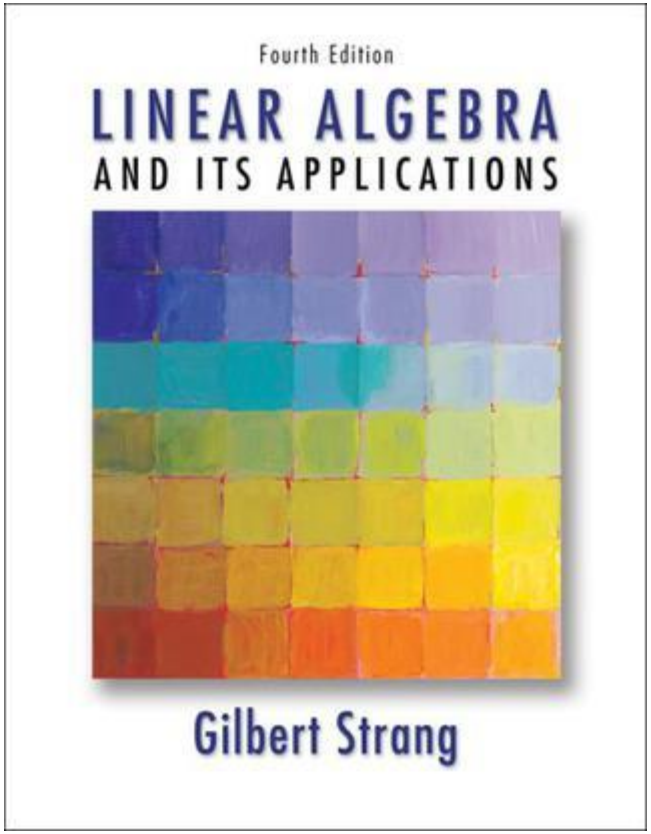
REVIEW - Midterm

Matrices and Gaussian Elimination

Vector Spaces

Orthogonality



Matrices and Gaussian Elimination

Linear system $\xrightarrow{\text{Gaussian Eliminations}}$ Equivalent system

# of Solutions:
Inconsistent: 0
Consistent $\begin{cases} \infty \\ 1 \end{cases}$

Augmented matrix $[\boldsymbol{A}\ \ \boldsymbol{b}]$ $\xrightarrow[r_i \leftrightarrow r_j,\, kr_i (k \neq 0),\, r_j + kr_i]{\text{Elementary row operations}}$ Row echelon form $[\boldsymbol{U}\ \ \boldsymbol{c}]$; Reduced echelon form $[\boldsymbol{R}\ \ \boldsymbol{d}]$

Matrix operations
$\boldsymbol{A}+\boldsymbol{B}, k\boldsymbol{A}$
$\boldsymbol{AB}, \boldsymbol{A}^{-1}, \boldsymbol{A}^T$

rank

LU factorization
LDU factorization

One linear system = Two triangular system

Determine $\boldsymbol{A}$ is invertible;
Find $\boldsymbol{A}^{-1}$;
rank($\boldsymbol{A}$);
$\boldsymbol{A} = \boldsymbol{LU}$; $\boldsymbol{A} = \boldsymbol{LDU}$; …
Solve $\boldsymbol{Ax} = \boldsymbol{0}$; $\boldsymbol{Ax} = \boldsymbol{b}$;
Find bases;
Dependency
… … …

Matrix

- $\boldsymbol{A}+\boldsymbol{B}, k\boldsymbol{A}, \boldsymbol{AB}, \boldsymbol{A}^T$
  - operations and properties
- four fundamental subspaces
- Elementary operations
- Rank
  - Properties; full column/row rank; rank-nullity theorem
- Matrix factorization
  - $\boldsymbol{A}=\boldsymbol{LU}$; $\boldsymbol{A}=\boldsymbol{LDU}$
  - $\boldsymbol{PA}=\boldsymbol{LDU}$ ($\boldsymbol{P}$: permutation matrix)
- Square matrix
  - Invertible: $\boldsymbol{A}^{-1}$ **(**$\boldsymbol{A}$ is invertible is equivalent to …; Gauss-Jordan method**)**
  - $\boldsymbol{A}^k$ (special matrices: $\boldsymbol{A}=a\boldsymbol{I}+\boldsymbol{B}$; $\boldsymbol{A}=\boldsymbol{uv}^{\mathrm{T}}$; elementary matrices; matrices related to transformations in geometry; …)

Vector Spaces

Vector Spaces

Vectors:

Operations $\begin{cases} \text{Addition;} \\ \text{Scalar multiplication.} \end{cases}$

Linear combinations;

Linear dependence and linear independence.

