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1. (a) **Solution:**
There are only 2 possible cases while picking up 2 from total k prime numbers: different or same. So we can add the counts of the above 2 cases as the total combinations.
Answer:
There are
$\binom{k}{2} + \binom{k}{1} = \frac{k!}{2!(k-2)!} + k = \frac{k(k-1)}{2} + k$
ways of choosing 2 prime numbers less than N.

(b) **Solution:**
To solve this problem, we can create a model which has r idential divider and k sorted items (prime numbers). The number just after each divider would be considered as one of the selected numbers. The First part of the polynomial represents the counts of the items + dividers combination. Second part represents the counts of the cases that at least one divider locates at the end of the queue.
Answer:
There are
$\binom{k+r}{k,r} - \binom{k+(r-1)}{k,(r-1)} = \frac{(k+r)!}{r!k!} - \frac{(k+r-1)!}{(r-1)!k!} = \frac{k(k+r-1)!}{r!k!}$
ways of choosing r prime numbers less than N.

(c) **Solution:**
Since the prime numbers set only includes distinct items in the beginning. We can just use combination of k prime numbers taken r at a time.
Answer:
There are
$\binom{k}{r} = \frac{k!}{(k-r)!r!}$
ways of choosing r distinct prime numbers less than N.

2. (a) **Solution:**
This is a permutation of all the 26 distinct letters.
Answer:
There are
26!
ways for the 26 letters to be ordered if each letter appears exactly once and there are no other restrictions.

(b) **Solution:**
We can treat Q and U as a bundle and then the solution is the permutation of 25 items (24 letters and 1 bundle) multiply by permutation of 2 letters which are close to each other.
ways for the 26 letters to be ordered if each letter appears exactly once and the letters Q and U must be next to each other (but in any order).
Answer:
There are
25!2!
ways for the 26 letters to be ordered if each letter appears exactly once and the letters Q and U must be next to each other (but in any order).

(c) **Solution:**
There are $26 - 5 + 1$ gaps in queue of all the consonants, the 5 vowels can be considered 5 dividers been inerst into those gaps but one vowel at most in each gap. There are $\binom{26-5+1}{5}5!$ ways to select and permute 5 gaps to fit 5 vowels. The final result will be the permutation of vowels multiply by the permutation of consonants $(26 - 5)!$.

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Answer:
There are
$\binom{26-5+1}{5}5!(26-5)! = \binom{22}{5}5!(26-5)! = \frac{22!}{17!5!}5!21! = \frac{22!21!}{17!}$
ways for the 26 letters to be ordered if each letter appears exactly once and no two vowels can be next to each other.

(d) **Solution:**
We can make all the 5 vowels as a group and the permutation of the 26 consonants with that group is $(26-5+1)!$. The permutation within the vowels group is 5!.
Answer:
There are
$(26-5+1)!5!$
ways for the 26 letters to be ordered if each letter appears exactly ones and 5 vowels must be next to each other.

(e) **Solution:**
If the position of the three most common letters are fixed. For each case, for example, "E" at position 1, "A" at position 2 and "T" at position 3. The number of the permutation is decided by the permutation of the $26-3=23$ rest letters.
Answer:
There are
$(26-3)! = 23!$
ways for the 26 letters to be ordered if the position of the three most coomon letters (E,T and A) are fixed.

3. (a) **Solution:**
For each distinct card, there are 4 duplications. The answer should be the number of possible divisions of 208 total positions into 52 distinct groups of size 4.
Answer:
There are
$\frac{208!}{(4!)^{52}}$
distinct ways for the cards to be ordered.

(b) **Solution:**
There a 2 mutual exclusive cases for the combination of 2 cards, 2 different cards or same cards. For the 2 different cards case, there are $\binom{52}{2}$ ways of being dealt 2 cards. For the same cards case, there are just 52 ways.
Answer:
There are
$\binom{52}{2} + 52 = \frac{52!}{2!50!} + 52$
distinct ways of being dealt two cards.

