Answer TD2 Portfolio Optimization

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We consider $d \geq 1$ risky assets $S_1, \ldots, S_d$ and a risk-free asset S_0. We note $R_T = (R_T^1, \ldots, R_T^d)'$ the random vector for the returns of $S_1, \ldots, S_d$ over the period $[0, T]$, M the expectation of R_T and R_0 the deterministic return of S_0. We assume that the variance-covariance matrix Σ of R_T is invertible and that $\mathbb{E}(R_T) \neq R_0 1_d$. We note $\Pi = (\pi_0, \pi)'$ the vector of allocations for an investment portfolio where $\pi \in \mathbb{R}^d$ defines the allocation in the risky assets and π_0 in the risk-free asset. We denote by $R_\pi = \pi' R_T$ the return of the risky part of the portfolio.

Exercise 2.1. Consider a portfolio $\Pi = (\pi_0, \pi)' \in \mathbb{R}^{d+1}$, $d \geq 1$.
1) Give the expression of the terminal wealth $W_T^\Pi(x_0)$ at time T for the buy and hold strategy Π and the initial capital x_0.
We consider now the optimisation problem

$$(P_\sigma) \qquad \begin{cases} \sup_{\pi \in \mathbb{R}^d} \mathbb{E}(W_T^\Pi(x_0)) \\ \operatorname{var}(W_T^\Pi(x_0)) = x_0^2 \sigma^2 \end{cases}$$

where $\sigma > 0$ is fixed.
2) Show that (P_σ) is equivalent to

$$\begin{cases} \sup_{\pi \in \mathbb{R}^d} \mathbb{E}(W_T^\Pi(x_0)) \\ \operatorname{var}(W_T^\Pi(x_0)) \leq x_0^2 \sigma^2 \end{cases}$$

3) For $\lambda > 0$, we define the Lagrangian

$$L_\lambda(\pi) := \mathbb{E}[R_\pi] - \lambda(\operatorname{Var}(R_\pi) - \sigma^2).$$

(a) Show that $\sup_{\pi \in \mathbb{R}^d} L_\lambda(\pi) < +\infty$ and that the supremum is attained for a particular value of π that we note π_λ.
(b) Can you choose $\lambda(\sigma) > 0$ such that $\operatorname{Var}(R_{\pi_{\lambda(\sigma)}}) = \sigma^2$?
(c) Explain why the efficient frontier is $\{(\sigma(\pi_{\lambda(\sigma)}), \mathbb{E}(R_{\pi_{\lambda(\sigma)}})), \, \sigma > 0\}$.

Réponse:

1. The terminal wealth $W_T^\Pi(x_0)$ at time T for the buy and hold strategy $\Pi = (\pi_0, \pi)'$ with initial capital x_0 can be expressed as:

$$W_T^\Pi(x_0) = x_0 \pi_0 R^0 + x_0 \pi' R_T$$

2. Calculation the variance, we have

$$\operatorname{var}(W_T^\Pi(x_0)) = x_0^2 \operatorname{var}(\pi' R_T) = x_0^2 \pi' \Sigma \pi$$

$\pi' \Sigma \pi$ is a quadratic form and Σ is positive definite. We have $\pi' \Sigma \pi \geq 0$ and the equality holds if and only if $\pi = 0$. Thus, we have that the constraint set $\{\pi \in \mathbb{R}^d : \operatorname{var}(W_T^\Pi(x_0)) = x_0^2 \sigma^2\} = \{\pi \in \mathbb{R}^d : \pi' \Sigma \pi = \sigma^2\}$ is a compact set (closed and bounded ellipsoid in $\mathbb{R}^d$).

Since the objective function $\mathbb{E}(W_T^\Pi(x_0))$ is continuous and linear in π, and we are maximizing over a compact set, the maximum is attained by the **Extreme Value Theorem**.

Moreover, since the constraint set is the boundary of the ellipsoid $\{\pi : \pi' \Sigma \pi \leq \sigma^2\}$, any point in the interior would give a strictly smaller variance. By the scaling argument (if $\pi' \Sigma \pi < \sigma^2$, then $k\pi$ with $k > 1$ appropriately chosen gives $(k\pi)' \Sigma (k\pi) = \sigma^2$ and higher expected return), the maximum over the inequality constraint is achieved on the boundary, i.e., where the equality constraint holds.