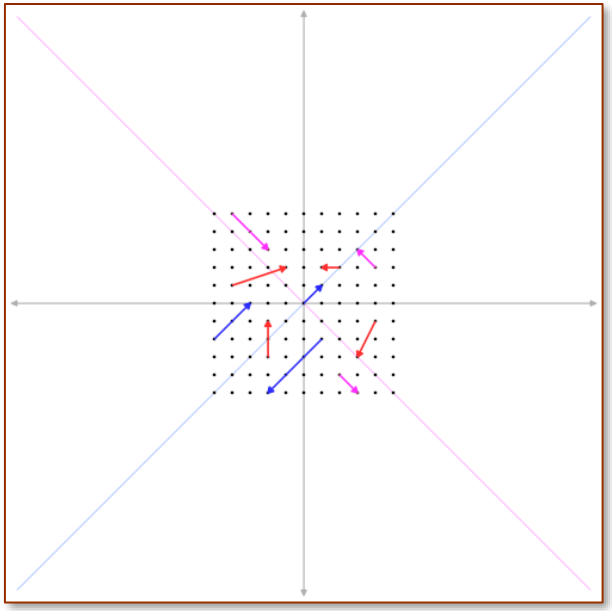
5 Eigenvalues and Eigenvectors

5 Eigenvalues and Eigenvectors (特征值与特征向量)

5.1

EIGENVALUES AND EIGENVECTORS

- Introduction
- Definition
- Calculations
- Properties
- Application

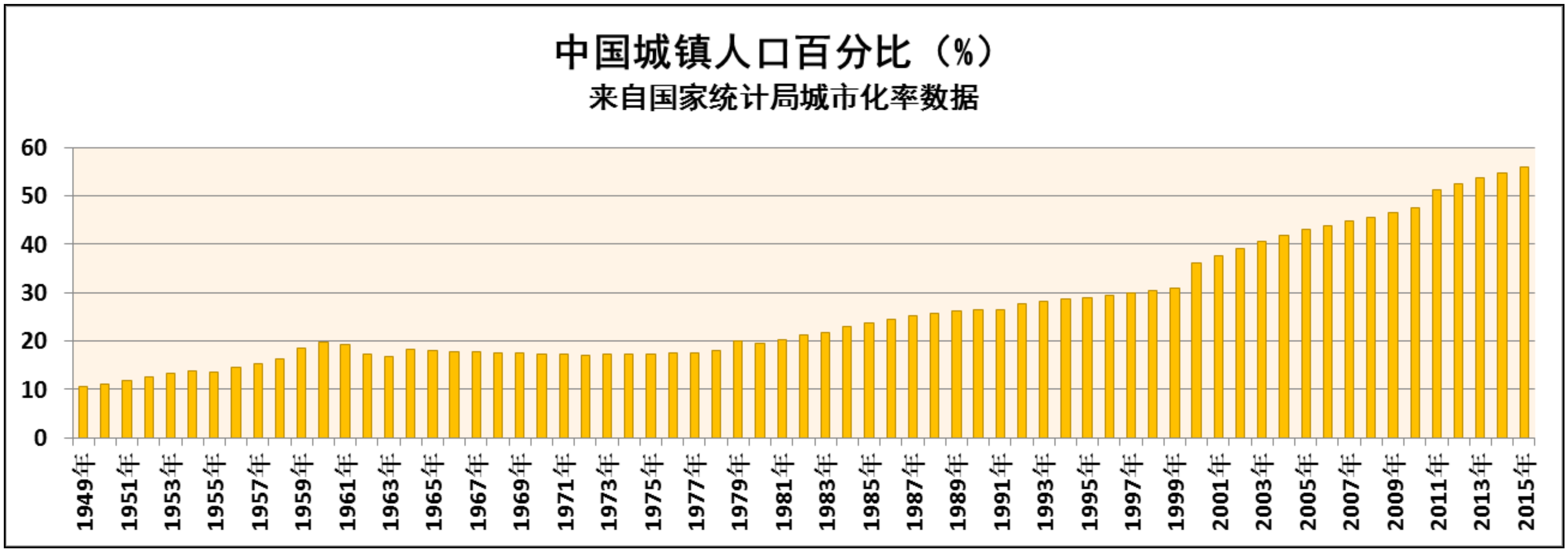


I. Introductory example - 人口流动问题

人口流动

城市建设　家庭经济条件

农村劳动力　留守儿童　空巢老人　春运　城市管理



发展趋势?

简化模型

- 假设:

  从事农业工作的人员中每年有四分之一转为从事非农业工作,

  从事非农业工作的人员中每年有六分之一转为从事农业工作.

- 人口总数不变.

预测多年之后劳动力从业情况的发展趋势.

分析 设最初农业人员和非农业人员的数量分别为 y_0, z_0,

第 1 年末数量为 y_1, z_1,

$$\begin{cases} y_1 = \dfrac{3}{4} y_0 + \dfrac{1}{6} z_0, \\ z_1 = \dfrac{1}{4} y_0 + \dfrac{5}{6} z_0. \end{cases} \quad \Longrightarrow \quad \begin{bmatrix} y_1 \\ z_1 \end{bmatrix} = \begin{bmatrix} \dfrac{3}{4} & \dfrac{1}{6} \\ \dfrac{1}{4} & \dfrac{5}{6} \end{bmatrix} \begin{bmatrix} y_0 \\ z_0 \end{bmatrix}$$

分析 设最初农业人员和非农业人员的数量分别为 y_0, z_0, 第 1 年末数量为 y_1, z_1,

$$\begin{bmatrix} y_1 \\ z_1 \end{bmatrix} = \boxed{\begin{bmatrix} \frac{3}{4} & \frac{1}{6} \\ \frac{1}{4} & \frac{5}{6} \end{bmatrix}} \begin{bmatrix} y_0 \\ z_0 \end{bmatrix}$$

矩阵 A

分析 设最初农业人员和非农业人员的数量分别为 y_0, z_0，第 k 年末数量为 y_k, z_k，

$$\begin{bmatrix} y_1 \\ z_1 \end{bmatrix} = \boxed{\begin{bmatrix} \frac{3}{4} & \frac{1}{6} \\ \frac{1}{4} & \frac{5}{6} \end{bmatrix}} \begin{bmatrix} y_0 \\ z_0 \end{bmatrix}$$

矩阵 A

如何计算方阵 A 的幂 A^k？

$$\begin{bmatrix} y_k \\ z_k \end{bmatrix} = \begin{bmatrix} \frac{3}{4} & \frac{1}{6} \\ \frac{1}{4} & \frac{5}{6} \end{bmatrix} \begin{bmatrix} y_{k-1} \\ z_{k-1} \end{bmatrix} = \cdots = \begin{bmatrix} \frac{3}{4} & \frac{1}{6} \\ \frac{1}{4} & \frac{5}{6} \end{bmatrix}^k \begin{bmatrix} y_0 \\ z_0 \end{bmatrix} = \boldsymbol{A}^k \begin{bmatrix} y_0 \\ z_0 \end{bmatrix}$$

最简单的方阵$\boldsymbol{Q}$： $\boldsymbol{Q} = \begin{bmatrix} \lambda_1 & \\ & \lambda_2 \end{bmatrix}$， $\boldsymbol{Q}^k = \begin{bmatrix} \lambda_1{}^k & \\ & \lambda_2{}^k \end{bmatrix}$.