(c) **Solution:**
There are only 5 good cards and the combination of them are very limited. Just as the previous question, we can consider 2 cases and the total number of choice is reduced from 52 to 5.
Answer:
There are
$\binom{5}{2} + 5 = 10 + 5 = 15$
ways for you to get two "good" cards.

4. (a) **Solution:**
To arrive the destination, the robot must move to the up $n-1$ times and move to the right $(m-1)$

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times. Then the question can be an equivalence to a permutation of $(n-1)$ "move to the up" items and $m-1$ "move to the right" items.
Answer:
There are
$\binom{(n-1)+(m-1)}{(n-1),(m-1)} = \frac{(n+m-2)!}{(n-1)!(m-1)!}$
distinct paths for the robot to take to the desination in cell (n, m).

(b) **Solution:**
If the first step of the robot must be moving to the right, then the number of paths equals to the case with $(n, m-1)$ grid.
Answer:
There are
$\binom{(n-1)+(m-2)}{(n-1),(m-2)} = \frac{(n+m-3)!}{(n-1)!(m-2)!}$
distinct paths for the robot to take to the destination in cell (n, m), if it must start by moving to the right.

(c) **Solution:**
There are 2 possible directions for the robot at the beginning, up or right. We can analyze the case that the robot start by moving to the right. Since the total direction change number is odd, that means the last direction change must be "right to up" and happens in the last column. In any cases, there is no other path but go straight to the destination after the last turn. So we can just consider the number of the combinations of the first 2 turns. The first turn must happen in one of the $m-2$ columns, which exclude the first and last column; the second turn must happen in one of the $n-2$ rows, which also exclude the first and last row. Then the total combinations of possible paths are $(m-2)(n-2)$. The situation of the case that the robot start by moving to the up is similar.
Answer:
There are
$(m-2)(n-2) + (n-2)(m-2) = 2(m-2)(n-2)$
distinct paths for the robot to take to destination in cell (n, m), if the robot changes direction exactly 3 times.

5. (a) **Solution:**
Since there are minimal investments for each company and each company must be invested, \$1, \$2, \$3, \$4 million can be assign to the respective company at first and then work out strategies based on rest of the \$10 million. Each \$1 million can choose any one of the total 4 companies.
Answer:
There are
$(20-10)^4$
different investment strategies are available if an investment must be made in each company.

(b) **Solution:**
We can consider 5 cases including 4 different cases that there is one distinct company have no investment and one more case that all the companies have investment.
Answer:
There are
$(20-10)^4 + \sum_{i=6}^{9}(20-i)^3$
different investment strategies are available if investments must be made in at least 3 of the 4 companies.

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6. **Solution:**
Define summation of all the vectors $m = \sum_{i=1}^{n} X_i$. Since $m \leq k$, there are $k+1$ possible value of m. For each distinct m, we can consider there are $n-1$ dividers insert into m items. The total possible combination of particular m is $\binom{m+n-1}{n-1,m+1} = \binom{m+n-1}{n-1,m}$. Then for all the possible m, we can have $\sum_{i=0}^{k} \binom{m+n-1}{n-1,m}$ different vectors.
Answer:
The number of vectors is
$\sum_{i=0}^{k} \binom{m+n-1}{n-1,m} = \sum_{i=0}^{k} \frac{(m+n-1)!}{(n-1)!(m)!}$

7. **Solution:**
The total number of requests which can be "pre-assigned"is $k = \sum_{i=1}^{r} m_i$. The rest of identical requests are distributed to r servers and the number of ways can be calculated by using $(r-1)$ dividers inserted into the queue of $(n-k)$ requests.
Answer:
There are
$\binom{n-k+r-1}{n-k,r-1} = \frac{(n-k+r-1)!}{(n-k)!(r-1)!}$ where $k = \sum_{i=1}^{r} m_i$
ways for the requests to be distributed.

