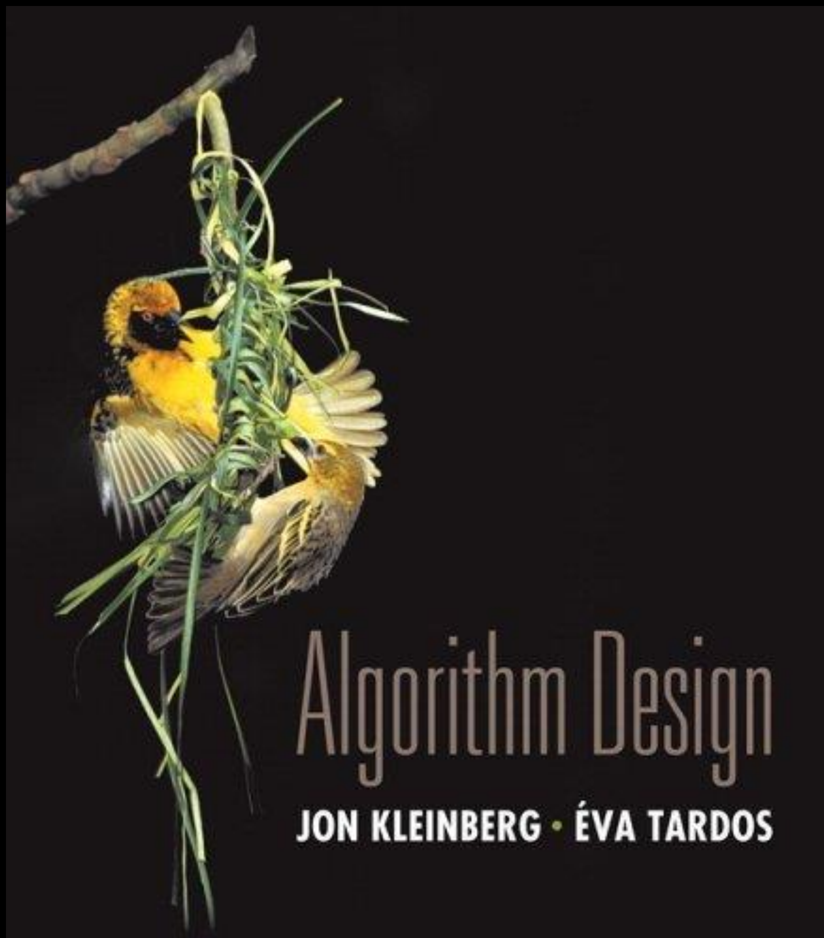


Chapter 3

Graphs

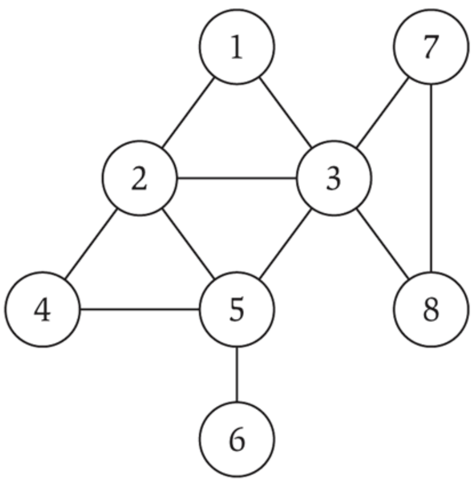
Slides by Kevin Wayne.

3.1 Basic Definitions and Applications

Undirected Graphs

Undirected graph. $G = (V, E)$

- V = nodes.
- E = edges between pairs of nodes.
- Captures pairwise relationship between objects.
- Graph size parameters: $n = |V|$, $m = |E|$.



$V = \{ 1, 2, 3, 4, 5, 6, 7, 8 \}$

$E = \{ 1\text{-}2, 1\text{-}3, 2\text{-}3, 2\text{-}4, 2\text{-}5, 3\text{-}5, 3\text{-}7, 3\text{-}8, 4\text{-}5, 5\text{-}6 \}$

$n = 8$

$m = 11$

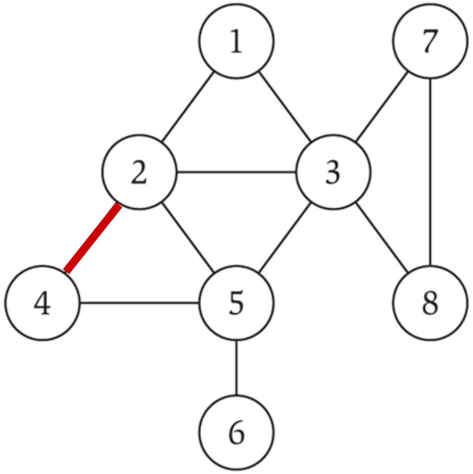
Some Graph Applications

Graph	Nodes	Edges
transportation	street intersections	highways
communication	computers	fiber optic cables
World Wide Web	web pages	hyperlinks
social	people	relationships
food web	species	predator-prey
software systems	functions	function calls
scheduling	tasks	precedence constraints
circuits	gates	wires

Graph Representation: Adjacency Matrix

Adjacency matrix. n-by-n matrix with A_{uv} = 1 if (u, v) is an edge.