Subspaces

Spanning sets and spanned subspaces

Basis:

$\begin{cases} \text{Spanning set} \\ \text{Linear independent} \end{cases}$

A matrix $\boldsymbol{A}_{m\times n}$:

row space

column space

nullspace

left nullspace

Basis?

Dimension?

Linear transformation $T: \mathbb{R}^n \to \mathbb{R}^m$ $\longleftrightarrow$ $\boldsymbol{A}_{m\times n}$

Rotation

Reflection

Projections

$T(\boldsymbol{u}) = \boldsymbol{A}\boldsymbol{u}$

range$(T) = C(\boldsymbol{A})$

kernel$(T) = N(\boldsymbol{A})$

Vector Spaces and Vectors

- Vector space
- Subspaces: Spanning sets and spanned subspaces
- Orthogonal Subspaces
- Orthogonal complement
- Linear combinations
- Linear dependence and linear independence
- Basis
- Inner product
- Length
- Cosines
- Orthogonal vectors

Linear Transformation & Matrix Representation

$E=\{\boldsymbol{v}_1,\boldsymbol{v}_2,\cdots,\boldsymbol{v}_n\}$ are a basis for V,

$F=\{\boldsymbol{w}_1,\boldsymbol{w}_2,\cdots,\boldsymbol{w}_m\}$ are a basis for W.

Each linear transformation T from V to W is represented by a matrix $\boldsymbol{A}$.

$$T(\boldsymbol{v}_j)=\boldsymbol{A}\boldsymbol{v}_j=a_{1j}\boldsymbol{w}_1+a_{2j}\boldsymbol{w}_2+\cdots+a_{mj}\boldsymbol{w}_m$$

$$\boldsymbol{A}=\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}$$

$$\downarrow \qquad \downarrow \qquad \downarrow$$

$$[T(\boldsymbol{v}_1)]_F \quad [T(\boldsymbol{v}_2)]_F \quad [T(\boldsymbol{v}_n)]_F$$

where

$$T(\boldsymbol{v}_1)=a_{11}\boldsymbol{w}_1+a_{21}\boldsymbol{w}_2+\cdots+a_{m1}\boldsymbol{w}_m$$

$$T(\boldsymbol{v}_2)=a_{12}\boldsymbol{w}_1+a_{22}\boldsymbol{w}_2+\cdots+a_{m2}\boldsymbol{w}_m$$

$$\vdots$$

$$T(\boldsymbol{v}_n)=a_{1n}\boldsymbol{w}_1+a_{2n}\boldsymbol{w}_2+\cdots+a_{mn}\boldsymbol{w}_m$$

Orthogonal Subspaces

$\boldsymbol{C}(A) =$ column space of A; dimension r. $\subseteq \mathbf{R}^m$

$\boldsymbol{N}(A) =$ nullspace of A; dimension $n-r$. $\subseteq \mathbf{R}^n$

$\boldsymbol{C}(A^{\mathrm{T}}) =$ row space of A; dimension r. $\subseteq \mathbf{R}^n$

$\boldsymbol{N}(A^{\mathrm{T}}) =$ left nullspace of A; dimension $m-r$. $\subseteq \mathbf{R}^m$

$r + (n - r) = n$

$r + (m - r) = m$

$\mathrm{N}(\boldsymbol{A}) = (\mathrm{C}(\boldsymbol{A}^{\mathrm{T}}))^{\perp}$ $\quad$ $\mathrm{N}(\boldsymbol{A}^{\mathrm{T}}) = (\mathrm{C}(\boldsymbol{A}))^{\perp}$

