Monte Carlo Methods Graded Lab 2

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1 Importance Sampling Estimator using Exponential Proposal

The integral we want to estimate is $p = 1 - 2q$ where $q = \mathbb{P}(Z > K)$ where $Z \sim \mathcal{N}(0,1)$. Formalizing q as the form of an expectation, we have

$$q = \mathbb{E}_{Z\sim\mathcal{N}(0,1)}\left[\mathbb{I}_{\{Z>K\}}\right] = \int_K^{+\infty} \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{z^2}{2}\right) dz.$$

We choose the exponential distribution with rate λ shifted by K as the proposal distribution, i.e., $Y \sim \mathcal{E}(\lambda) + K$ with PDF

$$f_Y(y) = \lambda \exp\left(-\lambda(y-K)\right), \quad y \geq K.$$

The importance sampling estimator for q using this proposal distribution is

$$\hat{q}_N^{IS}(\lambda) = \frac{1}{N}\sum_{i=1}^{N} \frac{\mathbb{I}_{\{Y_i>K\}} \cdot f_Z(Y_i)}{f_Y(Y_i)} = \frac{1}{N}\sum_{i=1}^{N} \frac{\frac{1}{\sqrt{2\pi}}\exp\left(-\frac{Y_i^2}{2}\right)}{\lambda\exp\left(-\lambda(Y_i-K)\right)}, \quad Y_i \sim \mathcal{E}(\lambda)+K.$$

The variance of the importance sampling estimator is

$$\mathrm{Var}(\hat{q}_N^{IS}(\lambda)) = \frac{1}{N}\mathrm{Var}\left(\frac{\mathbb{I}_{\{Y>K\}} \cdot f_Z(Y)}{f_Y(Y)}\right) = \frac{1}{N}\left(\mathbb{E}\left[\left(\frac{\mathbb{I}_{\{Y>K\}} \cdot f_Z(Y)}{f_Y(Y)}\right)^2\right] - q^2\right).$$

An optimal choice of λ would minimize this variance, and we found this variance using a grid search over [0.1, 6] with a step size of (6 - 0.1)/1000 = 0.0059. As shown in Figure 2, and the R executions in Listings 1, the optimal λ is approximately 3.507708.

```
> lam_opt <- lam_values[which.min(se_values)]
> lam_opt
[1] 3.507708
```

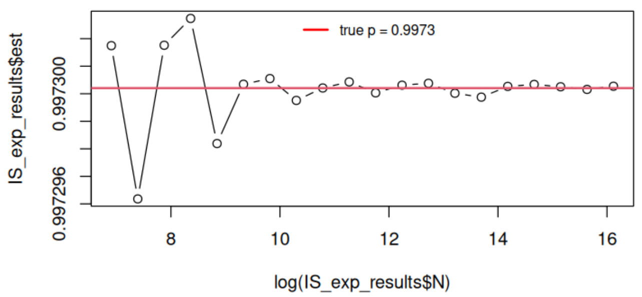

Figure 1: IS estimator with increasing N.

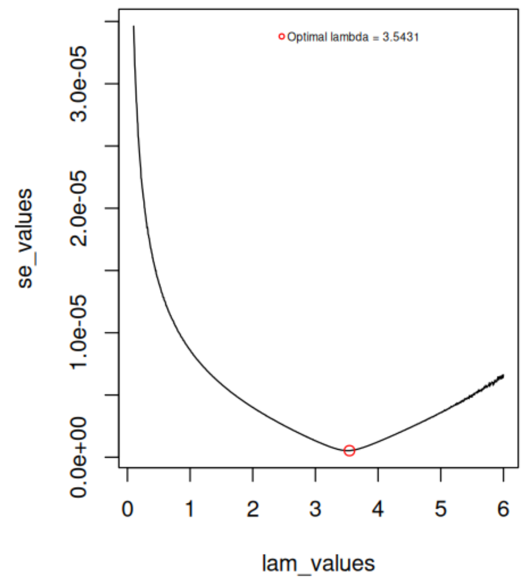

Figure 2: Variance as a function of λ.