有无可能？

设想 $\boxed{A = PQP^{-1}}$ $\longrightarrow$ $A^k = PQ^kP^{-1}$

(其中Q为对角阵)

$$A^2 = AA = PQP^{-1}PQP^{-1} = PQ^2P^{-1}$$

$$A^3 = AAA = PQ\cancel{P^{-1}}\cancel{P}Q\cancel{P^{-1}}\cancel{P}QP^{-1} = PQ^3P^{-1}$$

$$AP = PQ$$

$$\Updownarrow$$

$$A[P_1 \;\vdots\; P_2] = [P_1 \;\vdots\; P_2]\begin{bmatrix} \lambda_1 & \\ & \lambda_2 \end{bmatrix} \Leftrightarrow [AP_1 \;\; AP_2] = [\lambda_1 P_1 \;\; \lambda_2 P_2]$$

(其中P_1, P_2为2维列向量)

$$\Updownarrow$$

$$\boxed{AP_1 = \lambda_1 P_1,\;\; AP_2 = \lambda_2 P_2}$$

$$\boxed{A_{2\times 2}\boldsymbol{x}_{2\times 1} = \lambda \boldsymbol{x}_{2\times 1}}$$

II. Eigenvalues and Eigenvectors – Definition & Calculation

Definition 1 Let $\boldsymbol{A}$ be a *square matrix* of degree n.

If there exist a non-zero vector $\boldsymbol{x}$ and a scalar λ such that

$$\boldsymbol{A}\,\boldsymbol{x} = \lambda\,\boldsymbol{x}\,,$$

then λ is called an **eigenvalue (特征值)** of $\boldsymbol{A}$, and $\boldsymbol{x}$ is called an **eigenvector (特征向量)**, *corresponding to* the eigenvalue λ.

David Hilbert
1862-1943

德语 *eigen* (有特征的, 自身的, 个体的)

$\boldsymbol{A}$ 的特征值与特征向量有何直观含义?

$$\begin{aligned} R^n &\to R^n \\ \boldsymbol{A}:\ \boldsymbol{x} &\to \boldsymbol{A}\boldsymbol{x} \end{aligned}$$

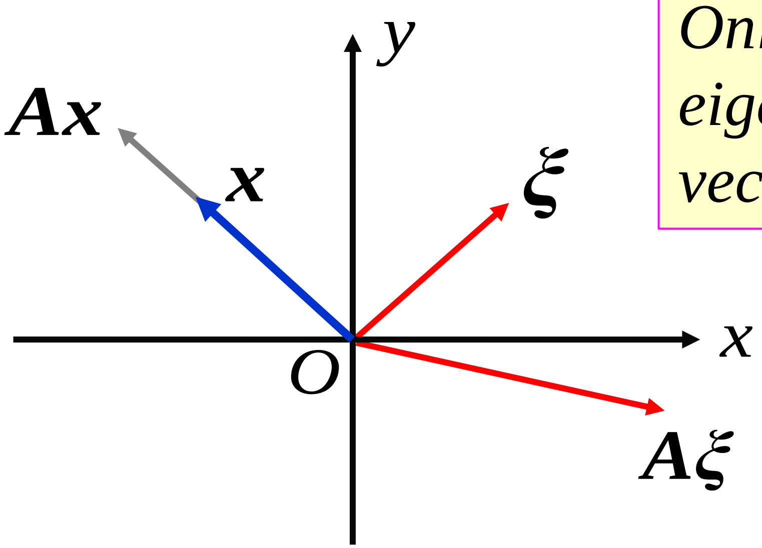
$$\begin{aligned} R^n &\to R^n \\ \lambda:\ \boldsymbol{x} &\to \lambda\boldsymbol{x} \end{aligned}$$

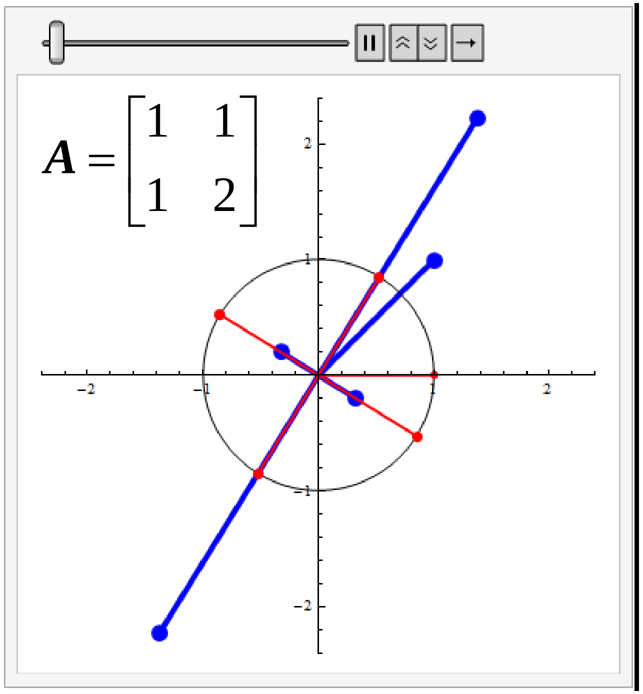
给定方阵$\boldsymbol{A} \in \mathbf{R}^{n\times n}$, 一般说来，对于$\xi \in R^n$, 向量 $\boldsymbol{A}\xi$ 与 ξ 不在同一方向上.

但也可能存在向量 $\boldsymbol{x}$, 使得 $\boldsymbol{A}\boldsymbol{x}$在$\boldsymbol{x}$的方向上.

$\boldsymbol{A}\boldsymbol{x}$ is a multiple of $\boldsymbol{x}$.

Only certain special numbers are eigenvalues, and only certain special vectors are eigenvectors.





$\boldsymbol{A}$ 的特征值与特征向量的直观含义:

特征向量: 与 $\boldsymbol{A}$ 左乘下的像 “共线”

特 征 值: “伸缩比例”

For example, $\boldsymbol{A}=\begin{bmatrix}1 & 3\\ 2 & 2\end{bmatrix}$

$$\begin{bmatrix}1 & 3\\ 2 & 2\end{bmatrix}\begin{bmatrix}1\\ 1\end{bmatrix}=\begin{bmatrix}4\\ 4\end{bmatrix}=4\begin{bmatrix}1\\ 1\end{bmatrix},$$

which shows that 4 is an eigenvalue of $\boldsymbol{A}$, $(1,1)^{\mathrm{T}}$ is an eigenvector for 4 (an eigenvector corresponding to 4).

$$\begin{bmatrix}1 & 3\\ 2 & 2\end{bmatrix}\begin{bmatrix}1\\ -\dfrac{2}{3}\end{bmatrix}=\begin{bmatrix}-1\\ \dfrac{2}{3}\end{bmatrix}=(-1)\begin{bmatrix}1\\ -\dfrac{2}{3}\end{bmatrix},$$

which tells us that -1 is also an eigenvalue of $\boldsymbol{A}$, and $(1,-2/3)^{\mathrm{T}}$ is an eigenvector corresponding to -1.

