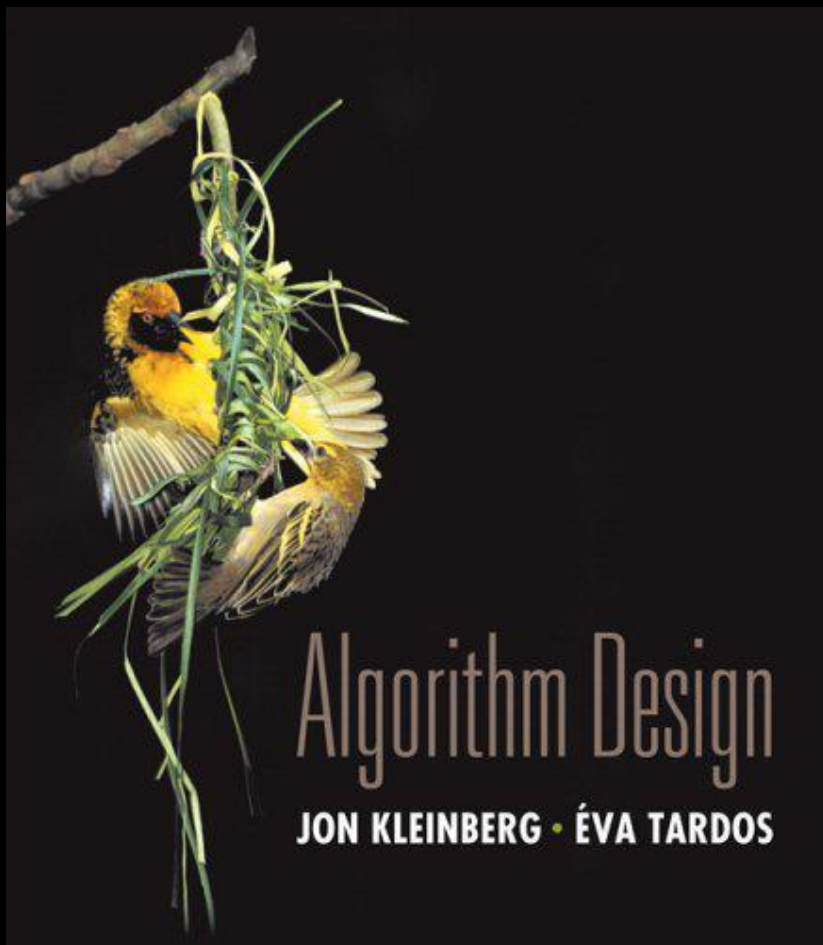


Chapter 7

Network Flow



Slides by Kevin Wayne.
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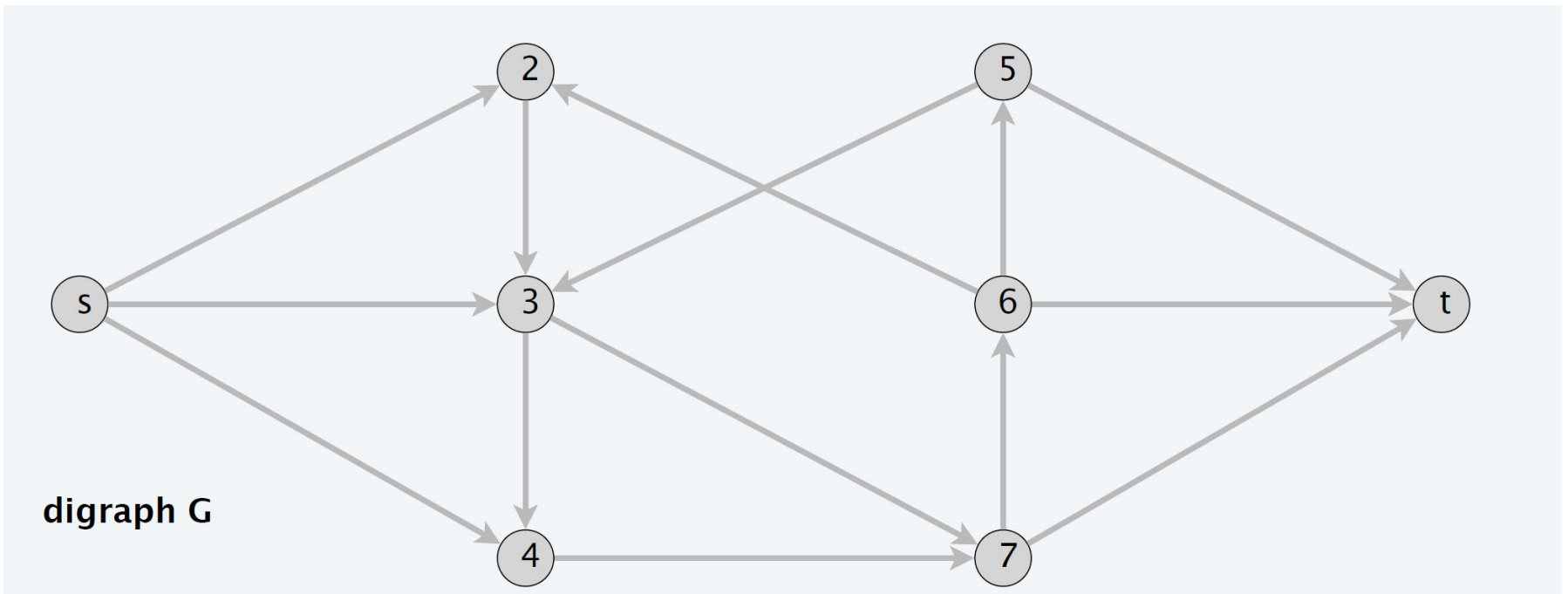
7.6 Disjoint Paths in Directed and Undirected Graphs

