Portfolio Management

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Overview

Laws of Returns of Financial Assets

Returns and Rate of Returns

- Return of an investment: $R_T = \frac{W_T}{W_0} - 1$ (initial wealth W_0)
- log (or continuous) Return of an investment: $R_T^c = ln(\frac{W_T}{W_0})$
- Total Return of an asset P_T: $R_T = \frac{P_T + Div(0,T)}{P_0} - 1$
- Price Return of an asset P_T: $R_T = \frac{P_T}{P_0} - 1$
- Different ways to define a rate of return:
  - monetary rate: $1 + r_m \times T = 1 + R_T$
  - actuarial rate: $(1 + r_a)^T = 1 + R_T$
  - continuous rate: $exp(r_c \times T) = 1 + R_T$ i.e $r_c \times T = R_T^c$
- Different ways to define T: 30/360, Act/360, Act/Act, Act/365

Returns and Rate of Returns

- For modelisation in continuous time, continuous rates make calculations simpler:
  - simplicity to compound interests:
    $exp(r_c T_1) exp(r_c T_2) = exp(r_c (T_1 + T_2))$
  - well suited for modelisation in continuous time
    $dP_t = r_c P_t dt \Longrightarrow P_T = P_0 e^{r_c T}$
  - enable to conserve the property of normality when assuming the independence of the returns:
    $r_c(0, T_1 + T_2) = \frac{1}{T_1 + T_2}(T_1 r_c(0, T_1) + T_2 r_c(T_1, T_1 + T_2))$
- For portfolio analysis as we will see later, using monetary rates and returns make calculations simpler as the monetary return of a portfolio is equal to the weighted average of the monetary returns of its components.

Probabilist Definitions and Empirical Statistics

- **Probabilist** definitions for a r.v X
  - Expectation: $E[X]$
  - Variance: $Var[X] = E[X^2] - E[X]^2$
  - Standard deviation: $\sigma[X] = \sqrt{Var[X]}$
  - Skew: $Skew[X] = E\left[(\frac{X-E(X)}{\sigma(X)})^3\right]$
  - Kurtosis: $Kur[X] = E\left[(\frac{X-E(X)}{\sigma(X)})^4\right]$(some authors substract 3 in the definition, others call this new quantity the excess kurtosis)

Probabilist Definitions and Empirical Statistics

- **Empirical** definitions for a sample $x = (x_1, x_2, \cdots x_n)$
  - Sample Mean: $\overline{x} = \hat{E}(x) = \frac{1}{n}\sum_{i=1}^{i=n} x_i$
  - Sample Variance : $\widehat{Var}(x) = \frac{1}{n}\sum_{i=1}^{i=n}(x_i - \overline{x})^2$
  - Sample Standard Deviation: $\widehat{\sigma}(x) = \sqrt{\widehat{Var}(x)}$
  - Sample Skew: $\widehat{Skew}(x) = \frac{1}{n}\sum_{i=1}^{i=n}(\frac{x_i - \overline{x}}{\hat{\sigma}(x)})^3$
  - Sample Kurtosis: $\widehat{Kur}(x) = \frac{1}{n}\sum_{i=1}^{i=n}(\frac{x_i - \overline{x}}{\hat{\sigma}(x)})^4$
- **Remark**: the empirical quantities can be obtained from the probabilist definitions by taking $P(X_i = x_i) = \frac{1}{n}$ instead of P_X in the expectations, and therefore can be called "plug-in" estimators.

Probabilistic Definitions and Empirical Statistics

- **Properties** skewness:
  - anti-symmetry: $Skew(-X) = -Skew(X)$
  - scale invariance: $if\, \lambda > 0, Skew(\lambda X) = Skew(X)$
  - location invariance: $\forall \lambda$, $Skew(X + \lambda) = Skew(X)$
  - for $X \sim N(m, \sigma^2)$, $Skew(X) = 0$
- **Properties** kurtosis:
  - symmetry: $Kur(-X) = Kur(X)$
  - scale invariance: $if\, \lambda \neq 0, Kur(\lambda X) = Kur(X)$
  - location invariance: $\forall \lambda$, $Kur(X + \lambda) = Kur(X)$
  - for $X \sim N(m, \sigma^2)$, $Kur(X) = 3$
- **Definition**: if $Kur(X) > 3$ the distribution is leptokurtic if $Kur(X) < 3$ the distribution is platykurtic
  Remark: (calculation trick): integration by parts proves that for any integer $n > 0$ $\int_{-\infty}^{+\infty} x^n e^{-\frac{x^2}{2}} dx = (n-1) \int_{-\infty}^{+\infty} x^{n-2} e^{-\frac{x^2}{2}} dx$

- **Remark**: the properties above are true for the empirical quantities as well
- **Exercise**: demonstrate the properties above
- **Remark**: for a mixture of normal distributions we have kurtosis > 3

Goodness of Fit Tests

Theorem (admitted) and the Berra and Jarque test

Let $X_1, X_2, \cdots X_n$ be i.i.d $N(m, \sigma^2)$ and $X = (X_1, X_2, \cdots, X_n)$ then asymptotically:

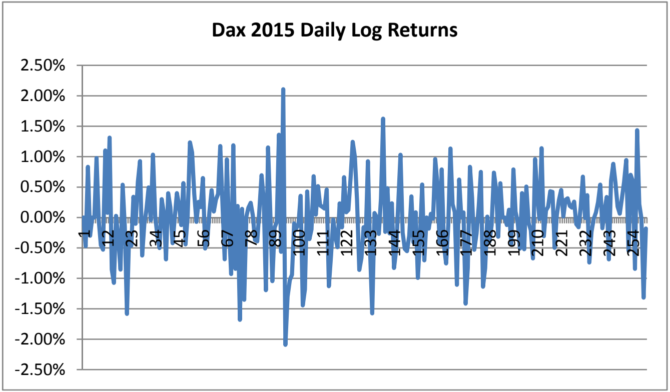
- $\sqrt{n}\widehat{Skew}(X) \sim N(0, 6)$
- $\sqrt{n}[\widehat{Kur}(X) - 3] \sim N(0, 24)$

Let $\widehat{BJ}(X) = \frac{n}{6}[\widehat{Skew}(X)]^2 + \frac{n}{24}[\widehat{Kur}(X) - 3]^2$ then asymptotically:

- $\widehat{BJ}(X) \approx \chi^2(2)$

Let $\chi^2_{1-\alpha}(2)$ be such that $P(\chi^2(2) > \chi^2_{1-\alpha}(2)) = 1 - \alpha$
Then for n large enough the Berra and Jarque test rejects at confidence level α the normality hypothesis iff: $\widehat{BJ}(x) > \chi^2_{1-\alpha}(2)$

Goodness of Fit Tests



Remark: the DAX is a total return index, i.e it is calculated with dividends reinvested

Goodness of Fit Tests

Exemple: We calculate the daily log returns of the DAX for year 2015

- number of daily returns 260, $x = (x_1, x_2, \cdots x_{260})$
- $\overline{x} = 0.02\%$, $\widehat{\sigma}(x) = 0.63\%$
- $\widehat{Skew}(x) = -0.15$, $\widehat{Kur}(x) = 3.56$ (fat tails)
- $\widehat{BJ}(x) = 4.32$, $\chi^2_{5\%}(2) = 5.99$
  So we accept the normality hypothesis at confidence level 95%

Remarks: The Berra and Jarque's test is very sensitive to outliers.
Here, $\frac{n}{6}[\widehat{Skew}(X)]^2 = 0.94$ and $\frac{n}{24}[\widehat{Kur}(X) - 3]^2 = 3.38$
The volatility is defined as $vol\sqrt{\Delta T} = \widehat{\sigma}(x)$. Here $\Delta T = \frac{1}{260}$ as we have here 260 returns observed in a 1 year period, so the estimate of the volatility is 10%.

Further Statistical Tests

Tests Based on Density Function Estimates

Theorem of Parzen Rosenblatt (admitted)

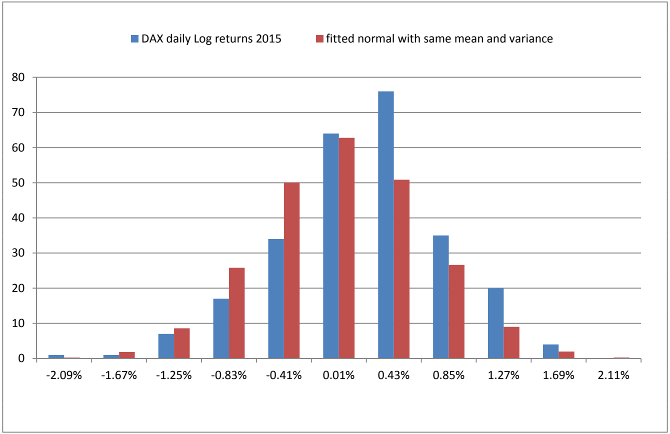
Let X be a random variable of density function $f(x)$
Let X_i be i.i.d variables of the same law as X
Let K be positive of integral 1 and $(h_n)_{n\in N}$ such that $h_n \to 0$ and $nh_n \to \infty$
Let $f_n(x)$ be defined by $f_n(x) = \frac{1}{n}\sum_{i=1}^{n}\frac{1}{h_n}K(\frac{X_i - x}{h_n})$ then $f_n(x)$ is a density and $\sqrt{nh_n}[f_n(x) - f(x)] \xrightarrow{Law} N(0, f(x)\int_{-\infty}^{+\infty}K^2(x)dx)$

Exemples of Kernels

- Rectangular Kernel $K(u) = \frac{1}{2}1_{|u|<1}$
- Gaussian Kernel $K(u) = \frac{1}{\sqrt{2\pi}}exp(-\frac{u^2}{2})$

Remark: "visual test" where the estimated density is usually compared to the density of a normal distribution with the same mean and variance as the empirical mean and variance of the sample.

Tests Based on Density Function Estimates



histogram for the 260 observations fitted into 10 buckets

Tests Based on Cumulative Distribution Function Estimates

Let X and $(X_i)_{i \in [1,n]}$ be i.i.d r.v with the same laws

Let $F_n(x) = \frac{1}{n} \sum_{i=1}^{i=n} 1_{X_i \leq x}$ and $\|F_n(x) - F(x)\|_\infty = \sup_x | F_n(x) - F(x) |$

Law of Large Numbers and Central Limit Theorem

$\forall x,\ F_n(x) \rightarrow F(x)$ p.s and $\sqrt{n}(F_n(x) - F(x)) \xrightarrow{Law} N(0, F(x)[1 - F(x)])$

Glivenko Cantelli Theorem (admitted)

$\|F_n(x) - F(x)\|_\infty \rightarrow 0$ p.s (which is stronger than the Law of Large Numbers)

Kolmogorov Smirnov Theorem (admitted)

$\sqrt{n}\|F_n(x) - F(x)\|_\infty \xrightarrow{Law} K$ when n is large
K is the Kolmogorov's law and is independant from F

Tests Based on Cumulative Distribution Function Estimates

Several Goodness of fit tests are based on Kolmogorov Smirnov's theorem with F being a normal cdf with the same mean and variance as the sample observed. Amongst these tests (available in SAS and with excel extended libraries):

- Kolmogorov Smirnov's test
- Cramer von Mises's test
- Anderson Darling's test

Remark: The CLT result is not well adapted to test the normality hypothesis as it is not optimal to test an hypothesis based on the value of the empirical repartition function on one single point. The Kolmogorov Smirnov result is much more interesting for this matter.

Tests Based on Order Statistics

Theorem and Definition:
if Z is a random variable and if we note $q_Z(\alpha) = \inf_x \{x, P(Z \leq x) \geq \alpha\}$
then $P(Z \leq q_Z(\alpha)) \geq \alpha$ and $q_Z(\alpha)$ is called the α-quantile of Z

Remark 1: if $Z \sim N(0,1)$: $q_Z(0.5) = 0$ and $q_Z(97.5\%) = 1.96$

Remark 2: (definition derived for a sample via the empirical distribution)
if $z = (z_1, z_2, \cdots, z_n)$ is a sample with frequency $(f_1, f_2, \cdots f_n)$ the α quantile of z, $\hat{q}_z(\alpha)$ is the quantile of the r.v Z defined by the empirical probability $P(Z = z_i) = f_i$

Remark 3: if we have 10 observations: $\{z_i = i\}_{i \in [1,10]}$ then
$\hat{q}_z(1) = 10$ and $\forall i \in \{1, \cdots, 9\}$ $\forall \alpha \in]\frac{i}{10}, \frac{i+1}{10}]$ $\hat{q}_z(\alpha) = i + 1$

Tests Based on Order Statistics

Exercice 1: Show that if F_Z is invertible then: $q_Z(\alpha) = F_Z^{-1}(\alpha)$

Exercice 2: Show that if F_Z is invertible then: $F_Z(Z) \sim U([0,1])$

Exercice 3: Show that if $Z_1 \sim N(m_1, \sigma_1^2)$ and $Z_2 \sim N(m_2, \sigma_2^2)$ then: $\{(q_{Z_1}(\alpha), q_{Z_2}(\alpha)), \alpha \in [0,1]\}$ is a line.

