

1. (a) **Solution:**

Define event

$J = \{\text{engineer program in Java}\}$

$C = \{\text{engineer program in C++}\}$

Then  $P(C|J) = \frac{P(CJ)}{P(J)}$

$$0.24 = \frac{P(CJ)}{0.36}$$

**Answer:**

The probability that a randomly selected engineer programs in Java and C++ is

$$P(CJ) = 0.0864$$

(b) **Solution:**

$$P(J|C) = \frac{P(CJ)}{P(C)}$$

**Answer:**

The probability that a randomly selected engineer programs in Java given that he/she programs in C++ is

$$P(J|C) = \frac{0.0864}{0.33} = 0.2618$$

2. (a) **Solution:**

$$P(E) = \frac{4 \times 3}{52 \times 51} = 0.0181$$

The Ace of Spades can be either the first card or the second card

$$P(F) = \frac{1 \times 51 + 51 \times 1}{52 \times 51} = 0.0385$$

$$P(EF) = \frac{1 \times 3 + 3 \times 1}{52 \times 51} = 0.00226$$

**Answer:**

$$P(E|F) = \frac{P(EF)}{P(F)} = 0.0588$$

(b) **Solution:**

Since event  $G$  must happen when event  $E$  happens

$$P(G|E) = 1$$

$$1 = \frac{P(GE)}{P(E)}$$

$$P(GE) = P(E)$$

We can calculate the complement of event  $G$

$$P(G^c) = \frac{48 \times 47}{52 \times 51} = 0.851$$

$$P(G) = 1 - P(G^c) = 0.149$$

**Answer:**

$$P(E|G) = \frac{P(EG)}{P(G)} = \frac{P(E)}{P(G)} = 0.121$$

3. (a) **Solution:**

Define event  $E_i = \{\text{a user likes movie } M_i\}, T = \{\text{a user like the Tearjerker genre}\}$

$$P(E_i|T) = p_i$$

**Answer:**

$$P((E_1 \cap E_2 \cap E_3)|T) = P(E_1|T) \cap P(E_2|T) \cap P(E_3|T)$$

Since all the  $E_i|T$  are conditionally independent, the probability that a user likes all three movies  $M_1, M_2$  and  $M_3$  given that they like the Tearjerker genre is

$$P((E_1 \cap E_2 \cap E_3)|T) = P(E_1|T)P(E_2|T)P(E_3|T) = p_1p_2p_3$$

(b) **Answer:**

$$P((E_1 \cup E_2 \cup E_3)|T) = P(E_1|T) \cup P(E_2|T) \cup P(E_3|T) = p_1 + p_2 + p_3 - (p_1p_2 + p_3p_2 + p_3p_1)$$

(c) **Solution:**

Define event  $E_{all} = \{\text{user likes all the 3 movie}\}$

$$P(E_{all}|T) = p_1p_2p_3$$

$$\begin{aligned}P(E_{all}|T^c) &= q_1 q_2 q_3 \\P(E_{all}|T) &= P(E_{all}|T)P(T) = 0.6p_1 p_2 p_3 \\P(E_{all}|T^c) &= P(E_{all}|T^c)P(T) = (1 - 0.6)q_1 q_2 q_3 = 0.4q_1 q_2 q_3 \\P(E_{all}) &= P(E_{all}|T) + P(E_{all}|T^c)\end{aligned}$$

**Answer:**

The probability that they like the Tearjerker genre that they like  $M_1$ ,  $M_2$  and  $M_3$  is

$$P(T|E_{all}) = \frac{P(T E_{all})}{P(E)} = \frac{0.6p_1 p_2 p_3}{0.6p_1 p_2 p_3 + 0.4q_1 q_2 q_3}$$

4. (a) **Solution:**

We can calculate the probability of event  $F = \{\text{text all the 5 servers failed in one year}\}$

$$P(F) = (1 - p)^5$$

**Answer:**

The probability that at least 1 server is still working after on year is

$$P(E_1) = 1 - P(F) = 1 - (1 - p)^5$$

(b) **Solution:**

We can consider each particular combination of the 3 servers that are still working.

$$P(G_i) = p^3(1 - p)^2$$

Since all the events are mutually exclusive **Answer:**

The probability that exactly 3 server is still working after on year is

$$P(E_3) = \binom{5}{3}P(G_i) = 6p^3(1 - p)^2$$

(c) **Solution:**

We can consider 3 situations: exactly 3, 4, 5 servers are still working after one year and combine them together.

**Answer:**

The probability that at least 3 server is still working after on year is

$$P(E) = P(E_3) + P(E_4) + P(E_5) = \sum_{i=3}^5 \binom{5}{i} p^i (1 - p)^{5-i}$$

5. **Solution:**

The probability of all the bit in a  $n$  bit string is

$$P(F) = (1 - p)^n$$

Then the probability that at least one 1 in the string is

$$P(E) = 1 - P(F) = 1 - (1 - p)^n$$

**Answer:**

$$P(E) > 0.7$$

The  $n$  requirement for the probability that there is at least one 1 in the string is at least 0.7 is

$$n > \log_{1-p}(0.3)$$

6. (a) **Solution:**

$F_i = \{\text{at least one string hashed into i-th bucket}\}$

$$P(E) = 1 - P((F_1 F_2 F_3 F_4)^c) = 1 - P(F_1^c \cup F_2^c \cup F_3^c \cup F_4^c)$$

Since all the  $F_i^c$  are not mutually exclusive, the answer will be very complex before expansion. Because of the limited number of buckets and strings, we can try to use another way to get the answer.

$G = \{\text{All the buckets have at least 1 string}\}$

$H = \{\text{No string in bucket 5, all the first 4 buckets have at least one or 2 strings}\}$

$I = \{\text{No string in bucket 5, all the first 4 buckets have at least one or 3 strings}\}$

$$P(E) = P(G) + P(G_5) + P(G_6)$$

For each of event  $G$ , there is only one bucket can have 2 strings in it and we can add up all the 5

possible situation to get the total probability.

$$P(G) = \frac{6!}{2!} \sum_{i=1}^5 (p_i \prod_{j=1}^5 p_j) = 360 \sum_{i=1}^5 p_i \prod_{j=1}^5 p_j = 360 \prod_{j=1}^5 p_j$$

For each of event  $H$ , there are 2 bucket have 2 strings in it.

$$P(H) = \frac{6!}{2!2!} (p_1 p_2 p_3^2 p_4^2 + p_1 p_2^2 p_3 p_4^2 \dots) = 180 (p_1 p_2 + p_1 p_3 + p_1 p_4 + p_2 p_3 + p_2 p_4 + p_3 p_4) p_1 p_2 p_3 p_4$$

For each of event  $I$ , there are 1 bucket have 3 strings in it.

$$P(I) = \frac{6!}{3!} (p_1^3 p_2 p_3 p_4 + p_1 p_2^3 p_3 p_4 + p_1 p_2 p_3^3 p_4 + p_1 p_2 p_3 p_4^3) = 120 (p_1^2 + p_2^2 + p_3^2 + p_4^2) p_1 p_2 p_3 p_4$$

**Answer:**

$$P(E) = 60(6p_5 + 3(p_1 p_2 + p_1 p_3 + p_1 p_4 + p_2 p_3 + p_2 p_4 + p_3 p_4) + 2(p_1^2 + p_2^2 + p_3^2 + p_4^2)) p_1 p_2 p_3 p_4$$

(b) **Answer:**

After substitute all the  $p_i$  values, we can get

$$P(E) = 60 \times (6 \times 0.1 + 3 \times 0.2925 + 2 \times 0.225) \times 0.001875 = 0.2168$$