- Two representations of each edge.
- Space proportional to n^2.
- Checking if (u, v) is an edge takes $\Theta(1)$ time.
- Identifying all edges takes $\Theta(n^2)$ time.



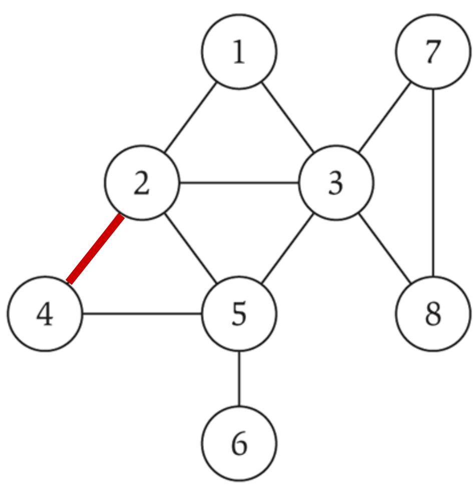
	1	2	3	4	5	6	7	8
1	0	1	1	0	0	0	0	0
2	1	0	1	1	1	0	0	0
3	1	1	0	0	1	0	1	1
4	0	1	0	1	1	0	0	0
5	0	1	1	1	0	1	0	0
6	0	0	0	0	1	0	0	0
7	0	0	1	0	0	0	0	1
8	0	0	1	0	0	0	1	0

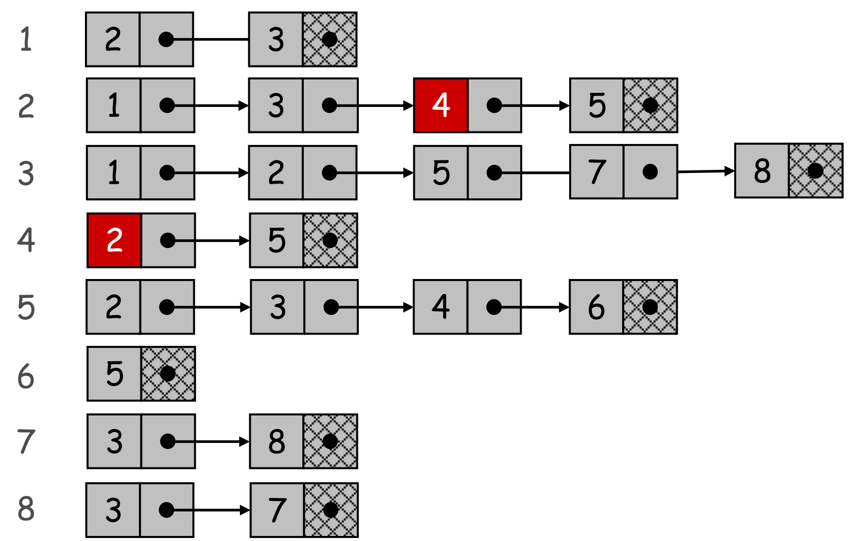
Graph Representation: Adjacency List

Adjacency list. Node indexed array of lists.

- Two representations of each edge.
- Space proportional to m + n.
- Checking if (u, v) is an edge takes O(deg(u)) time.
- Identifying all edges takes Θ(m + n) time.

degree = number of neighbors of u



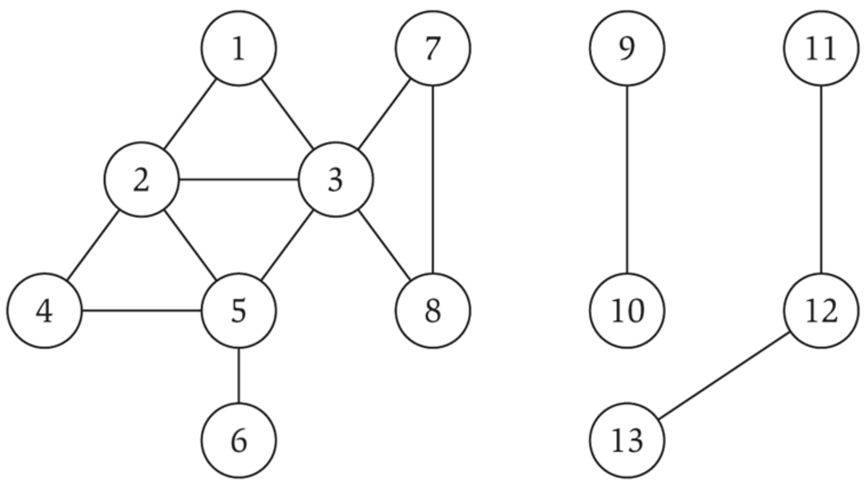


Paths and Connectivity

Def. A path in an undirected graph $G = (V, E)$ is a sequence P of nodes $v_1, v_2, \ldots, v_{k-1}, v_k$ with the property that each consecutive pair v_i, v_{i+1} is joined by an edge in E.