Theorem: order statistics (admitted) and QQ-Plot test

Let $N \sim N(0,1)$ and $z(n) = (z_1, z_2, \cdots, z_n)$ be a sample for the r.v Z we want to test. Then when n is large:

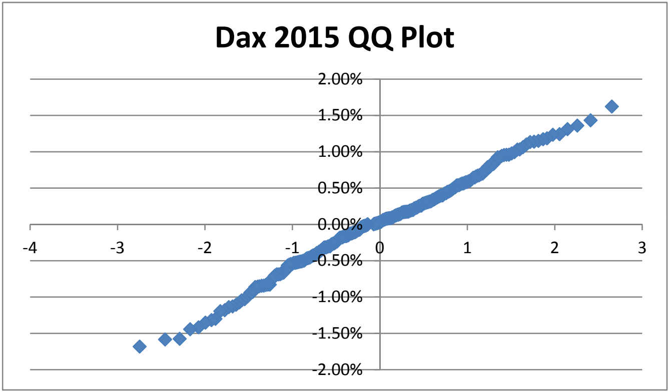
- $\hat{q}_{z(n)}(\frac{i}{n}) \approx q_Z(\frac{i}{n})$ and

if Z is a normal Law:

- The points $(q_N(\frac{i}{n}), \hat{q}_{z(n)}(\frac{i}{n}))$ should be "concentrated" around a line (called the Henry's line)

Tests Based on Order Statistics

Remark 1: The test can be used "visually" but can also be quantified through the Shapiro-Wilk test
Exemple: QQ plot DAX daily returns for 2015



Parameter Estimations and Confidence Intervals

(Reminder) Definition

- if $X \sim N(0,1)$ then $X^2 \sim \chi^2(1)$
- if $X_i \sim N(0,1)$ i.i.d then $\sum_{i=1}^{i=n} X_i^2 \sim \chi^2(n)$

(Reminder) Theorem and Definition

If $X_1 \sim N(M_1, Id_{\mathbb{R}^d})$ and $X_2 \sim N(M_2, Id_{\mathbb{R}^d})$ then:
$\|M_1\| = \|M_2\| \Longrightarrow \|X_1\|^2$ and $\|X_1\|^2$ have the same law and this law is called $\chi^2(d, \|M_1\|^2)$

demonstration:
$\|M_1\| = \|M_2\| \Longrightarrow \exists$ A othonormal such that $AM_1 = M_2$
Let's consider AX_1 then:

- X_1 has a Normal law $\Longrightarrow AX_1$ has a Normal law
- $E[AX_1] = AE[X_1] = AM_1 = M_2$
- $Var[AX_1] = Cov(AX_1, AX_1) = ACov(X_1, X_1)A^{'} = AId_{\mathbb{R}^d}A^{'} = Id_{\mathbb{R}^d}$

so, AX_1 and X_2 are both normal with the same mean and variance
so, $AX_1 \sim X_2$ and consequently $\|AX_1\|^2 \sim \|X_2\|^2$.
As $\|AX_1\|^2 = \|X_1\|^2$ (because A is orthonormal) by transitivity $\|X_1\|^2 \sim \|X_2\|^2$. Q.E.D

Exercise: Show that $Cov(AX, BY) = ACov(X, Y)B'$ (when A and B are matrices with the adequate dimensions) starting from the definition $Cov(X, Y) = E(XY') - E(X)E(Y)'$

Parameter Estimations and Confidence Intervals

(Reminder) Definition

- if $X \sim N(0,1)$ and $Z \sim \chi^2(d)$ are independent then $\frac{X}{\sqrt{\frac{Z}{d}}}$ is called a Student Law or t-distribution and is noted $t(d)$

Theorem: Student Law for Confidence Intervals

Let $X = (X^1, X^2, \cdots, X^n)$ with $X_i \sim N(m, \sigma^2)$ i.i.d

Let $\overline{X} = \frac{1}{n}\sum_{i=1}^{i=n} X^i$ and $\hat{\sigma}(X) = \sqrt{\frac{1}{n}\sum_{i=1}^{i=n}(X^i - \overline{X})^2}$ then:

- $\overline{X}$ and $\hat{\sigma}(X)$ are independant
- $\sqrt{n}(\frac{\overline{X}-m}{\sigma}) \sim N(0,1)$
- $n(\frac{\hat{\sigma}(X)}{\sigma})^2 \sim \chi^2(n-1)$ that we can write as $\frac{\|X-\overline{X}1_n\|^2}{\sigma^2} \sim \chi^2(n-1)$
- $\frac{\overline{X}-m}{\frac{\hat{\sigma}(X)}{\sqrt{n-1}}} \sim t(n-1)$ that we can write as $m = \overline{X} - \frac{\hat{\sigma}(X)}{\sqrt{n-1}} t(n-1)$

Parameter Estimations and Confidence Intervals

Sketch of the proof: we just need to show the result for $X^i \sim N(0,1)$ i.i.d

Let $h : \mathbb{R}^n \longrightarrow \mathbb{R}^n \times \mathbb{R}^n$ be defined by $h : \begin{pmatrix} X^1 \\ \vdots \\ X^n \end{pmatrix} \longrightarrow \begin{pmatrix} \overline{X} \\ X - \overline{X}1_n \end{pmatrix}$ then:

- X Gaussian and h linear $\Rightarrow h(X)$ is Gaussian
- $\forall i \ Cov(\overline{X}, X^i - \overline{X}) = 0 \Rightarrow \overline{X}$ and $X - \overline{X}1_n$ are independent because for Gaussian vectors zero covariance means independence
- $< X - \overline{X}1_n, 1_n >= 0 \Rightarrow X - \overline{X}1_n \in (\mathbb{R}1_n)^{\perp}$

Note that $\forall u \in (\mathbb{R}1_n)^{\perp}$

$Var(< u, X - \overline{X}1_n >) = Var(< u, X > -\overline{X} < u, 1_n, >)$

$= Var(< u, X >) = u' Var(X) u = \|u\|^2$

so if we call $\begin{pmatrix} 0 \\ Z \end{pmatrix}$ the components of $X - \overline{X}1_n$ in an orthonormal basis of $\mathbb{R}^n$ whose first vector is in $\mathbb{R}1_n$ we have $Z \sim N(0, Id_{\mathbb{R}^{n-1}})$

and so $\|X - \overline{X}1_n\|_n^2 = \|Z\|_{n-1}^2 \sim \chi^2(n-1)$ Q.E.D.

Parameter Estimations and Confidence Intervals

Exemple: GDAX returns for 2015

- $\overline{X} = 0.02\%$
- $\frac{\hat{\sigma}(X)}{\sqrt{n-1}} = 0.04\%$

So at confidence level 95% the expected daily rate of return is in the interval $[-0.06\%, 0.09\%]$

Remarks:

- $t(n-1)$ is a symmetric distribution
- from $\frac{\sqrt{n-1}(\overline{X}-m)}{\hat{\sigma}(X)} \sim t(n-1)$ and the law of large numbers we can deduct that $t(n-1) \xrightarrow{Law} N(0,1)$ (which means that when n is large the law/shape of a student distribution is very similar to the a normal distribution)

Utility Functions

Preferred investments

Definition: Utility functions

A utility function is any function $u : \mathbb{R} \longrightarrow \mathbb{R}$ such that
u is continuous, strictly increasing and two times differentiable.

Remark: based on the definition u is invertible

Definition: Preferred Investment and Risk Premium

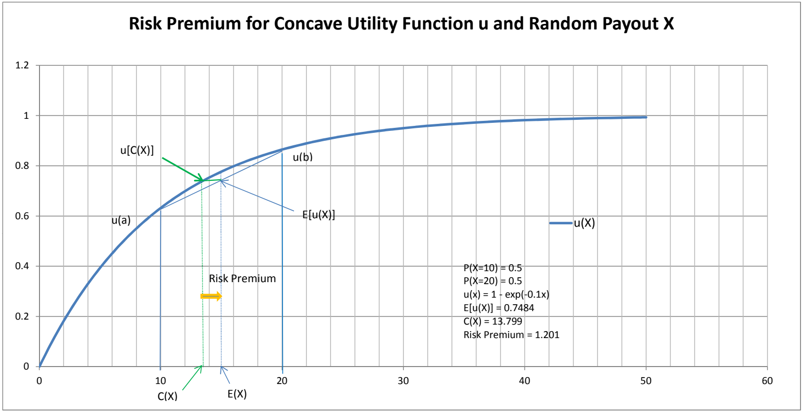
If X and Y are two random payoffs:

- $X \succ Y$ (X is preferred to Y for u) $\Leftrightarrow E[u(X)] > E[u(Y)]$

Let $C_u(X)$ be the constant defined by $u[C_u(X)] = E[u(X)]$

- $C_u(X)$ is called the certain equivalent to X
- $\Pi^u(X) = E[X] - C_u(X)$ is called the risk premium for X

Preferred investments



Preferred investments

Properties (exercise)

- u convex $\Leftrightarrow \forall X, u(E[X]) \leq E[u(X)] \Leftrightarrow \forall X, \Pi^u(X) \leq 0 \Leftrightarrow$ risk taker
- u concave $\Leftrightarrow \forall X, u(E[X]) \geq E[u(X)] \Leftrightarrow \forall X, \Pi^u(X) \geq 0 \Leftrightarrow$ risk adverse
- u affine $\Leftrightarrow \forall X, u(E[X]) = E[u(X)] \Leftrightarrow \forall X, \Pi^u(X) = 0 \Leftrightarrow$ risk neutral

Theorem: Risk Aversion Measure

Let u and v be two utility functions strictly concave or convex then:
$(\forall X \text{ discrete r.v}, \Pi^u(X) \geq \Pi^v(X)) \Leftrightarrow (\forall a, -\frac{u''(a)}{u'(a)} \geq -\frac{v''(a)}{v'(a)})$

Remarks:

- $-\frac{u''}{u'}$ defines the risk aversion/concavity of u
- $\forall \lambda \neq 0, \lambda u$ and u have the same risk aversion
- the concavity measure we arrive at here differs from the geometric definition of curvature which is given by $\frac{u''}{(1+u'^2)^{\frac{3}{2}}}$

Demonstration theorem:

Let u be a utility function two times differentiable and strictly convex or concave (i.e $u^{''} \neq 0$).

Demonstration $\Rightarrow$:

Let X_h^a be defined by: $P(X_h^a = a) = \frac{1}{2}$ and $P(X_h^a = a+h) = \frac{1}{2}$

Let $C_u^a(h) = C_u(X_h^a)$ i.e $u(C_u^a(h)) = E[u(X_h^a)] = \frac{1}{2}u(a) + \frac{1}{2}u(a+h)$

by derivation on h of $u(C_u^a(h)) = \frac{1}{2}u(a) + \frac{1}{2}u(a+h)$ we obtain:

- $C_u^a(0) = a$ from $u(C_u^a(0)) = \frac{1}{2}u(a) + \frac{1}{2}u(a)$
- $C_u^{a'}(0) = \frac{1}{2}$ from $u^{'}(C_u^a(0))C_u^{a'}(0) = \frac{1}{2}u^{'}(a)$
- $C_u^{a''}(0) = \frac{1}{4}\frac{u^{''}(a)}{u^{'}(a)}$ from deriving $u^{'}(C_u^a(h))C_u^{a'}(h) = \frac{1}{2}u^{'}(a+h)$

Preferred investments

Now, $\Pi^u \geq \Pi^v \Rightarrow \forall a, \forall h, \Pi^u(X_h^a) \geq \Pi^v(X_h^a)$
$\Rightarrow \forall h, E[X] - C_u^a(h) \geq E[X] - C_v^a(h)$
$\Rightarrow -C_u^{a''}(0) \geq -C_v^{a''}(0)$ (as in 0 the value and first derivatives are equal)
$\Rightarrow -\frac{u''(a)}{u'(a)} \geq -\frac{v''(a)}{v'(a)}$ (and this is true for all a). Q.E.D

Demonstration $\Leftarrow$
Let X be a discrete variable with $P(X = a_i) = p_i$
$\Pi^u(X) \geq \Pi^v(X) \Leftrightarrow C_v(X) - C_u(X) \geq 0$
$\Leftrightarrow v^{-1}[E(v(X))] - u^{-1}[E(u(X))] \geq 0$
$\Leftrightarrow v^{-1}\left(\sum_{i=1}^{i=n} p_i v(a_i)\right) - u^{-1}\left(\sum_{i=1}^{i=n} p_i u(a_i)\right) \geq 0$
$\Leftrightarrow \sum_{i=1}^{i=n} p_i v(a_i) \geq v \circ u^{-1}\left(\sum_{i=1}^{i=n} p_i u(a_i)\right)$ (as v is increasing)
which should be true if $v \circ u^{-1}$ is convex.