$A x = \lambda x$

How to get?

!dea Equivalently, an eigenvalue λ and a corresponding eigenvector $\boldsymbol{x}$ satisfy:

$$(\boldsymbol{A} - \lambda \boldsymbol{I})\, \boldsymbol{x} = \boldsymbol{0}.$$

Since $\boldsymbol{x}$ is a *non-zero* vector, the matrix $\boldsymbol{A} - \lambda \boldsymbol{I}$ is not invertible, and thus the determinant $|\, \boldsymbol{A} - \lambda \boldsymbol{I}\, |$ equals 0.

注: 特征向量 $\boldsymbol{x}$ 是非零向量, 是齐次线性方程组

$$(\boldsymbol{A} - \lambda \boldsymbol{I})\, \boldsymbol{x} = \boldsymbol{0}$$

的非零解. λ 应满足

$$|\, \boldsymbol{A} - \lambda \boldsymbol{I}\, | = 0.$$

(characteristic equation: 特征方程)

We note that the determinant $|\boldsymbol{A}-\lambda\boldsymbol{I}|$ is a polynomial of degree n in λ, called the **characteristic polynomial** of $\boldsymbol{A}$.

Definition 2 Let $\boldsymbol{A}=[a_{ij}]_{n\times n}$ be a *square matrix* of degree n. Then

$$f(\lambda)=|\boldsymbol{A}-\lambda\boldsymbol{I}|=\begin{vmatrix} a_{11}-\lambda & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22}-\lambda & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn}-\lambda \end{vmatrix}$$

is called the **characteristic polynomial (特征多项式)** of ***A*.**

n 阶矩阵**A**的特征多项式是λ的 n 次多项式.

Lemma 1 *A scalar λ is an eigenvalue of* ***A*** *if and only if λ is a root of the characteristic polynomial.*

(λ 是 ***A*** 的特征值, 当且仅当 λ 是特征多项式的根.)

n 阶矩阵 A 的特征多项式是λ的 n 次多项式.

Gauss
1777-1855

代数基本定理(Fundamental theorem of algebra)

在复数范围内每个 n 次复系数方程恰有 n 个根.

注释：n 阶方阵 A 在复数范围内有 n 个特征值.

n 阶矩阵 A 的特征多项式在复数域上的 n 个根都是矩阵 A 的特征值, 其 k 重根叫做 k 重特征值.

Galois 1811-1832

当 $n \geq 5$ 时, 特征多项式没有一般的求根公式.

(Galois and Abel proved that there can be no algebraic formula for the roots of a fifth-degree polynomial).

即使是三阶矩阵, 一般也难以求根.

Abel 1802-1829

解决方法： “计算方法（数值分析）”

Example 1 Let $\boldsymbol{A} = \begin{bmatrix} 4 & -5 \\ 2 & -3 \end{bmatrix}$. Then

$$|\boldsymbol{A} - \lambda \boldsymbol{I}| = \begin{vmatrix} 4-\lambda & -5 \\ 2 & -3-\lambda \end{vmatrix} = \lambda^2 - \lambda - 2.$$

The roots are $\lambda_1 = -1$ and $\lambda_2 = 2$. Eigenvectors of $\boldsymbol{A}$ can be obtained as follows.

For $\lambda_1 = -1$, we solve the system of linear equations $(\boldsymbol{A} - \lambda_1 \boldsymbol{I})\boldsymbol{x} = (\boldsymbol{A} + \boldsymbol{I})\boldsymbol{x} = \boldsymbol{0}$, i.e.,

$$\begin{bmatrix} 5 & -5 \\ 2 & -2 \end{bmatrix} \boldsymbol{x} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

It follows that $\boldsymbol{x}_1 = k_1(1,1)^{\mathrm{T}}$ ($k_1 \in \mathbf{R}$, $k_1 \neq 0$) are eigenvectors corresponding to the eigenvalue $\lambda_1 = -1$.

For $\lambda_2 = 2$, we solve the system of linear equations $(\boldsymbol{A} - \lambda_2 \boldsymbol{I})\boldsymbol{x} = (\boldsymbol{A} - 2\boldsymbol{I})\boldsymbol{x} = \boldsymbol{0}$, i.e.,

$$\begin{bmatrix} 2 & -5 \\ 2 & -5 \end{bmatrix} \boldsymbol{x} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

It follows that $\boldsymbol{x}_2 = k_2(5,2)^{\mathrm{T}}$ ($k_2 \in \mathbf{R}$, $k_2 \neq 0$) are eigenvectors corresponding to the eigenvalue $\lambda_2 = 2$.

This example illustrates a method for finding eigenvalues and eigenvectors of a given matrix, which is important in the area of matrix theory and many applications.

*A **Process** for finding eigenvalues and eigenvectors of a matrix **A** :*

1. ***Compute the determinant of*** $\boldsymbol{A} - \lambda \boldsymbol{I}$.

With λ subtracted along the diagonal, this determinant is a polynomial of degree n. It starts with $(-\lambda)^n$.

2. ***Find the roots of this polynomial***.

The n roots are the eigenvalues of $\boldsymbol{A}$.

3. ***For each eigenvalue solve the equation*** $(\boldsymbol{A} - \lambda \boldsymbol{I})\, \boldsymbol{x} = \mathbf{0}$.

Since the determinant is zero, there are solutions other than $\boldsymbol{x} = \mathbf{0}$. Those are the eigenvectors.

特征值与特征向量的求解 步骤

第一步　计算A的特征多项式;

第二步　求出特征多项式的全部根, 即得A的全部特征值;

第三步　将每一个特征值代入相应的线性方程组进行求解, 即得该特征值的特征向量.

$$\boldsymbol{A} \rightarrow \underbrace{|\boldsymbol{A}-\lambda \boldsymbol{I}|=0}_{\text{求特征值}\lambda_i} \rightarrow \underbrace{(\boldsymbol{A}-\lambda_i \boldsymbol{I})\boldsymbol{x}=\boldsymbol{0}}_{\text{求特征向量}}$$

Remark: 另一种等价求法

$$\boldsymbol{A} \rightarrow \underbrace{|\lambda \boldsymbol{I}-\boldsymbol{A}|=0}_{\text{求特征值}\lambda_i} \rightarrow \underbrace{(\lambda_i \boldsymbol{I}-\boldsymbol{A})\boldsymbol{x}=\boldsymbol{0}}_{\text{求特征向量}}$$

Example 2 Everything is clear when $\boldsymbol{A}$ is a ***diagonal matrix***:

$$\boldsymbol{A} = \begin{bmatrix} 3 & 0 \\ 0 & 2 \end{bmatrix} \text{ has } \lambda_1 = 3 \text{ with } \boldsymbol{x}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix},$$

$$\lambda_2 = 2 \text{ with } \boldsymbol{x}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}.$$

On each eigenvector $\boldsymbol{A}$ acts like: $\boldsymbol{A}\boldsymbol{x}_1 = 3\boldsymbol{x}_1$ and $\boldsymbol{A}\boldsymbol{x}_2 = 2\boldsymbol{x}_2$.

