Solutions to Math 1013 Tutorial 8 (Revision Exercises for Mid Term Exam)

1) $f(x)=\sqrt{x}$, $g(x)=x^3-2$

(a) $f\circ g = f(g(x)) = f(x^3-2)$
$= \sqrt{x^3-2}$ #

(b) $g\circ f = g(f(x)) = g(\sqrt{x})$
$= (\sqrt{x})^3-2$
$= x^{\frac{3}{2}}-2$ #

(c) $f\circ f = f(f(x)) = f(\sqrt{x})$
$= \sqrt{\sqrt{x}} = (x^{\frac{1}{2}})^{\frac{1}{2}}$
$= x^{\frac{1}{2}\times\frac{1}{2}} = x^{\frac{1}{4}}$ #

(d) $g\circ g = g(g(x)) = g(x^3-2)$
$= (x^3-2)^3-2$
$= x^9-3(2)(x^6)+3(2)^2(x^3) + (-2)^3-2$
$= x^9-6x^6+12x^3-10$ #

2(a) $g(x)=(x^2-4)\sqrt{x+5}$

Domain: $x \geq -5$

For $x>-5$, $x+5>0$

$g'(x) = (x^2-4)\left(\frac{1}{2\sqrt{x+5}}\right)+(\sqrt{x+5})(2x)$
$= \frac{(x^2-4)+4x(x+5)}{2\sqrt{x+5}}$
$= \frac{5x^2+20x-4}{2\sqrt{x+5}} \overset{\text{set}}{=} 0$

$\Rightarrow x = \frac{-10\pm 2\sqrt{30}}{5}$

$x=\frac{-10-2\sqrt{30}}{5}$ is a maximum.

$x=\frac{-10+2\sqrt{30}}{5}$ is a minimum.

$\therefore$ Range of $g(x)$ is
$\{y \mid y \geq g(\frac{-10+2\sqrt{30}}{5})\}$
$=\{y \mid y\geq -9.03039\}$

2(b) $f(x) = (9-x^2)^{\frac{3}{2}}$

$\Rightarrow 9-x^2 \geq 0$
$\Rightarrow -(x-3)(x+3)\geq 0$
$\Rightarrow (x+3)(x-3)\leq 0$

$\therefore$ Domain $=\{x \mid -3\leq x\leq 3\}$

$\max_{x\in[-3,3]} (9-x^2)^{\frac{3}{2}} = 9^{\frac{3}{2}} = 3^3$
$=27$

$\therefore$ Range of $f(x)$
$=\{y \mid 0\leq y\leq 27\}$
$=[0,27]$ #

2(c) $g(y) = \frac{y+1}{(y+2)(y-3)}$

Domain $=\{y \mid -\infty<y<-2,\ -2<y<+3,\ +3<y<\infty\}$
$=(-\infty,-2)\cup(-2,3)\cup(3,\infty)$.

2(c)

Range of $g(y)$
$=\{z\in\mathbb{R} \mid z\leq -2\sqrt{2}+3 \text{ or } z\geq 3+2\sqrt{2}\}$

$\uparrow$

Find $g'(y)=0$
and $g''(y_0)>0$
$g''(y_0)<0$

To find the max. and min. of $g(y)$.

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2(d) $F(w) = (2-w)^{\frac{1}{4}}$

$2-w\geq 0$
$\Rightarrow w\leq 2$

Domain $=\{w\in\mathbb{R} \mid w\leq 2\}$
$=(-\infty,2]$

Range $=[0,\infty)$ #

2(e) $f(x) = \frac{x^2-1}{x^3\sqrt{\ln x+1}}$

$x\neq 0$ and
$\ln x+1>0 \Rightarrow \ln x>-1$
$\Rightarrow x>e^{-1}=\frac{1}{e}$

$\therefore$ Domain $=(\frac{1}{e},\infty)$

Range $=\{y\in\mathbb{R} \mid y\leq \frac{1.48872}{4.73927}\}$

(But why?)

3) If $f^{-1}(x)$ exist,
f must be bijective.
i.e. f is both 1-to-1 and onto.

(4) $f(x)=\begin{cases}\frac{x^2-x-2}{x-2} & x\neq 2\\ 4 & x=2\end{cases}$

$\Rightarrow f(x)=\begin{cases}\frac{(x+1)(x-2)}{x-2} & x\neq 2\\ 4 & x=2\end{cases}$

$\Rightarrow f(x)=\begin{cases}x+1 & x\neq 2\\ 4 & x=2\end{cases}$

(a) $\lim_{x\to 2} f(x) = \lim_{x\to 2}(x+1) = 3$

(b) As $\lim_{x\to 2} f(x) = 3 \neq 4 = f(2)$
$\therefore f$ is discontinuous at $x=2$

(c) If f is discontinuous at $x=2$
$\Rightarrow f$ is not differentiable at $x=2$ #

(5) $g(x) = \frac{x-100}{\sqrt{x}-10} = \frac{(\sqrt{x}-10)(\sqrt{x}+10)}{(\sqrt{x}-10)}$

$\therefore g(x)=\sqrt{x}+10 \quad \forall x\neq 10$

(a) $\lim_{x\to 100^-} g(x) = \lim_{x\to 100^-}(\sqrt{x}+10) = 20$

(b) $\lim_{x\to 100^+} g(x) = \lim_{x\to 100^+}(\sqrt{x}+10) = 20$

(c) Let $g(100)=20$,
$\lim_{x\to 100} g(x) = 20 = g(100) \Rightarrow g(x)$ is continuous at $x=100$. #

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6(a) $\lim_{h\to0}\frac{3}{\sqrt{16+3h}+4}$