Let's calculate $(v \circ u^{-1})''$ to prove that $v \circ u^{-1}$ is convex
$(v \circ u^{-1})' = (v' \circ u^{-1})(u^{-1})'$
$(v \circ u^{-1})'' = (v'' \circ u^{-1})[(u^{-1})']^2 + (v' \circ u^{-1})(u^{-1})''$
Now, $(u^{-1})' = \frac{1}{u' \circ u^{-1}}$ and $\left(\frac{1}{u' \circ u^{-1}}\right)' = -\frac{(u'' \circ u^{-1})(u^{-1})'}{(u' \circ u^{-1})^2} = -\frac{(u'' \circ u^{-1})}{(u' \circ u^{-1})^3}$
so, $(v \circ u^{-1})'' \geq 0 \Leftrightarrow \frac{(v'' \circ u^{-1})}{(u' \circ u^{-1})^2} - (v' \circ u^{-1})\frac{(u'' \circ u^{-1})}{(u' \circ u^{-1})^3} \geq 0$
$\Leftrightarrow (v'' \circ u^{-1}) \geq (v' \circ u^{-1})\frac{(u'' \circ u^{-1})}{(u' \circ u^{-1})}$
$\Leftrightarrow -\frac{(v'' \circ u^{-1})}{(v' \circ u^{-1})} \leq -\frac{(u'' \circ u^{-1})}{(u' \circ u^{-1})}$ Q.E.D as we assumed here $-\frac{v''}{v'} \leq -\frac{u''}{u'}$

Remarks:

If there is aversion to risk we use u concave to modelize
For $u(a) = 1 - exp(-\lambda a)$ (which is often used) $-\frac{u''}{u'} = \lambda$

Gaussian Laws and Mean Variance Implications

When investing W_0 the utility function criteria picks strategies π which maximizes: $E[u(W_T^\pi)]$ where $\frac{W_T^\pi}{W_0} = 1 + R_T^\pi$
If we assume $R_T^\pi \sim N(m_\pi, \sigma_\pi^2)$ then: $E[u(W_T^\pi)] = E[u(W_0 + W_0 R_T^\pi)]$
$= E[u(W_0 + m_\pi W_0 + \sigma_\pi W_0 Z)]$ with $Z \sim N(0,1)$
If we define $U(m, \sigma) = E[u(W_0 + mW_0 + \sigma W_0 Z)]$ then:
$\max_\pi E[u(W_T^\pi)] = \max_\pi U(m_\pi, \sigma_\pi)$
So in practice the model consists in maximizing $\max_\pi U(m_\pi, \sigma_\pi)$ where U is derived from a utility function.

Remarks: For the DAX, if we assume that we are in an equilibrium where investors are therefore indifferent (when applying their utility functions to the expected returns) between investing into a 1 year zero coupon bond which yields approx 0% or into the DAX, which is expected to return approx 0.02%x260, then the Risk Premium is 5.20%...

From now on we will modelize directly with a function U and we will assume that U is increasing in m and decreasing in σ which means that:

- if the investor has the choice between two strategies with the same expected returns he prefers the one with the less variance on the returns
- if the investor has the choice between two strategies with the same variance on the returns he prefers the one with the higher expected returns

In practice an investor will define the level of risks he accepts and based on this will find the efficient portfolio (which maximizes the expected return). Based on this we make the following definition

Efficient Investment Strategies

Definition: Efficient investment strategy

An investment strategy π_e is efficient iff for all π:
$E[R^\pi] > E[R^{\pi_e}] \Rightarrow \sigma(R^\pi) > \sigma(R_e^\pi)$

Proposition:

π_e is a solution of $\begin{cases} \sup\limits_{\pi} E(R^\pi) \\ \sigma(R^\pi) \leq \sigma(R^{\pi_e}) \end{cases}$

Demonstration: If the solution was not reached in π_e then we could find π such that $\sigma(R^\pi) \leq \sigma(R^{\pi_e})$ and $E[R^\pi] > E[R^{\pi_e}]$ which would be in contradiction with the definition of efficiency for π_e. QED

Remarks: Definitions and Geometric interpretation

If we note:

- $\mathcal{E} = \{\pi_e \text{ efficient}\}$ and
- $\mathcal{E}(\sigma, m) = \left\{ \begin{pmatrix} \sigma(R^{\pi_e}) \\ E(R^{\pi_e}) \end{pmatrix}, \pi_e \in \mathcal{E} \right\}$

Then any point $\begin{pmatrix} \sigma(R^{\pi}) \\ E(R^{\pi}) \end{pmatrix}$ is either on or on the right of $\mathcal{E}(\sigma, m)$

Markowitz: The mean variance framework

Notations and Definitions

We note:

- S_t^i : the value of asset i at time t. Here we consider only two periods (0 and T)
- q_i : the number of shares i held $(q_i > 0)$ or shorted $(q_i < 0)$ at time 0
- R_T^i the return of asset i between 0 and T, i.e $R_T^i = \frac{S_T^i}{S_0^i} - 1$
- $R_T = \begin{pmatrix} R_T^1 \\ \vdots \\ R_T^n \end{pmatrix}$ the vector of returns of the n shares between 0 and T.

Definition:

At inception,

An investment portfolio is a portfolio for which $\sum_{i=1}^{i=n} q_i S_0^i > 0$

A self financing portfolio is a portfolio for which $\sum_{i=1}^{i=n} q_i S_0^i = 0$

Notations and Definitions

Theorem and Definition:

If for an investment portfolio we define $x_0 = \sum_{i=1}^{i=n} q_i S_0^i$ and $\forall i$, $\pi_i = \frac{q_i S_0^i}{x_0}$

then :

- $(x_0, \pi_1, \cdots, \pi_n)$ defines in a unique way the investment portfolio
- x_0 is the initial value of the portfolio
- $\sum_{i=1}^{i=n} \pi_i = 1$
- π_i is the percentage of the value of the portfolio invested in asset i at time 0 and $(\pi_1, \cdots, \pi_n)$ is called the allocation.

Notations and Definitions

Theorem and Definition:

All self financing portfolios can be represented by a (n+1)-uplet $(x_0, \pi_1, \cdots, \pi_n)$ such that :

- $\forall i,\ q_i = \frac{x_0 \pi_i}{S_0^i}$
- $\sum_{i=1}^{i=n} \pi_i = 0$
- $x_0 > 0$

The representation is not unique as $\forall \lambda > 0$ $(\frac{1}{\lambda} x_0, \lambda\pi_1, \cdots, \lambda\pi_n)$ represents the same self financing portfolio as $(x_0, \pi_1, \cdots, \pi_n)$.
For any chosen representation $(x_0, \pi_1, \cdots, \pi_n)$ of the self financing portfolio, x_0 is called the Notional and $(\pi_1, \cdots, \pi_n)$ the allocation.

Notations and Definitions

Fom now on we will describe investment portfolios and self financing portfolios by pairs (x_0, π) where $\pi = \begin{pmatrix} \pi_1 \\ \vdots \\ \pi_n \end{pmatrix}$

Theorem and Definition:

We note $W_t(x_0, \pi)$ the value at time t of the portfolio (x_0, π) and we have:

- $W_0(x_0, \pi) = x_0$ for an investment portfolio
- $W_0(x_0, \pi) = 0$ for a self-financing portfolio
- $W_T(x_0, \pi) = \sum_{i=1}^{i=n} x_0 \pi_i \frac{S_T^i}{S_0^i}$ for a portfolio of any type

Remark:

$\pi^i < 0$ means that the portfolio has a short position in asset i

S^i is called a risky asset iff R_T^i is not determinist i.e $Var(R_T^i) \neq 0$

Notations and Definitions

Example: when we consider various allocations (x_0, π) between two shares $S_0^1 = 100$ $S_0^2 = 50$ we get the following table:

x_0	π_1	π_2	q^1	q^2	W_0
100	1	-1	1	-2	0
1000	1	-1	10	-20	0
100	0.5	0.5	0.5	1	100
1000	0.5	0.5	5	10	1000

Notations and Definitions

Theorem and Definition:

We define the return of a portfolio (x_0, π) as: $R_T = \frac{W_T(x_0,\pi) - W_0(x_0,\pi)}{x_0}$

- For an investment portfolio this definition corresponds to the definition of the return of an asset
- For a self financing portfolio this quantity equals $\frac{W_T(x_0,\pi)}{x_0}$

Proposition

For any investment or self financing portfolio (x_0, π) the return R_T verifies:

$$R_T = \sum_{i=1}^{i=n} \pi_i R_T^i.$$

As this expression depends only on π and not on x_0 we note it R_T^π.

Notations and Definitions

Proposition:

For any investment or self-financing portfolio (x_0, π) we have:

- $R_T^{\pi} = \pi^{'} R_T$
- $E(R_T^{\pi}) = E(\pi^{'} R_T) = \pi^{'} E[R_T]$
- $Cov(R_T^{\pi_1}, R_T^{\pi_2}) = Cov(\pi_1^{'} R_T, \pi_2^{'} R_T) = \pi_1^{'} Cov[R_T, R_T]\pi_2$

Exercise: demonstrate the proposition

Notations: from now on we note:

- $\sigma^{\pi} = \sigma(R_T^{\pi})$
- $m^{\pi} = E(R_T^{\pi})$

Markowitz's Framework

We are going two consider two cases, each based on the existence or not of a risk-free investment strategy:

First case:

There are n risky assets $(S^i)_{i \in [1,n]}$ and no risk-free asset.

Second case:

There are n risky assets $(S^i)_{i \in [1,n]}$ and one risk-free asset S^0.

Remarks:

In both cases it is natural to assume that the R^i_T are such as :

◦ (H1) there is no way to build a risk-free investment portfolio based on the $(S^i)_{i \in [1,n]}$

◦ (H2) there is no way to build a risk-free self-financing portfolio (other than the null portfolio) based on the $(S^i)_{i \in [1,n]}$

Comments:

The reason why we assume (H1) in the first case is because otherwise we could build a risk-free asset. The reason why we assume (H1) in the second case is because otherwise we could build a second risk-free asset and then:

- either this second risk-free asset has the same return as the risk-free asset and in this case one risky asset could be replicated (and then should be disregarded).
- or this second risk-free asset has a different return from the risk-free asset, in which case there would be some arbitrage opportunities.

The reason why we assume (H2) is because for a risk-free self-financing strategy:

- if the return is different from zero there are some arbitrage opportunities
- if the return is zero one risky asset can be replicated (and then should be be disregarded)

We Note :

- $\Sigma = Cov[R_T, R_T]$
- $M = E[R_T]$

Proposition:

(H1) and (H2) $\Longrightarrow \Sigma$ is invertible

Demonstration:

If Σ was not invertible we could find $\pi \neq 0$ such that $\pi^{'}\Sigma\pi = 0$ and then there would be two cases:
either $\pi^{'}1_n = 0$ in which case we could find a self-financing portfolio made of the risky assets without risk or
$\pi^{'}1_n \neq 0$ in which case we could find an investment portfolio made of the risky assets without risk.
in both cases this would be in contradiction with either (H1) or (H2).

Markowitz: Opimization without a risk-free asset

Opimization without a risk-free asset

From the utility framework and in the context of normal distributions assumptions the efficient investment portfolios (x_0, π) are the solutions of

$$(P)\begin{cases} \min_{\pi} \pi^{'}\Sigma\pi \\ \pi^{'}M = m \\ \pi^{'}1_n = 1 \end{cases}$$ where 1_n is the vector of $\mathbb{R}^n$ with components equal to 1

We define a new scalar product in $\mathbb{R}^n$ by $< x, y >_{\Sigma^{-1}} = x^{'}\Sigma^{-1}y$

We can write (P) as

$$(P)\begin{cases} \min_{\pi} < \Sigma\pi, \Sigma\pi >_{\Sigma^{-1}} \\ < \Sigma\pi, M >_{\Sigma^{-1}} = m \\ < \Sigma\pi, 1_n >_{\Sigma^{-1}} = 1 \end{cases}$$

(P) can be solved either by writing its Lagrangien or geometrically by noticing that $\Sigma\pi$ must be in $Vect(M, 1_n)$ in order to mimimize its $<,>_{\Sigma^{-1}}$ norm while satisfying the constraints. We note $\mathcal{F}$ all the investment portfolios solutions of (P) for any possible value of m.

Opimization without a risk-free asset

We note:

- $a =< 1_n, 1_n >_{\Sigma^{-1}}$ and
- $b =< M, 1_n >_{\Sigma^{-1}}$

Excluding for now the case $M - \frac{b}{a}1_n = 0$ and noticing that $< M - \frac{b}{a}1_n, 1_n >_{\Sigma^{-1}} = 0$ we get that $M - \frac{b}{a}1_n$ and 1_n form an orthogonal basis of $Vect(M, 1_n)$. So we can write:
$\Sigma\pi_e = \lambda 1_n + \nu(M - \frac{b}{a}1_n)$ or $\pi_e = \lambda\Sigma^{-1}1_n + \nu\Sigma^{-1}(M - \frac{b}{a}1_n)$
We can now renormalize this decomposition.

