Notes

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1 Joint distribution

Let X and Y be two random variables. If X and Y are linearly related, i.e., there exist $a \in \mathbb{R}$ such that $Y = aX$. We are interested in determining the joint distribution of (X, Y), given the distribution of X.
Let δ be the Dirac function,

$$\delta(x) = \begin{cases} +\infty, & x = 0, \\ 0, & x \neq 0, \end{cases} \quad \text{and} \quad \int_{-\infty}^{+\infty} \delta(x)\, dx = 1.$$

An important property of the Dirac function is that for any measurable function g,

$$\int_{-\infty}^{+\infty} g(x)\, \delta(x - a)\, dx = g(a).$$

Let $f_X(x)$ be the probability density function (PDF) of X. We calculate the probability of the event $(X, Y) \in A \times B$ for measurable sets A and B as follows:

$$\mathbb{P}\big((X, Y) \in A \times B\big) = \iint 1_{A\times B}(x, y)\, f_X(x)\, \delta\big(y - ax\big)\, dx\, dy = \int_A 1_B(ax)\, f_X(x)\, dx.$$

For $a \neq 0$, the marginal of Y is

$$f_Y(y) = \int f_X(x)\, \delta\big(y - ax\big)\, dx = \frac{1}{|a|}\, f_X\left(\frac{y}{a}\right).$$

The joint distribution of (X, Y) is concentrated on the line $y = ax$ and can be written (in the distributional sense) as

$$f_{X,Y}(x, y) = f_X(x)\, \delta\big(y - ax\big).$$