$=\frac{3}{\sqrt{16+0}+4}$

$=\frac{3}{4+4}=\frac{3}{8}$ #

(b) $\lim_{h\to0}\frac{\frac{1}{5+h}-\frac{1}{5}}{h}$ $\left(=\frac{0}{0}\text{ form}\right)$

$=\lim_{h\to0}\frac{5-(5+h)}{(5+h)(5)h}$

$=\lim_{h\to0}\frac{-h}{(5+h)(5)h}$

$=\lim_{h\to0}\frac{-1}{5(5+h)}$

$=\frac{-1}{25}$ #

$\left(\frac{d(x^{-1})}{dx}\Big|_{x=5}=(-1)(x^{-2})\Big|_{5}=\frac{-1}{25}\right)$

(c) $\lim_{x\to4}\frac{3(x-4)\sqrt{x+5}}{3-\sqrt{x+5}}$ $\left(=\frac{0}{0}\text{ form}\right)$

$=\lim_{x\to4}\frac{3(x-4)\sqrt{x+5}}{3-\sqrt{x+5}}\times\left(\frac{3+\sqrt{x+5}}{3+\sqrt{x+5}}\right)$

$=\lim_{x\to4}\frac{\left(3(x-4)\sqrt{x+5}\right)\left(3+\sqrt{x+5}\right)}{3^2-(x+5)}$

$=\lim_{x\to4}\frac{\left(3(x-4)\sqrt{x+5}\right)\left(3+\sqrt{x+5}\right)}{(4-x)}$

$=\lim_{x\to4}\left\{(-3)\sqrt{x+5}\,(3+\sqrt{x+5})\right\}$

$=(-3)(3)(3+3)$

$=-54$ #

6(d) As $-1\le\sin x\le1$

$\frac{-1}{\sqrt{x}}\le\frac{\sin x}{\sqrt{x}}\le\frac{1}{\sqrt{x}}$

$\forall x>1$

But $\lim_{x\to\infty}\frac{-1}{\sqrt{x}}=\lim_{x\to\infty}\frac{1}{\sqrt{x}}=0$

$\therefore\lim_{x\to\infty}\frac{\sin x}{\sqrt{x}}=0$

(By Squeeze thm)

Hence

$\lim_{x\to\infty}\left(5+\frac{\sin x}{\sqrt{x}}\right)$

$=\lim_{x\to\infty}5+\lim_{x\to\infty}\frac{\sin x}{\sqrt{x}}$

$=5+0=5$ #

6(e) Set $t=-x$

$\lim_{x\to-\infty}\frac{10x^3-3x^2+8}{\sqrt{25x^6+x^4+2}}$

$=\lim_{t\to\infty}\frac{-10t^3-3t^2+8}{\sqrt{25t^6+t^4+2}}$

$=\lim_{t\to\infty}\frac{-10-\frac{3}{t}+\frac{8}{t^3}}{\sqrt{25+\frac{1}{t^2}+\frac{2}{t^6}}}$

$=\frac{-10}{\sqrt{25}}=\frac{-10}{5}=-2$ #

6(f) Let $y=\left(\frac{1}{x}\right)^x$

$\ln y=x\ln\frac{1}{x}$

$=\frac{-\ln x}{\frac{1}{x}}$

$\lim_{x\to0^+}(\ln y)=\lim_{x\to0^+}\frac{-\ln x}{\frac{1}{x}}$ $\left(=\frac{\infty}{\infty}\text{ form}\right)$

$=\lim_{x\to0^+}\frac{-\frac{1}{x}}{-\frac{1}{x^2}}=\lim_{x\to0^+}(x)=0.$

$\therefore\lim_{x\to0^+}(\ln y)=0$

But $\ln x$ is a continuous function

$\therefore\lim_{x\to0^+}\ln y$

$=\ln\left(\lim_{x\to0^+}y\right)$

$\Rightarrow\lim_{x\to0^+}y=e^0=1$

$\Rightarrow\lim_{x\to0^+}\left(\frac{1}{x}\right)^x=1$ #

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7) $f(x)=\begin{cases}0, & x\le-5\\ \sqrt{25-x^2}, & -5<x<5\\ 3x, & x\ge5\end{cases}$

(a) $\lim_{x\to-5^-}f(x)$

$=\lim_{x\to-5^-}(0)=0$ #

(b) $\lim_{x\to-5^+}f(x)$

$=\lim_{x\to-5^+}\left(\sqrt{25-x^2}\right)$

$=\sqrt{25-25}$

$=0$ #

(c) $\because\lim_{x\to-5^+}f(x)=\lim_{x\to-5^-}f(x)=0$

$\therefore\lim_{x\to-5}f(x)=0$ #

(d) $\lim_{x\to5^-}f(x)=\lim_{x\to5^-}\sqrt{25-x^2}$

$=\sqrt{25-5^2}=0$

(e) $\lim_{x\to5^+}f(x)$

$=\lim_{x\to5^+}(3x)=(3)(5)$

$=15$

(f) As $\lim_{x\to5^-}f(x)=0$

$\neq15=\lim_{x\to5^+}f(x)$

$\therefore\lim_{x\to5}f(x)$ does not exist. #

8(a) $f(x)=\frac{(x^6+8)^{\frac{1}{3}}}{4x^2+\sqrt{3x^4+1}}$

$\lim_{x\to\infty}f(x)=\lim_{x\to\infty}\frac{\left(1+\frac{8}{x^6}\right)^{\frac{1}{3}}}{4+\sqrt{3+\frac{1}{x^4}}}$

