Assignments & Tuturial

- **Assignments**
  - 3.3
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  - 3.22 (a) -> Figs. (b) (d) (f)
  - 3.24
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- **Tutorial problems**
  - Basic Problems wish Answers 3.8
  - Basic Problems 3.34
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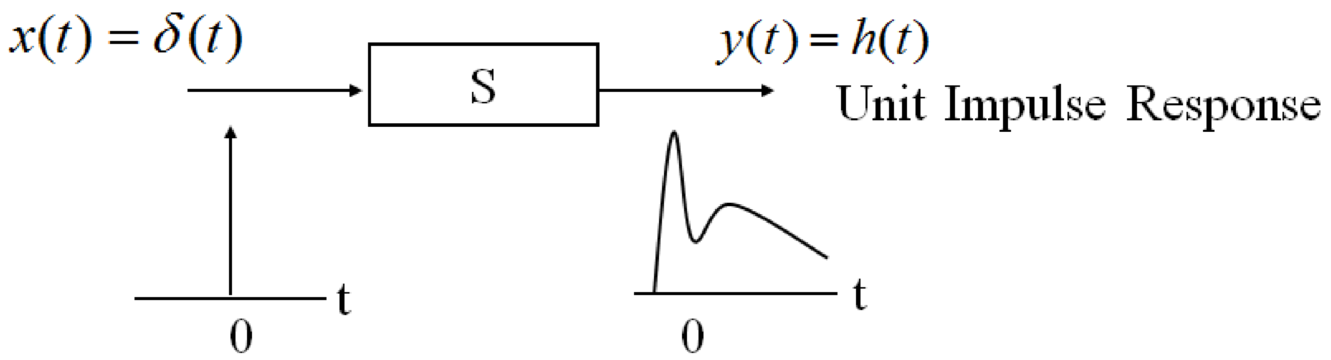
Review for Chapter 2

- **Why to introduce unit impulse response?**

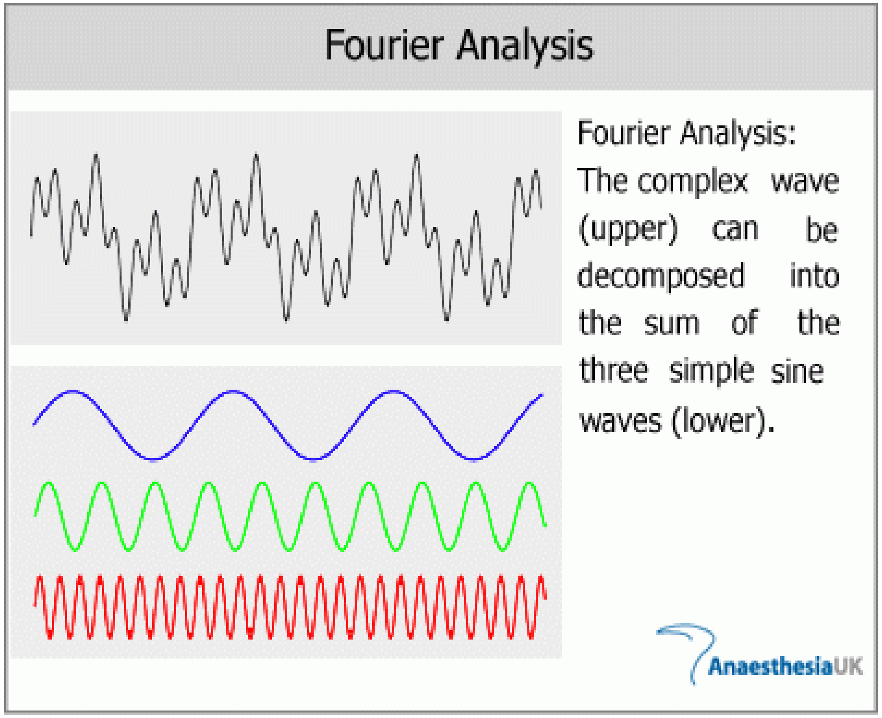


- **Why to introduce convolution?**
  - **(DT or CT) Signal can be represented by a linear combination of unit impulse response**
  - **When it goes through the system, the output is computed via convolution of input signal and unit impulse response**