Other vectors like $\boldsymbol{x} = \begin{bmatrix} 1 \\ 5 \end{bmatrix}$ are mixtures $\boldsymbol{x}_1 + 5\boldsymbol{x}_2$ of the two eigenvectors,

and $\boldsymbol{A}$ acts like: $\boldsymbol{A}(\boldsymbol{x}_1 + 5\boldsymbol{x}_2) = 3\boldsymbol{x}_1 + 10\boldsymbol{x}_2 = \begin{bmatrix} 3 \\ 10 \end{bmatrix}$.

This is $\boldsymbol{A}\boldsymbol{x}$ for a typical vector $\boldsymbol{x}$—not an eigenvector. But the action of $\boldsymbol{A}$ is determined by its eigenvectors and eigenvalues.

Remark: 1. The eigenvalues are on the main diagonal when $\boldsymbol{A}$ is ***diagonal***.

(n 阶对角矩阵$\boldsymbol{A}$的特征值是它的n个主对角元 $a_{11}, a_{22}, \cdots, a_{nn}$.)

This is true since the characteristic polynomial of $\boldsymbol{A}$ is

$$|\boldsymbol{A}-\lambda \boldsymbol{I}|=(a_{11}-\lambda)(a_{22}-\lambda)\cdots(a_{nn}-\lambda).$$

2. The eigenvalues are on the main diagonal when $\boldsymbol{B}$ is ***triangular***.

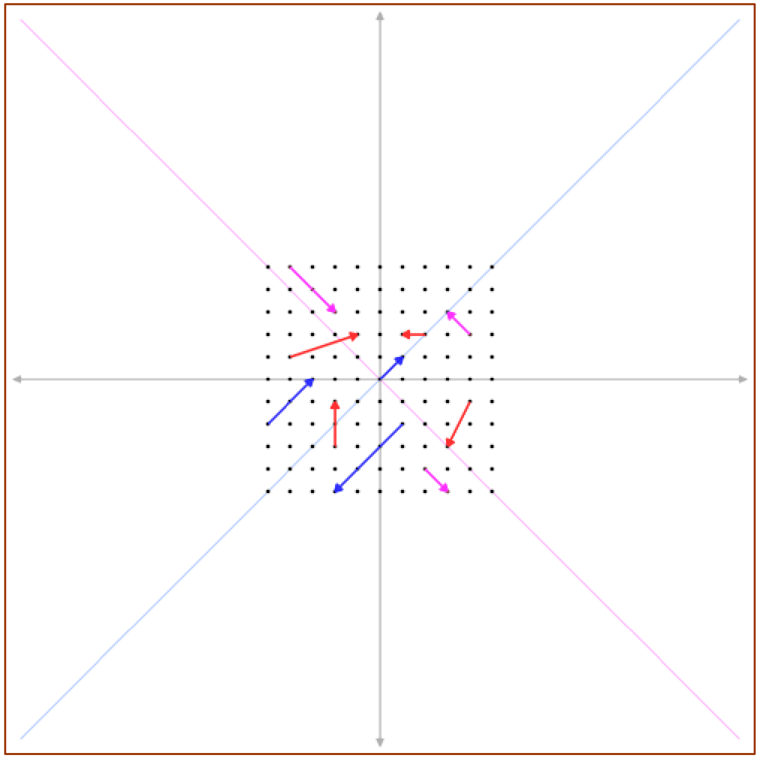
(n 阶上(下)三角形矩阵$\boldsymbol{B}$的特征值也是它的 n 个主对角元 b_{11}, $b_{22}, \cdots, b_{nn}$.)

Exercise

Find the eigenvalues and eigenvectors of $\boldsymbol{A}=\begin{bmatrix}2 & 1\\ 1 & 2\end{bmatrix}$.

Example 3 Find the eigenvalues and eigenvectors of the matrix

$$\boldsymbol{A}=\begin{bmatrix}2 & 1\\ 1 & 2\end{bmatrix}$$



http://en.wikipedia.org/wiki/File:Eigenvectors-extended.gif

The Eigenvectors

$$k_1\begin{bmatrix}1\\ -1\end{bmatrix} \qquad k_2\begin{bmatrix}1\\ 1\end{bmatrix}$$

$$(k_1 \neq 0) \qquad (k_2 \neq 0)$$

corresponding respectively to

the Eigenvalues:

$$\lambda_1=1 \qquad \lambda_2=3$$

https://www.geogebra.org/m/c3be6neg

Example 4 If a square matrix $\boldsymbol{A}$ satisfies $\boldsymbol{A}^2 = \boldsymbol{A}$. Show that the only possible eigenvalues of $\boldsymbol{A}$ are 0 or 1.

Proof Let λ be the eigenvalue of $\boldsymbol{A}$, then $\boldsymbol{A}\boldsymbol{x} = \lambda\boldsymbol{x}$. And

$$\boldsymbol{A}^2\boldsymbol{x} = \boldsymbol{A}(\boldsymbol{A}\boldsymbol{x}) = \boldsymbol{A}(\lambda\boldsymbol{x}) = \lambda\boldsymbol{A}\boldsymbol{x} = \lambda\lambda\boldsymbol{x} = \lambda^2\boldsymbol{x}$$

So $(\lambda^2 - \lambda)\boldsymbol{x} = \boldsymbol{0}.$

Since $\boldsymbol{x} \neq \boldsymbol{0}$, we have $\lambda^2 - \lambda = 0$, and $\lambda = 0$ or $\lambda = 1$.

For example,

$$\boldsymbol{A}_1 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, \boldsymbol{A}_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \boldsymbol{A}_3 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$$

Eigenvalues: 0 1 1, 0

Remark: The eigenvalues of a *projection matrix* are 1 or 0.
(投影矩阵的特征值为0或1)

Remarks: Give a matrix $\boldsymbol{A}$ of degree n.

- $\boldsymbol{A}$ has exactly n eigenvalues, some of them might be *repeated*.
- Some eigenvalues of $\boldsymbol{A}$ may be *complex* numbers; some matrices may have no *real* eigenvalue, for instance, rotation matrices.

$$\boldsymbol{A}_1 = \begin{bmatrix} 0 & -2 & -2 \\ 2 & -4 & -2 \\ -2 & 2 & 0 \end{bmatrix}$$

has eigenvalues: $\lambda_1=0$, $\lambda_2= -2$ (二重特征值, ***a root of multiplicity 2***).

$$|\boldsymbol{A}_1 - \lambda \boldsymbol{I}| = -\lambda(\lambda+2)^2$$

$$\boldsymbol{A}_2 = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

has eigenvalues: $\lambda = \cos\theta \pm i\sin\theta$.