$=\frac{1}{4+\sqrt{3}}$ #

$\lim_{x\to-\infty}f(x)=\lim_{t\to\infty}f(-t)$

$=\lim_{t\to\infty}\frac{(t^6+8)^{\frac{1}{3}}}{4t^2+\sqrt{3t^4+1}}$

$=\frac{1}{4+\sqrt{3}}$ #

8(b) $f(x)=4x\left(3x-\sqrt{9x^2+1}\right)$

$\lim_{x\to\infty}f(x)=\lim_{x\to\infty}4x\left(3x-\sqrt{9x^2+1}\right)$

$=\lim_{x\to\infty}4x\left(3x-\sqrt{9x^2+1}\right)\left(\frac{3x+\sqrt{9x^2+1}}{3x+\sqrt{9x^2+1}}\right)$

$=\lim_{x\to\infty}\frac{4x\left(9x^2-(9x^2+1)\right)}{3x+\sqrt{9x^2+1}}$

$=\lim_{x\to\infty}\frac{-4x}{3x+\sqrt{9x^2+1}}$

$=\lim_{x\to\infty}\frac{-4}{3+\sqrt{9+\frac{1}{x^2}}}$

$=\frac{-4}{3+3}=\frac{-2}{3}$ #

$\lim_{x\to-\infty}f(x)=\lim_{t\to\infty}f(-t)$

$=\lim_{t\to\infty}\left(-4t\left(-3t-\sqrt{9t^2+1}\right)\right)$

$=\lim_{t\to\infty}(4t)\left(3t+\sqrt{9t^2+1}\right)$

$=\infty\cdot\infty=\infty$ #

9) $g(x)=\begin{cases}5x-2, & x<1\\ a, & x=1\\ ax^2+bx, & x\geq1\end{cases}$

If $g(x)$ is continuous at $x=1$,

$\Rightarrow\lim_{x\to1^-}g(x)$

$=\lim_{x\to1^+}g(x)$

$=g(1).$

Now $g(1)=a$

$\lim_{x\to1^-}g(x)$

$=\lim_{x\to1^-}(5x-2)$

$=5-2=3.$

$\therefore a=3$ #

$\lim_{x\to1^+}g(x)$

$=\lim_{x\to1^+}(ax^2+bx)$

$=a+b=3+b$

$=g(1)=a=3$

$\therefore b=0.$

$\therefore a=3,\ b=0$ #

11(a) $y=e^{x^2+1}\sin x^3$

$$\frac{dy}{dx}=e^{x^2+1}\frac{d\sin x^3}{dx}+\sin x^3\frac{de^{x^2+1}}{dx}$$

$$=e^{x^2+1}\frac{d\sin x^3}{dx^3}*\frac{dx^3}{dx}+$$
$$+\sin x^3\frac{de^{x^2+1}}{d(x^2+1)}\cdot\frac{dx^2+1}{dx}$$

$$=e^{x^2+1}\cos x^3(3x^2)$$
$$+\sin x^3 e^{x^2+1}\cdot(2x)$$

$$=e^{x^2+1}\left[3x^2\cos x^3+2x\sin x^3\right]\ \#$$

(b) $y=(1+2\tan x)^{15}$

$$\frac{dy}{dx}=15(1+2\tan x)^{14}(2\sec^2 x)$$

(c) $y=\sin^5(\sin e^x)$

Let $w=\sin e^x$

$$\frac{dw}{dx}=\frac{d\sin e^x}{de^x}\cdot\frac{de^x}{dx}$$
$$=(\cos e^x)(e^x)$$

$y=\sin^5 w$

$$\frac{dy}{dx}=\frac{d\sin^5 w}{d\sin w}\times\frac{d\sin w}{dw}\times\frac{dw}{dx}$$
$$=5\sin^4 w(\cos w)(e^x\cos e^x)$$
$$=5\sin^4(\sin e^x)\left(\cos(\sin e^x)\right)\left[e^x\cos e^x\right]\ \#$$

11(d) $y=\tan\left(e^{\sqrt{3x}}\right)$

$$\frac{dy}{dx}=\sec^2\left(e^{\sqrt{3x}}\right)\frac{de^{\sqrt{3x}}}{dx}$$
$$=\sec^2\left(e^{\sqrt{3x}}\right)\frac{de^{\sqrt{3x}}}{d\sqrt{3x}}\cdot\frac{d\sqrt{3x}}{dx}$$
$$=\sec^2\left(e^{\sqrt{3x}}\right)e^{\sqrt{3x}}\cdot\sqrt{3}\left(\frac{1}{2\sqrt{x}}\right)$$
$$=\frac{\sqrt{3}}{2}\left(\frac{1}{\sqrt{x}}\right)e^{\sqrt{3x}}\sec^2\left(e^{\sqrt{3x}}\right)\ \#$$

11(f) $y=\sin^2\left(e^{3x+1}\right)$

Let $w=e^{3x+1}$

$$\frac{dw}{dx}=\frac{de^{3x+1}}{d(3x+1)}\cdot\frac{d(3x+1)}{dx}$$
$$=\left(e^{3x+1}\right)(3)$$

$y=\sin^2 w$

$$\frac{dy}{dx}=\frac{d\sin^2 w}{d\sin w}\cdot\frac{d\sin w}{dw}\cdot\frac{dw}{dx}$$
$$=(2\sin w)(\cos w)\left(3e^{3x+1}\right)$$
$$=6\sin e^{3x+1}\cdot\cos e^{3x+1}\cdot e^{3x+1}$$

