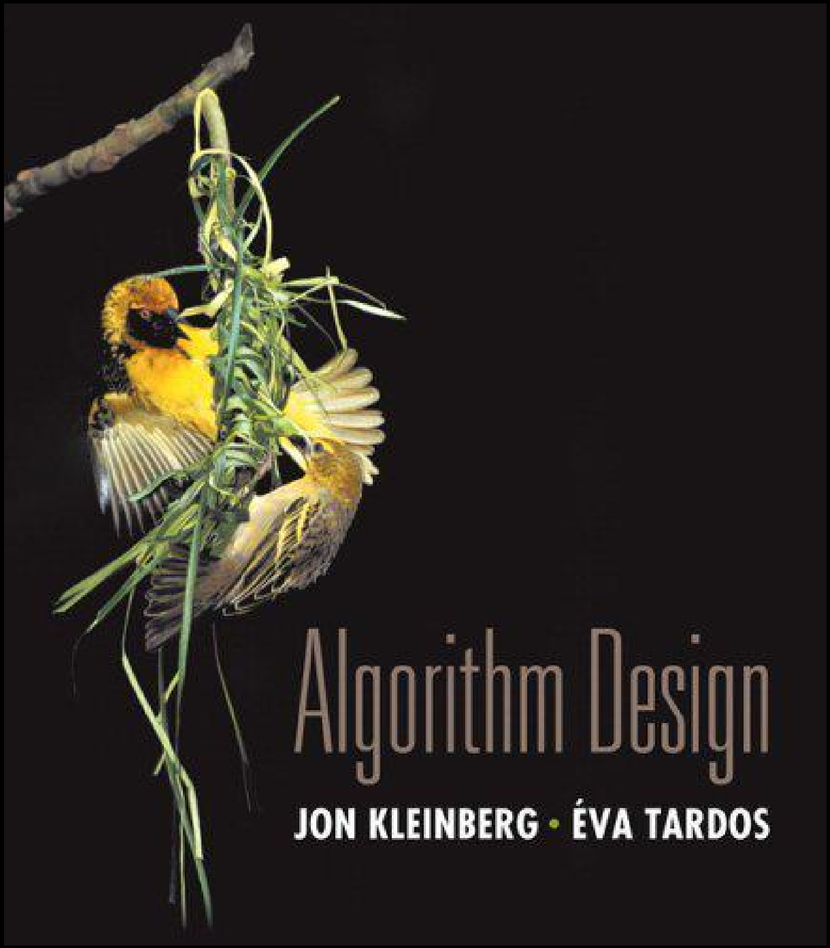


Chapter 7

Network Flow

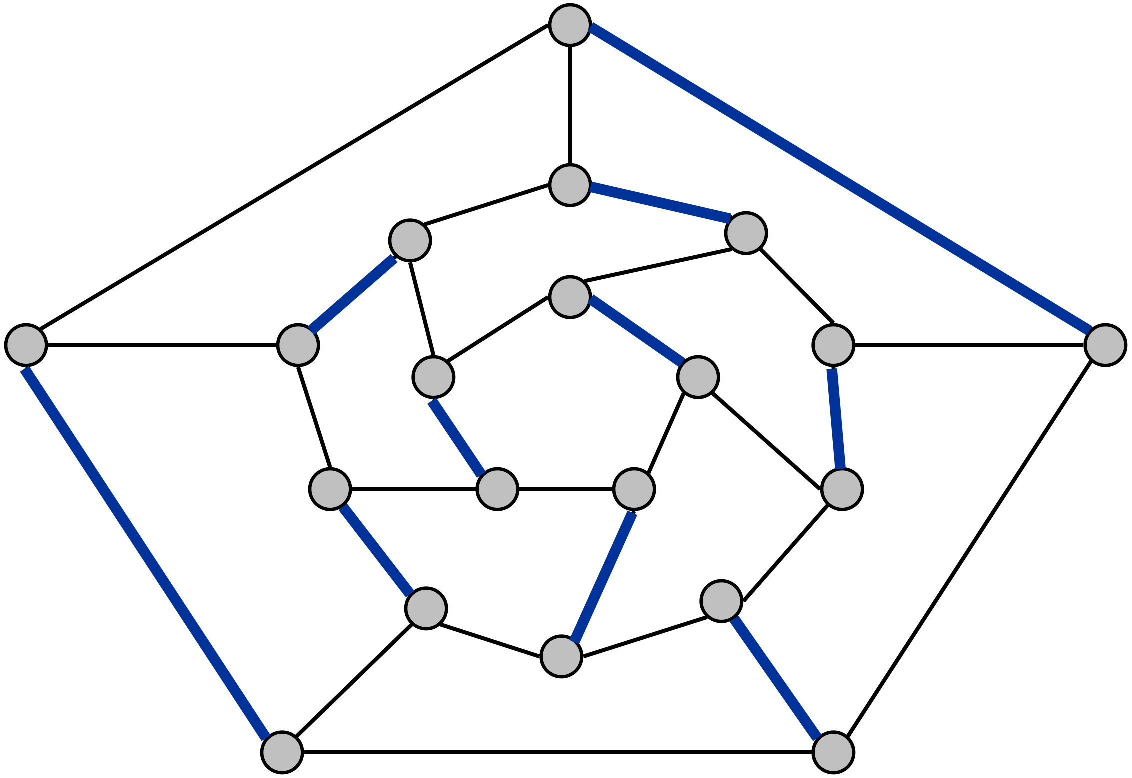
Slides by Kevin Wayne.

