Discrete Processes TD1

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Exercise 1.1. Let X and Y be two independent random variables with Poisson distribution of parameters λ and μ respectively.
1. What is the distribution of $X+Y$?
2. Compute $\mathbb{E}(X \mid X+Y)$.

Réponse:

1. X and Y are independent Poisson random variables with respectively parameters λ and μ. The sum of two independent Poisson random variables is a Poisson random variable with parameter equal to the sum of the parameters. Thus, $X+Y \sim \text{Poisson}(\lambda+\mu)$.

This can be shown using the characteristic function or the moment generating function. And by definition, they are:

$$\mathbb{E}[e^{tX}] = \exp(\lambda(e^t - 1)), \quad \mathbb{E}[e^{tY}] = \exp(\mu(e^t - 1)).$$

2. By the expression of the conditional expectation for discrete random variables, we have

$$\mathbb{E}(X \mid X+Y=n) = \sum_{k=0}^{n} k\,\mathbb{P}(X=k \mid X+Y=n).$$

Moreover, the complete expression for $\mathbb{E}(X \mid X+Y)$ is given by

$$\mathbb{E}(X \mid X+Y) = \sum_{n=0}^{\infty} \mathbb{E}(X \mid X+Y=n)\,\mathbb{1}_{\{X+Y=n\}}.$$

Using the definition of conditional probability and the independence of X and Y, we get

$$\mathbb{P}(X=k \mid X+Y=n) = \frac{\mathbb{P}(X=k, Y=n-k)}{\mathbb{P}(X+Y=n)} = \frac{\mathbb{P}(X=k)\,\mathbb{P}(Y=n-k)}{\mathbb{P}(X+Y=n)}.$$

Substituting the probability mass functions of X and Y, we have

$$\mathbb{P}(X=k) = e^{-\lambda}\frac{\lambda^k}{k!}, \quad \mathbb{P}(Y=n-k) = e^{-\mu}\frac{\mu^{n-k}}{(n-k)!}.$$

The probability mass function of $X+Y$ is given by

$$\mathbb{P}(X+Y=n) = e^{-(\lambda+\mu)}\frac{(\lambda+\mu)^n}{n!}.$$

Therefore,

$$\mathbb{P}(X = k \mid X+Y=n) = \frac{e^{-\lambda}\frac{\lambda^k}{k!}e^{-\mu}\frac{\mu^{n-k}}{(n-k)!}}{e^{-(\lambda+\mu)}\frac{(\lambda+\mu)^n}{n!}} = \binom{n}{k}p^k(1-p)^{n-k},$$

where $p = \frac{\lambda}{\lambda+\mu}$. This shows that given $X+Y=n$, the random variable X follows a Binomial distribution with parameters n and p. Thus,

$$\mathbb{E}(X \mid X+Y=n) = np = n\frac{\lambda}{\lambda+\mu}.$$

Finally, we have

$$\mathbb{E}(X \mid X+Y) = \frac{\lambda}{\lambda+\mu}(X+Y).$$

Remark

It is interesting to note that $X \mid Y=y$ and $\mathbb{E}(X \mid Y)$ are different random variables. **An intuitive example is**, if X and Y are independent, then $\mathbb{E}(X \mid Y) = \mathbb{E}(X)$ is a constant, while $X \mid Y=y$ has the same distribution as X.
In this exercise, $X \mid X+Y=n$ is a Binomial random variable, while $\mathbb{E}(X \mid X+Y)$ is a random variable that takes values in $\{0, p, \ldots, np\}$ with probabilities given by $\mathbb{P}(X+Y=z)$ for $z = 0, 1, \ldots, n$.

Exercise 1.2. Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space and $\{A_1, \ldots, A_n\}$ be a finite partition of Ω. We define $\mathcal{G} = \sigma(A_1, \ldots, A_n)$ the σ-algebra generated by this partition.
1. Describe the σ-field $\mathcal{G}$.
2. Let X be an integrable random variable. Show that

$$\mathbb{E}(X \mid \mathcal{G})(\omega) = \sum_{j:\,\mathbb{P}(A_j)>0} \frac{\mathbb{E}(X1_{A_j})}{\mathbb{P}(A_j)} 1_{A_j}(\omega).$$

Réponse:

1. The σ-algebra $\mathcal{G}$ generated by the partition $\{A_1, \ldots, A_n\}$ consists of all possible unions of the sets in the partition. Since the sets A_i are disjoint and cover the entire sample space Ω, any event in $\mathcal{G}$ can be expressed as a union of some subset of the A_i. Therefore, we have:

$$\mathcal{G} = \left\{ \bigcup_{j\in J} A_j : J \subseteq \{1, 2, \ldots, n\} \right\}.$$

This includes the empty set (when $J = \emptyset$) and the entire space Ω (when $J = \{1, 2, \ldots, n\}$).

This can be shown rigorously by using the double inclusion. The easy inclusion is that any union of the sets A_j is in $\mathcal{G}$ by definition.

