

4 Determinants

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4

Determinants (行列式)

4.1-2

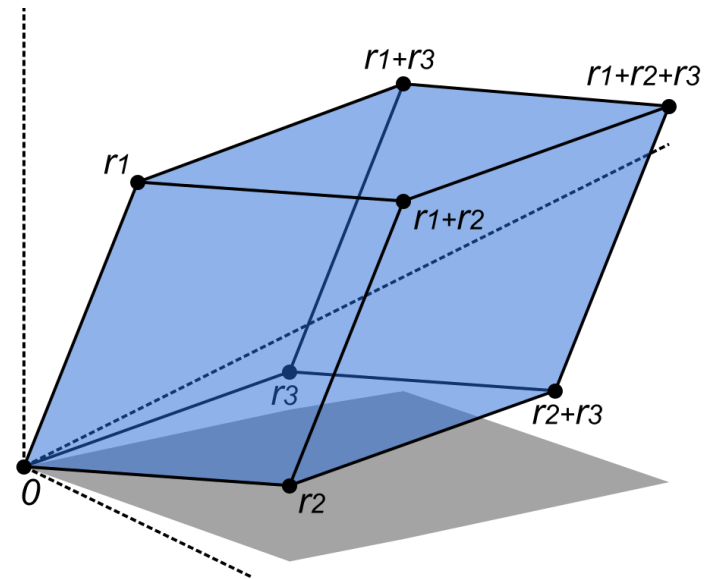
DETERMINANTS AND PROPERTIES

Introduction

Definition

Properties

Calculations



I. Introduction

Using elimination to solve the system of linear equations:

$$\begin{cases} a_{11}x_1 + a_{12}x_2 = b_1, \\ a_{21}x_1 + a_{22}x_2 = b_2. \end{cases}$$

If $a_{11}a_{22} - a_{12}a_{21} \neq 0$, then the solution is

$$x_1 = \frac{b_1a_{22} - a_{12}b_2}{a_{11}a_{22} - a_{12}a_{21}}, \quad x_2 = \frac{a_{11}b_2 - b_1a_{21}}{a_{11}a_{22} - a_{12}a_{21}}.$$

The denominator can be determined by the 4 numbers.

$$D = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = a_{11}a_{22} - a_{12}a_{21}$$

is the determinant of this 2 by 2 coefficient matrix.

$$\begin{cases} a_{11}x_1 + a_{12}x_2 = b_1, \\ a_{21}x_1 + a_{22}x_2 = b_2. \end{cases}$$

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$$D = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}$$

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$$D_1 = \begin{vmatrix} b_1 & a_{12} \\ b_2 & a_{22} \end{vmatrix}$$

$$D_2 = \begin{vmatrix} a_{11} & b_1 \\ a_{21} & b_2 \end{vmatrix}$$

$$x_1 = \frac{b_1 a_{22} - a_{12} b_2}{a_{11} a_{22} - a_{12} a_{21}}, \quad x_2 = \frac{a_{11} b_2 - b_1 a_{21}}{a_{11} a_{22} - a_{12} a_{21}}.$$

So the solution is $x_1 = \frac{D_1}{D}, \quad x_2 = \frac{D_2}{D}.$

For a system of linear equations in 3 unknowns:

$$\begin{cases} a_{11}x_1 + a_{12}x_2 + a_{13}x_3 = b_1, \\ a_{21}x_1 + a_{22}x_2 + a_{23}x_3 = b_2, \\ a_{31}x_1 + a_{32}x_2 + a_{33}x_3 = b_3, \end{cases}$$

