TD 2: Martingales

Exercise 1. Let $(X_n)_{n\geq 0}$ be a martingale with respect to a filtration $(\mathcal{F}_n)_{n\geq 0}$. Show that (X_n) is also a martingale with respect to its natural filtration.

Exercise 2. Let $(Y_{n,k})_{n\geq 0, k\geq 1}$ be a collection of i.i.d. $\mathbb{N}$–valued random variables such that

$$0 \ < \ m \ := \ \mathbb{E}(Y_{n,k}) \ < \ +\infty.$$

We define the process $(X_n)_{n\geq 0}$ by

$$\begin{cases} X_0 \ = \ 1, \\ X_{n+1} \ = \ \sum_{k=1}^{X_n} Y_{n,k}, \quad n \geq 0, \end{cases}$$

with the convention $\sum_{k=1}^{0} \cdots = 0$. We can think of X_n as the size of a population at time n, in which each individual k, $1 \leq k \leq X_n$, is replaced at time $n+1$ by a random number $Y_{n,k}$ of children. The reference filtration is $(\mathcal{F}_n)_{n\geq 0}$ the natural filtration of $(X_n)_{n\geq 0}$.

1. Show that for all $n \in \mathbb{N}$,

$$\mathbb{E}\left[X_{n+1} \,|\, \mathcal{F}_n\right] = mX_n$$

2. Deduce that $X_n \in L^1$ and find a deterministic sequence $(c_n)_{n\geq 0}$ such that $Z_n = c_n X_n$ is a martingale.

Exercise 3. Let S and T be stopping times. Show that $\min(S, T)$ and $\max(S, T)$ are also stopping times.

Exercise 4 (The martingale)**.** Originally the term *martingale* referred to the strategy consisting, in a series of coin flip games, to double the bet as long as the player looses and to stop at the first success. The goal of this exercise is to analyze this strategy.

Let $(X_n)_{n\geq 1}$ be a sequence of i.i.d. random variables taking values 1 and -1 with probability 1/2. We consider a process $(M_n)_{n\geq 0}$ defined by

$$M_0 = 0, \qquad M_n = \sum_{k=1}^{n} 2^{k-1} X_k, \quad n \geq 1,$$

and a random variable

$$T = \inf\{n \geq 1;\ X_n = 1\}.$$

Let the natural filtration of (X_n) be the reference filtration.

1. Show that (M_n) is a martingale and that T is a stopping time.
2. Deduce the value of $\mathbb{E}(M_{n\wedge T})$ for all $n \in \mathbb{N}$.
3. Show that T is finite. What is the value of M_T? Comment.
4. Show that $\mathbb{E}(M_{T-1}) = -\infty$. Comment.

Exercise 5. We consider a random walk on $\mathbb{Z}$ with bias to the left. Let $(B_n)_{n\geq 1}$ be a sequence of i.i.d. random variables taking values in $\{-1, 1\}$ with $\mathbb{P}(B_1 = 1) = p < 1/2$. Let us also denote $q = 1 - p$. We define the process $(X_n)_{n\geq 0}$ by

$$X_0 = 0, \qquad X_n = \sum_{k=1}^{n} B_k, \quad n \geq 1.$$

Our goal is to determine the law of $S = \sup_{n\geq 0} X_n$. The reference filtration is $\mathcal{F}_n = \sigma(B_1, \ldots, B_n)$, $n \geq 1$, and $\mathcal{F}_0 = \{\emptyset, \Omega\}$.

1. Show that $S \geq 0$ a.s. and $S < +\infty$ a.s..
   Hint: use the law of large numbers.

2. Show that $((q/p)^{X_n})_{n\geq 0}$ is a martingale.

3. Denote $T_x = \min\{n \geq 0 : X_n = x\}$ for $x \in \mathbb{Z}$. Let $a \leq -1$ and $b \geq 1$ be two integers. Show that $T = T_a \wedge T_b$ is a finite stopping time.