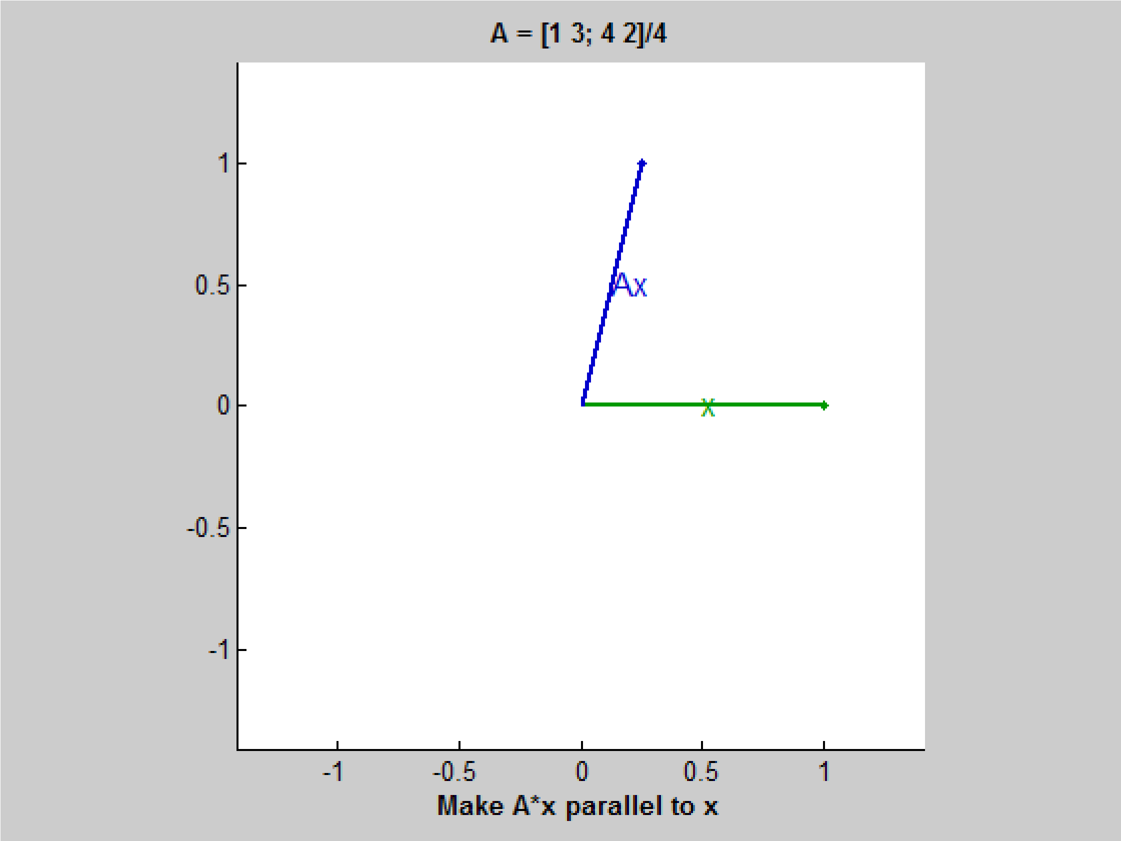
$$|\boldsymbol{A}_2 - \lambda \boldsymbol{I}| = \lambda^2 - (2\cos\theta)\lambda + 1$$

https://www.geogebra.org/m/zen7ncsy

Matlab demo: eigshow

$$\boldsymbol{A} = \frac{1}{4}\begin{bmatrix} 1 & 3 \\ 4 & 2 \end{bmatrix}$$

$$\lambda_1 = -0.5,$$
$$\lambda_2 = 1.25.$$

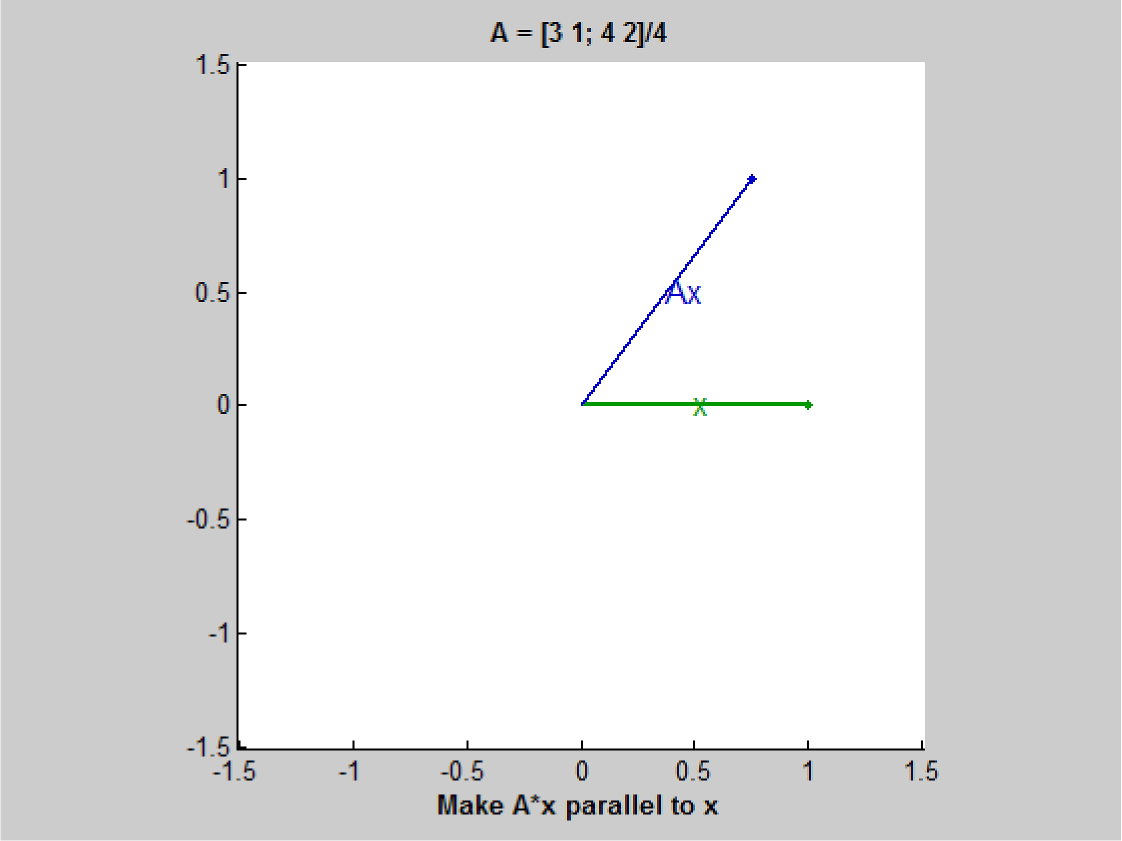


https://blogs.mathworks.com/cleve/2013/07/08/eigshow-week-1/#f96996aa-ef86-4137-b343-07584baff36c