7.5 Bipartite Matching

Matching

Matching.

- Input: undirected graph $G = (V, E)$.
- $M \subseteq E$ is a matching if each node appears in at most one edge in M.
- Max matching: find a max cardinality matching.

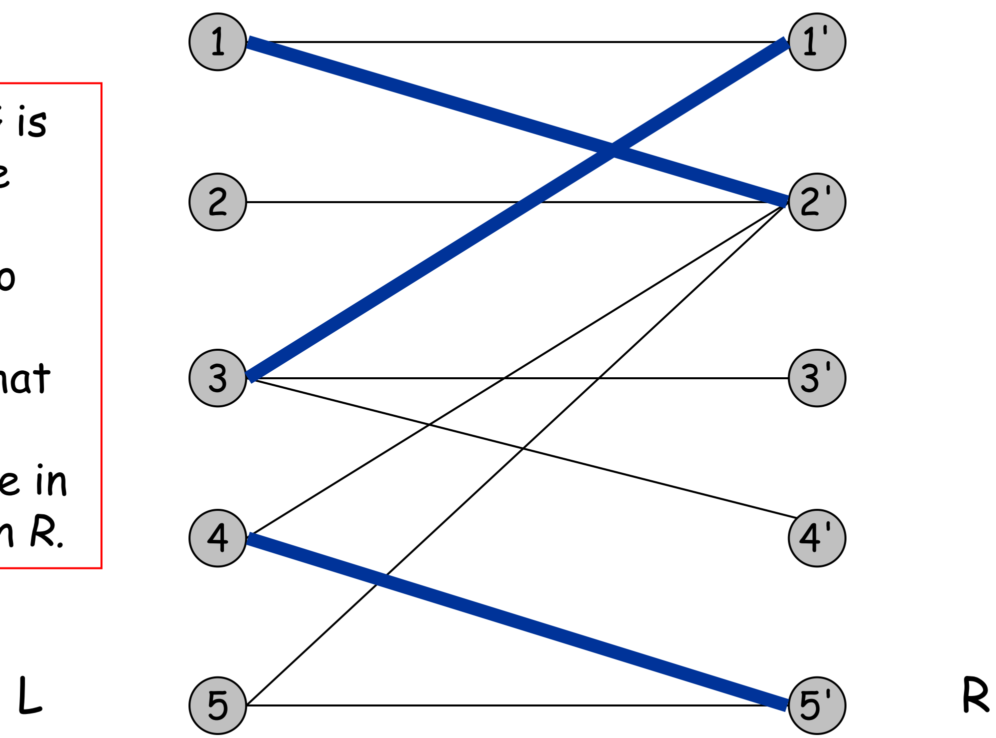


Bipartite Matching

Bipartite matching.

- Input: undirected, bipartite graph $G = (L \cup R, E)$.
- $M \subseteq E$ is a matching if each node appears in at most one edge in M.
- Max matching: find a max cardinality matching.

Def. A graph G is bipartite if the nodes can be partitioned into two subsets L and R such that every edge connects a node in L with a node in R.



