Optimization TD1

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Exercise 1.1. Let $A \in S_d(\mathbb{R})$.

1. Letting $\lambda_1(A) \leq \lambda_2(A) \leq \cdots \leq \lambda_d(A)$ be the eigenvalues of A, show that

$$\lambda_1(A) = \inf_{\|z\|_2=1} \langle Az, z\rangle.$$

2. Show that for any two $A, B \in S_d(\mathbb{R})$ there holds

$$|\lambda_1(A) - \lambda_1(B)| \leq \|A - B\|_{\text{op}}$$

where $\|\cdot\|_{\text{op}}$ stands for the standard operator norm on the set of matrices.

Réponse:

1. Let $(e_1 ... e_d)$ be a basis of unit eigenvectors of A such that $Ae_i = \lambda_i e_i$ for all i. From the properties of eigenvectors and eigenvalues, we know that the eigenvectors corresponding to distinct eigenvalues are orthogonal. Therefore, we have:

$$\langle e_i, e_j\rangle = 0 \quad \text{for } i \neq j$$

and

$$\langle e_i, e_i\rangle = 1.$$

For any $z \in \mathbb{R}^d$, $\|z\|_2 = 1$, we can express z in terms of the basis $(e_1, \ldots, e_d)$:

$$z = \sum_{i=1}^{d} \langle z, e_i\rangle e_i,\ \|z\|_2^2 = \sum_{i=1}^{d} \langle z, e_i\rangle^2 = 1$$

Any vector z can be decomposed into its components along the eigenvectors of A. We can then compute $\langle Az, z\rangle$:

$$\langle Az, z\rangle = \langle A\left(\sum_{i=1}^{d} \langle z, e_i\rangle e_i\right), \sum_{j=1}^{d} \langle z, e_j\rangle e_j\rangle$$

Using the linearity of A and the fact that $Ae_i = \lambda_i e_i$, we get:

$$\langle Az, z\rangle = \langle\sum_{i=1}^{d} \langle z, e_i\rangle \lambda_i e_i, \sum_{j=1}^{d} \langle z, e_j\rangle e_j\rangle$$

By the orthogonality of the eigenvectors, this simplifies to:

$$\langle Az, z\rangle = \sum_{i=1}^{d} \langle z, e_i\rangle^2 \lambda_i$$

Since $\sum_{i=1}^{d} \langle z, e_i\rangle^2 = 1$ and each λ_i is bounded between $\lambda_{\min}$ and $\lambda_{\max}$, we have:

$$\inf_{\|z\|_2=1} \langle Az, z\rangle = \inf_{\|z\|_2=1} \sum_{i=1}^{d} \langle z, e_i\rangle^2 \lambda_i \leq \sum_{i=1}^{d} \langle z, e_i\rangle^2 \lambda_i$$

and

$$\sup_{\|z\|_2=1} \langle Az, z\rangle = \sup_{\|z\|_2=1} \sum_{i=1}^{d} \langle z, e_i\rangle^2 \lambda_i \geq \sum_{i=1}^{d} \langle z, e_i\rangle^2 \lambda_i$$

Assume that $c_1, c_2 \geq 0$ such that $c_1 + c_2 = 1$ and $a, b \in \mathbb{R}$ such that $a \leq b$.

$$a = c_1 a + c_2 a \leq c_1 a + c_2 b \leq c_1 b + c_2 b = b$$

Generalizing the above result to more variables, we have:

$$\lambda_1(A) = \inf_{\|z\|_2=1} \langle Az, z\rangle \leq \langle Az, z\rangle = \sum_{i=1}^{d} \langle z, e_i\rangle^2 \lambda_i \leq \sup_{\|z\|_2=1} \langle Az, z\rangle = \lambda_d(A).$$

2. Using the variational characterization from part (1), we have:

$$\lambda_1(A) = \inf_{\|z\|_2=1} \langle Az, z\rangle \quad \text{and} \quad \lambda_1(B) = \inf_{\|z\|_2=1} \langle Bz, z\rangle$$

For any unit vector z with $\|z\|_2 = 1$, we can write:

$$\langle Az, z\rangle - \langle Bz, z\rangle = \langle (A - B)z, z\rangle$$

By the Cauchy-Schwarz inequality and the definition of operator norm:

$$|\langle (A - B)z, z\rangle| \leq \|(A - B)z\|_2 \|z\|_2 \leq \|A - B\|_{\text{op}} \|z\|_2^2 = \|A - B\|_{\text{op}}$$

Therefore:

$$\langle Az, z\rangle \leq \langle Bz, z\rangle + \|A - B\|_{\text{op}}$$

Taking the infimum over all unit vectors z on the left side:

$$\lambda_1(A) = \inf_{\|z\|_2=1} \langle Az, z\rangle \leq \inf_{\|z\|_2=1} (\langle Bz, z\rangle + \|A - B\|_{\text{op}}) = \lambda_1(B) + \|A - B\|_{\text{op}}$$