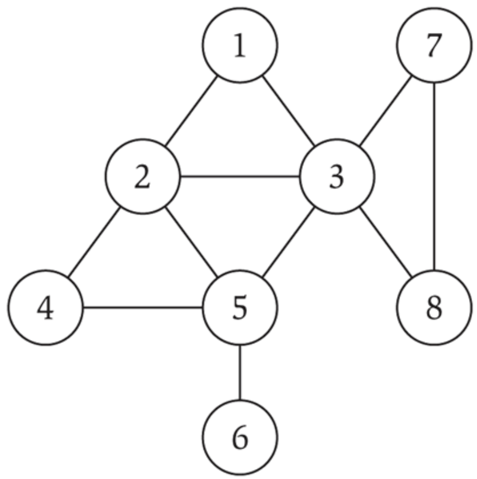
Def. A path is simple if all nodes are distinct.

Def. An undirected graph is connected if for every pair of nodes u and v, there is a path between u and v.



Cycles

Def. A cycle is a path $v_1, v_2, \ldots, v_{k-1}, v_k$ in which $v_1 = v_k$, $k > 2$, and the first $k-1$ nodes are all distinct.



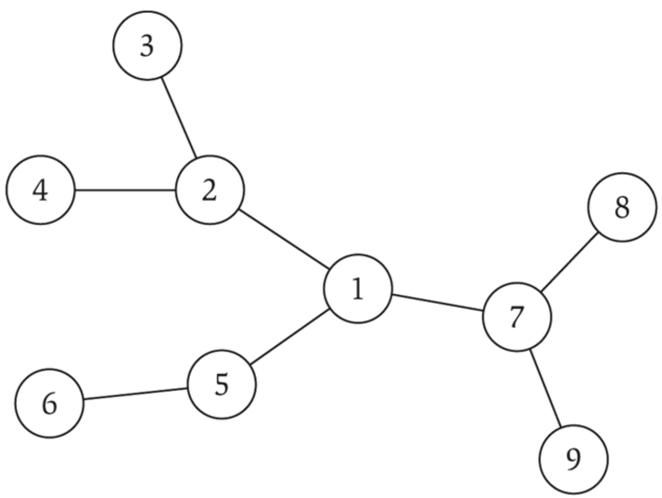
cycle C = 1-2-4-5-3-1

Trees

Def. An undirected graph is a tree if it is connected and does not contain a cycle.

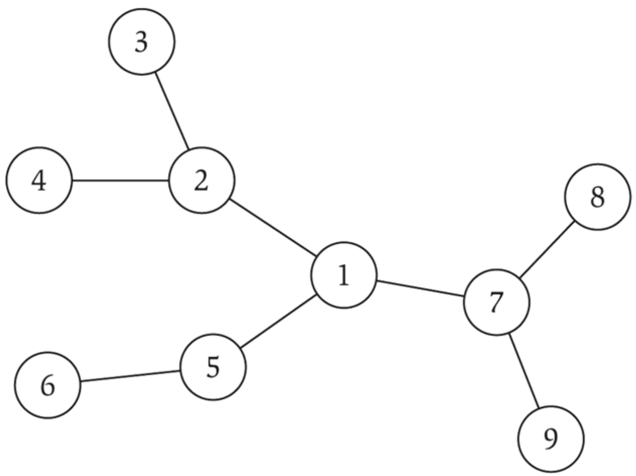
Theorem. Let G be an undirected graph on n nodes. Any two of the following statements imply the third.

- G is connected.
- G does not contain a cycle.
- G has n-1 edges.

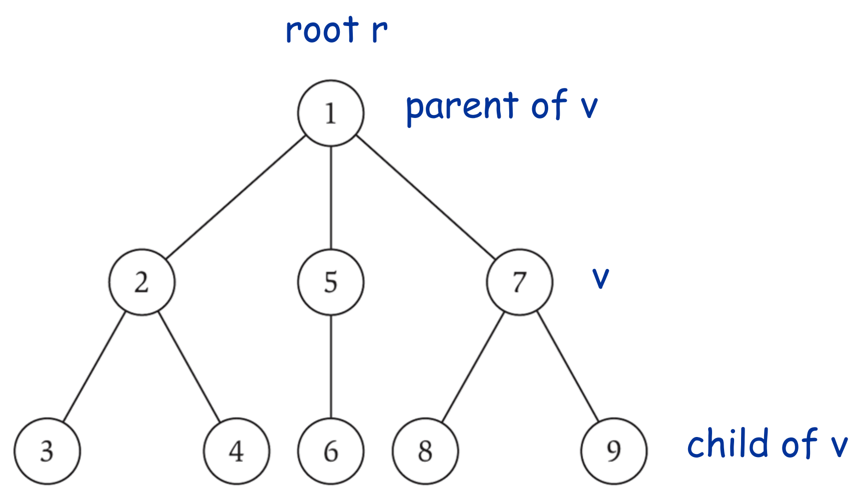


Rooted Trees

Rooted tree. Given a tree T, choose a root node r and orient each edge away from r.