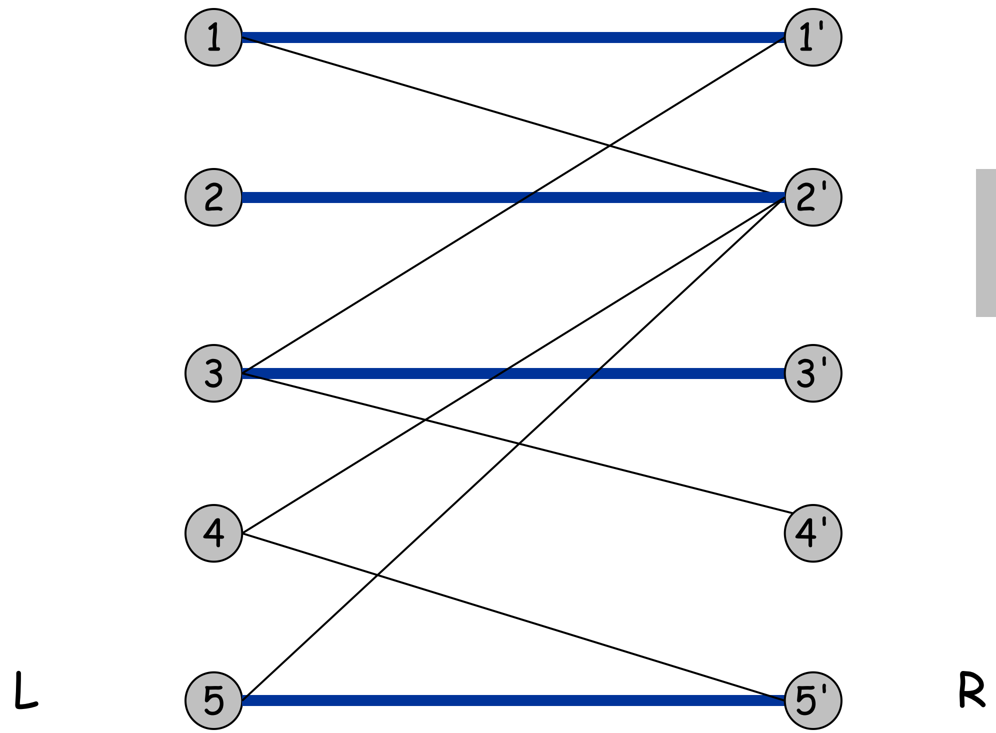
matching

1-2', 3-1', 4-5'

Bipartite Matching

Bipartite matching.

- Input: undirected, bipartite graph $G = (L \cup R, E)$.
- $M \subseteq E$ is a matching if each node appears in at most one edge in M.
- Max matching: find a max cardinality matching.

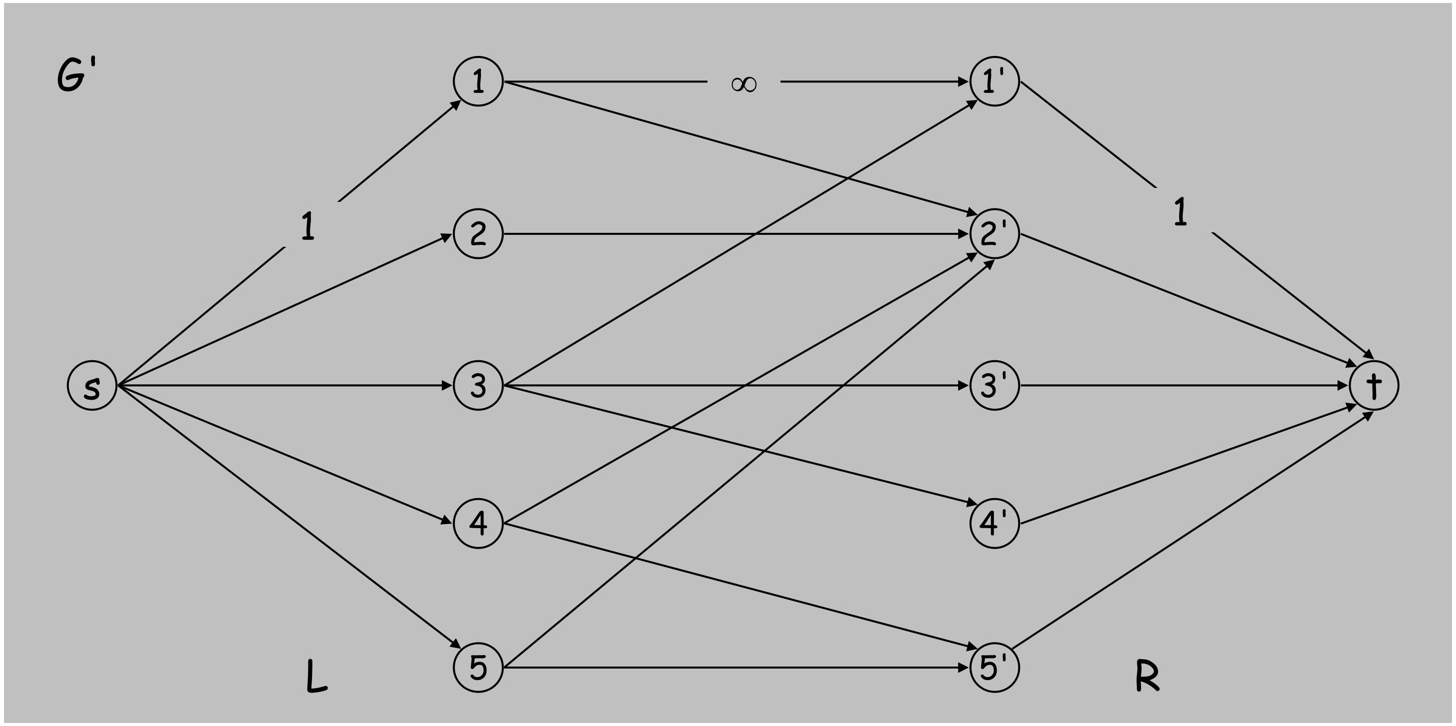


max matching

1-1', 2-2', 3-3' 4-4'

Max flow formulation.