We note:

- $\pi_a = \frac{1}{a}\Sigma^{-1}1_n$ which satisfies $\pi_a^{'}1_n = 1$ and
- $\omega_{a,b} = \frac{\Sigma^{-1}(M - \frac{b}{a}1_n)}{\|M - \frac{b}{a}1_n\|_{\Sigma^{-1}}}$ which satisfies $\omega_{a,b}^{'}1_n = 0$ and $var[R_T^{\omega_{a,b}}] = 1$

Opimization without a risk-free asset

Proposition

- $\pi_a^{'} 1_n = 1$ (so π_a is an investment portfolio).
- $m^{\pi_a} = \frac{b}{a}$
- $\sigma^{\pi_a} = \frac{1}{\sqrt{a}}$
- $\omega_{a,b}^{'} 1_n = 0$ (so $\omega_{a,b}$ is a self-financing portfolio).
- $m^{\omega_{a,b}} = \|M - \frac{b}{a} 1_n\|_{\Sigma^{-1}}$
- $\sigma^{\omega_{a,b}} = 1$
- $cov(R^{\pi_a}, R^{\omega_{a,b}}) = 0$

Opimization without a risk-free asset

Demonstration:

$\pi_a^{'} 1_n = 1_n^{'} \pi_a = \frac{1}{a} 1_n^{'} \Sigma^{-1} 1_n = 1$

$m^{\pi_a} = M^{'} \pi_a = \frac{1}{a} M^{'} \Sigma^{-1} 1_n = \frac{b}{a}$

$\sigma^{\pi_a} = [\pi_a{}^{'} \Sigma \pi_a]^{\frac{1}{2}} = [\frac{1}{a^2} 1_n^{'} \Sigma^{-1} \Sigma \Sigma^{-1} 1_n]^{\frac{1}{2}} = [\frac{a}{a^2}]^{\frac{1}{2}} = \frac{1}{\sqrt{a}}$

$\omega_{a,b}^{'} 1_n = \frac{1}{\|M - \frac{b}{a} 1_n\|_{\Sigma^{-1}}} 1_n^{'} \Sigma^{-1} (M - \frac{b}{a} 1_n) = 0$

$m^{\omega_{a,b}} = M^{'} \omega_{a,b} = M^{'} \frac{\Sigma^{-1}(M - \frac{b}{a} 1_n)}{\|M - \frac{b}{a} 1_n\|_{\Sigma^{-1}}}$

using $1_n^{'} \Sigma^{-1} (M - \frac{b}{a} 1_n) = 0$ we get:

$M^{'} \Sigma^{-1} (M - \frac{b}{a} 1_n) = (M - \frac{b}{a} 1_n)^{'} \Sigma^{-1} (M - \frac{b}{a} 1_n)$ and so

$m^{\omega_{a,b}} = \frac{\|M - \frac{b}{a} 1_n\|^2_{\Sigma^{-1}}}{\|M - \frac{b}{a} 1_n\|_{\Sigma^{-1}}} = \|M - \frac{b}{a} 1_n\|_{\Sigma^{-1}}$

$(\sigma^{\omega_{a,b}})^2 = \frac{1}{\|M - \frac{b}{a} 1_n\|^2_{\Sigma^{-1}}} (M - \frac{b}{a} 1_n)^{'} \Sigma^{-1} \Sigma \Sigma^{-1} (M - \frac{b}{a} 1_n) = 1$

$cov(R^{\pi_a}, R^{\omega_{a,b}}) = \frac{1}{a} 1_n^{'} \Sigma^{-1} \Sigma \Sigma^{-1} (M - \frac{b}{a} 1_n) \frac{1}{\|M - \frac{b}{a} 1_n\|^2_{\Sigma^{-1}}} = 0$ Q.E.D

Opimization without a risk-free asset

Proposition

a) $\mathcal{F} = \{\pi_a + \lambda\omega_{a,b}, \lambda \in \mathbb{R}\}$
b) $R^{\pi_a+\lambda\omega_{a,b}} = R^{\pi_a} + \lambda R^{\omega_{a,b}}$
c) $m^{\pi_a+\lambda\omega_{a,b}} = m^{\pi_a} + \lambda m^{\omega_{a,b}}$
d) $(\sigma^{\pi_a+\lambda\omega_{a,b}})^2 = (\sigma^{\pi_a})^2 + \lambda^2$
e) $\min_{\pi\in\mathcal{F}} \sigma^{\pi} = \sigma^{\pi_a}$

Demonstration:

a) We know that a solution of (P) is of the form $\alpha\pi_a + \beta\omega_{a,b}$ as it must be in $Vect(\pi_a, \omega_{a,b})$. The condition $\pi^{'}1_n = 1$ implies $\alpha\pi_a{}'1_n = 1$ which implies $\alpha = 1$. So the solutions of (P) are all of the forms $\pi_a + \lambda\omega_{a,b}$. Reciprocally the investment portfolio $\pi_a + \lambda\omega_{a,b}$ is the solution of (P) for $m = m^{\pi_a} + \lambda m^{\omega_{a,b}}$.

Optimization without a risk-free asset

b) $R^{\pi_a+\lambda\omega_{a,b}} = (\pi_a + \lambda\omega_{a,b})^{'} R_T = \pi_a^{'} R_T + \lambda\omega_{a,b}^{'} R_T = R^{\pi_a} + \lambda R^{\omega_{a,b}}$.
Q.E.D

c) by definition $m^{\pi_a+\lambda\omega_{a,b}} = E[R^{\pi_a+\lambda\omega_{a,b}}]$ so c) is derived from b) by taking expectations on both side of equality b).

d) by definition $(\sigma^{\pi_a+\lambda\omega_{a,b}})^2 = Var[R^{\pi_a+\lambda\omega_{a,b}}]$ and according to b) this quantity equals $Var[R^{\pi_a} + \lambda R^{\omega_{a,b}}]$. As the covariance between these two variables is zero, the expression is $Var[R^{\pi_a}] + \lambda^2 Var[R^{\omega_{a,b}}]$ and as seen previously $Var[R^{\omega_{a,b}}] = 1$. Q.E.D

e) this results from d)

Optimization without a risk-free asset

Corollary

If $\pi_e \in \mathcal{F}$

- $\pi_e = \pi_a + \frac{m^{\pi_e} - m^{\pi_a}}{m^{\omega_{a,b}}} \omega_{a,b}$
- $(\sigma^{\pi_e})^2 = (\sigma^{\pi_a})^2 + (\frac{m^{\pi_e} - m^{\pi_a}}{m^{\omega_{a,b}}})^2$
- $m^{\pi_e} = m^{\pi_a} + m^{\omega_{a,b}} \sqrt{(\sigma^{\pi_e})^2 - (\sigma^{\pi_a})^2}$ if $m^{\pi_e} > m^{\pi_a}$
- $m^{\pi_e} = m^{\pi_a} - m^{\omega_{a,b}} \sqrt{(\sigma^{\pi_e})^2 - (\sigma^{\pi_a})^2}$ if $m^{\pi_e} < m^{\pi_a}$

Demonstration:

Trivial when writing π_e as $\pi_a + \lambda \omega_{a,b}$ and eliminating λ between the equations.

Optimization without a risk-free asset

Remarks:

- The frontier $\mathcal{F}(\sigma, m)$ is an hyperbole
- For all assets and investment portfolios: (σ, m) is on or inside the hyperbole
- Geometrically: the line $y = a + bx$ is above the hyperbole $y = a + b\sqrt{x^2 - 1}$ and is "asymptotically tangent" as:
  - $a + bx - (a + b\sqrt{x^2 - 1}) \xrightarrow{x \to \infty} 0$ (points of the curves converging)and
  - $b/[b\frac{x}{\sqrt{x^2-1}}] \xrightarrow{x \to \infty} 1$ (slopes of the curves converging)

Definitions:

We note:

- $\mathcal{F}^+ = \{\pi_a + \lambda\omega_{a,b}, \lambda \geq 0\}$
- $\mathcal{F}^- = \{\pi_a + \lambda\omega_{a,b}, \lambda \leq 0\}$
- $\mathcal{F}^+(\sigma, m) = \{(\sigma^\pi, m^\pi), \pi \in \mathcal{F}^+\}$ and call it the Efficient Frontier
- $\mathcal{F}^-(\sigma, m) = \{(\sigma^\pi, m^\pi), \pi \in \mathcal{F}^-\}$ and call it the Inefficient Frontier

Two Funds Theorem:

All the efficient portfolios can be built from two fixed distinct efficient investment portfolios π_1 and π_2.
For this reason to analyse the efficient frontier we just need to analyse a model based on two risky assets!

Demonstration:

Let $\pi_1 = \pi_a + \lambda_1 \omega_{a,b}$ and $\pi_2 = \pi_a + \lambda_2 \omega_{a,b}$ be distinct on $\mathcal{F}$
Let $\pi = \pi_a + \lambda \omega_{a,b}$ be on $\mathcal{F}$.
For any $\alpha \in \mathbb{R}$, $\alpha\pi_1 + (1-\alpha)\pi_2$ is an investment portfolio (as $\alpha\pi_1^{'}1_n + (1-\alpha)\pi_2^{'}1_n = 1$) and if we choose $\alpha = \frac{\lambda-\lambda_2}{\lambda_1-\lambda_2}$ we get $\alpha\pi_1 + (1-\alpha)\pi_2 = \pi_a + \lambda\omega_{a,b} = \pi$. Q.E.D

Optimization without a risk-free asset

Exercise 1:

We consider 2 risky assets S^1, S^2:

- $m_1 = 5\%$, $\sigma_1 = 15\%$
- $m_2 = 20\%$, $\sigma_2 = 30\%$

We note ρ the correlation between R^1 and R^2.

a) Plot $\mathcal{F}$ for: $\rho = -1$, $\rho = 0$, $\rho = 0.5$, $\rho = 1$.

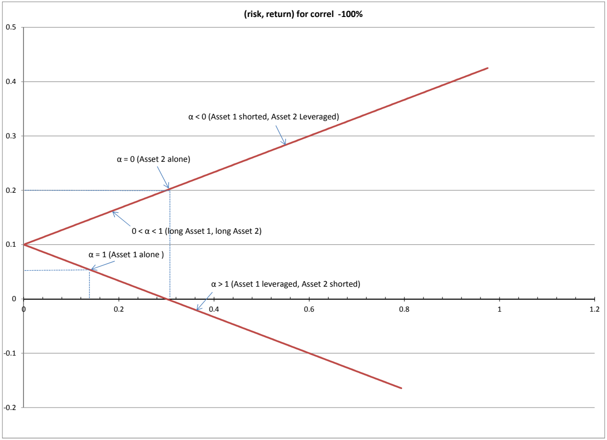
b) What is the portfolio of minimum standard deviation in each case?

If we consider the portfolio $\pi_\alpha = \alpha\pi_1 + (1-\alpha)\pi_2$

c) In which region of $\mathcal{F}(\sigma, m)$ is $(\sigma^{\pi_\alpha}, m^{\pi_\alpha})$ in the following cases:

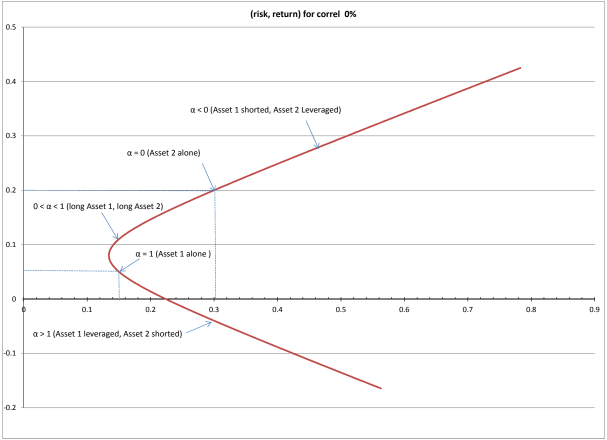
- $\alpha < 0$
- $0 \leq \alpha \leq 1$
- $\alpha > 1$

Optimization without a risk-free asset



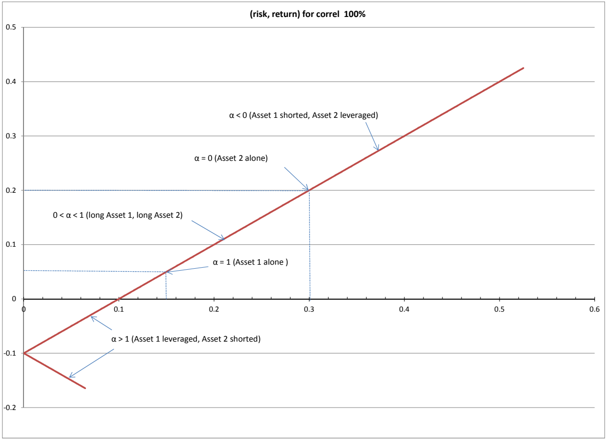
Note that Σ is not invertible here and a risk-free asset can be constructed

Optimization without a risk-free asset



Note that Σ is invertible here and we obtain the "usual" hyperbole

Optimization without a risk-free asset



Note that Σ is not invertible here and a risk-free asset can be constructed

Optimization without a risk-free asset

Proposition:

