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1. (a) **Solution:**
Define event
$J = \{\text{engineer program in Java}\}$
$C = \{\text{engineer program in C++}\}$
Then $P(C|J) = \frac{P(CJ)}{P(J)}$
$0.24 = \frac{P(CJ)}{0.36}$
Answer:
The probability that a randomly selected engineer programs in Java and C++ is
$P(CJ) = 0.0864$

(b) **Solution:**
$P(J|C) = \frac{P(CJ)}{P(C)}$
Answer:
The probability that a randomly selected engineer programs in Java given that he/she programs in C++ is
$P(J|C) = \frac{0.0864}{0.33} = 0.2618$

2. (a) **Solution:**
$P(E) = \frac{4\times3}{52\times51} = 0.00452$
The Ace of Spades can be either the first card or the second card
$P(F) = \frac{1\times51+51\times1}{52\times51} = 0.0385$
$P(EF) = \frac{1\times3+3\times1}{52\times51} = 0.00226$
Answer:
$P(E|F) = \frac{P(EF)}{P(F)} = 0.0588$

(b) **Solution:**
Since event G must happen when event E happens
$P(G|E) = 1$
$1 = \frac{P(GE)}{P(E)}$
$P(GE) = P(E)$
We can calculate the complement of event G
$P(G^c) = \frac{48\times47}{52\times51} = 0.851$
$P(G) = 1 - P(G^c) = 0.149$
Answer:
$P(E|G) = \frac{P(EG)}{P(G)} = \frac{P(E)}{P(G)} = 0.0303$

3. (a) **Solution:**
Define event $E_i = \{\text{a user likes movie} M_i\}$,$T = \{\text{a user like the Tearjerker genre}\}$
$P(E_i|T) = p_i$
Answer:
$P((E_1 \cap E_2 \cap E_3)|T) = P(E_1|T) \cap P(E_2|T) \cap P(E_3|T)$
Since all the $E_i|T$ are conditionally independent, the probability that a user likes all three movies M_1, M_2 and M_3 given that they like the Tearjerker genre is
$P((E_1 \cap E_2 \cap E_3)|T) = P(E_1|T)P(E_2|T)P(E_3|T) = p_1p_2p_3$

(b) **Answer:**
$P((E_1 \cup E_2 \cup E_3)|T) = P(E_1|T) \cup P(E_2|T) \cup P(E_3|T) = p_1 + p_2 + p_3 - (p_1p_2 + p_3p_2 + p_3p_1) + p_1p_2p_3$

(c) **Solution:**
Define event $E_{all} = \{\text{user likes all the 3 movie}\}$
$P(E_{all}|T) = p_1p_1p_3$

$P(E_{all}|T^c) = q_1q_2q_3$
$P(E_{all}T) = P(E_{all}|T)P(T) = 0.6p_1p_2p_3$
$P(E_{all}T^c) = P(E_{all}|T^c)P(T) = (1-0.6)q_1q_2q_3 = 0.4q_1q_2q_3$
$P(E_{all}) = P(E_{all}T) + P(E_{all}T^c)$
Answer:
The probability that they like the Tearjerker genre that they like M_1, M_2 and M_3 is
$P(T|E_{all}) = \frac{P(TE_{all})}{P(E)} = \frac{0.6p_1p_2p_3}{0.6p_1p_2p_3+0.4q_1q_2q_3}$

4. (a) **Solution:**
We can calculate the probability of event $F = \{textallthe5serversfailedinoneyear\}$
$P(F) = (1-p)^5$
Answer:
The probability that at least 1 server is still working after on year is
$P(E_1) = 1 - P(F) = 1 - (1-p)^5$

(b) **Solution:**
We can consider each particular combination of the 3 servers that are still working.
$P(G_i) = p^3(1-p)^2$
Since all the events are mutually exclusive **Answer:**
The probability that exactly 3 server is still working after on year is
$P(E_3) = \binom{5}{3}P(G_i) = 10p^3(1-p)^2$

(c) **Solution:**
We can consider 3 situations: exactly 3, 4, 5 servers are still working after one year and combine them together.
Answer:
The probability that at least 3 server is still working after on year is
$P(E) = P(E_3) + P(E_4) + P(E_5) = \sum\limits_{i=3}^{5} \binom{5}{i}p^i(1-p)^{5-i}$

5. **Solution:**
The probability of all the bit is 0 in a n bit string is
$P(F) = (1-p)^n$
Then the probability that at least one 1 in the string is
$P(E) = 1 - P(F) = 1 - (1-p)^n$
Answer:
$P(E) > 0.7$
The n requirment for the probability that there is at least one 1 in the string is at least 0.7 is
$n > log_{1-p}(0.3)$