11(g) $y=\ln\left(e^{\sin 3x}+x^2+7\right)$

Let $w=e^{\sin 3x}$

$$\frac{dw}{dx}=\frac{de^{\sin 3x}}{d\sin 3x}\cdot\frac{d\sin 3x}{d(3x)}\cdot\frac{d3x}{dx}$$
$$=\left(e^{\sin 3x}\right)(\cos 3x)(3)$$

$y=\ln(w+x^2+7)$

$$\frac{dy}{dx}=\frac{d\ln(w+x^2+7)}{d(w+x^2+7)}\ \frac{d(w+x^2+7)}{dx}$$
$$=\left(\frac{1}{w+x^2+7}\right)\left(\frac{dw}{dx}+2x\right)$$
$$=\left(\frac{1}{e^{\sin 3x}+x^2+7}\right)\left(3e^{\sin 3x}\cdot\cos 3x+2x\right)\ \#$$

11(h) $f(x)=x^{\ln x}$

Let $w=\ln x$

$$\frac{df(x)}{dx}=\frac{dx^w}{dx}=?$$

Let $y=x^{\ln x}$

$$\ln y=(\ln x)(\ln x)$$
$$=(\ln x)^2$$

$$\frac{d\ln y}{dx}=\frac{d(\ln x)^2}{dx}$$

$$\Rightarrow\frac{1}{y}\frac{dy}{dx}=2(\ln x)\left(\frac{1}{x}\right)$$

$$\therefore\frac{dy}{dx}=\frac{2y}{x}\ln x$$
$$=2\left(x^{\ln x}\right)\left(\frac{\ln x}{x}\right)\ \#$$

11(i) $y=f(x)=\dfrac{(x+1)^{\frac{3}{2}}(x-4)^{\frac{5}{2}}}{(5x+3)^{\frac{2}{3}}}$

$$\ln y=\frac{3}{2}\ln(x+1)+\frac{5}{2}\ln(x-4)$$
$$-\frac{2}{3}\ln(5x+3)$$

$$\therefore\frac{1}{y}\frac{dy}{dx}=\frac{3}{2(x+1)}+\frac{5}{2}\left(\frac{1}{x-4}\right)$$
$$-\frac{2}{3}\left(\frac{1}{5x+3}\right)(5)$$

$$\therefore\frac{dy}{dx}=\frac{(x+1)^{\frac{3}{2}}(x-4)^{\frac{5}{2}}}{(5x+3)^{\frac{2}{3}}}\left\{\frac{3}{2(x+1)}+\frac{5}{2(x-4)}-\frac{10}{3(5x+3)}\right\}$$

12) $f(1)=3,\ f'(1)=1,$
$g(1)=2,\ g'(1)=0,$
$h(1)=2,\ h'(1)=5.$

(a) $w(x)=\dfrac{f(x)}{g(x)}$

$$w'(1)=\frac{g(1)f'(1)-f(1)g'(1)}{(g(1))^2}$$
$$=\frac{2(1)-3(0)}{2^2}$$
$$=\frac{2}{4}=\frac{1}{2}\ \#$$

(b) $S(x)=\dfrac{g(x)}{(f(x))^2}$

$$S'(x)=\frac{(f(x))^2g'(x)-g(x)\left(2f(x)\right)f'(x)}{(f(x))^4}$$

$$\therefore S'(1)=\frac{(f(1))^2g'(1)-g(1)(2f(1))f'(1)}{(f(1))^4}$$
$$=\frac{9(0)-2(6)(1)}{3^4}$$
$$=\frac{-12}{81}=\frac{-4}{27}\ \#$$

(c) $m(x)=f(x)g(x)h(x)$

$$m'(x)=f(x)g(x)h'(x)+f'(x)g(x)h(x)$$
$$+g'(x)f(x)h(x)$$

$$\therefore m'(x)=f(1)g(1)h'(1)+f'(1)g(1)h(1)$$
$$+g'(1)f(1)h(1)$$
$$=(3)(2)(5)+(1)(2)(2)$$
$$+0(3)(2)$$
$$=30+4+0$$
$$=34\ \#$$

12(d) $n(x)=\dfrac{(f(x))^{\frac{5}{3}}(g(x))^{30}}{(h(x))^{\frac{1}{2}}}$

$$\ln(n(x))$$
$$=\frac{5}{3}\ln(f(x))$$
$$+30\ln(g(x))$$
$$-\frac{1}{2}\ln(h(x))$$

$$\Rightarrow\frac{1}{n(x)}n'(x)$$
$$=\frac{5}{3}\left(\frac{f'(x)}{f(x)}\right)+30\left(\frac{g'(x)}{g(x)}\right)$$
$$-\frac{1}{2}\left(\frac{h'(x)}{h(x)}\right)$$

$$\therefore n'(1)=n(1)\left\{\frac{5}{3}\frac{f'(1)}{f(1)}+30\frac{g'(1)}{g(1)}-\frac{1}{2}\frac{h'(1)}{h(1)}\right\}$$

Now n

$$n(1)=\frac{(f(1))^{\frac{5}{3}}(g(1))^{30}}{(h(1))^{\frac{1}{2}}}$$
$$=\frac{3^{\frac{5}{3}}\cdot 2^{30}}{\sqrt{2}}$$

$$\therefore n'(1)$$
$$=\frac{3^{\frac{5}{3}}\cdot 2^{30}}{\sqrt{2}}\left(\frac{5}{3}\left(\frac{1}{3}\right)+30(0)+\frac{1}{2}\left(\frac{5}{2}\right)\right)$$

$$=\frac{3^{\frac{5}{3}}\cdot 2^{30}}{\sqrt{2}}\left(\frac{5}{9}+\frac{5}{4}\right)$$