- Create digraph $G' = (L \cup R \cup \{s, t\},\ E')$.
  - Direct all edges from L to R, and assign infinite (or unit) capacity.
  - Add source s, and unit capacity edges from s to each node in L.
  - Add sink t, and unit capacity edges from each node in R to t.

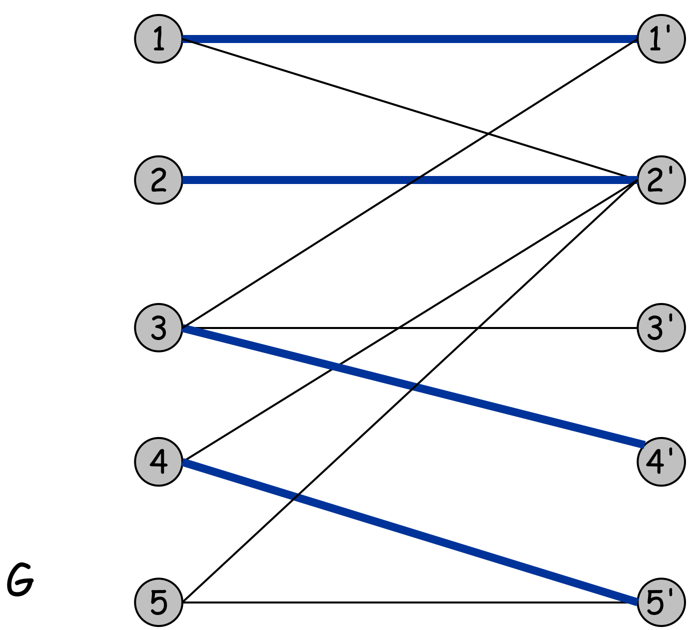


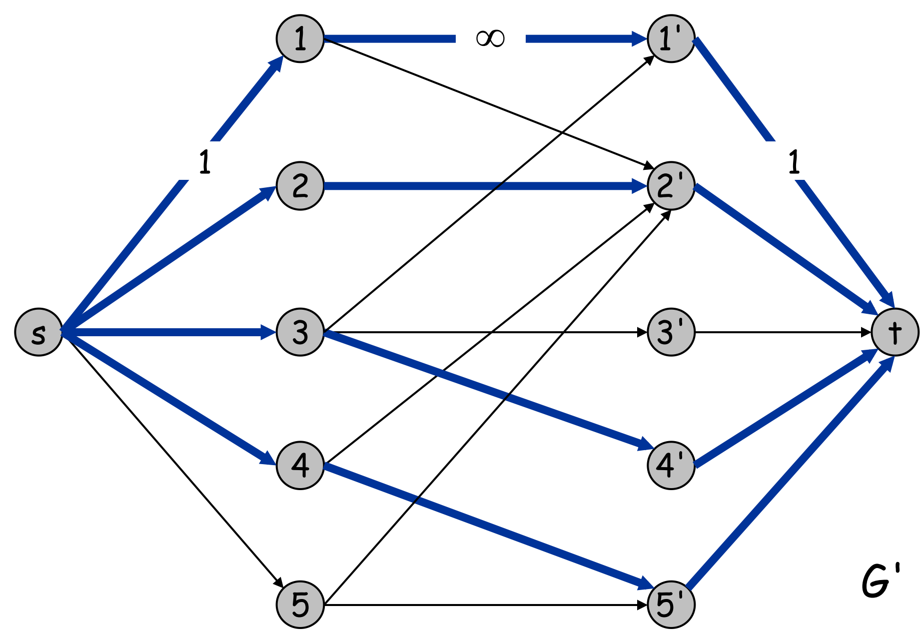
Bipartite Matching: Proof of Correctness

Theorem. value of max flow in G' = Max cardinality matching in G .

Pf. $\leq$

- Given max matching M of cardinality k.
- Consider flow f that sends 1 unit along each of k paths.
- f is a flow, and has cardinality k. ▪