6. (a) **Solution:**
$F_i = \{$at least one string hashed into i-th bucket$\}$
$P(E) = 1 - P((F_1F_2F_3F_4)^c) = 1 - P(F_1^c \cup F_2^c \cup F_3^c \cup F_4^c)$
Since all the F_i^c are not mutually exclusive, the answer will be very complex before expansion. Because of the limited number of buckets and strings, we can try to use another way to get the answer.
$G = \{$All the buckets have at least 1 string$\}$
$H = \{$No string in bucket 5, all the first 4 buckets have one or 2 strings$\}$
$I = \{$No string in bucket 5, all the first 4 buckets have one or 3 strings$\}$
$P(E) = P(G) + P(H) + P(I)$
For each of event G, there is only one bucket can have 2 strings in it and we can add up all the 5

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possible situation to get the total probability.
$P(G) = \frac{6!}{2!}\sum_{i=1}^{5}(p_i \prod_{j=1}^{5} p_j) = 360\sum_{i=1}^{5} p_i \prod_{j=1}^{5} p_j = 360\prod_{j=1}^{5} p_j$
For each of event H, there are 2 bucket have 2 strings in it.
$P(H) = \frac{6!}{2!2!}(p_1p_2p_3^2p_4^2 + p_1p_2^2p_3p_4^2 \ldots) = 180(p_1p_2 + p_1p_3 + p_1p_4 + p_2p_3 + p_2p_4 + p_3p_4)p_1p_2p_3p_4$
For each of event H, there are 1 bucket have 3 strings in it.
$P(I) = \frac{6!}{3!}(p_1^3p_2p_3p_4 + p_1p_2^3p_3p_4 + p_1p_2p_3^3p_4 + p_1p_2p_3p_4^3) = 120(p_1^2 + p_2^2 + p_3^2 + p_4^2)p_1p_2p_3p_4$
Answer:
$P(E) = 60(6p_5 + 3(p_1p_2 + p_1p_3 + p_1p_4 + p_2p_3 + p_2p_4 + p_3p_4) + 2(p_1^2 + p_2^2 + p_3^2 + p_4^2))p_1p_2p_3p_4$

(b) **Answer:**
After substitute all the p_i values, we can get
$P(E) = 60 \times (6 \times 0.1 + 3 \times 0.2925 + 2 \times 0.225) \times 0.001875 = 0.2168$

7. (a) **Solution:**
The probability that **fairRandom** returns 1 can be described as
$P(r2 = 1|r2 \neq r1) = \frac{P(\{r2=1, r1=0\})}{P(\{r2 \neq r1\})}$
Answer:
The probability that **fairRandom** returns 1 is
$P(E) = \frac{p(1-p)}{p(1-p)+(1-p)p} = 0.5$
So that **fairRandom** dose indeed return a 0 and a 1 with equal probability.

(b) **Solution:**
Based on the **simpleRandom**, the only chance that function can return is when $r2 \neq r1$. So we can find
$P(\{r2 = 1|r1 = 1\}) = 0$
$P(\{r2 = 1|r1 = 0\}) = 1$
The probability of $P($`simpleRandom` returns 1$)$ is
$P(\{r2 = 1\}) = P(\{r1 = 0\}) = 1 - p$
Answer:
We can not guarantee that **simpleRandom** generates 0's and 1's with equal probability unless **unknownRandom** returns 0's and 1's with equal probability $p = 0.5$.

(c) **Solution:**
After run the simulation, the probability that second player wins is
$P(\{\text{second player wins}\}) = 0.516$
Assuming all the random numbers are integers.
If the random number can be decimals, the simulation result is
$P(\{\text{second player wins}\}) = 0.528$
The difference between simulation results of simulation in integers and decimals is because of the probability that first and second player have the same number is more obvious when we use integers in simulation. The condition when first and second player start the game is not equivalent. The first player always starts at $S = 0$. The rule to the second player is equivalent to that stop when $S > 100$ but the start point is $S = X$, X is the residual of the first player. Since second player's turn ends by the larger numbers in the first trial is higher than the first player, the chance of the second player has larger last number is higher than first player.

8. **Solution:**
We can define the event $E = \{\text{window is detected}\}$. Then
$P(EL_1) = 0.8 \times 0.2 = 0.16$
$P(EL_2) = 0.2 \times 0.9 = 0.18$
Since L_1 and L_2 are complement to each other,
$P(E) = P(EL_1) + P(EL_2) = 0.34$
We can update the probability estimation based on the new information that the window is detected.