a tree



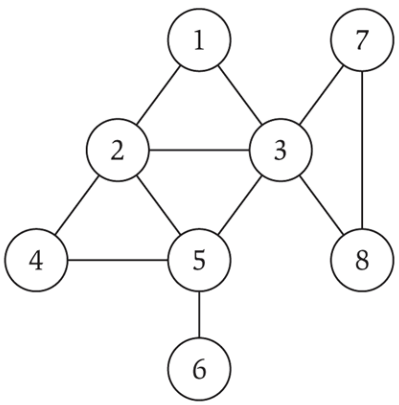
the same tree, rooted at 1

3.2 Graph Traversal

Connectivity

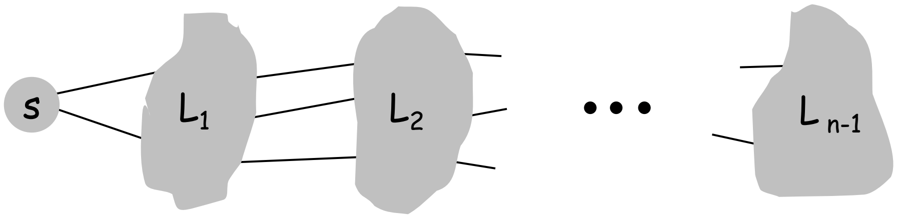
s-t connectivity problem. Given two node s and t, is there a path between s and t?

s-t shortest path problem. Given two node s and t, what is the length of the shortest path between s and t?



Breadth First Search

Explore outward from s in all possible directions, adding nodes one "layer" at a time.



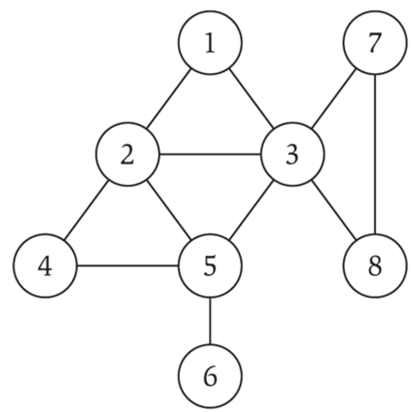
BFS algorithm.

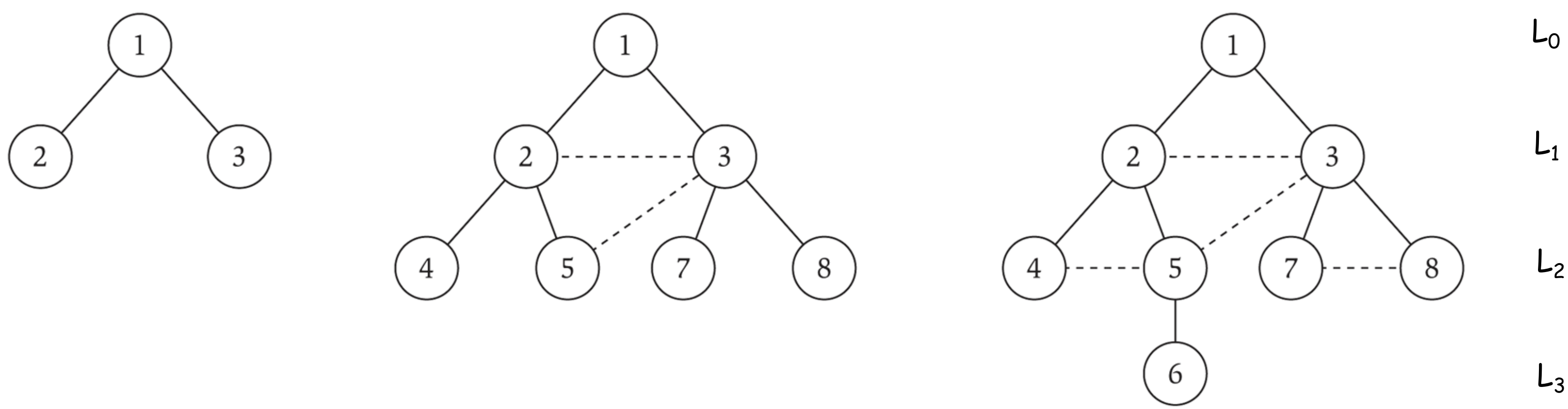
- $L_0 = \{ s \}$.
- L_1 = all neighbors of L_0.
- L_2 = all nodes that do not belong to L_0 or L_1, and that have an edge to a node in L_1.
- L_{i+1} = all nodes that do not belong to an earlier layer, and that have an edge to a node in L_i.

Theorem. For each i, L_i consists of all nodes at distance exactly i from s. There is a path from s to t iff t appears in some layer.

