

Chapter 4

Greedy Algorithms



Slides by Kevin Wayne.
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Coin Changing

Coin Changing

Goal. Given currency denominations: 1, 5, 10, 25, 100, devise a method to pay amount to customer using fewest number of coins.

Ex: 34¢.



Cashier's algorithm. At each iteration, add coin of the largest value that does not take us past the amount to be paid.

Ex: \$2.89.



Coin-Changing: Greedy Algorithm

CASHIERS – ALGORITHM(x, c_1, \dots, c_n)

```
1: SORT  $n$  coin denominations so that  $c_1 < c_2 < \dots < c_n$ .
2:  $S \leftarrow \emptyset$ 
3: while  $x > 0$  do
4:    $k \leftarrow$  largest coin denomination  $c_k$  such that  $c_k \leq x$ .
5:   if no such  $k$  then
6:     return “no solution”.
7:   else
8:      $x \leftarrow x - c_k, S \leftarrow S \cup \{k\}$ .
9:   end if
10: end while
11: return  $S$ .
```



Q. Is cashier's algorithm optimal?

Coin-Changing: Analysis of Greedy Algorithm

Theorem. Greedy is optimal for U.S. coinage: 1, 5, 10, 25, 100.

Pf. (by induction on x)

Consider optimal way to change $c_k \leq x < c_{k+1}$: greedy takes coin k .

We claim that any optimal solution must also take coin k .

- if not, it needs enough coins of type c_1, \dots, c_{k-1} to add up to x
- table below indicates no optimal solution can do this

Problem reduces to coin-changing $x - c_k$ cents, which, by induction, is optimally solved by greedy algorithm. ▀

k	c_k	All optimal solutions must satisfy	Max value of coins 1, 2, ..., $k-1$ in any OPT
1	1	$P \leq 4$	-
2	5	$N \leq 1$	4
3	10	$N + D \leq 2$	$4 + 5 = 9$
4	25	$Q \leq 3$	$20 + 4 = 24$
5	100	no limit	$75 + 24 = 99$

Coin-Changing: Analysis of Greedy Algorithm

Observation. Greedy algorithm is sub-optimal for US postal denominations: 1, 10, 21, 34, 70, 100, 350, 1225, 1500.

Counterexample. 140¢.

Greedy: 100, 34, 1, 1, 1, 1, 1, 1.

Optimal: 70, 70.



4.1 Interval Scheduling

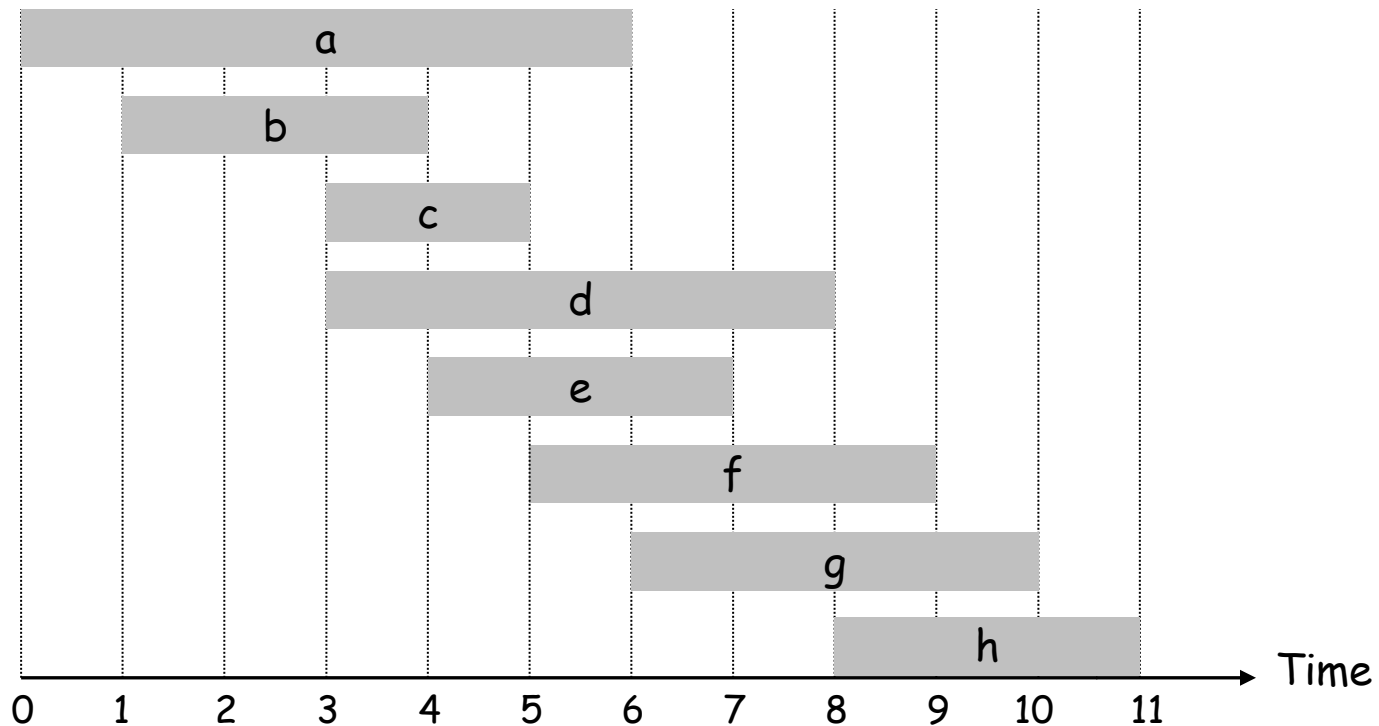
Interval Scheduling

Interval scheduling.