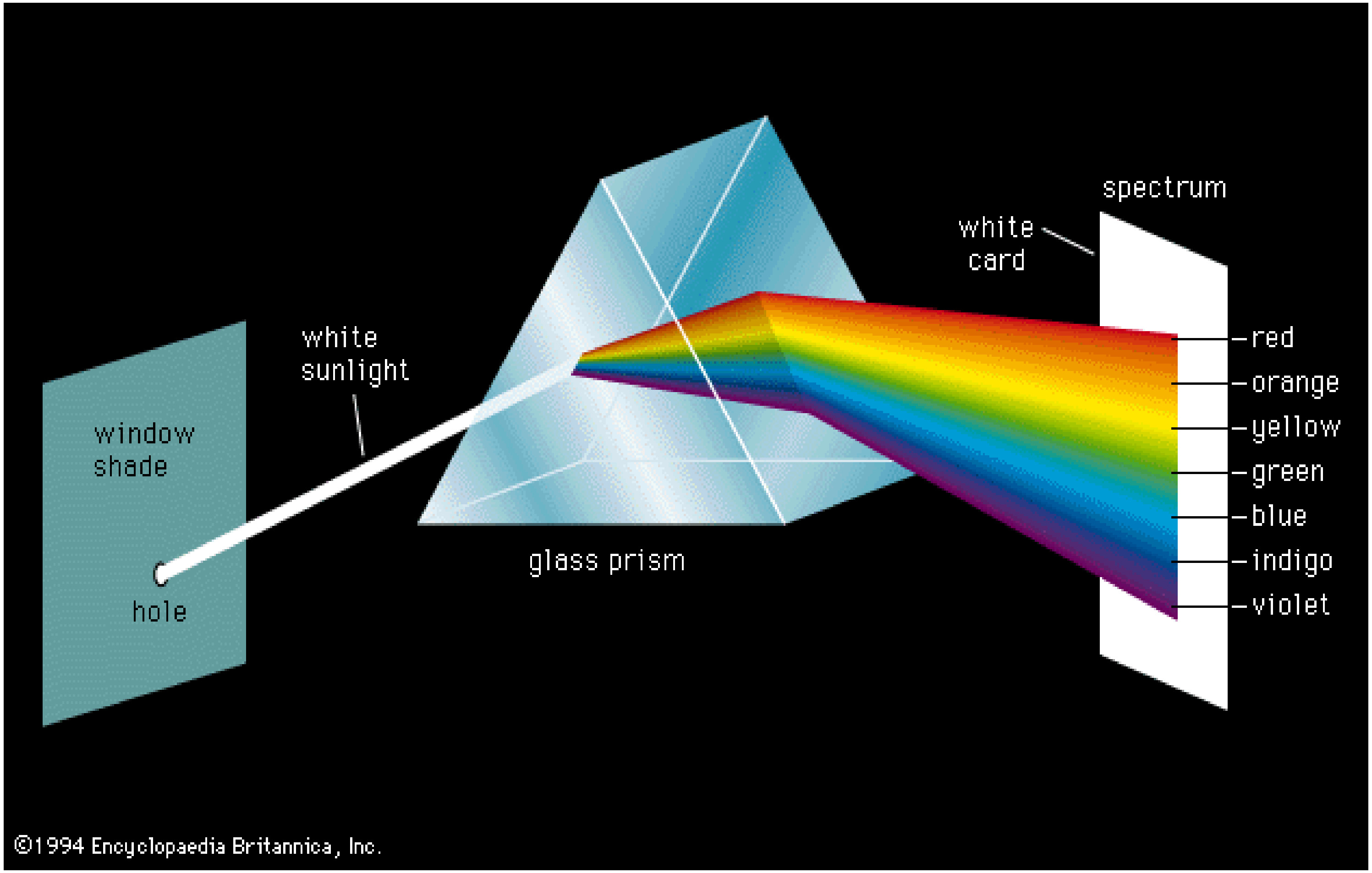
Chapter 3

Fourier Series Representation of Periodic Signals

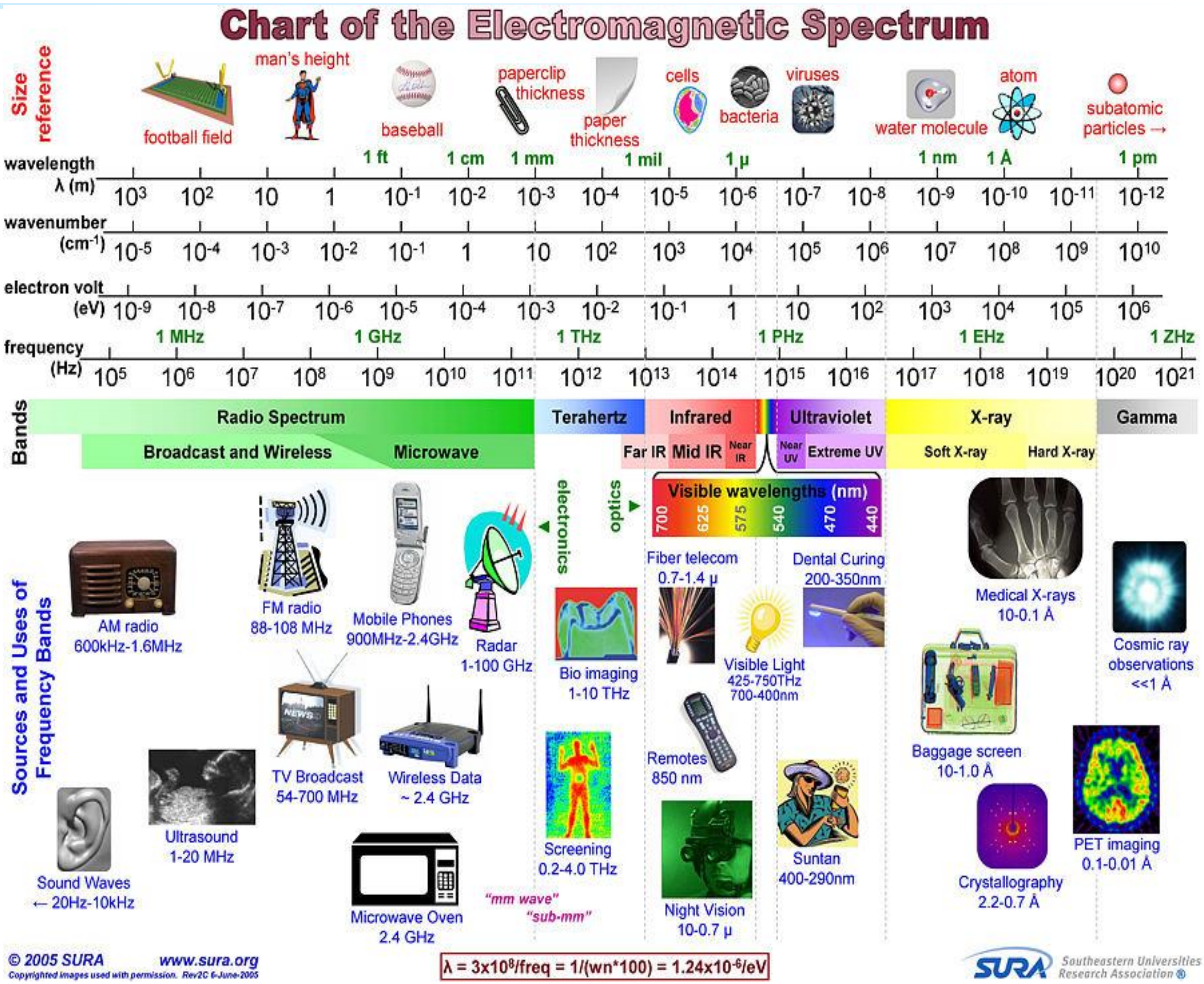
Joseph Fourier



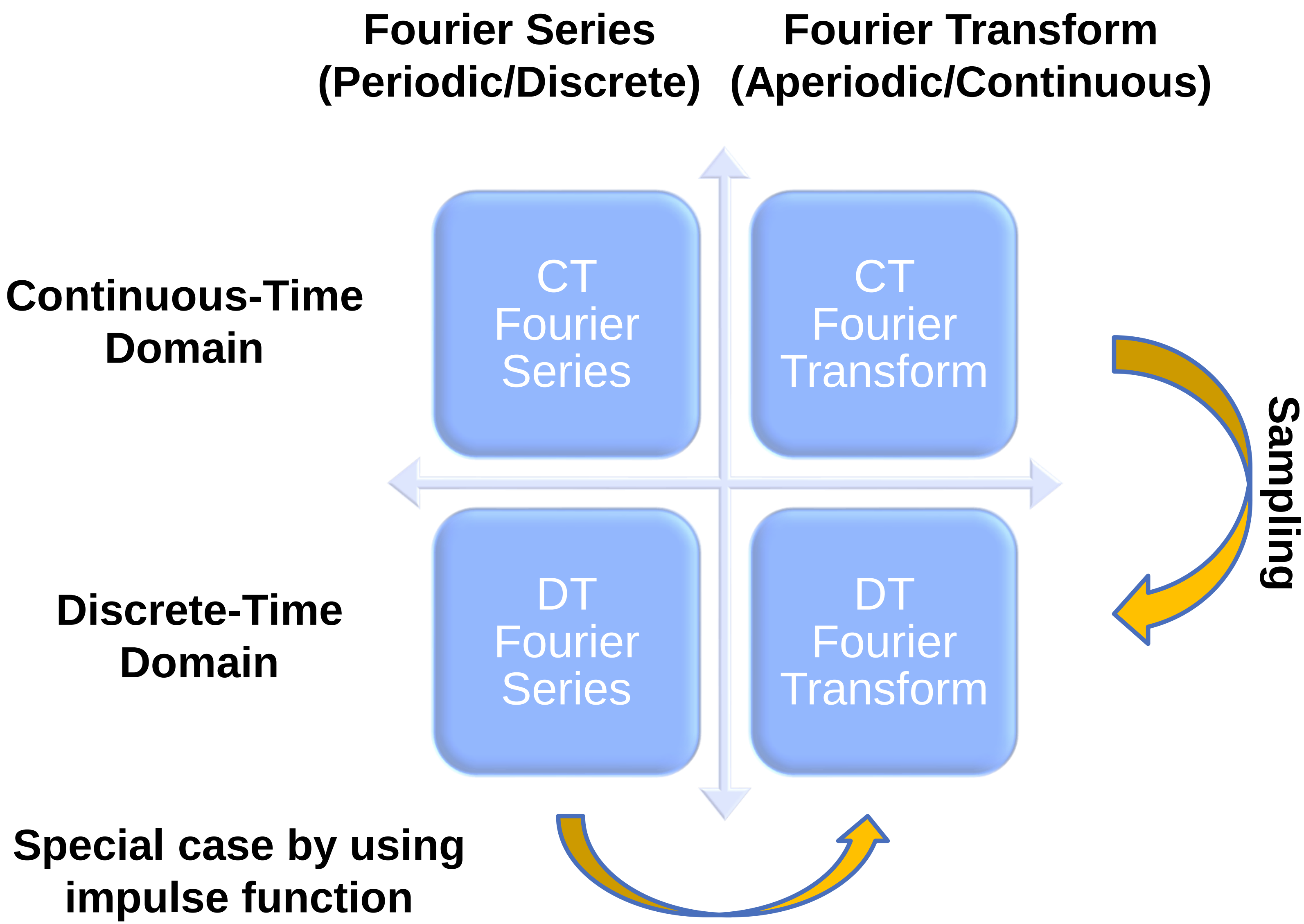
A Useful Analogy



Example: Radio Spectrum



Overview on Frequency Analysis



Two Special Signals for LTI Systems

$$x(t) = e^{st} \longrightarrow \boxed{h(t)} \longrightarrow$$

$$\begin{aligned} y(t) &= \int_{-\infty}^{+\infty} h(\tau) e^{s(t-\tau)} d\tau \\ &= \left[\int_{-\infty}^{+\infty} h(\tau) e^{-s\tau} d\tau\right] e^{st} \\ &= \underbrace{H(s)}_{\text{eigenvalue}} \underbrace{e^{st}}_{\text{eigenfunction}} \end{aligned}$$