This gives us:

$$\lambda_1(A) - \lambda_1(B) \leq \|A - B\|_{\text{op}}$$

By symmetry (swapping the roles of A and B), we also have:

$$\lambda_1(B) - \lambda_1(A) \leq \|B - A\|_{\text{op}} = \|A - B\|_{\text{op}}$$

Combining both inequalities:

$$|\lambda_1(A) - \lambda_1(B)| \leq \|A - B\|_{\text{op}}$$

Remark

spectral theorem: Let $A \in S_d(\mathbb{R})$ be a symmetric matrix. Then, there exists an orthonormal basis of $\mathbb{R}^d$ consisting of eigenvectors of A, and the eigenvalues can be ordered as $\lambda_1(A) \leq \lambda_2(A) \leq \cdots \leq \lambda_d(A)$.

Remark

The operator norm (or spectral norm) of a matrix $A \in \mathbb{R}^{d\times d}$ is defined as:

$$\|A\|_{\text{op}} = \sup_{\|x\|_2=1} \|Ax\|_2 = \sup_{x\neq 0} \frac{\|Ax\|_2}{\|x\|_2}$$

This norm measures the maximum amplification factor of the matrix when applied to unit vectors. For symmetric matrices, the operator norm equals the largest absolute eigenvalue: $\|A\|_{\text{op}} = \max_i |\lambda_i(A)|$. The operator norm is induced by the Euclidean norm and satisfies the submultiplicative property: $\|AB\|_{\text{op}} \leq \|A\|_{\text{op}}\|B\|_{\text{op}}$. It provides a measure of how much a linear transformation can stretch vectors and is fundamental in analyzing the conditioning and stability of linear systems.

Exercise 1.2. Let $A \in S_d(\mathbb{R})$ and $b \in \mathbb{R}^d$. We consider

$$f : x \mapsto \tfrac{1}{2}\langle Ax, x\rangle - \langle b, x\rangle.$$

1. Show that f is coercive if, and only if $A \in S_d^{++}(\mathbb{R})$.
2. Show that f is convex if, and only if $A \in S_d^{+}(\mathbb{R})$.
3. Show that f is strictly convex if, and only if $A \in S_d^{++}(\mathbb{R})$.

Réponse:

1. Let's show the contrapositive of " $\Longrightarrow$ " : if $A \notin S_d^{++}(\mathbb{R})$, then f is not coercive. We can safely omit $\langle b, x\rangle$ in the definition since it is not a part of the dominant term when $\|x\|_2 \to \infty$. If $A \notin S_d^{++}(\mathbb{R})$, then an eigenvalue $\lambda_d \leq 0$ (adopting the notation from last exercise). This implies that there exists a sequence $(x_n) \subset \mathbb{R}^d$ (up to choosing from the eigenspace E_{λ_d} associated with λ_d) such that $\|x_n\|_2 \to \infty$ and $\langle Ax_n, x_n\rangle \to -\infty$ or 0 which shows that f is not coercive.

As for the converse " $\Longleftarrow$ " : **We find a lower bound for** $\langle Ax, x\rangle$. Let $A \in S_d^{++}(\mathbb{R})$. Then all eigenvalues are positive, and we can find a constant $\alpha = \lambda_{\min}(A) > 0$ such that

$$\langle Ax, x\rangle \geq \alpha\|x\|_2^2 \quad \forall x \in \mathbb{R}^d.$$

This implies that

$$f(x) = \tfrac{1}{2}\langle Ax, x\rangle - \langle b, x\rangle \geq \tfrac{\alpha}{2}\|x\|_2^2 - \langle b, x\rangle.$$

Now, if $\|x\|_2 \to \infty$, the term $\frac{\alpha}{2}\|x\|_2^2$ dominates $-\langle b, x\rangle$, and we conclude that $f(x) \to \infty$. Thus, f is coercive.

2. We use the characterization of convexity through the Hessian matrix: f is convex if, and only if, $\nabla^2 f(x) \succeq 0$ for all $x \in \mathbb{R}^d$. The result is trivial since $\nabla^2 f(x) = A \succeq 0$ for all $x \in \mathbb{R}^d$.

3. We use the characterization of strict convexity through the Hessian matrix: f is strictly convex if, and only if, $\nabla^2 f(x) \succ 0$ for all $x \in \mathbb{R}^d$. The result is again trivial since $\nabla^2 f(x) = A \succ 0$ for all $x \in \mathbb{R}^d$.

Exercise 1.3. Classify the critical points (local minimisers, local maximisers, saddle points, indeterminate critical points) of the following functions:

1. $f_1 : (x, y) \mapsto (x-y)^2 + (x+y)^3$,
2. $f_2 : (x, y) \mapsto x^2 - 2y^2 + 3xy$,
3. $f_3 : (x, y) \mapsto x^4 + y^3 - 3y - 2$.

