Chapter 6: Time and Frequency Characterization of Signals and Systems

Department of Electrical & Electronic Engineering

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Outline

- Magnitude and phase of Fourier transform
- First and second order filters
- Non-ideal low-pass filter

Magnitude & Phase

- Magnitude of spectrum, $|X(j\omega)|$ or $|X(e^{j\omega})|$, determines the energy of frequency component
- Phase of spectrum affects the shape of time-domain signal

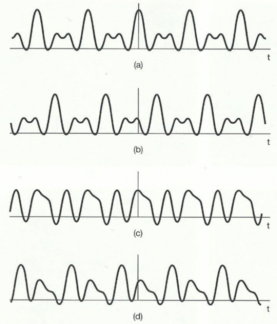


Figure: $x(t) = 1 + \frac{1}{2}cos(2\pi t + \phi_1) + cos(4\pi t + \phi_2) + \frac{2}{3}cos(6\pi t + \phi_3)$

- Case by case, the magnitude and phase information of Fourier transform may carry information of different importance.

Example: Magnitude & Phase of Image

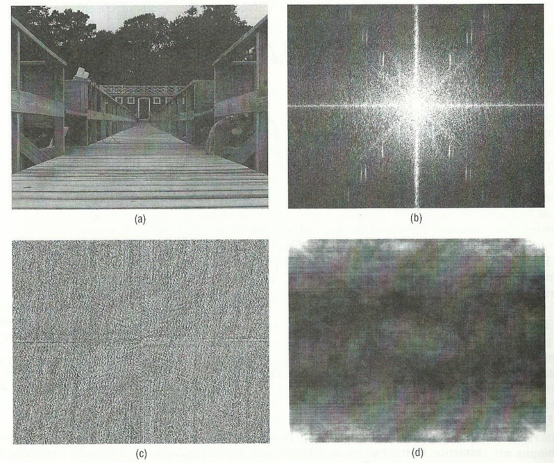


Figure: (a) Original image; (b) Magnitude; (c) Phase; (d) Set phase to zero

Example: Magnitude & Phase of Image

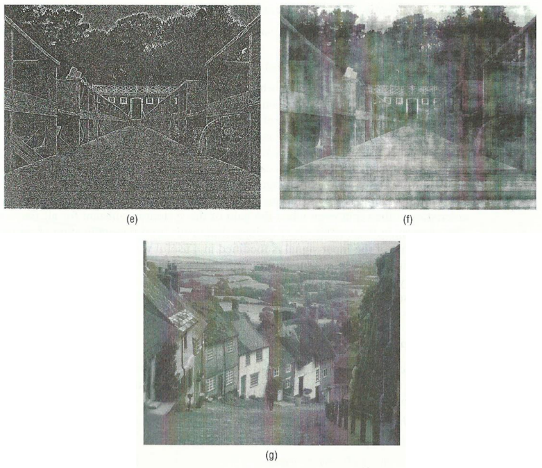
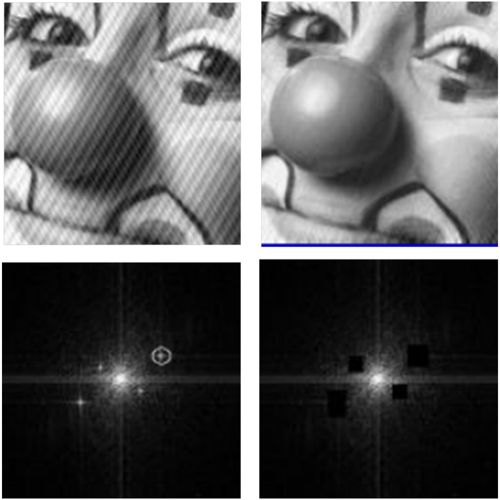


Figure: (e) Set magnitude to 1; (f) Original phase + (g)'s magnitude

Example: Remove Noise

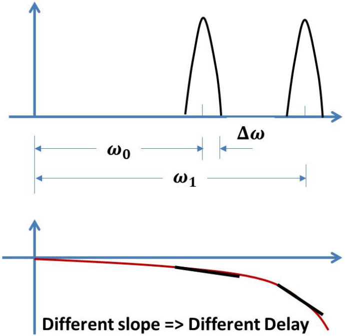


LTI Systems: Linear & Nonlinear Phase

- Linear phase response leads to delay

$$X(j\omega)H(j\omega) = X(j\omega)e^{-j\omega t_0} \longleftrightarrow x(t - t_0)$$

- Non-linear phase response leads to distortion: $e^{-j\omega^2 t_0}$
- Narrow-band signal: phase response can be approximated as linear



