

# Discrete Processes Midterm 2025 Autumn

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## 1 Tools for conditional expectation calculations

- 1. Case  $\mathbb{E}[X|\varphi(X)]$ :** Let  $X$  be a discrete random variable and  $\varphi$  a measurable function. We need to discuss how  $\sigma(\varphi(X))$  is related to  $\sigma(X)$ .

- If  $\varphi$  is bijective, then  $\varphi^{-1}$  exists and  $X = \varphi^{-1}(\varphi(X))$  which implies that  $X$  is measurable with respect to  $\sigma(\varphi(X))$ . Thus,  $\mathbb{E}[X|\varphi(X)] = X$ . Furthermore, in this case,  $\sigma(\varphi(X)) = \sigma(X)$ .
- If  $\varphi$  is not bijective, then there exist values  $x_1 \neq x_2$  such that  $\varphi(x_1) = \varphi(x_2)$ . In this case,  $X$  is not measurable with respect to  $\sigma(\varphi(X))$ . Thus,  $\mathbb{E}[X|\varphi(X)]$  is not simply  $X$  but rather a function that averages  $X$  over the pre-images of  $\varphi(X)$ . For example, for  $\mathbb{E}[X|X^2]$  and  $X$  symmetric we have:

$$\mathbb{E}[X|X^2 = t] = 0.5 \cdot \sqrt{t} + 0.5 \cdot (-\sqrt{t}) = 0. \quad (\text{Non rigorous})$$

The way to compute it rigorously is to use a property of conditional expectation: Let  $Y$  be the conditional expectation  $\mathbb{E}[X|\varphi(X)]$ . Then for any bounded measurable function  $g$ :

$$\mathbb{E}[Yg(\varphi(X))] = \mathbb{E}[Xg(\varphi(X))].$$

Using the symmetry of  $X$ , we can deduce that  $-X \sim X$  and if  $\varphi$  is such that  $\varphi(-X) = \varphi(X)$ , we have:

$$\mathbb{E}[Yg(\varphi(X))] = \mathbb{E}[Xg(\varphi(X))] = \mathbb{E}[-Xg(\varphi(X))].$$

This implies that  $Y = \mathbb{E}[X|\varphi(X)] = \mathbb{E}[-X|\varphi(X)] = -Y$ , hence  $Y = 0$ .

- Another useful result for  $X \sim Y$  (i.e.,  $X$  and  $Y$  have the same distribution) is that for any measurable function  $\varphi$ , we have  $\varphi(X) \sim \varphi(Y)$ . This can be used to simplify calculations of conditional expectations when dealing with symmetric distributions.

- 2. Case  $\mathbb{E}[f(X, Y)|g(X, Y)]$ :** Let  $(X, Y)$  be a pair of continuous random variables with a joint distribution and  $f, g$  measurable functions. We need to calculate in this case the joint distribution of  $(g(X, Y), f(X, Y))$  and then use the formula for conditional expectation for continuous random variables:

$$\mathbb{E}[f(X, Y)|g(X, Y) = t] = \int_{-\infty}^{\infty} s \cdot f_{f(X, Y)|g(X, Y)}(s|t) ds,$$

$$f_{f(X, Y)|g(X, Y)}(s|t) = \frac{f_{f(X, Y), g(X, Y)}(s, t)}{f_{g(X, Y)}(t)}.$$

To find the joint distribution of  $(g(X, Y), f(X, Y))$ , we can use the transformation technique. Let  $U = g(X, Y)$  and  $V = f(X, Y)$ . We need to find the Jacobian of the

transformation from  $(X, Y)$  to  $(U, V)$  and then compute the joint density  $f_{U,V}(u, v)$  using the change of variables formula.

Let  $T : (X, Y) \mapsto (U, V) = (g(X, Y), f(X, Y))$ . Assuming  $T$  is invertible and differentiable, we can find the inverse transformation  $T^{-1} : (U, V) \mapsto (X, Y)$ .

Remind that the change of variables formula integrating over a set  $B$  is given by (Since it's easier to solve  $(u, v) \in B$  transforming it back with  $T^{-1}$  then considering the conditions on  $(x, y)$ ):

$$\int_{T^{-1}(B)} \phi(T(x, y)) f_{X,Y}(x, y) dx dy = \int_B \phi(u, v) f_{U,V}(u, v) |\det J_{T^{-1}}(u, v)| du dv,$$

where  $J_{T^{-1}}(u, v)$  is the Jacobian matrix of the inverse transformation  $T^{-1}$ .