14(a) $xy + x^{\frac32}y^{-\frac12} = 2$

Diff. both sides of the equation w.r.t. x. we have

$$\frac{dxy}{dx} + \frac{dx^{\frac32}y^{-\frac12}}{dx} = 0$$

$$\left(x\frac{dy}{dx} + y\frac{dx}{dx}\right) + x^{\frac32}\frac{dy^{-\frac12}}{dx} + y^{-\frac12}\frac{dx^{\frac32}}{dx} = 0$$

$$\Rightarrow x\frac{dy}{dx} + y + x^{\frac32}\left(-\frac12 y^{-\frac32}\frac{dy}{dx}\right) + y^{-\frac12}\left(\frac32\right)x^{\frac12} = 0$$

$$\therefore \left(x - \frac{x^{\frac32}}{2y^{\frac32}}\right)\frac{dy}{dx} = -\left(y + \frac{3\sqrt{x}}{2\sqrt{y}}\right)$$

$$\therefore \frac{dy}{dx} = \frac{y + \frac32\sqrt{\frac{x}{y}}}{\frac12\left(\frac{x}{y}\right)^{\frac32} - x}$$

when $(x,y)=(1,1)$

$$\left.\frac{dy}{dx}\right|_{(1,1)} = \frac{1+\frac32}{\frac12 - 1}$$

$$= \frac{\frac52}{-\frac12} = -5$$

$=$ slope of tangent at $(1,1)$

$\therefore$ The Equation of the required tangent is

$$y-1 = (-5)(x-1)$$

$$y = -5x+6 \ \#$$

14(b) $xy^{\frac52} + x^{\frac32}y = 12$ at $(x,y)=(4,1)$

$$\frac{dxy^{\frac52}}{dx} + \frac{dx^{\frac32}y}{dx} = 0$$

$$\Rightarrow \left(x\frac{dy^{\frac52}}{dx} + y^{\frac52}\frac{dx}{dx}\right) + \left(x^{\frac32}\frac{dy}{dx} + y\frac{dx^{\frac32}}{dx}\right) = 0$$

$$\Rightarrow x\left(\frac52\right)y^{\frac32}\frac{dy}{dx} + y^{\frac52} + x^{\frac32}\frac{dy}{dx} + y\left(\frac32\right)x^{\frac12} = 0$$

$$\therefore \left(\frac{dy}{dx}\right)\left(\frac52 xy^{\frac32} + x^{\frac32}\right) = -\left(y^{\frac52} + \frac32 x^{\frac12}y\right)$$

$$\therefore \frac{dy}{dx} = \frac{-\left(y^{\frac52} + \frac32 x^{\frac12}\right)}{\frac52 xy^{\frac32} + x^{\frac32}}$$

Slope of tangent at $(4,1)$ is

$$\left.\frac{dy}{dx}\right|_{(4,1)} = \frac{-\left(1+\frac32(2)\right)}{\frac52(4)(1)+8} = \frac{-4}{18}$$

$$= \frac{-2}{9}$$

$\therefore$ the required tangent is

$$y - 1 = \frac{-2}{9}(x-4)$$

$$\Rightarrow y = \frac{-2}{9}x + \frac{17}{9} \ \#$$

14(c) $(x^2+y^2-2x)^2 = 2(x^2+y^2)$ (at $(2,2)$)

$$\cancel{2}(x^2+y^2-2x)\left(2x+2y\frac{dy}{dx}-2\right) = \cancel{2}\left(2x+2y\frac{dy}{dx}\right)$$

$$\Rightarrow \left(2y(x^2+y^2-2x) - 2y\right)\frac{dy}{dx} = 2x + (2-2x)(x^2+y^2-2x)$$

$$\therefore \frac{dy}{dx} = \frac{x + (1-x)(x^2+y^2-2x)}{y(x^2+y^2-2x-1)}$$

$$\left.\frac{dy}{dx}\right|_{(2,2)} = \frac{2+(-1)(4)}{2(3)} = \frac{-1}{3}$$

$\therefore$ The required Tangent is

$$y-2 = \frac{-1}{3}(x-2)$$

$$\Rightarrow y = \frac{-1}{3}x + \frac83 \ \#$$

14(d) $(3x^7+y^2)^{\frac12} = \sin^2 y + 100xy$ at $(1,0)$

$$\Rightarrow \frac12(3x^7+y^2)^{-\frac12}\left(21x^6 + 2y\frac{dy}{dx}\right)$$

$$= \underbrace{2\sin y\cos y}_{(\sin 2y)}\frac{dy}{dx} + 100x\frac{dy}{dx} + 100y$$

$$\therefore \left(\frac{y}{\sqrt{3x^7+y^2}} - \sin 2y - 100x\right)\frac{dy}{dx}$$

$$= \left(\frac{-21x^6}{2\sqrt{3x^7+y^2}}\right) + 100y$$

$$\therefore \frac{dy}{dx} = \frac{\frac{-21x^6}{2\sqrt{3x^7+y^2}} + 100y}{\frac{y}{\sqrt{3x^7+y^2}} - \sin 2y - 100x}$$

$$\left.\frac{dy}{dx}\right|_{(1,0)} = \left(\frac{-21}{2\sqrt3} + 0\right) \Big/ (0 - 0 - 100)$$

