Discrete Processes Midterm 2025 Autumn

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1 Tools for conditional expectation calculations

1. **Case $\mathbb{E}[X|\varphi(X)]$:** Let X be a discrete random variable and φ a measurable function. We need to discuss how $\sigma(\varphi(X))$ is related to $\sigma(X)$.

   - If φ is bijective, then φ^{-1} exists and $X = \varphi^{-1}(\varphi(X))$ which implies that X is measurable with respect to $\sigma(\varphi(X))$. Thus, $\mathbb{E}[X|\varphi(X)] = X$.
     Futhermore, in this case, $\sigma(\varphi(X)) = \sigma(X)$.
   - If φ is not bijective, then there exist values $x_1 \neq x_2$ such that $\varphi(x_1) = \varphi(x_2)$. In this case, X is not measurable with respect to $\sigma(\varphi(X))$. Thus, $\mathbb{E}[X|\varphi(X)]$ is not simply X but rather a function that averages X over the pre-images of $\varphi(X)$. For example, for $\mathbb{E}[X|X^2]$ and X symmetric we have:

     $$\mathbb{E}[X|X^2 = t] = 0.5 \cdot \sqrt{t} + 0.5 \cdot (-\sqrt{t}) = 0. \quad \text{(Non rigorous)}$$

     The way to compute it rigorously is to use a property of conditional expectation: Let Y be the conditional expectation $\mathbb{E}[X|\varphi(X)]$. Then for any bounded measurable function g:

     $$\mathbb{E}[Yg(\varphi(X))] = \mathbb{E}[Xg(\varphi(X))].$$

     Using the symmetry of X, we can deduce that $-X \sim X$ and if φ is such that $\varphi(-X) = \varphi(X)$, we have:

     $$\mathbb{E}[Yg(\varphi(X))] = \mathbb{E}[Xg(\varphi(X))] = \mathbb{E}[-Xg(\varphi(X))].$$

     This implies that $Y = \mathbb{E}[X|\varphi(X)] = \mathbb{E}[-X|\varphi(X)] = -Y$, hence $Y = 0$.
   - Another useful result for $X \sim Y$ (i.e., X and Y have the same distribution) is that for any measurable function φ, we have $\varphi(X) \sim \varphi(Y)$. This can be used to simplify calculations of conditional expectations when dealing with symmetric distributions.

2. **Case $\mathbb{E}[f(X,Y)|g(X,Y)]$:** Let (X,Y) be a pair of continuous random variables with a joint distribution and f, g measurable functions. We need to calculate in this case the joint distribution of $(g(X,Y), f(X,Y))$ and then use the formula for conditional expectation for continuous random variables:

   $$\mathbb{E}[f(X,Y)|g(X,Y) = t] = \int_{-\infty}^{\infty} s \cdot f_{f(X,Y)|g(X,Y)}(s|t)\, ds,$$

   $$f_{f(X,Y)|g(X,Y)}(s|t) = \frac{f_{f(X,Y),g(X,Y)}(s,t)}{f_{g(X,Y)}(t)}.$$

   To find the joint distribution of $(g(X,Y), f(X,Y))$, we can use the transformation technique. Let $U = g(X,Y)$ and $V = f(X,Y)$. We need to find the Jacobian of the transformation from (X, Y) to (U, V) and then compute the joint density $f_{U,V}(u, v)$ using the change of variables formula.

Let $T : (X, Y) \mapsto (U, V) = (g(X, Y), f(X, Y))$. Assuming T is invertible and differentiable, we can find the inverse transformation $T^{-1} : (U, V) \mapsto (X, Y)$.

Remind that the change of variables formula integrating over a set B is given by (Since it's easier to solve $(u, v) \in B$ transforming it back with T^{-1} then considering the conditions on (x, y)):

$$\int_{T^{-1}(B)} \phi(T(x, y)) f_{X,Y}(x, y)\, dx\, dy = \int_B \phi(u, v) f_{U,V}(u, v) \left|\det J_{T^{-1}}(u, v)\right|\, du\, dv,$$

where $J_{T^{-1}}(u, v)$ is the Jacobian matrix of the inverse transformation T^{-1}.