Breadth First Search

Property. Let T be a BFS tree of G = (V, E), and let (x, y) be an edge of G. Then the level of x and y differ by at most 1.





(a) (b) (c)

Breadth First Search: Analysis

Theorem. The above implementation of BFS runs in $O(m + n)$ time if the graph is given by its adjacency list representation.

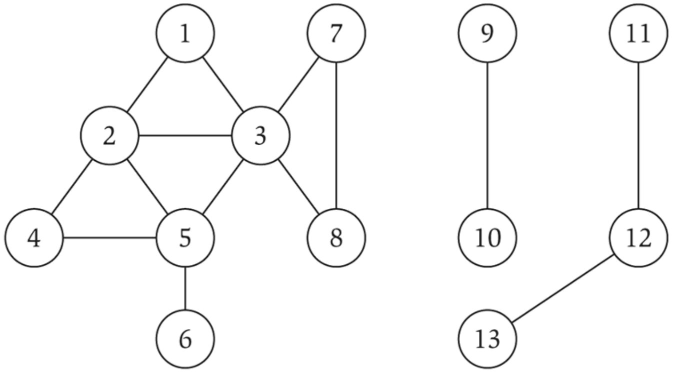
Pf.

- when we consider node u, there are deg(u) incident edges (u, v)
- total time processing edges is $\Sigma_{u \in V} \deg(u) = 2m$ ▪

each edge (u, v) is counted exactly twice
in sum: once in deg(u) and once in deg(v)

Connected Component

Connected component. Find all nodes reachable from s.



Connected component containing node 1 = { 1, 2, 3, 4, 5, 6, 7, 8 }.

3.4 Testing Bipartiteness

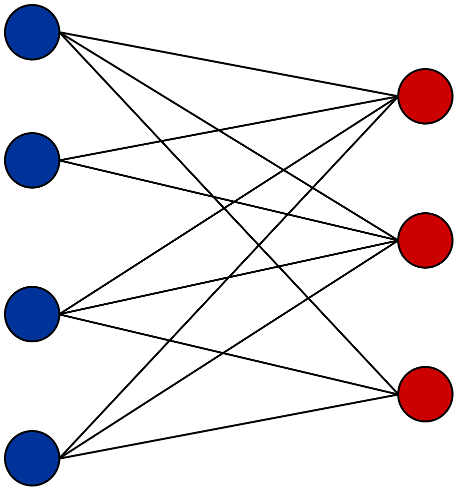
Bipartite Graphs

Def. An undirected graph G = (V, E) is bipartite if the nodes can be colored red or blue such that every edge has one red and one blue end.

Applications.

Stable marriage: men = red, women = blue.

Scheduling: machines = red, jobs = blue.



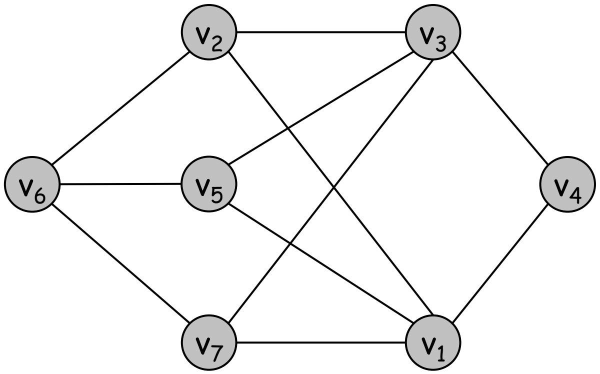
a bipartite graph

Testing Bipartiteness

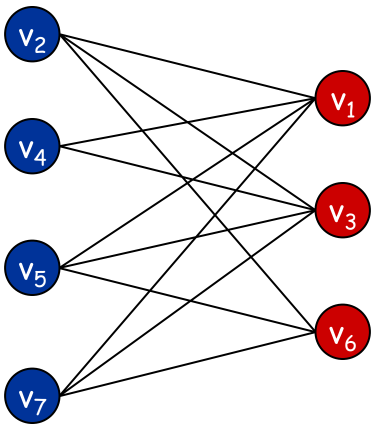
Testing bipartiteness. Given a graph G, is it bipartite?

Many graph problems become:

- easier if the underlying graph is bipartite (matching)
- tractable if the underlying graph is bipartite (independent set)



a bipartite graph G

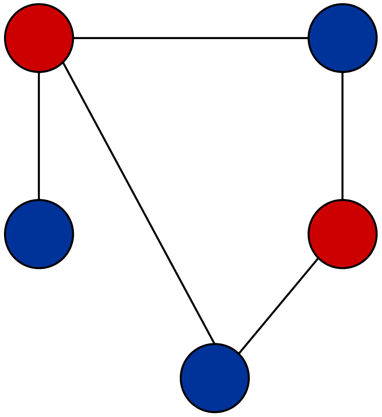


another drawing of G

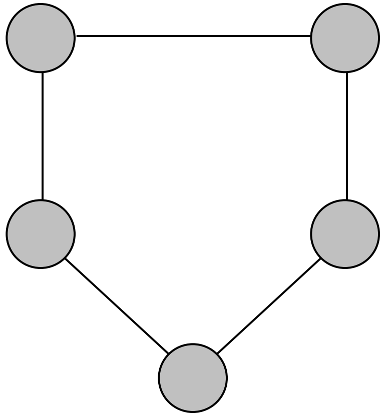
An Obstruction to Bipartiteness

Lemma. If a graph G is bipartite, it cannot contain an odd length cycle.