8. (a) **Solution:**
Sample set is
$S = \{\frac{52!}{47!}$ permutations of 5 cards from total 52$\}$

For each suit, the event set is
$E_n = \{\frac{13!}{8!}$ permutations of 5 cards from total 13 cards in same suit$\}$
and all the 4 event set are exclusive.
Answer:
The probability of being dealt a flush is
$P(E) = \frac{|E_1 \bigcup E_2 \bigcup E_3 \bigcup E_4|}{|S|} = \frac{(13\times14\times12\times11\times10\times9)\times4}{52\times51\times50\times49\times48} = \frac{617760}{311875200} = 0.00198$

(b) **Solution:**
For the number of elements of the event $E = \{$one pair in 5 cards$\}$, we can select 1 numeric value from 13 numbers and pick up 2 cards in the total 4. Then we can add another 3 cards from the rest 48 cards. After all the 5 cards are selected, we can calculate the permutation of the 5 cards.
$|E| = 13 \cdot \binom{4}{2} \cdot \binom{48}{3} \cdot 5!$
Answer:
The probability of being dealt a one pair is
$P(E) = \frac{|E|}{|S|} = \frac{13\times6\times48\times47\times46\times\frac{1}{6}\times5!}{311875200} = 0.519$

(c) **Solution:**
For the number of elements of the event $E = \{$two pair in 5 cards$\}$, we can select 2 numeric value from 13 numbers and pick up 2 cards in the total 4 of each value. Then we can add the last card from the rest 44 cards. After all the 5 cards are selected, we can calculate the permutation of the 5 cards.
$|E| = \binom{13}{2} \cdot \binom{4}{2}^2 \cdot 44 \cdot 5!$
Answer:
The probability of being dealt a two pair is

$P(E) = \frac{|E|}{|S|} = \frac{\frac{13\times 12}{2}\times 6^2\times 44\times 5!}{311875200} = 0.0475$

(d) **Solution:**
For the number of elements of the event $E = \{\text{three of a kind in 5 cards}\}$, we can select 1 numeric value from 13 numbers and pick up 3 cards in the total 4. Then we can add the last 2 cards from the rest 48 cards. After all the 5 cards are selected, we can calculate the permutation of the 5 cards.
$|E| = 13 \cdot \binom{4}{3} \cdot \binom{48}{2} \cdot 5!$
Answer:
The probability of being dealt three of a kind is
$P(E) = \frac{|E|}{|S|} = \frac{13\times 4\times 48\times 47\times \frac{1}{2}\times 5!}{311875200} = 0.0226$

(e) **Solution:**
For the number of elements of the event $E = \{\text{four of a kind in 5 cards}\}$, we can select 1 numeric value from 13 numbers and pick up all the cards. Then we can add the last 1 cards from the rest 48 cards. After all the 5 cards are selected, we can calculate the permutation of the 5 cards.
$|E| = 13 \cdot 48 \cdot 5!$
Answer:
The probability of being dealt four of a kind is
$P(E) = \frac{|E|}{|S|} = \frac{13\times 48\times 5!}{311875200} = 0.000240$

9. (a) Each die rolling have 6 possible resluts, the number of element in sample set
$|S| = 6^6$
For the event $E = \{\text{three different numbers and twice each}\}$, we can choose 3 different numbers from 6 numbers and then put those 3 item gourp, which has 2 element in each, into 6 buckets.
$|E| = \binom{6}{3}\binom{6}{2,2,2} = 1800$
Answer:
The possibility that we will roll three different numbers, twice each is
$P(E) = \frac{|E|}{|S|} = \frac{1800}{6^6} = 0.0386$