Matlab demo: eigshow

$$\boldsymbol{A} = \frac{1}{4}\begin{bmatrix} 3 & 1 \\ 4 & 2 \end{bmatrix}$$

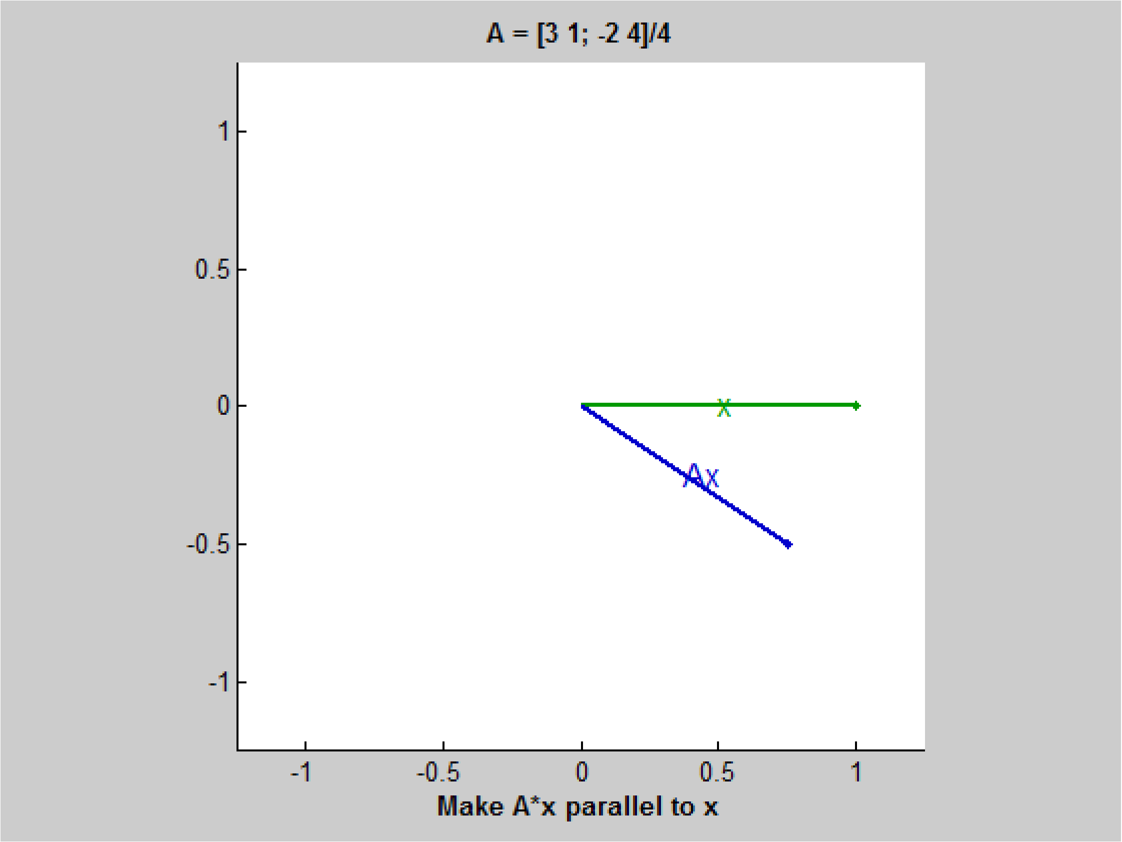
$$\lambda_1 = 1.14,$$
$$\lambda_2 = 0.11.$$



https://blogs.mathworks.com/cleve/2013/07/08/eigshow-week-1/#f96996aa-ef86-4137-b343-07584baff36c

Matlab demo: eigshow

$$A = \frac{1}{4}\begin{bmatrix} 3 & 1 \\ -2 & 4 \end{bmatrix} \qquad \lambda_1 = 0.8750 + 0.3307i, \quad \lambda_2 = 0.8750 - 0.3307i.$$



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III. Eigenvalues and Eigenvectors – Properties

Theorem 1 If $\boldsymbol{x}_1, \boldsymbol{x}_2$ are two eigenvectors of $\boldsymbol{A}$ corresponding to the eigenvalue λ_0, then $k_1\boldsymbol{x}_1 + k_2\boldsymbol{x}_2$ is also an eigenvector for λ_0, where k_1, k_2 are any numbers that make $k_1\boldsymbol{x}_1 + k_2\boldsymbol{x}_2 \neq \boldsymbol{0}$.

定理1 若$\boldsymbol{x}_1, \boldsymbol{x}_2$是$\boldsymbol{A}$属于$\lambda_0$的两个特征向量, 则$k_1\boldsymbol{x}_1 + k_2\boldsymbol{x}_2$也是$\boldsymbol{A}$属于$\lambda_0$的特征向量 (其中$k_1, k_2$是任意常数, 但$k_1\boldsymbol{x}_1 + k_2\boldsymbol{x}_2 \neq \boldsymbol{0}$).

Proof $\boldsymbol{x}_1, \boldsymbol{x}_2$ are solutions to the following homogeneous system of linear equations:

$$(\boldsymbol{A} - \lambda_0 \boldsymbol{I})\, \boldsymbol{x} = \boldsymbol{0},$$

So $k_1\boldsymbol{x}_1 + k_2\boldsymbol{x}_2$ is also a solution.

Therefore, nonzero vector $k_1\boldsymbol{x}_1 + k_2\boldsymbol{x}_2$ is also an eigenvector of $\boldsymbol{A}$ corresponding to the eigenvalue λ_0.

Theorem 2 Let $\boldsymbol{A}=[a_{ij}]_{n\times n}$, and $\lambda_1,\lambda_2,\cdots,\lambda_n$ be the n eigenvalues of $\boldsymbol{A}$.

Then

$$(1)\ \prod_{i=1}^{n}\lambda_i=|\boldsymbol{A}|\,(\text{i.e., }\det\boldsymbol{A}),$$

$$(2)\ \sum_{i=1}^{n}\lambda_i=\sum_{i=1}^{n}a_{ii}=\operatorname{trace}(\boldsymbol{A})(\text{i.e., }\operatorname{tr}(\boldsymbol{A})).$$

The sum of all diagonal entries of $\boldsymbol{A}$ is called the **trace of A (A 的迹)**.

*Some properties about the traces of matrices:

$\operatorname{tr}(\boldsymbol{A}_{n\times n}+\boldsymbol{B}_{n\times n})=\operatorname{tr}(\boldsymbol{A}_{n\times n})+\operatorname{tr}(\boldsymbol{B}_{n\times n})$, $\operatorname{tr}(\boldsymbol{A}_{m\times n}\boldsymbol{B}_{n\times m})=\operatorname{tr}(\boldsymbol{B}_{n\times m}\boldsymbol{A}_{m\times n})$.

Proof

(1) Let $\lambda_1, \lambda_2, \cdots, \lambda_n$ be the eigenvalues of the square matrix $\boldsymbol{A}$.

Then the characteristic polynomial of $\boldsymbol{A}$ can be expressed as

$$|\boldsymbol{A} - \lambda \boldsymbol{I}| = (\lambda_1 - \lambda)(\lambda_2 - \lambda)\cdots(\lambda_n - \lambda)$$

Let $\lambda = 0$ on both sides, we can immediately get

$$|\boldsymbol{A}| = \prod_{i=1}^{n} \lambda_i .$$

(make a clever choice of λ)

(2)
$$|\boldsymbol{A}-\lambda \boldsymbol{I}|=\begin{vmatrix} a_{11}-\lambda & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22}-\lambda & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn}-\lambda \end{vmatrix},$$

Among the expansion of the determinant, there is a term $(a_{11}-\lambda)(a_{22}-\lambda)\cdots(a_{nn}-\lambda)$, and the terms λ^n, λ^{n-1} in the characteristic polynomial *only* come from this term.

