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CS109

April 24, 2017

Problem Set #1 Solutions

1.

a. k^2. Think of it as a two step experiment: chose the first prime, then chose the second. Since the primes are used in distinct ways, the order of the choice matters.
b. k^r. Same as above, but now its an r step experiment.
c. $k!/(k-r)! = (k)(k-1)(k-2)\ldots(k-r-1)$

2.

a. 26!. Any arrangement of 26 letters is possible.

b. 2! · 25!. Since Q and U have to be together, you can think of them as "one" letter in the arrangement (giving 25!), then Q and U can be permuted in 2! ways.

c. (5!21!)*((17+5) choose 5) Split the problem into 3 parts

   1. Order 5: (5!)
   2. Order 21 consonants (21!)
   3. Split the 21 consonants into 6 groups with the middle 4 groups having at least one consonant. Since the middle four groups get one consonant there are 17 left to place: ((17+5) choose 5)

d. 22! · 5!. The 5 vowels have to be together can be thought of as "one" entity, so there are 22 entities (21 consonants and the vowel group) who can be permuted. The 5 vowels can then be permuted within their block.

e. 23! Since the three letters are fixed, it is as if you are permuting, but only with 23 letters.

3.

a. $208! / (4!)^{52}$. This is a permutation with repetitions question. There are 208 objects. For every card in the deck of 52, there is a group of four indistinguishable cards.

b. Sum of integers from 1 up to 52 = (52)(53)/2 = 1378. First, the answer is not (52 choose 2) = 1326 which is the number of ways of choosing two cards without replacement since we have multiple decks and you could get a hand which has two Ace of spaces (for example). Though you could take that number and add 52. Nor is it (52)*(52) since the order of the cards in your hand does not matter.

   In a grid which shows all 52^2 pairs of card values you want the upper triangle (as the lower triangle has matches that already exist in different orders in the upper triangle.
   Let us consider each card having a value 1 through 52. Card 1 can be matched with any of the other 52 values. Card 2 can be matched with 51 other values, since we have already counted the match of card 1 and card 2. And so on.

c. Sum of the integers 1 through 20 = (20)(21)/2 = 210. This is the same as part (b) but we only have 20 distinct good cards which are 10,J,Q,K,A (with each of the four suits).

4.

a. (n+m-2)!/(n-1)!(m-1)! We are going to order (m – 1) right moves and (n – 1) up moves. Right moves are indistinguishable from right moves and up moves are indistinguishable from up moves.

b. (n+m-3)!/(n-1)!(m-2)! Same as above with one less right move.

c. 2(n-3)(m-3). There are two possible cases: [Ups, Rights, Ups, Rights] or [Rights, Ups, Rights, Ups]. They are mutually exclusive so we can sum up the number of paths from the two cases.

Case 1: [Ups, Rights, Ups, Rights]
Lets fix a few Us (Ups) and Rs (Rights) to make sure we match the desired format:
U <more up moves> R <more right moves> U <more up moves> R <more right moves>

   a. Substep 1: place any number of the (n-3) up moves into the first slot for more up moves, the rest will go in the second slot: (n-3) ways.
   b. Substep 2: You can place any number of the (m-3) right moves into the first slot for more right moves, the rest will go in the second slot: (m-3).

   In total: (n-3)(m-3)

Case 2: [Rights, Ups, Rights, Ups]
Comes to the same number (n-3)(m-3) by a symmetric argument.

5.

d. $\binom{13}{3}$ = 286. Since you must invest the minimum in all the opportunities, you must invest 1+2+3+4 = \$10 million. Then you have \$10 million left to invest in the 4 opportunities, which has the same number of possibilities as solutions to $x_1 + x_2 + ... + x_4 = 10$.

e. $\binom{13}{3}+\binom{13}{2}+\binom{14}{2}+\binom{15}{2}+\binom{16}{2}$ = 680. First, you still need to consider all the cases where you invest in all 4 opportunities (same as in part (a)), then you can use a similar analysis from part (a) to consider the number of possibilities if you (i) didn't invest in company 1, (ii) didn't invest in company 2, (iii) didn't invest in company 3, and (iv) didn't invest in company 4, respectively. Summing all these possibilities give the complete answer.

6. $\sum_{j=0}^{k}\binom{j+n-1}{n-1}$. Since $\sum_{i=1}^{n} x_i \le k$, we consider all possibilities $\sum_{i=1}^{n} x_i = j$, where $0 \le j \le k$. For any particular value j, the number of non-negative integer solutions to $x_1 + x_2 + ... + x_n = j$ is given by $\binom{j+n-1}{n-1}$. To count all possible vectors, we sum this quantity over all values of j from 0 to k.

