Appendix: Portfolio Management [1]

1 Lagrangian Method

1.1 Equality constraints

Let (P) be the optimisation problem under constraints

$$(P)\begin{cases} \inf\limits_{x\in\mathbf{R}^d} f(x) \\ g_1(x)=0 \\ \qquad\vdots \\ g_k(x)=0 \end{cases}$$

where $f(.)$ and the $g_i(.)$ are differentiable functions from $\mathbf{R}^d$ into $\mathbf{R}$.
We note, $\mathcal{D}=\{x\in\mathbf{R}^d, g_1(x)=0,\cdots,g_k(x)=0\}$.

Definition .1. *The Lagrangian of (P) is the function from $\mathbf{R}^{d+k}$ into $\mathbf{R}$ defined by*

$$\mathcal{L}(x,\lambda)=f(x)-\sum_{i=1}^{i=k}\lambda_i g_i(x)$$

with $\lambda=(\lambda_1,\cdots,\lambda_k)'$.

The following proposition explains how the Lagrangian method transforms an optimisation problem under constraints into an optimisation problem without constraint.

Proposition .1. *Solving the problem (P) is equivalent to solving*

$$(Q)\ \inf_{x\in\mathbf{R}^d}\sup_{\lambda\in\mathbf{R}^k}\mathcal{L}(x,\lambda).$$

Proof. $x\notin\mathcal{D}\Longrightarrow\exists i, g_i(x)\neq 0\Longrightarrow\sup\limits_{\lambda\in\mathbf{R}^k}\mathcal{L}(x,\lambda)=+\infty$ as the term $-\lambda_i g_i(x)$ tends to $+\infty$ when λ_i tends to $+\infty$ or $-\infty$ (depending on the sign of $g_i(x)$) . So,

$$\inf_{x\in\mathbf{R}^d}\sup_{\lambda\in\mathbf{R}^k}\mathcal{L}(x,\lambda)=\inf_{x\in\mathcal{D}}\sup_{\lambda\in\mathbf{R}^k}\mathcal{L}(x,\lambda)$$

Now,
$x\in\mathcal{D}\Longrightarrow\forall\lambda\in\mathbf{R}^k,\mathcal{L}(x,\lambda)=f(x)\Longrightarrow\sup\limits_{\lambda\in\mathbf{R}^k}\mathcal{L}(x,\lambda)=f(x)$.
So,

$$\inf_{x\in\mathcal{D}}\sup_{\lambda\in\mathbf{R}^k}\mathcal{L}(x,\lambda)=\inf_{x\in\mathcal{D}}f(x)$$

which finishes the proof. ∎

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Definition .2. *Dual Problem*

$$(Q^*) \sup_{\lambda \in \mathbf{R}^k} \inf_{x \in \mathbf{R}^d} \mathcal{L}(x, \lambda).$$

is called the dual problem of (Q).

Usually, the dual problem (Q^*) is easier to solve than the initial problem (Q). In the dual problem the sup is calculated for a concave function as demonstrated below.

Property .1. *The function:* $\phi : \lambda \longrightarrow \inf_{x \in \mathbf{R}^d} \mathcal{L}(x, \lambda)$ *is concave*

Proof. Let $t \in [0,1]$ and λ_1 and $\lambda_2 \in \mathbf{R}^k$.

$$\forall z \in \mathbf{R}^d, \inf_{x \in \mathbf{R}^d} \mathcal{L}(x, \lambda_1) \leq \mathcal{L}(z, \lambda_1) \text{ and } \inf_{x \in \mathbf{R}^d} \mathcal{L}(x, \lambda_2) \leq \mathcal{L}(z, \lambda_2)$$

$$\Longrightarrow \forall z \in \mathbf{R}^d, t \inf_{x \in \mathbf{R}^d} \mathcal{L}(x, \lambda_1) + (1-t) \inf_{x \in \mathbf{R}^d} \mathcal{L}(x, \lambda_2) \leq t\mathcal{L}(z, \lambda_1) + (1-t)\mathcal{L}(z, \lambda_2)$$

and as $\mathcal{L}(x, \lambda)$ is an affine function of λ we get

$$\Longrightarrow \forall z \in \mathbf{R}^d, t \inf_{x \in \mathbf{R}^d} \mathcal{L}(x, \lambda_1) + (1-t) \inf_{x \in \mathbf{R}^d} \mathcal{L}(x, \lambda_2) \leq \mathcal{L}(z, \lambda_1 + (1-t)\lambda_2)$$

$$\Longrightarrow t \inf_{x \in \mathbf{R}^d} \mathcal{L}(x, \lambda_1) + (1-t) \inf_{x \in \mathbf{R}^d} \mathcal{L}(x, \lambda_2) \leq \inf_{z \in \mathbf{R}^d} \mathcal{L}(z, \lambda_1 + (1-t)\lambda_2)$$

$$\Longrightarrow t\phi(\lambda_1) + (1-t)\phi(\lambda_2) \leq \phi(\lambda_1 + (1-t)\lambda_2)$$

which proves the concavity of ϕ. □

The initial problem (Q) is called the primal problem. A link between the primal problem and the dual problem (Q^*) is given by the minimax theorem.