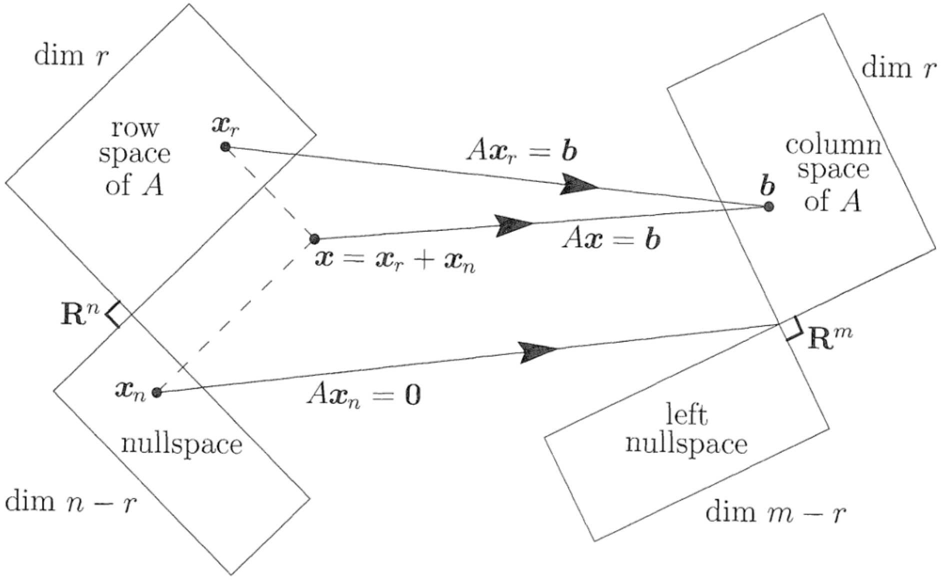


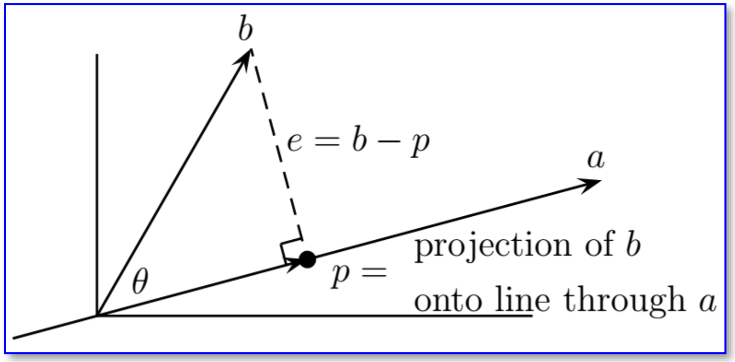
Figure 3.4: The true action $Ax = A(x_{\text{row}} + x_{\text{null}})$ of any m by n matrix.

Projection onto a Line

$$\cos\theta = \frac{\boldsymbol{a}^{\mathrm{T}}\boldsymbol{b}}{\|\boldsymbol{a}\|\|\boldsymbol{b}\|}.$$

The projection $\mathrm{proj}_{\boldsymbol{a}}$ *satisfies*

$$\mathrm{proj}_{\boldsymbol{a}}(\boldsymbol{b}) = \frac{\boldsymbol{a}^{\mathrm{T}}\boldsymbol{b}}{\boldsymbol{a}^{\mathrm{T}}\boldsymbol{a}}\boldsymbol{a}.$$



Projection onto a line is carried out by a ***projection matrix*** $\boldsymbol{P}$:

$$\boldsymbol{P} = \frac{\boldsymbol{a}\boldsymbol{a}^{\mathrm{T}}}{\boldsymbol{a}^{\mathrm{T}}\boldsymbol{a}}.$$

The ***projection matrix P*** is symmetric and idempotent.

