Homework & Tutorial Problems

- Tutorial
  - Property of DTFT: 5.24
  - Difference equation of LTI systems: 5.36
- Homework
  - 5.23, 5.29, 5.33

Convolution Property & LTI Systems (1/2)

- Let $h[n]$ be the impulse response of certain LTI system
- The output $y[n]$ of input $x[n]$ is given by $y[n] = x[n] * h[n]$
- For input signal $x[n] = e^{j\omega n}$,

$$y[n] = \sum_{k=-\infty}^{\infty} h[k] e^{j\omega(n-k)} = e^{j\omega n} \underbrace{\sum_{k=-\infty}^{\infty} h[k] e^{-j\omega k}}_{\text{Frequency Response } H(e^{j\omega})}$$

- For periodic input signal $x[n] = \sum_{k=<N>} a_k e^{jk(\frac{2\pi}{N})n}$:

$$y[n] = \sum_{k=<N>} a_k H(e^{jk(\frac{2\pi}{N})}) e^{jk(\frac{2\pi}{N})n}$$

$$b_k = a_k H(e^{jk(\frac{2\pi}{N})})$$

Convolution Property & LTI Systems (2/2)

Convolution Property

If $y[n] = x_1[n] * x_2[n]$, *then*

$$Y(e^{j\omega}) = X_1(e^{j\omega})X_2(e^{j\omega})$$

- For general input signal $x[n]$:

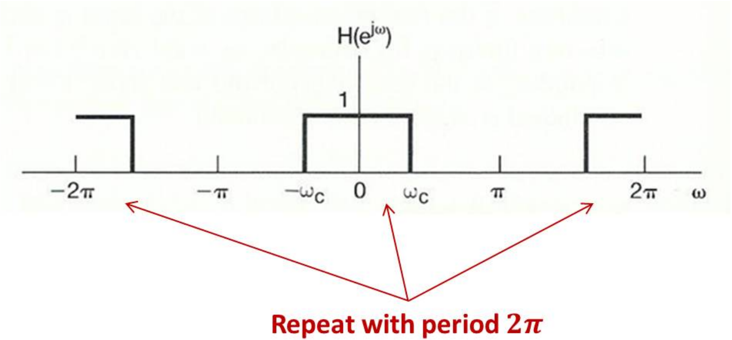
$$y[n] = x[n] * h[n] \longleftrightarrow Y(e^{j\omega}) = X(e^{j\omega})H(e^{j\omega})$$

- Observation: It's easier to evaluate LTI systems in frequency domain
- Drawback: Not every LTI system has frequency response
  - $h[n] = a^n u[n]$ $(a > 1)$
  - Stable LTI system has frequency response, because

$$\sum_{n=-\infty}^{\infty} |h[n]| < \infty$$

Example: Ideal Low-Pass Filter (1/2)

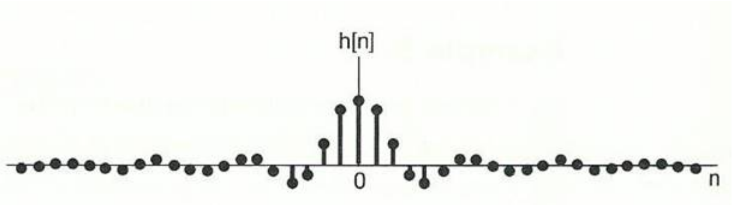
- What's ideal low-pass filter?
- Perfectly maintain the low-frequency component
- Perfectly cancel the high-frequency component



Example: Ideal Low-Pass Filter (2/2)

- Impulse response:

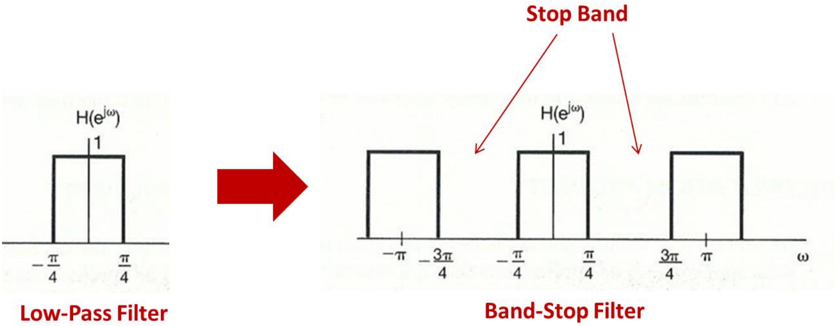
$$h[n] = \frac{1}{2\pi}\int_{-\pi}^{\pi} H(e^{j\omega})e^{j\omega n}d\omega = \frac{1}{2\pi}\int_{-\omega_c}^{\omega_c} e^{j\omega n}d\omega = \frac{sin\omega_c n}{\pi n}$$



- Pros: no distortion in frequency domain
- Cons: non-causal
- See textbook, Example 5.12

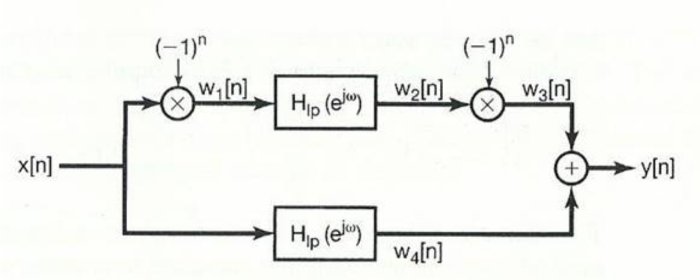
Example: Band-Stop Filter (1/2)

- Given low-pass filter $H_{lp}(e^{j\omega})$, how to generate band-stop effect from low-pass filter?



Example: Band-Stop Filter (2/2)

- Two branches: low-pass + high-pass



- $(-1)^n = e^{j\pi n} \Rightarrow W_1(e^{j\omega}) = X(e^{j(\omega-\pi)})$

$$\Rightarrow W_2(e^{j\omega}) = H_{lp}(e^{j\omega})X(e^{j(\omega-\pi)})$$

$$\Rightarrow W_3(e^{j\omega}) = H_{lp}(e^{j(\omega-\pi)})X(e^{j(\omega-2\pi)}) = H_{lp}(e^{j(\omega-\pi)})X(e^{j\omega})$$

- See textbook, Example 5.14

Multiplication Property

Multiplication Property

Let $y[n] = x_1[n]x_2[n]$, *then*

$$Y(e^{j\omega}) = \frac{1}{2\pi}\int_{2\pi} X_1(e^{j\theta})X_2(e^{j(\omega-\theta)})d\theta,$$

which is periodic convolution of $X_1(e^{j\omega})$ *and* $X_2(e^{j\omega})$

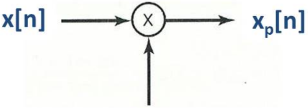
- Comparison: Multiplication of periodic signals

$$x_1[n] \longleftrightarrow a_k \quad \text{and} \quad x_2[n] \longleftrightarrow b_k$$

$$\Rightarrow x_1[n]x_2[n] \longleftrightarrow \sum_{k=<N>} a_k b_{n-k}$$ discrete-time periodic convolution

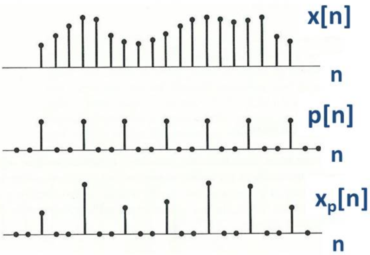
Example: Sampling on Discrete-Time Signals