Theorem .1. *mini-max theorem*
For any domains $\mathcal{Y}$ and $\mathcal{Z}$ and real function g defined on $\mathcal{Y} \times \mathcal{Z}$:
$\sup_{z \in \mathcal{Z}} \left[\inf_{y \in \mathcal{Y}} g(y,z) \right] \leq \inf_{y \in \mathcal{Y}} \left[\sup_{z \in \mathcal{Z}} g(y,z) \right]$

Proof. $\inf_{y \in \mathcal{Y}} g(y,z) \leq g(y,z) \Rightarrow \sup_{z \in \mathcal{Z}} \left[\inf_{y \in \mathcal{Y}} g(y,z) \right] \leq \sup_{z \in \mathcal{Z}} g(y,z)$ (1)
As (1) is true for all y, the inequality stands for the inf of the right term of (1).
So, $\sup_{z \in \mathcal{Z}} \left[\inf_{y \in \mathcal{Y}} g(y,z) \right] \leq \inf_{y \in \mathcal{Y}} \left[\sup_{z \in \mathcal{Z}} g(y,z) \right]$ Q.E.D.
□

Definition .3. *Duality*
The difference between $\inf_{y \in \mathcal{Y}} \left[\sup_{z \in \mathcal{Z}} g(y,z) \right]$ *and* $\sup_{z \in \mathcal{Z}} \left[\inf_{y \in \mathcal{Y}} g(y,z) \right]$ *is called the duality gap. When the* ***duality gap*** *is zero, we say that there is* ***strong duality*** *or* ***no duality gap****.*

There are some mathematical conditions which guarantee that the duality gap is zero (Slater's conditions and non-empty interior conditon).

Remark .1. *Example with duality gap.*
In this example:

g(y,z)	y=1	y=2	y=3
z=3	3	3	1
z=2	2	1	3
z=1	1	2	3

$\sup\limits_{z\in\mathcal{Z}}\left[\inf\limits_{y\in\mathcal{Y}} g(y,z)\right] = 1$
$\inf\limits_{y\in\mathcal{Y}}\left[\sup\limits_{z\in\mathcal{Z}} g(y,z)\right] = 3$ *and the duality gap is* 2.

Here, with our notations we can write

$$\text{(dual problem) } d^* = \sup_{\lambda\in\mathbf{R}^k}\inf_{x\in\mathbf{R}^d}\mathcal{L}(x,\lambda) \leq \inf_{x\in\mathbf{R}^d}\sup_{\lambda\in\mathbf{R}^k}\mathcal{L}(x,\lambda) = p^* \text{ (primal problem)}$$

To solve $\sup\limits_{\lambda\in\mathbf{R}^k}\inf\limits_{x\in\mathbf{R}^k}\mathcal{L}(x,\lambda)$ we can consider the following approach:

- solve $\inf\limits_{x\in\mathbf{R}^k}\mathcal{L}(x,\lambda)$ and for this solve $\frac{\partial\mathcal{L}}{\partial x}(x,\lambda) = 0$ and find a solution $x^*(\lambda)$ dependent on λ then,
- solve $\sup\limits_{\lambda\in\mathbf{R}^k}\mathcal{L}(x^*(\lambda),\lambda)$, and for this solve $\frac{\partial\mathcal{L}}{\partial\lambda}(x^*(\lambda),\lambda) = 0$ to obtain λ^*.
- from this get the solution $x^*(\lambda^*)$.

In practice, instead of solving $\frac{\partial\mathcal{L}}{\partial\lambda}(x^*(\lambda),\lambda) = 0$ we solve

$$\forall i \in [\![1,k]\!], g(x^*(\lambda)) = 0$$

the two being equivalent, as demonstrated in the following proposition which assumes some differentiability properties for $x^*(\lambda)$.

Property .2. .
$\{\lambda \in \mathbf{R}^k, \frac{\partial\mathcal{L}}{\partial\lambda}(x^*(\lambda),\lambda) = 0\} = \{\lambda \in \mathbf{R}^k \textit{such that } \forall i \in [\![1,k]\!], g_i(x^*(\lambda)) = 0\}$.