(b) For the event $E = \{\text{some number exactly 4 times}\}$, we can choose 1 number from 6 numbers and then choose the other 2 from the rest of 5 numbers. After choose all the numbers, put them into 6 bucket.
$|E| = \binom{6}{1}\binom{5}{2}\binom{6}{4,1,1} = 1800$
Answer:
The possiblility that we will roll some number exacly 4 times is
$P(E) = \frac{|E|}{|S|} = \frac{1800}{6^6} = 0.0386$

10. (a) **Solution:**
The sample space size equals to the number of all permutations of n integers, $|S| = n!$. The event space is the permutations that make the BSD have completely degenerate structure. Since we can only choose the maximum or the minimum from the rest of all the numbers for each node in sequence. The event space size is $|E| = 2^{n-1}$
Answer:
The possibility that the resulting BST will have completely degenerate structure is
$P(E) = \frac{|E|}{|S|} = \frac{2^{n-1}}{n!}$

(b) **Solution:**
Usually, we can just use program to calculate the possibility of forming a completely degenerate

BST of each n and find the first one have the possibility lower than 0.01. Without help of computer, we can still estimate it. First, $n!$ must higher than 100, then $n \geqslant 6$, $P(E) = 0.0127$ when $n = 7$,$P(E) = 0.0032$ when $n = 8$.

Answer:

The smallest value of n for which the probability of forming a completely degenerate BST is less than 0.01 is

$n = 8$ while $P(E) = 0.0032$

11. (a) **Solution:**

The sample space is all the distinct password permutations, then $|S| = n!$. If the first successful login will be exactly on her k-th try, that means the event space is the rest $(n-1)$ distinct password permutation. The event space size is $|E| = (n-1)!$.

Answer:

The probability that her first successful login will be exactly on her k-th try is

$P(E) = \frac{|E|}{|S|} = \frac{(n-1)!}{n!}$

(b) **Solution:**

Since the passwords are never deleted, the probability of the event that the hacker login successfully in the i-th try is $P(E_i) = \frac{1}{n}$. The probability of login successfully on exactly k-th equals to the probability of all the first $k-1$ tries are failed and the k-th try is successful.

Answer:

The probability of $E_{successful} = \{$the hacker first successful login will be exactly on her k-th try$\}$ is

$P(E_{successful}) = P(E_1^c E_2^c \dots E_{k-1}^c E_k) = P(E_1^c)P(E_2^c) \dots P(E_k^c - 1)P(E_k) = (\frac{n-1}{n})^{k-1}\frac{1}{n}$

12. (a) **Solution:**

From the description in the question, we can define

$J = \{$Students in Java class$\}$

$C = \{$Students in C++ class$\}$

$Y = \{$Students in Python class$\}$

Then we can calculate the probability of the student in each exclusive condition

$P(JYC^c) = P(JY) - P(JYC) = \frac{5}{100} - \frac{3}{100} = 0.02$

$P(JCY^c) = P(JC) - P(JYC) = 0.09$

$P(YCJ^c) = P(YC) - P(JYC) = 0.04$

$P(YC^cJ^c) = P(Y) - P(YCJ^c \cup YJC^c \cup YJC)$

$= P(Y) - (P(YCJ^c) + P(YJC^c) + P(YJC)) = \frac{18}{100} - (0.04 + 0.02 + 0.03) = 0.09$

$P(CY^cJ^c) = P(C) - P(CYJ^c \cup CJY^c \cup YJC) = 0.10$

$P(JY^cC^c) = P(J) - P(JYC^c \cup JCY^c \cup YJC) = 0.13$

The the probability of a student chooses at least one of the programming classes is

$P(J \cup Y \cup C) = P(JYC^c) + P(JCY^c) + P(YCJ^c) + P(YC^cJ^c) + P(CY^cJ^c) + P(JY^cC^c) + P(JCY) = 0.5$

So the probability of a student do not choose any one of the 3 classes is

$P(Y^cJ^cC^c) = 1 - P(J \cup Y \cup C) = 0.5$ We can also use the proposition about the probability of the union of multiple events