Projection onto C(A)

Theorem. *If a system* $\boldsymbol{Ax} = \boldsymbol{b}$ *is inconsistent* (*has no solution*), *its least-squares solution minimizes* $||\boldsymbol{Ax} - \boldsymbol{b}||^2$:

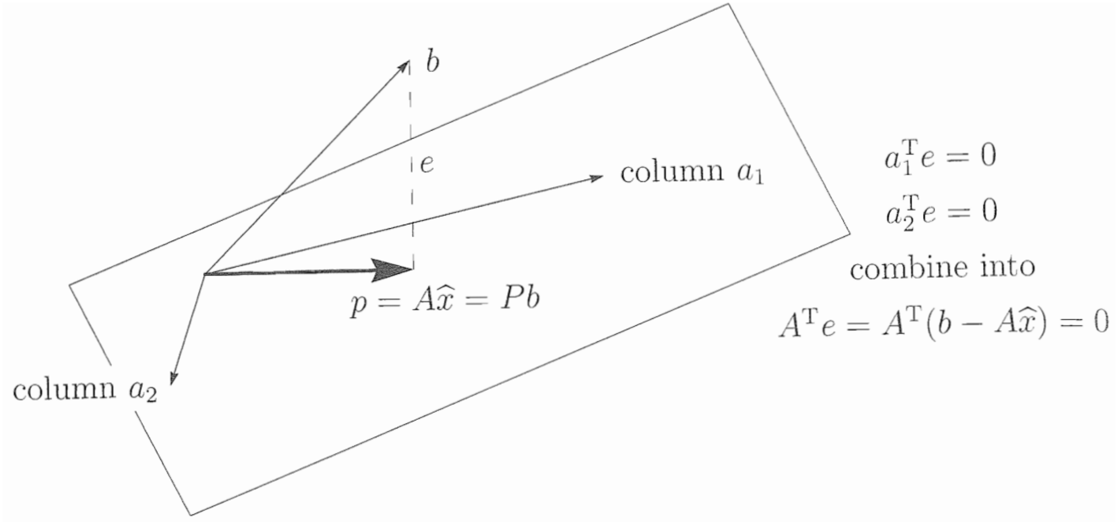
$$\boldsymbol{A}^{\mathrm{T}}\boldsymbol{A}\widehat{\boldsymbol{x}} = \boldsymbol{A}^{\mathrm{T}}\boldsymbol{b}. \qquad \textbf{(Normal equations)}$$

Moreover, if $\boldsymbol{A}^{\mathrm{T}}\boldsymbol{A}$ *is invertible, then*

$$\widehat{\boldsymbol{x}} = \left(\boldsymbol{A}^{\mathrm{T}}\boldsymbol{A}\right)^{-1}\boldsymbol{A}^{\mathrm{T}}\boldsymbol{b}. \qquad \textbf{(Best estimate)}$$

The projection of $\boldsymbol{b}$ *onto the column space is the nearest point* $\boldsymbol{A}\widehat{\boldsymbol{x}}$:

$$\boldsymbol{p} = \boldsymbol{A}\widehat{\boldsymbol{x}} = \boldsymbol{A}\left(\boldsymbol{A}^{\mathrm{T}}\boldsymbol{A}\right)^{-1}\boldsymbol{A}^{\mathrm{T}}\boldsymbol{b}. \qquad \textbf{(Projection)}$$



Projection onto the column space of a 3 by 2 matrix

For a system of linear equations: $\boldsymbol{Ax} = \mathbf{0}$

- Always has zero solution
- May or may not has non-zero solution
  - always has non-zero solution when $n > m$
  - find special solutions to span $\mathrm{N}(\boldsymbol{A})$

Solve $\boldsymbol{Ax} = \mathbf{0}$
& $\boldsymbol{Ax} = \boldsymbol{b}$
(with parameters)

For a system of linear equations: $\boldsymbol{Ax} = \boldsymbol{b}$

- Consistent
  - It has a unique solution.
  - It has infinitely many solutions:

$$\boldsymbol{x} = \boldsymbol{x}_p + \boldsymbol{x}_n$$

  where $\boldsymbol{x}_n \in \mathrm{N}(\boldsymbol{A})$.

- Inconsistent
  - Find $\widehat{\boldsymbol{x}}$: Least Squares solutions
  - The best $\widehat{\boldsymbol{x}}$ is the vector that minimizes the squared error

$$E^2 = \|\boldsymbol{Ax} - \boldsymbol{b}\|^2.$$