Job j starts at s_j and finishes at f_j .

Two jobs **compatible** if they don't overlap.

Goal: find maximum subset of mutually compatible jobs.



Interval Scheduling: Greedy Algorithms

Greedy template. Consider jobs in some order. Take each job provided it's compatible with the ones already taken.

[Earliest start time] Consider jobs in ascending order of start time s_j .

[Earliest finish time] Consider jobs in ascending order of finish time f_j .

[Shortest interval] Consider jobs in ascending order of interval length $f_j - s_j$.

[Fewest conflicts] For each job, count the number of conflicting jobs c_j . Schedule in ascending order of conflicts c_j .

Interval Scheduling: Greedy Algorithms

Greedy template. Consider jobs in some order. Take each job provided it's compatible with the ones already taken.



breaks earliest start time



breaks shortest interval



breaks fewest conflicts

Interval Scheduling: Greedy Algorithm

Greedy algorithm. Consider jobs in increasing order of finish time.
Take each job provided it's compatible with the ones already taken.

EARLIEST – FINISH – TIME – FIRST($n, s_1, \dots, s_n, f_1, \dots, f_n$)

```
1: SORT jobs by finish time so that  $f_1 \leq f_2 \leq \dots \leq f_n$ .
2:  $A \leftarrow \emptyset$ 
3: for  $j = 1$  to  $n$  do
4:   if job  $j$  is compatible with  $A$  then
5:      $A \leftarrow A \cup \{j\}$ .
6:   end if
7: end for
8: return  $A$ .
```

Implementation. $O(n \log n)$.

Remember job j^* that was added last to A .

Job j is compatible with A if $s_j \geq f_{j^*}$.

Interval Scheduling: Analysis

Theorem. Greedy algorithm is optimal.

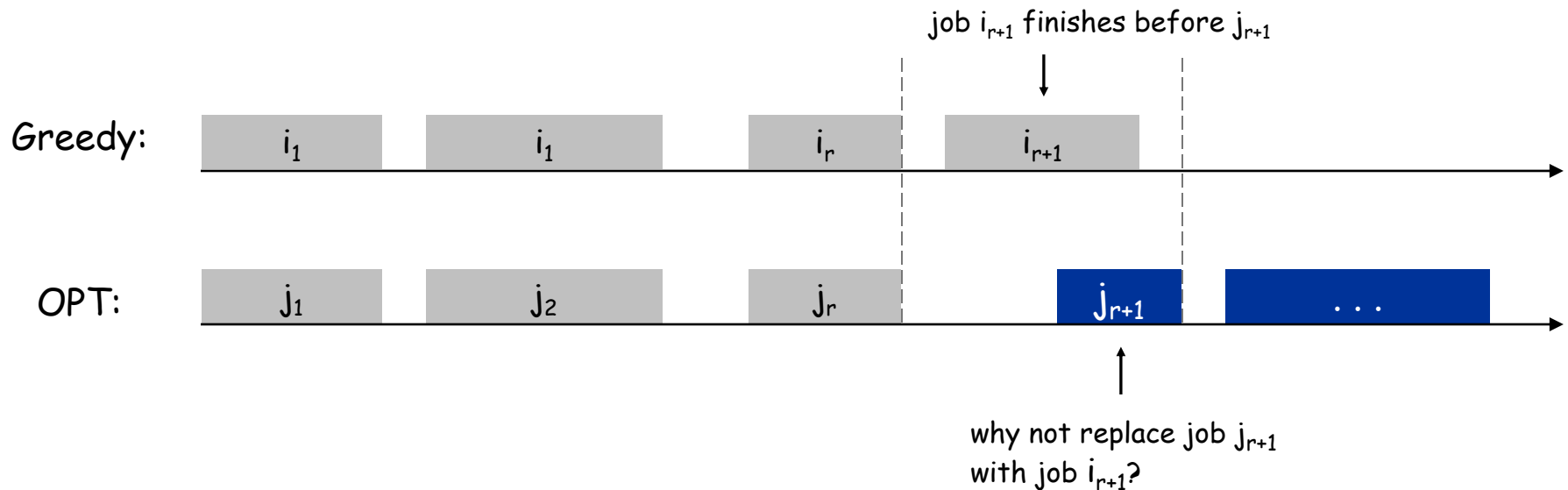
Pf. (by contradiction)

Assume greedy is not optimal, and let's see what happens.

Let i_1, i_2, \dots, i_k denote set of jobs selected by greedy.

Let j_1, j_2, \dots, j_m denote set of jobs in the optimal solution with

$i_1 = j_1, i_2 = j_2, \dots, i_r = j_r$ for the largest possible value of r .