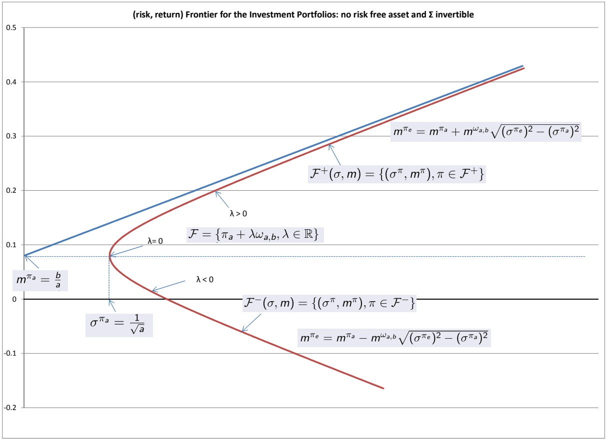
$$\mathcal{F} = \left\{\frac{1}{b-ma}\Sigma^{-1}(M - m1_n), m \neq \frac{b}{a}\right\} \bigcup \left\{\frac{1}{a}\Sigma^{-1}1_n\right\}$$

Demonstration:
We know that $\mathcal{F} = \left\{\frac{1}{a}\Sigma^{-1}1_n + \lambda\Sigma^{-1}(M - \frac{b}{a}1_n), \lambda \in R\right\}$
for $\lambda = 0$ we obtain the portfolio $\frac{1}{a}\Sigma^{-1}1_n$
for $\lambda \neq 0$ we can write λ in the form $\lambda = \frac{1}{b-ma}$ and by doing so we obtain

$$\frac{1}{a}\Sigma^{-1}1_n + \frac{1}{b-ma}\Sigma^{-1}(M - \frac{b}{a}1_n)$$
$$= \frac{1}{b-ma}\Sigma^{-1}(M - m1_n) + \frac{1}{b-ma}\Sigma^{-1}(m - \frac{b}{a})1_n + \frac{1}{a}\Sigma^{-1}1_n$$
$$= \frac{1}{b-ma}\Sigma^{-1}(M - m1_n) + \frac{1}{b-ma}\Sigma^{-1}(am - b)\frac{1}{a}1_n + \frac{1}{a}\Sigma^{-1}1_n$$
$$= \frac{1}{b-ma}\Sigma^{-1}(M - m1_n) \text{ Q.E.D}$$

Remarks: We will demonstrate later that the parameter m can be interpreted geometrically, by showing that the tangent to $\mathcal{F}$ at point (σ, m) intersects the axe $\{\sigma = 0\}$ at point $(0, m)$.

Optimization without a risk-free asset



case where $m^{\omega_{a,b}} > 0$

Remarks: We have assumed so far that $M - \frac{b}{a} 1_n \neq 0$. We analyse here what would happen if this was not the case.
If $M - \frac{b}{a} 1_n = 0$ then all portfolios would have the same returns equal to $\frac{b}{a}$.
In this case (P) would be a problem of minimizing, for an investment portfolio,the standard deviation of the return, i.e to solve:

$$(P) \begin{cases} \min\limits_{\pi} < \Sigma\pi, \Sigma\pi >_{\Sigma^{-1}} \\ < \Sigma\pi, 1_n >_{\Sigma^{-1}} = 1 \end{cases}$$

As previously, geometrically we see that the solution should verify $\Sigma\pi_e \in Vect(1_n)$, so $\pi_e = \lambda\Sigma^{-1} 1_n$. The only π_e of this form satisfying $(\pi_e)' 1_n = 1$ is $\pi_e = \frac{1}{a}\Sigma^{-1} 1_n$

Markowitz: Opimization with a risk-free asset

Optimization with a risk-free asset

We note $\Pi = \begin{pmatrix} \pi^0 \\ \pi \end{pmatrix}$ the allocation between the risk-free asset and the n risky assets.

We note $\Pi_0 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ the risk-free asset of return r_0.

For any investment portfolios Π we must have $\pi^0 + (\pi)^{'} 1_n = 1$.
Replacing π^0 by $1 - \pi^{'} 1_n$ the problem we have to solve, to find the efficient portfolios, can now be written as:

$$(Q)\begin{cases} \min\limits_{\pi} \pi^{'} \Sigma \pi \\ \pi^{'} M + (1 - \pi^{'} 1_n) r_0 = m \end{cases}$$

Which we can also write as:

$$(Q)\begin{cases} \min\limits_{\pi} < \Sigma\pi, \Sigma\pi >_{\Sigma^{-1}} \\ < \Sigma\pi, M - r_0 1_n >_{\Sigma^{-1}} = m - r_0 \end{cases}$$

Optimization with a risk-free asset

Geometrically, the solution has to be of the form $\Sigma\pi = \lambda(M - r_0 1_n)$
or equivalently $\pi = \lambda\Sigma^{-1}(M - r_0 1_n)$.
Note that $1_n^{'}\Sigma^{-1}(M - r_0 1_n) = (b - r_0 a)$.
Until the end of this section we will consider that $b - r_0 a \neq 0$
To renormalize the problem we define: $\pi_T = \frac{1}{b - r_0 a}\Sigma^{-1}(M - r_0 1_n)$

Note that $\Pi_T = \begin{pmatrix} 0 \\ \pi_T \end{pmatrix}$ is an investment portfolio as $\pi_T^{'} 1_n = 1$

Remark : for any portfolio

$$\Pi_P = \begin{pmatrix} 1 - \pi_P' 1_d \\ \pi_P \end{pmatrix} \Longrightarrow m_P = \begin{pmatrix} r_o \\ M \end{pmatrix}' \begin{pmatrix} 1 - \pi_P' 1_d \\ \pi_P \end{pmatrix}$$

$$\Longrightarrow m_P - r_o = \pi_P'(M - r_0 1_d)$$

Theorem: Capital Market Line

The portfolios solutions of (Q) are the portfolios: $\lambda\Pi_T + (1 - \lambda)\Pi_0$ with $\lambda \in \mathbb{R}$. We note $\mathcal{C} = \{\lambda\Pi_T + (1 - \lambda)\Pi_0, \lambda \in \mathbb{R}\}$

Optimization with a risk-free asset

Corollaries and "Market Portfolio"

1. The portfolios Π of $\mathcal{C}$ (i.e the solutions of (Q)) verify:
   - $m^{\Pi} = \lambda m^{\Pi_T} + (1-\lambda) m^{\Pi_0}$
   - $\sigma^{\Pi} = |\lambda| \sigma^{\Pi_T}$

   so their risk parameters (σ, m) are on a cone (i.e $\mathcal{C}(\sigma, m)$ is a cone).
2. All the efficient portfolios are built by allocating money only between Π_0 and Π_T. For this reason to study optimal investments we just need to use a model with one single risky asset!
3. CAPM/MEDAF: if all the market participants are allocating efficiently and with the same parameters, then:
   - all the risky investments are in Π^T and
   - in Π^T the weight in risky asset i is the % of the total Market Capitalization of risky assets that asset i represents.

   For this reason Π^T should correspond to the (risky) "Market Portfolio".

Optimization with a risk-free asset

Corollaries:

1. Usually in the model $\frac{b}{a} > r_0$ otherwise all the efficient portfolios would short Π_T which practically would not make sense.
2. Assuming $\frac{b}{a} > r_0$ we define:
   - $\mathcal{C}^+ = \{\lambda m^{\Pi_T} + (1-\lambda)m^{\Pi_0}, \lambda \geq 0\}$ and call it the Cone Efficient Frontier (or Capital Market Line)
   - $\mathcal{C}^- = \{\lambda m^{\Pi_T} + (1-\lambda)m^{\Pi_0}, \lambda \leq 0\}$ and call it the Cone Inefficient Frontier
   - All the assets and portfolios we can built have their risk parameters (σ, m) within the cone $\mathcal{C}(\sigma, m)$ and in particular $\mathcal{F}(\sigma, m)$ is included in $\mathcal{C}(\sigma, m)$.

Demonstration:

Straightforward when writing $\Pi_e = \lambda\Pi_T + (1-\lambda)\Pi_0$

Exercise:

Draw $\mathcal{F}$ and $\mathcal{C}$ in a model where there are two risky assets and a risk free asset and see what happens when you are changing the correlation

Tangent Portfolio

Lemma: Tangent Portfolio

The "Tangent Portfolio" Π_T, which has no allocation in the risk-free asset, is also a solution of the mean/variance optimization problem (P) where there was no risk-free asset. So we can write $\Pi_T \in \mathcal{F}$

Demonstration:

Geometrically: it is obvious as otherwise there would be some portfolios more efficient (above) than those on the Capital Market Line.

Algebraically:

$\pi_T = \frac{1}{b-r_0 a}\Sigma^{-1}(M - r_0 1_n) = \frac{1}{b-r_0 a}\Sigma^{-1}[(M - \frac{b}{a}1_n) + (\frac{b}{a}1_n - r_0 1_n)]$

$= \frac{1}{b-r_0 a}(\frac{b}{a} - r_0)\Sigma^{-1}1_n + \Sigma^{-1}(M - \frac{b}{a}1_n)$

$= \frac{1}{a}\Sigma^{-1}1_n + \frac{1}{b-r_0 a}\Sigma^{-1}(M - \frac{b}{a}1_n)$

$= \pi_a + \lambda\omega_{a,b}$ which is the form of the portfolios of $\mathcal{F}$. Q.E.D

Until the end of this section we assume that $\frac{b}{a} > r_0$

Tangent Portfolio

Theorem: Tangent Portfolio

$\mathcal{C}(\sigma, m)$ is tangent to $\mathcal{F}(\sigma, m)$ at the point $(\sigma^{\Pi_T}, m^{\Pi_T})$

Demonstration:
We know that Π_T is on $\mathcal{C}$ and that Π_T is on $\mathcal{F}$.
Geometrically: If a line and an hyperbole have a contact point either they are tangent on this contact point or they cross each other. The situation where they cross each other is not possible here as it would imply that some portfolios of $\mathcal{F}$ are more efficient than any portfolios of $\mathcal{C}$.
Remark:
Equivalently, we can say that the tangent to the Efficient Frontier $\mathcal{F}$ at the point Π_T intersects the $\sigma = 0$ axis at the point $\begin{pmatrix} 0 \\ r_0 \end{pmatrix}$

More Geometric Properties

Corollary: Geometry of the Efficient Frontier

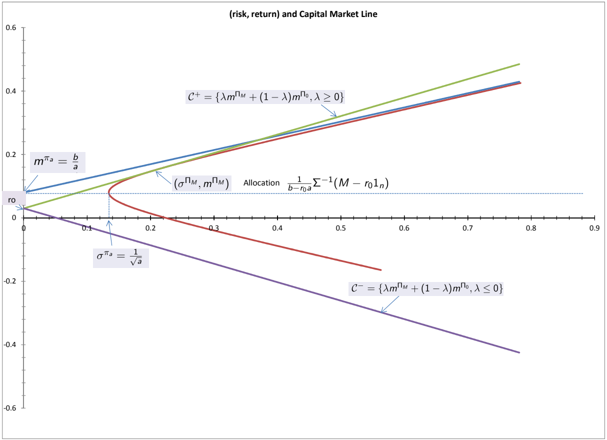
- For any risky efficient investment portfolio $\pi = \frac{1}{b-ma}\Sigma^{-1}(M - m1_n)$ of $\mathcal{F}$, the tangent to $\mathcal{F}(\sigma, m)$ at (σ^π, m^π) intersects the $\{\sigma = 0\}$ axis at the point $\begin{pmatrix} 0 \\ m \end{pmatrix}$.
- For the risky efficient investment portfolio $\pi = \frac{1}{a}\Sigma^{-1}1_n$ of $\mathcal{F}$, the tangent to $\mathcal{F}(\sigma, m)$ at (σ^π, m^π) is parallel to the $\{\sigma = 0\}$ axis.

  These results mean that there is a bijection between $\mathcal{F} - \{\frac{1}{a}\Sigma^{-1}1_n\}$ and $\mathbb{R} - \{\frac{b}{a}\}$ and that is it the tangents to the the $\mathcal{F}(\sigma, m)$ curve which establish the bijection between the portfolios and their parameters m.

Demonstration:

This result for a portfolio of parameter m corresponds to the result of the preceding corollary when taking $r_0 = m$. Q.E.D

More Geometric Properties



Security Market Line

Security Market Line

Theorem: Security Market Line

Let Π_T be the Tangent Portfolio as defined previously
Let Π_P be any investment portfolio composed of the risk-free and risky assets. Then:

- $m_P - r_0 = (m_T - r_0)\rho(R_P, R_T)\frac{\sigma_P}{\sigma_T}$ (SML equation) and
- $R_P - r_0 = (R_T - r_0)\rho(R_P, R_T)\frac{\sigma_P}{\sigma_T} + \epsilon$ with ϵ normal independant from R_T and centered.

Demonstration:

Let Π_P be an investment portfolio then:
$cov(R_T, R_P) = \pi_T^{'}\Sigma\pi_P = \frac{1}{b-r_0a}(M - r_0 1_n)^{'}\pi_P = \frac{m_P - r_0}{b-r_0a}$
If we apply the same calculation to Π_T then:
$cov(R_T, R_T) = \frac{m_T - r_0}{b-r_0a}$
From this we get:
$cov(R_T, R_P) = \frac{m_P - r_0}{m_T - r_0}cov(R_T, R_T)$

Security Market Line

From which we get : $m_P - r_0 = (m_T - r_0)\rho(R_T, R_P)\frac{\sigma_P}{\sigma_T}$
We now want to show a relationship for the r.v and not only for their expectations.