Edge-disjoint paths

Def. Two paths are **edge-disjoint** if they have no edge in common.

Edge-disjoint paths problem. Given a digraph $G = (V, E)$ and two nodes s and t , find the max number of edge-disjoint $s \rightsquigarrow t$ paths.

Ex. Communication networks.

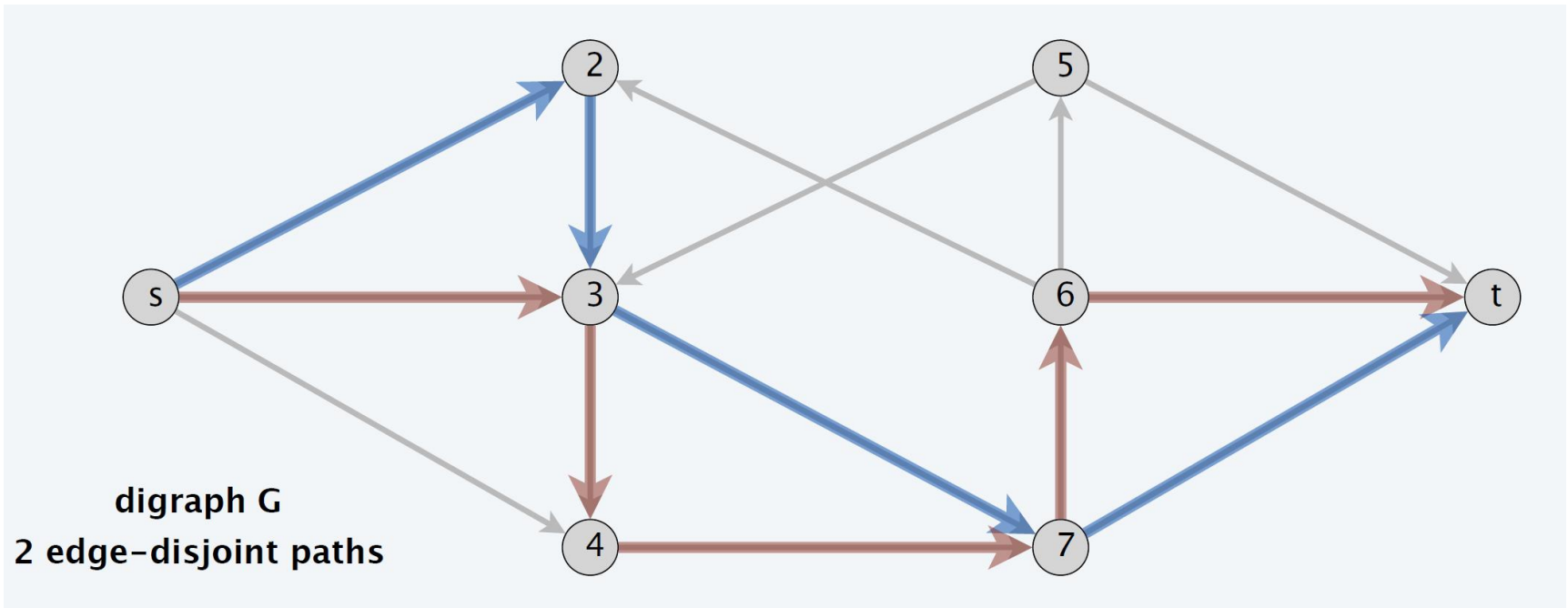


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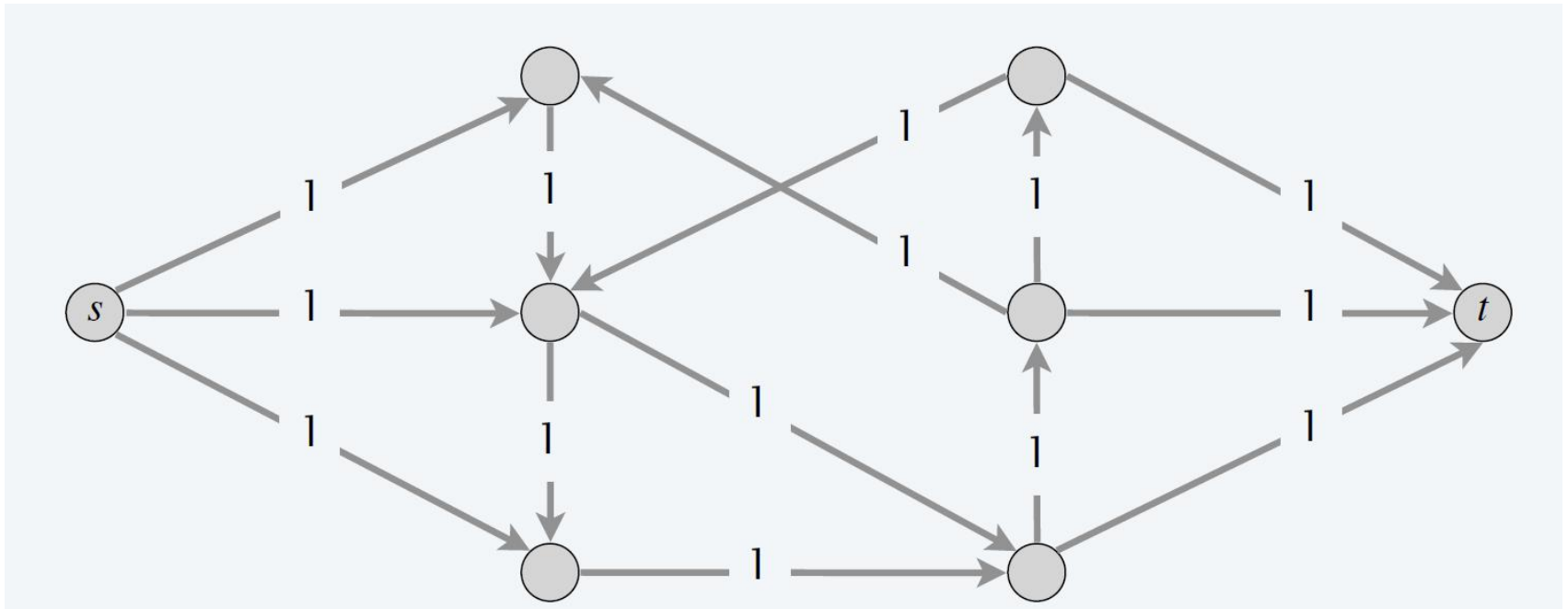
Edge-disjoint paths

Max-flow formulation. Assign unit capacity to every edge.

Theorem. Max number of edge-disjoint $s \rightsquigarrow t$ paths = value of max flow.

Pf. \geq

- Suppose there are k edge-disjoint $s \rightsquigarrow t$ paths P_1, \dots, P_k .
- Set $f(e) = 1$ if e participates in some path P_j ; else set $f(e) = 0$.
- Since paths are edge-disjoint, f is a flow of value k . ▀



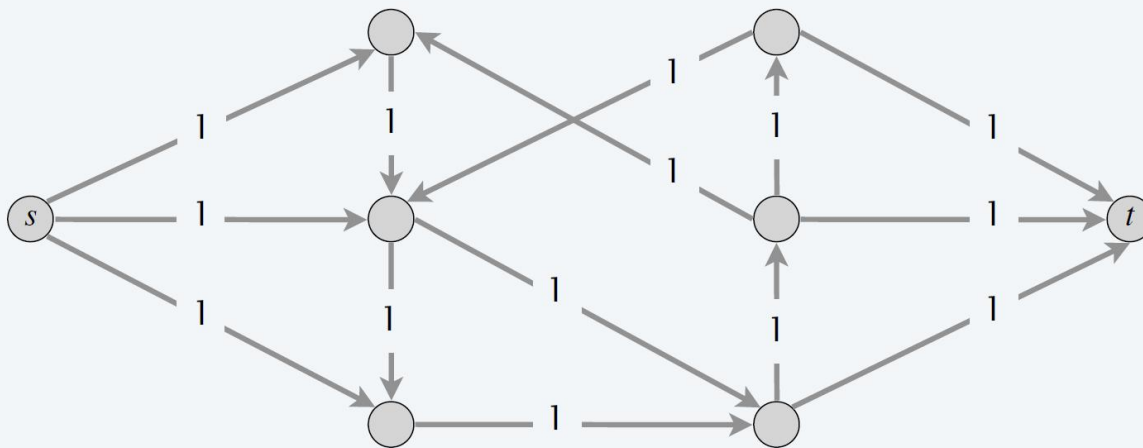
Edge-disjoint paths

Max-flow formulation. Assign unit capacity to every edge.