Interval Scheduling: Analysis

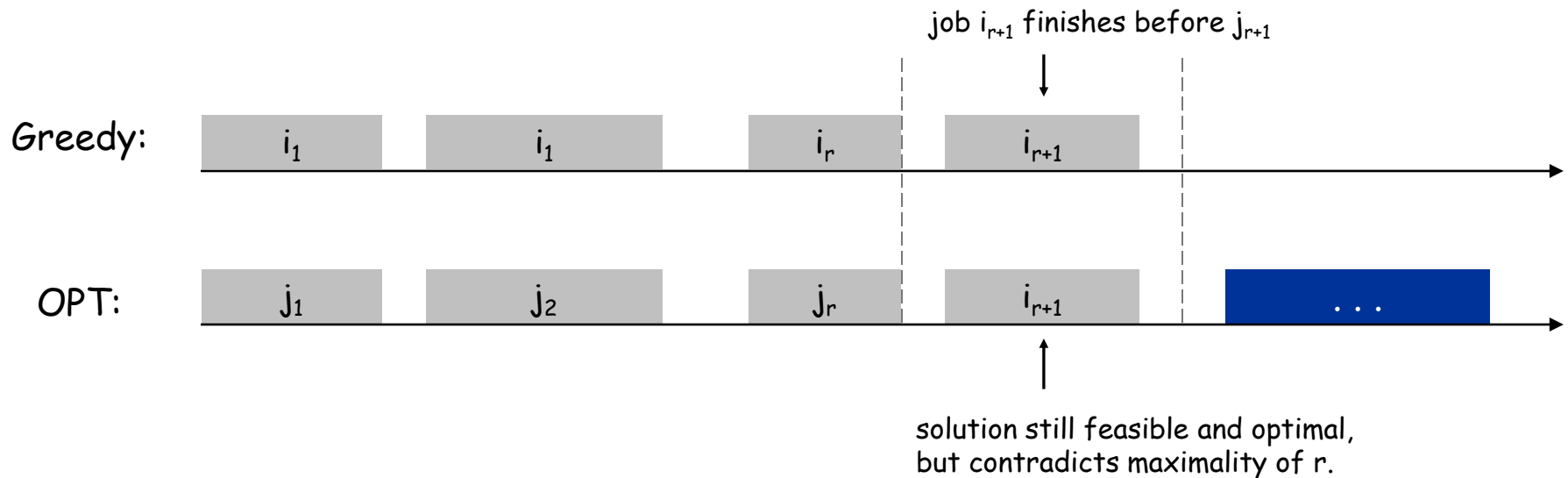
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4.1 Interval Partitioning

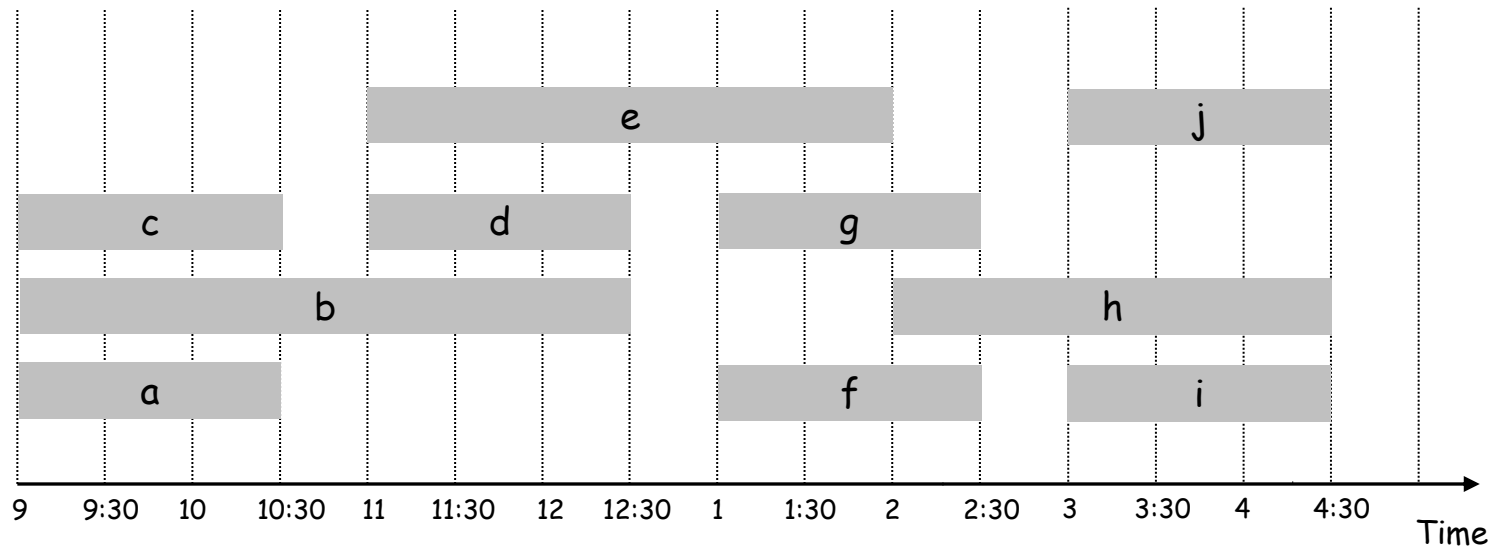
Interval Partitioning

Interval partitioning.

Lecture j starts at s_j and finishes at f_j .

Goal: find minimum number of classrooms to schedule all lectures so that no two occur at the same time in the same room.

Ex: This schedule uses 4 classrooms to schedule 10 lectures.