$\begin{pmatrix} (R_P - r_0) - (R_T - r_0)\rho(R_T, R_P)\frac{\sigma_P}{\sigma_T} \\ R_T \end{pmatrix}$ is Gaussian because it is an affine transformation of the vector of the returns of the risky assets which is assumed to be a Gaussian vector. Thus, to show that the first variable that we call ϵ is independent from the second one we just need to show that the covariance is zero.

Indeed, $cov(\epsilon, R_T) = cov(R_P - R_T\rho(R_T, R_P)\frac{\sigma_P}{\sigma_M}, R_T)$
$= cov(R_P, R_T) - \rho(R_T, R_P)\frac{\sigma_P}{\sigma_T}cov(R_T, R_T) = 0$
The fact that ϵ is centered i.e $E(\epsilon) = 0$, results from the previous result.
Q.E.D

Security Market Line

Definition:

The quantity $\rho(R_T, R_P)\frac{\sigma_P}{\sigma_T}$ is noted $\beta_T(P)$ and is called the beta of the asset Π_P (in respect to the Tangent Portfolio Π_T).

Remarks:

The equation: $m_P - r_0 = (m_T - r_0)\beta_T(P)$ (SML)

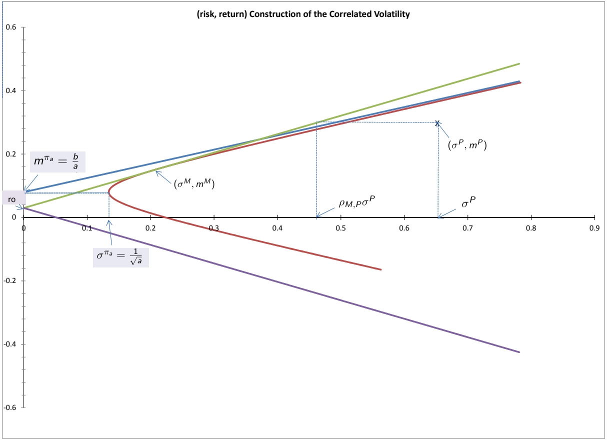
- is valid for all investment portfolios and not only for efficient ones.
- shows that only the risk correlated with the Tangent Portfolio is remunerated.
- is used in capital budgeting / CAPM to determine the price an asset should have based on its expected returns and beta with the sector.
- the beta can be estimated by statistical regression of the observed excess returns $R_P - r_o$ over $R_T - r_o$ over several periods.

Definition:

When we write $R_P = r_0 + (R_T - r_0)\beta_T(P) + \epsilon$ we have
$\sigma_P^2 = \sigma_T^2 \beta_T(P)^2 + \sigma_\epsilon^2$

- $\sigma_T|\beta_T(P)|$ is called the systematic risk. "Economically", the fact that it is remunerated is explained by the fact that it cannot be "reduced by diversification".
- σ_ϵ is called the idiosyncratic risk or specific risk. "Economically", the fact that it is not remunerated is interpreted as the fact that a risk that can be "reduced by diversification" does not need to be remunerated.

Security Market Line



We can read on the graph, for any portfolio, the portion of the volatility correlated with the movements of the Market Portfolio.

Security Market Line and "Arbitrage" Detections

Remarks: The (β, m) of the assets we consider investing it according to the SML should be on a line.

- The beta will usually be calculated in relation to a broader index to which these stocks belong
- The expected returns will be based either on some historical estimates or some analysts predictions

In practice, the points will not be perfectly aligned and a regression line will be calculated.

- The assets over the line will look cheap
- The assets below the line will look expensive

In "pair-trading", strategies will be considered consisting in:

- Selling assets lying below the line
- Buying assets lying above the line (usually in the same sector)

Exercice: Show that if we consider an investment portfolio Π_P with risk parameters (σ_P, m_P) and if we call in the $\{(\sigma, m)\}$ plane, (x, m_P) the intersection of the SML and the line $\{m = m_P\}$ then $x = \beta_T(P)\sigma_T$. Conclude that we can read in the $\{(\sigma, m)\}$ plane the decomposition between systematic risk and idiosyncratic risk.

Performance Indicators

Definition: Sharp Ratio

The Sharpe Ratio of an investment portfolio P is defined as: $\frac{m_P - r_0}{\sigma_P}$

Remarks:

Under the Markowitz's framework:

- The Ratio should be maximal for portfolios belonging to the CML
- All the portfolios belonging to the CML have the same Ratio
- Wealth should be allocated:
  - First by determining a portfolio with the maximum Sharpe Ratio that can be built
  - Then by allocating all the wealth between this portfolio and the risk free asset
- The Sharpe Ratio is independent from the leverage has $\lambda\Pi_P + (1-\lambda)\Pi_0$ has the same Sharpe Ratio as Π_P for any $\lambda > 0$. So the indicator is really "intrinsic to the fund".
- The Sharpe Ratio is usually estimated by: $\frac{\hat{m}_P - r_0}{\widehat{\sigma}_P}$

Remarks: An investor choosing a mutual fund to represent a large portion of his/her wealth should be concerned by the full risk of the fund and should look at the Sharpe Ratio.

Exercise 1: Show that the Sharpe Ratio is independent from the leverage

Exercise 2: How do you read in a $\{(\sigma, m)\}$ representation the Sharpe Ratio of a fund as the slope of a particular line?

Definition: Jensen Index

The Jensen Index of an investment portfolio P is defined as:
$m_P - [r_0 + \beta_{M,P}(m_M - r_0)]$

Remarks:

Under the Markowitz's framework this quantity should be zero according to the SML

In practice all the portfolios considered for investment are represented in the (β, m) plane where they should form a line (the SML) and where:

- the *beta* are estimated historically
- the expected returns are either historical estimates or analyst predictions

Then a regression line is calculated and the portfolios above the line could be considered for addition to the investment portfolio as:

- their systematic risk is remunerated more than expected
- their idiosyncratic risk should disappear via diversification

Remarks:
A large pension fund which allocates money amongst many asset managers may assume that the idiosyncratic risk is going to be reduced/cancelled through diversification and in this case may be concerned only by the remuneration of the non diversifiable risk and by the Jensen Index of each Asset Manager's funds.

Exercise 1: Show that the Jensen Ratio is dependent on leverage

Exercise 2: How do you read in a $\{(\beta, m)\}$ representation the Jensen Ratio of a fund as the distance above a particular line ?

Definition: Treynor Index

The Treynor Index of an investment portfolio P is defined as: $\frac{m_P - r_0}{\beta_M(P)}$

Remarks:

Under the Markowitz's framework the Treynor Index should be constant according to the SML.
The Treynor Index is similar to the Jensen index in its objectives to detect funds for which there is an excess of remuneration of the systematic risk.
The excess is usually called the α !
Compared to the Jensen Index the advantage of the Treynor Index is that it does not dependent on leverage and thus is a more intrinsic measure.

Remarks:

Show that the Treynor Index does not depend on leverage.

Factor Model

Factor Model

We revisit here the SML equation for risky assets and investment portfolios: $r^i(t) = r_0 + b^i(R_T(t) - r_0) + \epsilon^i(t)$ because:

- in practice the $\epsilon^i(t)$ and $\epsilon^j(t)$ appear to be correlated and to represent a significant portion of the variance of the assets.
- by exhibiting additional factors we aim at identifying better the common sources of risks (even when they may not be remunerated) and to end up with smaller non explained residual specific risks.
- we want to determine the remunerations linked to all sources of risks through a non arbitrage argument.
- we add here the time parameter t to show, in a times series analysis perspective, which parameters are assumed to be fixed and which ones are supposed to vary with time (on top of their randomness character when time is fixed).

Factor Model

Definition: K-factors model

For all assets $i \in [\![1, N]\!]$ and for all instants $t \in [\![1, T]\!]$

$r^i(t) = a^i + \sum_{j=1}^{j=K} \beta^i_j f^j(t) + \epsilon^i(t)$ or matricially $R(t) = A + BF(t) + \mathcal{E}(t)$

with the assumptions that:

$\forall t \in [\![1, T]\!]$, $Var(F(t))$ is invertible, $E(\mathcal{E}(t)) = 0$ and $Cov(F(t), \mathcal{E}(t)) = 0$

Remark 1:

If $(F(t), \mathcal{E}(t))$ is assumed to be a Gaussian vector then $R(t)$ is a Gaussian vector and we are still in the "Markowitz framework".

Remark 2:

$$R(t) = \begin{pmatrix} r^1(t) \\ \vdots \\ r^N(t) \end{pmatrix} \quad A = \begin{pmatrix} a^1 \\ \vdots \\ a^N \end{pmatrix} \quad B = \begin{pmatrix} \beta_1^1 & \vdots & \beta_K^1 \\ \vdots & \vdots & \vdots \\ \beta_1^N & \vdots & \beta_K^N \end{pmatrix}$$

$$F(t) = \begin{pmatrix} f^1(t) \\ \vdots \\ f^K(t) \end{pmatrix} \text{ and } \mathcal{E}(t) = \begin{pmatrix} \epsilon^1(t) \\ \vdots \\ \epsilon^N(t) \end{pmatrix}$$

Remark 3: additional assumptions are often made that

$E(F(t)) = 0$

$F(t)$ independent from $F(t^{'})$ for $t \neq t^{'}$

$\mathcal{E}(t)$ independent from $\mathcal{E}(t^{'})$ for $t \neq t^{'}$

$\mathcal{E}(.)$ independent from $F(.)$

$Var(\mathcal{E}(t)) = diag(\sigma_i^2)$ independent from t.

Factor Model

Remark 4: Σ_F is assumed to be symmetric definite positive.
If this was not the case, we could find a vector $u \neq 0$ such that $u^{'} Var[F(t)]u = 0$ which would imply that $Var[u^{'} F(t)] = 0$ and $u^{'} F(t) = Cte$. This would imply that some of the factors would be redundant (could be "cointegrated").

Remark 5:

$$Var(R(t)) = cov(BF(t) + \mathcal{E}(t), BF(t) + \mathcal{E}(t))$$
$$= cov(BF(t), BF(t)) + cov(\mathcal{E}(t), \mathcal{E}(t))$$
$$= Bcov(F(t), F(t))B^{'} + diag(\sigma_i^2)$$
$$= B\Sigma_F B^{'} + diag(\sigma_i^2)$$

Factor Model

Remark 7: If we assume $(F(t), \mathcal{E}(t))$ Gaussian, then a factor model is a "Markowitz model", where the vector of returns of the risky assets over the period $[t-1, t]$ follows a Gaussian law of expectation A and variance-covariance matrix $\Sigma = B\Sigma_F B^{'} + diag(\sigma_i^2)$

Remark 8: In the SML approach the tangent portfolio is a particular factor, explaining the expected remuneration, while in a factor analysis the focus is more on identifying the various sources of risks of the assets, which generate the correlations between them.

Factor Model

Remark 9: Factor models were introduced by Charles Spearman in 1904 in psychometrics.

Remark 10: In financial econometrics, the factors used are either:

- Macroeconomics factors: ex GDP, inflation rate, unemployment rate..etc, in this case the $F(t)$ are "exogene" i.e given and observable.
- Fundamental factors: ex market cap, leverage, book/price ...etc which are as well exogene.
- Statistical factors: in this case the $F(t)$ are "endogene" / hidden factors of the model and the aim is to determine these $F(t)$ as well as the corresponding sensibilities (i.e B). In this approach the factors can be interpreted as returns of some investments and self financing portfolios which are not correlated.

Factor Model - Example

Numerical Example:

Consider the 3 factors model $R = A + BF + \mathcal{E}$ with 3 risky assets where:

$$E(R) = \begin{pmatrix} 5\% \\ 4\% \\ 6\% \end{pmatrix},\ B = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix} \text{ and } P = \begin{pmatrix} 100\% & 0 & 50\% \\ 0 & 100\% & 0 \\ 50\% & 0 & 100\% \end{pmatrix}$$

is the matrix of correlations for the factors and where:

- $\sigma(f_1) = 15\%$, $\sigma(f_2) = 10\%$, $\sigma(f_3) = 10\%$ are the standard deviations of the factors
- $(F, \mathcal{E})$ is Gaussian with $Cov(F, \mathcal{E}) = 0$ and $Var(\mathcal{E})$ diagonal
- $\sigma(\epsilon_1) = 5\%$, $\sigma(\epsilon_2) = 5\%$, $\sigma(\epsilon_3) = 5\%$ are the standard deviations of the specific risks
- there is a risk free asset with $r_0 = 2\%$

After calculating the law of R and applying Markowitz's results we find that:

Factor Model - Example

a) the investment portfolio of minimum variance π_a verifies:
$E(R_{\pi_a}) = 3.84\%$ and $\sigma(R_{\pi_a}) = 10.69\%$
b) the tangent portfolio verifies π_T verifies:

$$\pi_T = \begin{pmatrix} 0.684 \\ 0.353 \\ -0.037 \end{pmatrix} \text{ with } E(R_{\pi_T}) = 4.61\% \text{ and } \sigma(R_{\pi_T}) = 12.74\%$$

c)$R_{\pi_T} = 0.046 + 0.647F_1 + 0.353F_2 - 0.037F_3 + 0.684\epsilon_1 + 0.353\epsilon_2 - 0.037\epsilon_3$
d) from c)we can derive the β of the three risky assets to the tangent portfolio and find: $\beta_T(1) = 1.15$, $\beta_T(2) = 0.77$, $\beta_T(3) = 1.53$
e) we can verify that for the three assets the SML is satisfied as:
$5\% = 2\% + 1.15x(4.6\% - 2\%)$
$4\% = 2\% + 0.77x(4.6\% - 2\%)$
$6\% = 2\% + 1.53x(4.6\% - 2\%)$

Factor Model - Example

f) the residual e_i of the returns of the tree risky assets in the SML model $r^i = r_0 + \beta_i(r_T - r_0) + e_i$ have a variance-covariance matrix of:

$$\begin{pmatrix} 0.004 & -0.007 & 0.001 \\ -0.007 & 0.013 & -0.002 \\ 0.001 & -0.002 & 0.012 \end{pmatrix}$$

and verify: $\sigma(e_1) = 5.98\%$, $\sigma(e_2) = 11.39\%$, $\sigma(e_3) = 10.92\%$

As we can see in this example the 3 factors model enables a better explanation of the risks than the (one factor) SML model because:

- in the 3 factors model the residuals risks have lower variances than in the SML model
- in the 3 factors model the residuals are uncorrelated, proving that all common sources of risks have been identified.

