

1. (a) **Solution:**

We calculate after how many rounds, the game provide will use up all their money.

$$m = \lfloor \log_2 X \rfloor$$

Then we can derive the expectation of payoff of the game by considering 2 parts, normal payoff and the limited payoff after provider run out of money, after you win more or equal to 2^m , no matter what happen, you will win X

$$E[Y] = \sum_{n=0}^m \left(\frac{1}{2}\right)^{n+1} 2^n + \frac{1}{2}^{m+1} X = (m+1)\frac{1}{2} + \frac{1}{2}^{m+1} X$$

Answer:

$$m = 2$$

$$E[Y] = 3 \times \frac{1}{2} + \frac{1}{2}^3 \times 5 = 2.125$$

(b) **Answer:**

$$m = 8$$

$$E[Y] = 9 \times \frac{1}{2} + \frac{1}{2}^8 \times 500 = 5.477$$

Answer:

$$m = 12$$

$$E[Y] = 13 \times \frac{1}{2} + \frac{1}{2}^{13} \times 4096 = 7$$

2. **Solution:**

The expectation of new users in 5 minutes is $E(X) = 2$. The expectation of gain for the trip can be preseted as

$$E[(X+1) \times \$6 - \$7] = E[X] \times \$6 - \$1$$

Answer:

The expectation of Lyft to make in this trip is

$$E[X] \times \$6 - \$1 = \$11$$

3. (a) **Solution:**

The possibility that x bit are corrupted is

$$P(X = x) = \binom{2n}{x} p^x (1-p)^{(2n-x)}$$

Answer:

The probability that the message ss is received without any corruption is

$$P(X = 0) = (1 - 0.05)^8 = 0.663$$

(b) **Solution:**

For each first string s which has any corruption bit, if the second time the corruption bit(s) are exactly the same, we can not detect that problem

Answer:

The probability that we can not detect the corrupted bit(s)

$$P(E) = \sum_{x=1}^n \binom{n}{x} (p^x (1-p)^{(n-x)})^2$$

$$P(E) = \sum_{x=1}^4 \binom{4}{x} (0.05^x (1-0.05)^{(4-x)})^2 = 0.0074$$

(c) **Solution:**

The probability that recipient can detect the corruption took place is

$$P(F) = 1 - P(X = 0) - P(E)$$

Answer:

$$P(F) = 1 - (1-p)^{2n} - \sum_{x=1}^n \binom{n}{x} (p^x (1-p)^{(n-x)})^2 = 1 - 0.663 - 0.0074 = 0.329$$

4. **Solution:**

There are 2 cases that the jury renders a correct decision, votes guilty when the defendants are actually guilty or votes innocent and the defendants are actually innocent.

The probability of the defendants are actually guilty is

$$P(G) = 0.75$$

For each juror, the probability that votes guilty $P(H)$ in different conditions is

$$P(H|G^c) = 0.1$$

$$P(H^c|G) = 0.2$$

The probability that the jury decides the defendants are guilty is $P(F)$

Then the probability make correct decision when the defendant is actually guilty (less than 4 juror votes innocent) is

$$P(F|G) = \sum_{n=0}^3 \binom{12}{n} P(H^c|G)^n P(H|G)^{12-n}$$

$$P(F|G) = \sum_{n=0}^3 \binom{12}{n} 0.2^n 0.8^{12-n} = 0.7946$$

$$P(FG) = P(F|G)P(G) = 0.5959$$

Then the probability make correct decision when the defendant is actually innocent (1 - P(less than 4 juror votes innocent)) is

$$P(F^c|G^c) = 1 - \sum_{n=0}^4 \binom{12}{n} P(H^c|G^c)^n P(H|G^c)^{12-n}$$

$$P(F^c|G^c) = 1 - \sum_{n=0}^4 \binom{12}{n} 0.9^n 0.1^{12-n} = 1 - 3.4 \times 10^{-6} = 1$$

$$P(F^c) = P(F^c|G^c)P(G^c) = 0.25$$

Answer:

The probability that the jury renders a correct decision is

$$P(FG) + P(F^cG^c) = 0.5959 + 0.25 = 0.8459$$

The percentage of defendant found guilty by the jury is

$$P(F) = P(FG) + P(FG^c) = 0.5959 + 0 = 59.6\%$$

5. **Solution:**

Define event

$X_i = \{\text{A person computer crash } i \text{ times in the month}\}$

$E = \{\text{The patch has had an effect on the user's computer}\}$

Then, based on the Poisson distribution

$$P(X_2|E) = e^{-3} \frac{3^2}{2!} = 0.224$$

$$P(X_2|E^c) = e^{-5} \frac{5^2}{2!} = 0.084$$

Answer:

$$P(X_2E) = P(X_2|E)P(E) = 0.224 \times 0.75 = 0.168$$

The probability that the patch has had an effect on the user's computer is

$$P(E|X_2) = \frac{P(X_2E)}{P(X_2)} = \frac{0.168}{0.224+0.084} = 0.545$$

6. **Solution:**

$$E\left[\sum_{i=1}^k X_i\right] = \sum_{i=1}^k E[X_i]$$

$$E[X_i] = 1 \times P(X_i = 1) + 0 \times P(X_i = 0) = 1 \times (1 - (1 - p_i)^n) + 0 \times (1 - p_i)^n$$

Answer:

The expected number of buckets that have at least one string hashed into them is

$$E\left[\sum_{i=1}^k X_i\right] = \sum_{i=1}^k (1 - (1 - p_i)^n)$$

7. **Solution:**

The CDF of given uniformly random distribution is

$$F(a) = \begin{cases} 0 & a \leq 0 \\ \frac{a}{n} & 0 < a < n \\ 1 & a \geq n \end{cases}$$

Only in the cases that $x < 0.2n$ or $x > 0.8n$, the shorter piece is less than $1/4$ th of the longer one.

Answer:

$$P(x < 0.2n) + P(x > 0.8n) = F(0.2n) + (1 - F(0.8n)) = \frac{0.2n}{n} + (1 - \frac{0.8n}{n}) = 0.4$$

8. (a) **Solution:**

$$1 = \int_{-\infty}^{+\infty} f(x)dx = \int_{-1}^{+1} c(3 - 2x^2)dx$$

$$1 = c(3x - \frac{2}{3}x^3) \Big|_{-1}^{+1} = \frac{14}{3}c$$

Answer:

$$c = \frac{3}{14}$$

(b) **Answer:**

$$F(a) = PX < a = \int_{-\infty}^a f(x)dx$$

$$F(a) = \begin{cases} 0 & a \leq -1 \\ (3a - \frac{2}{3}a^3)c - (-\frac{7}{3}c) & -1 < a < 1 \\ 1 & a \geq 1 \end{cases}$$

$$F(a) = \begin{cases} 0 & a \leq -1 \\ (\frac{9}{14}a - \frac{1}{7}a^3) + \frac{1}{2} & -1 < a < 1 \\ 1 & a \geq 1 \end{cases}$$

(c) **Answer:**

$$E[X] = \int_{-\infty}^{+\infty} xf(x)dx$$

$$E[X] = \int_{-1}^{+1} x \frac{3}{14}(3 - 2x^2)dx$$

$$E[X] = \frac{9}{28}x^2 - \frac{3}{28}x^4 \Big|_{-1}^{+1} = 0$$

9. **Solution:**

$$P(A) = \alpha$$

$$P(B) = 1 - \alpha$$

Since, $P(A) = P(B^c)$

$$P(R) = P(RA) + P(RB) = P(R|A)P(A) + P(R|B)P(B)$$

$$P(R) = (f_A(x)dx|x=5)\alpha + (f_B(x)dx|x=5)(1 - \alpha)$$

$$P(R|A) = \frac{1}{\sqrt{2\pi \times 3}} e^{-(5-6)^2/(2 \times 9)} dx = 0.1311dx$$