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$P(L_1|E) = \frac{P(EL_1)}{P(E)} = 0.47$
$P(L_2|E) = \frac{P(EL_2)}{P(E)} = 0.53$
Answer:
The robot new values for $P(L_1)$ and $P(L_2)$ is
$P(L_1') = P(L_1|E) = 0.47$
$P(L_2') = P(L_2|E) = 0.53$

9. **Solution:**
For each location we can define the probability of the cell at that location is $P(L_i)$; the probability of a cell records two bars is $P(B)$
$P(B|L_i) = \frac{P(BL_i)}{P(L_i)}$
$P(BL_i) = P(L_i)P(B|L_i)$
Since the summation of the probability that records two bars at all the locations is the total probability that the cell can have two bars.
$P(Li|B) = \frac{P(BL_i)}{P(B)} = \frac{P(L_i)P(B|L_i)}{\sum\limits_{i=1}^{n} P(L_i)P(B|L_i)}$
In the program, we can calculate $P(BL_i)$ in each location and get the summation of them.
Answer:
The probabilities of all 16 cells is
$$\begin{bmatrix} 0.0744 & 0.1885 & 0.0744 & 0.0050 \\ 0.0050 & 0.1488 & 0.0942 & 0.0744 \\ 0.0010 & 0.0050 & 0.1488 & 0.0942 \\ 0.0010 & 0.0010 & 0.0010 & 0.0744 \end{bmatrix}$$

10. (a) **Solution:**
Assume A is the blue-eyed genes and B is the brown-eyed gene. Since William's sister have blue eye and William and both of his parents have brown eyes, we can know that both of William's parents are (AB) combination. The possible gene combinations of William is (AA), (AB), (BA), (BB). Define $E = \{\text{William possesses a blue-eyed gene}\}$, $F = \{\text{William has brown eyes}\}$.
$P(F) = \frac{3}{4}$
$P(E) = \frac{2}{4} = \frac{1}{2}$
Answer:
The probability that William possesses a blue-eyed gene is
$P(G) = P(E|F) = \frac{P(EF)}{P(F)} = \frac{0.5}{0.75} = \frac{2}{3}$

(b) **Solution:**
William's wife have blue eyes, that means her gene combination is (AA).
Define $H = \{\text{William's first child have blue eyes}\}$
$P(H) = P(HG) + P(HG^c) = P(H|G)P(G) + P(H|G^c)P(G^c)$
Answer:
The probability that their first child will have blue eyes is
$P(H) = \frac{1}{2} \times \frac{2}{3} + \frac{1}{3} \times 0 = \frac{1}{3}$

(c) **Solution:**
Define $I = \{\text{William's next child have brown eyes}\}$, so $I^c = \{\text{William's next child have blue eyes}\}$
We should update event $G = \{\text{William possesses a blue-eyed gen}\}$, based on knowing that his first child have brown eyes.
$P(G') = P(G|H^c) = \frac{P(GH^c)}{P(H^c)} = \frac{\frac{1}{3}}{\frac{2}{3}} = \frac{1}{2}$
Answer:
By using the previous conclusion
$P(I^c) = P(I^c|G')P(G) + P(I^c|G'^c)P(G'^c) = \frac{1}{2} \times \frac{1}{2} + 0 \times \frac{1}{3} = \frac{1}{4}$

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The probability that their next child will also have brown eyes is
$P(I) = 1 - P(I^c) = \frac{3}{4}$

11. (a) **Solution:**
From the description, we can find
$P(T_1|G^c) = 0$
$P(T_2|G^c) = 0$
$P(T_1T_2|G^c) = 0.72$
Answer:
$P(T_1|G)P(T_2|G) = P(T_1T_2|G) = 0.72$
So, T_1 and T_2 are conditionally independent given G.

(b) **Answer:**
$P(T_1|G) = P(T_2|G) = 0$
Since both of the probability is 0, T_1 and T_2 conditionally independence is undifined ginve G^c

(c) **Answer:**
$P(T_1) = P(T_1|G)P(G) + P(T_1|G^c)P(G^c) = 0.8 \times 0.6 = 0.48$

(d) **Answer:**
$P(T_2) = P(T_2|G)P(G) + P(T_2|G^c)P(G^c) = 0.9 \times 0.6 = 0.54$
Answer:
$P(T_1T_2) = P(T_1T_2|G)P(G) = 0.72 \times 0.6 = 0.432$
$P(T_1)P(T_2) = 0.48 \times 0.52 = 0.2592$
$P(T_1)P(T_2) \neq P(T_1T_2)$
T_1 and T_2 are NOT independent (gene G is the relation ship between them.)

