Reminders for TD2 – Time Series

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1 Linear Processes and Autocovariance

1.1 Definition: Linear Process

Statement: If $Z \sim \mathrm{BB}(0,1)$ and $\alpha \in \ell^1(\mathbb{Z})$, then $X = F_\alpha(Z)$ is a stationary process with mean 0 and autocovariance function:

$$\gamma_X(h) = \sum_{k\in\mathbb{Z}} \alpha_k \alpha_{k+h}$$

Relevant for: Exercise 1

1.2 Summability and Decorrelation

Properties:

- The autocovariance function γ_X of a stationary process is bounded: $|\gamma_X(h)| \leq \gamma_X(0)$ for all $h \in \mathbb{Z}$.
- For a linear process $X = F_\alpha(Z)$ with $\alpha \in \ell^1(\mathbb{Z})$, the autocovariance $\gamma_X(h) = \sum_{k\in\mathbb{Z}} \alpha_k \alpha_{k+h}$ is summable.
- Cauchy-Schwarz inequality: $|\gamma_X(h)|^2 \leq \gamma_X(0)^2$ and $\sum_{h\in\mathbb{Z}} |\gamma_X(h)| \leq \left(\sum_{k\in\mathbb{Z}} |\alpha_k|\right)^2 < \infty$.

Relevant for: Exercise 1

1.3 Theorem: Properties of Autocovariance (Theorem 1.1)

The autocovariance function γ_X is always:

1. **Symmetric:** $\gamma_X(s,t) = \gamma_X(t,s)$ for all $s, t \in \mathbb{Z}$
2. **Positive definite:** For all $n \geq 1$, $(t_1, \ldots, t_n) \in \mathbb{Z}^n$, $(\lambda_1, \ldots, \lambda_n) \in \mathbb{R}^n$:

$$\sum_{i=1}^{n} \sum_{j=1}^{n} \lambda_i \lambda_j \gamma_X(t_i, t_j) \geq 0$$

Relevant for: Exercise 1

1.4 Corollary: Autocovariance of Stationary Processes (Corollary 1.1)

When X is stationary, $\gamma_X : \mathbb{Z} \to \mathbb{R}$ is:

1. **Even:** $\gamma_X(h) = \gamma_X(-h)$ for all $h \in \mathbb{Z}$
2. **Positive definite:** $\sum_{i=1}^n \sum_{j=1}^n \lambda_i \lambda_j \gamma_X(t_i - t_j) \geq 0$
3. **Bounded:** $|\gamma_X(h)| \leq \gamma_X(0)$ for all $h \in \mathbb{Z}$

Relevant for: Exercise 1

2 Filtering and L^2 Convergence

2.1 Theorem: Filtering of Bounded Processes (Theorem 2.1)

Given $\alpha = (\alpha_k)_{k \in \mathbb{Z}} \in \ell^1(\mathbb{Z})$, $p \in [1, \infty[$, and process X bounded in L^p:

$$\sup_{t \in \mathbb{Z}} \|X_t\|_{L^p} < \infty$$

Then:

1. For all $t \in \mathbb{Z}$, the sum $Y_t := \sum_{k \in \mathbb{Z}} \alpha_k X_{t-k}$ is a.s. absolutely convergent
2. Process $Y = (Y_t)_{t \in \mathbb{Z}}$ is bounded in L^p
3. For all $t \in \mathbb{Z}$:
$$\sum_{k \in [-n,n]} \alpha_k X_{t-k} \xrightarrow[n \to \infty]{a.s., L^p} Y_t$$

Relevant for: Exercise 2

2.2 Weak Filtering with Bilateral Series

Key idea: Even when $\alpha \notin \ell^1(\mathbb{Z})$, if $\alpha \in \ell^2(\mathbb{Z})$ and Z is a centered white noise with unit variance, the series $\sum_{k \in \mathbb{Z}} \alpha_k Z_{t-k}$ can still converge in L^2.
Method: For fixed $t \in \mathbb{Z}$, consider the partial sums $S_n = \sum_{k=-n}^n \alpha_k Z_{t-k}$. Since (Z_t) is a white noise:

$$\mathbb{E}[S_n^2] = \sum_{k=-n}^{n} \alpha_k^2 \mathbb{E}[Z_{t-k}^2] = \sum_{k=-n}^{n} \alpha_k^2$$

If $\alpha \in \ell^2(\mathbb{Z})$, then (S_n) is Cauchy in L^2, hence converges to some $X_t \in L^2$.
Relevant for: Exercise 2

2.3 Stationarity via Filtering

Corollary 2.1: If $\alpha \in \ell^1(\mathbb{Z})$ and X is stationary second-order, then $Y = F_\alpha(X)$ is well-defined, second-order, stationary with:

$$\mu_Y = \mu_X \sum_{k\in\mathbb{Z}} \alpha_k, \quad \gamma_Y(h) = \sum_{j\in\mathbb{Z}} \sum_{k\in\mathbb{Z}} \alpha_j \alpha_k \gamma_X(h+j-k)$$