Therefore, $|\boldsymbol{A}-\lambda \boldsymbol{I}|$
$$=(-\lambda)^n+\boxed{(a_{11}+a_{22}+\cdots+a_{nn})(-\lambda)^{n-1}}+\cdots+|\boldsymbol{A}|.$$

On the other hand
$$|\boldsymbol{A}-\lambda \boldsymbol{I}|=(\lambda_1-\lambda)(\lambda_2-\lambda)\cdots(\lambda_n-\lambda)$$
$$=(-\lambda)^n+\boxed{(\lambda_1+\lambda_2+\cdots+\lambda_n)(-\lambda)^{n-1}}\cdots+\prod_{i=1}^{n}\lambda_i,$$

Finally, $\displaystyle\sum_{i=1}^{n}\lambda_i=\sum_{i=1}^{n}a_{ii}.$ **(find the coefficient of $(-\lambda)^{n-1}$ and compare)**

Remark: Zero is an eigenvalue of $\boldsymbol{A}$ if and only if $\boldsymbol{A}$ is not invertible.

矩阵$\boldsymbol{A}$可逆的充要条件是 $\boldsymbol{A}$的任意一个特征值不等于零.

$\boldsymbol{A}$为奇异(singular)矩阵的充要条件是$\boldsymbol{A}$至少有一个特征值等于零.

Note: A certain eigenvector of $\boldsymbol{A}$ cannot be corresponding to different eigenvalues. （$\boldsymbol{A}$的一个特征向量不能属于不同的特征值.）

If $\boldsymbol{x}$ were an eigenvector of $\boldsymbol{A}$ corresponding to different eigenvalues $\lambda_1, \lambda_2\,(\lambda_1 \neq \lambda_2)$,

i.e., $\boldsymbol{A}\boldsymbol{x} = \lambda_1 \boldsymbol{x}$ and $\boldsymbol{A}\boldsymbol{x} = \lambda_2 \boldsymbol{x}$,

then $\lambda_1 \boldsymbol{x} = \lambda_2 \boldsymbol{x}$, that is, $(\lambda_1 - \lambda_2)\boldsymbol{x} = \boldsymbol{0}$.

Since $\lambda_1 - \lambda_2 \neq 0$, then $\boldsymbol{x} = \boldsymbol{0}$, which is impossible for an eigenvector.

Property 1 If $\lambda_1, \lambda_2, \ldots, \lambda_n$ are the eigenvalues of $\boldsymbol{A}$, then $\lambda_1^k, \lambda_2^k, \ldots, \lambda_n^k$ are the eigenvalues of $\boldsymbol{A}^k$, where k is a positive integer, and k may equal -1 if $\boldsymbol{A}$ is invertible.

Moreover, if $\boldsymbol{x}$ *is an eigenvector of* $\boldsymbol{A}$ *corresponding to* λ*, then* $\boldsymbol{x}$ *is also an eigenvector of* $\boldsymbol{A}^k$ *corresponding to* λ^k.

Proof. Notice that, if $\boldsymbol{x}$ is an eigenvector of $\boldsymbol{A}$ corresponding to λ_i, then

$$\boldsymbol{A}^2\boldsymbol{x} = \boldsymbol{A}(\boldsymbol{A}\boldsymbol{x}) = \boldsymbol{A}(\lambda_i\boldsymbol{x}) = \lambda_i\boldsymbol{A}\boldsymbol{x} = \lambda_i(\lambda_i\boldsymbol{x}) = \lambda_i^2\boldsymbol{x},$$

The proposition then follows.

性质1 若λ是$\boldsymbol{A}$的特征值, $\boldsymbol{x}$ 是$\boldsymbol{A}$的属于λ 的特征向量. 则

(1) $k\lambda$ 是$k\boldsymbol{A}$的特征值（k为任意常数）；

(2) λ^m是$\boldsymbol{A}^m$的特征值;

(3) 若$\boldsymbol{A}$可逆，则λ^{-1}为$\boldsymbol{A}^{-1}$的一个特征值;

且 $\boldsymbol{x}$ 仍然是矩阵$k\boldsymbol{A}$, $\boldsymbol{A}^m$和$\boldsymbol{A}^{-1}$的分别对应于特征值 $k\lambda$, λ^m 和 λ^{-1} 的特征向量.

Example 5 Suppose a 3×3 matrix $\boldsymbol{A}$ has eigenvalues 1, −1, 2. And $\boldsymbol{B}=\boldsymbol{A}^3-5\boldsymbol{A}^2$. Find $|\boldsymbol{B}|$.

Solution The eigenvalues of $\boldsymbol{B}$ are: $1^3-5\cdot1^2=-4$,

$$(-1)^3-5\cdot(-1)^2=-6,\qquad 2^3-5\cdot2^2=-12.$$

So $|\boldsymbol{B}|=(-4)(-6)(-12)=-288.$

Property 2 The matrices $\boldsymbol{A}$ and $\boldsymbol{A}^{\mathrm{T}}$ have same eigenvalues.

(**性质2** 矩阵 $\boldsymbol{A}$ 和 $\boldsymbol{A}^{\mathrm{T}}$ 的特征值相同.)

Proof. $\det(\boldsymbol{A}-\lambda\boldsymbol{I})=\det(\boldsymbol{A}-\lambda\boldsymbol{I})^{\mathrm{T}}$

$$=\det(\boldsymbol{A}^{\mathrm{T}}-(\lambda\boldsymbol{I})^{\mathrm{T}})=\det(\boldsymbol{A}^{\mathrm{T}}-\lambda\boldsymbol{I}).$$

Thus $\boldsymbol{A}$ and $\boldsymbol{A}^{\mathrm{T}}$ have the same characteristic polynomials.

The proposition then follows.

回顾：人口流动问题

Possible? Yes!

$$A = PQP^{-1} \longrightarrow A^k = PQ^kP^{-1} \quad (Q\text{: diagonal})$$

$$AP = PQ \iff A\begin{bmatrix}2\\3\end{bmatrix} = 1\times\begin{bmatrix}2\\3\end{bmatrix}, \quad A\begin{bmatrix}1\\-1\end{bmatrix} = \frac{7}{12}\begin{bmatrix}1\\-1\end{bmatrix}$$

$$A = \begin{bmatrix}\frac{3}{4} & \frac{1}{6}\\ \frac{1}{4} & \frac{5}{6}\end{bmatrix} \qquad Q = \begin{bmatrix}1 & \\ & \frac{7}{12}\end{bmatrix} \qquad P = \begin{bmatrix}2 & 1\\ 3 & -1\end{bmatrix}$$

$$\begin{bmatrix} y_k \\ z_k \end{bmatrix} = \boldsymbol{A}^k \begin{bmatrix} y_0 \\ z_0 \end{bmatrix} = \boldsymbol{P}\boldsymbol{Q}^k \boldsymbol{P}^{-1} \begin{bmatrix} y_0 \\ z_0 \end{bmatrix}$$

$$= \begin{bmatrix} \frac{2}{5}(y_0+z_0)+\frac{1}{5}\times\left(\frac{7}{12}\right)^k(3y_0-2z_0) \\ \frac{3}{5}(y_0+z_0)+\frac{1}{5}\times\left(\frac{7}{12}\right)^k(2z_0-3y_0) \end{bmatrix} \to \begin{bmatrix} \frac{2}{5}(y_0+z_0) \\ \frac{3}{5}(y_0+z_0) \end{bmatrix}, \quad 当k\to\infty时.$$

农业与非农业人口比例的趋势为 2 : 3.