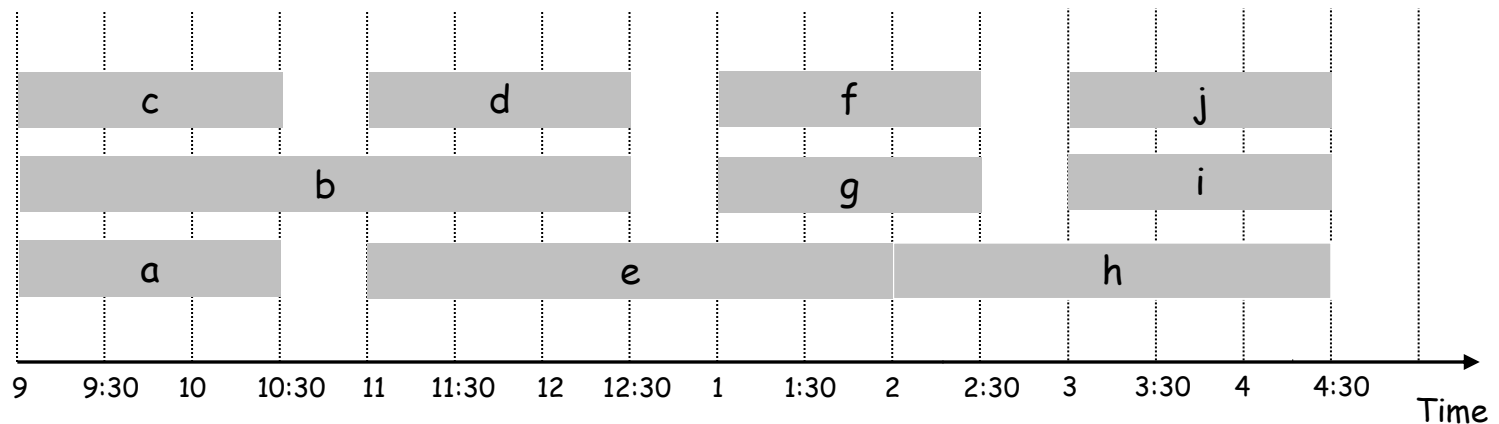
Interval Partitioning

Interval partitioning.

Lecture j starts at s_j and finishes at f_j .

Goal: find minimum number of classrooms to schedule all lectures so that no two occur at the same time in the same room.

Ex: This schedule uses only 3.



Interval Partitioning: Lower Bound on Optimal Solution

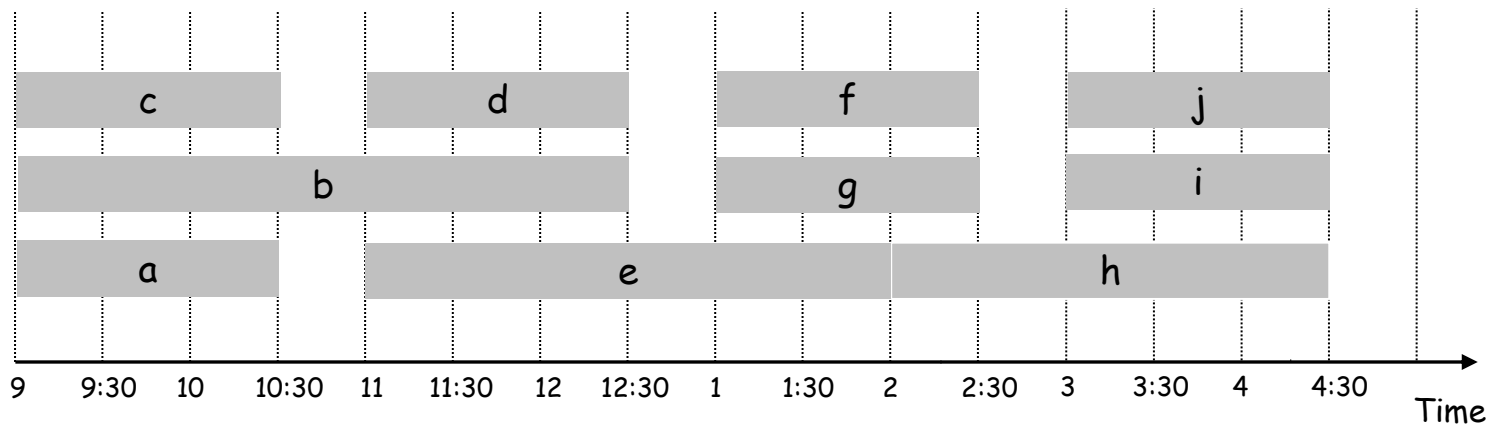
Def. The **depth** of a set of open intervals is the maximum number that contain any given time.

Key observation. Number of classrooms needed \geq depth.

Ex: Depth of schedule below = 3 \Rightarrow schedule below is optimal.

↑
a, b, c all contain 9:30

Q. Does there always exist a schedule equal to depth of intervals?



Interval Partitioning: Greedy Algorithm

EARLIEST – START – TIME – FIRST($n, s_1, \dots, s_n, f_1, \dots, f_n$)

```
1: SORT lectures by start time so that  $s_1 \leq s_2 \leq \dots \leq s_n$ .
2:  $d \leftarrow 0$            % number of allocated classrooms
3: for  $j = 1$  to  $n$  do
4:   if lecture  $j$  is compatible with some classroom then
5:     Schedule lecture  $j$  in any such classroom  $k$ .
6:   else
7:     Allocate a new classroom  $d + 1$ .
8:     Schedule lecture  $j$  in classroom  $d + 1$ .
9:      $d \leftarrow d + 1$ 
10:  end if
11: end for
12: return schedule.
```

Implementation. $O(n \log n)$.

Interval Partitioning: Greedy Analysis

Observation. Greedy algorithm never schedules two incompatible lectures in the same classroom.

Theorem. Greedy algorithm is optimal.

Pf.

Let d = number of classrooms that the greedy algorithm allocates. Classroom d is opened because we needed to schedule a job, say j , that is incompatible with all $d-1$ other classrooms.

Since we sorted by start time, all these incompatibilities are caused by lectures that start no later than s_j .

Thus, we have d lectures overlapping at time $s_j + \varepsilon$.

Key observation \Rightarrow all schedules use $\geq d$ classrooms. ▀

4.2 Scheduling to Minimize Lateness

Scheduling to Minimizing Lateness

Minimizing lateness problem.

Single resource processes one job at a time.