Extension to ℓ^2 case: Even if $\alpha \in \ell^2(\mathbb{Z}) \setminus \ell^1(\mathbb{Z})$, the filtered process (X_t) defined as the L^2 limit is still stationary because:

- $\mathbb{E}[X_t] = 0$ (by linearity of expectation and orthogonality)
- $\gamma_X(t,s) = \mathrm{Cov}(X_t, X_s)$ depends only on $|t-s|$ due to the stationarity of (Z_t)

Relevant for: Exercise 2

3 Autoregressive (AR) Processes

3.1 Theorem: AR(1) Processes (Theorem 3.1)

Equation: $X_t = \phi X_{t-1} + Z_t$ where $Z \sim \mathrm{BB}(0,1)$
Case $|\phi| = 1$: No stationary solution
Case $|\phi| < 1$: Unique stationary solution:

$$X_t = \sum_{k=0}^{\infty} \phi^k Z_{t-k}$$

- Linear, **causal** (depends only on past noise)
- $\mu_X = 0$, $\gamma_X(h) = \frac{\phi^{|h|}}{1-\phi^2}$

Case $|\phi| > 1$: Unique stationary solution:

$$X_t = -\sum_{k=1}^{\infty} \frac{Z_{t+k}}{\phi^k}$$

- Linear, **non-causal** (depends on future noise)
- $\mu_X = 0$, $\gamma_X(h) = \frac{\phi^{-|h|}}{\phi^2-1}$

Relevant for: Exercise 3

3.2 General AR(p) Processes and Stationarity

For the general autoregressive equation:

$$X_t = \sum_{i=1}^{n} a_i X_{t-i} + Z_t$$

Define the **characteristic polynomial**:

$$\Phi(z) = 1 - \sum_{i=1}^{n} a_i z^i$$

Criterion for stationary solution: The equation admits a stationary solution if and only if $\Phi(z)$ has no roots of modulus 1.
Key observation: If $\sum_{i=1}^{n} a_i = 1$, then $\Phi(1) = 0$, so $z = 1$ is a root. Similarly, if $\sum_{i=1}^{n} (-1)^i a_i = 1$, then $\Phi(-1) = 0$, so $z = -1$ is a root. Since both $|1| = 1$ and $|-1| = 1$, no stationary solution exists.
Relevant for: Exercise 3

3.3 Causality of AR Processes

Definition: A process X is **causal** if $X_t \in \overline{\mathrm{Vect}}(Z_t, Z_{t-1}, Z_{t-2}, \ldots)$ for all $t \in \mathbb{Z}$ (closure in L^2).
Criterion: The stationary solution of $\Phi(B)X_t = Z_t$ is causal if and only if $\Phi(z)$ has no roots in the closed unit disk $\{z \in \mathbb{C} : |z| \leq 1\}$.
Relevant for: Exercise 3

4 Filter Invertibility

4.1 Power Series Representation

For $\alpha \in \ell^1(\mathbb{Z})$, define:

$$P_\alpha(z) := \sum_{k \in \mathbb{Z}} \alpha_k z^k$$

This is continuous on the unit circle $\mathbb{U} = \{z \in \mathbb{C} : |z| = 1\}$.
Key properties (Proposition 2.2):

- $P_{\alpha * \beta}(z) = P_\alpha(z) \times P_\beta(z)$
- $P_e(z) = 1$ where $e = (\mathbb{1}_{k=0})_{k \in \mathbb{Z}}$
- **Injectivity:** $P_\alpha = P_\beta \Rightarrow \alpha = \beta$

Relevant for: Exercise 4

4.2 Theorem: Inversion of Polynomial Filters (Theorem 2.3)

Let $\alpha \in \ell^1(\mathbb{Z})$ be a polynomial filter. The following are equivalent:

1. Filter α is invertible (i.e., $\exists \beta \in \ell^1(\mathbb{Z})$ such that $\alpha * \beta = e$)
2. Polynomial $P_\alpha(z) = \sum_{k \in \mathbb{Z}} \alpha_k z^k$ has no roots of modulus 1
3. The rational fraction $1/P_\alpha$ is expandable as a power series on the unit circle: $\exists \beta \in \ell^1(\mathbb{Z})$ such that

$$\forall z \in \mathbb{U}, \quad \frac{1}{P_\alpha(z)} = \sum_{k \in \mathbb{Z}} \beta_k z^k$$

When this holds: $\alpha^{-1} = \beta$
Relevant for: Exercise 4

4.3 Practical Inversion Formulas

For polynomial filters, use partial fraction decomposition. If $P_\alpha(z) = (z-a)^k$ for some $a \in \mathbb{C} \setminus \{0\}$:
If $|a| > 1$:

$$\frac{1}{(z-a)^k} = \frac{1}{(-a)^k} \sum_{n=0}^{\infty} \binom{n+k-1}{k-1} z^n$$

(The inverse is causal: $\beta_k = 0$ for $k < 0$)
If $|a| < 1$:

$$\frac{1}{(z-a)^k} = \sum_{n=0}^{\infty} \binom{n+k-1}{k-1} \frac{a^n}{z^{n+k}}$$

(The inverse is anti-causal: $\beta_k = 0$ for $k > 0$)
If $|a| = 1$: The filter is not invertible in $\ell^1(\mathbb{Z})$.
Relevant for: Exercise 4

4.4 Examples for Exercise 4

Case 1: $\alpha_0 = 2$, $\alpha_1 = -1$, $\alpha_k = 0$ otherwise.