Now it is left to show that $\mathcal{G} \subseteq \{\bigcup_{j\in J} A_j : J \subseteq \{1, 2, \ldots, n\}\}$. The trick here is to **show that** $\{\bigcup_{j\in J} A_j : J \subseteq \{1, 2, \ldots, n\}\}$ **is a σ-algebra.** Then the inclusion follows from the definition of $\mathcal{G}$ as the smallest σ-algebra containing the sets $A_1, \ldots, A_n$.

2. To show that

$$\mathbb{E}(X \mid \mathcal{G})(\omega) = \sum_{j:\, \mathbb{P}(A_j)>0} \frac{\mathbb{E}(X 1_{A_j})}{\mathbb{P}(A_j)} 1_{A_j}(\omega),$$

we start by noting that $\mathbb{E}(X \mid \mathcal{G})$ is $\mathcal{G}$-measurable. This means that it is constant on each set A_j of the partition. Therefore, for each j, there exists a constant c_j such that:

$$\mathbb{E}(X \mid \mathcal{G})(\omega) = c_j, \quad \text{for } \omega \in A_j.$$

To find c_j, we use the property of conditional expectation:

$$c_j = \mathbb{E}(X \mid A_j) = \frac{\mathbb{E}(X 1_{A_j})}{\mathbb{P}(A_j)}.$$

Thus, we can express $\mathbb{E}(X \mid \mathcal{G})(\omega)$ as:

$$\mathbb{E}(X \mid \mathcal{G})(\omega) = \sum_{j=1}^{n} c_j 1_{A_j}(\omega) = \sum_{j:\, \mathbb{P}(A_j)>0} \frac{\mathbb{E}(X 1_{A_j})}{\mathbb{P}(A_j)} 1_{A_j}(\omega).$$

This completes the proof.

Exercise 1.3. Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space and $\mathcal{G}$ be a sub-σ-algebra of $\mathcal{F}$. Let X and Y be two square integrable random variables. Show that

$$\mathbb{E}(X\, \mathbb{E}(Y \mid \mathcal{G})) = \mathbb{E}(Y\, \mathbb{E}(X \mid \mathcal{G})).$$

Exercise 1.4. Let $X_1, \ldots, X_n$ be i.i.d. integrable random variables. Determine the following conditional expectations:
1. $\mathbb{E}[X_1 + X_2 + \cdots + X_n \mid X_1]$,
2. $\mathbb{E}[X_1 \mid X_1 + X_2 + \cdots + X_n]$.

Exercise 1.5. 1. Let X, Y be two i.i.d. random variables uniformly distributed on $[0, 1]$. Compute $\mathbb{E}(X \mid XY)$.
2. Let $X \sim \mathcal{N}(0, 1)$. Compute $\mathbb{E}(X^2 \mid X)$, $\mathbb{E}(X \mid X^2)$ and $\mathbb{E}(X^3 \mid X^2)$.
3. Let X and Y be i.i.d. random variables uniformly distributed on $[-\pi/2, \pi/2]$. Compute

$$\mathbb{E}(\sin X \mid \cos X), \quad \mathbb{E}(X \mid e^X), \quad \mathbb{E}(\cos X \mid \sin Y), \quad \mathbb{E}(\sin X \mid \cos(X + 2Y)).$$

Exercise 1.6. Let (X, Y) be a random vector with density

$$p_{X,Y}(x, y) = \alpha\beta\, y \exp\left(-\frac{\alpha x}{y} - \beta y\right) 1_{x>0} 1_{y>0},$$

where $\alpha > 0$ and $\beta > 0$ are parameters. Determine $\mathbb{E}(X \mid Y)$ and deduce $\mathbb{E}(X)$.

Exercise 1.7. Let Z be a random variable exponentially distributed with parameter 1 and let $t > 0$. We set $X = \min(Z, t)$ and $Y = \max(Z, t)$. Compute $\mathbb{E}[Z \mid X]$ and $\mathbb{E}[Z \mid Y]$.

Exercise 1.8. Let X be a square integrable random variable defined on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and let $\mathcal{G}$ a sub-σ-algebra of $\mathcal{F}$. We set

$$\operatorname{var}(X \mid \mathcal{G}) = \mathbb{E}[X^2 \mid \mathcal{G}] - \mathbb{E}[X \mid \mathcal{G}]^2.$$

Show that

$$\operatorname{var}(X) = \mathbb{E}[\operatorname{var}(X \mid \mathcal{G})] + \operatorname{var}(\mathbb{E}[X \mid \mathcal{G}]).$$

Exercise 1.9. Let $(X_0, X_1, \ldots, X_n)$ be a Gaussian random vector with mean zero and nondegenerate covariance matrix Γ. Show that there exist real numbers $\lambda_1, \ldots, \lambda_n$ such that

$$\mathbb{E}[X_0 \mid X_1, \ldots, X_n] = \sum_{i=1}^{n} \lambda_i X_i,$$

and determine the weights λ_i as a function of Γ.
Hint: The coordinates of a Gaussian vector are independent if and only if their covariance is zero.