TD 1: Conditional Expectation

Exercise 1. Let X and Y be two independent random variables with Poisson distribution of parameters λ and μ respectively.

1. What is the distribution of $X+Y$?
2. Compute $\mathbb{E}(X|X+Y)$.

Exercise 2. Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space and $\{A_1, \ldots, A_n\}$ be a finite partition of Ω. We define $\mathcal{G} = \sigma(A_1, \ldots, A_n)$ the σ-algebra generated by this partition.

1. Describe the σ-field $\mathcal{G}$.
2. Let X be an integrable random variable. Show that

$$\mathbb{E}(X|\mathcal{G})(\omega) = \sum_{j:\mathbb{P}(A_j)>0} \frac{\mathbb{E}(X 1_{A_j})}{\mathbb{P}(A_j)} 1_{A_j}(\omega).$$

Exercise 3. Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space and $\mathcal{G}$ be a sub-σ-algebra of $\mathcal{F}$. Let X and Y be two square integrable random variables. Show that

$$\mathbb{E}(X\mathbb{E}(Y|\mathcal{G})) = \mathbb{E}(Y\mathbb{E}(X|\mathcal{G})).$$

Exercise 4. Let $X_1, \ldots, X_n$ be i.i.d. integrable random variables. Determine the following conditional expectations:

1. $\mathbb{E}[X_1 + X_2 + \cdots + X_n \mid X_1]$,
2. $\mathbb{E}[X_1 \mid X_1 + X_2 + \cdots + X_n]$.

Exercise 5.

1. Let X, Y be two i.i.d. random variables uniformly distributed on $[0, 1]$. Compute $\mathbb{E}(X|XY)$.
2. Let $X \sim \mathcal{N}(0, 1)$. Compute $\mathbb{E}(X^2|X)$, $\mathbb{E}(X|X^2)$ and $\mathbb{E}(X^3|X^2)$.
3. Let X and Y be i.i.d. random variables uniformly distributed on $[-\pi/2, \pi/2]$. Compute

$$\begin{aligned} &\mathbb{E}(\sin X|\cos X), \quad \mathbb{E}\left(X|\mathrm{e}^X\right), \\ &\mathbb{E}(\cos X|\sin Y), \quad \mathbb{E}(\sin X|\cos(X+2Y)). \end{aligned}$$

Exercise 6. Let (X, Y) be a random vector with density

$$p_{X,Y}(x, y) = \frac{\alpha\beta}{y} \exp\left\{-\frac{\alpha x}{y} - \beta y\right\} \mathbf{1}_{x>0}\mathbf{1}_{y>0},$$

where $\alpha > 0$ and $\beta > 0$ are parameters. Determine $\mathbb{E}(X|Y)$ and deduce $\mathbb{E}(X)$.

Exercise 7. Let Z be a random variable exponentially distributed with parameter 1 and let $t > 0$. We set $X = \min(Z, t)$ and $Y = \max(Z, t)$. Compute $\mathbb{E}[Z \mid X]$ and $\mathbb{E}[Z \mid Y]$.

Exercise 8. Let X be a square integrable random variable defined on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and let $\mathcal{G}$ a sub-σ-algebra of $\mathcal{F}$. We set

$$\operatorname{var}(X \mid \mathcal{G}) = \mathbb{E}[X^2 \mid \mathcal{G}] - \mathbb{E}[X \mid \mathcal{G}]^2.$$

Show that

$$\operatorname{var}(X) = \mathbb{E}\left[\operatorname{var}(X \mid \mathcal{G})\right] + \operatorname{var}\left(\mathbb{E}[X \mid \mathcal{G}]\right).$$

Exercise 9. Let $(X_0, X_1, \ldots, X_n)$ be a Gaussian random vector with mean zero and nondegenerate covariance matrix Γ. Show that there exist real numbers $\lambda_1, \ldots, \lambda_n$ such that

$$\mathbb{E}[X_0 \mid X_1, \ldots, X_n] = \sum_{i=1}^{n} \lambda_i X_i$$

and determine the weights λ_i as a function of Γ.
Hint: The coordinates of a Gaussian vector are independent if and only if their covariance is zero.