Group Delay

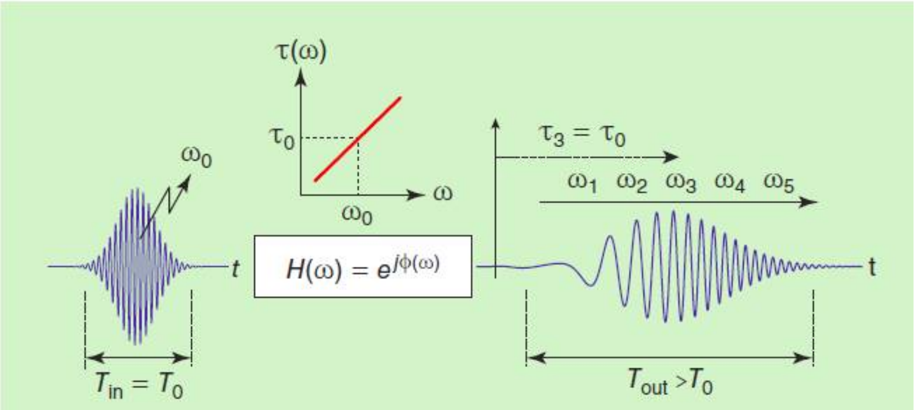
Suppose two narrow-band signals X_0 and X_1 are delivered into a system $H(j\omega) = e^{-j\omega^2 t_0}$:

$$-j\omega^2 t_0 \approx -2j\omega_0 t_0 \omega + j\omega_0^2 t_0$$

$$\begin{aligned} X_0(j\omega)H(j\omega) &= X_0(j\omega)e^{-j\omega^2 t_0} \\ &\approx X_0(j\omega)e^{-2j\omega_0 t_0 \omega} e^{j\omega_0^2 t_0} \\ &\leftrightarrow e^{j\omega_0^2 t_0} x_0(t - 2\omega_0 t_0) \end{aligned}$$

$$X_1(j\omega)H(j\omega) \leftrightarrow e^{j\omega_1^2 t_0} x_1(t - 2\omega_1 t_0)$$

Group delay: different frequency components have different delay


τ(ω)
τ0
ω
ω0
ω0
t
H(ω) = e^{jϕ(ω)}
τ3 = τ0
ω1 ω2 ω3 ω4 ω5
t
Tin = T0
Tout > T0

First-Order Continuous-Time Systems

- Differential equation:

$$\tau \frac{dy(t)}{dt} + y(t) = x(t) \qquad \tau > 0 \quad \text{(why positive?)}$$

- Frequency response:

$$H(j\omega) = \frac{1}{j\omega\tau + 1}$$

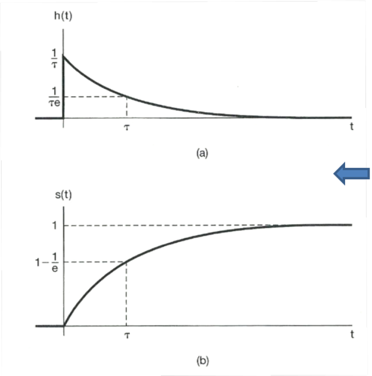
- Impulse response:

$$h(t) = \frac{1}{\tau} e^{-t/\tau} u(t)$$

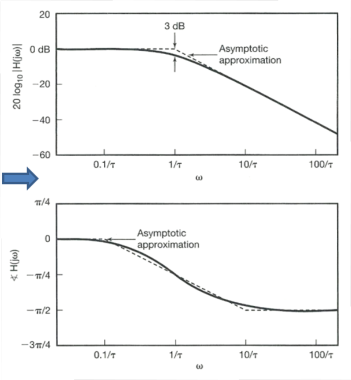
- Step response:

$$s(t) = [1 - e^{-t/\tau}] u(t)$$

First-Order Continuous-Time Systems: Plots

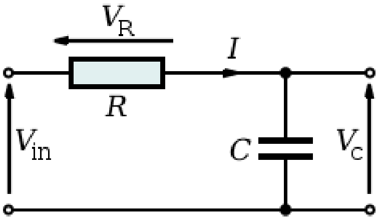


Trade-off



Magnitude: $20log_{10}|H(j\omega)| \approx \begin{cases} 0 & \text{if } \omega < 1/\tau \\ -20log_{10}(\omega) - 20log_{10}(\tau) & \text{if } \omega > 1/\tau. \end{cases}$

Example: RC Circuit



$$RC\frac{dV_c(t)}{dt} + V_c(t) = V_{in}(t)$$

$$\frac{V_c(j\omega)}{V_{in}(j\omega)} = \frac{1}{RCj\omega + 1} \text{ and } \frac{V_R(j\omega)}{V_{in}(j\omega)} = 1 - \frac{V_c(j\omega)}{V_{in}(j\omega)} = \frac{RCj\omega}{RCj\omega + 1}$$