- In digital signal processing, we may sample the discrete-time signals
- Example: image, audio, video compression
- Its mathematical model is give below:



$$p[n] = \sum_{k=-\infty}^{+\infty} \delta[n - kN]$$

$$x_p[n] = \sum_{k=-\infty}^{+\infty} x[kN]\delta[n - kN]$$



Review: Discrete-Time Impulse Chain

- What's the Fourier transform of $x[n] = \sum_{k=-\infty}^{+\infty} \delta[n-kN]$?
- First of all, we calculate the Fourier series:

$$
\begin{aligned}
a_k &= \frac{1}{N}\sum_{n=<N>} x[n]e^{-jk\frac{2\pi}{N}n} \\
&= \frac{1}{N}\sum_{n=<N>}\sum_{k=-\infty}^{+\infty} \delta[n-kN]e^{-jk\frac{2\pi}{N}n} \\
&= \frac{1}{N}\sum_{n=<N>} \delta[n]e^{-jk\frac{2\pi}{N}n} = \frac{1}{N}
\end{aligned}
$$

- Then, we have

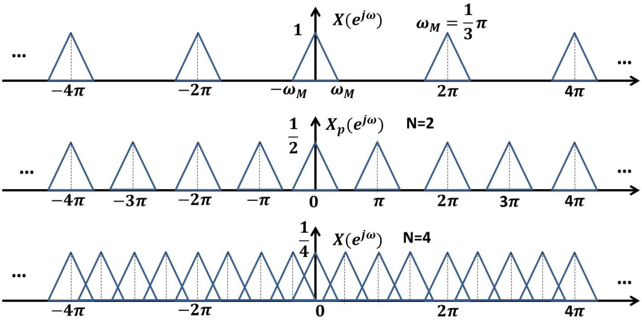
$$X(e^{j\omega}) = \sum_{l=-\infty}^{\infty}\sum_{k=0}^{N-1}\frac{2\pi}{N}\delta(\omega - k(2\pi/N) - 2\pi l) = \frac{2\pi}{N}\sum_{k=-\infty}^{\infty}\delta(\omega - \frac{2\pi k}{N})$$

- Time domain period $\times$ Frequency domain period $=$?

Example: Frequency Analysis

$$P(e^{j\omega}) = \frac{2\pi}{N}\sum_{k=-\infty}^{\infty}\delta(\omega - k\omega_s) \quad \textit{where } \omega_s = 2\pi/N$$

$$X_p(e^{j\omega}) = \frac{1}{2\pi}\int_{2\pi} P(e^{j\theta})X(e^{j(\omega-\theta)})d\theta = \frac{1}{N}\sum_{k=0}^{N-1} X(e^{j(\omega-k\omega_s)})$$



Duality in DTFS

- Analysis and synthesis equations of DTFS share a similar form:

$$
\begin{aligned}
x[n] &= \sum_{k=<N>} a_k e^{jk(2\pi/N)n} \\
a_k &= \frac{1}{N} \sum_{k=<N>} x[n] e^{-jk(2\pi/N)n}
\end{aligned}
\tag{1}
$$

- Every of DTFS has a dual (See Table 3.2)
  - Time shift v.s. Frequency shift
  - Time multiplication v.s. Frequency multiplication
- If $g[n] \longleftrightarrow f[k]$, then $f[n] \longleftrightarrow \frac{1}{N} g[-k]$

Dualtiy Between CTFS and DTFT

- DTFT transform pair is given by

$$\begin{aligned} x[n] &= \tfrac{1}{2\pi}\int_{2\pi} X(e^{j\omega})e^{j\omega n}d\omega \\ X(e^{j\omega}) &= \textstyle\sum_{n=-\infty}^{\infty} x[n]e^{-j\omega n} \end{aligned}$$

- CTFS transform pair is given by

$$\begin{aligned} x(t) &= \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t} \\ a_k &= \frac{1}{T}\int_T x(t)e^{-jk\omega_0 t}dt \end{aligned}$$

- Frequency Domain of DTFT $\Leftrightarrow$ Time Domain of CTFS
- Frequency Domain of CTFS $\Leftrightarrow$ Time Domain of DTFT
- Discrete-Time Low-pass Filter v.s. Continuous-Time Square Wave