$$x[n] = z^{n} \longrightarrow \boxed{h[n]} \longrightarrow$$

$$\begin{aligned} y[n] &= \sum_{m=-\infty}^{\infty} h[m] z^{n-m} \\ &= \left[\sum_{m=-\infty}^{\infty} h[m] z^{-m}\right] z^{n} \\ &= \underbrace{H(z)}_{\text{eigenvalue}} \underbrace{z^{n}}_{\text{eigenfunction}} \end{aligned}$$

System Functions *H*(*s*) or *H*(*z*)

CT:

$$x(t) \xrightarrow{e^{st}} \boxed{h(t)} \xrightarrow{H(s)e^{st}} y(t)$$

$$H(s) = \int_{-\infty}^{\infty} h(t)e^{-st}dt$$

$$x(t) = \sum a_k e^{s_k t} \longrightarrow y(t) = \sum H(s_k) a_k e^{s_k t}$$

DT:

$$x[n] \xrightarrow{z^n} \boxed{h[n]} \xrightarrow{H(z)z^n} y[n]$$

$$H(z) = \sum_{n=-\infty}^{\infty} h[n]z^{-n}$$

$$x[n] = \sum a_k z_k^n \longrightarrow y[n] = \sum H(z_k) a_k z_k^n$$

Fourier and Beyond

Observation: if one signal can be written as the linear combination of $e^{st} or\ z^n$, we need NOT to calculate the convolution for the LTI output.

When $s = j\omega, z = e^{j\omega}$

$$\Rightarrow\ e^{j\omega t}, e^{j\omega n}:\ Fourier\ Series$$

When s or z is general complex number

$$\Rightarrow Laplace\ Transform\ \&\ Z\ Transform$$

A “Special” Class of Periodic Signals

CT Fourier Series: one periodic CT signal with period T can be written as

$$x(t) = \sum_{k=-\infty}^{+\infty} a_k e^{jk\omega_0 t}$$

where $\omega_0 = 2\pi/T$.

$\{a_k\}$: Fourier series coefficients, which represent the strength of the component $e^{jk\omega_0 t}$.

$e^{jk\omega_0 t}$ is a signal with pure frequency $k\omega_0 \Rightarrow x(t)$ is a periodic signal with period T, it consists of components with different frequencies $k\omega_0$ and different weights.

Convergence of CT Fourier Series

What kind of periodic signals have Fourier series expansion?

Define $e(t) = x(t) - \sum_{k=-\infty}^{+\infty} a_k e^{jk\omega_0 t}$,

- Fourier series expansion exists <=> $\exists\{a_k\}, \mathrm{e(t)} = 0$ (D1)

Relaxation:

- Fourier series expansion exists <=> $\exists\{a_k\}, \int_T \, |e(t)|^2 = 0$ (D2)
- $D1 \subset D2$

- Two sufficient conditions for D2
  - $\int_T \, |x(t)|^2 dt < \infty$
  - Dirichlet condition

Dirichlet Conditions

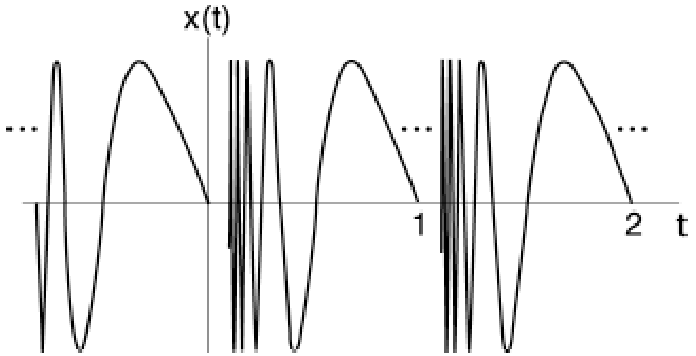
Condition 1. $x(t)$ is *absolutely integrable* over one period, i. e.

$$\int_T |x(t)|\, dt < \infty$$

Condition 2. In a finite time interval, $x(t)$ has a *finite* number of maxima and minima.

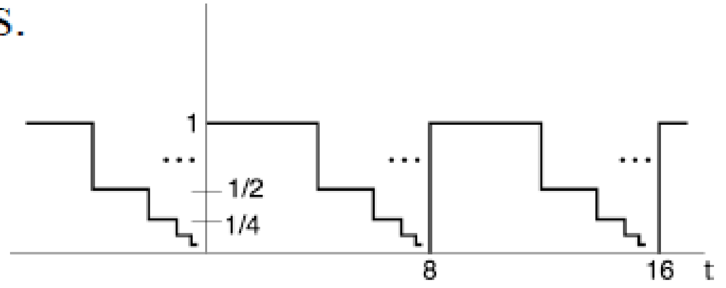
Ex. An example that violates Condition 2.