思考

1. y_0 与 z_0 的比例对稳定的趋势有多大影响?
2. 稳定的趋势与 $\boldsymbol{A}$ 的什么特征有联系?
3. 如果人口总数是变化的，如何建模?

$$A=\begin{bmatrix}\frac{3}{4} & \frac{1}{6}\\ \frac{1}{4} & \frac{5}{6}\end{bmatrix}.$$

$$=PQP^{-1}.$$

$$\begin{bmatrix}2 & 1\\ 3 & -1\end{bmatrix}\begin{bmatrix}1 & \\ & 7/12\end{bmatrix}$$

对 $A_{n\times n}$,

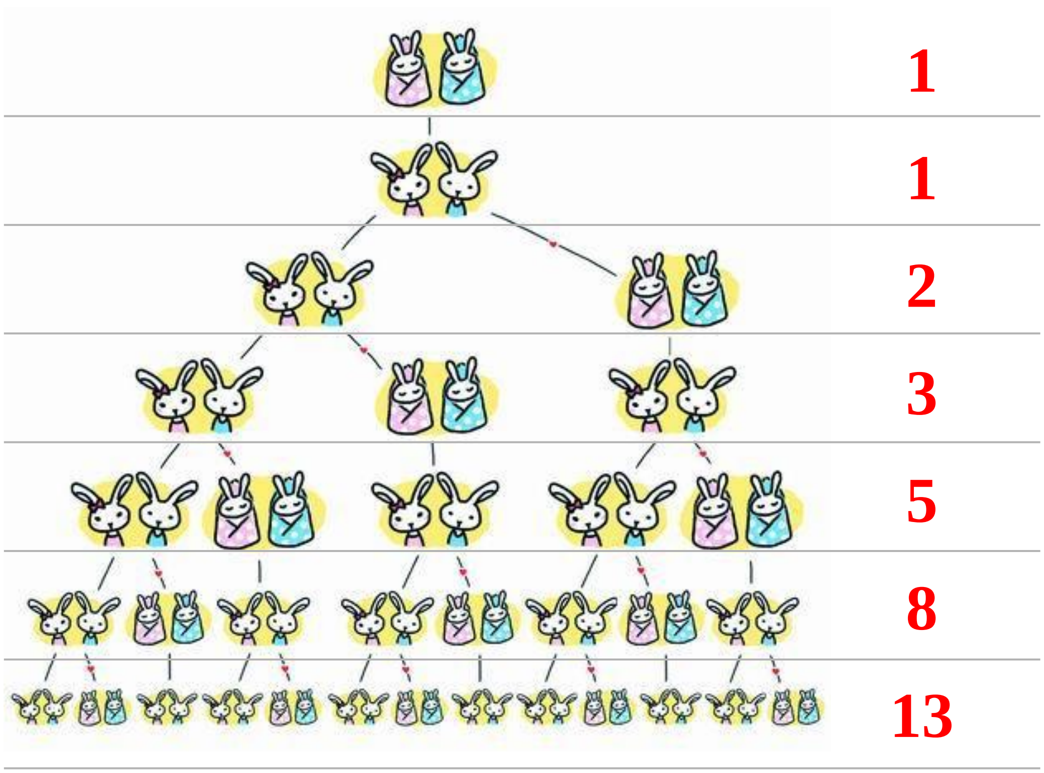
- 不同的特征值对应的特征向量线性无关吗?
- 是否存在可逆矩阵 P 和对角矩阵 Q, 使得 $A=PQP^{-1}$?

Come back later in 5.2:
***Diagonalization* (对角化) *of a Matrix*.**

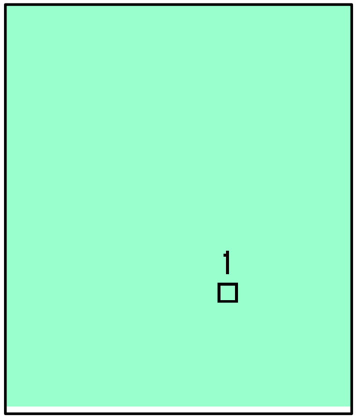
Another interesting application…

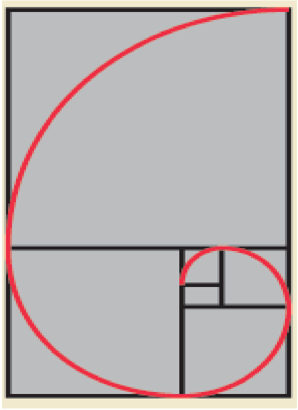
Fibonacci
1175-1250

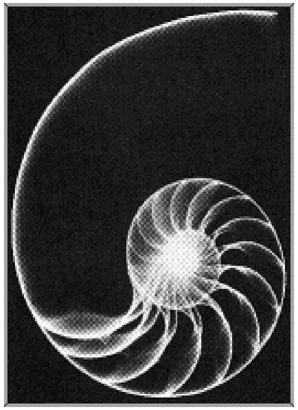
用矩阵方法求*Fibonacci*数列的通项公式



1
1
2
3
5
8
13
21
⋮







*Fibonacci*数列的递推关系为

$$F_0=0,\ F_1=1,\quad F_{n+2}=F_{n+1}+F_n,\quad n=0,1,2,\ldots$$

首先构造一组恒等式

$$\begin{cases} F_{n+2}=F_{n+1}+F_n, \\ F_{n+1}=F_{n+1}. \end{cases} \Rightarrow \begin{bmatrix} F_{n+2} \\ F_{n+1} \end{bmatrix} = \underbrace{\begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} F_{n+1} \\ F_n \end{bmatrix}}_{\boldsymbol{u}_n}.$$

$$\Rightarrow \boxed{\boldsymbol{u}_n = A\boldsymbol{u}_{n-1} = A^n\boldsymbol{u}_0.}$$

A的特征值为

$$\lambda_1=\frac{1-\sqrt{5}}{2},\ \lambda_2=\frac{1+\sqrt{5}}{2}.$$

进而求出数列通项

$$\boxed{F_n=\frac{\sqrt{5}}{5}\left[\left(\frac{1+\sqrt{5}}{2}\right)^n-\left(\frac{1-\sqrt{5}}{2}\right)^n\right].}$$

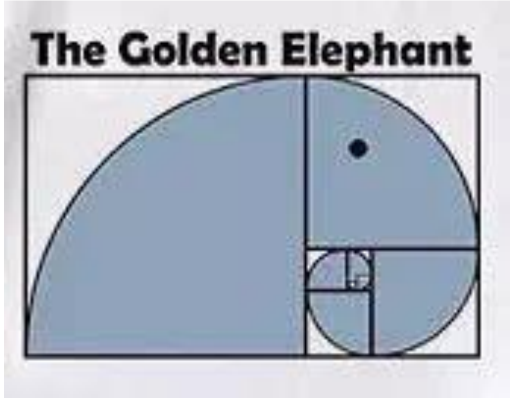
Key words:

Definition

Calculation

Properties

Examples



Homework

See Blackboard