Summary of Duality

	Continuous time		Discrete time	
	Time domain	**Frequency domain**	**Time domain**	**Frequency domain**
Fourier Series	$x(t) = \sum_{k=-\infty}^{+\infty} a_k e^{jk\omega_0 t}$ continuous time periodic in time	$a_k = \frac{1}{T_0}\int_{T_0} x(t) e^{-jk\omega_0 t}$ discrete frequency aperiodic in frequency	$x[n] = \sum_{k=\langle N\rangle} a_k e^{jk(2\pi/N)n}$ discrete time periodic in time	$a_k = \frac{1}{N}\sum_{k=\langle N\rangle} x[n] e^{-jk(2\pi/N)n}$ discrete frequency periodic in frequency
Fourier Transform	$x(t) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} X(j\omega) e^{j\omega t} d\omega$ continuous time aperiodic in time	$X(j\omega) = \int_{-\infty}^{+\infty} x(t) e^{-j\omega t} dt$ continuous frequency aperiodic in frequency	$x[n] = \frac{1}{2\pi}\int_{2\pi} X(e^{j\omega}) e^{j\omega n} d\omega$ discrete time aperiodic in time	$X(e^{j\omega}) = \sum_{n=-\infty}^{+\infty} x[n] e^{-j\omega n}$ continuous frequency periodic in frequency

- See textbook, Table 5.3

LTI by Difference Equation

- A number of DT LTI systems can be written as the following linear constant-coefficient difference equation

$$\sum_{k=0}^{N} a_k y[n-k] = \sum_{k=0}^{M} b_k x[n-k]$$

What's the frequency response?

- Taking Fourier transform on both side, we have

$$\begin{aligned} \textstyle\sum_{k=0}^{N} a_k e^{-jk\omega} Y(e^{j\omega}) &= \textstyle\sum_{k=0}^{M} b_k e^{-jk\omega} X(e^{j\omega}) \\ \Rightarrow \qquad H(e^{j\omega}) = \frac{Y(e^{j\omega})}{X(e^{j\omega})} &= \frac{\sum_{k=0}^{M} b_k e^{-jk\omega}}{\sum_{k=0}^{N} a_k e^{-jk\omega}} \end{aligned}$$

What's the impulse response?

Example: Difference Equation

Please calculate the frequency and impulse response of the following LTI systems

$$y[n] - \frac{3}{4}y[n-1] + \frac{1}{8}y[n-2] = 2x[n].$$

Solution

According to the last slide,

$$H(e^{j\omega}) = \frac{2}{1 - \frac{3}{4}e^{-j\omega} + \frac{1}{8}e^{-2j\omega}}.$$

Moreover, by partial fraction expansion

$$H(e^{j\omega}) = \frac{4}{1 - \frac{1}{2}e^{-j\omega}} - \frac{2}{1 - \frac{1}{4}e^{-j\omega}} \Rightarrow h[n] = 4(\frac{1}{2})^n u[n] - 2(\frac{1}{4})^n u[n]$$

Discrete Fourier Transform (DFT)

- In practice, there is a huge demand on processing finite duration signals
- Given a finite duration signal $\{x[0], x[1], ..., x[N-1]\}$, its Fourier transform is

$$X(e^{j\omega}) = \sum_{n=0}^{N-1} x[n]e^{-j\omega n}$$

- Drawback: The spectrum of DTFT is continuous $\Rightarrow$ Cannot be handle by computer.
- Discrete Fourier Transfrom (DFT) is developed for digital processing of finite duration signals

$$\widetilde{X}[k] = \frac{1}{N}\sum_{n=0}^{N-1} x[n]e^{-jk(2\pi/N)n} \quad k =< N >$$

- DFT: frequency sampling of DTFT

$$\widetilde{X}[k] = \frac{1}{N}X(e^{j2k\pi/N}) \quad k =< N >$$

Inverse DFT

- Equation system of DFT:

$$\begin{pmatrix} \widetilde{X}[0] \\ \widetilde{X}[1] \\ \cdots \\ \widetilde{X}[N-1] \end{pmatrix} = \underbrace{\frac{1}{N}\begin{pmatrix} e^{-j0} & e^{-j0} & \cdots & e^{-j0} \\ e^{-j0} & e^{-j2\pi/N} & \cdots & e^{-j2(N-1)\pi/N} \\ \cdots & \cdots & \cdots & \cdots \\ e^{-j0} & e^{-j2(N-1)\pi/N} & \cdots & e^{-j2(N-1)(N-1)\pi/N} \end{pmatrix}}_{\mathbf{F}} \begin{pmatrix} x[0] \\ x[1] \\ \cdots \\ x[N-1] \end{pmatrix}$$

