1 Optimal bounding constant M

1. `rinvgamma_trunc2` uses the the inverse-gamma distribution as the proposal distribution to sample from a truncated inverse-gamma distribution.

   Let $f_{\alpha,\beta}(x)$ be the PDF of an inverse-gamma distribution and let $f^{tr}_{\alpha,\beta}(x)$ be the PDF of a truncated inverse-gamma distribution. We need to calculate M such that

$$M = \sup_x \frac{f^{tr}_{\alpha,\beta}(x)}{f_{\alpha,\beta}(x)}$$

   for all x in the support of the truncated inverse-gamma distribution.

$$M = \sup_x \frac{f_{\alpha,\beta}}{f_{\alpha,\beta} \times (F_{\alpha,\beta}(b) - F_{\alpha,\beta}(0))} = \frac{1}{F_{\alpha,\beta}(b) - F_{\alpha,\beta}(0)}$$

   where $F_{\alpha,\beta}$ is the CDF of the inverse-gamma distribution.

2. `rinvgamma_trunc3` uses the uniform distribution $g \sim \mathcal{U}(0, b)$ as the proposal distribution to sample from a truncated inverse-gamma distribution. Choosing the uniform distribution whose support covers the support of the truncated inverse-gamma distribution is essential for the rejection sampling to work, since otherwise there will be regions where the target distribution has non-zero density but the proposal distribution has zero density, making it impossible to sample from those regions. With the same notation as above, M is calculated such that

$$M = \sup_x \frac{f^{tr}_{\alpha,\beta}(x)}{g(x)} = \sup_{x\in[0,b]} \frac{f^{tr}_{\alpha,\beta}(x)}{1/b} = \frac{b}{F_{\alpha,\beta}(b) - F_{\alpha,\beta}(0)} \times \frac{\beta^\alpha}{\Gamma(\alpha)} \sup_{x\in[0,b]} x^{-\alpha-1} \exp\left(-\frac{\beta}{x}\right)$$

   Let $h(x) = x^{-\alpha-1}\exp(-\beta/x)$, we have $h'(x) = h(x)x^{-1}(\beta x^{-1} - \alpha - 1)$. Setting $h'(x) = 0$, we have a critical point at $x = \frac{\beta}{\alpha+1}$. Since $h'(x) > 0$ for $x < \frac{\beta}{\alpha+1}$ and $h'(x) < 0$ for $x > \frac{\beta}{\alpha+1}$, $h(x)$ is increasing on $(0, \frac{\beta}{\alpha+1})$ and decreasing on $(\frac{\beta}{\alpha+1}, +\infty)$. Therefore, the supremum of $h(x)$ on $[0, b]$ is achieved either at the right endpoint b (if $b \le \frac{\beta}{\alpha+1}$) or at the critical point $\frac{\beta}{\alpha+1}$ (if $b > \frac{\beta}{\alpha+1}$). Thus,

$$M = \begin{cases} \frac{b}{F_{\alpha,\beta}(b)-F_{\alpha,\beta}(0)} \times \frac{\beta^\alpha}{\Gamma(\alpha)} b^{-\alpha-1} \exp\left(-\frac{\beta}{b}\right), & b \le \frac{\beta}{\alpha+1} \\ \frac{b}{F_{\alpha,\beta}(b)-F_{\alpha,\beta}(0)} \times \frac{\beta^\alpha}{\Gamma(\alpha)} \left(\frac{\beta}{\alpha+1}\right)^{-\alpha-1} \exp\left(-\frac{\alpha+1}{1}\right), & b > \frac{\beta}{\alpha+1} \end{cases}$$

```r
const <- b/(pinvgamma(b,alpha,beta) - pinvgamma(0,alpha,beta)) * beta^alpha/gamma(alpha)
M <- ifelse(b <= beta/(alpha+1),
            const * b^(-alpha-1) * exp(-beta/b),
            const * (beta/(alpha+1))^(-alpha-1) * exp(-alpha-1))
```