2 Antithetic Variates with IS Exponential Proposal

We use the antithetic variates method to further reduce the variance of the importance sampling estimator with the optimal λ found in Section 1. We propose to use the inverse transform method to generate exponential random variables. Given a uniform random variable $U \sim \mathrm{Uniform}(0,1)$, the following transformations yield the same exponential distribution with rate λ:

$$\begin{aligned}
Y &= -\frac{1}{\lambda}\ln(U) \\
A(Y) &= -\frac{1}{\lambda}\ln(1-U) \\
\mathrm{Cov}(Y, A(Y)) &= \mathrm{Cov}(-\frac{1}{\lambda}\ln(U), -\frac{1}{\lambda}\ln(1-U)) \\
&= \frac{1}{\lambda^2}\mathrm{Cov}(\ln(U), \ln(1-U)) \\
&= \frac{1}{\lambda^2}\left(\mathbb{E}[\ln(U)\ln(1-U)] - \mathbb{E}[\ln(U)]\mathbb{E}[\ln(1-U)]\right) \\
&= \frac{1}{\lambda^2}\left(2 - \frac{\pi^2}{6} - 1\right) = \frac{1}{\lambda^2}\left(1 - \frac{\pi^2}{6}\right) < 0.
\end{aligned}$$

To implement the antithetic variates method, for each generated uniform random variable U_i, we also consider its antithetic counterpart $1 - U_i$. This leads to two exponential random variables:

$$Y_i = -\frac{1}{\lambda}\ln(U_i) \quad \text{and} \quad A(Y_i) = -\frac{1}{\lambda}\ln(1-U_i).$$

The antithetic variates estimator for the integral can then be expressed as:

$$\begin{aligned}
\hat{q}_N^{AV}(\lambda^*) &= \frac{1}{2N}\sum_{i=1}^{N}\left(\frac{h(Y_i)f_Z(Y_i)}{f_Y(Y_i)} + \frac{h(A(Y_i))f_Z(A(Y_i))}{f_Y(A(Y_i))}\right), \\
&= \frac{1}{2N}\sum_{i=1}^{N}\left(\frac{\mathbb{I}_{\{Y_i>K\}} \cdot f_Z(Y_i)}{f_Y(Y_i)} + \frac{\mathbb{I}_{\{A(Y_i)>K\}} \cdot f_Z(A(Y_i))}{f_Y(A(Y_i))}\right)
\end{aligned}$$

We compare the performance of the antithetic variates estimator with the standard importance sampling estimator using the optimal λ^*. As shown in Figure 3 and the R executions in Listings 2, the antithetic variates method consistently yields a lower standard error compared to the standard importance sampling estimator, demonstrating its effectiveness in variance reduction.

```
> IS_exp_results[1:5,]
     N       est           se
1 1000 0.9973017 5.105095e-06
2 1624 0.9972962 3.926113e-06
3 2637 0.9973018 3.215584e-06
4 4281 0.9973027 2.624856e-06
5 6952 0.9972982 1.997997e-06
> IS_exp_ant_results[1:5,]
     N       est           se
1 1000 0.9972996 3.556453e-06
2 1624 0.9972999 2.695311e-06
3 2637 0.9973014 2.150150e-06
4 4281 0.9972997 1.600106e-06
5 6952 0.9973002 1.254755e-06
```

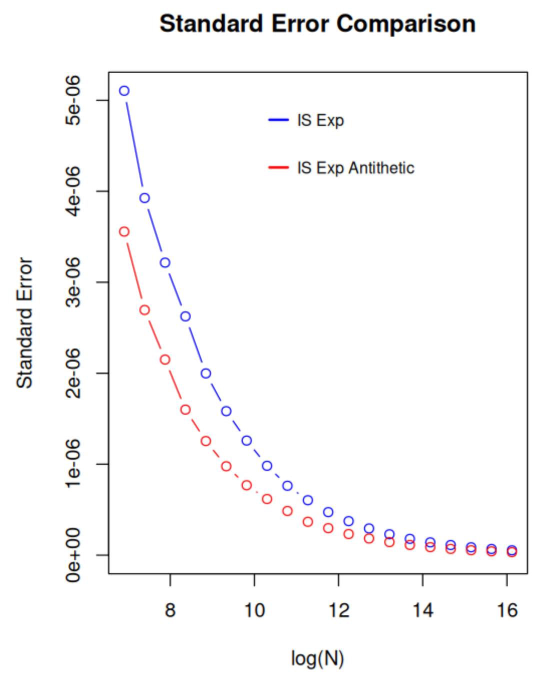

Figure 3: SE comparison between IS and IS with AV.