$$= \frac{21}{200\sqrt3}$$

$\therefore$ The required tangent is

$$y - 0 = \frac{21}{200\sqrt3}(x-1)$$

$$y = \frac{21}{200\sqrt3}x - \frac{21}{200\sqrt3} \ \#$$

15) $f(x) = \sqrt{3x^2+4}$

$$f'(2) = \lim_{h\to0}\frac{f(2+h)-f(2)}{h}$$

$$= \lim_{h\to0}\frac{\sqrt{3(2+h)^2+4} - \sqrt{16}}{h}$$

$$= \lim_{h\to0}\frac{\sqrt{16+12h+h^2} - 4}{h}$$

$$= \lim_{h\to0}\frac{\left(\sqrt{16+12h+h^2}-4\right)}{h} \times \frac{\sqrt{16+12h+h^2}+4}{\sqrt{16+12h+h^2}+4}$$

$$= \lim_{h\to0}\frac{16+12h+h^2-16}{h\left(\sqrt{16+12h+h^2}+4\right)}$$

$$= \lim_{h\to0}\frac{h(12+h)}{h\left(\sqrt{16+12h+h^2}+4\right)}$$

$$= \frac{12}{4+4} = \frac32 \ \#$$

Now $f'(0) = \lim_{h\to0}\frac{f(0+h)-f(0)}{h}$

$$= \lim_{h\to0}\frac{f(h)-f(0)}{h}$$

$$= \lim_{h\to0}\frac{\sqrt{3h^2+4}-2}{h}$$

$$= \lim_{h\to0}\frac{3h^2+4-4}{h\left(\sqrt{3h^2+4}+2\right)}$$

$$= \lim_{h\to0}\frac{3h}{\sqrt{3h^2+4}+2} = 0 \ \#$$

16(a) As $\frac{da^x}{dx} = (\ln a)a^x$

$$\therefore \frac{d}{dx}(x^\pi + \pi^x) = \frac{dx^\pi}{dx} + \frac{d\pi^x}{dx}$$

$$= \pi x^{\pi-1} + (\ln\pi)\pi^x \ \#$$

(b) Let $y = \left(1+\frac1x\right)^x$

$$\ln y = x\ln\left(1+\frac1x\right)$$

$$\Rightarrow \frac1y\frac{dy}{dx} = \ln\left(1+\frac1x\right) + x\left(\frac{1}{1+\frac1x}\right)\left(\frac{-1}{x^2}\right)$$

$$= \ln\left(1+\frac1x\right) - \frac1x\left(\frac{1}{\frac{1+x}{x}}\right)$$

$$= \ln\left(1+\frac1x\right) - (1+x)$$

$$\therefore \frac{dy}{dx} = \left(1+\frac1x\right)^x\left[\ln\left(1+\frac1x\right) - (1+x)\right] \ \#$$

16(c) $y=(1+x^2)^{\sin x}$

$\ln y = \sin x \ln(1+x^2)$

$\frac{d\ln y}{dx} = \sin x \frac{d\ln(1+x^2)}{dx} + \ln(1+x^2)\frac{d\sin x}{dx}$

$\frac{1}{y}\frac{dy}{dx} = \sin x\left(\frac{1}{1+x^2}\right)(2x) + (\cos x)\ln(1+x^2)$

$\therefore \frac{dy}{dx} = y\left(\frac{2x\sin x}{1+x^2}+\cos x\ln(1+x^2)\right)$

$= (1+x^2)^{\sin x}\left(\frac{2x\sin x}{1+x^2}+\cos x \ln(1+x^2)\right)$ #

16(d) $y = x^{(x^{10})}$

$\ln y = x^{10}\ln x$

$\frac{1}{y}\frac{dy}{dx} = x^{10}\frac{d\ln x}{dx} + \ln x\frac{dx^{10}}{dx}$

$= x^{10}\left(\frac{1}{x}\right)+10(\ln x)(x^9)$

$\therefore \frac{dy}{dx} = y\left(x^9 + 10x^9\ln x\right)$

$= \left(x^{x^{10}}\right)(x^9)(1+10\ln x)$ #

16(e) $y = (\ln x)^{x^2}$

$\ln y = \{\ln(\ln x)\}\{x^2\}$

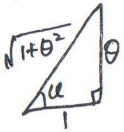
$\frac{1}{y}\frac{dy}{dx} = x^2\frac{d\ln(\ln x)}{dx} + 2x\ln(\ln x)$

$= x^2\left(\frac{1}{\ln x}\right)\left(\frac{1}{x}\right) + 2x\ln(\ln x)$

$\therefore \frac{dy}{dx} = y\left(\frac{x}{\ln x} + 2x\ln(\ln x)\right)$

$= (\ln x)^{x^2}\left(\frac{x}{\ln x} + 2x\ln(\ln x)\right)$ #

17(a) Let $u = \tan^{-1}\theta$



$\tan u = \theta$

$\frac{d\tan u}{d\theta} = \frac{d\theta}{d\theta} = 1$

$\sec^2 u\frac{du}{d\theta} = 1$

$\therefore \frac{du}{d\theta} = \cos^2 u = \frac{1}{1+\theta^2}$

$\therefore \frac{d\tan^{-1}\theta}{d\theta} = \frac{1}{1+\theta^2}$

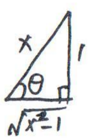
Now $y = \tan^{-1}(e^{4x})$

$\frac{dy}{dx} = \frac{d\tan^{-1}(e^{4x})}{de^{4x}} \times \frac{de^{4x}}{d4x}\cdot\frac{d4x}{dx}$

$= \frac{1}{1+(e^{4x})^2}\cdot e^{4x}\cdot 4$

$= \frac{4e^{4x}}{1+e^{8x}}$ #

17(b) Let $\theta = \csc^{-1}x$



$\csc\theta = x$

$\Rightarrow \frac{d\csc\theta}{dx} = \frac{dx}{dx} = 1$

$-\csc\theta\cot\theta\frac{d\theta}{dx} = 1$

$\Rightarrow \frac{d\theta}{dx} = -\tan\theta\sin\theta = \frac{-1}{\sqrt{x^2-1}}\times\frac{1}{x}$