Pf. Not possible to 2-color the odd cycle.



bipartite
(2-colorable)



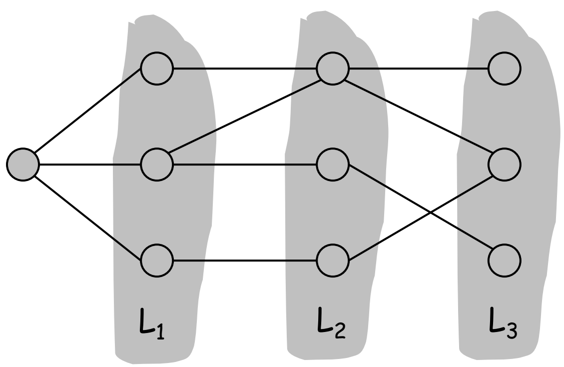
not bipartite
(not 2-colorable)

Bipartite Graphs

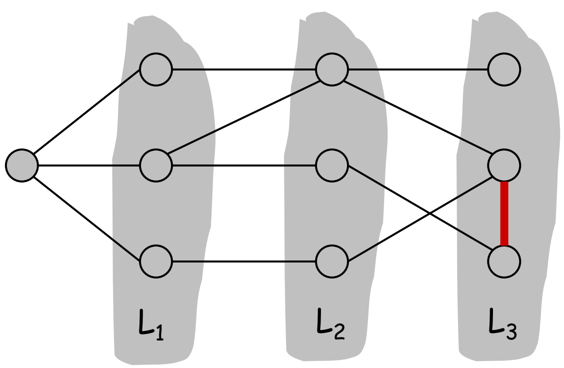
Lemma. Let G be a connected graph, and let $L_0, \ldots, L_k$ be the layers produced by BFS starting at node s. Exactly one of the following holds.

(i) No edge of G joins two nodes of the same layer, and G is bipartite.

(ii) An edge of G joins two nodes of the same layer, and G contains an odd-length cycle (and hence is not bipartite).



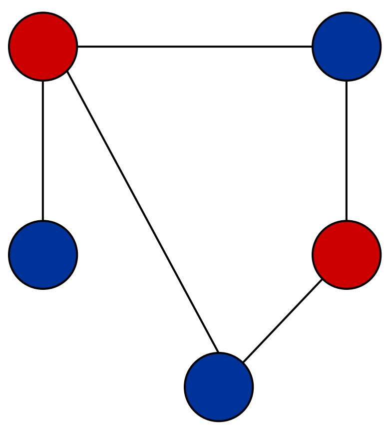
Case (i)



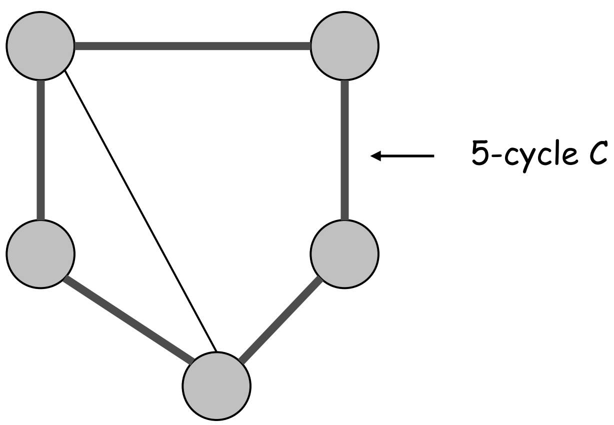
Case (ii)

Obstruction to Bipartiteness

Corollary. A graph G is bipartite iff it contain no odd length cycle.



bipartite
(2-colorable)



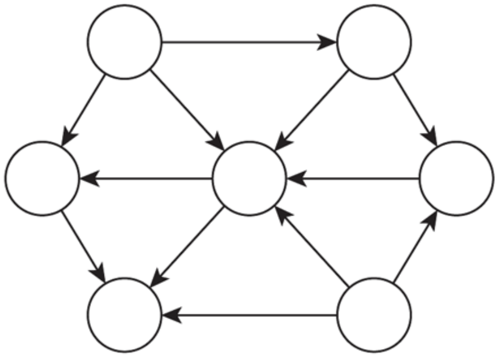
not bipartite
(not 2-colorable)

3.5 Connectivity in Directed Graphs

Directed Graphs

Directed graph. G = (V, E)

Edge (u, v) goes from node u to node v.



Ex. Web graph - hyperlink points from one web page to another.

Directedness of graph is crucial.

Modern web search engines exploit hyperlink structure to rank web pages by importance.

