University Paris Dauphine-PSL

Mido M1

Portfolio Management 2025-2026 [1]

TD 2: Optimisation and portfolio selection

We consider $d \geq 1$ risky assets $S^1, \cdots, S^d$ and a risk-free asset S^0. We note $R_T = (R_T^1, \cdots, R_T^d)'$ the random vector for the returns of $S^1, \cdots, S^d$ over the period $[0, T]$, M the expectation of R_T and R^0 the deterministic return of S^0. We assume that the variance-covariance matrix Σ of R_T is invertible and that $\mathbf{E}(R_T) \neq R^0 \mathbf{1}_d$. We note $\Pi = (\pi^0, \pi)'$ the vector of allocations for an investment portfolio where $\pi \in \mathbf{R}^d$ defines the allocation in the risky assets and π^0 in the risk-free asset.

Exercise 1.

Assume that $d = 2$ and that investments are made only in the risky assets, i.e. $\pi^0 = 0$.

1) Does the expected return of a risky portfolio determines in a unique way the variance of its return ?

2) Express π for the portfolio of minimum variance as a function of σ_1, σ_2 and the covariance of the returns of the two risky assets noted $\rho_{1,2}$.

Exercise 2.

Consider a portfolio $\Pi = (\pi^0, \pi)' \in \mathbf{R}^{d+1}$, $d \geq 1$

1) Give the expression of the terminal wealth $W_T^\Pi(x_0)$ at time T for the buy and hold strategy Π and the initial capital x_0.

   We consider now the optimisation problem

   $$(P_\sigma) \begin{cases} \sup_{\pi \in \mathbf{R}^d} \mathbf{E}(W_T^\Pi(x_0)) \\ \mathrm{var}(W_T^\Pi(x_0)) = x_0^2 \sigma^2 \end{cases} \text{ where } \sigma > 0 \text{ is fixed}$$

2) Show that (P_σ) is equivalent to

   $$\begin{cases} \sup_{\pi \in \mathbf{R}^d} \mathbf{E}(W_T^\Pi(x_0)) \\ \mathrm{var}(W_T^\Pi(x_0)) \leq x_0^2 \sigma^2 \end{cases}$$

[1] Pierre Brugière, Philippe Bergault, Emmanuel Lepinette

3) For $\lambda > 0$, we define the Lagrangian $Ł^{\lambda}(\pi) := \mathbf{E}[R^{\pi}] - \lambda(\mathbf{Var}(R^{\pi}) - \sigma^2)$.

   (a) show that $\sup_{\pi \in \mathbf{R}^d} Ł^{\lambda}(\pi) < +\infty$ and that the sup is attained for a particular value of π that we note π_{λ}

   (b) can you choose $\lambda(\sigma) > 0$ such that $\mathbf{Var}(R^{\pi_{\lambda(\sigma)}}) = \sigma^2$?

   (c) explain why the efficient frontier is $\{(\sigma(\pi_{\lambda(\sigma)}), E(R^{\pi_{\lambda(\sigma)}}), \sigma > 0)\}$.

Exercise 3.
Let L be a random variable. The quantile of L is defined by:
$q_L(\alpha) = \inf_x \{x, P(L \leq x) \geq \alpha\}$.
In finance L usually represents the distributions of possible losses ($L > 0$ means a loss) and $q_L(\alpha)$ is called the α-value at risk and is noted $\mathrm{VAR}_{\alpha}(L)$

1) Show that $q_L(\alpha) = \sup_x \{x, P(L \leq x) < \alpha\}$ and $P(L \leq q_L(\alpha)) \geq \alpha$.

2) Show that $P\left(L \leq \mathrm{VAR}_{95\%}(L)\right) \geq 95\%$ and $P\left(L > \mathrm{VAR}_{95\%}(L)\right) \leq 5\%$.

   Let X_1, X_2, ν_1, ν_2 be independent variables with $X_i \sim N(0,1)$. We assume that $P(\nu_i = 10) = 0.02$ and $P(\nu_i = 0) = 0.98$. We note $L_i = X_i + \nu_i$.

3) Show that $P(L_i > 4) < 3\%$ and $P(\frac{1}{2}L_1 + \frac{1}{2}L_2 > 4) > 3\%$

4) Deduct from 3) that $\mathrm{VAR}_{97\%}(\frac{1}{2}L_1 + \frac{1}{2}L_2) > \frac{1}{2}\mathrm{VAR}_{97\%}(L_1) + \frac{1}{2}\mathrm{VAR}_{97\%}(L_2)$

5) How do you interpret the previous result and what problem does it show about the utilization of the VAR as a risk measure?