12. (a) **Answer:**
$P(T) = 0.30079$

(b) **Answer:**
$P(G_1) = 0.70028$
$P(G_2) = 0.30076$
$P(G_3) = 0.5009$
$P(G_4) = 0.8016$
$P(G_5) = 0.32705$

(c) **Solution:**
$P(G_1T) = 0.21210$ vs. $P(G1)P(T) = 0.21124$
$P(G_2T) = 0.09089$ vs. $P(G2)P(T) = 0.09047$
$P(G_3T) = 0.29212$ vs. $P(G3)P(T) = 0.15067$
$P(G_4T) = 0.29703$ vs. $P(G4)P(T) = 0.24112$
$P(G_5T) = 0.29434$ vs. $P(G5)P(T) = 0.09837$
Answer:
G_1 and G_2 independent of T, G_4 is likely independent of T

(d) **Answer:**
$P(T|G_1) = \frac{P(G_1T)}{P(G_3)} = 0.302$ independent

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$P(T|G_2) = \frac{P(G_2T)}{P(G_3)} = 0.302$ independent

$P(T|G_3) = \frac{P(G_3T)}{P(G_3)} = 0.583$ double the chance to have trait

$P(T|G_4) = \frac{P(G_4T)}{P(G_4)} = 0.371$ likely independent

$P(T|G_5) = \frac{P(G_5T)}{P(G_5)} = 0.900$ important fact to have trait

(e) **Answer:**

For the previous results we can find

1. G_5 is assumed to be a main course of the trait
2. G_3 is assumed to increase the probability of the trait
3. G_4 is assumed to have some relationship between the trait

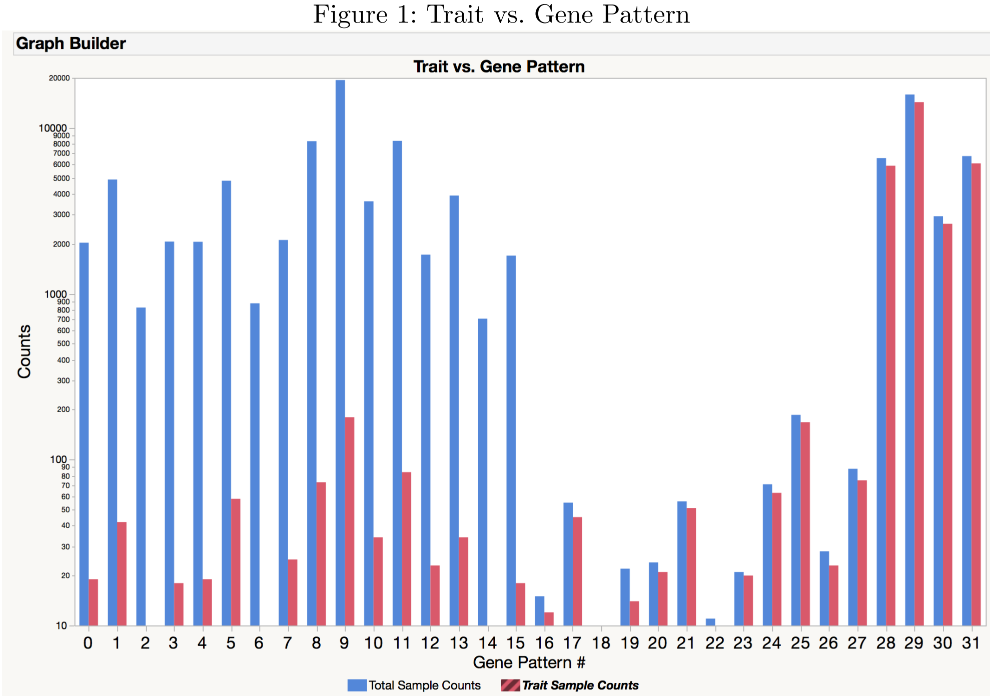
(f) **Solution:**

We can always create a program to ergodic the relationships between all the combination of genes and automatically figure out the independences. Based on the limited size of gene that we are interested in, we can have a other way to find the hidden rule to guide our analysis.

We create a pattern number based on the combination of gene. For all the 6 gene, we can have pattern 0 to 31, and each pattern can represent one of the distinct gene combination. For example, combination $G_5G_4^cG_3G_2G_1^c$ and be represented as b10110 = 22.

Following is the plot of counts of samples in different gene pattern.

Figure 1: Trait vs. Gene Pattern



Becasue of the high dynamic range of the sample counts in different gene pattern, the Y scale of the plot is set to log. We can find that G_5 only exist when both G_3 and G_4 exist.

$P(G_5|G_3G_4) = 0.800$

$P(G_3G_4|G_5) = 0.982$

$G_3 \cap G_4$ is the necessity condition of G_5

$P(G_3G_4) = 0.401$ vs.$P(G_4)P(G_4) = 0.402$

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G_3 and G_4 are independent.
Given $P(T|G_5) = 0.900$, not all the bats with T have G_5. If we still believe G_5 is necessity condition of T, we can analysis the fales positive rate of the experiment or test for trait.
If $P\{\text{false positive}\} > P(T|G_5^c) = 0.049$, it is very likely the trait without G_5 is casued by fales positive in the test.
Same as the samples have G_5 but not trait.
If $P\{\text{false negative}\} > P(G_5|T) = 0.020$, it is very likely that the sample have G_5 but not have T is casued by false negative in the test.

Figure 2: Gene and Trait Relationship