$$P(R|B) = \frac{1}{\sqrt{2\pi \times 2}} e^{-(5-4)^2/(2 \times 2)} dx = 0.1933dx$$

$$P(RA) = 0.1311dx \alpha$$

$$P(RB) = 0.1933(1 - \alpha)$$

$$P(R) = P(RA) + P(RB) = 0.1933 - 0.0622\alpha dx$$

Answer:

$$P(A|R) = \frac{P(AR)}{P(R)} = \frac{0.1311\alpha dx}{0.1933 - 0.0622\alpha dx} = \frac{0.1311\alpha}{0.1933 - 0.0622\alpha}$$

$$\frac{0.1311\alpha}{0.1933 - 0.0622\alpha} = 0.5$$

$$\alpha = 0.596$$

10. (a) **Solution:**

For each return number from 1 to 10 we can derive the probability

$$P(X = i) = 0.1$$

$$P(X = -1) = 0$$

Answer:

$$E[X] = \sum_{i=0}^9 0.1i = 4.5$$

(b) **Solution:**

The only chance that the function can return is when $\text{arr}[\text{mid}] = \text{key}$. So the result would be same as the previous

$$P(X = i) = 0.1 | 1 \leq i \leq 10$$

Answer:

$$E[X] = \sum_{i=0}^9 0.1i = 4.5$$

11. (a) **Solution:**

Since the total hash trial of the strings $3m = 72000$ are very high and the probability for each string hashed into certain bucket is very low $\frac{1}{8000}$. We can use binomial distribution function to derive the probability of each number of strings that hashed into a certain bucket. $PX = i \approx e^{-\lambda} \frac{\lambda^i}{i!}$
 $\lambda = 3m \frac{1}{n} = 72000 \times \frac{1}{8000} = 9$

Answer:

The probability that the first bucket has 0 strings hashed into it is

$$PX = 0 \approx e^{-9} \frac{9^0}{0!} = 1.234 \times 10^{-4}$$

(b) **Answer:**

The probability that the first bucket has 10 or fewer strings hashed to it is

$$PX \leq 10 = \sum_{i=0}^{10} e^{-9} \frac{9^i}{i!} = 0.706$$

(c) **Solution:**

For a bloom filter which have i bits are 1s, the probability $P(E)$ of a string that is reported in the set incorrectly is

$$P(E|\{X = i\}) = \left(\frac{i}{n}\right)^3$$

$$P(E\{X = i\}) = \frac{i^3}{n^3} \frac{e^{-\lambda} \lambda^i}{i!}$$

$$P(E) = \sum_{i=0}^{+\infty} P(E\{X = i\}) = \frac{1}{n^3} \sum_{i=0}^{+\infty} \frac{i^3 e^{-\lambda} \lambda^i}{i!}$$

$$P(E) = \frac{\lambda}{n^3} \sum_{i=1}^{+\infty} \frac{i^2 e^{-\lambda} \lambda^{(i-1)}}{(i-1)!} = \frac{\lambda}{n^3} \sum_{i=1}^{+\infty} \left(\frac{(i-1)^2 e^{-\lambda} \lambda^{(i-1)}}{(i-1)!} + \frac{2(i-1) e^{-\lambda} \lambda^{(i-1)}}{(i-1)!} + \frac{e^{-\lambda} \lambda^{(i-1)}}{(i-1)!} \right)$$

$$P(E) = \frac{\lambda}{n^3} (E[X^2] + 2E[X] - 1) = \frac{\lambda}{n^3} (\lambda(\lambda + 1) + 2\lambda + 1) = \frac{1}{n^3} (\lambda^3 + 3\lambda^2 + \lambda)$$

Answer:

$$P(E) = \frac{1}{8000^3} (9^3 + 3 \times 9^2 + 9) = 1.916 \times 10^{-9}$$