Alternatively, an equivalent answer is: $\binom{k+n}{n}$. This answer comes from conceptually adding an extra element x_{n+1} to the vector to represent $k - \sum_{i=1}^{n} x_i$, and then counting the number of non-negative integer solutions to $x_1 + x_2 + ... + x_n + x_{n+1} = k$. Note that if we were to subtract x_{n+1} from both sides of that equation, we get $x_1 + x_2 + ... + x_n = k - x_{n+1}$, which is equivalent to $x_1 + x_2 + ... + x_n \le k$, since x_{n+1} is a non-negative integer $\le k$.

7. $\binom{n-k+r-1}{r-1}$, letting $k = \sum_{i=1}^{r} m_i$. If you consider that k requests must be allocated according to the constraints that the i-th server receives at least m_i requests, it leaves $n - k$ requests to distribute in the r servers. This number of ways to do this is the same as finding the number of solutions to $x_1 + x_2 + ... + x_r = n - k$, where each of the $x_i \ge 0$.

8.

a. $\binom{4}{1}\binom{13}{5} \Big/ \binom{52}{5}$. Choose 1 of the 4 suits for the flush, then choose 5 of the 13 cards in that suit.

b. $\binom{13}{1}\binom{4}{2}\binom{12}{3}\binom{4}{1}^3 \Big/ \binom{52}{5}$. Choose 1 of the 13 card ranks for the rank of the pair. Then choose 2 of the 4 cards of that rank to form the pair. Then choose 3 of the remaining 12 ranks, and choose 1 of the 4 cards at each of those ranks to form the rest of the hand.

c. $\binom{13}{2}\binom{4}{2}^2\binom{44}{1} \Big/ \binom{52}{5}$. Choose 2 of the 13 card ranks for the ranks for the two pairs, and choose 2 of the 4 cards of each rank to form each pair. Then choose 1 of the 44 (=52–8) cards that does not share the same rank as one of the 2 chosen pairs to complete the hand.

d. $\binom{13}{1}\binom{4}{3}\binom{12}{2}\binom{4}{1}^2 \Big/ \binom{52}{5}$. Choose 1 of the 13 card ranks for the rank of the three of a kind, and choose 3 of the 4 cards of that rank to form the three of a kind. Then choose 2 of the remaining 12 ranks, and choose 1 of the 4 cards at each of those ranks to form the rest of the hand.

e. $\binom{13}{1}\binom{4}{4}\binom{48}{1} \Big/ \binom{52}{5}$. Choose 1 of the 13 card ranks for the rank of the four of a kind, and choose 4 of the 4 cards of that rank to form the four of a kind. Then choose 1 of the 48 (=52–4) cards that does not share the same rank as the four of a kind to complete the hand.

9.

a. $\binom{6}{3}\binom{6}{2,2,2}\Big/6^6 = \binom{6}{3}\binom{6}{2}\binom{4}{2}\binom{2}{2}\Big/6^6$. First, select the 3 different numbers (out of 6) that are each rolled twice. Then, we have a multinomial coefficient $\binom{6}{2,2,2}$, representing that in the 6 rolls there are 3 sets of 2 indistinguishable rolls (i.e., each number is rolled twice). Since the die rolls are distinct, there are 6^6 total outcomes for rolling the die 6 times.

b. $\dfrac{\binom{6}{1}\binom{6}{4}5^2}{6^6} = \dfrac{6\binom{6}{4}}{6^4}\cdot\dfrac{5^2}{6^2}$. First, we select 1 of the 6 numbers that will appear exactly 4 times (call this number x). Then, we select the 4 rolls (out of 6) where x is rolled. The other 2 rolls can be any one of the 5 other numbers on the die that are not x. Similar to part (a), there are 6^6 total outcomes for rolling the die 6 times.

10.

a. $\dfrac{2^{n-1}}{n!}$. First, note that there are n! orderings of the values 1 through n we insert into the BST. Now, the only way to produce a completely degenerate BST is to have every successive insertion be either the minimal or maximal value of the values remaining to be inserted. This means we have one of 2 choices for every successive element we insert into the BST (minimal or maximal), except for the last element inserted (since it is both the minimal and maximal remaining element). Since we insert n elements total, we have 2^{n-1} insertion orderings that can produce a completely degenerate BST.

b. n = 8 is the smallest value such that $\dfrac{2^{n-1}}{n!} < 0.01$. Specifically, $\dfrac{2^7}{8!} \approx 0.003175$. Note how rapidly the probability of producing a degenerate BST decreases with the number of elements inserted.