Proof. $\frac{\partial\mathcal{L}}{\partial\lambda}(x^*(\lambda),\lambda) = 0$
$\iff \forall j \in [\![1,k]\!]$,

$$\sum_{i=1}^{i=d}\frac{\partial f}{\partial x_i}(x^*(\lambda))\frac{\partial x_i^*}{\partial\lambda_j}(\lambda) - g_j(x^*(\lambda)) - \sum_{l=1}^{k}\left[\lambda_l\sum_{i=1}^{i=d}\frac{\partial g_l}{\partial x_i}(x^*(\lambda))\frac{\partial x_i^*}{\partial\lambda_j}(\lambda)\right] = 0$$

$\Longleftrightarrow \forall j \in [\![1, k]\!]$,

$$\sum_{i=1}^{i=d} \frac{\partial f}{\partial x_i}(x^*(\lambda)) \frac{\partial x_i^*}{\partial \lambda_j}(\lambda) - g_j(x^*(\lambda)) - \sum_{i=1}^{i=d} \Big[\sum_{l=1}^{k} \lambda_l \frac{\partial g_l}{\partial x_i}(x^*(\lambda)) \frac{\partial x_i^*}{\partial \lambda_j}(\lambda) \Big] = 0$$

$\Longleftrightarrow \forall j \in [\![1, k]\!]$,

$$\sum_{i=1}^{i=d} \Big[\frac{\partial f}{\partial x_i}(x^*(\lambda)) - \sum_{l=1}^{k} \lambda_l \frac{\partial g_l}{\partial x_i}(x^*(\lambda)) \Big] \frac{\partial x_i^*}{\partial \lambda_j}(\lambda) - g_j(x^*(\lambda)) = 0$$

but $x^*(\lambda)$, solves $\frac{\partial \mathcal{L}}{\partial x}(x^*(\lambda), \lambda) = 0$ and so for the first term in bracket above

$$\forall i \in [\![1, d]\!], \frac{\partial f}{\partial x_i}(x^*(\lambda)) - \sum_{l=1}^{k} \lambda_l \frac{\partial g_l}{\partial x_i}(x^*(\lambda)) = 0$$

Therefore, $\frac{\partial \mathcal{L}}{\partial \lambda}(x^*(\lambda), \lambda) = 0 \Longleftrightarrow \forall j \in [\![1, k]\!], g_j(x^*(\lambda)) = 0$. Q.E.D.

□

So, in practice to solve (P) first we solve in x the equation $\frac{\partial \mathcal{L}}{\partial x}(x, \lambda) = 0$ which gives a solution $x^*(\lambda)$ dependent on λ. Then we solve in λ the equations $g_i(x^*(\lambda)) = 0$ which gives the solution $x^*(\lambda^*)$ for which the optimum is reached. The next section illustrates the method.

1.2 Solution of the Markowitz problem

We solve,

$$(P) \begin{cases} \inf_{\pi \in \mathbf{R}^d} \pi^{'} \Sigma \pi \\ \pi^{'} M = m \\ \pi^{'} 1_d = 1 \end{cases}$$ where 1_d is the vector of $\mathbb{R}^d$ with components equal to 1

The Lagrangian is, $\mathcal{L}(\pi, \lambda) = \pi^{'} \Sigma \pi - \lambda_1 (\pi^{'} M - m) - \lambda_2 (\pi^{'} 1_d - 1)$

We solve,

$$\frac{\partial \mathcal{L}}{\partial \pi}(\pi, \lambda) = 0$$
$$\Longleftrightarrow 2 \Sigma \pi - \lambda_1 M - \lambda_2 1_d = 0$$
$$\Longleftrightarrow \pi = \frac{\lambda_1}{2} \Sigma^{-1} M + \frac{\lambda_2}{2} \Sigma^{-1} 1_d$$

to satisfy the second constraint we get,

$$\pi^{'} 1_d = 1 \Longleftrightarrow \frac{\lambda_1}{2} b + \frac{\lambda_2}{2} a = 1$$

so, $\pi = \frac{\lambda_1}{2} \Sigma^{-1} M + (\frac{1}{a} - \frac{\lambda_1}{2} \frac{b}{a}) \Sigma^{-1} 1_d = \frac{1}{a} \Sigma^{-1} 1_d + \frac{\lambda_1}{2} \Sigma^{-1} (M - \frac{b}{a} 1_d)$
The second constraint $\pi^{'} M = m$ determines λ_1 and the portfolio in a unique way as a function of m,

$$\pi^{'} M = m \Longleftrightarrow \frac{1}{a} b + \frac{\lambda_1}{2} \| M - \frac{b}{a} 1_d \|^2_{\Sigma^{-1}} = m$$

$$\iff \frac{\lambda_1}{2} = \frac{m - \frac{b}{a}}{\|M - \frac{b}{a}1_d\|^2_{\Sigma^{-1}}}$$