Implementation of adjustable first-order filter

Second-Order Continuous-Time Systems

- Differential equation

$$\frac{d^2y(t)}{dt^2} + 2\zeta\omega_n \frac{dy(t)}{dt} + \omega_n^2 y(t) = \omega_n^2 x(t)$$

- Frequency response (relation with first-order system?)

$$H(j\omega) = \frac{\omega_n^2}{(j\omega - c_1)(j\omega - c_2)} = \frac{M}{j\omega - c_1} - \frac{M}{j\omega - c_2}$$

where $c_1 = -\zeta\omega_n + \omega_n\sqrt{\zeta^2 - 1}$, $c_2 = -\zeta\omega_n - \omega_n\sqrt{\zeta^2 - 1}$ and $M = \frac{\omega_n}{2\sqrt{\zeta^2-1}}$

- Impulse response

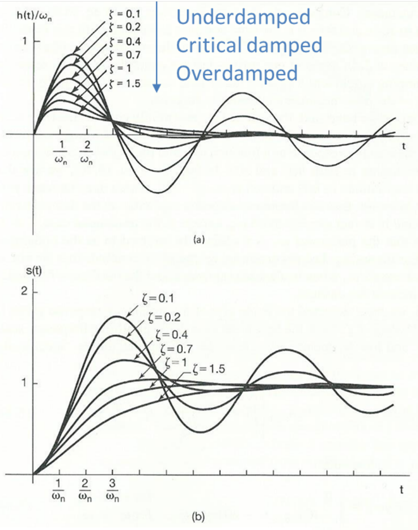
$$h(t) = M[e^{c_1 t} - e^{c_2 t}]u(t)$$

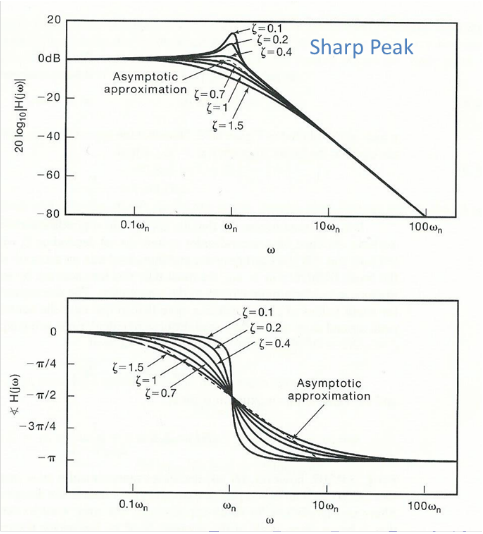
- Step response

$$s(t) = \left\{1 + M\left[\frac{e^{c_1 t}}{c_1} - \frac{e^{c_2 t}}{c_2}\right]\right\}u(t)$$

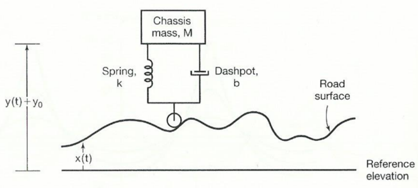
- ζ damping ratio; ω_n undamped natural frequency

Second-Order Continuous-Time Systems: Plots





Example: Automobile Suspension System



- Suspension system of automobile: high-pass or low-pass?
- $\zeta = 1$; ω_n: sport or normal mode;

$$H(j\omega) = \frac{\omega_n^2 + 2\zeta\omega_n(j\omega)}{(j\omega)^2 + 2\zeta\omega_n(j\omega) + \omega_n^2} \text{ where } \omega_n = \sqrt{\frac{k}{M}}, \zeta = \frac{b}{2\omega_n M}$$

Rational Frequency Responses Filter

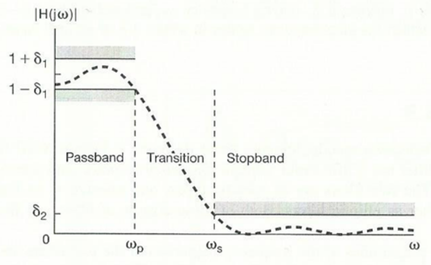
Question

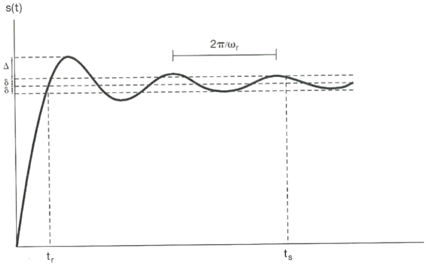
For general causal stable filter with rational frequency response, can it be constructed by first and second order filters?