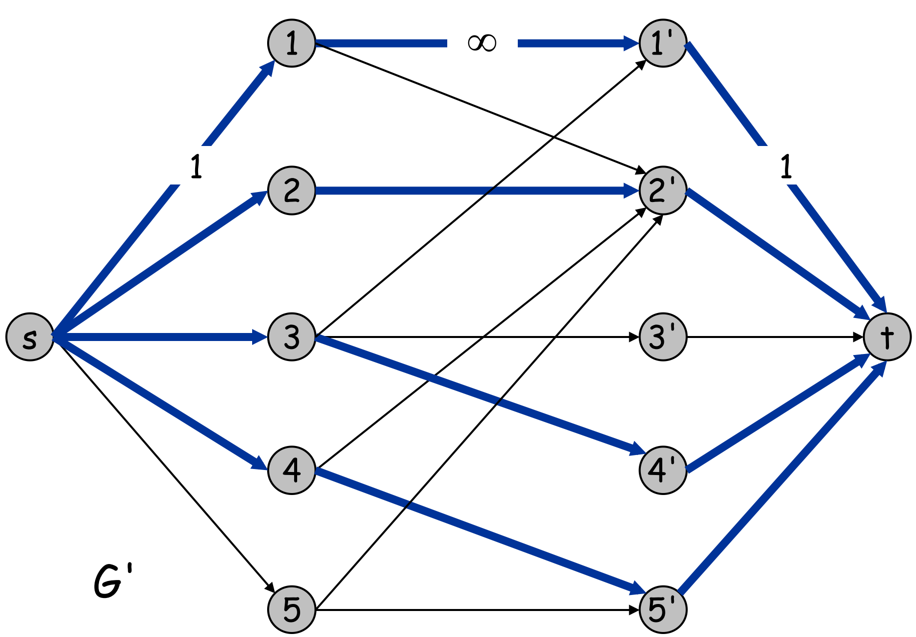
Bipartite Matching: Proof of Correctness

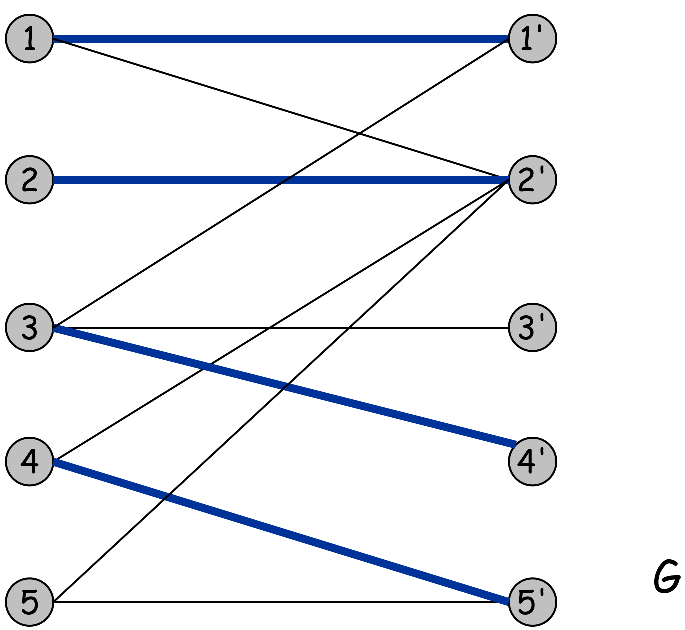
Theorem. value of max flow in G' = Max cardinality matching in G .

Pf. $\geq$

- Let f be a max flow in G' of value k.
- Integrality theorem $\Rightarrow$ k is integral and can assume f is 0-1.
- Consider M = set of edges from L to R with f(e) = 1.
  - each node in L and R participates in at most one edge in M
  - |M| = k: apply flow-value lemma to cut $(L \cup s, R \cup t)$ ▪

Flow value lemma. Let f be any flow, and let (A, B) be any s-t cut. Then, the net flow sent across the cut is equal to the amount leaving s.





Perfect Matching

Def. A matching $M \subseteq E$ is perfect if each node appears in exactly one edge in M.

Q. When does a bipartite graph have a perfect matching?

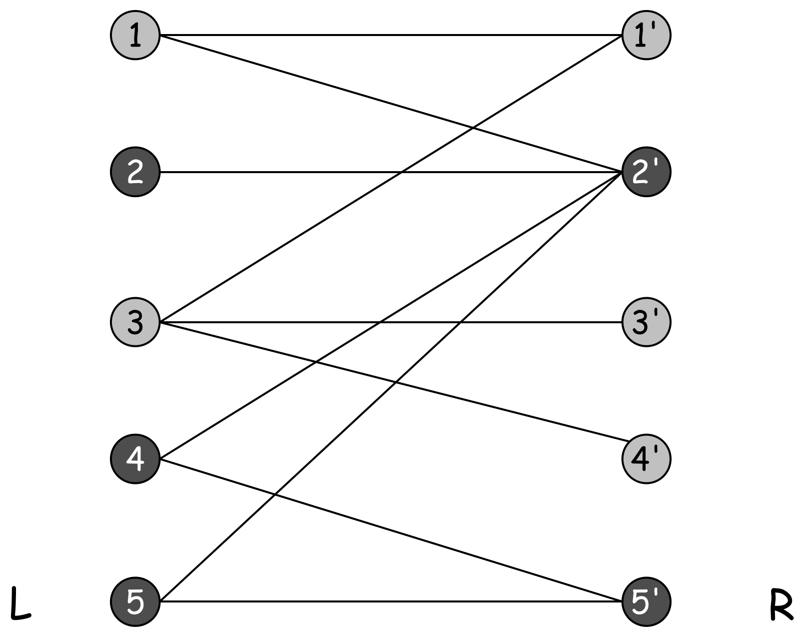
Structure of bipartite graphs with perfect matchings.