Factor Model - Standard Form

Reminder: Diagonalisation Theorem

If Σ is symmetric definite positive in $(\mathbb{R}^k, < . >)$ we can find $V_1, V_2, \cdots, V_k$ in $\mathbb{R}^k$ and $\lambda_1, \lambda_2, \cdots, \lambda_k$ strictly positive such that:

- $\Sigma V_i = \lambda_i V_i$
- $< V_i, V_j >= \delta_{i,j}$ (orthonormal basis)

Matricially, if we note V the matrix whose vectors columns are the V_i then:

- $V^{'} V = Id_k$ (orthonormal basis)
- $V^{'} \Sigma V = diag(\lambda_i)$
- $\Sigma = \sum_{i=1}^{i=k} \lambda_i V_i V_i^{'}$

Here the λ_i are the eigenvalues of Σ and the V_i are the eigenvectors.

Factor Model - Standard Form

Standard Form Theorem (normalization of the factors)

We can re-write the factor model in the form: $R(t) = A + DH(t) + \mathcal{E}(t)$
with $Var[H(t)] = Id_K$
This form is called the Standard Form of the Factor Model

Demonstration:

$A + BF(t) + \mathcal{E}(t)$
$= A + BVV'F(t) + \mathcal{E}(t)$
$= A + BVdiag(\sqrt{\lambda_i})diag(\frac{1}{\sqrt{\lambda_i}})V'F(t) + \mathcal{E}(t))$
$= A + \left(BVdiag(\sqrt{\lambda_i})\right)\left(diag(\frac{1}{\sqrt{\lambda_i}})V'F(t)\right) + \mathcal{E}(t) = A + DH(t) + \mathcal{E}(t)$
with $D = BVdiag(\sqrt{\lambda_i})$ and $H(t) = diag(\frac{1}{\sqrt{\lambda_i}})V'F(t)$
Now, $Var[H(t)] = diag(\frac{1}{\sqrt{\lambda_i}})Var[V'F(t)]diag(\frac{1}{\sqrt{\lambda_i}})$ and
$Var[V'F(t)] = V'\Sigma_F V = diag(\lambda_i)$
so $Var[H(t)] = diag(\frac{1}{\sqrt{\lambda_i}})diag(\lambda_i)diag(\frac{1}{\sqrt{\lambda_i}}) = Id_K$. Q.E.D

Factor Model - Factor Decomposition

Exercise: Prove the previous result by writing

$$R(t) = A + B\left(\sum_{j=1}^{j=k} \langle F(t), \frac{u_j}{\sqrt{\lambda_j}} \rangle \sqrt{\lambda_j} u_j\right) + \mathcal{E}(t)$$

Factor Model - Absence of Arbitrage Opportunity

Definition: Absence of Arbitrage Opportunity

There is no **Arbitrage Opportunity** in a model if and only if:

- all investment portfolios with no risk have the same return
- all self financing portfolios with no risk have a return of zero

Proposition

If a model is such that :

- there is only one risk free investment rate
- the vector of return R for the risky assets satisfies: $Var(R)$ invertible

Then, there is no **Arbitrage Opportunity** in the model

Remark :

In fact,

•All self financing portfolios with no risk have a return of zero implies

•All investment portfolios with no risk have the same return.

So,

the AOA conditions can be reduced to:

•All self financing portfolios with no risk have a return of zero

Factor Model - APT Theorem

Definition: Asset Pricing Theory conditions

Let $r^i(t) = a^i + \sum_{j=1}^{j=K} b_j^i f^j(t) + \epsilon^i(t)$ $(i \in [\![1, N]\!])$ or in matrix terms $R(t) = A + BF(t) + \mathcal{E}(t)$ be a K-factors model with

- $K < N$
- $E(\mathcal{E}(t)) = 0$, $Var(F(t))$ invertible and $Cov(F(t), \mathcal{E}(t)) = 0$

The **APT conditions** are said to be satisfied if and only if:
There is no **Arbitrage Opportunity** in the reduced model $R(t) = A + BF(t)$ (where the "diversifiable" risk $\mathcal{E}(t)$ is neglected)

If the APT conditions are satisfied we say that the model is an **APT Model**

APT Theorem

The APT conditions are satisfied if and only if:

$$\exists \lambda^0, \lambda^1, \lambda^2, \cdots, \lambda^K \text{ such that } E(R) = \lambda^0 \begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix} + B \begin{pmatrix} \lambda^1 \\ \vdots \\ \lambda^K \end{pmatrix}$$

that we can also note as

$$E(R) = \lambda^0 1_N + B\lambda \qquad (1)$$

Also, assuming that the decomposition (1) is valid then:

- If it is possible to build a risk-free investment portfolio in the reduced model then, λ^0 is this risk-free rate
- If it is not possible to build a risk-free investment portfolio in the reduced model then, the decomposition (1) is not unique and we can take $\lambda^0 = 0$.

Remark 1: If there is a risk free rate r^0 in the economy and a a risk free investment portfolio in the reduced model then usually it is expected that $\lambda^0 = r^0$

Remark 2:
As $K < N$ it is possible to build in the reduced model a risk-free portfolio by choosing $\pi \neq 0$ such that $\pi' B = 0$
a) If we can find such π such that $\pi' 1_N \neq 0$ we are able to build a risk-free investment portfolio
b) If for all such π, $\pi' 1_N = 0$ we are only able to build risk-free self financing portfolios

Lemma 1

Let us define λ_0 by:

- $\lambda_0 = 0$ if there is no risk-free investment portfolio in the reduced model
- λ_0 is the risk-free rate of the risk-free investment portfolios if such portfolios exist in the reduced model

Then, if there is no arbitrage in the reduced model we have:
$\forall \pi \in \mathbb{R}^N \ \pi^{'} B = [0, 0, \cdots, 0] \Longrightarrow \pi^{'}(A - \lambda^0 1_N) = 0$

Demonstration Lemma 1:
Let $\pi \in \mathbb{R}^N \setminus \{0\}$ be such that $\pi^{'} B = [0, 0, \cdots, 0]$. ($(vectB)^{\perp}$ is of dimension $N - K$).

- if $\pi^{'} 1_N \neq 0$ then $\tilde{\pi} = \frac{\pi}{\pi^{'} 1_N}$ is an investment portfolio which is without risk in the reduced model as $R^{\tilde{\pi}} = \tilde{\pi}^{'} A + \tilde{\pi}^{'} BF = \tilde{\pi}^{'} A$

therefore as there is no Arbitrage Opportunity in the reduced model we should have $\tilde{\pi}^{'}A = \lambda^0$ and thus $\tilde{\pi}'(A - \lambda^0 1_N) = 0$ and $\pi'(A - \lambda^0 1_N) = 0$

- if $\pi^{'}1_N = 0$ then π is a self-financing portfolio without risk which should therefore satisfy $\pi^{'}A = 0$ and also if λ^0 is defined $\pi'(A - \lambda^0 1_N) = 0$ as $\pi^{'}1_N = 0$

So in all cases:
$\pi^{'}B = [0, 0, \cdots, 0] \Longrightarrow \pi^{'}(A - \lambda^0 1_N) = 0$ Q.E.D

Lemma 2:

If $\forall \pi \in \mathbb{R}^N\ \pi^{'}B = [0, 0, \cdots, 0]\ (1) \Longrightarrow \pi^{'}(A - \lambda^0 1_N) = 0\ (2)$
then $\exists \mu^1, \mu^2, \cdots \mu^K,\ A - \lambda^0 1_N = \sum_{i=1}^{i=K} \mu^i b_i$

Demonstration Lemma 2:
Let us note $B = [b_1, b_2, \cdots, b_K]$ and $Vect\{b_1, b_2, \cdots, b_K\}$ the vector space generated by $b_1, b_2, \cdots, b_K$.
$(1) \iff Vect\{b_1, b_2, \cdots, b_K\}^{\perp} \subset Vect\{A - \lambda^0 1_N\}^{\perp}$
$\implies Vect\{A - \lambda^0 1_N\} \subset Vect\{b_1, b_2, \cdots, b_K\}$
$\implies A - \lambda^0 1_N \in Vect\{b_1, b_2, \cdots, b_K\}$
This proves Lemma 2.

Demonstration APT Theorem: (First implication)
According to Lemma 1 and 2 the AOA in the reduced model implies that $A = \lambda^0 + B\mu$. Therefore as $E(R) = A + BE(F)$ we get
$E(R) = \lambda^0 + B\mu + BE(F) = \lambda^0 + B\lambda$ if we define λ by $\lambda = \mu + E(F)$
Q.E.D

Factor Model - APT Theorem

(second implication)
We assume now that $\exists \lambda^0, \lambda$ such that $E(R) = \lambda^0 1_N + B\lambda$ and want to prove that the AOA conditions are satisfied in the reduced model.
So, let π be a portfolio without risk in the reduced model

- π investment portfolio without risk in the reduced model
  $\Longrightarrow \pi' B = 0 \Longrightarrow E(R_\pi) = \pi' E(R) = \lambda^0 \pi' 1_N = \lambda^0$
  so the first AOA condition that all risk free investment portfolios have the same return is satisfied
- π self financing portfolio without risk in the reduced model
  $\Longrightarrow \pi' B = 0 \Longrightarrow E(R_\pi) = \pi' E(R) = \lambda^0 \pi' 1_N = 0$
  so the second AOA condition that all risk free self financing portfolios have a return of zero is satisfied
  This finishes the proof of the APT theorem

Numerical Example :

We consider the 2 factor model $R = A + BF + \mathcal{E}$ with 3 assets where:

$E(R) = \begin{pmatrix} 5\% \\ 8\% \\ 5\% \end{pmatrix}$ and $B = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ \frac{1}{3} & \frac{1}{3} \end{pmatrix}$ with $E(\mathcal{E}) = 0$ and $cov(F, \mathcal{E}) = 0$ and $Var(F)$ invertible.

According to the Fundamental APT theorem there is no arbitrage in the reduced model i.i.f we can find λ^0, λ^1, λ^2 such that:

$$\begin{cases} 5\% = \lambda^0 + \lambda^1 \\ 8\% = \lambda^0 + \lambda^2 \\ 5\% = \lambda^0 + \frac{1}{3}\lambda^1 + \frac{1}{3}\lambda^2 \end{cases}$$

Indeed we find a unique solution $\lambda^0 = 2\%$, $\lambda^1 = 3\%$, $\lambda^2 = 6\%$

In the reduced Model $\pi = (-1, -1, 3)'$ is an investment portfolio without risk and when we calculate its expected return based on its components we find $-5\% - 8\% + 15\% = 2\%$ which is consistent with $\lambda^0 = 2\%$

Factor Model - APT Theorem

Proposition

The two following propositions are equivalent

- there is no risk free investment portfolio in the reduced model $R = A + BF$ (1)
- $Vect\{b_1, b_2, \cdots, b_K\} = Vect\{1_N, b_1, b_2, \cdots, b_K\}$ (2)

Demonstration:

$(1) \iff (x'B = 0 \implies x'1_N = 0)$

$\iff Vect\{b_1, b_2, \cdots, b_K\}^{\perp} \subset Vect\{1_N\}^{\perp}$

$\iff Vect\{1_N\} \subset Vect\{b_1, b_2, \cdots, b_K\}$

$\iff$ (2) Q.E.D

Remark: the proposition above shows that when the APT conditions are satisfied and that there is no risk free investment portfolio in the reduced economy we can take in fact any value for λ_0 (and not only the value zero)

Factor Model - APT Theorem

Remark 1: *Unicity of the decomposition of a K-factor model*
The equation $R = A + BF + \mathcal{E}$ where F is given with Σ_F invertible $E(\mathcal{E}) = 0$ and $Cov(F, \mathcal{E}) = 0$ defines A and B in a unique way by $B = Cov(R, F)\Sigma_F^{-1}$ and $A = E(R) - BE(F)$

Proposition: Standard Form of an APT model

According to the APT theorem any APT model can be written in the standard form: $R = \lambda^0 1_N + BG + \mathcal{E}$ where $Var(G)$ is invertible, $E(\mathcal{E}) = 0$ and $cov(G, \mathcal{E}) = 0$

Demonstration: $R = A + BF + \mathcal{E}$ with $E(R) = \lambda_0 1_N + B\lambda$ implies that $A + BE(F) = \lambda_0 1_N + B\lambda \implies R = \lambda_0 1_N + B(\lambda - E(F)) + BF + \mathcal{E}$
$\implies R = \lambda_0 1_N + B(\lambda - E(F) + F) + \mathcal{E}$ Q.E.D.

