1. 半地转框架下 Hoskins 形式 Omega 方程推导

SG 半地转+非地转局地变化框架：

$$\frac{\partial u_a}{\partial t}+\frac{\partial u_g}{\partial t}+u\frac{\partial u_g}{\partial x}+v\frac{\partial u_g}{\partial y}+w\frac{\partial u_g}{\partial z}=fv_a+D_V(u)+D_H(u) \tag{1}$$

$$\frac{\partial v_a}{\partial t}+\frac{\partial v_g}{\partial t}+u\frac{\partial v_g}{\partial x}+v\frac{\partial v_g}{\partial y}+w\frac{\partial v_g}{\partial z}=-fu_a+D_V(v)+D_H(v) \tag{2}$$

$$\frac{\partial b}{\partial t}+u\frac{\partial b}{\partial x}+v\frac{\partial b}{\partial y}+w\frac{\partial b}{\partial z}=D_V(b) \tag{3}$$

$$\frac{\partial u_a}{\partial x}+\frac{\partial v_a}{\partial y}+\frac{\partial w}{\partial z}=0 \tag{4}$$

热风平衡：

$$f\frac{\partial v_g}{\partial z}=\frac{\partial b}{\partial x}$$

$$f\frac{\partial u_g}{\partial z}=-\frac{\partial b}{\partial y}$$

地转流没有散度：

$$\frac{\partial u_g}{\partial x}+\frac{\partial v_g}{\partial y}=0$$

$\partial(1)/\partial z$:

$$\frac{\partial}{\partial z}\frac{\partial u_a}{\partial t}+\frac{\partial}{\partial t}\frac{\partial u_g}{\partial z}+\frac{\partial u}{\partial z}\frac{\partial u_g}{\partial x}+u\frac{\partial}{\partial x}\frac{\partial u_g}{\partial z}+\frac{\partial v}{\partial z}\frac{\partial u_g}{\partial y}+v\frac{\partial}{\partial y}\frac{\partial u_g}{\partial z}+\frac{\partial w}{\partial z}\frac{\partial u_g}{\partial z}+w\frac{\partial}{\partial z}\frac{\partial u_g}{\partial z}$$
$$=f\frac{\partial v_a}{\partial z}+\frac{\partial}{\partial z}\left(D_V(u)\right)+\frac{\partial}{\partial z}(D_H(u))$$

$\partial(3)/\partial y$:

$$\frac{\partial}{\partial t}\frac{\partial b}{\partial y}+\frac{\partial u}{\partial y}\frac{\partial b}{\partial x}+u\frac{\partial}{\partial x}\frac{\partial b}{\partial y}+\frac{\partial v}{\partial y}\frac{\partial b}{\partial y}+v\frac{\partial}{\partial y}\frac{\partial b}{\partial y}+\frac{\partial w}{\partial y}\frac{\partial b}{\partial z}+w\frac{\partial}{\partial z}\frac{\partial b}{\partial y}=\frac{\partial}{\partial y}(D_V(b))$$

代入热风平衡:

$$-f\frac{\partial}{\partial t}\frac{\partial u_g}{\partial z}+\frac{\partial u}{\partial y}\frac{\partial b}{\partial x}-fu\frac{\partial}{\partial x}\frac{\partial u_g}{\partial z}+\frac{\partial v}{\partial y}\frac{\partial b}{\partial y}-fv\frac{\partial}{\partial y}\frac{\partial u_g}{\partial z}+\frac{\partial w}{\partial y}\frac{\partial b}{\partial z}-fw\frac{\partial}{\partial z}\frac{\partial u_g}{\partial z}=\frac{\partial}{\partial y}(D_V(b))$$