Graph Search

Directed reachability. Given a node s, find all nodes reachable from s.

Directed s-t shortest path problem. Given two node s and t, what is the length of the shortest path between s and t?

Graph search. BFS extends naturally to directed graphs.

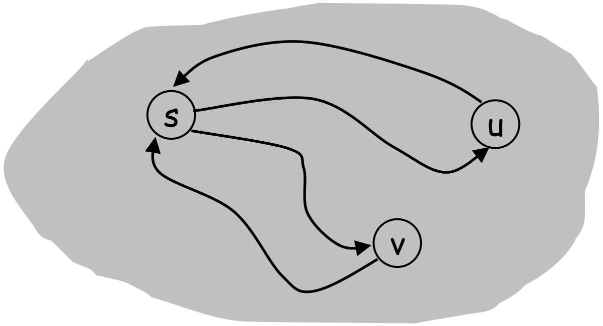
Web crawler. Start from web page s. Find all web pages linked from s, either directly or indirectly.

Strong Connectivity

Def. Node u and v are mutually reachable if there is a path from u to v and also a path from v to u.

Def. A graph is strongly connected if every pair of nodes is mutually reachable.

Lemma. Let s be any node. G is strongly connected iff every node is reachable from s, and s is reachable from every node.



Strong Connectivity: Algorithm

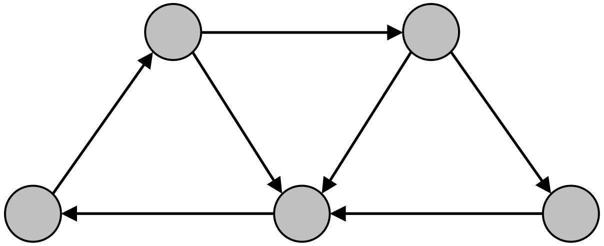
Theorem. Can determine if G is strongly connected in $O(m + n)$ time.

Pf.

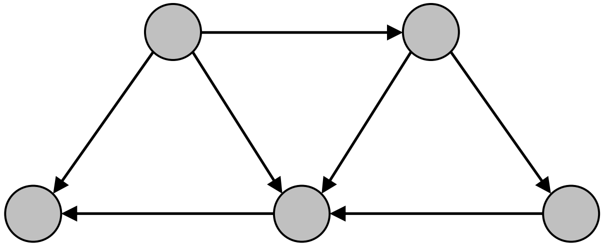
- Pick any node s.
- Run BFS from s in G.
- Run BFS from s in G^{rev}.

reverse orientation of every edge in G

- Return true iff all nodes reached in both BFS executions.
- Correctness follows immediately from previous lemma. ▪



strongly connected



not strongly connected

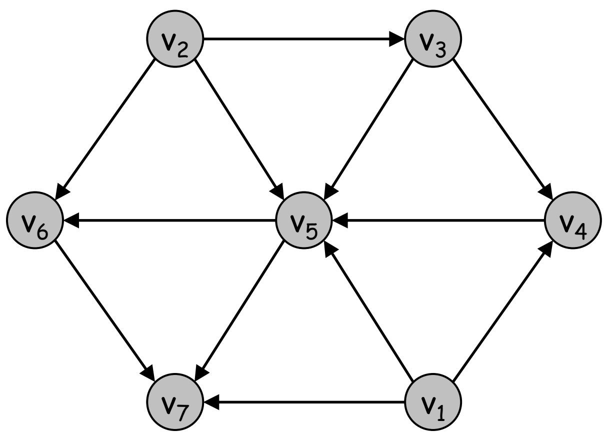
3.6 DAGs and Topological Ordering

Directed Acyclic Graphs

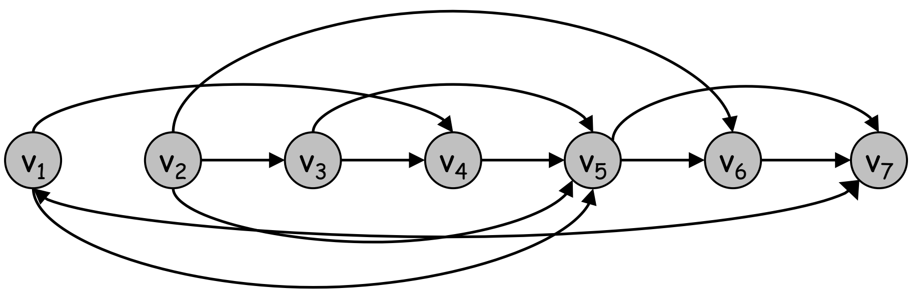
Def. An DAG is a directed graph that contains no directed cycles.

Ex. Precedence constraints: edge (v_i, v_j) means v_i must precede v_j.

Def. A topological order of a directed graph $G = (V, E)$ is an ordering of its nodes as $v_1, v_2, \ldots, v_n$ so that for every edge (v_i, v_j) we have $i < j$.



a DAG



a topological ordering

Directed Acyclic Graphs

Lemma. If G has a topological order, then G is a DAG.

