Monte Carlo Methods Graded Lab 3

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Control Variates

The probability density function of the random variable X is given by

$$f(x) = \frac{3}{4}(1 - x^2)\mathbb{1}_{[-1,1]}(x)$$

To apply the control variates method, we first need to compute the even moments of X:

We can find a general formula for the even moments as follows:

$$\begin{aligned}\mathbb{E}(X^{2k}) &= \int_{-\infty}^{+\infty} x^{2k} f(x)\,dx = \frac{3}{4}\int_{-1}^{1} x^{2k}(1-x^2)\,dx \\ &= \frac{3}{4}\cdot 2\int_0^1 x^{2k}(1-x^2)\,dx \\ &= \frac{3}{2}\int_0^1 (x^{2k} - x^{2k+2})\,dx \\ &= \frac{3}{2}\left[\frac{1}{2k+1} - \frac{1}{2k+3}\right] = \frac{3}{(2k+1)(2k+3)}\end{aligned}$$

Therefore, we have:

$\mathbb{E}(X^2) = \frac{1}{5}$,

$\mathbb{E}(X^4) = \frac{3}{35}$,

$\mathbb{E}(X^6) = \frac{1}{21}$.

$\mathbb{E}(X^8) = \frac{1}{33}$,

$\mathbb{E}(X^{12}) = \frac{1}{65}$.

Using $h_{0,1} = x^4$ and $h_{0,2} = x^6$ as control variables, we compute the optimal coefficients:

$$\beta_1^* = \frac{\mathrm{Cov}(f(X), h_{0,1}(X))}{\mathrm{Var}(h_{0,1}(X))} = \frac{\mathbb{E}(X^2X^4) - \mathbb{E}(X^2)\mathbb{E}(X^4)}{\mathbb{E}(X^8) - (\mathbb{E}(X^4))^2} = \frac{\frac{1}{21} - \frac{1}{5}\cdot\frac{3}{35}}{\frac{1}{33} - \left(\frac{3}{35}\right)^2} = \frac{77}{58} \approx 1.33$$

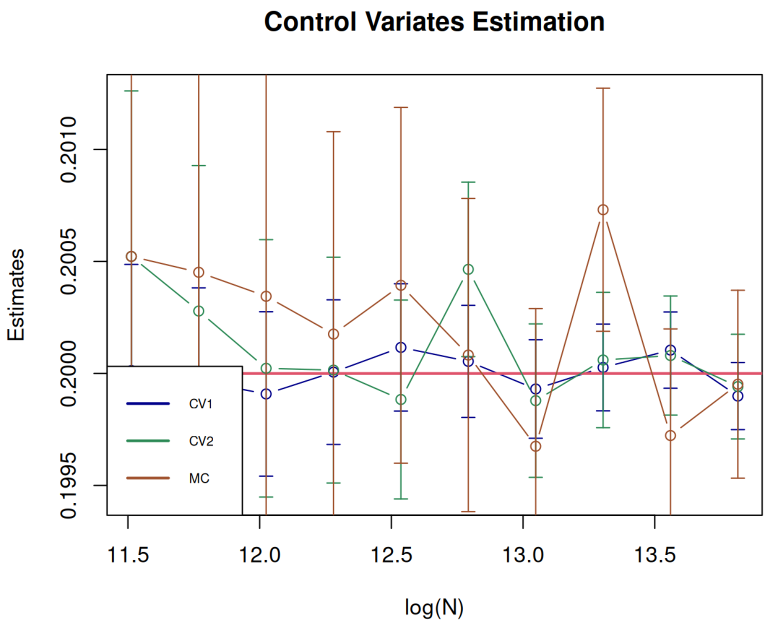
$$\beta_2^* = \frac{\mathrm{Cov}(f(X), h_{0,2}(X))}{\mathrm{Var}(h_{0,2}(X))} = \frac{\mathbb{E}(X^2X^6) - \mathbb{E}(X^2)\mathbb{E}(X^6)}{\mathbb{E}(X^{12}) - (\mathbb{E}(X^6))^2} = \frac{\frac{1}{33} - \frac{1}{5}\cdot\frac{1}{21}}{\frac{1}{65} - \left(\frac{1}{21}\right)^2} = \frac{819}{517} \approx 1.58.$$

Comparison of estimators

The true value of the integral is equal to

$$\mathbb{E}[X^2] = \frac{1}{5}$$

Now let us compare values of the estimators with the true value from $n = 10^5$ to $n = 10^6$.



The calculation with the highest n yields

$$|\hat{I}_n^{MC} - \hat{I}_n^{CV1}| \approx 15 \times 10^{-4}$$

$$|\hat{I}_n^{MC} - \hat{I}_n^{CV2}| \approx 2 \times 10^{-3}$$

$$|\hat{I}_n^{CV1} - \hat{I}_n^{CV2}| \approx 5 \times 10^{-4}$$

Let us compare efficiency of estimators. We will do it by estimating R:

$$R_{1,2} = \frac{cost1 * var1}{cost2 * var2}$$

Thus we obtain following values

$$R_{CV1,MC} = 0.1608956$$

We can conclude that CV1 estimator is more efficient than MC as $R < 1$.

$$R_{CV2,MC} = 0.3900529$$

We can conclude that CV2 estimator is more efficient than MC as $R < 1$.