$$
\begin{aligned}
H(j\omega) &= \frac{a_N(j\omega)^N + a_{N-1}(j\omega)^{N-1} + \ldots + a_0}{b_M(j\omega)^M + b_{M-1}(j\omega)^{M-1} + \ldots + b_0} \quad M \geq N \\
&= \frac{a_N(j\omega)^N + a_{N-1}(j\omega)^{N-1} + \ldots + a_0}{b_M(j\omega + d_1)^{k_1} \ldots (j\omega + d_{M'})^{k_{M'}}} \\
&= \frac{a_N}{b_M}\left[\frac{A_{11}}{(j\omega + d_1)^{k_1}} + \ldots + \frac{A_{1k_1}}{(j\omega + d_1)} + \ldots + \right. \\
&\quad \left. \frac{A_{M'1}}{(j\omega + d_{M'})^{k_{M'}}} + \ldots + \frac{A_{M'k_{M'}}}{(j\omega + d_{M'})}\right]
\end{aligned}
$$

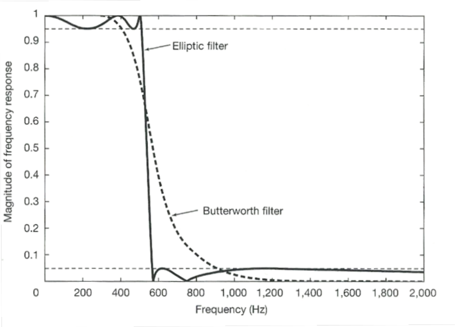
Non-Ideal Low-pass Filter

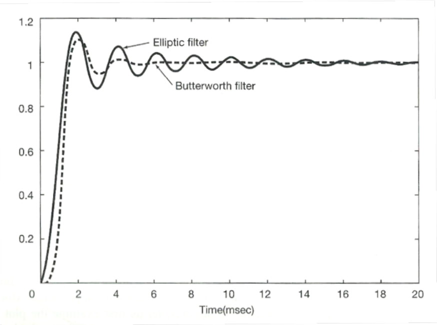
- Ideal low-pass filter: non-causal, sharp edge, implementation issue
- Non-ideal filter: passband (with tolerant ripple) → transition band → stopband
- Step response: rise time, overshot, ringing frequency, setting time





Example: Elliptic Filter v.s. Butterworth Filter



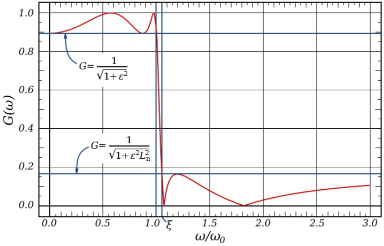


- See textbook, Example 6.3

Elliptic Filter

$$G_n(j\omega) = \frac{1}{\sqrt{1+\epsilon^2 R_n^2(\xi, \omega/\omega_0)}}$$

- where R_n is the nth-order elliptic rational function, ω_0 is the cut-off frequency, ϵ is the ripple factor, ξ is the selectivity factor
- "No other filter of equal order can have a faster transition in gain between the passband and the stopband, for the given values of ripple"



Butterworth Filter

$$G_n(j\omega) = \sqrt{\frac{1}{1+\omega^{2n}}}$$

- "have as flat a frequency response as possible in the passband"
- Maximally flat magnitude filter

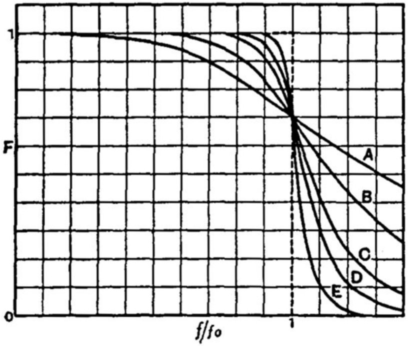


Fig. 3.

Problem

7.29. Figure P7.29(a) shows the overall system for filtering a continuous-time signal using a discrete-time filter. If $X_c(j\omega)$ and $H(e^{j\omega})$ are as shown in Figure P7.29(b), with $1/T = 20$ kHz, sketch $X_p(j\omega)$, $X(e^{j\omega})$, $Y(e^{j\omega})$, $Y_p(j\omega)$, and $Y_c(j\omega)$.

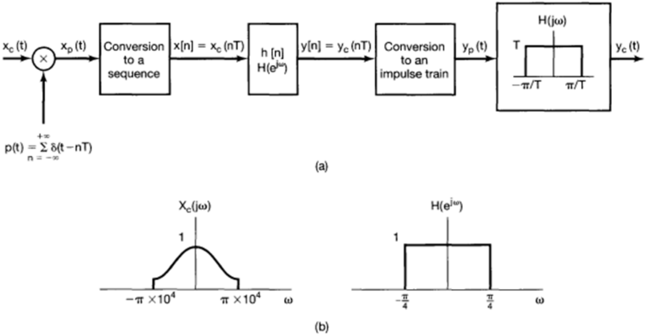


Figure P7.29