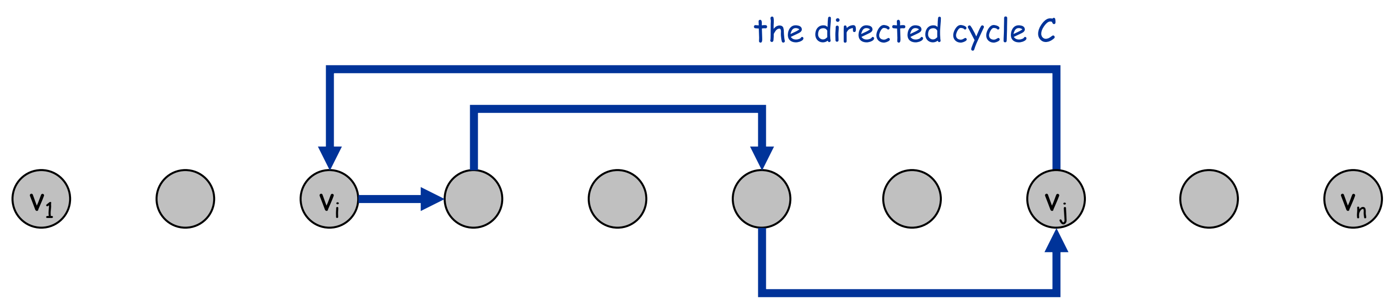
Pf. (by contradiction)

Suppose that G has a topological order $v_1, \ldots, v_n$ and that G also has a directed cycle C. Let's see what happens.

Let v_i be the lowest-indexed node in C, and let v_j be the node just before v_i in C; thus (v_j, v_i) is an edge.

By our choice of i, we have $i < j$.

On the other hand, since (v_j, v_i) is an edge and $v_1, \ldots, v_n$ is a topological order, we must have $j < i$, a contradiction. ▪



the supposed topological order: $v_1, \ldots, v_n$

Lemma. If G is a DAG, then G has a node with no incoming edges.

Pf. (by contradiction)

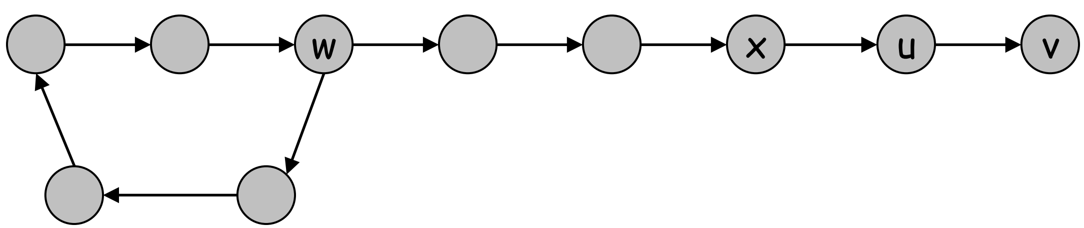
Suppose that G is a DAG and every node has at least one incoming edge. Let's see what happens.

Pick any node v, and begin following edges backward from v. Since v has at least one incoming edge (u, v) we can walk backward to u.

Then, since u has at least one incoming edge (x, u), we can walk backward to x.

Repeat until we visit a node, say w, twice.

Let C denote the sequence of nodes encountered between successive visits to w. C is a cycle. ▪



Directed Acyclic Graphs

Lemma. If G is a DAG, then G has a topological ordering.

Pf. (by induction on n)

Base case: true if $n = 1$.

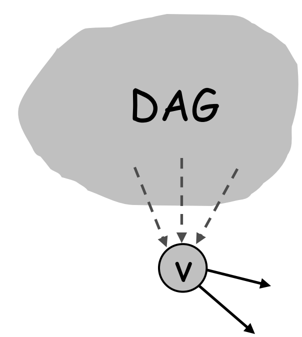
Given DAG on $n > 1$ nodes, find a node v with no incoming edges.

$G - \{ v \}$ is a DAG, since deleting v cannot create cycles.

By inductive hypothesis, $G - \{ v \}$ has a topological ordering.

Place v first in topological ordering; then append nodes of $G - \{ v \}$ in topological order. This is valid since v has no incoming edges. ▪

1: Find a node v with no incoming edges and order it first.
2: Delete v from G.
3: Recursively computer a topological ordering of $G - \{v\}$ and append this order after v.



Topological Sorting Algorithm: Running Time

Theorem. Algorithm finds a topological order in $O(m + n)$ time.

Pf.

Maintain the following information:

- `count[w]` = remaining number of incoming edges
- S = set of remaining nodes with no incoming edges

Initialization: $O(m + n)$ via single scan through graph.

Update: to delete v

- remove v from S
- decrement `count[w]` for all edges from v to w, and add w to S if `count[w]` hits 0
- this is $O(1)$ per edge. ▪

Homework

- Read Chapter 3 of the textbook.

- Exercises 2, 5, 6 & 8 in Chapter 3.