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1. (a) **Solution:**
We calculate after how many rounds, the game provide will use up all their money.
$m = \lfloor log_2 X \rfloor$
Then we can derive the expectation of payoff of the game by considering 2 parts, normal payoff and the limited payoff after provider run out of money, after you win more or equal to 2^m, no matter what happend, you will win X
$E[Y] = \sum_{n=0}^{m} (\frac{1}{2})^{n+1} 2^n + \frac{1}{2}^{n+1} X = (m+1)\frac{1}{2} + \frac{1}{2}^{m+1} X$
Answer:
$m = 2$
$E[Y] = 3 \times \frac{1}{2} + \frac{1}{2}^3 \times 5 = 2.125$

(b) **Answer:**
$m = 8$
$E[Y] = 9 \times \frac{1}{2} + \frac{1}{2}^8 \times 500 = 5.477$

Answer:
$m = 12$
$E[Y] = 13 \times \frac{1}{2} + \frac{1}{2}^{13} \times 4096 = 7$

2. **Solution:**
The expectation of new users in 5 minutes is $E(X) = 2$. The expectation of gain for the trip can be preseted as
$E[(X + 1) \times \$6 - \$7] = E[X] \times \$6 - \$1$
Answer:
The expectation of Lyft to make in this trip is
$E[X] \times \$6 - \$1 = \$11$

3. (a) **Solution:**
The possibility that x bit are corrupted is
$P(X = x) = \binom{2n}{x} p^x (1-p)^{(2n-x)}$
Answer:
The probability that the message ss is received without any corruption is
$P(X = 0) = (1 - 0.05)^8 = 0.663$

(b) **Solution:**
For each first string s which has any corruption bit, if the second time the corruption bit(s) are exactly the same, we can not detect that problem
Answer:
The probability that we can not detect the corrupted bit(s)
$P(E) = \sum_{x=1}^{n} \binom{n}{x} (p^x (1-p)^{(n-x)})^2$
$P(E) = \sum_{x=1}^{4} \binom{4}{x} (0.05^x (1-0.05)^{(4-x)})^2 = 0.0074$

(c) **Solution:**
The probability that recipient can detect the corruption took place is
$P(F) = 1 - P(X = 0) - P(E)$
Answer:
$P(F) = 1 - (1-p)^{2n} - \sum_{x=1}^{n} \binom{n}{x} (p^x (1-p)^{(n-x)})^2 = 1 - 0.663 - 0.0074 = 0.329$

4. **Solution:**
There are 2 cases that the jury renders a correct decision, votes guilty when the defendants are actually guilty or votes innocent and the defendants are actally innocent.
The probability of the defendants are actually guilty is
$P(G) = 0.75$
For each juror, the probability that votes guilty $P(H)$ in different conditions is
$P(H|G^c) = 0.1$
$P(H^c|G) = 0.2$
The probability that the jury decides the defendants are guilty is $P(F)$
Then the probability make correct decision when the defenant is actually guilty (less than 4 juror votes innocent) is
$P(F|G) = \sum_{n=0}^{3} \binom{12}{i} P(H^c|G)^i P(H|G)^{12-i}$
$P(F|G) = \sum_{n=0}^{3} \binom{12}{i} 0.2^i 0.8^{12-i} = 0.7946$
$P(FG) = P(F|G)P(G) = 0.5959$
Then the probability make correct decision when the defenant is actually innocent (1 - P(less than 4 juror votes innocent)) is
$P(F^c|G^c) = 1 - \sum_{n=0}^{4} \binom{12}{i} P(H^c|G^c)^i P(H|G^c)^{12-i}$
$P(F^c|G^c) = 1 - \sum_{n=0}^{4} \binom{12}{i} 0.9^i 0.1^{12-i} = 1 - 3.4 \times 10^{-6} = 1$
$P(F^c) = P(F^c|G^c)P(G^c) = 0.25$
Answer:
The probability that the jury renders a correct decision is
$P(FG) + P(F^cG^c) = 0.5959 + 0.25 = 0.8459$
The percentage of defendant found guilty by the jury is
$P(F) = P(FG) + P(FG^c) = 0.5959 + 0 = 59.6\%$

5. **Solution:**
Define event
$X_i = \{$A person computer crash i times in the month$\}$
$E = \{$The patch has had an effect on the user's computer$\}$
Then, based on the Poisson distribution
$P(X_2|E) = e^{-3}\frac{3^2}{2!} = 0.224$
$P(X_2|E^c) = e^{-5}\frac{5^2}{2!} = 0.084$
Answer:
$P(X_2E) = P(X_2|E)P(E) = 0.224 \times 0.75 = 0.168$
The probability that the patch has had an effect on the user's computer is
$P(E|X_2) = \frac{P(X_2E)}{P(X_2)} = \frac{0.168}{0.224+0.084} = 0.545$

