Monte Carlo Methods Graded Lab 3

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Control Variates

The probability density function of the random variable X is given by

$$f(x) = \frac{3}{4}(1 - x^2)\mathbb{1}_{[-1,1]}(x)$$

To apply the control variates method, we first need to compute the even moments of X:

We can find a general formula for the even moments as follows:

$$\begin{aligned}
\mathbb{E}(X^{2k}) &= \int_{-\infty}^{+\infty} x^{2k} f(x)\,dx = \frac{3}{4}\int_{-1}^{1} x^{2k}(1 - x^2)\,dx \\
&= \frac{3}{4} \cdot 2 \int_0^1 x^{2k}(1 - x^2)\,dx \\
&= \frac{3}{2}\int_0^1 (x^{2k} - x^{2k+2})\,dx \\
&= \frac{3}{2}\left[\frac{1}{2k+1} - \frac{1}{2k+3}\right] = \frac{3}{(2k+1)(2k+3)}
\end{aligned}$$

Therefore, we have:

$\mathbb{E}(X^2) = \frac{1}{5}$,

$\mathbb{E}(X^4) = \frac{3}{35}$,

$\mathbb{E}(X^6) = \frac{1}{21}$.

$\mathbb{E}(X^8) = \frac{1}{33}$,

$\mathbb{E}(X^{12}) = \frac{1}{65}$.

Using $h_{0,1} = x^4$ and $h_{0,2} = x^6$ as control variates, we compute the optimal coefficients:

$$\beta_1^* = \frac{\mathrm{Cov}(f(X), h_{0,1}(X))}{\mathrm{Var}(h_{0,1}(X))} = \frac{\mathbb{E}(X^2X^4) - \mathbb{E}(X^2)\mathbb{E}(X^4)}{\mathbb{E}(X^8) - (\mathbb{E}(X^4))^2} = \frac{\frac{1}{21} - \frac{1}{5}\cdot\frac{3}{35}}{\frac{1}{33} - \left(\frac{3}{35}\right)^2} = \frac{77}{58} \approx 1.33$$

$$\beta_2^* = \frac{\mathrm{Cov}(f(X), h_{0,2}(X))}{\mathrm{Var}(h_{0,2}(X))} = \frac{\mathbb{E}(X^2X^6) - \mathbb{E}(X^2)\mathbb{E}(X^6)}{\mathbb{E}(X^{12}) - (\mathbb{E}(X^6))^2} = \frac{\frac{1}{33} - \frac{1}{5}\cdot\frac{1}{21}}{\frac{1}{65} - \left(\frac{1}{21}\right)^2} = \frac{819}{517} \approx 1.58.$$