Réponse:

1. The critical points of f_1 can be found by computing the gradient and setting it to zero:

$$\nabla f_1(x, y) = \begin{pmatrix} 2(x-y) + 3(x+y)^2 \\ 2(y-x) + 3(x+y)^2 \end{pmatrix} = 0.$$

This gives us a system of equations to solve for the critical points. Simplifying the equations,

$$\begin{cases} 3(x+y)^2 = 2(x-y) \\ 3(x+y)^2 = -2(x-y) \end{cases}$$

Which leads to $x = y = 0$ as the only critical point. This critical point is a saddle point, and we only need to take the direction $\vec{d} = (1, 1)$.

2. For f_2, it is worth noticing that f_2 can be expressed with a quadratic form:

$$f_2(x, y) = \frac{1}{2} \begin{pmatrix} x \\ y \end{pmatrix}^T \begin{pmatrix} 2 & 3 \\ 3 & -4 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$

we compute the gradient:

$$\nabla f_2(x, y) = \begin{pmatrix} 2 & 3 \\ 3 & -4 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$

Again, we have a system of equations to solve. The unique solution is given by:

$$\begin{pmatrix} 2 & 3 \\ 3 & -4 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = 0$$

which leads to $x = 0$ and $y = 0$. To classify this critical point, we compute the Hessian matrix:

$$\nabla^2 f_2(x, y) = \begin{pmatrix} 2 & 3 \\ 3 & -4 \end{pmatrix}$$

The eigenvalues of this Hessian matrix can be found by solving the characteristic polynomial:

$$\det\left(\nabla^2 f_2(x, y) - \lambda I\right) = 0$$

which simplifies to:

$$\det \begin{pmatrix} 2-\lambda & 3 \\ 3 & -4-\lambda \end{pmatrix} = 0$$

The characteristic polynomial is given by:

$$(2-\lambda)(-4-\lambda) - 9 = 0$$

which leads to:

$$\lambda^2 + 2\lambda - 17 = 0$$

The eigenvalues are:

$$\lambda_{1,2} = -1 \pm \sqrt{18} = -1 \pm 3\sqrt{2}$$

Since one eigenvalue is positive and the other is negative, the critical point is a saddle point.

3. Finally, for f_3:

$$\nabla f_3(x, y) = \begin{pmatrix} 4x^3 \\ 3y^2 - 3 \end{pmatrix} = 0.$$

We can solve these equations to find the critical points. The solutions are:

$$x = 0, \quad y = \pm 1$$

To classify these critical points, we compute the Hessian matrix:

$$\nabla^2 f_3(x, y) = \begin{pmatrix} 12x^2 & 0 \\ 0 & 6y \end{pmatrix}$$

The eigenvalues of this Hessian matrix are given by the diagonal elements:

$$\lambda_1 = 12x^2, \quad \lambda_2 = 6y$$

At both critical points the Hessian is degenerate in the x-direction (entry $12x^2 = 0$), so the usual second derivative test is inconclusive and we use higher-order expansion.

1. Point $(0, 1)$. Write $x = x - 0,\ y = y - 1$ (move first critical point to O):

$$f_3(x, y) = x^4 + y^3 - 3y^2$$

We examine $f(x, y) - f(0, 0) = x^4 + y^3 - 3y^2$ along various paths through the origin:

(a) **Along the x-axis** ($y = 0$):

$$f(x, 0) = x^4 \geq 0$$

with equality only at $x = 0$.

(b) **Along the y-axis** ($x = 0$):

$$f(0, y) = y^3 - 3y^2 = y^2(y - 3)$$

For small $|y| > 0$: since $y - 3 < 0$, we have $f(0, y) < 0$.

(c) **Along the parabola** $y = x^2$:

$$f(x, x^2) = x^4 + x^6 - 3x^4 = x^6 - 2x^4 = x^4(x^2 - 2)$$

For small $|x| > 0$: since $x^2 - 2 < 0$, we have $f(x, x^2) < 0$.

Since the function takes both positive values (along the x-axis) and negative values (along the y-axis and the parabola $y = x^2$) in every neighborhood of $(0, 0)$, we conclude that:

$$\boxed{(0, 0) \text{ is a } \textbf{saddle point} \text{ of } f_3(x, y) = x^4 + y^3 - 3y^2}$$

2. Point $(0, -1)$. Write $y = -1 + s$:

$$f_3(x, -1 + s) = x^4 + (-1 + s)^3 - 3(-1 + s) - 2 = x^4 - 3s^2 + s^3.$$

Thus

$$f_3(x, -1 + s) - f_3(0, -1) = x^4 - 3s^2 + s^3.$$

Along $s = 0$, $x \neq 0$: difference $= x^4 > 0$. Along $x = 0$, $0 < s < 3$: difference $= -3s^2 + s^3 = -3s^2(1 - s/3) < 0$. Hence values of both signs occur arbitrarily close to $(0, -1)$: $(0, -1)$ is a saddle point.