Job j requires t_j units of processing time and is due at time d_j .

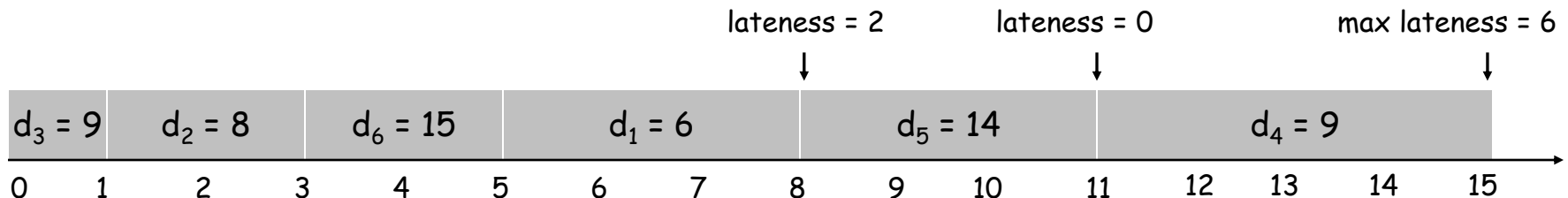
If j starts at time s_j , it finishes at time $f_j = s_j + t_j$.

Lateness: $\ell_j = \max \{ 0, f_j - d_j \}$.

Goal: schedule all jobs to minimize **maximum** lateness $L = \max \ell_j$.

Ex:

	1	2	3	4	5	6
t_j	3	2	1	4	3	2
d_j	6	8	9	9	14	15



Minimizing Lateness: Greedy Algorithms

Greedy template. Consider jobs in some order.

[Shortest processing time first] Consider jobs in ascending order of processing time t_j .

[Earliest deadline first] Consider jobs in ascending order of deadline d_j .

[Smallest slack] Consider jobs in ascending order of slack $d_j - t_j$.

Minimizing Lateness: Greedy Algorithms

Greedy template. Consider jobs in some order.

[Shortest processing time first] Consider jobs in ascending order of processing time t_j .

	1	2
t_j	1	10
d_j	100	10

counterexample

[Smallest slack] Consider jobs in ascending order of slack $d_j - t_j$.

	1	2
t_j	1	10
d_j	2	10

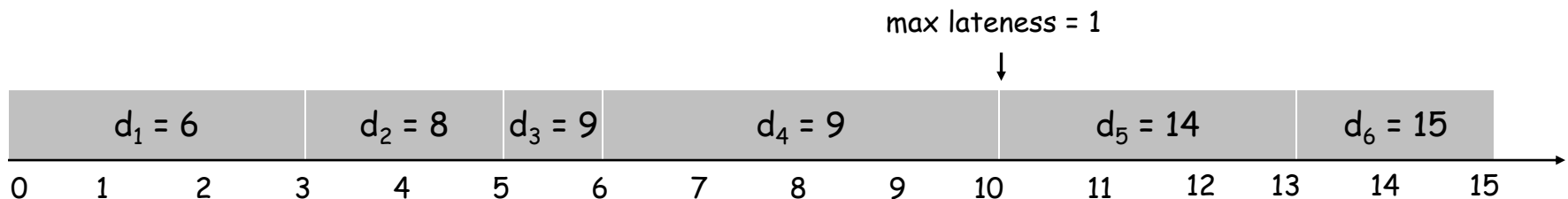
counterexample

Minimizing Lateness: Greedy Algorithm

Greedy algorithm. Earliest deadline first.

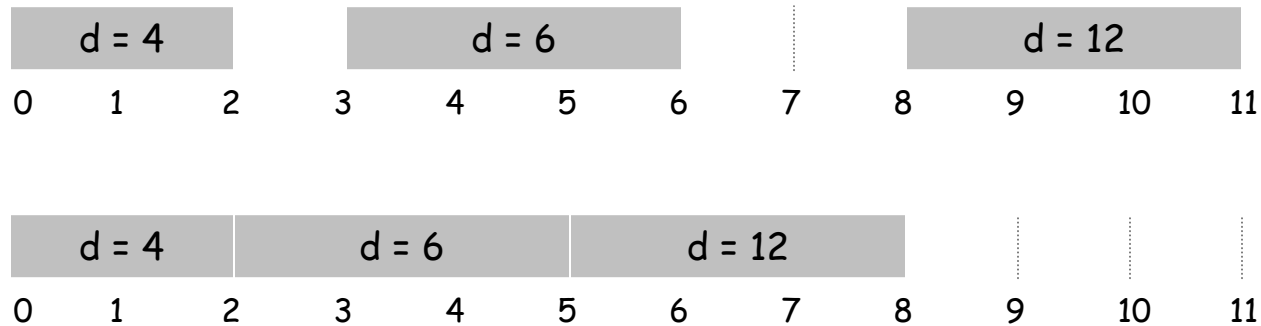
EARLIEST – DEADLINE – FIRST($n, t_1, \dots, t_n, d_1, \dots, d_n$)

```
1: SORT jobs so that  $d_1 \leq d_2 \leq \dots \leq d_n$ .
2:  $t \leftarrow 0$ 
3: for  $j = 1$  to  $n$  do
4:   Assign job  $j$  to interval  $[t, t + t_j]$ .
5:    $s_j \leftarrow t$ ;  $f_j \leftarrow t + t_j$ 
6:    $t \leftarrow t + t_j$ 
7: end for
8: return Intervals  $[s_1, f_1], [s_2, f_2], \dots, [s_n, f_n]$ .
```