6. **Solution:**
$E[\sum_{i=1}^{k} X_i] = \sum_{i=1}^{k} E[X_i]$
$E[X_i] = 1 \times P(X_i = 1) + 0 \times P(X_i = 0) = 1 \times (1 - (1 - p_i)^n) + 0 \times (1 - p_i)^n$
Answer:
The expected number of buckets that have at least one string hashed into them is
$E[\sum_{i=1}^{k} X_i] = \sum_{i=1}^{k} (1 - (1 - p_i)^n)$

7. **Solution:**
The CDF of given uniformly random distribution is
$$F(a) = \begin{cases} 0 & a \le 0 \\ \frac{a}{n} & 0 < a < n \\ 1 & a \ge n \end{cases}$$
Only in the cases that $x < 0.2n$ or $x > 0.8n$, the shorter piece is less than 1/4th of the longer one.
Answer:
$P(x < 0.2n) + P(x > 0.8n) = F(0.2n) + (1 - F(0.8n)) = \frac{0.2n}{n} + (1 - \frac{0.8n}{n}) = 0.4$

8. (a) **Solution:**
$1 = \int_{-\infty}^{+\infty} f(x)dx = \int_{-1}^{+1} c(3 - 2x^2)dx$
$1 = c(3x - \frac{2}{3}x^3)\Big|_{-1}^{1} = \frac{14}{3}c$
Answer:
$c = \frac{3}{14}$

(b) **Answer:**
$F(a) = PX < a = \int_{-\infty}^{a} f(x)dx$
$$F(a) = \begin{cases} 0 & a \le -1 \\ (3a - \frac{2}{3}a^3)c - (-\frac{7}{3}c) & -1 < a < 1 \\ 1 & a \ge 1 \end{cases}$$
$$F(a) = \begin{cases} 0 & a \le -1 \\ (\frac{9}{14}a - \frac{1}{7}a^3) + \frac{1}{2} & -1 < a < 1 \\ 1 & a \ge 1 \end{cases}$$

(c) **Answer:**
$E[X] = \int_{-\infty}^{+\infty} xf(x)dx$
$E[X] = \int_{-1}^{+1} x\frac{3}{14}(3 - 2x^2)dx$
$E[X] = \frac{9}{28}x^2 - \frac{3}{28}x^4\Big|_{-1}^{1} = 0$

9. **Solution:**
$P(A) = \alpha$
$P(B) = 1 - \alpha$
Since, $P(A) = P(B^c)$
$P(R) = P(RA) + P(RB) = P(R|A)P(A) + P(R|B)P(B)$
$P(R) = (f_A(x)dx|x = 5)\alpha + (f_B(x)dx|x = 5)(1 - \alpha)$
$P(R|A) = \frac{1}{\sqrt{2\pi}\times 3}e^{-(5-6)^2/(2\times 9)}dx = 0.1311dx$
$P(R|B) = \frac{1}{\sqrt{2\pi}\times 2}e^{-(5-4)^2/(2\times 2)}dx = 0.1933dx$
$P(RA) = 0.1311dx\,\alpha$
$P(RB) = 0.1933(1 - \alpha)$
$P(R) = P(RA) + P(RB) = 0.1933 - 0.0622\alpha\,dx$
Answer:
$P(A|R) = \frac{P(AR)}{P(R)} = \frac{0.1311\alpha\,dx}{0.1933 - 0.0622\alpha\,dx} = \frac{0.1311\alpha}{0.1933 - 0.0622\alpha}$
$\frac{0.1311\alpha}{0.1933 - 0.0622\alpha} = 0.5$
$\alpha = 0.596$

10. (a) **Solution:**
For each return number from 1 to 10 we can derive the probability
$P(X = i) = 0.1$

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$P(X=-1)=0$

Answer:

$E[X]=\sum_{i=0}^{9} 0.1i = 4.5$

(b) **Solution:**

The only chance that the founction can return is when arr[mid] = key. So the result would as same as the previous

$P(X=i)=0.1|1 \le i \le 10$

Answer:

$E[X]=\sum_{i=0}^{9} 0.1i = 4.5$

11. (a) **Solution:**

Since the total hash trial of the strings $3m = 72000$ are very high and the probability for each string hashed into certain bucket is very low $\frac{1}{8000}$. We can use binomial distribution function to derive the probability of each number of strings that hashed into a certain bucket. $P\{X=i\} \approx e^{-\lambda}\frac{\lambda^i}{i!}$