4. Using the optional stopping theorem, compute $\mathbb{P}(X_T = b)$.

5. Deduce the value of $\mathbb{P}(T_b < +\infty)$.
   Hint: observe that $\mathbb{P}(X_T = b) = \mathbb{P}(T_b < T_a)$.

6. Determine the distribution of S.

Exercise 6. Let $x \in \mathbb{R}$ and $(\Delta_n)_{n\geq 1}$ be a sequence of square integrable random variables. Suppose that the sequence $(X_n)_{n\geq 0}$ defined by

$$X_0 \ = \ x, \quad X_n \ = \ x + \Delta_1 + \cdots + \Delta_n, \quad n \geq 1,$$

is a martingale. Show that, for all $n \geq 1$, $\mathbb{E}(\Delta_n) = 0$ and

$$\mathbb{E}(X_n^2) \ = \ x^2 + \mathbb{E}(\Delta_1^2) + \cdots + \mathbb{E}(\Delta_n^2).$$

Exercise 7. Let $(Y_n)_{n\geq 1}$ be a sequence of random variables in L^1. Suppose that the sequence $(P_n)_{n\geq 0}$ defined by

$$P_0 \ = \ 1, \quad P_n \ = \ Y_1 \cdots Y_n, \quad n \geq 1,$$

is a martingale such that $P_n \neq 0$ a.s.. Show that

$$E(Y_n) \ = \ 1, \qquad \forall n \geq 1,$$

and deduce that

$$\mathbb{E}(Y_1 \cdots Y_n) \ = \ \mathbb{E}(Y_1) \cdots \mathbb{E}(Y_n).$$

What happens if we no longer assume that $P_n \neq 0$ a.s.?

Exercise 8. Let $(B_n)_{n\geq 1}$ be a sequence of i.i.d. random variables such that $\mathbb{P}(B_n = 1) = \mathbb{P}(B_n = -1) = 1/2$ for all $n \geq 1$. Let also $L \geq 1$ be an integer and $D : \mathbb{Z} \to \{1, \ldots, L\}$ be a function. Let us define the process $(X_n)_{n\geq 0}$ by

$$\begin{cases} X_0 = 0, \\ X_{n+1} = X_n + D(X_n)B_{n+1}, \quad n \geq 0. \end{cases}$$

The reference filtration is $\mathcal{F}_n = \sigma(B_1, \ldots, B_n)$, $n \geq 1$, and $\mathcal{F}_0 = \{\emptyset, \Omega\}$.

1. Show that $(X_n)_{n\geq 0}$ is a martingale.
2. Show that, for all $n \geq 1$,
$$n \;\leq\; \mathbb{E}(X_n^2) \;=\; \sum_{k=1}^{n} \mathbb{E}(D^2(X_{k-1})) \;\leq\; L^2 n.$$
3. Let $\lambda > 0$. Show that, for all $n \geq 1$,
$$\mathbb{E}(\mathrm{e}^{\lambda X_n} | \mathcal{F}_{n-1}) \;=\; \mathrm{e}^{\lambda X_{n-1}} \cosh(\lambda D(X_{n-1})).$$
Using the bound $\cosh(x) \leq \mathrm{e}^{x^2/2}$, $x \in \mathbb{R}$, conclude that
$$\mathbb{E}(\mathrm{e}^{\lambda X_n}) \;\leq\; \mathrm{e}^{\lambda^2 L^2 n/2}.$$
4. Show the Azuma–Hoeffding inequality: for all $a > 0$ and $n \geq 1$,
$$\mathbb{P}(|X_n| \geq a) \;\leq\; 2\mathrm{e}^{-a^2/2L^2 n}.$$
Hint: use $\mathbb{P}(X_n \geq a) = \mathbb{P}(\mathrm{e}^{\lambda X_n} \geq \mathrm{e}^{\lambda a})$ *and optimize over* $\lambda > 0$.
5. Deduce that $X_n/n \to 0$ a.s..
6. Repeat the last two questions replacing $|X_n|$ with $\max_{0\leq k\leq n} |X_k|$.

