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1. (a) **Solution:**
We calculate after how many rounds, the game provide will use up all their money.
$m = \lfloor log_2 X \rfloor$
Then we can derive the expectation of payoff of the game by considering 2 parts, normal payoff and the limited payoff after provider run out of money, after you win more or equal to 2^m, no matter what happend, you will win X
$E[Y] = \sum_{n=0}^{m} (\frac{1}{2})^{n+1} 2^n + \frac{1}{2}^{n+1} X = (m+1)\frac{1}{2} + \frac{1}{2}^{m+1} X$
Answer:
$m = 2$
$E[Y] = 3 \times \frac{1}{2} + \frac{1}{2}^3 \times 5 = 2.125$

(b) **Answer:**
$m = 8$
$E[Y] = 9 \times \frac{1}{2} + \frac{1}{2}^8 \times 500 = 5.477$

Answer:
$m = 12$
$E[Y] = 13 \times \frac{1}{2} + \frac{1}{2}^{13} \times 4096 = 7$

2. **Solution:**
The expectation of new users in 5 minutes is $E(X) = 2$. The expectation of gain for the trip can be preseted as
$E[(X+1) \times \$6 - \$7] = E[X] \times \$6 - \$1$
Answer:
The expectation of Lyft to make in this trip is
$E[X] \times \$6 - \$1 = \$11$

3. (a) **Solution:**
The possibility that x bit are corrupted is
$P(X = x) = \binom{2n}{x} p^x (1-p)^{(2n-x)}$
Answer:
The probability that the message ss is received without any corruption is
$P(X = 0) = (1 - 0.05)^8 = 0.663$

(b) **Solution:**
For each first string s which has any corruption bit, if the second time the corruption bit(s) are exactly the same, we can not detect that problem
Answer:
The probability that we can not detect the corrupted bit(s)
$P(E) = \sum_{x=1}^{n} \binom{n}{x} (p^x (1-p)^{(n-x)})^2$
$P(E) = \sum_{x=1}^{4} \binom{4}{x} (0.05^x (1-0.05)^{(4-x)})^2 = 0.0074$

(c) **Solution:**
The probability that recipient can detect the corruption took place is
$P(F) = 1 - P(X = 0) - P(E)$
Answer:
$P(F) = 1 - (1-p)^{2n} - \sum_{x=1}^{n} \binom{n}{x} (p^x (1-p)^{(n-x)})^2 = 1 - 0.663 - 0.0074 = 0.329$

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4. **Solution:**
There are 2 cases that the jury renders a correct decision, votes guilty when the defendants are actually guilty or votes innocent and the defendants are actally innocent.
The probability of the defendants are actually guilty is
$P(G) = 0.75$
For each juror, the probability that votes guilty $P(H)$ in different conditions is
$P(H|G^c) = 0.1$
$P(H^c|G) = 0.2$
The probability that the jury decides the defendants are guilty is $P(F)$
Then the probability make correct decision when the defenant is actually guilty (less than 4 juror votes innocent) is
$P(F|G) = \sum\limits_{n=0}^{3} \binom{12}{i} P(H^c|G)^i P(H|G)^{12-i}$
$P(F|G) = \sum\limits_{n=0}^{3} \binom{12}{i} 0.2^i 0.8^{12-i} = 0.7946$
$P(FG) = P(F|G)P(G) = 0.5959$
Then the probability make correct decision when the defenant is actually innocent (1 - P(less than 4 juror votes innocent)) is
$P(F^c|G^c) = 1 - \sum\limits_{n=0}^{4} \binom{12}{i} P(H^c|G^c)^i P(H|G^c)^{12-i}$
$P(F^c|G^c) = 1 - \sum\limits_{n=0}^{4} \binom{12}{i} 0.9^i 0.1^{12-i} = 1 - 3.4 \times 10^{-6} = 1$
$P(F^c) = P(F^c|G^c)P(G^c) = 0.25$
Answer:
The probability that the jury renders a correct decision is
$P(FG) + P(F^cG^c) = 0.5959 + 0.25 = 0.8459$
The percentage of defendant found guilty by the jury is
$P(F) = P(FG) + P(FG^c) = 0.5959 + 0 = 59.6\%$