Conclusion:

$$\boxed{\text{Both } (0, 1) \text{ and } (0, -1) \text{ are } \textbf{saddle points} \text{ of } f_3(x, y) = x^4 + y^3 - 3y - 2}$$

Exercise 1.4 (Distance between two sets)**.** Let A and B be two closed, nonempty subsets of $\mathbb{R}^d$.

1. Show that if A is compact, then the problem

$$\min_{a \in A,\, b \in B} \|a - b\|$$

has a solution (at least one).

2. Show with a counter-example that this problem need not have a solution if neither A nor B is assumed compact, even if A and B are convex.

Réponse:

1. Let $d_B(a) = \inf_{b \in B} \|a - b\|$ be the distance from the point a to the set B. Since B is closed, the infimum is attained at some point $b^* \in B$, and it depends on choice of a i.e.,

$$d_B(a) = \|a - b_a^*\|.$$

Since A is compact, the function d_B is continuous on A and attains its minimum at some point $a^* \in A$. Therefore, a minimizer of the original problem exists.

2. If neither A nor B is compact,

Exercise 1.5. Give an example of a strictly convex function $\varphi : \mathbb{R}^d \to \mathbb{R}$ such that the equation

$$\nabla^2 \varphi(x) = 0$$

has infinitely many solutions.

Exercise 1.6 (Carathéodory theorem)**.** Let $\Omega \subset \mathbb{R}^d$. We call the convex hull of Ω the smallest convex set containing Ω, denoted $C(\Omega)$.

1. Show that

$$C(\Omega) = \left\{ \sum_{i=0}^{N} t_i x_i \;\middle|\; N \in \mathbb{N},\ t_i \in [0,1],\ \sum_{i=0}^{N} t_i = 1,\ x_i \in \Omega \right\}.$$

2. Prove the Carathéodory theorem: for any $x \in C(\Omega)$, there exist $t_0, \ldots, t_d \in [0,1]$ and $x_0, \ldots, x_d \in \Omega$ such that

$$\sum_{i=0}^{d} t_i = 1, \quad x = \sum_{i=0}^{d} t_i x_i.$$

   (a) Using an example, show why one needs at least $(d+1)$ points.

   (b) Prove the theorem by descending induction, starting from a representation with $d+2$ points and eliminating one.

   (c) Deduce that if Ω is compact, then so is $C(\Omega)$.

Exercise 1.7 (Extreme points I: projection on closed convex sets)**.** Let $K \subset \mathbb{R}^d$ be a closed convex set. Show that there exists a unique $z \in K$, denoted by $\Pi_K(x)$ and called the orthogonal projection of x on K, such that

$$\|x - \Pi_K(x)\| = \min_{z \in K} \|x - z\|$$

and that

$$\forall y \in K,\ \langle x - \Pi_K(x),\, y - \Pi_K(x) \rangle \leq 0.$$

Show that Π_K is 1-Lipschitz.

Exercise 1.8 (Extreme points II: The Krein-Milman theorem)**.**

1. Give an example of a convex set $K \subset \mathbb{R}^d$ that has no extreme points.

2. Assume K is closed. Prove that K has extreme points.

3. Prove the finite-dimensional Krein-Milman theorem: any $x \in K$ is a convex combination of extreme points of K.

   (a) Let $x \in \partial K$. Show that there exists a supporting hyperplane $H = \{\varphi = 0\}$ with $\varphi \in (\mathbb{R}^d)^*, \varphi \neq 0$, such that $x \in H$ and $\varphi(K) \subset (-\infty, 0]$.

   (b) Show that if $x \in H$ for some supporting hyperplane of K, then x is an extreme point of K iff it an extreme point of $H \cap K$.

   (c) Conclude the theorem by induction on the dimension.

Exercise 1.9 (Polyak–Łojasiewicz Inequality)**.** Let $f : \mathbb{R}^d \to \mathbb{R}$ be an α-strongly convex function and let x^* be a minimiser of f.

1. Prove that
$$\forall x \in \mathbb{R}^d, \quad \|x - x^*\|^2 \leq \tfrac{2}{\alpha}\big(f(x) - f(x^*)\big).$$
2. Show that
$$\forall x \in \mathbb{R}^d, \quad f(x) - f(x^*) \leq \tfrac{1}{2\alpha}\|\nabla f(x)\|^2.$$
3. Deduce that
$$\forall x \in \mathbb{R}^d, \quad \|x - x^*\| \leq \tfrac{1}{\alpha}\|\nabla f(x)\|.$$