$$x(t) = \sin(2\pi / t) \quad 0 < t \le 1$$

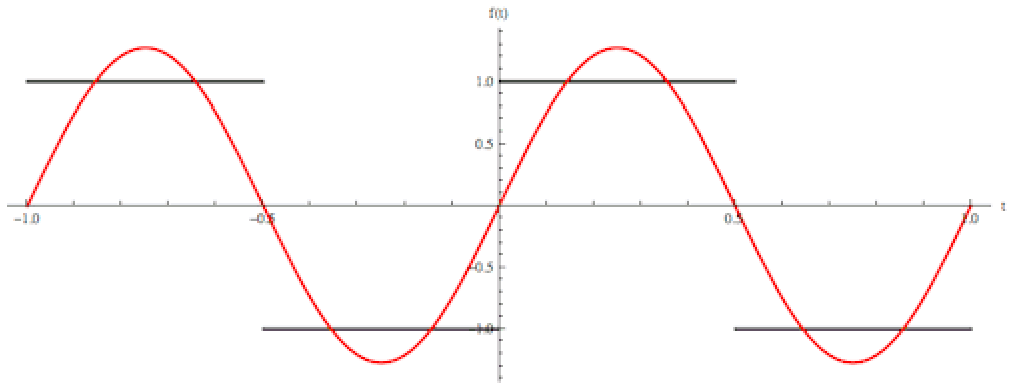


Condition 3. In a finite time interval, $x(t)$ has only a *finite* number of discontinuities.

Ex. An example that violates Condition 3.



Almost all the periodic signals in practice have Fourier series expansion!



$f(t) = \frac{4\,\mathrm{Sin}[2\pi t]}{\pi}$

Question: How do we find the Fourier coefficients?

Let's first take a detour by studying a three-dimensional vector:

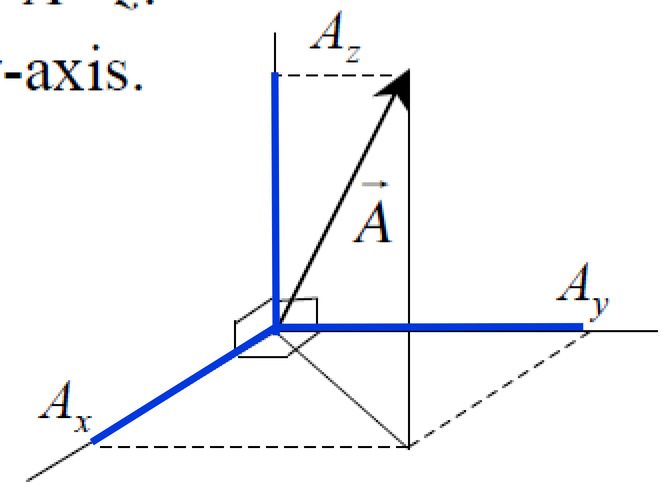
$$\vec{A} = A_x\hat{x} + A_y\hat{y} + A_z\hat{z},$$

$$\hat{x}, \hat{y}, \textit{and } \hat{z} \textit{ are unit vectors.}$$

How do we find the coefficients A_x, A_y, and A_z?

Easy, $A_x = \vec{A} \bullet \hat{x},\ A_y = \vec{A} \bullet \hat{y},\ and\ A_z = \vec{A} \bullet \hat{z}.$

Project the vector onto the *x*-, *y*-, and *z*-axis.



Why does it work this way?

Orthogonality:

$$\boxed{\hat{y} \bullet \hat{x} = \hat{z} \bullet \hat{y} = \hat{x} \bullet \hat{z} \equiv 0}$$

Inner Product of Exponential Signals

- Define inner product as

$$< e^{jk\omega_0 t} \cdot e^{jn\omega_0 t} > = \frac{1}{T}\int_T e^{jk\omega_0 t}\left(e^{jn\omega_0 t}\right)^* dt$$

- We have

$$< e^{jk\omega_0 t} \cdot e^{jn\omega_0 t} > = 1\ (\mathrm{k} = \mathrm{n})$$
$$< e^{jk\omega_0 t} \cdot e^{jn\omega_0 t} > = 0\ (\mathrm{k} \neq n)$$

- $\{e^{jk\omega_0 t} | \forall integer\ k\}$ is similar to basis of vector space

How to Obtain Fourier Coefficients

- $x(t) = \sum_{k=-\infty}^{+\infty} a_k e^{jk\omega_0 t} \Rightarrow \vec{A} = A_x\hat{x} + A_y\hat{y} + A_z\hat{z}$
- Notice