- $P_\alpha(z) = 2 - z = -z(1 - 2/z)$
- Root: $z = 2$ (modulus $|2| = 2 > 1$)
- Invertible, inverse is causal

Case 2: $\alpha_0 = 1$, $\alpha_1 = 2$, $\alpha_k = 0$ otherwise.

- $P_\alpha(z) = 1 + 2z$
- Root: $z = -1/2$ (modulus $|-1/2| = 1/2 < 1$)
- Invertible, inverse is anti-causal

Case 3: $\alpha_0 = 1$, $\alpha_1 = -1$, $\alpha_k = 0$ otherwise.

- $P_\alpha(z) = 1 - z$
- Root: $z = 1$ (modulus $|1| = 1$)
- **Not invertible** in $\ell^1(\mathbb{Z})$

Relevant for: Exercise 4

5 Abstract Filtering Theory

5.1 Banach Spaces for Time Series

Definition: E is the space of processes (X_t) bounded in L^2:

$$\|X\|_E = \sup_{t \in \mathbb{Z}} \|X_t\|_2 < \infty$$

E with this norm is a Banach space.
Backward shift operator: $B \in L(E)$ defined by:

$$BX = (X_{t-1})_{t\in\mathbb{Z}}$$

Relevant for: Exercise 5

5.2 Theorem: B is an Isometry

The backward shift operator B on E satisfies:

1. B is linear and bijective
2. $\|BX\|_E = \|X\|_E$ for all $X \in E$ (isometry property)
3. B^{-1} is the forward shift: $(B^{-1}X)_t = X_{t+1}$

Consequence: $\|B\|_{L(E)} = 1$ and B^n is also an isometry for all $n \in \mathbb{Z}$.
Relevant for: Exercise 5

5.3 Convergence of Operator Series

Proposition: If $\alpha \in \ell^1(\mathbb{Z})$, then the series $\sum_{n\in\mathbb{Z}} \alpha_n B^n$ converges in $L(E)$.
Proof idea: Since $\|B^n\|_{L(E)} = 1$ for all n:

$$\left\|\sum_{n\in\mathbb{Z}} \alpha_n B^n\right\|_{L(E)} \leq \sum_{n\in\mathbb{Z}} |\alpha_n|\|B^n\|_{L(E)} = \|\alpha\|_1 < \infty$$

We denote: $\phi(\alpha) := \sum_{n\in\mathbb{Z}} \alpha_n B^n \in L(E)$
Relevant for: Exercise 5

5.4 Morphism Property

Theorem: The map $\phi : \ell^1(\mathbb{Z}) \to L(E)$ is an algebra homomorphism:

$$\phi(\alpha * \beta) = \phi(\alpha) \circ \phi(\beta)$$

Proof sketch: Expand both sides and use the fact that $B^m \circ B^n = B^{m+n}$.
Consequence: If α is invertible in $\ell^1(\mathbb{Z})$ (i.e., $\exists \beta \in \ell^1(\mathbb{Z})$ with $\alpha * \beta = e$), then $\phi(\alpha)$ is invertible in $L(E)$ with:

$$\phi(\alpha)^{-1} = \phi(\beta) = \phi(\alpha^{-1})$$

Relevant for: Exercise 5

5.5 Injectivity of ϕ

Theorem: The map $\phi : \ell^1(\mathbb{Z}) \to L(E)$ is injective.
Proof strategy: Consider a white noise $Z = (Z_t)_{t\in\mathbb{Z}}$ with $\mathbb{E}[Z_t] = 0$ and $\mathbb{E}[Z_t^2] = 1$. Then $Z \in E$.
For $\alpha \in \ell^1(\mathbb{Z})$, we have:

$$\phi(\alpha)Z = \sum_{n\in\mathbb{Z}} \alpha_n B^n Z = \sum_{n\in\mathbb{Z}} \alpha_n (Z_{t-n})_{t\in\mathbb{Z}} = (Y_t)_{t\in\mathbb{Z}}$$

where $Y_t = \sum_{n \in \mathbb{Z}} \alpha_n Z_{t-n}$.
Key observation: The map $\alpha \mapsto Y$ is injective because we can recover α from Y via:

$$\alpha_n = \mathbb{E}[Y_0 Z_{-n}]$$

(using orthogonality of white noise).
Thus, if $\phi(\alpha) = 0$, then $Y = 0$, so $\alpha = 0$.
Relevant for: Exercise 5