Monte Carlo Methods Graded Lab 1

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1 Optimal bounding constant M

1. `rinvgamma_trunc2` uses the the inverse-gamma distribution as the proposal distribution to sample from a truncated inverse-gamma distribution.

Let $f_{\alpha,\beta}(x)$ be the PDF of an inverse-gamma distribution and let $f^{tr}_{\alpha,\beta}(x)$ be the PDF of a truncated inverse-gamma distribution. We need to calculate M such that

$$M = \sup_x \frac{f^{tr}_{\alpha,\beta}(x)}{f_{\alpha,\beta}(x)}$$

for all x in the support of the truncated inverse-gamma distribution.

$$M = \sup_x \frac{f_{\alpha,\beta}}{f_{\alpha,\beta} \times (F_{\alpha,\beta}(b) - F_{\alpha,\beta}(0))} = \frac{1}{F_{\alpha,\beta}(b) - F_{\alpha,\beta}(0)}$$

where $F_{\alpha,\beta}$ is the CDF of the inverse-gamma distribution.

2. `rinvgamma_trunc3` uses the uniform distribution $g \sim \mathcal{U}(0,b)$ whose support covers the support of the truncated inverse-gamma distribution is essential for the rejection sampling to work, since otherwise there will be regions where the target distribution has non-zero density but the proposal distribution has zero density, making it impossible to sample from those regions. With the same notation as above, M is calculated such that

$$M = \sup_x \frac{f^{tr}_{\alpha,\beta}(x)}{g(x)} = \sup_{x\in[0,b]} \frac{f^{tr}_{\alpha,\beta}(x)}{1/b} = \frac{b}{F_{\alpha,\beta}(b) - F_{\alpha,\beta}(0)} \times \frac{\beta^\alpha}{\Gamma(\alpha)} \sup_{x\in[0,b]} x^{-\alpha-1} \exp\left(-\frac{\beta}{x}\right)$$

We find that:

$$M = \begin{cases} \frac{b}{F_{\alpha,\beta}(b) - F_{\alpha,\beta}(0)} \times \frac{\beta^\alpha}{\Gamma(\alpha)} b^{-\alpha-1} \exp\left(-\frac{\beta}{b}\right), & b \le \frac{\beta}{\alpha+1} \\[2ex] \frac{b}{F_{\alpha,\beta}(b) - F_{\alpha,\beta}(0)} \times \frac{\beta^\alpha}{\Gamma(\alpha)} \left(\frac{\beta}{\alpha+1}\right)^{-\alpha-1} \exp\left(-\frac{\alpha+1}{1}\right), & b > \frac{\beta}{\alpha+1} \end{cases}$$

2 Comparing the two accept-reject samplers
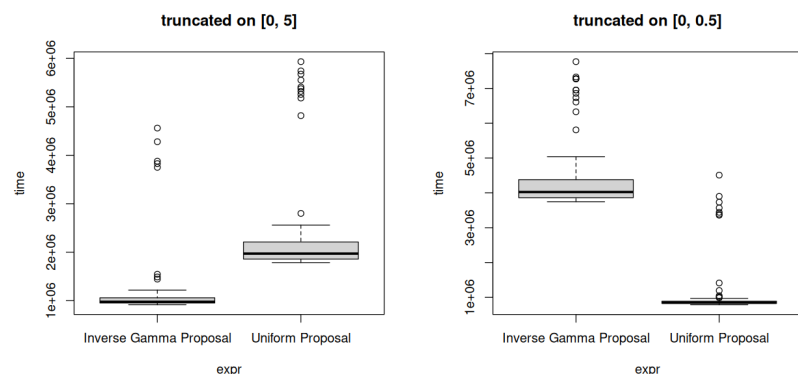
For number of samples $n = 10^4$, we can see that when $b = 5$ (left), the inverse-gamma proposal sampler performs better than the uniform proposal sampler, using shorter time to generate samples and the generation time variance is also smaller. When $b = 0.5$ (right), the uniform proposal sampler performs better, with both shorter time and smaller variance in time. This is explained by the optimal bounding constant M we calculated in Section 1.

```
1  > (pinvgamma(5,3,2) - pinvgamma(0,3,2))^(-1)
2  [1] 1.00799 ## inverse-gamma proposal b = 5
3  > const <- 5/(pinvgamma(5,3,2) - pinvgamma(0,3,2)) * 2^3/gamma(3)
4  > const * (2/(3+1))^(-3-1) * exp(-3-1)
5  [1] 5.907832 ## uniform proposal b = 5
6
```

```
7  > (pinvgamma(0.5,3,2) - pinvgamma(0,3,2))^(-1)
8  [1] 4.199858 ## inverse-gamma proposal b = 0.5
9  > const <- 0.5/(pinvgamma(0.5,3,2) - pinvgamma(0,3,2)) * 2^3/gamma(3)
10 > const * 0.5^(-3-1) * exp(-2/0.5)
11 [1] 2.461538 ## uniform proposal b = 0.5
```

A smaller M leads to a higher acceptance rate in the rejection sampling, thus less time is needed to generate the required number of samples. Our calculations results is thus consistent with the observed performance of the two samplers in the experiments.

