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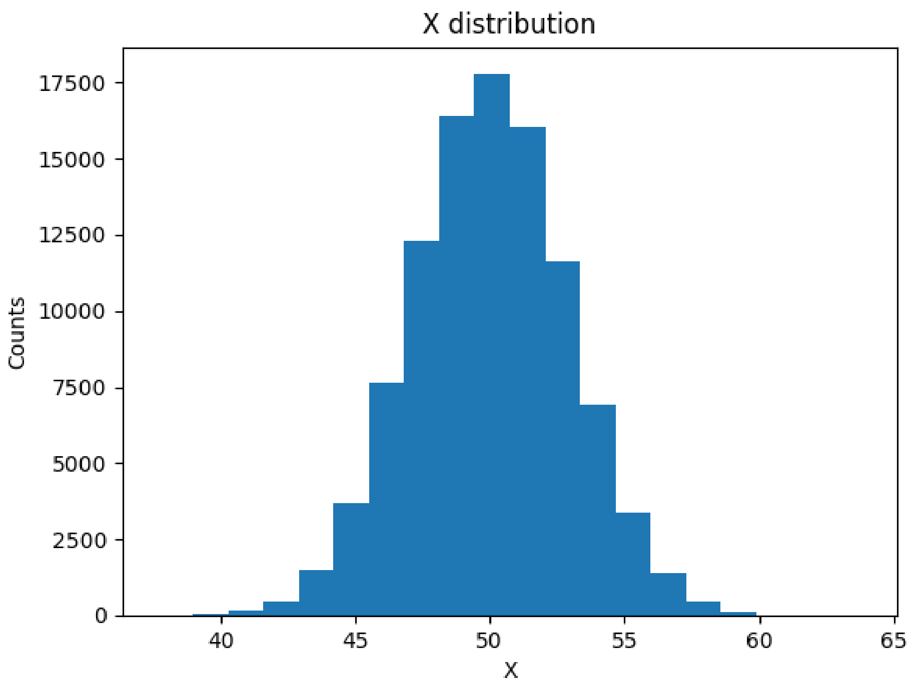
1. (a) **Answer:**
The after trial distribution is:
$f(x) = Beta(2+7, 2+2) = Beta(9, 4)$

(b) **Answer:**
$F_{Beta}(0.5) = 0.073$
$P(\text{drug having effect} \geq 0.5) = 1 - F_{Beta}(0.5) = 0.927$

2. (a) **Answer:**
$P(35 \leq X \leq 40) = 0.00022$
$P(40 \leq X \leq 45) = 0.04192$
$P(60 \leq X \leq 65) = 0.00022$



(b) **Solution:**
From the simulation results, we can find
$E[X] = 50.00$
$Var(X) = 8.336$
Then we can use normal distribution to represent the distribution.
Answer:
$X \sim N(50.00, 8.336)$
$f(x) = \frac{1}{\sqrt{2\pi 8.336}} e^{-(x-50)^2/(2\times 8.336)}$

(c) **Solution:**
$P\{40 \leq X \leq 45\} = F(45) - F(40) = \Phi(\frac{(45-50)}{\sqrt{8.336}}) - \Phi(\frac{(40-50)}{\sqrt{8.336}})$
$\text{text}P\{40 \leq X \leq 45\} = \Phi(\frac{10}{2.89}) - \Phi(\frac{5}{2.89}) = 0.9997 - 0.9582 = 0.0415$

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3. (a) **Answer:**
The expected amount of money that each person gives is
$E[X] = 5.95$

(b) **Answer:**
$Var(X) = E[X^2] - (E[X])^2 = 23.19$

(c) **Solution:**
$E[n\overline{X}] = nE[X] = 5.95 \times 50 = 279.5$
$Var(n\overline{X}) = nVar(X) = 1159.5$
Answer:
$\sum_{i=1}^{50} X_i \sim N(279.5, 1159.5)$

(d) **Answer:**
$Y = \sum_{i=1}^{50} X_i$
$P\{Y \geq 350\} = 1 - P\{Y < 350\} = 1 - \Phi(\frac{350-279.5}{\sqrt{1159.5}}) = 1 - 0.9808 = 0.02$

4. **Solution:**
$Cov(X,Y) = Cov(X,X^2) = E[X^3] - E[X]E[X^2]$
$E[X^3] = \frac{1}{6}\sum_{i=1}^{6} i^3 = 73.5$
$E[X^2] = \frac{1}{6}\sum_{i=1}^{6} i^2 = 15.17$
$E[X] = \frac{1}{6}\sum_{i=1}^{6} i = 3.5$
Answer:
$Cov(X,Y) = 20.65$

5. **Answer:**
$2X + Y \sim N(2+1, 2\times 2+2)$
$2X + Y \sim N(3,6)$

6. (a) **Answer:**
Let X be the true distance of the satellite
$f_X(x) = \frac{1}{4\sqrt{2\pi}} e^{-(x-98)^2/32}$

(b) **Answer:**
Let Y be the measured distance
$f_{Y|X}(y=100|x=t) = \frac{1}{2\sqrt{2\pi}} e^{-(100-t)^2/8}$

(c) **Solution:**
Let X' be the posterior belief of the true distance of satellite
$f_{XY}(x, y=100) = f_{Y|X}(y=100|x) f_X(x) = \frac{1}{2\sqrt{2\pi}} e^{-(100-x)^2/8} \times \frac{1}{4\sqrt{2\pi}} e^{-(x-98)^2/32}$

$$f_{X'}(x') = f_{X|Y}(x=x'|y=100) = \frac{f_{XY}(y=100, x=x')}{f_Y(y=100)} = \frac{f_{XY}(x=x', y=100)}{\int_{-\infty}^{+\infty} f_{XY}(y=100)dx}$$