Theorem. Max number of edge-disjoint $s \rightsquigarrow t$ paths = value of max flow.

Pf. \leq

- Suppose max flow value is k .
- Integrality theorem \Rightarrow there exists 0-1 flow f of value k .
- Consider edge (s, u) with $f(s, u) = 1$.
 - by flow conservation, there exists an edge (u, v) with $f(u, v) = 1$
 - continue until reach t , always choosing a new edge
- Produces k (not necessarily simple) edge-disjoint paths

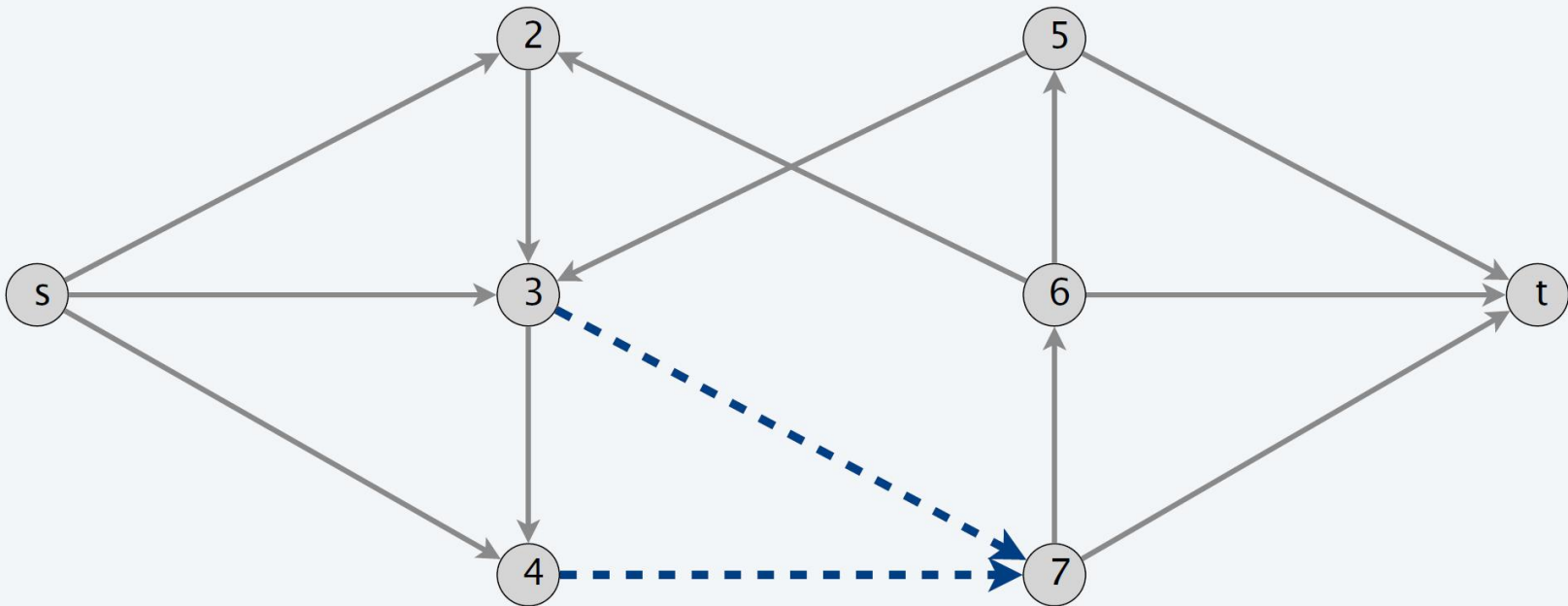


can eliminate cycles
to get simple paths
in $O(mn)$ time if desired
(flow decomposition)

Network connectivity

Def. A set of edges $F \subseteq E$ **disconnects** t from s if every $s \rightsquigarrow t$ path uses at least one edge in F .

Network connectivity. Given a digraph $G = (V, E)$ and two nodes s and t , find min number of edges whose removal disconnects t from s .