Minimizing Lateness: No Idle Time

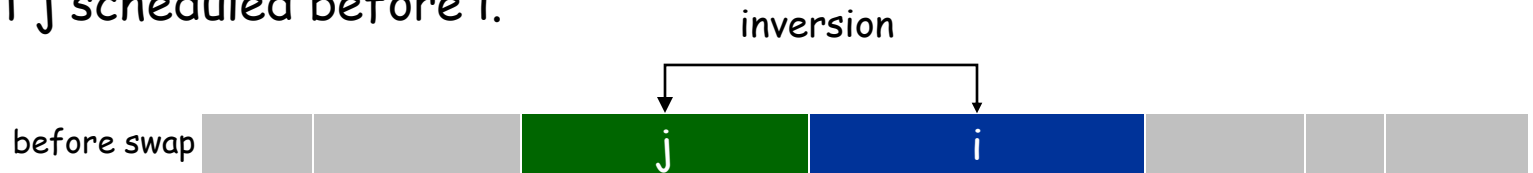
Observation. There exists an optimal schedule with no **idle time**.



Observation. The greedy schedule has no idle time.

Minimizing Lateness: Inversions

Def. An **inversion** in schedule S is a pair of jobs i and j such that: $i < j$ but j scheduled before i .

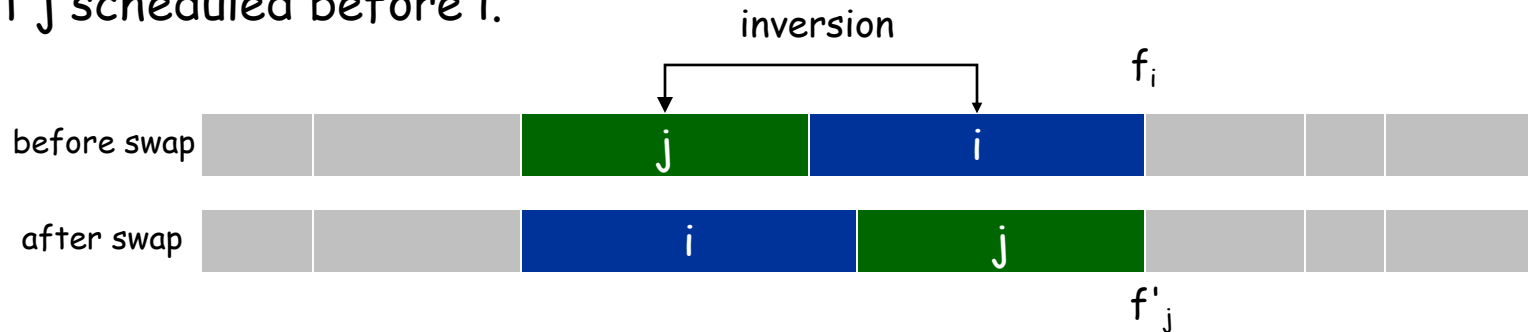


Observation. Greedy schedule has no inversions.

Observation. If a schedule (with no idle time) has an inversion, it has one with a pair of inverted jobs scheduled consecutively.

Minimizing Lateness: Inversions

Def. An **inversion** in schedule S is a pair of jobs i and j such that: $i < j$ but j scheduled before i .



Claim. Swapping two adjacent, inverted jobs reduces the number of inversions by one and does not increase the max lateness.

Pf. Let ℓ be the lateness before the swap, and let ℓ' be it afterwards.

$$\ell'_k = \ell_k \text{ for all } k \neq i, j$$

$$\ell'_i \leq \ell_i$$

If job j is late:

$$\begin{aligned} \ell'_j &= f'_j - d_j && \text{(definition)} \\ &= f_i - d_j && \text{(j finishes at time } f_i) \\ &\leq f_i - d_i && (i < j) \\ &\leq \ell_i && \text{(definition)} \end{aligned}$$

Minimizing Lateness: Analysis of Greedy Algorithm

Theorem. Greedy schedule S is optimal.

Pf. Define S^* to be an optimal schedule that has the fewest number of inversions, and let's see what happens.

Can assume S^* has no idle time.

If S^* has no inversions, then $S = S^*$.

If S^* has an inversion, let i - j be an adjacent inversion.

- swapping i and j does not increase the maximum lateness and strictly decreases the number of inversions
- this contradicts definition of S^* ■

Greedy Analysis Strategies

Greedy algorithm stays ahead. Show that after each step of the greedy algorithm, its solution is at least as good as any other algorithm's.

Exchange argument. Gradually transform any solution to the one found by the greedy algorithm without hurting its quality.

Structural. Discover a simple "structural" bound asserting that every possible solution must have a certain value. Then show that your algorithm always achieves this bound.

4.3 Optimal Caching

Optimal Offline Caching

Caching.

Cache with capacity to store k items.

Sequence of m item requests d_1, d_2, \dots, d_m .

Cache hit: item already in cache when requested.

Cache miss: item not already in cache when requested: must bring requested item into cache, and evict some existing item, if full.

Goal. Eviction schedule that minimizes number of cache misses.

Ex: $k = 2$, initial cache = ab ,
requests: a, b, c, b, c, a, a, b .

Optimal eviction schedule: 2 cache misses.

a	a	b
b	a	b
c	c	b
b	c	b
c	c	b
a	a	b
a	a	b
b	a	b
requests	cache	

Optimal Offline Caching: Farthest-In-Future

Farthest-in-future. Evict item in the cache that is not requested until farthest in the future.

current cache:

a	b	c	d	e	f
---	---	---	---	---	---

future queries: g a b c e d a b b a c d e a f a d e f g h ...