- Clearly we must have $|L| = |R|$.
- What other conditions are necessary?
- What conditions are sufficient?

Notation. Let S be a subset of nodes, and let N(S) be the set of nodes adjacent to nodes in S.

Observation. If a bipartite graph $G = (L \cup R, E)$, has a perfect matching, then $|N(S)| \geq |S|$ for all subsets $S \subseteq L$.

Pf. Each node in S has to be matched to a different node in N(S).



No perfect matching:

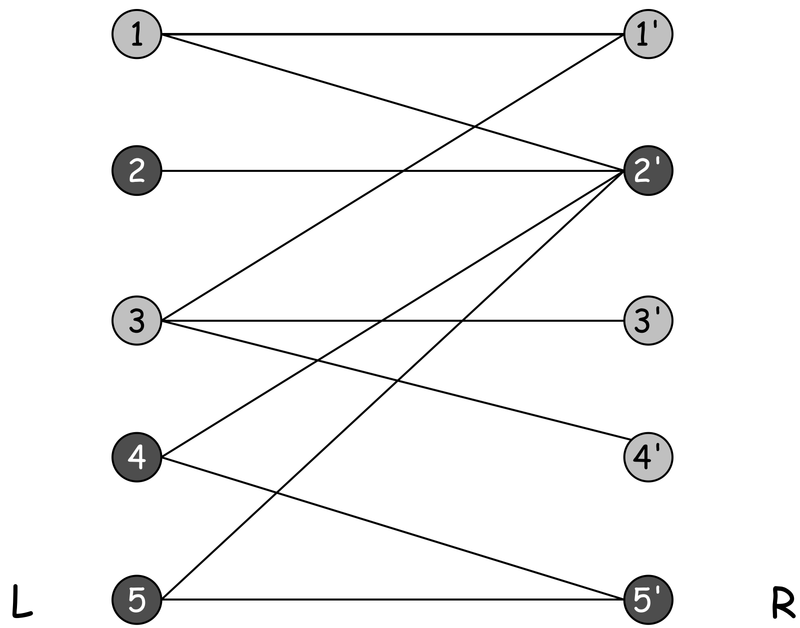
$S = \{ 2, 4, 5 \}$

$N(S) = \{ 2', 5' \}$.

Hall's marriage theorem

Marriage Theorem. [Frobenius 1917, Hall 1935] Let $G = (L \cup R, E)$ be a bipartite graph with $|L| = |R|$. Then, graph G has a perfect matching iff $|N(S)| \geq |S|$ for all subsets $S \subseteq L$.

Pf. $\Rightarrow$ This was the previous observation.

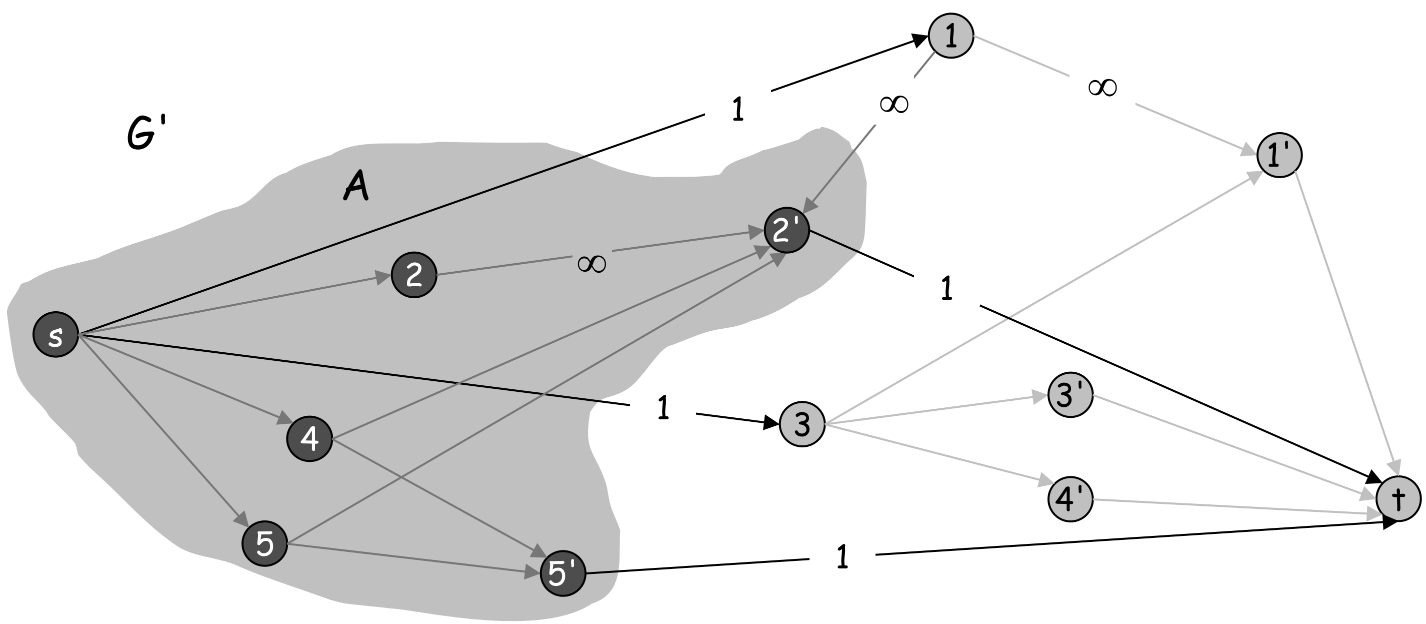


No perfect matching:
$S = \{ 2, 4, 5 \}$
$N(S) = \{ 2', 5' \}$.