$$A_x = \vec{A} \cdot \hat{x}$$

- Similarly, we guess

$$a_k = < x(t) \cdot e^{jk\omega_0 t} >$$

Let's double-check

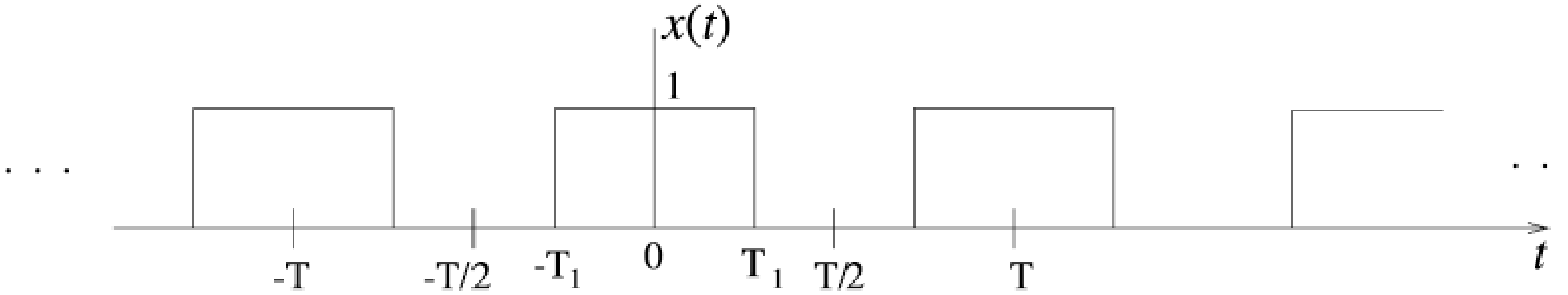
$$< x(t) \cdot e^{jk\omega_0 t} >$$
$$= \sum_{n=-\infty}^{+\infty} a_n < e^{jn\omega_0 t} \cdot e^{jk\omega_0 t} >$$
$$= a_k$$

CT Fourier Series Pair

CT Fourier Series Pair $(\omega_o = 2\pi / T)$

$$\boxed{\begin{aligned} x(t) &= \sum_{k=-\infty}^{+\infty} a_k e^{jk\omega_o t} && \text{(Synthesis equation)} \\ a_k &= \frac{1}{T}\int_T x(t) e^{-jk\omega_o t} dt && \text{(Analysis equation)} \end{aligned}}$$

Example 3.5: Periodic Square Wave



$$a_o = \frac{1}{T}\int_{-T/2}^{T/2} x(t)dt = \frac{2T_1}{T}$$

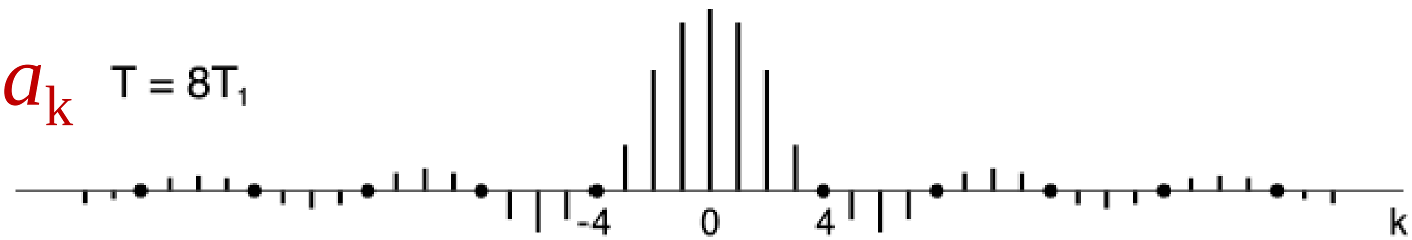
$k \neq 0$

$$a_k = \frac{1}{T}\int_{-T/2}^{T/2} x(t)e^{-jk\omega_0 t}dt = \frac{1}{T}\int_{-T_1}^{T_1} e^{-jk\omega_0 t}dt$$

$(\omega_o = \frac{2\pi}{T})$:

$$= -\frac{1}{jk\omega_o T}e^{-jk\omega_0 t}\Big|_{-T_1}^{T_1} = \frac{\sin(k\omega_o T_1)}{k\pi}$$