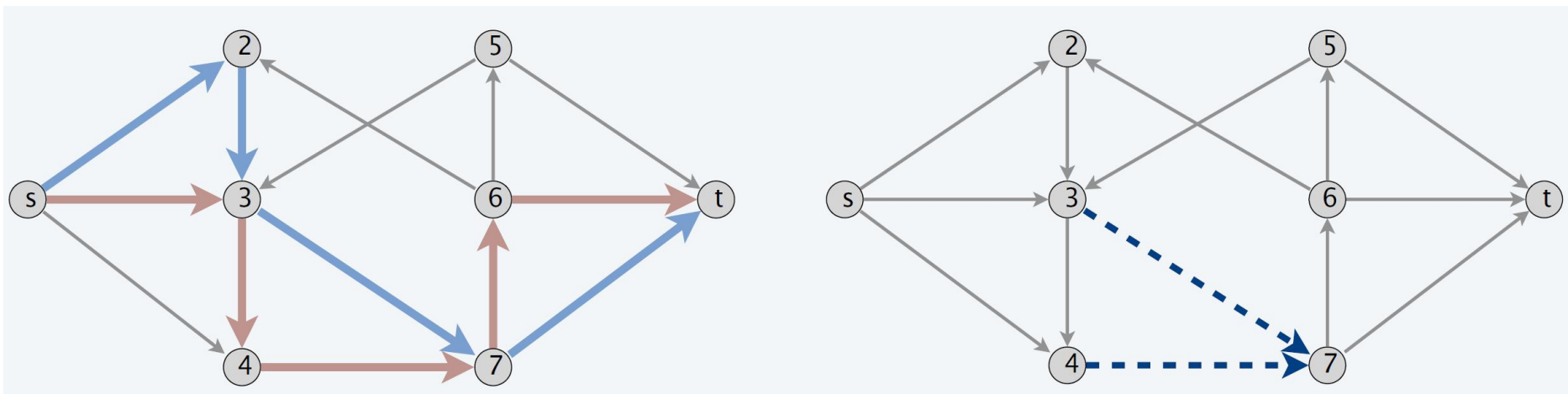


Menger's theorem

Theorem. [Menger 1927] The max number of edge-disjoint $s \rightsquigarrow t$ paths equals the min number of edges whose removal disconnects t from s .

Pf. \leq

- Suppose the removal of $F \subseteq E$ disconnects t from s , and $|F| = k$.
- Every $s \rightsquigarrow t$ path uses at least one edge in F .
- Hence, the number of edge-disjoint paths is $\leq k$.