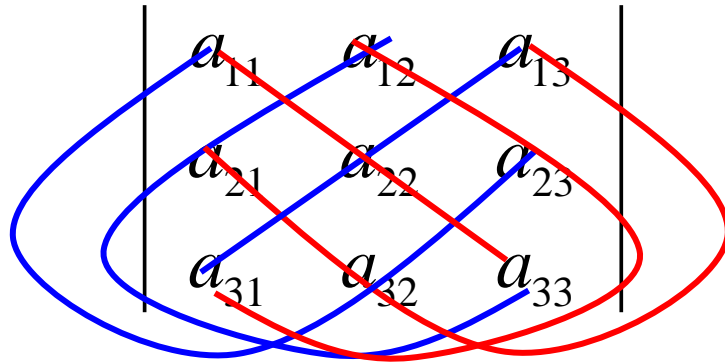
By eliminating x_2, x_3 ,

$$\begin{aligned} & (a_{11}a_{22}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32} \\ & - a_{13}a_{22}a_{31} - a_{12}a_{21}a_{33} - a_{11}a_{23}a_{32})x_1 \\ & = b_1a_{22}a_{33} + b_3a_{12}a_{23} + b_2a_{13}a_{32} \\ & - b_3a_{13}a_{22} - b_2a_{12}a_{33} - b_1a_{23}a_{32}. \end{aligned}$$

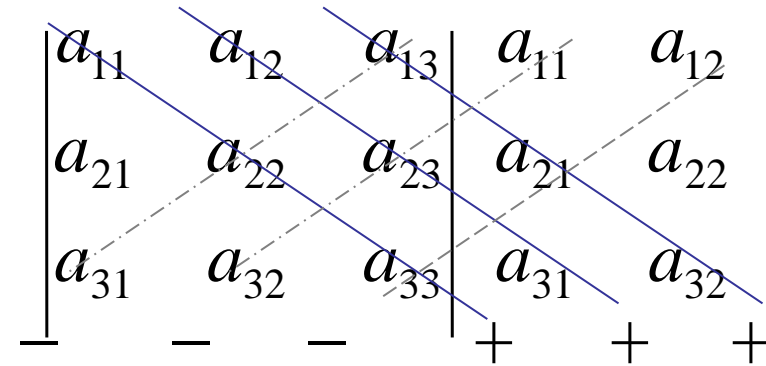
$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$$

$$\begin{aligned} & = a_{11}a_{22}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32} \\ & - a_{11}a_{23}a_{32} - a_{12}a_{21}a_{33} - a_{13}a_{22}a_{31} \end{aligned}$$

(1) 对角线法则 (又称 沙路法, *Sarrus' rule*)



$$= a_{11}a_{22}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32} \\ - a_{13}a_{22}a_{31} - a_{12}a_{21}a_{33} - a_{11}a_{23}a_{32}$$



- The determinant is a number.

- 6 terms (+ or -)

For example, the matrix

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 1 & 1 & 1 \\ 2 & 1 & 1 \end{bmatrix}$$

has determinant $|A|$
 $= (1 + 4 + 3) - (6 + 2 + 1) = -1.$

(2) 展开法则 (*expansion rule*)

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = \underbrace{a_{11}a_{22}a_{33}} + \underbrace{a_{12}a_{23}a_{31}} + \underbrace{a_{13}a_{21}a_{32}} \\ - \underbrace{a_{11}a_{23}a_{32}} - \underbrace{a_{12}a_{21}a_{33}} - \underbrace{a_{13}a_{22}a_{31}}$$

$$\begin{aligned} &= a_{11}(a_{22}a_{33} - a_{23}a_{32}) - a_{12}(a_{21}a_{33} - a_{23}a_{31}) \\ &\quad + a_{13}(a_{21}a_{32} - a_{22}a_{31}) \end{aligned}$$

$$= a_{11} \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - a_{12} \begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + a_{13} \begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}.$$

Use determinants to solve the system of linear equations:

If the determinant $D = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} \neq 0$,

the system has unique solution.

$$\begin{cases} a_{11}x_1 + a_{12}x_2 + a_{13}x_3 = b_1, \\ a_{21}x_1 + a_{22}x_2 + a_{23}x_3 = b_2, \\ a_{31}x_1 + a_{32}x_2 + a_{33}x_3 = b_3, \end{cases}$$

Let $D_1 = \begin{vmatrix} b_1 & a_{12} & a_{13} \\ b_2 & a_{22} & a_{23} \\ b_3 & a_{32} & a_{33} \end{vmatrix}$.