相加整理：

$$-f^2\frac{\partial v_a}{\partial z}+\frac{\partial w}{\partial y}\frac{\partial b}{\partial z}-\frac{\partial w}{\partial z}\frac{\partial b}{\partial y}=$$
$$-f\left(\frac{\partial u_a}{\partial z}\frac{\partial u_g}{\partial x}+\frac{\partial v_a}{\partial z}\frac{\partial u_g}{\partial y}\right)-2\left(\frac{\partial v_g}{\partial y}\frac{\partial b}{\partial y}+\frac{\partial u_g}{\partial y}\frac{\partial b}{\partial x}\right)-\left(\frac{\partial u_a}{\partial y}\frac{\partial b}{\partial x}+\frac{\partial v_a}{\partial y}\frac{\partial b}{\partial y}\right)$$
$$+f\frac{\partial}{\partial z}\left(D_V(u)\right)+f\frac{\partial}{\partial z}\left(D_H(u)\right)+\frac{\partial}{\partial y}\left(D_V(b)\right)-f\frac{\partial}{\partial z}\frac{\partial u_a}{\partial t} \tag{5}$$

$\partial(2)/\partial z$:

$$\frac{\partial}{\partial z}\frac{\partial v_a}{\partial t}+\frac{\partial}{\partial t}\frac{\partial v_g}{\partial z}+\frac{\partial u}{\partial z}\frac{\partial v_g}{\partial x}+u\frac{\partial}{\partial x}\frac{\partial v_g}{\partial z}+\frac{\partial v}{\partial z}\frac{\partial v_g}{\partial y}+v\frac{\partial}{\partial y}\frac{\partial v_g}{\partial z}+\frac{\partial w}{\partial z}\frac{\partial v_g}{\partial z}+w\frac{\partial}{\partial z}\frac{\partial v_g}{\partial z}$$
$$=-f\frac{\partial u_a}{\partial z}+\frac{\partial}{\partial z}\left(D_V(v)\right)+\frac{\partial}{\partial z}(D_H(v))$$

$\partial(3)/\partial x$:

$$\frac{\partial}{\partial t}\frac{\partial b}{\partial x}+\frac{\partial u}{\partial x}\frac{\partial b}{\partial x}+u\frac{\partial}{\partial x}\frac{\partial b}{\partial x}+\frac{\partial v}{\partial x}\frac{\partial b}{\partial y}+v\frac{\partial}{\partial x}\frac{\partial b}{\partial y}+\frac{\partial w}{\partial x}\frac{\partial b}{\partial z}+w\frac{\partial}{\partial z}\frac{\partial b}{\partial x}=\frac{\partial}{\partial x}(D_V(b))$$

相减整理：

$$-f^2\frac{\partial u_a}{\partial z}+\frac{\partial w}{\partial x}\frac{\partial b}{\partial z}-\frac{\partial w}{\partial z}\frac{\partial b}{\partial x}=$$
$$-2\left(\frac{\partial v_g}{\partial x}\frac{\partial b}{\partial y}+\frac{\partial u_g}{\partial x}\frac{\partial b}{\partial x}\right)+f\left(\frac{\partial v_a}{\partial z}\frac{\partial v_g}{\partial y}+\frac{\partial u_a}{\partial z}\frac{\partial v_g}{\partial x}\right)-\left(\frac{\partial u_a}{\partial x}\frac{\partial b}{\partial x}+\frac{\partial v_a}{\partial x}\frac{\partial b}{\partial y}\right)$$
$$-f\frac{\partial}{\partial z}\left(D_V(v)\right)-f\frac{\partial}{\partial z}\left(D_H(v)\right)+\frac{\partial}{\partial x}\left(D_V(b)\right)+f\frac{\partial}{\partial z}\frac{\partial v_a}{\partial t} \tag{6}$$