Proof of Marriage Theorem

Pf. $\Leftarrow$ Suppose G does not have a perfect matching.

- Formulate as a max flow problem and let (A, B) be min cut in G'.
- By max-flow min-cut, cap(A, B) < | L |.
- Define $L_A = L \cap A$, $L_B = L \cap B$, $R_A = R \cap A$.
- cap(A, B) = $|L_B| + |R_A|$.



$L_A = \{2, 4, 5\}$
$L_B = \{1, 3\}$
$R_A = \{2', 5'\}$
$N(L_A) = \{2', 5'\}$

Proof of Marriage Theorem

Pf. $\Leftarrow$ Suppose G does not have a perfect matching.

- Formulate as a max flow problem and let (A, B) be min cut in G'.
- By max-flow min-cut, cap(A, B) < | L |.
- Define $L_A = L \cap A$, $L_B = L \cap B$, $R_A = R \cap A$.
- cap(A, B) = $| L_B | + | R_A |$.
- Since min cut can't use ∞ edges: $N(L_A) \subseteq R_A$.

The two ends of the edge (x, y) will be on different sides of the cut, but this edge does not add to the capacity of the cut, as it goes from B to A

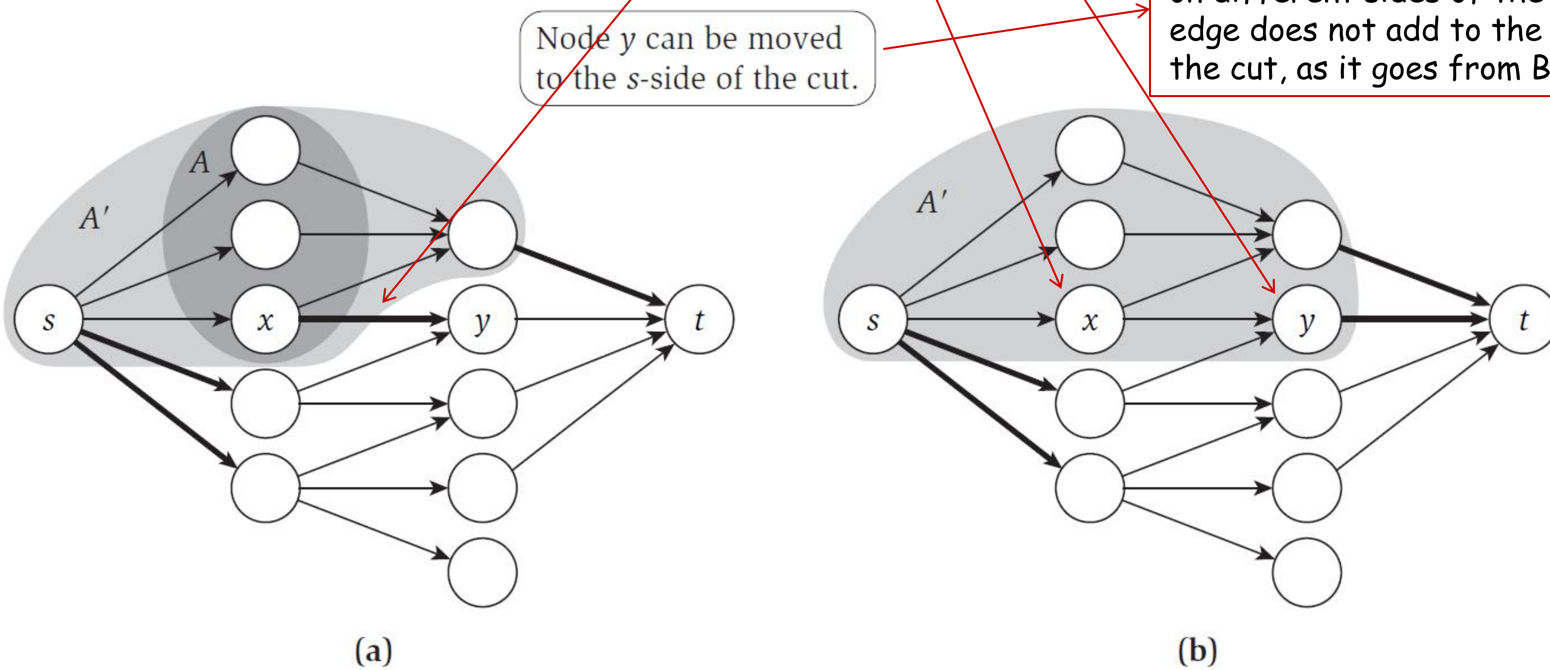
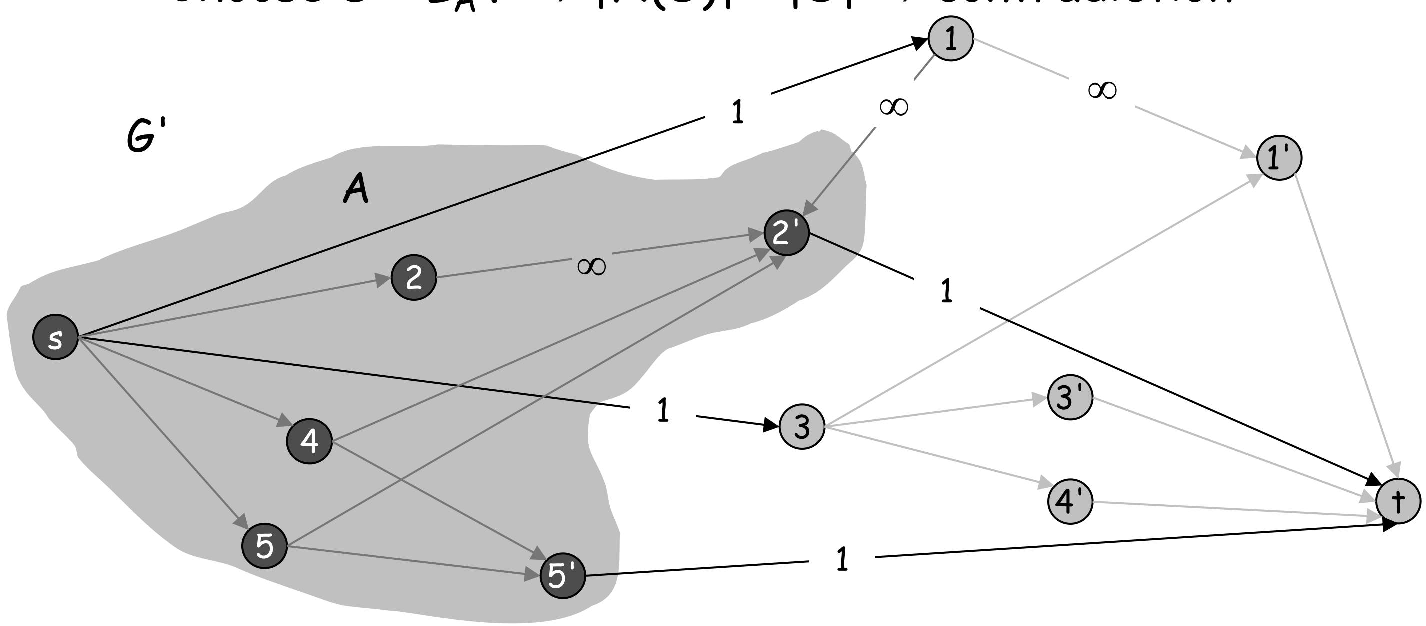


Figure 7.11 (a) A minimum cut in proof of (7.40). (b) The same cut after moving node y to the A' side. The edges crossing the cut are dark.