5. **Solution:**
Define event
$X_i = \{$A person computer crash i times in the month$\}$
$E = \{$The patch has had an effect on the user's computer$\}$
Then, based on the Poisson distribution
$P(X_2|E) = e^{-3}\frac{3^2}{2!} = 0.224$
$P(X_2|E^c) = e^{-5}\frac{5^2}{2!} = 0.084$
Answer:
$P(X_2E) = P(X_2|E)P(E) = 0.224 \times 0.75 = 0.168$
The probability that the patch has had an effect on the user's computer is
$P(E|X_2) = \frac{P(X_2E)}{P(X_2)} = \frac{0.168}{0.224+0.084} = 0.545$

6. **Solution:**
$E[\sum\limits_{i=1}^{k} X_i] = \sum\limits_{i=1}^{k} E[X_i]$
$E[X_i] = 1 \times P(X_i = 1) + 0 \times P(X_i = 0) = 1 \times (1 - (1 - p_i)^n) + 0 \times (1 - p_i)^n$
Answer:
The expected number of buckets that have at least one string hashed into them is
$E[\sum\limits_{i=1}^{k} X_i] = \sum\limits_{i=1}^{k} (1 - (1 - p_i)^n)$

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7. **Solution:**
The CDF of given uniformly random distribution is
$$F(a) = \begin{cases} 0 & a \leq 0 \\ \frac{a}{n} & 0 < a < n \\ 1 & a \geq n \end{cases}$$
Only in the cases that $x < 0.2n$ or $x > 0.8n$, the shorter piece is less than 1/4th of the longer one.
Answer:
$P(x < 0.2n) + P(x > 0.8n) = F(0.2n) + (1 - F(0.8n)) = \frac{0.2n}{n} + (1 - \frac{0.8n}{n}) = 0.4$

8. (a) **Solution:**
$1 = \int_{-\infty}^{+\infty} f(x)dx = \int_{-1}^{+1} c(3 - 2x^2)dx$
$1 = c(3x - \frac{2}{3}x^3) \Big|_{-1}^{1} = \frac{14}{3}c$
Answer:
$c = \frac{3}{14}$

(b) **Answer:**
$F(a) = PX < a = \int_{-\infty}^{a} f(x)dx$
$$F(a) = \begin{cases} 0 & a \leq -1 \\ (3a - \frac{2}{3}a^3)c - (-\frac{7}{3}c) & -1 < a < 1 \\ 1 & a \geq 1 \end{cases}$$
$$F(a) = \begin{cases} 0 & a \leq -1 \\ (\frac{9}{14}a - \frac{1}{7}a^3) + \frac{1}{2} & -1 < a < 1 \\ 1 & a \geq 1 \end{cases}$$

(c) **Answer:**
$E[X] = \int_{-\infty}^{+\infty} xf(x)dx$
$E[X] = \int_{-1}^{+1} x\frac{3}{14}(3 - 2x^2)dx$
$E[X] = \frac{9}{28}x^2 - \frac{3}{28}x^4 \Big|_{-1}^{1} = 0$

9. **Solution:**
$P(A) = \alpha$
$P(B) = 1 - \alpha$
Since, $P(A) = P(B^c)$
$P(R) = P(RA) + P(RB) = P(R|A)P(A) + P(R|B)P(B)$
$P(R) = (f_A(x)dx|x = 5)\alpha + (f_B(x)dx|x = 5)(1 - \alpha)$
$P(R|A) = \frac{1}{\sqrt{2\pi}\times 3}e^{-(5-6)^2/(2\times 9)}dx = 0.1311dx$
$P(R|B) = \frac{1}{\sqrt{2\pi}\times 2}e^{-(5-4)^2/(2\times 2)}dx = 0.1933dx$
$P(RA) = 0.1311dx\,\alpha$
$P(RB) = 0.1933(1 - \alpha)$
$P(R) = P(RA) + P(RB) = 0.1933 - 0.0622\alpha\,dx$
Answer:
$P(A|R) = \frac{P(AR)}{P(R)} = \frac{0.1311\alpha\,dx}{0.1933 - 0.0622\alpha\,dx} = \frac{0.1311\alpha}{0.1933 - 0.0622\alpha}$
$\frac{0.1311\alpha}{0.1933 - 0.0622\alpha} = 0.5$
$\alpha = 0.596$

