has dimensions $r = R/2$ and $h = H/2$.

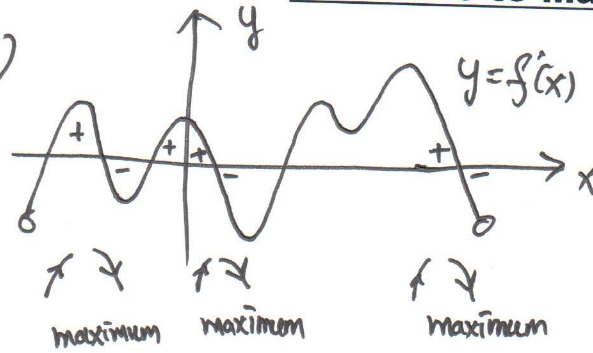
21. Following the hint, place two points P and Q above the midpoint of the base of the square, at distances x and y to the sides (see figure), where $0 \le x, y \le 1/2$. Then join the bottom vertices of the square to P, the upper vertices to Q and join P to Q. This road system has total length $L = 2\sqrt{x^2+\frac{1}{4}} + 2\sqrt{y^2+\frac{1}{4}} + (1-x-y) = 1 + \left(\sqrt{4x^2+1} - x\right) + \left(\sqrt{4y^2+1} - y\right)$. We can minimize the contributions from x and y separately; the critical points of the function $f(x) = \sqrt{4x^2+1} - x$ satisfy $f'(x) = \frac{4x}{\sqrt{4x^2+1}} - 1 = 0$ which gives $\sqrt{4x^2+1} = 4x$, so $12x^2 = 1$ and $x = 1/(2\sqrt{3})$. By the First (or Second) Derivative Test, this critical point corresponds to a local minimum, and by Theorem 4.5, this solitary local minimum is also the absolute minimum on the interval $[0, 1/2]$. The minimum value of $f(x)$ on this interval is $f(1/(2\sqrt{3})) = \sqrt{3}/2$, so the shortest road system has length $L = 1 + 2\cdot\frac{\sqrt{3}}{2} = 1 + \sqrt{3}$.



22. Let x be the distance between the point on the track nearest your initial position to the point where you catch the train. If you just catch the back of the train, then the train will have travelled $x + 1/3$ miles, which will require time $T = \frac{\text{distance}}{\text{rate}} = \frac{x+\frac{1}{3}}{20}$. The distance you must run is $\sqrt{x^2 + 1/(16)}$, so your running speed must be $v = \frac{\text{distance}}{\text{time}} = \frac{20\sqrt{x^2+\frac{1}{16}}}{x+\frac{1}{3}}$. We wish to minimize this function for $x \ge 0$. The derivative of $v(x)$ can be written $v'(x) = \left(\frac{x}{x^2+\frac{1}{16}} - \frac{1}{x+\frac{1}{3}}\right)v(x)$, so the critical points of $v(x)$ satisfy $\frac{x}{x^2+\frac{1}{16}} = \frac{1}{x+\frac{1}{3}}$ so $x\left(x+\frac{1}{3}\right) = x^2 + \frac{1}{16}$ which gives $x = 3/16$ mi. By the First (or Second) Derivative Test, this critical point corresponds to a local minimum, and by Theorem 4.5, this solitary local minimum is also the absolute minimum on the interval $[0, \infty)$. The minimum running speed is $v\left(\frac{3}{16}\right) = \frac{20\sqrt{\left(\frac{3}{16}\right)^2+\frac{1}{16}}}{\frac{3}{16}+\frac{1}{3}} = \frac{60\sqrt{9+16}}{9+16} = 12$ mph.

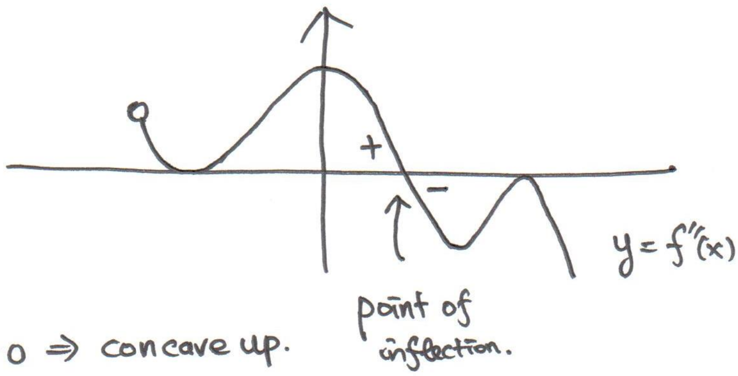
Solutions to Math 1013 (Tutorial 13)

23)



$\Rightarrow$ There are 3 local maximums. #

24)



$f''(x) > 0 \Rightarrow$ concave up.

$f''(x) < 0 \Rightarrow$ concave down.