That is, $\begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} D = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$

$$D = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}, \quad D_1 = \begin{vmatrix} b_1 & a_{12} & a_{13} \\ b_2 & a_{22} & a_{23} \\ b_3 & a_{32} & a_{33} \end{vmatrix},$$

$$D_2 = \begin{vmatrix} a_{11} & b_1 & a_{13} \\ a_{21} & b_2 & a_{23} \\ a_{31} & b_3 & a_{33} \end{vmatrix}, \quad D_3 = \begin{vmatrix} a_{11} & a_{12} & b_1 \\ a_{21} & a_{22} & b_2 \\ a_{31} & a_{32} & b_3 \end{vmatrix},$$

Then the solution to this system is

$$x_1 = \frac{D_1}{D}, \quad x_2 = \frac{D_2}{D}, \quad x_3 = \frac{D_3}{D}.$$

II. Definition and expansion (行列式的定义与展开法则)

Definition 1 We now study *the determinant of a square matrix*.
For

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix},$$

the **determinant** (行列式) of A is defined as $ad - bc$, denoted by $|A|$ or $\det(A)$.

For a 3×3 matrix

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix},$$

the determinant $|A|$ equals

$$\begin{aligned} & a_{11}a_{22}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32} \\ & - a_{31}a_{22}a_{13} - a_{21}a_{12}a_{33} - a_{11}a_{32}a_{23}. \end{aligned}$$

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = a_{11} \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - a_{12} \begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + a_{13} \begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}.$$

Let A be a matrix of degree n :

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix} = [a_{ij}]_{n \times n}$$

For $1 \leq i, j \leq n$, let \mathbf{M}_{ij} be the $(n - 1) \times (n - 1)$ matrix resulted from deleting the i -th row and j -th column of A .

Definition 2 Define the **determinant** of a matrix A of degree n as


$$|A| = a_{11}|\mathbf{M}_{11}| - a_{12}|\mathbf{M}_{12}| + \cdots + (-1)^{1+n}a_{1n}|\mathbf{M}_{1n}|.$$

We make an observation. If $|\mathbf{A}|$ is of the form

$$\begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ 0 & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & a_{nn} \end{vmatrix} \quad \text{or} \quad \begin{vmatrix} a_{11} & 0 & \cdots & 0 \\ a_{21} & a_{22} & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{vmatrix} \quad \text{or} \quad \begin{vmatrix} a_{11} & 0 & \cdots & 0 \\ 0 & a_{22} & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & a_{nn} \end{vmatrix}$$

then $|\mathbf{A}| = a_{11}a_{22} \cdots a_{nn}$.

If \mathbf{A} is an *upper triangular* matrix or *lower triangular* matrix (or *diagonal matrix*), then $|\mathbf{A}|$ equals the product of diagonal entries of the matrix.



$$\begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ a_{n1} & 0 & \cdots & 0 \end{vmatrix} = ? \quad \text{(reverse-triangular matrix)}$$

Matrix & Determinant

Matrix

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}$$

Determinant

$$\begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{vmatrix}$$

	Matrix A	Determinant $ A $
外观	行数为 m , 列数为 n	行数与列数相等
	中括号	竖线
本质	表示 mn 个数	表示1个数

Expansion of $\det A$ in cofactors (A的行列式用代数余子式展开)

The definition of determinant is expanded along **row 1**. Actually it can be extended along any row, or any column, resulting in same value of the determinant.

Theorem 1 *The determinant of A can be calculated by expanding along row i ,*

$$|A| = (-1)^{i+1} a_{i1} |M_{i1}| + (-1)^{i+2} a_{i2} |M_{i2}| + \cdots + (-1)^{i+n} a_{in} |M_{in}|,$$

and by expanding along **column j** ,

$$|A| = (-1)^{1+j} a_{1j} |M_{1j}| + (-1)^{2+j} a_{2j} |M_{2j}| + \cdots + (-1)^{n+j} a_{nj} |M_{nj}|.$$

Note: The determinant of the submatrix M_{ij} with the correct sign is also called the **cofactor** (代数余子式), denoted by $C_{ij} = (-1)^{i+j} |M_{ij}|$.

Pay attention to the sign!

For example,

$$D = \begin{vmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{vmatrix},$$

+	-	+	-
-	+	-	+
+	-	+	-
-	+	-	+

Example 1 Let $A = \begin{bmatrix} 1 & 0 & 3 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 3 & 0 & 2 & 3 \end{bmatrix}$.