↑
cache miss

↑
eject this one

Theorem. [Bellady, 1960s] FF is optimal eviction schedule.

Pf. Algorithm and theorem are intuitive; proof is subtle.

Reduced Eviction Schedules

Def. A **reduced** schedule is a schedule that only inserts an item into the cache in a step in which that item is requested.

Intuition. Can transform an unreduced schedule into a reduced one with no more cache misses.

a	a	b	c
a	a	x	c
c	a	d	c
d	a	d	b
a	a	c	b
b	a	x	b
c	a	c	b
a	a	b	c
a	a	b	c

an unreduced schedule

a	a	b	c
a	a	b	c
c	a	b	c
d	a	d	c
a	a	d	c
b	a	d	b
c	a	c	b
a	a	c	b
a	a	c	b

a reduced schedule

Claim. Given any unreduced schedule S , can transform it into a reduced schedule S' with no more cache misses.

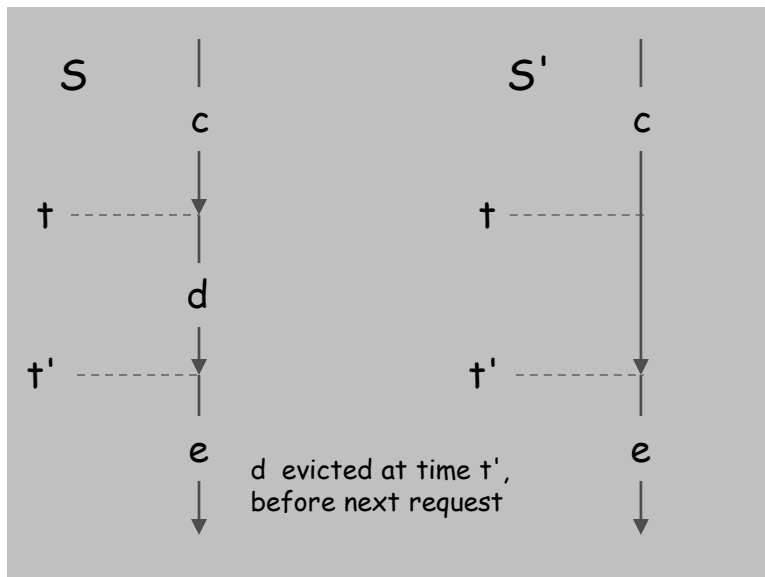
Pf.

Suppose S brings d into the cache at time t , without a request.

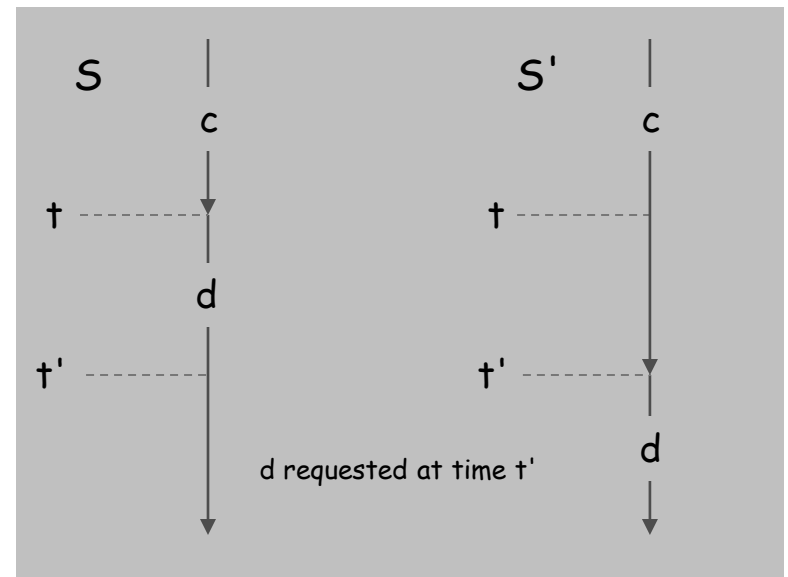
Let c be the item S evicts when it brings d into the cache.

Case 1: d evicted at time t' , before next request for d .

Case 2: d requested at time t' before d is evicted. ▀



Case 1



Case 2

Farthest-In-Future: Analysis

Theorem. FF is optimal eviction algorithm.

Pf. (by induction on number of requests j)

Invariant: There exists an optimal reduced schedule S that makes the same eviction schedule as S_{FF} through the first $j+1$ requests.

Let S be reduced schedule that satisfies invariant through j requests.

We produce S' that satisfies invariant after $j+1$ requests.

Consider $(j+1)^{st}$ request $d = d_{j+1}$.

Since S and S_{FF} have agreed up until now, they have the same cache contents before request $j+1$.

Case 1: (d is already in the cache). $S' = S$ satisfies invariant.

Case 2: (d is not in the cache and S and S_{FF} evict the same element).

$S' = S$ satisfies invariant.

j

same	e	f
------	---	---

S

same	e	f
------	---	---

S'

j+1

same	e	d
------	---	---

S

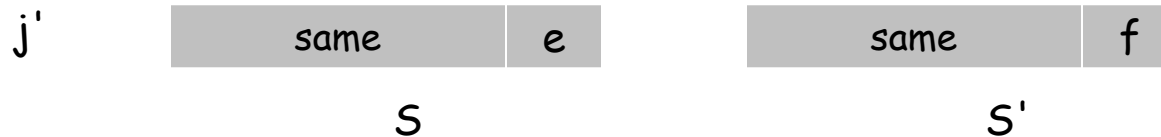
same	d	f
------	---	---

S'

Farthest-In-Future: Analysis

Let j' be the **first** time after $j+1$ that S and S' take a different action, and let g be item requested at time j' .