$P(Y \cup J \cup C) = P(Y) + P(J) + P(C) - P(YJ) - P(YC) - P(JC) + P(YJC) = 0.5$

The reason that all the individual probabilities are calculated is just for the convenience to the following questions. to get the probability of a student that chooses at least on of the classes

Answer:

The probability that he or she is not in any of the 3 programming classes is

$P(Y^cJ^cC^c) = 1 - P(J \cup Y \cup C) = 0.5$

(b) **Solution:**

We can use the conclusion of the previous question directly

Answer:
The probability that he or she is taking exactly one of the three programming classes is
$P(E) = P(JY^cC^c) + P(CY^cJ^c) + P(YJ^cC^c) = 0.32$

(c) **Solution:**
The probability of each student that is taking at least one programming class is
$P(E_i) = P(Y^cJ^cC^c) = 1 - P(J \cup Y \cup C) = 0.5$
Answer:
The probability that at least on of the chosen students is taking at least one programming class is
$P(E_1 \cup E_2) = P(E_1) + P(E_2) - P(E_1E_2) = 0.75$

13. **Solution:**
The permutations of all the $(M+N)$ bits can be a sample space having equally likely outcomes.
$|S| = \binom{M+N}{M,N} = \frac{(M+N)!}{M!N!}$
The event space can be determined by 2 parts, permutation fo the first r bits and the permutation of the rest bits.
For the first part, the event space size can be considered as choose k positions to fit '1' in them from the total r positions $|E_1| = \binom{r}{k}$.
For the second part, for each particular case in E_1, the event space size can be considered as to choose $N-k$ positions to fit '1' in them from the total $M+N-r$ positions $|E_2| = \binom{M+N-r}{N-k}$
$|E| = |E_1| \cdot |E_2| = \binom{r}{k}\binom{M+N-r}{N-k}$
Answer:
The probability that the first r bit of the received messaged contain exactly k 1's is
$P(E) = \frac{|E|}{|S|} = \frac{\binom{r}{k}\binom{M+N-r}{N-k}}{\binom{M+N}{M,N}}$

14. **Solution:**
There are 3 possible situation when generate 2 integers in the range 1 to 12
$E_g = \{$The second value is greater than the first one$\}$
$E_s = \{$The second value is smaller than the first one$\}$
$E_e = \{$The second value is equal to the first one$\}$
It is easy to calculate
$P(E_e) = \frac{12}{12^2} = \frac{1}{12}$
Since the probability of E_g and E_s are the same we can have following equations
$P(E_g) + P(E_s) + P(E_e) = 1$
$P(E_g) = P(E_s)$
Answer:
The Probability that the second randomly generated integer has a value that is greater than the first is
$P(E_g) = \frac{1-P(E_e)}{2} = 0.458$

15. **Solution:**
For the event space size we can consider it in 2 parts, pickup k strings in the first bucket from m strings $\binom{m}{k}$. And then put the rest of strings into $(N-1)$ buckets.
$|E| = \binom{m}{k}(N-1)^{m-k}$
Answer:
$P(E) = \frac{|E|}{|S|} = \frac{\binom{m}{k}(N-1)^{m-k}}{N^m}$

More Discussion
We can solve this question in an other perspective. We can build events
$E_n=$ All the elements in particular group with k strings go into first bucket and the rest N-k

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string go into other buckets
$P(E) = P(E_1 \cup E_2 \cup \ldots \cup E_n)$
Since all the E_n are mutually exclusive.
$P(E) = \sum\limits_{i=1}^{n} P(E_i)$
$P(E_i) = (\frac{1}{N})^k(\frac{N-1}{N})^{m-k}$
$n = \binom{m}{k}$
$P(E) = nP(E_i) = \binom{m}{k}(\frac{1}{N})^k(\frac{N-1}{N})^{m-k}$
which is as same as the previous conclusion.