Notice that $a_{12} = a_{14} = 0$. The determinant

$$|A| = |M_{11}| + 3|M_{13}|$$

$$\begin{aligned}
 &= \begin{vmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 2 & 3 \end{vmatrix} + 3 \begin{vmatrix} 0 & 1 & 0 \\ 0 & 1 & 1 \\ 3 & 0 & 3 \end{vmatrix} \\
 &= (-2 - 3) + 3 \times (3) = 4.
 \end{aligned}$$

Method: To find the value of determinant, choose a row or a column which has most entries equal to 0 to expand.

Theorem 2

A permutation matrix has determinant 1 or -1 .

Recall that for a permutation matrix (若干初等对换矩阵的乘积), each row and each column has exactly one non-zero entry, which is 1.

$$\mathbf{I} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad |\mathbf{I}| = 1.$$

$$\mathbf{P}_1 = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} \quad \begin{aligned} \mathbf{P}_1 &= \mathbf{P}_{13}. \\ |\mathbf{P}_1| &= -1. \end{aligned}$$

$$\mathbf{P}_2 = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix} \quad \begin{aligned} \mathbf{P}_2 &= \mathbf{P}_{23}\mathbf{P}_{12}. \\ |\mathbf{P}_2| &= 1. \end{aligned}$$

III. Properties of Determinants (行列式的性质)

$$\text{Let } \mathbf{A} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}, \mathbf{B} = \begin{bmatrix} x & y \\ z & w \end{bmatrix},$$

$$\begin{aligned} \text{then } |\mathbf{AB}| &= \begin{vmatrix} ax + bz & ay + bw \\ cx + dz & cy + dw \end{vmatrix} \\ &= (ax + bz)(cy + dw) - (ay + bw)(cx + dz) \\ &= adxw + bcyz - adyz - bcxw. \end{aligned}$$

On the other hand, we have

$$\begin{aligned} |\mathbf{A}||\mathbf{B}| &= (ad - bc)(xw - yz) \\ &= adxw + bcyz - adyz - bcxw. \end{aligned}$$

$$\text{Thus } |\mathbf{AB}| = |\mathbf{A}||\mathbf{B}|.$$

This is actually true for the general case (not only for the degree 2 case).

Theorem 3: Product (矩阵乘积的行列式)

$$|\mathbf{AB}| = |\mathbf{A}||\mathbf{B}|, \text{ where } \mathbf{A} \text{ and } \mathbf{B} \text{ are both } n \text{ by } n \text{ matrix.}$$

- Recall that for a square matrix A ,
 either $A = LU$ (LU factorization without permutations),
 or $PA = LU$ (LU factorization with permutations).

By Theorem 2, either $|A| = |L||U|$, or $-|A| = |L||U|$.

Then A is invertible if and only if U is invertible.

(determinant = \pm product of the pivots)

Theorem 4

A matrix A is invertible if and only if $|A| \neq 0$.

If A is invertible, then $|A^{-1}| = 1/|A|$.

Remark $|A^k| = |A|^k$.

Let

$$D = \begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{vmatrix}, \quad D^T = \begin{vmatrix} a_{11} & a_{21} & \cdots & a_{n1} \\ a_{12} & a_{22} & \cdots & a_{n2} \\ \vdots & \vdots & & \vdots \\ a_{1n} & a_{2n} & \cdots & a_{nn} \end{vmatrix},$$

Obviously, $(D^T)^T = D$.

Actually, we also have $D^T = D$.

Property 1 The transpose of A has the same determinant as A itself. (行列式与它的转置行列式相等.)

(注 行列式中行与列具有同等的地位, 因此行列式的性质, 凡是对行成立的对列也同样成立.)

- If Q is an orthogonal matrix (i.e. $Q^T Q = I$), then $|Q|$ is either 1 or -1 .