(5)与(6)整理：利用连续方程消去 $\frac{\partial w}{\partial z}$这一项以保证 Omega 方程形式：

$$-f^2\frac{\partial v_a}{\partial z}+\frac{\partial w}{\partial y}\frac{\partial b}{\partial z}=$$
$$-2\left(\frac{\partial v_g}{\partial y}\frac{\partial b}{\partial y}+\frac{\partial u_g}{\partial y}\frac{\partial b}{\partial x}\right)-f\left(\frac{\partial u_a}{\partial z}\frac{\partial u_g}{\partial x}+\frac{\partial v_a}{\partial z}\frac{\partial u_g}{\partial y}\right)-\left(\frac{\partial u_a}{\partial y}\frac{\partial b}{\partial x}+2\frac{\partial v_a}{\partial y}\frac{\partial b}{\partial y}+\frac{\partial u_a}{\partial x}\frac{\partial b}{\partial y}\right)$$
$$+f\frac{\partial}{\partial z}\left(D_V(u)\right)+f\frac{\partial}{\partial z}\left(D_H(u)\right)+\frac{\partial}{\partial y}\left(D_V(b)\right)-f\frac{\partial}{\partial z}\frac{\partial u_a}{\partial t}$$

$$-f^2\frac{\partial u_a}{\partial z}+\frac{\partial w}{\partial x}\frac{\partial b}{\partial z}=$$
$$-2\left(\frac{\partial v_g}{\partial x}\frac{\partial b}{\partial y}+\frac{\partial u_g}{\partial x}\frac{\partial b}{\partial x}\right)+f\left(\frac{\partial v_a}{\partial z}\frac{\partial v_g}{\partial y}+\frac{\partial u_a}{\partial z}\frac{\partial v_g}{\partial x}\right)-\left(\frac{\partial v_a}{\partial y}\frac{\partial b}{\partial x}+2\frac{\partial u_a}{\partial x}\frac{\partial b}{\partial x}+\frac{\partial v_a}{\partial x}\frac{\partial b}{\partial y}\right)$$
$$-f\frac{\partial}{\partial z}\left(D_V(v)\right)-f\frac{\partial}{\partial z}\left(D_H(v)\right)+\frac{\partial}{\partial x}\left(D_V(b)\right)+f\frac{\partial}{\partial z}\frac{\partial v_a}{\partial t}$$

$$\boldsymbol{Q}_{tg}\begin{cases}Q_{tgx}=-2\left(\frac{\partial v_g}{\partial x}\frac{\partial b}{\partial y}+\frac{\partial u_g}{\partial x}\frac{\partial b}{\partial x}\right)\\ Q_{tgy}=-2\left(\frac{\partial v_g}{\partial y}\frac{\partial b}{\partial y}+\frac{\partial u_g}{\partial y}\frac{\partial b}{\partial x}\right)\end{cases}$$

$$\boldsymbol{Q}_{tag}\begin{cases}Q_{tagx}=-\left(\frac{\partial v_a}{\partial y}\frac{\partial b}{\partial x}+2\frac{\partial u_a}{\partial x}\frac{\partial b}{\partial x}+\frac{\partial v_a}{\partial x}\frac{\partial b}{\partial y}\right)\\ Q_{tagy}=-\left(\frac{\partial u_a}{\partial y}\frac{\partial b}{\partial x}+2\frac{\partial v_a}{\partial y}\frac{\partial b}{\partial y}+\frac{\partial u_a}{\partial x}\frac{\partial b}{\partial y}\right)\end{cases}$$

$$\boldsymbol{Q}_{dag}\begin{cases}Q_{dagx}=f\left(\frac{\partial v_a}{\partial z}\frac{\partial v_g}{\partial y}+\frac{\partial u_a}{\partial z}\frac{\partial v_g}{\partial x}\right)\\ Q_{dagy}=-f\left(\frac{\partial u_a}{\partial z}\frac{\partial u_g}{\partial x}+\frac{\partial v_a}{\partial z}\frac{\partial u_g}{\partial y}\right)\end{cases}$$

$$
\boldsymbol{Q}_{th}\begin{cases} Q_{thx} =- f\dfrac{\partial}{\partial z}\big(D_V(v) + D_H(v)\big) \\ Q_{thy} = f\dfrac{\partial}{\partial z}\big(D_V(u) + D_H(u)\big) \end{cases}
$$

$$
\boldsymbol{Q}_{dm}\begin{cases} Q_{dmx} = \dfrac{\partial}{\partial x}\big(D_V(b)\big) \\ Q_{dmy} = \dfrac{\partial}{\partial y}\big(D_V(b)\big) \end{cases}
$$

$$
\boldsymbol{Q}_{tr}\begin{cases} Q_{trx} = f\dfrac{\partial}{\partial z}\dfrac{\partial v_a}{\partial t} \\ Q_{try} =- f\dfrac{\partial}{\partial z}\dfrac{\partial u_a}{\partial t} \end{cases}
$$