Proof of Marriage Theorem

Pf. $\Leftarrow$ Suppose G does not have a perfect matching.

- Formulate as a max flow problem and let (A, B) be min cut in G'.
- By max-flow min-cut, cap(A, B) < | L |.
- Define $L_A = L \cap A$, $L_B = L \cap B$, $R_A = R \cap A$.
- cap(A, B) = $| L_B | + | R_A |$.
- Since min cut can't use ∞ edges: $N(L_A) \subseteq R_A$.
- $|N(L_A)| \leq | R_A | = \text{cap}(A, B) - | L_B | < | L | - | L_B | = | L_A |$.
- Choose $S = L_A$. $\Rightarrow |N(S)| < |S| \Rightarrow$ contradiction ▪



$L_A = \{2, 4, 5\}$

$L_B = \{1, 3\}$

$R_A = \{2', 5'\}$

$N(L_A) = \{2', 5'\}$

Bipartite Matching: Running Time

Which max flow algorithm to use for bipartite matching?

- Generic augmenting path: $O(mn\ val(f^*)) = O(mnC)$.
- Capacity scaling: $O(m^2 \log C)$.

Non-bipartite matching.

- Structure of non-bipartite graphs is more complicated, but well-understood. [Tutte-Berge, Edmonds-Galai]
- Blossom algorithm: $O(n^4)$. [Edmonds 1965]
- Best known: $O(m\, n^{1/2})$. [Micali-Vazirani 1980]