Property 2 The determinant changes sign when two rows (or columns) are exchanged. (互换行列式的两行(列), 行列式变号.)

$$D = \begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ \vdots & \vdots & & \vdots \\ a_{s1} & a_{s2} & \cdots & a_{sn} \\ \vdots & \vdots & & \vdots \\ a_{t1} & a_{t2} & \cdots & a_{tn} \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{vmatrix}, \quad D_1 = \begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ \vdots & \vdots & & \vdots \\ a_{t1} & a_{t2} & \cdots & a_{tn} \\ \vdots & \vdots & & \vdots \\ a_{s1} & a_{s2} & \cdots & a_{sn} \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{vmatrix}.$$

$$D = -D_1.$$

Exercise

$$\begin{vmatrix} 6 & 7 & 7 & 7 & 6 \\ 6 & 7 & 7 & 7 & 6 \\ 5 & 4 & 3 & 2 & 1 \\ 1 & 7 & 2 & 5 & 6 \\ 1 & 1 & 1 & 1 & 1 \end{vmatrix} = ?$$

$$\begin{vmatrix} 5 & 5 & 10 & 10 & 10 \\ 1 & 1 & 2 & 2 & 2 \\ 5 & 4 & 3 & 2 & 1 \\ 4 & 3 & 6 & 1 & 0 \\ 1 & 4 & 2 & 3 & 1 \end{vmatrix} = ?$$

Corollary If two rows (or columns) of A are equal, then $|A| = 0$.
(如果行列式有两行(列)完全相同, 则此行列式为零.)

Property 3 Scalar multiplication: 行列式的某一行(列)中所有的元素都乘以同一数 k , 等于用数 k 乘此行列式.

$$\begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ \vdots & \vdots & & \vdots \\ \textcolor{red}{ka}_{l1} & \textcolor{red}{ka}_{l2} & \cdots & \textcolor{red}{ka}_{ln} \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{vmatrix} = \textcolor{red}{k} \begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ \vdots & \vdots & & \vdots \\ a_{l1} & a_{l2} & \cdots & a_{ln} \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{vmatrix}.$$

Be careful!

$$|kA| = k|A| ?? \quad \textcolor{red}{\bigotimes}$$

$$|kA| = k^n |A| \quad \textcolor{green}{\bigotimes}$$

Property 4 Vector addition: 若行列式的某一列(行)的元素都是两数之和:

$$D = \begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1i} + a'_{1i} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2i} + a'_{2i} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots & & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{ni} + a'_{ni} & \cdots & a_{nn} \end{vmatrix},$$

则 D 等于下列两个行列式之和:

$$D = \begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1i} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2i} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots & & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{ni} & \cdots & a_{nn} \end{vmatrix} + \begin{vmatrix} a_{11} & a_{12} & \cdots & a'_{1i} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a'_{2i} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots & & \vdots \\ a_{n1} & a_{n2} & \cdots & a'_{ni} & \cdots & a_{nn} \end{vmatrix}.$$

Property 5 倍加变换: 将行列式的某一行(列)的各元素乘以同一数然后加到另一行(列)对应的元素上去, 行列式不变.

$$= \begin{vmatrix} a_{11} & \cdots & a_{1i} & \cdots & a_{1j} & \cdots & a_{1n} \\ a_{21} & \cdots & a_{2i} & \cdots & a_{2j} & \cdots & a_{2n} \\ \vdots & & \vdots & & \vdots & & \vdots \\ a_{n1} & \cdots & a_{ni} & \cdots & a_{nj} & \cdots & a_{nn} \end{vmatrix} \quad \begin{matrix} \text{---} \\ \text{---} \\ \text{---} \\ \text{---} \end{matrix} \begin{matrix} \text{---} \\ \text{---} \\ \text{---} \\ \text{---} \end{matrix} \begin{matrix} \text{---} \\ \text{---} \\ \text{---} \\ \text{---} \end{matrix}$$

Summary of

The properties of Determinant

(可用于计算)

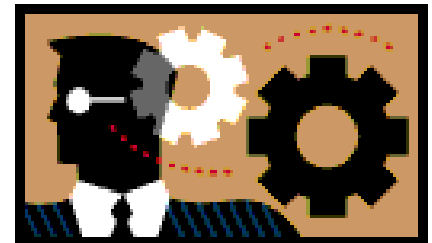
转置不改

换行反号

因子能提

行列可拆

倍加不变



We consider the effects of **elementary operations** on determinants.

Example 2 Compute the determinant of the matrix \mathbf{A} , where

$$\mathbf{A} = \begin{bmatrix} 1 & -4 & 2 \\ -2 & 8 & -9 \\ -1 & 7 & 0 \end{bmatrix}.$$

Solution The **strategy** is to reduce \mathbf{A} to echelon form and then to use the fact that the determinant of a triangular matrix is the product of the diagonal entries.