3 Monte Carlo estimation

We note

$$f^{tr}_{\alpha,\beta} \sim \mathrm{InvGamma}_{[0,b]}(\alpha,\beta) \quad f_{\alpha,\beta} \sim \mathrm{InvGamma}(\alpha,\beta)$$

The integral of interest is

$$\mathbb{E}_{f_Y}\left[1 - Y^2\right] \quad \text{where } Y \sim \mathrm{InvGamma}_{[0,b]}(\alpha,\beta)$$

1. If we sample $X \sim \mathrm{InvGamma}(\alpha,\beta)$, then

$$\mathbb{E}_{f_Y}\left[1 - Y^2\right] = \int_{-\infty}^{\infty} (1 - y^2) \frac{f_{\alpha,\beta}(y)}{F_{\alpha,\beta}(b) - F_{\alpha,\beta}(0)} \, dy = \frac{\mathbb{E}_{f_X}\left[(1 - X^2)\right]}{F_{\alpha,\beta}(b) - F_{\alpha,\beta}(0)}$$

Thus we can estimate $\mathbb{E}_{f_Y}\left[1 - Y^2\right]$ by sampling $X_i \sim \mathrm{InvGamma}(\alpha,\beta)$ and computing

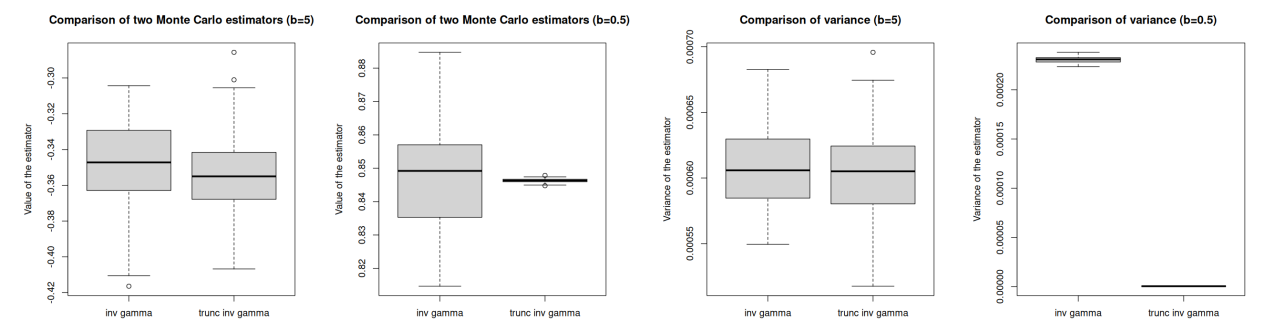$$\hat\theta_n = \frac{1}{(F_{\alpha,\beta}(b) - F_{\alpha,\beta}(0))n} \sum_{i=1}^n (1 - X_i^2)$$

2. If we sample $X \sim \mathrm{InvGamma}_{[0,b]}(\alpha,\beta)$, then

$$\mathbb{E}_{f_Y}\left[1 - Y^2\right] = \int_0^b (1 - y^2) f^{tr}_{\alpha,\beta}(y) \, dy = \mathbb{E}_{f_X}\left[1 - X^2\right]$$

Thus we can estimate $\mathbb{E}_{f_Y}\left[1 - Y^2\right]$ by sampling $X_i \sim \mathrm{InvGamma}_{[0,b]}(\alpha,\beta)$ and computing

$$\hat\theta_n = \frac{1}{n} \sum_{i=1}^n (1 - X_i^2)$$

3. The variance and the computation time of the two estimators are compared below for $b = 0.5, 5$.

From the results, we can see that for $b = 5$, the two estimators have similar variance, the variance of each estimate is also similarly distributed. However, for the case of $b = 0.5$, the truncated inverse-gamma based estimator has significantly lower variance compared to the full inverse-gamma based estimator. We figure out that when b is small, a large portion of the distribution's mass lies outside the truncation interval $[0,b]$. As a result, samples drawn from the full inverse-gamma distribution often fall outside this interval, leading to higher variability in the estimates. In contrast, samples drawn from the truncated inverse-gamma distribution are confined within $[0,b]$, resulting in more consistent and reliable estimates with lower variance.