$\therefore \frac{d\csc^{-1}x}{dx} = \frac{-1}{x\sqrt{x^2-1}}$

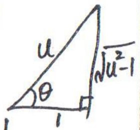
Now $y = f(u) = \csc^{-1}(2u+1)$

$f'(u) = \frac{dy}{du} = \frac{d\csc^{-1}(2u+1)}{d(2u+1)}\times\frac{d(2u+1)}{du}$

$= \frac{-1}{(2u+1)\sqrt{(2u+1)^2-1}}\ (2)$

$= \frac{-2}{(2u+1)\sqrt{4u^2+4u}}$ #

17(c) Let $\theta = \sec^{-1}u$



$\sec\theta = u$

$\frac{d\sec\theta}{du} = \frac{du}{du} = 1$

$\sec\theta\tan\theta\frac{d\theta}{du} = 1$

$\frac{d\theta}{du} = \cos\theta\cot\theta$

$= \left(\frac{1}{u}\right)\left(\frac{1}{\sqrt{u^2-1}}\right)$

$\Rightarrow \frac{d\sec^{-1}u}{du} = \frac{1}{u\sqrt{u^2-1}}$

Now

$y = f(x) = \sec^{-1}\sqrt{x}$

$\frac{dy}{dx} = \frac{d\sec^{-1}\sqrt{x}}{d\sqrt{x}}\cdot\frac{d\sqrt{x}}{dx}$

$= \frac{1}{\sqrt{x}\sqrt{x-1}}\cdot\frac{1}{2\sqrt{x}}$

$= \frac{1}{2x\sqrt{x-1}}$ #

17(d) As $\frac{d\tan^{-1}t}{dt} = \frac{1}{1+t^2}$

Now $y = f(t) = \ln(\tan^{-1}t)$

$\therefore \frac{dy}{dt} = f'(t) = \frac{d\ln(\tan^{-1}t)}{dt}$

$= \frac{d\ln(\tan^{-1}t)}{d(\tan^{-1}t)}\times\frac{d\tan^{-1}t}{dt}$

$= \left(\frac{1}{\tan^{-1}t}\right)\left(\frac{1}{1+t^2}\right)$

$= \frac{1}{(1+t^2)\tan^{-1}t}$ #

18 (a) $y = f^{-1}(x)$ passes through the point $(4,7)$

$\Rightarrow 7 = f^{-1}(4)$

$\Rightarrow$ the curve $y = f(x)$ pass through the point $(7,4)$.

i.e. $4 = f(7)$

For $y = f^{-1}(x)$

$f(y) = x$

$\frac{d f(y)}{dx} = \frac{dx}{dx} = 1$

$\therefore \frac{df(y)}{dy}\times\frac{dy}{dx} = 1$

$\Rightarrow \frac{dy}{dx} = \frac{1}{\frac{df(y)}{dy}}$

$\Rightarrow \frac{d f^{-1}(x)}{dx} = \frac{1}{\frac{df(y)}{dy}}$ $\left(= \frac{1}{\frac{dx}{dy}}\right)$

$\Rightarrow f^{-1\prime}(x) = \frac{1}{f'(y)}$

Hence $f'(y) = \frac{1}{f^{-1\prime}(x)}$

Now $f^{-1\prime}(4) = \frac{4}{5}$

$\Rightarrow f'(7) = \frac{1}{f^{-1\prime}(4)} = \frac{5}{4}$ #

(b)

For $f(x) = x^3+x+1 = y$

when $y=3 \Rightarrow x^3+x+1=3 \Rightarrow x^3+x-2=0$

$\Rightarrow (x^3-1)+(x-1)=0$

$\Rightarrow (x-1)(x^2+x+1)+(x-1)=0$

$\Rightarrow (x-1)(x^2+x+2)=0 \Rightarrow x=1.$

Now $f'(1) = 3x^2+1\big|_{x=1} = 4$

$\therefore (f^{-1})'(3) = \frac{1}{f'(1)} = \frac{1}{4}$ #

19)(a) $f(x)=\frac{x}{x+5}=y$

interchage x and y,

$$x=\frac{y}{y+5}$$

$$xy+5x=y$$

$$(x-1)y=-5x$$

$$\therefore y=\frac{5x}{1-x}=f^{-1}(x)$$

$$\frac{dy}{dx}=\frac{(1-x)(5)-5x(-1)}{(1-x)^2}$$

$$=\frac{5}{(1-x)^2}$$

$$\therefore \frac{df^{-1}(x)}{dx}=\frac{5}{(1-x)^2} \quad \#$$

19(b) $f(x)=x^3+3=y$

interchange x and y,

$$x=y^3+3$$

$$\therefore y=(x-3)^{\frac{1}{3}}$$

$$\frac{dy}{dx}=\frac{1}{3}(x-3)^{-\frac{2}{3}}$$

$$=\frac{1}{3(x-3)^{\frac{2}{3}}}$$

$$\therefore \frac{df^{-1}(x)}{dx}=\frac{1}{3(x-3)^{\frac{2}{3}}} \quad \#$$

19(c) $f(x)=|x+2|$ for $x\geqslant -2$

$$f(-2)=|-2+2|=0$$

As $x\geqslant -2$, $f(x)=x+2 \overset{\text{set}}{=\!=} y$

interchange of x and y,

$$x=y+2$$

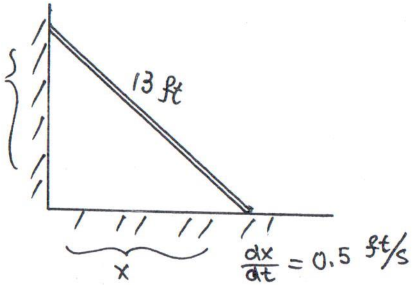
$$\therefore y=x-2$$

$$\therefore y=f^{-1}(x)=x-2, \quad x\geqslant 0 \quad \left(\because f(x)\geqslant 0 \text{ for all } x\geqslant -2\right)$$

$$\frac{dy}{dx}=1, \quad \forall x\geqslant 0$$

$$\Rightarrow \frac{d f^{-1}(x)}{dx}=1 \quad \#$$

20)