$\lambda = 3m\frac{1}{n} = 72000 \times \frac{1}{8000} = 9$

Answer:

The probability that the first bucket has 0 strings hashed into it is

$P\{X=0\} = \approx e^{-9}\frac{9^0}{0!} = 1.234 \times 10^{-4}$

(b) **Answer:**

The probability that the first bucket has 10 or fewer strings hashed to it is

$P\{X \le 10\} = \sum_{i=0}^{10} e^{-9}\frac{9^i}{i!} = 0.706$

(c) **Solution:**

For a bloom filter which have $X=i$ bits are 0s, the probability $P(E)$ of a string that is reported in the set incorrectly is

$P(E|\{X=i\}) = (1-\frac{i}{n})^3$

For each bit, the probability that remains to 0s after 24000 strings added is

$p = 1.234 \times 10^{-4}$

Let $\lambda = np = 0.9873$

The probability that $X=i$ bucket have no string been hashed into is

$P(\{X=i\}) = \frac{e^{-\lambda}\lambda^i}{i!}$

$P(E\{X=i\}) = \frac{(n-i)^3}{n^3}\frac{e^{-\lambda}\lambda^i}{i!}$

$P(E) = \sum_{i=0}^{+\infty} P(E\{X=i\}) = \sum_{i=0}^{+\infty}\frac{e^{-\lambda}\lambda^i}{i!} - \frac{1}{n}\sum_{i=0}^{+\infty}\frac{3ie^{-\lambda}\lambda^i}{i!} + \frac{1}{n^2}\sum_{i=0}^{+\infty}\frac{3i^2e^{-\lambda}\lambda^i}{i!} - \frac{1}{n^3}\sum_{i=0}^{+\infty}\frac{i^3e^{-\lambda}\lambda^i}{i!}$

$P(E) = 1 - \frac{3}{n}E[X] + \frac{3}{n^2}E[X^2] - \frac{1}{n^3}E[X^3]$

$\frac{1}{n^3}E[X^3] = \frac{\lambda}{n^3}\sum_{i=1}^{+\infty}\frac{i^2e^{-\lambda}\lambda^{(i-1)}}{(i-1)!} = \frac{\lambda}{n^3}\sum_{i=1}^{+\infty}(\frac{(i-1)^2e^{-\lambda}\lambda^{(i-1)}}{(i-1)!} + \frac{2(i-1)e^{-\lambda}\lambda^{(i-1)}}{(i-1)!} + \frac{e^{-\lambda}\lambda^{(i-1)}}{(i-1)!})$

$\frac{1}{n^3}E[X^3] = \frac{\lambda}{n^3}(E[X^2] + 2E[X] - 1) = \frac{\lambda}{n^3}(\lambda(\lambda+1) + 2\lambda + 1) = \frac{1}{n^3}(\lambda^3 + 3\lambda^2 + \lambda)$

$P(E) = 1 - \frac{3}{n}\lambda + \frac{3}{n^2}\lambda(\lambda+1) - \frac{1}{n^3}(\lambda^3 + 3\lambda^2 + \lambda)$

$P(E) \approx 1 - \frac{3}{n}\lambda$

Answer:

$P(E) \approx 1 - \frac{3}{8000} \times 0.9873 = 0.9996$

(d) **Solution:**

Let $n = 4000$, then $\lambda = e^{-18} \times 4000$

Answer:

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$P(E) \approx 1 - \frac{3}{4000} \times e^{-18} \times 4000 \approx 1 - 4.57 \times 10^{-8} \approx 1$

12. (a) **Solution:**
Since all the polls are equivalent, we can just add all the samples that vote A and divide by the total sample number.
Answer:
The probability that a random person in France votes for candidate A is
$P(A) = \frac{4881}{7453} = 0.655$

(b) **Solution:**
Since the total number of people is too large, we can use 64888 total population and iterate the experiment for 10000 times.
The result is all the 10000 experiment show that A wins.
Answer:
$P(A) \approx 1.00$
The distribution of votes for A is a Poisson distribution with $\lambda = np = 42502158.76$
Another way to solve this problem is, we can treat the distribution of votes for A as a normal distribution which has $\mu = \lambda$,$\sigma^2 = \lambda$
$F(0.5\lambda) = \Phi(\frac{0.5\lambda - \lambda}{\sqrt{\lambda}}) = 1 - \Phi(\frac{0.5\lambda}{\sqrt{\lambda}})$
Since λ is very large
$\Phi(\frac{0.5\lambda}{\sqrt{\lambda}}) = 0$
$P(A) = F(0.5\lambda) \approx 1$