a_k T = $8T_1$



Example: Synthesis

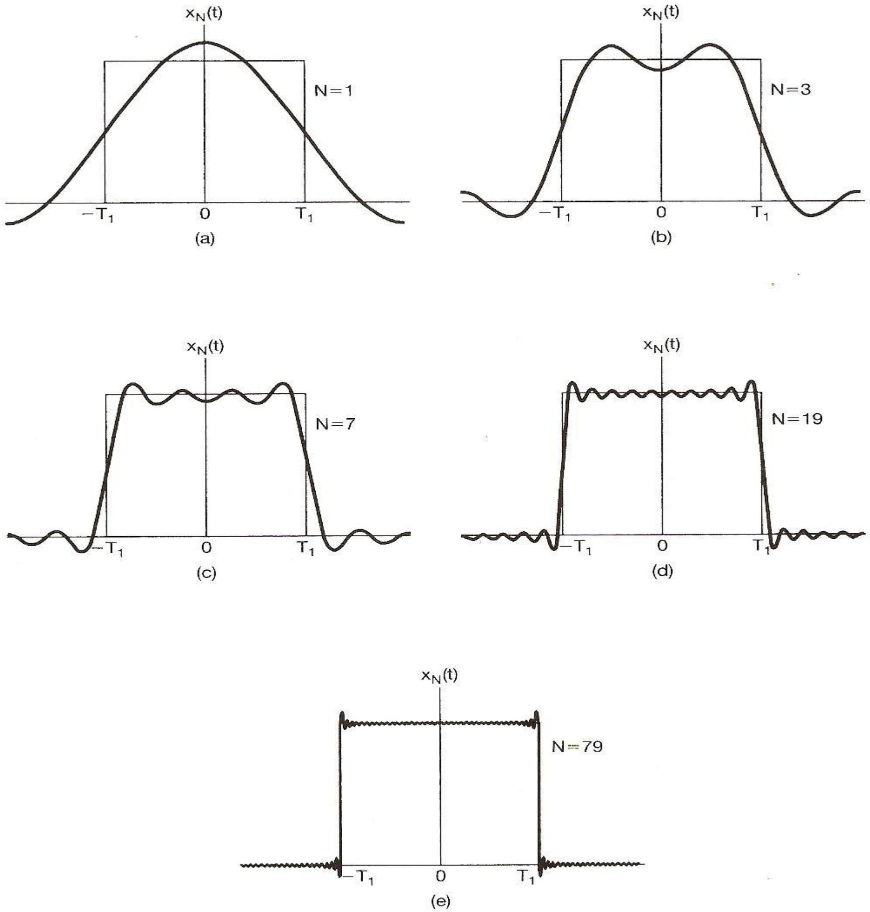


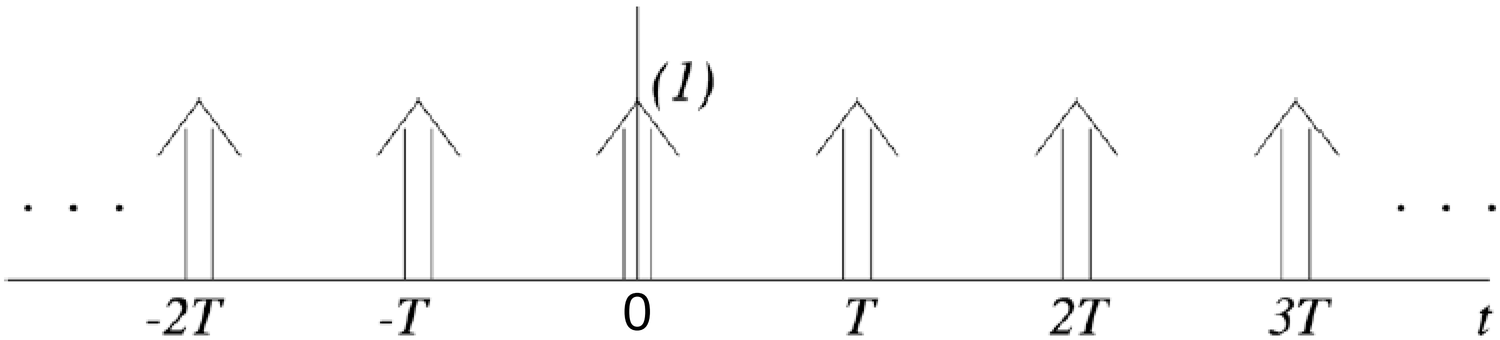
Figure 3.9 Convergence of the Fourier series representation of a square wave: an illustration of the Gibbs phenomenon. Here, we have depicted the finite series approximation $x_N(t) = \sum_{k=-N}^{N} a_k e^{jk\omega_0 t}$ for several values of N.

- **Gibbs Phenomenon: the partial sum in the vicinity of the discontinuity exhibits ripples whose amplitude does not seem to decrease with increasing *N***

Periodic Impulse Train

$$x(t) = \sum_{n=-\infty}^{\infty} \delta(t - nT)$$

— Sampling function, important for sampling later



$$a_k = \frac{1}{T}\int_{-T/2}^{T/2} x(t)e^{-jk\omega_0 t}dt = \frac{1}{T}\int_{-T/2}^{T/2} \delta(t)e^{-jk\omega_0 t}dt$$

$$= \frac{1}{T} \qquad \text{for } \textit{all} \;\; k\,!$$

$$\Downarrow$$

$$x(t) = \frac{1}{T}\sum_{k=-\infty}^{+\infty} e^{jk\omega_0 t}$$

— All components have:
(1) the same amplitude,
&
(2) the same initial phase.

Properties of Fourier Series

- Linearity

$$x(t)\ and\ y(t)\ are\ with\ the\ same\ period,$$
$$x(t)\ \leftrightarrow a_k\quad and\quad y(t) \leftrightarrow b_k,$$
$$\Downarrow$$
$$\alpha x(t) + \beta y(t) \leftrightarrow \alpha a_k + \beta b_k$$

- Time shift (delay leads to linear phase shift)

$$x(t)\ \leftrightarrow a_k$$
$$\Downarrow$$
$$x(t-t_0) \leftrightarrow b_k = a_k e^{-j\omega_0 t_0}$$

- Conjugate symmetry

$$x(t)\ \ is\ \ real\ \ and\ \ x(t) \leftrightarrow a_k \ \ \Longrightarrow \ \ a_{-k} = a_k^*$$

Proof: $a_{-k} = \frac{1}{T}\int_T x(t)e^{jk\omega_o t}dt = \left[\frac{1}{T}\int_T x^*(t)e^{-jk\omega_o t}dt\right]^* = a_k^*$

$\boxed{\mathrm{Re}\{a_{-k}\} + j\,\mathrm{Im}\{a_{-k}\}}$ $\qquad$ $\boxed{\mathrm{Re}\{a_k\} - j\,\mathrm{Im}\{a_k\}}$

$$\Downarrow \quad a_k = \mathrm{Re}\{a_k\} + j\,\mathrm{Im}\{a_k\}$$

$$= \ |a_k|\, e^{j\angle a_k}$$

$$\mathrm{Re}\{a_k\} \text{ is even },\quad \mathrm{Im}\{a_k\} \text{ is odd}$$

or $\quad |a_k|$ is even, $\ \angle a_k$ is odd

- Time reversal

$$x(t) \leftrightarrow a_k \quad \Longrightarrow \quad x(-t) \leftrightarrow b_k = a_{-k}$$