10. (a) **Solution:**
For each return number from 1 to 10 we can derive the probability
$P(X = i) = 0.1$

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$P(X=-1)=0$
Answer:
$E[X]=\sum_{i=0}^{9}0.1i=4.5$

(b) **Solution:**
The only chance that the founction can return is when arr[mid] = key. So the result would as same as the previous
$P(X=i)=0.1|1\leq i\leq 10$
Answer:
$E[X]=\sum_{i=0}^{9}0.1i=4.5$

11. (a) **Solution:**
Since the total hash trial of the strings $3m=72000$ are very high and the probability for each string hashed into certain bucket is very low $\frac{1}{8000}$. We can use binomial distribution function to derive the probability of each number of strings that hashed into a certain bucket. $P\{X=i\}\approx e^{-\lambda}\frac{\lambda^i}{i!}$
$\lambda=3m\frac{1}{n}=72000\times\frac{1}{8000}=9$
Answer:
The probability that the first bucket has 0 strings hashed into it is
$P\{X=0\}=\approx e^{-9}\frac{9^0}{0!}=1.234\times10^{-4}$

(b) **Answer:**
The probability that the first bucket has 10 or fewer strings hashed to it is
$P\{X\leq 10\}=\sum_{i=0}^{10}e^{-9}\frac{9^i}{i!}=0.706$

(c) **Solution:**
For a bloom filter which have $X=i$ bits are 0s, the probability $P(E)$ of a string that is reported in the set incorrectly is
$P(E|\{X=i\})=(1-\frac{i}{n})^3$
For each bit, the probability that remains to 0s after 24000 strings added is
$p=1.234\times10^{-4}$
Let $\lambda=np=0.9873$
The probability that $X=i$ bucket have no string been hashed into is
$P(\{X=i\})=\frac{e^{-\lambda}\lambda^i}{i!}$
$P(E\{X=i\})=\frac{(n-i)^3}{n^3}\frac{e^{-\lambda}\lambda^i}{i!}$
$P(E)=\sum_{i=0}^{+\infty}P(E\{X=i\})=\sum_{i=0}^{+\infty}\frac{e^{-\lambda}\lambda^i}{i!}-\frac{1}{n}\sum_{i=0}^{+\infty}\frac{3ie^{-\lambda}\lambda^i}{i!}+\frac{1}{n^2}\sum_{i=0}^{+\infty}\frac{3i^2e^{-\lambda}\lambda^i}{i!}-\frac{1}{n^3}\sum_{i=0}^{+\infty}\frac{i^3e^{-\lambda}\lambda^i}{i!}$

$P(E)=1-\frac{3}{n}E[X]+\frac{3}{n^2}E[X^2]-\frac{1}{n^3}E[X^3]$
$\frac{1}{n^3}E[X^3]=\frac{\lambda}{n^3}\sum_{i=1}^{+\infty}\frac{i^2e^{-\lambda}\lambda^{(i-1)}}{(i-1)!}=\frac{\lambda}{n^3}\sum_{i=1}^{+\infty}(\frac{(i-1)^2e^{-\lambda}\lambda^{(i-1)}}{(i-1)!}+\frac{2(i-1)e^{-\lambda}\lambda^{(i-1)}}{(i-1)!}+\frac{e^{-\lambda}\lambda^{(i-1)}}{(i-1)!})$
$\frac{1}{n^3}E[X^3]=\frac{\lambda}{n^3}(E[X^2]+2E[X]-1)=\frac{\lambda}{n^3}(\lambda(\lambda+1)+2\lambda+1)=\frac{1}{n^3}(\lambda^3+3\lambda^2+\lambda)$
$P(E)=1-\frac{3}{n}\lambda+\frac{3}{n^2}\lambda(\lambda+1)-\frac{1}{n^3}(\lambda^3+3\lambda^2+\lambda)$
$P(E)\approx 1-\frac{3}{n}\lambda$
Answer:
$P(E)\approx 1-\frac{3}{8000}\times0.9873=0.9996$

(d) **Solution:**
Let $n=4000$, then $\lambda=e^{-18}\times4000$
Answer:

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$P(E) \approx 1 - \frac{3}{4000} \times e^{-18} \times 4000 \approx 1 - 4.57 \times 10^{-8} \approx 1$