Factor Model - APT Theorem

Proposition : Alternative Definition of an APT model

Let $R = A + BF + \mathcal{E}$ be a K-factor model with the assumptions

- $K < N$
- $Var(F)$ invertible, $E(\mathcal{E}) = 0$ and $cov(F, \mathcal{E}) = 0$

then the two following hypothesis are equivalent:
(H1) there is no arbitrage in the reduced model $R = A + BF$
(H2) $\exists \lambda^0$ such that $\forall \pi$ portfolio $Cov(R_\pi, F) = 0 \Longrightarrow E(R_\pi) = \lambda^0 \pi' 1_N$

Demonstration: Let's assume $(H2)$
It is easy to see $(H2) \Longleftrightarrow (\tilde{H}2)$ where
$(\tilde{H}2)$: $\exists \lambda^0, \forall x \in \mathbb{R}^N, x' B \Sigma_F = 0 \Longrightarrow x' E(R) = \lambda^0 x' 1_N$
$\Longleftrightarrow \exists \lambda^0, \forall x \in \mathbb{R}^N, x' B = 0 \Longrightarrow x'(E(R) - \lambda^0 1_N) = 0$
$\Longleftrightarrow \exists \lambda^0, Vect\{b_1, b_2, \cdots b_K\}^\perp \subset Vect\{E(R) - \lambda^0 1_N\}^\perp$
$\Longleftrightarrow \exists \lambda^0, Vect\{(E(R) - \lambda^0 1_N\} \subset Vect\{b_1, b_2, \cdots b_K\}$
$\Longleftrightarrow \exists \lambda^0, \lambda, E(R) - \lambda^0 1_N = B\lambda \Longleftrightarrow (H1)$ according to the APT theorem

Principal Components Analysis

Principal Components Analysis

Lemma:

Let $x = \begin{pmatrix} x^1 \\ \cdots \\ x^d \end{pmatrix}$ and $y = \begin{pmatrix} y^1 \\ \cdots \\ y^d \end{pmatrix}$ be in $\mathbb{R}^d$ then $x^{'}y = Tr(xy^{'})$

demonstration: trivial

Proposition:

Let $Z = \begin{pmatrix} Z^1 \\ \vdots \\ Z^d \end{pmatrix}$ be a random variable in $\mathbb{R}^d$ then:

$E\left[\|Z - E[Z]\|^2\right] = Tr\left[Var(Z)\right] = \sum_{i=1}^{i=d} \lambda_i$ where Tr is the trace operator and the λ_i are the eigenvalues of $Var[Z]$

Principal Components Analysis

demonstration: $E\left[\|Z-E[Z]\|^2\right]=E\left[(E-E[Z])'(Z-E[Z])\right]$
$=E\left[Tr\left((E-E[Z])(Z-E[Z])'\right)\right]=Tr\left(E\left[(Z-E[Z])(Z-E[Z])'\right]\right)$
$=Tr\left[Var(Z)\right]$ Q.E.D

Definition:

We call $E\left[\|Z-E[Z]\|^2\right]$ the dispersion of Z and in dimension 1 this definition corresponds to the usual definition of variance.

Principal Components Analysis

Proposition:

Let Z be a random variable in $\mathbb{R}^d$, $Var(Z)$ its matrix of variance covariance and $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n \geq 0$ the eigenvalues of $Var(Z)$.
For any $k \leq d$ let $\mathcal{B}_k$ the set of all orthonormal families of k vectors of $\mathbb{R}^d$.

Then for any $k \in [1, d]$, $\sup\limits_{(x_i)_{i\in[1,k]}\in\mathcal{B}_k} \sum\limits_{i=1}^{i=k} x_i' Var(Z) x_i = \sum\limits_{i=1}^{i=k} \lambda_i$

Demonstration:

Let $(x_i)_{i\in[1,k]}$ be the basis of a vector space V on which the random variable Z is projected as $p_V(Z)$. The Trace of $Var(p_V(Z))$ is equal to

$$\sum_{i=1}^{i=k} x_i' Var(p_V(Z)) x_i = \sum_{i=1}^{i=k} Var(x_i' p_V(Z))$$

$$= \sum_{i=1}^{i=k} Var(x_i' Z) = \sum_{i=1}^{i=k} x_i' Var(Z) x_i$$

We therefore solve the maximization problem,

$$(P)\begin{cases} \sup\limits_{(x_i)_{i\in[1,k]}\in\mathcal{B}_k} \sum\limits_{i=1}^{i=k} x_i' Var(Z) x_i \\ \forall i \in [\![1,k]\!],\ x_i'x_i = 1 \\ \forall i,j \in [\![1,k]\!],\ i<j \Longrightarrow x_i'x_j = 0 \end{cases}$$

The Lagrangian is $\sum\limits_{i=1}^{i=k} x_i' Var(Z) x_i - \sum\limits_{i<j} \lambda_{i,j} x_i' x_j - \sum\limits_{i=1}^{i=k} \lambda_{i,i}(x_i'x_i - 1)$

$$\nabla_{x_i} L = 0 \Longleftrightarrow 2Var(Z)x_i - 2\lambda_{i,i}x_i - \sum_{j\neq i} \lambda_{i,j} x_j = 0$$

$$\Longleftrightarrow 2Var(Z)x_i = -2\lambda_{i,i}x_i - \sum_{j\neq i} \lambda_{i,j} x_j.$$

As a consequence $Var(Z)V \subset V$ and therefore V must be a vector space generated by eigenvectors of $Var(Z)$ and $Trace(Var(pH(Z)))$ is equal to the sum of these k eigenvalues.

Now, to get a maximum value these eigenvalues must be taken amongsts the k biggest eigenvalues of $Var(Z)$. Q.E.D

Corollary

Let H_k be a sub vector space of $\mathbb{R}^d$ of dimension k $(k \geq d)$
Let $\mathcal{H}_k$ be the set of all sub vector spaces H_k
Let p_{H_k} be the orthogonal projerction on H_k then,

$$\max_{H_k \in \mathcal{H}_k} E(\|p_{H_k}(Z - E(Z))\|^2) = \sum_{i=1}^{k} \lambda_i$$

Demonstration: Let $(x_i)_{i \in [1,k]}$ be an orthonormal basis of H_k

$p_{H_k}(E - EZ) = \sum_{i=1}^{k} x_i'(Z - E(Z))x_i$ and

$$\|p_{H_k}(Z - E(Z))\|^2 = \sum_{i=1}^{k}(x_i'(Z - E(Z)))^2 = \sum_{i=1}^{k} x_i'(Z - E(Z)(Z - E(Z)'x_i$$

so,

$$E(\|p_{H_k}(Z - E(Z))\|^2) = \sum_{i=1}^{k} x_i' E[(Z - E(Z)(Z - E(Z)')x_i = \sum_{i=1}^{k} x_i' Var(Z) x_i$$

So the result follows from the previous proposition.

Exercise 1:

Show that $\min_{H_k} E[\|p_{H_k}(Z-EZ)\|^2] = \sum_{i=n-k+1}^{i=n} \lambda_i$

Exercise 2:

Let Φ be a scalar product of $\mathbb{R}^d$. We use the intrinsic notation $Z \sim \mathcal{N}(0,\Phi)$, where Φ is a quadratic form to mean that the Gaussian variable Z has a variance-covariance defined by $Var(x'Z) = \Phi(x,x)$.
Let H be a sub-space of $\mathbb{R}^d$ and $\Phi_{|H}$ be the restriction of Φ to H.
a) Show that: $min\lambda_{\Phi} \leq min\lambda_{\Phi_{|H}} \leq max\lambda_{\Phi_{|H}} \leq max\lambda_{\Phi}$.
b) Let $Z \sim \mathcal{N}(0,\Phi)$ be a Gaussian variable of $\mathbb{R}^d$.
Show that $p_H(Z)$ is a Gaussian variable of H which verifies $p_H(Z) \sim \mathcal{N}(0,\Phi_{|H})$.
c) Study the eigenvalues of Φ and $\Phi_{|H}$, when $Mat(\Phi) = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}$ and $H = Vect(e_1+e_2)$.

Principal Components Analysis - Numerical Example

Hint:

a)

$$min \lambda_{\Phi_{|H}} = \min_{x \in H, \|x\|^2=1} \Phi_{|H}(x,x) = \min_{x \in H, \|x\|^2=1} \Phi(x,x) \geq \min_{x \in \mathbb{R}^d, \|x\|^2=1} \Phi(x,x)$$

b) Let $Z \sim \mathcal{N}(0, \Phi)$ in $\mathbb{R}^d$.

As p_H is a linear transformation, $p_H(Z)$ is Gaussian and

- $E(p_H(Z)) = p_H(E(Z)) = 0$
- $\forall x \in H, Var(x' p_H(Z)) = Var(p_H(x)' Z) = Var(x'Z)$
  $= \Phi(x,x) = \Phi_{|H}(x,x)$

Therefore, $p_H(Z) \sim \mathcal{N}(0, \Phi_{|H})$, in H.

c) $Var(Z)$ has for eigenvalues 1 and 2 and $Var(p_H(Z))$ has for value (eigenvalue) $\frac{3}{2}$ and as expected $1 \leq \frac{3}{2} \leq 2$.

Numerical Example:

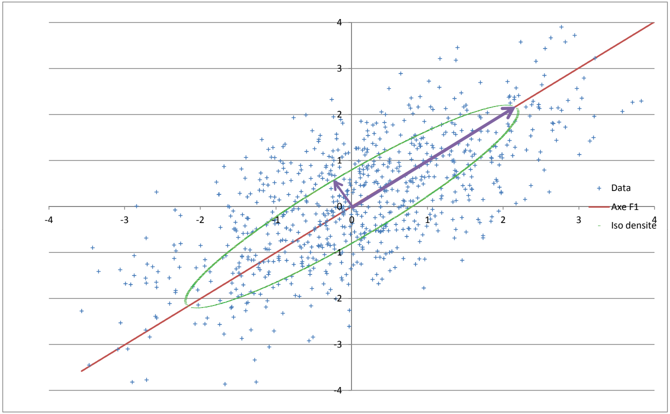
We consider an economy with two risky assets whose returns r_1 and r_2 follow the following one factor model:

$$\begin{pmatrix} r_1 \\ r_2 \end{pmatrix} = \begin{pmatrix} 1 & 0.6 & 0 \\ 1.2 & 0 & 0.6 \end{pmatrix} \begin{pmatrix} f \\ e_1 \\ e_2 \end{pmatrix}$$

with f, e_1 and e_2 being independent of variance 1 and expectation zero.

We plot on the graph below the 800 simulations of $\begin{pmatrix} r_1 \\ r_2 \end{pmatrix}$

Principal Components Analysis - Numerical Example



in red the axis corresponding to the eigenvector of maximum eigenvalue.

Principal Components Analysis - Numerical Example

We have the following results:

- the theoretical eigenvalues are 2.80 and 0.36. From our sample and empirical variance-covariance matrix we obtain 3.06 and 0.57
- the theoretical measure of dispersion is 3.16 and from our sample 3.63
- $\begin{pmatrix} 1.20 \\ 1.44 \end{pmatrix}$ and $\begin{pmatrix} 1.20 \\ -1.00 \end{pmatrix}$ are theoretical eigenvectors
- $\begin{pmatrix} 1.24 \\ 1.25 \end{pmatrix}$ and $\begin{pmatrix} 1.25 \\ -1.24 \end{pmatrix}$ are empirical eigenvectors
- on the graph the red line corresponds to the axe generated by the sample eigenvector of the highest eigenvalue

We have the following results:

- the green ellipse on the chart represents points for which the density function of the model is constant equal to the value derived for the point $\begin{pmatrix} 1.20 \\ 1.44 \end{pmatrix}$.
- the lengths of the axis (in purple) of the ellipse are proportional to the eigenvalues
- when we project the points on the red axis, the dispersion of the points projected is 3.06 which is what we expected, as we have projected on an axe corresponding to the (observed) eigenvalue 3.06.

References

Idris Kharroubi, Imen Ben Tahar (2014, 2013)
Gestion de Portefeuilles
Course Manual University Paris 9 Dauphine, pp.76

David Ruppert (2004)
Statistics and Finance, an introduction
Springer Texts in Statistics

Pierre Brugière (2020)
Quantitative Portfolio Management, pp.217
https://www.springer.com/gp/book/9783030377397

The End