Let

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$$C = \frac{1}{\int_{-\infty}^{+\infty} f_{XY}(x, y = 100)dx}$$

Answer:

$$f_{X'}(x') = C\frac{1}{16\pi^2}e^{\frac{-(4(100-x')^2+(x'-98)^2)}{32}}$$

7. (a) **Solution:**
   Let X_n be the total value of n rolling before any value ≥ 3 is rolled
   $E[X] = \sum_{i=0}^{\infty} E[X_i] + 4.5$
   Let Y be the value when for each rolling when value < 3
   $E[Z] = 1.5$
   $E[X_i] = (\frac{1}{3})^i(iE[Z]) = 1.5i(\frac{1}{3})^i$
   Answer:
   $$E[X] = \sum_{i=0}^{\infty} 1.5i(\tfrac{1}{3})^i + 4.5 = 1.5\frac{\frac{1}{3}}{(1-\frac{1}{3})^2} + 4.5 = 5.625$$

   (b) **Answer:**
   $$E[Y] = \sum_{i=0}^{\infty} i(\tfrac{1}{3})^i + 1 = \frac{\frac{1}{3}}{(1-\frac{1}{3})^2} + 1 = 1.75$$

8. **Solution:**
   $E[Y^2] = E[Y^2|X = 1]P(X = 1) + E[Y^2|X = 2]P(X = 2) + E[Y^2|X = 2]P(X = 2)$
   $E[Y^2] = E[3^2]\frac{1}{3} + E[(Y + 5)^2]\frac{1}{3} + E[(Y + 7)^2]\frac{1}{3}$
   $3E[Y^2] = 9 + E[Y^2] + 10E[Y] + 25 + E[Y^2] + 14E[Y] + 49$
   $E[Y^2] = 443$
   Answer:
   $$Var(Y) = E[Y^2] - E[Y]^2 = 443 - 225 = 218$$

9. (a) **Solution:**
   $E[n\overline{A}] = n\mu_A = 20 \times 50 = 1000$
   $Var(n\overline{A}) = n^2\frac{\sigma_A^2}{n} = n\sigma_A^2 = 2000$
   $P(n\overline{A} < 950) = \Phi(\frac{950-1000}{\sqrt{2000}}) = 1 - \Phi(\frac{50}{44.72}) = 1 - 0.8686$
   $$P(n\overline{A} < 950) = 0.1314$$

   (b) $E[n\overline{B}] = n\mu_B = 20 \times 52 = 1040$
   $Var(n\overline{B}) = n^2\frac{\sigma_B^2}{n} = n\sigma_B^2 = 4000$
   $P(n\overline{B} < 950) = \Phi(\frac{950-1040}{\sqrt{4000}}) = 1 - \Phi(\frac{90}{44.72}) = 1 - 0.9207$
   $$P(n\overline{B} < 950) = 0.0793$$

   (c) Let $C = A - B$
   $E[n\overline{C}] = n\mu_C = \mu_A - \mu_B = -40$
   $Var(n\overline{C}) = n\sigma_C^2 = n(\sigma_A^2 + \sigma_B^2) = 6000$
   Answer:
   $$P(C < 0) = F_C(0) = \Phi(\frac{0-(-40)}{\sqrt{6000}}) = 0.6985$$

10. **Solution:**
    Let $Y = \sum_{i=1}^{79} X_i$

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$E[Y] = 79E[X] = 79 \times 3.5 = 276.5$
$Var(Y) = 79Var(X)$
$Var(X) = E[X^2] - E[X]^2 = 2.92$
$Var(Y) = 230.68$
$Y \sim N(276.5, 230.68)$
When the first 79 rolls have sum lower than 300, at least 80 rolls are necessary to reach a sum that exceeds 300.
$P(Y \leq 299.5) = F_Y(299.5) = \Phi(\frac{299.5-276.5}{sqrt230.68})$
Answer:
$P(Y \leq 299.5) = 0.9332$

11. (a) **Answer:**
$P(X \geq 85) \leq \frac{E[X]}{85} = 0.882$

(b) **Solution:**
$P(65 \leq X \leq 85) > P(|X - 75| < 10) = 1 - P(|X - 75| \geq 10)$
$P(|X - 75| \geq 10) \leq \frac{Var(X)}{10^2} = 0.25$
$0.75 \leq 1 - P(|X - 75| \geq 10) < P(65 \leq X \leq 85)$
Answer:
$P(65 \leq X \leq 85) > 0.75$

(c) **Solution:**
Let $Y = \overline{X}$
$E[Y] = E[X] = 75$
$Var(Y) = \frac{Var(X)}{n} = \frac{25}{n}$ $P(|Y - 75| \geq 5) \leq \frac{Var(Y)}{5^2} = \frac{\frac{25}{n}}{25} \leq 0.1$
Answer:
$\frac{1}{n} \leq 0.1$
The number of students would have to take the midterm in order to ensure 90% probability that the class average would be within 5 of 75 is
$n \geq 10$

(d) **Solution:**
$P(-5 \leq \overline{X} - E[X] \leq 5) = \Phi(\frac{5}{\frac{\sigma}{\sqrt{n}}}) - \Phi(\frac{-5}{\frac{\sigma}{\sqrt{n}}}) = 2\Phi(\frac{5}{\frac{5}{\sqrt{n}}}) - 1 \geq 0.9$
$\Phi(\sqrt{n}) > 0.95$ $\sqrt{n} > 1.65$
Answer:
The number of students would have to take the midterm in order to ensure 90% probability that the class average would be within 5 of 75 is
$n = 3 > 2.72$