- Transform matrix **F** is full rank.
- Observation: $\{\widetilde{X}[k] | k =< N >\}$ maintains all the information of $\{x[0], x[1], ..., x[N-1]\}$
- Inverse DFT is feasible:

$$x[n] = \sum_{k=0}^{N-1} \widetilde{X}[k] e^{jk(2\pi/N)n} \quad n = 0, 1, ..., N-1$$

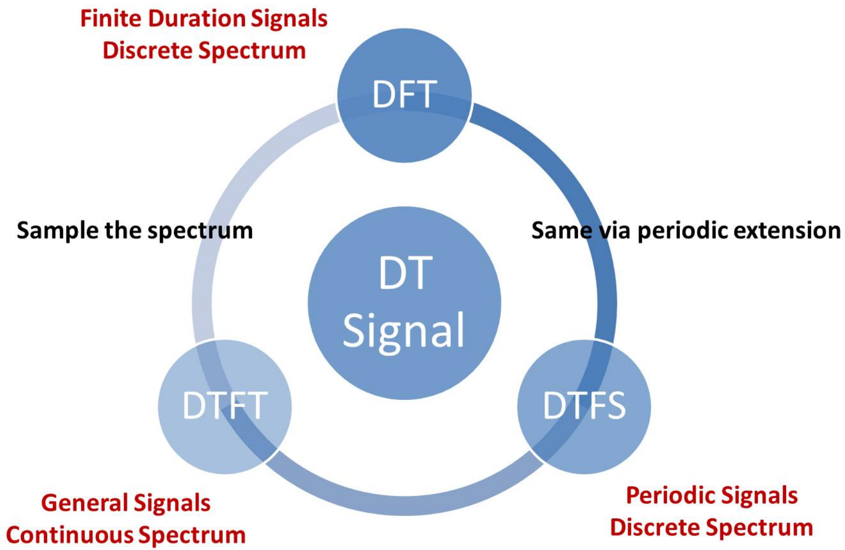
DFT and DTFS

- DFT is for finite duration signals; DTFS is for periodic signals
- Define $\widetilde{x}[n]$ as the periodic extension of $x[n]$: Repeat $x[n]$ with period N
- Fourier series of $\widetilde{x}[n]$:

$$\frac{1}{N}\sum_{n=0}^{N-1}\widetilde{x}[n]e^{-jk(2\pi/N)n} = \frac{1}{N}\sum_{n=0}^{N-1}x[n]e^{-jk(2\pi/N)n} \Rightarrow \text{ DFT of } x[n]$$

- DFT of a finite-duration signal = DTFS of its periodic extension
- Reference on DFT:
  - Textbook: Problem 5.53, 5.54
  - http://en.wikipedia.org/wiki/Discrete_Fourier_transform

Comparison


Finite Duration Signals
Discrete Spectrum
DFT
Sample the spectrum
Same via periodic extension
DT
Signal
DTFT
DTFS
General Signals
Continuous Spectrum
Periodic Signals
Discrete Spectrum

Periodic Convolution of Finite Duration Signals

Periodic Convolution

Let x and y be two finite duration signals with duration N, $\tilde{x}$ and $\tilde{y}$ be the associated periodic extension, then the periodic convolution of finite duration signals is defined as

$$x[n] \circledast y[n] := \widetilde{x}[n] \circledast \widetilde{y}[n] = \sum_{k=<N>} \tilde{x}[k]\tilde{y}[n-k] \quad n = 0, 1, 2..., N-1$$

Convoluation Property of DFT

Time domain periodic convolution is equivalent to frequency domain multiplication, thus,

- $x[n] \circledast y[n] \longleftrightarrow N\widetilde{X}[k]\widetilde{Y}[k]$