The first two row replacements in column 1 do not change the determinant: (倍加不变)

$$|\mathbf{A}| = \begin{vmatrix} 1 & -4 & 2 \\ -2 & 8 & -9 \\ -1 & 7 & 0 \end{vmatrix} = \begin{vmatrix} 1 & -4 & 2 \\ 0 & 0 & -5 \\ -1 & 7 & 0 \end{vmatrix} = \begin{vmatrix} 1 & -4 & 2 \\ 0 & 0 & -5 \\ 0 & 3 & 2 \end{vmatrix}$$

An interchange of rows 2 and 3 reverses the sign of the determinant

(换行反号), so $|\mathbf{A}| = - \begin{vmatrix} 1 & -4 & 2 \\ 0 & 3 & 2 \\ 0 & 0 & -5 \end{vmatrix} = -(1)(3)(-5) = 15.$

多种方法可以根据需要进行选择.

Example 3 Find the determinant of the matrix \mathbf{A} , where

$$\mathbf{A} = \begin{bmatrix} 1 & 1 & 3 & 0 \\ 1 & 1 & 4 & 5 \\ 0 & 1 & 0 & 1 \\ 3 & 0 & 10 & 3 \end{bmatrix}.$$

Solution

$$\mathbf{A} \rightarrow \mathbf{B} = \begin{bmatrix} 1 & 1 & 3 & 0 \\ 0 & 0 & 1 & 5 \\ 0 & 1 & 0 & 1 \\ 0 & -3 & 1 & 3 \end{bmatrix},$$

Thus

$$|\mathbf{A}| = |\mathbf{B}| = \begin{vmatrix} 0 & 1 & 5 \\ 1 & 0 & 1 \\ -3 & 1 & 3 \end{vmatrix} = \begin{vmatrix} 0 & 1 & 5 \\ 1 & 0 & 1 \\ 0 & 1 & 6 \end{vmatrix} = - \begin{vmatrix} 1 & 5 \\ 1 & 6 \end{vmatrix} = -1.$$

Remark 任何行列式总可以利用三种**行变换**把行列式化为上三角形行列式或下三角形行列式.

任何行列式总可以利用三种**列变换**把行列式化为上三角形行列式或下三角形行列式.

三角化法
(Using elementary operations to find determinants)

Exercise

$$D = \begin{vmatrix} 4 & 1 & 2 & 4 \\ 1 & 2 & 0 & 2 \\ 10 & 5 & 2 & 0 \\ 1 & 1 & 1 & 7 \end{vmatrix}$$

Example 4 Find the determinant:

$$D_n = \begin{vmatrix} a & b & b & \cdots & b \\ b & a & b & \cdots & b \\ b & b & a & \cdots & b \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ b & b & b & \cdots & a \end{vmatrix}_{n \times n} .$$

Solution

$$D_n = \begin{vmatrix} a & b & \cdots & b \\ b & a & \cdots & b \\ \vdots & \vdots & \ddots & \vdots \\ b & b & \cdots & a \end{vmatrix}$$

D_n 的每行元素之和均为 $a+(n-1)b$
把各列加到第1列

$$D_n = \begin{vmatrix} a+(n-1)b & b & \cdots & b \\ a+(n-1)b & a & \cdots & b \\ \vdots & \vdots & \ddots & \vdots \\ a+(n-1)b & b & \cdots & a \end{vmatrix} = [a+(n-1)b] \begin{vmatrix} 1 & b & \cdots & b \\ 1 & a & \cdots & b \\ \vdots & \vdots & \ddots & \vdots \\ 1 & b & \cdots & a \end{vmatrix}$$

$$= [a+(n-1)b] \begin{vmatrix} 1 & b & \cdots & b \\ & a-b & \cdots & 0 \\ & & \ddots & \vdots \\ & & & a-b \end{vmatrix}$$

$$= [a+(n-1)b](a-b)^{n-1}$$

将第1行乘 (-1) 加到
其余各行，化为上
三角行列式

Key words:

Definition (Expansion)

Properties

Using elementary operations to find determinants

Homework

See Blackboard