So, the solution (investment) portfolio solution of (P) is

$$\pi^* = \frac{1}{a}\Sigma^{-1}1_d + \frac{m - \frac{b}{a}}{\|M - \frac{b}{a}1_d\|^2_{\Sigma^{-1}}}\Sigma^{-1}(M - \frac{b}{a}1_d)$$

1.3 Inequality constraints

Let (P) be the optimisation problem under constraints

$$(P)\begin{cases} \inf\limits_{x\in\mathbf{R}^d} f(x) \\ g_1(x) = 0 \\ \vdots \\ g_k(x) = 0 \\ h_1(x) \geq 0 \\ \vdots \\ h_m(x) \geq 0 \end{cases}$$

where $f(.)$ the $g_i(.)$ and the $h_j(.)$ are differentiable functions from $\mathbf{R}^d$ into $\mathbf{R}$. We note, $\mathcal{D} = \{x \in \mathbf{R}^d, g_1(x) = 0, \cdots, g_k(x) = 0, h_1(x) = 0, \cdots, h_m(x) = 0\}$.

Definition .4. *The Lagrangian of (P) is the function from $\mathbf{R}^{d+k}$ into $\mathbf{R}$ defined by*

$$\mathcal{L}(x, \lambda, \mu) = f(x) - \sum_{i=1}^{i=k} \lambda_i g_i(x) - \sum_{j=1}^{j=m} \mu_i h_j(x)$$

with $\lambda = (\lambda_1, \cdots, \lambda_k)'$ and $\mu = (\mu_1, \cdots, \mu_m)'$.

The following proposition explains how the Lagrangian method transforms an optimisation problem under constraints into an optimisation problem without constraint.

Proposition .2. *Solving the problem (P) is equivalent to solving*

$$(Q) \inf_{x\in\mathbf{R}^d} \sup_{\lambda\in\mathbf{R}^k, \mu\in(\mathbf{R}^+)^m} \mathcal{L}(x, \lambda, \mu)$$

Proof. Let $x \notin \mathcal{D}$,
if $\exists i, g_i(x) \neq 0$ then, $\sup\limits_{\lambda\in\mathbf{R}^k} \mathcal{L}(x, \lambda) = +\infty$ as the term $-\lambda_i g_i(x)$ tends to $+\infty$ when λ_i tends to $+\infty$ or $-\infty$ (depending on the sign of $g_i(x)$).
if $\exists j, h_j(x) < 0$ then, $\sup\limits_{\lambda\in\mathbf{R}^k} \mathcal{L}(x, \lambda) = +\infty$ as the term $-\lambda_i h_j(x)$ tends to $+\infty$ when μ_j tends to $+\infty$.

Now, if $x \in \mathcal{D}$, $\sup\limits_{\lambda\in\mathbf{R}^k, \mu\in(\mathbf{R}^+)^m} \mathcal{L}(x, \lambda, \mu)$ is obtained for $\mu = 0$

Therefore,

$$\begin{aligned} &\inf_{x\in\mathbf{R}^d} \sup_{\lambda\in\mathbf{R}^k,\mu\in(\mathbf{R}^+)^m} \mathcal{L}(x,\lambda,\mu) \\ &= \inf_{x\in\mathcal{D}} \sup_{\lambda\in\mathbf{R}^k,\mu\in(\mathbf{R}^+)^m} \mathcal{L}(x,\lambda,\mu) \\ &= \inf_{x\in\mathcal{D}} f(x) \end{aligned}$$

□

and as before we solve the dual problem.

Remark .2. *We do not need the equality constraints $g_i(x) = 0$ to describe the general case as such constraints can be expressed as $-g_i(x) \geq 0$ and $g_i(x) \geq 0$.*

Property .3. *Complementary Slackness*
If we assume that,

- $d^* = p^*$ *(no duality gap)*
- $\exists x^* \in \mathcal{D}, p^* = f(x^*)$ *(there is an optimal point for the primal problem)*
- $\exists \mu^* \in (\mathbf{R}^+)^m, \sup_{\mu\in(\mathbf{R}^+)^m} \inf_{x\in\mathbf{R}^d} \mathcal{L}(x,\mu) = \inf_{x\in\mathbf{R}^d} \mathcal{L}(x,\mu^*)$ *(there is an optimal point for the dual problem).*

Then, $\forall i \in [\![1,m]\!], \mu_i^* h_i(x^*) = 0$.

Proof. $f(x^*) = \inf_{x\in\mathbf{R}^d} \mathcal{L}(x,\mu^*) \Longrightarrow f(x^*) \leq \mathcal{L}(x^*,\mu^*)$

$$\Longrightarrow f(x^*) \leq f(x^*) - \sum_{i=1}^{i=m} \mu_i^* h_i(x^*)$$

but all the terms $-\mu_i^* h_i(x^*)$ are negative so equality can be otained only iff all these terms are zero. Q.E.D. □