Exercise 9. Let $X, \xi_1, \xi_2, \ldots$ be independent variables such that $X \sim \mathcal{N}(0, 1)$ and $\xi_n \sim \mathcal{N}(0, \varepsilon_n^2)$ for all $n \geqslant 1$. For $n \geq 1$, we set $Y_n = X + \xi_n$ and we denote byl $(\mathcal{F}_n)$ the natural filtration of (Y_n). We define the process $(X_n)_{n\geq 1}$ by

$$X_n = \mathbb{E}[X \mid \mathcal{F}_n], \quad n \geq 1.$$

The question is whether X_n converges to X or not.

We can think of the variable X as a unknown quantity that we seek to estimate. At each unit of time we measure X by committing an error ξ_n assumed to be Gaussian and independent of past mistakes. We therefore observe $Y_n = X + \xi_n$. The variable $X_n = \mathbb{E}[X \mid \mathcal{F}_n]$ is our best (in the L^2 sense) estimate of X at time n.

1. Show that the process (X_n) converges a.s. and in L^2. We denote by X_∞ the limit.

2. Express X_n in terms of Y_i and ε_i.
   Hint: Note that $(X, Y_1, \ldots, Y_n)$ is a Gaussian vector.

3. Deduce the value of $\|X - X_n\|_2$.

4. Show that $X_\infty = X$ if and only if

$$\sum_{i \geq 1} \varepsilon_i^{-2} = +\infty.$$

Exercise 10 (Doob's maximal inequality)**.** Let $p > 1$ and $(Y_n)_{n \geq 0}$ be a nonnegative submartingale. The aim is to show that for all $n \geq 0$,

$$\mathbb{E}\left[\left(\max_{0 \leq k \leq n} Y_k\right)^p\right] \leq \left(\frac{p}{p-1}\right)^p \mathbb{E}\left[Y_n^p\right].$$

1. Let Z be an integrable nonnegative random variable. Prove the following two results: for all $a > 0$,

$$\mathbb{P}(Z \geq a) \leq \frac{\mathbb{E}(Z 1_{Z \geq a})}{a} \quad \text{and} \quad \mathbb{E}(Z) = \int_0^\infty \mathbb{P}(Z \geq a)\, da.$$

2. Denote $Y_n^\star = \max_{0 \leq k \leq n} Y_k$ for $n \geq 0$. Show that for all $a > 0$,

$$\mathbb{P}(Y_n^\star \geq a) \leq \frac{\mathbb{E}(Y_n 1_{Y_n^\star \geq a})}{a}.$$

3. Show that

$$\mathbb{E}\big[(Y_n^\star)^p\big] = \int_0^{+\infty} p\, a^{p-1} \mathbb{P}(Y_n^\star \geq a)\, da.$$

4. Deduce from the last two questions that

$$\mathbb{E}\big[(Y_n^\star)^p\big] \leq \frac{p}{p-1} \mathbb{E}\big[Y_n (Y_n^\star)^{p-1}\big].$$

5. Conclude by applying Hölder's inequality.

Exercise 11 (Convergence in L^p)**.** Let $p > 1$ and $(X_n)_{n \geq 0}$ be a martingale bounded in L^p, *i.e.*,

$$\sup_{n \in \mathbb{N}} \mathbb{E}\left[|X_n|^p\right] < +\infty.$$

1. Show that for all $n \geq 0$,

$$\mathbb{E}\left[\left(\max_{0 \leq k \leq n} |X_k|\right)^p\right] \leq \left(\frac{p}{p-1}\right)^p \mathbb{E}\left[|X_n|^p\right],$$

   and deduce that $\max_{k \geq 0} |X_k| \in L^p$.

2. Conclude that (X_n) converges a.s. and in L^p.