$\frac{dx}{dt}=0.5$ ft/s

when $x=5$ ft, $\frac{dy}{dt}=?$

As $x^2+y^2=13^2=169$

when $x=5 \Rightarrow y^2=169-25=144$

$\Rightarrow y=12$.

$$2x\frac{dx}{dt}+2\left(\frac{dy}{dt}\right)(y)=0$$

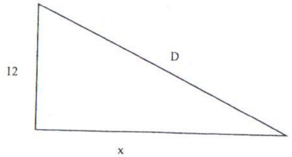
$$\therefore \frac{dy}{dt}=\frac{-x}{y}\frac{dx}{dt}=\left(\frac{-5}{12}\right)(0.5) \text{ ft/s}$$

$$=\frac{-5}{24} \text{ ft/s} \quad \#$$

So the top of the ladder slides down the wall at $\frac{5}{24}$ ft/s #

Solutions to Math 1013 Tutorial 8 (Revision Exercises for Mid Term Exam)

21) Let x be the distance between the fish and the fisherman's feet, and let D be the distance between the fish and the tip of the pole. Then $D^2 = x^2 + 144$, so $2D(dD/dt) = 2x(dx/dt)$. Note that $dD/dt = -1/3$ ft/sec, so when $x = 20$ ft, we have $dx/dt = \sqrt{400+144}/20 \cdot (-1/3) \approx -0.3887$ ft/sec ≈ -4.66 in/sec. The fish is moving toward the fisherman at about 4.66 in/sec.



22) As the height of the piston is decreasing,

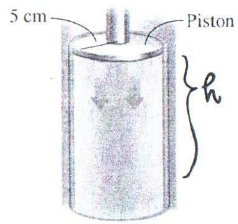
$$\therefore \frac{dh}{dt}=-3 \text{ cm/s}$$

Volume of the cyclinder $V=\pi(5)^2 h$

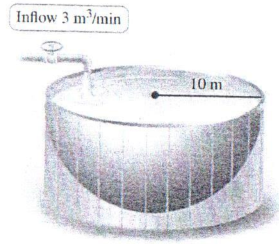
$$\frac{dV}{dt}=25\pi\frac{dh}{dt}$$

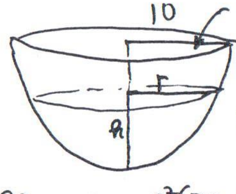
when $h=2$,

$$\frac{dV}{dt}=25\pi(-3)=-75\pi \text{ cm}^3/\text{s} \quad \#$$



23)





$\frac{dV}{dt}=3$ m³/min

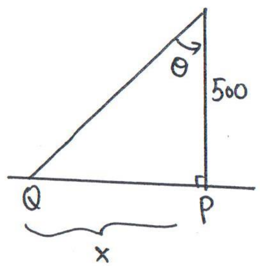
when $h=5$.

$\frac{dh}{dt}=?$

Given $V=\frac{\pi h^2}{3}(3r-h)$

The volume of a segment of water of height h within a hemisphere of radius 10 is given by $V = \frac{1}{3}\pi h^2(30-h) = 10\pi h^2 - \frac{1}{3}\pi h^3$. We have that $\frac{dV}{dt} = 20\pi h\frac{dh}{dt} - \pi h^2\frac{dh}{dt}$. We are given that $\frac{dV}{dt} = 3\,\text{m}^3/\text{min}$, so when $h=5$ we have $3 = (100\pi - 25\pi)\frac{dh}{dt}$, so $\frac{dh}{dt} = \frac{3}{75\pi}$ meters per minute.

24)



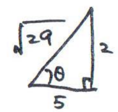
$$\frac{d\theta}{dt}=\frac{(2\pi)(4)}{60} \text{ rad/s}=\frac{2\pi}{15} \text{ rad/s}$$

$$\tan\theta=\frac{x}{500}$$

when $x=200$

$$\tan\theta=\frac{2}{5}$$

$$\therefore \sec^2\theta=\frac{29}{25}$$



Let θ be the angle RLP where L represents the lighthouse and R represents the point on the land where the light is currently hitting. Let s be the distance from the point P to the point R. We are given that $\frac{d\theta}{dt} = \frac{2\pi}{15}$ radians per second. Note that $\tan\theta = \frac{s}{500}$, so $\sec^2\theta \cdot \frac{d\theta}{dt} = \frac{1}{500}\frac{ds}{dt}$. When the light is at point Q, $\tan\theta = \frac{2}{5}$, so $\sec^2\theta = 1 + \frac{4}{25} = \frac{29}{25}$. Then

$$\frac{ds}{dt} = 500 \cdot \frac{2\pi}{15} \cdot \frac{29}{25} = \frac{232\pi}{3} \text{ m/s}.$$

The beam moves more slowly when R is near P, and more quickly when it is further away from P.