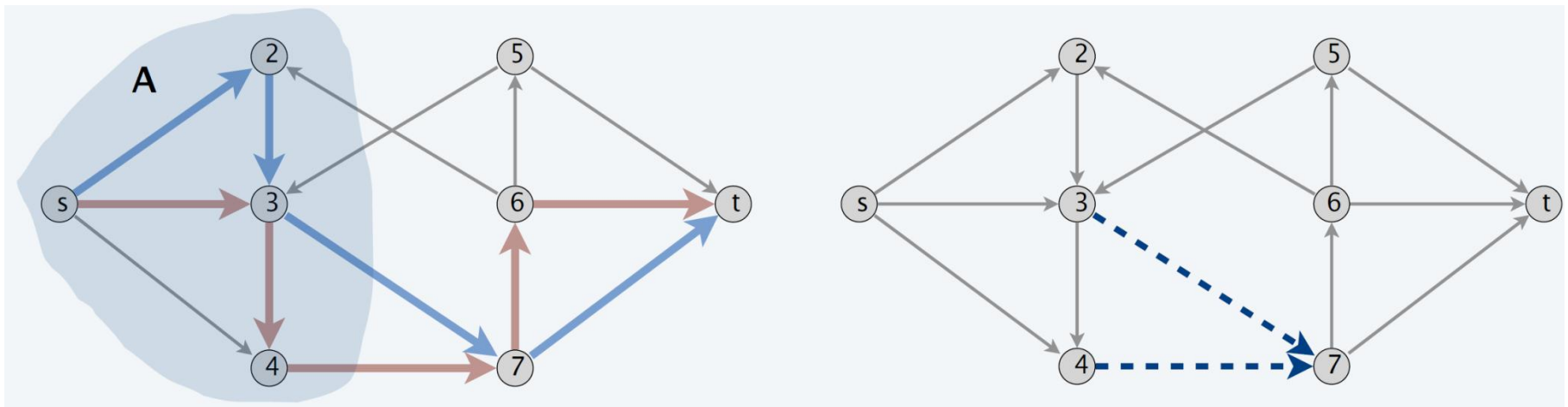


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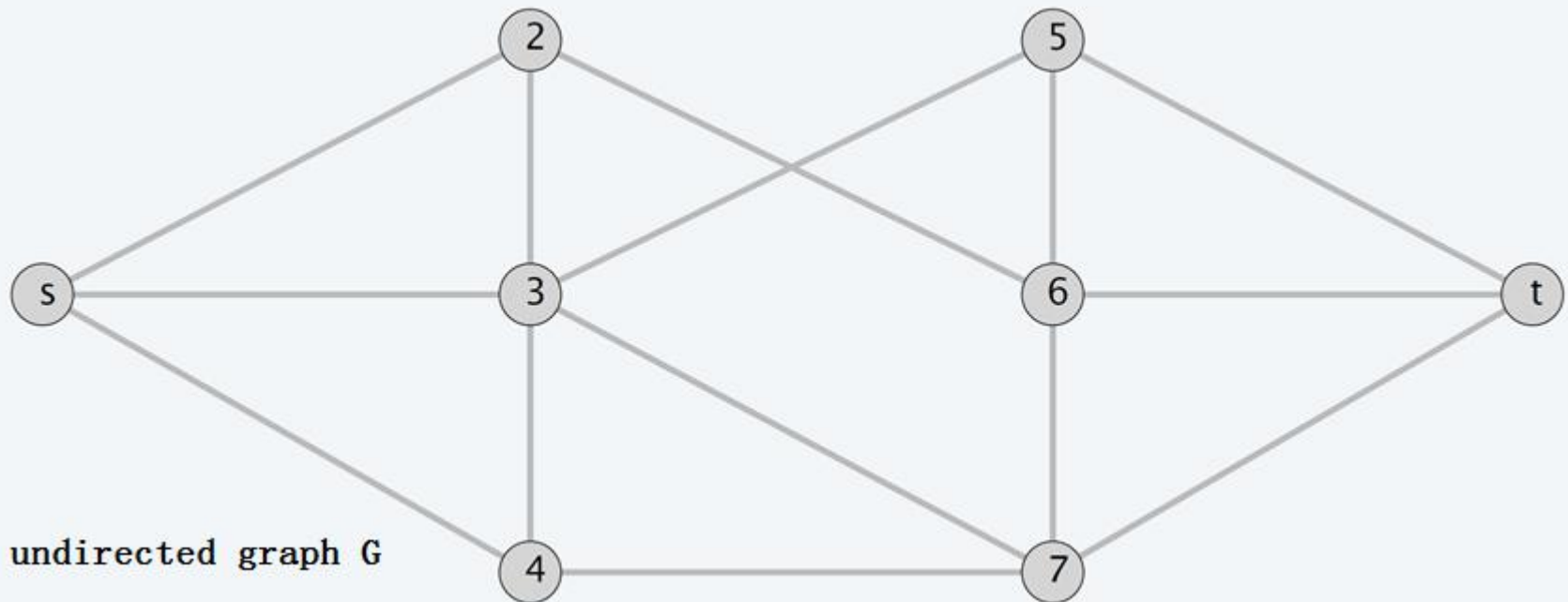
- Suppose max number of edge-disjoint paths is k .
- Then value of max flow = k .
- Max-flow min-cut theorem \Rightarrow there exists a cut (A, B) of capacity k .
- Let F be set of edges going from A to B .