Observation: the effect of sign change for $x(t)$ and a_k are identical

- Time scaling
  - α: positive real number
  - $x(\alpha t)$: periodic with period T/α and fundamental frequency $\omega_0 \alpha$

$$x(t) = \sum_{-\infty}^{\infty} a_k \, e^{jk\omega_0 t} \Longrightarrow x(\alpha t) = \sum_{-\infty}^{\infty} a_k \, e^{jk(\alpha w_0)t}$$

- Multiplication Property

$$x(t) \text{ and } y(t) \text{ are with the same period,}$$
$$x(t) \leftrightarrow a_k \quad and \quad y(t) \leftrightarrow b_k,$$
$$\Downarrow$$
$$x(t)\cdot y(t) \leftrightarrow c_k = \sum_{\mathrm{l}=-\infty}^{\infty} a_l b_{k-l} = a_k * b_k$$

Proof:

$$\sum_{l=-\infty}^{+\infty} a_l e^{jlw_0 t} \cdot \sum_{m=-\infty}^{+\infty} b_m e^{jmw_0 t}$$
$$= \sum_{l,m=-\infty}^{+\infty} a_l\, b_m e^{(\mathrm{l}+m)w_0 t} \xrightarrow{l+m=k} \sum_{k=-\infty}^{+\infty} [\sum_{l=-\infty}^{+\infty} a_l b_{k-l}] e^{jkw_0 t}$$

- Parseval Relation

$$\underbrace{\frac{1}{T}\int_T |x(t)|^2 dt}_{\textit{Average Signal Power}} = \sum_{k=-\infty}^{\infty} \underbrace{|a_k|^2}_{\textit{Power of component } e^{jk\omega_0 t}}$$

Observation: power is the same whether measured in the time-domain or the frequency-domain

Proof:

$$\frac{1}{T}\int_T |x(t)|^2 dt = < \sum_{l=-\infty}^{+\infty} a_l e^{jlw_0 t}, \sum_{k=-\infty}^{+\infty} a_k e^{jkw_0 t} >$$

$$= \sum_{k=-\infty}^{+\infty} < a_k e^{jkw_0 t}, a_k e^{jkw_0 t} > = \sum_{k=-\infty}^{+\infty} |a_k|^2$$

More Properties

- Frequency shifting

$$e^{jM\omega_0 t}x(t) \longleftrightarrow b_k = a_{k-M}$$

- Differentiation

$$\frac{dx(t)}{dt} \longleftrightarrow b_k = jk\omega_0 a_k$$

- Integration

$$\int_{-\infty}^{t} x(t)dt \longleftrightarrow b_k = (\frac{1}{jk\omega_0})a_k$$

TABLE 3.1 PROPERTIES OF CONTINUOUS-TIME FOURIER SERIES

Property	Section	Periodic Signal	Fourier Series Coefficients
		$\left.\begin{matrix} x(t) \\ y(t) \end{matrix}\right\}$ Periodic with period T and fundamental frequency $\omega_0 = 2\pi/T$	a_k b_k
Linearity	3.5.1	$Ax(t) + By(t)$	$Aa_k + Bb_k$
Time Shifting	3.5.2	$x(t - t_0)$	$a_k e^{-jk\omega_0 t_0} = a_k e^{-jk(2\pi/T)t_0}$
Frequency Shifting		$e^{jM\omega_0 t} = e^{jM(2\pi/T)t}x(t)$	a_{k-M}
Conjugation	3.5.6	$x^*(t)$	a^*_{-k}
Time Reversal	3.5.3	$x(-t)$	a_{-k}
Time Scaling	3.5.4	$x(\alpha t)$, $\alpha > 0$ (periodic with period T/α)	a_k
Periodic Convolution		$\int_T x(\tau)y(t-\tau)d\tau$	Ta_kb_k
Multiplication	3.5.5	$x(t)y(t)$	$\sum_{l=-\infty}^{+\infty} a_l b_{k-l}$
Differentiation		$\frac{dx(t)}{dt}$	$jk\omega_0 a_k = jk\frac{2\pi}{T}a_k$
Integration		$\int_{-\infty}^{t} x(t)\,dt$ (finite valued and periodic only if $a_0 = 0$)	$\left(\frac{1}{jk\omega_0}\right)a_k = \left(\frac{1}{jk(2\pi/T)}\right)a_k$
Conjugate Symmetry for Real Signals	3.5.6	$x(t)$ real	$\begin{cases} a_k = a^*_{-k} \\ \Re e\{a_k\} = \Re e\{a_{-k}\} \\ \Im m\{a_k\} = -\Im m\{a_{-k}\} \\ \lvert a_k \rvert = \lvert a_{-k} \rvert \\ \sphericalangle a_k = -\sphericalangle a_{-k} \end{cases}$
Real and Even Signals	3.5.6	$x(t)$ real and even	a_k real and even
Real and Odd Signals	3.5.6	$x(t)$ real and odd	a_k purely imaginary and odd
Even-Odd Decomposition of Real Signals		$\begin{cases} x_e(t) = \mathcal{E}v\{x(t)\} & [x(t) \text{ real}] \\ x_o(t) = \mathcal{O}d\{x(t)\} & [x(t) \text{ real}] \end{cases}$	$\Re e\{a_k\}$ $j\Im m\{a_k\}$

Parseval's Relation for Periodic Signals

$$\frac{1}{T}\int_T |x(t)|^2 dt = \sum_{k=-\infty}^{+\infty} |a_k|^2$$