整理成广义 omega 方程 Q vector 的形式：

$$
- f^2\frac{\partial u_a}{\partial z} + \frac{\partial w}{\partial x}\frac{\partial b}{\partial z} = Q_{tgx} + Q_{dagx} + Q_{tagx} + Q_{dmx} + Q_{thx} + Q_{trx} \tag{7}
$$

$$
- f^2\frac{\partial v_a}{\partial z} + \frac{\partial w}{\partial y}\frac{\partial b}{\partial z} = Q_{tgy} + Q_{dagy} + Q_{tagy} + Q_{dmy} + Q_{thy} + Q_{try} \tag{8}
$$

$\partial(7)/\partial x + \partial(8)/\partial y$:

$$
f^2\frac{\partial^2 w}{\partial z^2} + \nabla_h\big(N^2 \cdot \nabla_h w\big) = \nabla_h \cdot \boldsymbol{Q} \tag{9}
$$

混合项处理：

$$
D_V(u) = \frac{\partial}{\partial z}(A_v\frac{\partial u}{\partial z})
$$

$$
D_V(v) = \frac{\partial}{\partial z}(A_v\frac{\partial v}{\partial z})
$$

$$
D_H(u) = A_h(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2})
$$

$$
D_H(v) = A_h(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2})
$$

2. 离散化数值求解及超松弛迭代法求解

（9）式左端项为经典的斯特姆刘维尔问题，迭代法求解需要保证偏微分方程（PDE）为椭圆型，即需要保证$N^2 > 0$。此外，在其他求解方法中，例如在谱空间中展开求解，也需要满足方程为椭圆型。因此在代码中存在工程上的小手段，即将$N^2 < 0$ 的点均赋值为 0。

由于海洋网格常采用均匀的经纬网格，这种网格转化为单位为 m 的网格并不是均匀的，因此对于方程的离散需要采用非均匀网格的离散格式。离散的网格如图 1：

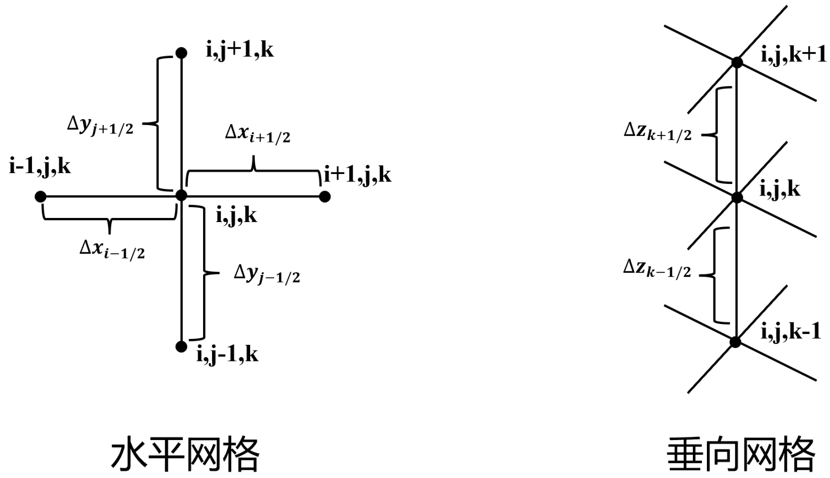

图 1 离散网格点

下面进行离散，网格间距的表示亦如图 1 所示:

$$f^2\frac{\partial^2 w}{\partial z^2}=\frac{2f_{i,j,k}^2}{\Delta z_{k+1/2}+\Delta z_{k-1/2}}(\frac{w_{i,j,k+1}-w_{i,j,k}}{\Delta z_{k+1/2}}-\frac{w_{i,j,k}-w_{i,j,k-1}}{\Delta z_{k-1/2}})$$