- $|F| = k$ and disconnects t from s .



Edge-disjoint paths in undirected graphs

Def. Two paths are **edge-disjoint** if they have no edge in common.

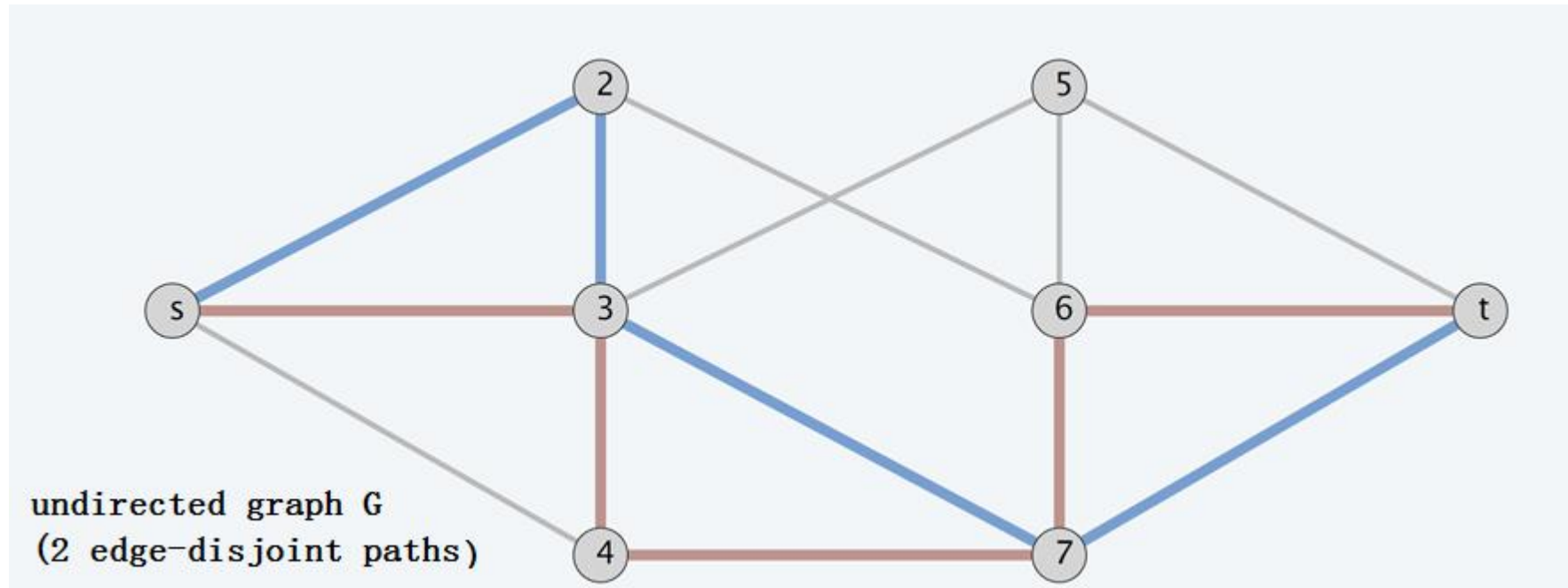
Edge-disjoint paths problem in undirected graphs. Given a graph $G = (V, E)$ and two nodes s and t , find the max number of edge-disjoint s - t paths.