$\therefore$ There is only one point of inflection. #

25) $f(x) = \sqrt{1+x} + \sin x$

$f(0.02) = f(0 + 0.02) = f(a + \Delta x)$

where $a = 0$, $\Delta x = 0.02$

The linear approximation is given by:

$$f(0.02) \approx f(0) + f'(0)\Delta x$$

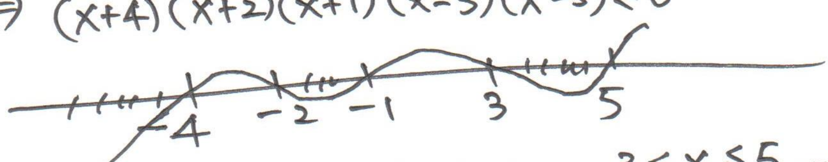
Now $f(0) = \sqrt{1+0} + \sin 0 = 1$

$f'(x) = \frac{1}{2\sqrt{1+x}} + \cos x$, $\quad f'(0) = \frac{1}{2} + 1 = \frac{3}{2}$

$\therefore \quad f(0.02) \approx 1 + \frac{3}{2}(0.02) = 1 + 0.03 = 1.03$ #

26 (a) $(x+1)(x+2)(x-3)(x+4)(x-5) \leq 0$

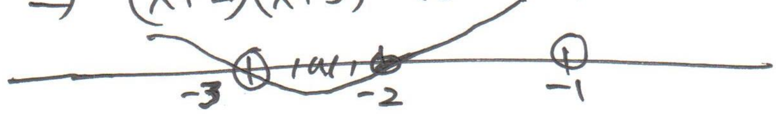
$\Rightarrow (x+4)(x+2)(x+1)(x-3)(x-5) \leq 0$



$\Rightarrow x \leq -4, \quad -2 \leq x \leq -1, \quad 3 \leq x \leq 5$ #

(b) $(x+1)^2 (x+2)(x+3)^3 < 0$

$\Rightarrow (x+2)(x+3) < 0$ and $x \neq -1$

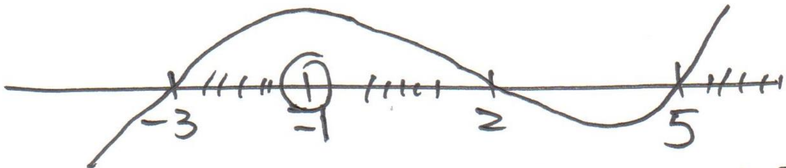


$-3 < x < -2$ #

Solutions to Math 1013 (Tutorial 13)

26(c) $\dfrac{(x+1)^2(x-2)(x+3)^3}{(x-5)^3} > 0$

$\Rightarrow (x+3)(x-2)(x-5) > 0$ and $x \neq -1$



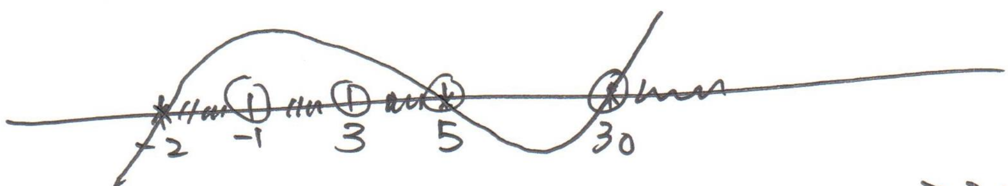
$\Rightarrow -3 < x < -1, \quad -1 < x < 2, \quad x > 5$

$\Rightarrow (-3,-1) \cup (-1,2) \cup (5,\infty)$

26(d) $\dfrac{(x+1)^{20}(x+2)^{2015}(x-3)^{\frac{300000}{4}}}{(x-5)^{27}(x-30)} \geq 0$

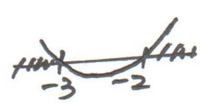
(As $\frac{300000}{4} = 3000 \times 25$ is an even integer.)

$\Rightarrow (x+2)(x-5)(x-30) \geq 0$ and $x \neq -1, 3, 5, 30$

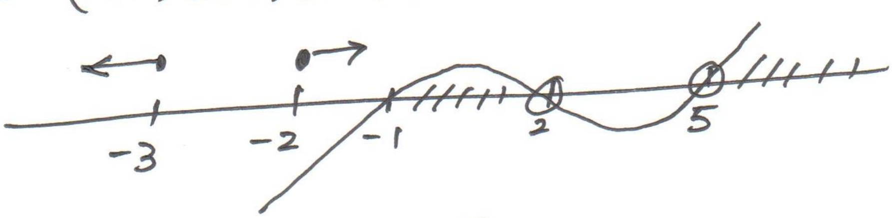


$\Rightarrow -2 \leq x < -1, \ -1 < x < 3, \ 3 < x < 5$ or $x > 30$ #

26(e) $\dfrac{(x+1)^{\frac{1}{3}}(x+2)^{\frac{1}{2}}(x+3)^{\frac{3}{4}}}{(x-5)^{27}(x-2)} \geq 0$



$\Rightarrow (x+1)(x-5)(x-2) \geq 0$ and $(x+3)(x+2) \geq 0$ and $x \neq 2, 5$



$\Rightarrow -1 \leq x < 2, \quad x > 5$

$\Rightarrow [-1, 2) \cup (5,\infty)$ #