$$\frac{\partial}{\partial x}\left(N^2\frac{\partial w}{\partial x}\right)=\frac{2}{\Delta x_{i+1/2}+\Delta x_{i-1/2}}[\frac{N_{i+1,j,k}^2+N_{i,j,k}^2}{2}(\frac{w_{i+1,j,k}-w_{i,j,k}}{\Delta x_{i+1/2}})-\frac{N_{i,j,k}^2+N_{i-1,j,k}^2}{2}(\frac{w_{i,j,k}-w_{i-1,j,k}}{\Delta x_{i-1/2}})]$$

$$\frac{\partial}{\partial y}\left(N^2\frac{\partial w}{\partial y}\right)=\frac{2}{\Delta y_{j+1/2}+\Delta y_{j-1/2}}[\frac{N_{i,j+1,k}^2+N_{i,j,k}^2}{2}(\frac{w_{i,j+1,k}-w_{i,j,k}}{\Delta y_{j+1/2}})-\frac{N_{i,j,k}^2+N_{i,j-1,k}^2}{2}(\frac{w_{i,j,k}-w_{i,j-1,k}}{\Delta y_{j-1/2}})]$$

利用迭代法求解，需要将其整理为如下形式:

$$C_{center}w_{i,j,k}=C_{left}w_{i-1,j,k}+C_{right}w_{i+1,j,k}+C_{up}w_{i,j+1,k}+C_{down}w_{i,j-1,k}+C_{top}w_{i,j,k+1}+C_{bottom}w_{i,j,k-1}-\nabla_h\cdot\boldsymbol{Q}$$

其中各项系数表达式如下:

$$C_{right}=\frac{N_{i+1,j,k}^2+N_{i,j,k}^2}{(\Delta x_{i+1/2}+\Delta x_{i-1/2})\Delta x_{i+1/2}}$$

$$C_{left}=\frac{N_{i,j,k}^2+N_{i-1,j,k}^2}{(\Delta x_{i+1/2}+\Delta x_{i-1/2})\Delta x_{i-1/2}}$$

$$C_{top}=\frac{2f_{i,j,k}^2}{(\Delta z_{k+1/2}+\Delta z_{k-1/2})\Delta z_{k+1/2}}$$

$$C_{bottom}=\frac{2f_{i,j,k}^2}{(\Delta z_{k+1/2}+\Delta z_{k-1/2})\Delta z_{k-1/2}}$$

$$C_{up}=\frac{N_{i,j+1,k}^2+N_{i,j,k}^2}{(\Delta y_{j+1/2}+\Delta y_{j-1/2})\Delta y_{j+1/2}}$$

$$C_{down}=\frac{N_{i,j,k}^2+N_{i,j-1,k}^2}{(\Delta y_{j+1/2}+\Delta y_{j-1/2})\Delta y_{j-1/2}}$$

$$C_{center}=C_{right}+C_{left}+C_{top}+C_{bottom}+C_{up}+C_{down}$$

超松弛迭代法：

$$w_{i,j,k}^{new} = (1-\alpha)w_{i,j,k}^{old} + \frac{\alpha}{C_{center}}(C_{left}w_{i-1,j,k}^{old} + C_{right}w_{i+1,j,k}^{old} + C_{up}w_{i,j+1,k}^{old} + C_{down}w_{i,j-1,k}^{old}$$
$$+ C_{top}w_{i,j,k+1} + C_{bottom}w_{i,j,k-1}^{old} - \nabla_h \cdot \boldsymbol{Q})$$

α为超松弛因子，一般取 1.5-1.8，若取 1 则变为高斯迭代法。迭代至这一步与上一步残差极小即可，代码中使用的为 L2 范数表示残差。

另外，边界条件侧边界只有无滑移边界条件，海表缸盖近似，最底层设置为 0。更高级边界条件如，Neumann 边界条件（dw/dn = 0）,辐射边界条件等随缘更新，已有边界条件结果本就不差。