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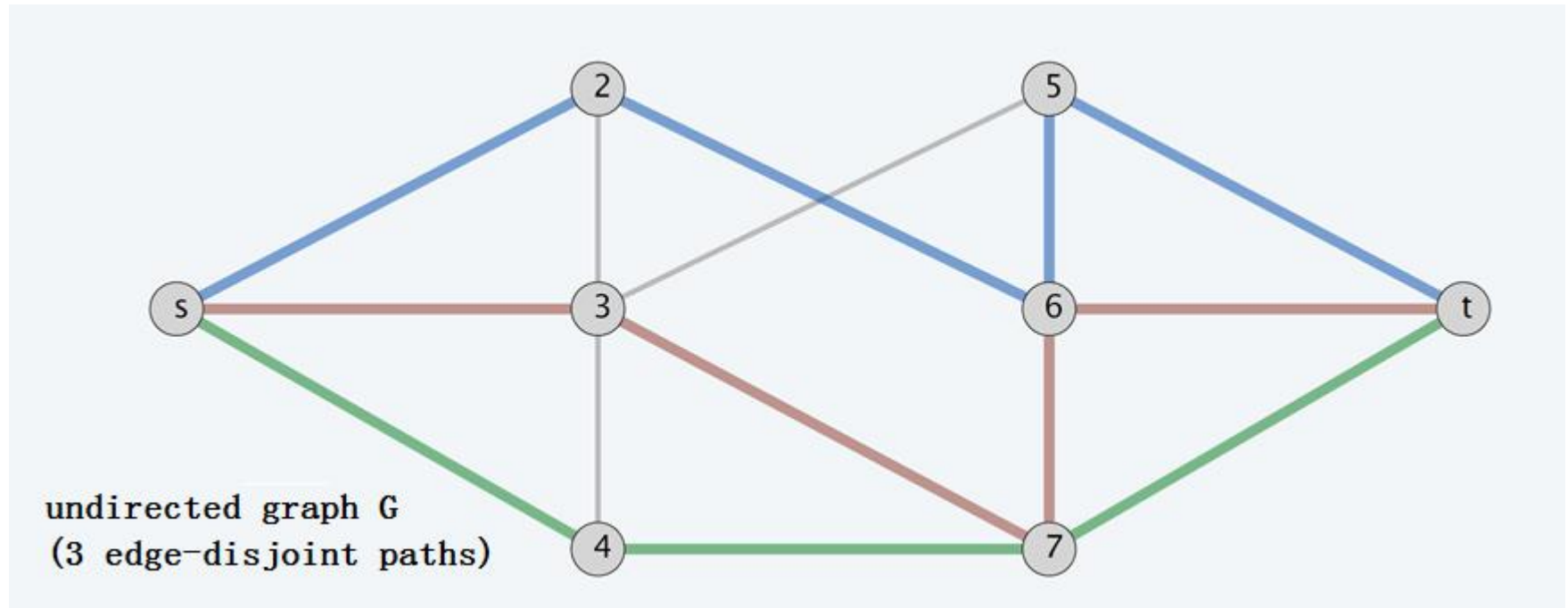
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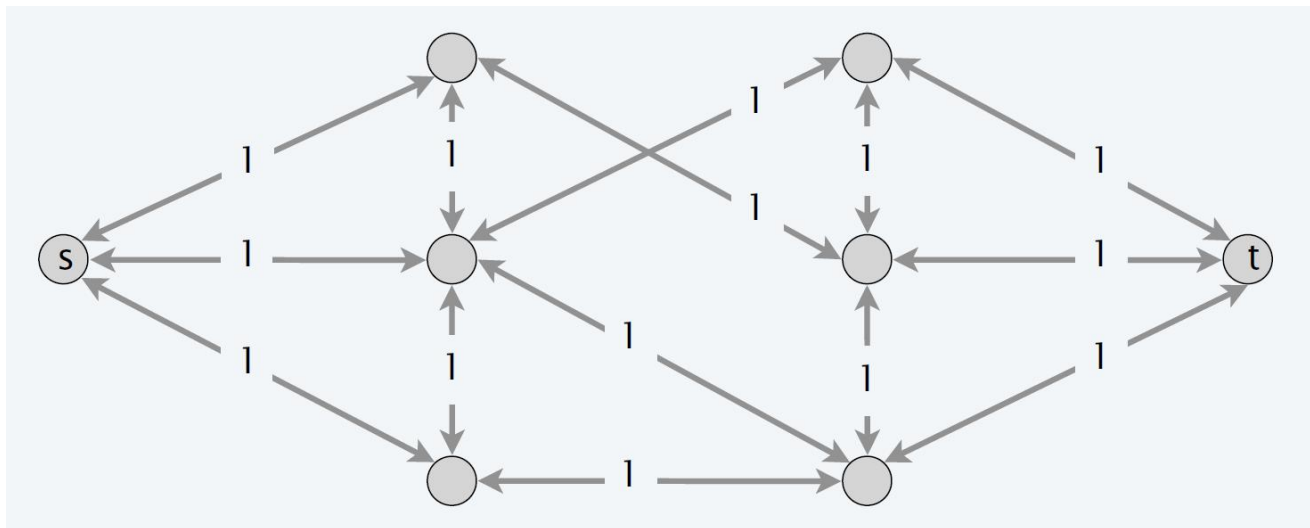


