

1. (a) **Solution:**

The number of users should be a poisson distribution which is

$X \sim Poi(\lambda)$ where $\lambda = 5.5$

The probability that more than 7 users will sign-up for the social networking site in the next minute is

$$P(E) = 1 - P(E^c) = 1 - P\{X \leq 7\} = 1 - \sum_{i=0}^7 \frac{e^{-5.5} 5.5^i}{i!}$$

Answer:

We can also use normal distribution to approximate the probability

$$P\{X > 7\} = 1 - \Phi\left(\frac{7-5.5}{\sqrt{5.5}}\right) = 1 - 0.7389 = 0.2611$$

(b) **Answer:**

The probability that more than 13 users will sign-up for the social networking site in the next 2 minute is

$$P\{X > 13\} = 1 - P\{X \leq 13\} = 1 - \Phi\left(\frac{13-5.5 \times 2}{\sqrt{5.5 \times 2}}\right) = 1 - 0.7257 = 0.2743$$

(c) **Answer:**

The probability that more than 15 users will sign-up for the social networking site in the next 3 minute is

$$P\{X > 15\} = 1 - P\{X \leq 15\} = 1 - \Phi\left(\frac{15-5.5 \times 3}{\sqrt{5.5 \times 3}}\right) = \Phi(0.3693) = 0.6433$$

2. (a) **Solution:**

$$\int_0^1 \int_0^x c \frac{y}{x} dy dx = \int_0^1 \frac{1}{2} c \frac{x^2}{x} dx = \frac{1}{4} c x^2 \Big|_0^1 = \frac{1}{4} c = 1$$

Answer:

$$c = 4$$

(b) **Solution:**

$$f_X(x) = \int_0^x 4 \frac{y}{x} dy = 2c \frac{y^2}{x} \Big|_0^x = 2x$$

$$f_Y(y) = \int_y^1 4 \frac{y}{x} dx = cy \ln(x) \Big|_y^1 = -4y \ln(y)$$

$$f_{XY}(x, y) \neq f_X(x) f_Y(y)$$

Answer:

X and Y are not independent

(c) **Answer:**

$$f_X(x) = 2x$$

(d) **Answer:**

$$f_Y(y) = -4y \ln(y)$$

(e) **Answer:**

$$E[X] = \int_0^1 2x^2 dx = \frac{2}{3} x^3 \Big|_0^1 = \frac{2}{3}$$

(f) **Answer:**

$$E[Y] = \int_0^1 -4y^2 \ln(y) dy = -4(\ln(y) \int_0^1 y^2 dy - \int_0^1 (\frac{1}{y} \frac{1}{3} y^3) dy)$$

$$E[Y] = -4(\ln(y) \frac{1}{3} y^3 - \frac{1}{9} y^3) \Big|_0^1 = \frac{4}{9}$$

3. (a) **Answer:**

$$P\{X_1 + X_2 \geq 5000\} = P\left\{\frac{X_1 + X_2 - 2200 \times 2}{\sqrt{52900 \times 2}} \geq \frac{5000 - 2200 \times 2}{\sqrt{52900 \times 2}}\right\} = 1 - \Phi(1.84) = 0.033$$

To be more accurate, we should use > 4999.5 instead of ≥ 5000 . The result in this particular case has no significant difference.

(b) **Solution:**

$$P\{X \geq 2000\} = 1 - \Phi\left(\frac{2000 - 2200}{\sqrt{52900}}\right) = \Phi(0.87) = 0.808$$

Answer:

The probability that the weekly number of visitors exceeds 2000 in at least 2 of the next 3 weeks is

$$\binom{3}{2}p^2(p-1) + \binom{3}{1}p^3 = 0.904$$

4. (a) **Answer:**

$$A \sim N(\mu_1 + \mu_2, \sigma_1^2 + \sigma_2^2)$$

(b) **Answer:**

$$B \sim N(5\mu_1 + 2, 25\sigma_1^2)$$

(c) **Solution:**

$$C_a = aX$$

$$C_b = -bY$$

$$C_c = c^2Z$$

$$C_a \sim N(a\mu_1, a^2\sigma_1^2)$$

$$C_b \sim N(-b\mu_1, b^2\sigma_2^2)$$

$$C_c \sim N(c^2\mu_1, c^4\sigma_3^2)$$

Answer:

$$C \sim N(a\mu_1 - b\mu_2 + c^2\mu_3, a^2\sigma_1^2 + b^2\sigma_2^2 + c^4\sigma_3^2)$$

5. (a) **Answer:**

$$f_{XY}(x, y) = \begin{cases} \frac{1}{6} \frac{1}{x} & x, y \in \{1, 2, 3, 4, 5, 6\} \\ 0 & x, y \notin \{1, 2, 3, 4, 5, 6\} \end{cases}$$

(b) **Answer:**

$$P(X = j | Y = i) = \begin{cases} \frac{1}{6} \frac{1}{j} & i \leq j \\ 0 & i > j \end{cases}$$

(c) **Solution:**

$$f_X(x) = \begin{cases} \frac{1}{6} & x \in \{1, 2, 3, 4, 5, 6\} \\ 0 & x \notin \{1, 2, 3, 4, 5, 6\} \end{cases}$$

$$f_Y(y) = \begin{cases} \frac{1}{6} \sum_{i=y}^6 \frac{1}{i} & y \in \{1, 2, 3, 4, 5, 6\} \\ 0 & y \notin \{1, 2, 3, 4, 5, 6\} \end{cases}$$

$$f_{XY}(x, y) \neq f_X(x)f_Y(y)$$

Answer:

X and Y are NOT independent

6. **Solution:**

Since the expectation of the distance are symmetrical when the package is dropped off in the 4 different quadrants. Let D' = the distance the robot travels to get to the package in first quadrants.

$$f'_D(d) = \frac{1}{4} \left(\frac{1}{5}\right)^2$$

$$E[D'] = \frac{1}{100} \int_0^5 \int_0^5 (x+y) dx dy = \frac{1}{100} \int_0^5 \frac{25}{2} + 5y dy = \frac{125}{100}$$

Answer:

$$E[D] = 4E[D'] = 4 \frac{125}{100} = 5$$

7. **Solution:**

The probability that "max update" is executed in each loop $p_i = P(E_i) = 2^{-i}$, because the probability that the current number is higher than each of the previous numbers are all equal to $\frac{1}{2}$

Let

$$I_i = \begin{cases} 1 & E_i \text{ occurs} \\ 0 & \text{otherwise} \end{cases}$$

Answer:

The expected number of times that "max update" is executed is

$$E[X] = \sum_{i=0}^{n-1} E[I_i] = \sum_{i=0}^{n-1} p_i = 2 - \left(\frac{1}{2}\right)^{n-1}$$

8. (a) **Solution:**

$$E[X] = \frac{a}{a+b} = 0.5$$

$$Var[X] = \frac{ab}{(a+b)^2(a+b+1)}$$

Answer:

$$a = b = \frac{2}{17}$$