11.

a. $1/n$. You can think of this as analogous to the passwords have been arranged linearly, with a numbering from 1 to n on the passwords. We want to determine the probability that the password that successfully logs in is given the number k. This is simply $1/n$, since the number k is equally likely to be assigned to any of the n passwords.

b. $\dfrac{(n-1)^{k-1}}{n^k}$. To successfully log in on exactly the k-th try, the hacker would have had to pick one of the wrong passwords (where the probability of picking one of the wrong passwords on a particular attempt is $\dfrac{n-1}{n}$) on each of the $k - 1$ previous tries, yielding

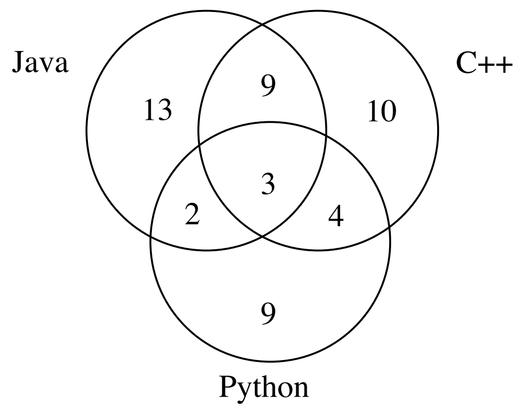
$\frac{(n-1)^{k-1}}{n^{k-1}}$. Then, on the k-th try, the hacker would have a $\frac{1}{n}$ probability of selecting the right password, giving the final answer.

12.

a. 1/2. We compute this by noting the probability that a student is in at least 1 programming class is P(Java ∪ C++ ∪ Python) = (27 + 26 + 18 – 12 – 5 – 7 + 3)/100 = 1/2. The probability that a student is not in any programming class is simply:

1 – P(Java ∪ C++ ∪ Python) = 1 – 1/2 = 1/2.

b. 32/100. We compute this by explicitly forming the Venn diagram of class enrollments (shown below), and then summing cell counts (13, 10, and 9) that are only a member of one set.



c. 149/198 ≈ 0.7525. As shown in part (a), the probability that a student is not taking any programming class is 1/2, which means there are 50 out of 100 students who are not taking any programming class. Now we randomly draw 2 students and determine the probability that *neither* one is taking a programming class as: (50/100) * (49/99). So, the probability that at least one of the drawn students is taking a programming class is:

1 – (50/100) * (49/99) = 1 – 49/198 = 149/198.

13. $\binom{N}{k}\binom{M}{r-k}\Big/\binom{M+N}{r}$ or equivalently $\binom{r}{k}\binom{M+N-r}{N-k}\Big/\binom{M+N}{N}$. The first expression gives the answer by focusing on forming the first r bits of the received message. If we were to order all the N 1's with integers from 1 to N, we could select k of them to be in the first r bits of the message, yielding $\binom{N}{k}$ possibilities. Similarly, if we were to order all the M 0's with integers from 1 to M, we could select $(r-k)$ of them to be in the first r bits of the message, yielding $\binom{M}{r-k}$ possibilities. Multiplying, we get $\binom{N}{k}\binom{M}{r-k}$ total ways of arranging the first r bits, having exactly k 1's. The number of ways to form the first r bits of the message (with no constraints), is simply to think of numbering the $M + N$ bits with integers from 1 to $(M + N)$, and then choosing r of these integers to form the first r bits, yielding $\binom{M+N}{r}$ for the denominator. Alternatively, we can get a mathematically equivalent answer to the problem (the second solution given above) by determining how to construct the whole binary string, so as to have k 1's in the first r bits. Consider the message being sent as have $M + N$ 'slots' to fill with 0s and 1s. The denominator is determined by choosing N of the $(M + N)$ slots to put 1's into. The numerator is determined by choosing k of the first r slots to put 1's in, then choosing $(N - k)$ of the remaining $(M + N - r)$ slots to put 1's in. Once all slots for 1's are determined, the 0's uniquely fill the remaining unfilled slots.

14. 66/144 = 11/24. One way we can solve this problem is to enumerate all the possibilities. Namely, if the first number generated is a 1 (with a 1/12 probability), then there are 11 possibilities for the second number to be greater (i.e., 11/12 probability). Similarly, if the first number is a 2, then there are 10 possibilities for the second number to be greater, etc. Summing all these possibilities yields: (1/12)[11/12 + 10/12 + ... + 1/12] = 66/144.

Another way to solve the problem is to use symmetry. We start with the equation:
1 = P(1st number > 2nd number) + P(2nd number > 1st number) + P(1st and 2nd are same)

By symmetry, we note that P(1st number > 2nd number) = P(2nd number > 1st number), so we rewrite the equation above as:
1 = 2[P(2nd number > 1st number)] + P(1st and 2nd are same)

Noting that P(1st and 2nd are same) = 1/12, we have:
1 = 2[P(2nd number > 1st number)] + 1/12
2[P(2nd number > 1st number)] = 11/12
P(2nd number > 1st number) = 11/24 = 66/144

This is clearly the same as above, but does not require us to sum a series. Good times.

15. $\dfrac{\binom{m}{k}(N-1)^{m-k}}{N^m}$. Choose k of the m strings to get hashed into the first bucket. Then each of the remaining $(m - k)$ strings can get hashed into any of the remaining $N - 1$ buckets. Recall that that all N^m arrangements are equally likely to get the denominator.