Edge-disjoint paths in undirected graphs

Max-flow formulation. Replace each edge with two antiparallel edges and assign unit capacity to every edge.

Observation. Two paths P_1 and P_2 may be edge-disjoint in the digraph but not edge-disjoint in the undirected graph.

if P_1 uses edge (u, v)
and P_2 uses its antiparallel edge (v, u)



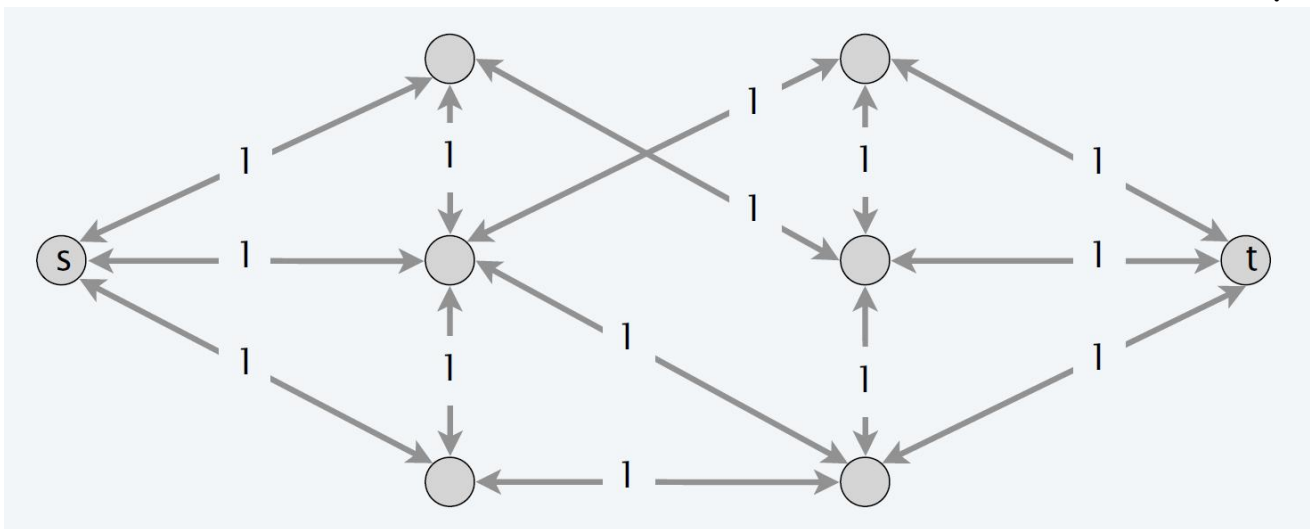
Edge-disjoint paths in undirected graphs

Max-flow formulation. Replace each edge with two antiparallel edges and assign unit capacity to every edge.

Lemma. In any flow network, there exists a maximum flow f in which for each pair of antiparallel edges e and e' : either $f(e) = 0$ or $f(e') = 0$ or both. Moreover, integrality theorem still holds.

Pf. [by induction on number of such pairs]

- Suppose $f(e) > 0$ and $f(e') > 0$ for a pair of antiparallel edges e and e' .
- Set $f(e) = f(e) - \delta$ and $f(e') = f(e') - \delta$, where $\delta = \min \{ f(e), f(e') \}$.
- f is still a flow of the same value but has one fewer such pair



Edge-disjoint paths in undirected graphs

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Lemma. In any flow network, there exists a maximum flow f in which for each pair of antiparallel edges e and e' : either $f(e) = 0$ or $f(e') = 0$ or both. Moreover, integrality theorem still holds.

Theorem. Max number of edge-disjoint $s \rightsquigarrow t$ paths = value of max flow.
Pf. Similar to proof in digraphs; use lemma.