↑
must involve e or f (or both)



Case 3a: $g = e$. Can't happen with Farthest-In-Future since there must be a request for f before e .

Case 3b: $g = f$. Element f can't be in cache of S , so let e' be the element that S evicts.

- if $e' = e$, S' accesses f from cache; now S and S' have same cache
- if $e' \neq e$, S' evicts e' and brings e into the cache; now S and S' have the same cache

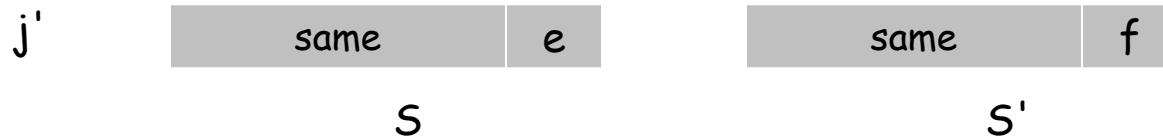
↑

Note: S' is no longer reduced, but can be transformed into a reduced schedule that agrees with S_{FF} through step $j+1$

Farthest-In-Future: Analysis

Let j' be the **first** time after $j+1$ that S and S' take a different action, and let g be item requested at time j' .

↑
must involve e or f (or both)

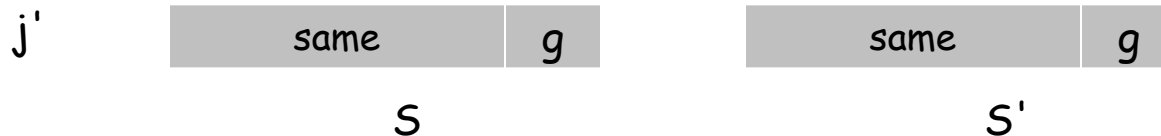


otherwise S' would take the same action



Case 3c: $g \neq e, f$. S must evict e .

Make S' evict f ; now S and S' have the same cache. ▀



Caching Perspective

Online vs. offline algorithms.

Offline: full sequence of requests is known a priori.

Online (reality): requests are not known in advance.

Caching is among most fundamental online problems in CS.

LIFO. Evict page brought in most recently.

LRU. Evict page whose most recent access was earliest.

↑
FF with direction of time reversed!

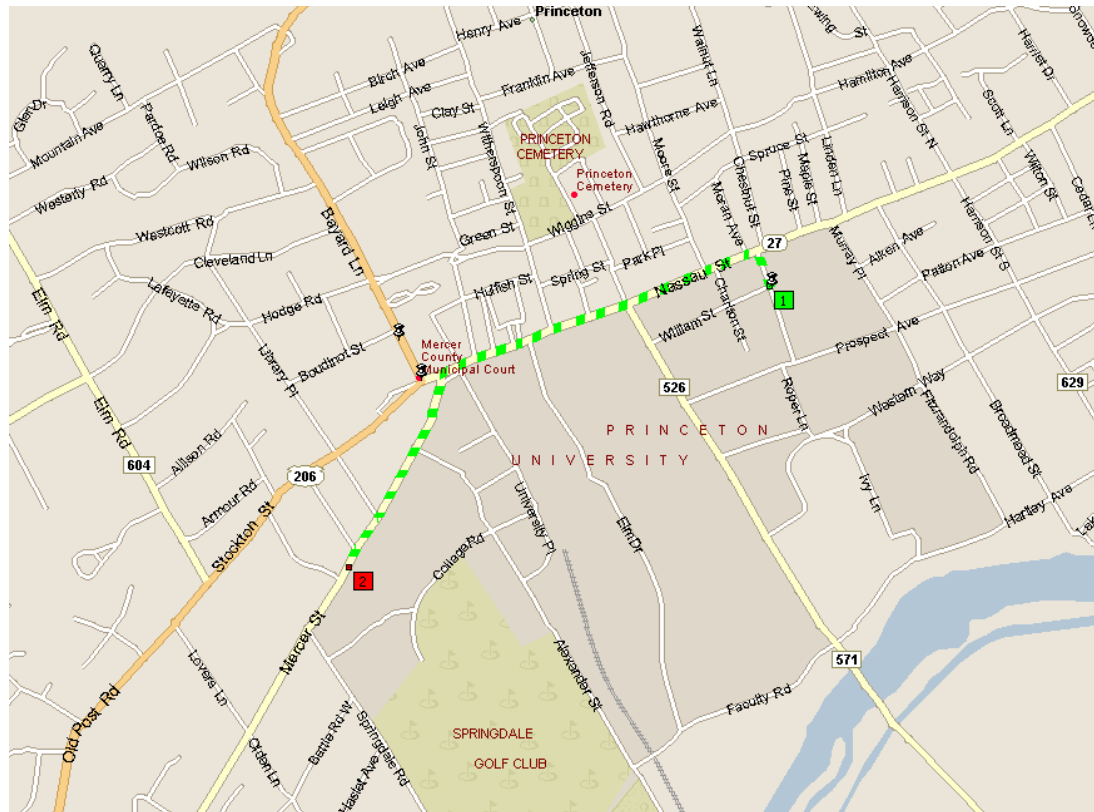
Theorem. FF is optimal offline eviction algorithm.

Provides basis for understanding and analyzing online algorithms.

LRU is k -competitive. [Section 13.8]

LIFO is arbitrarily bad.

4.4 Shortest Paths in a Graph



shortest path from Princeton CS department to Einstein's house

Shortest Path Problem

Shortest path network.

Directed graph $G = (V, E)$.

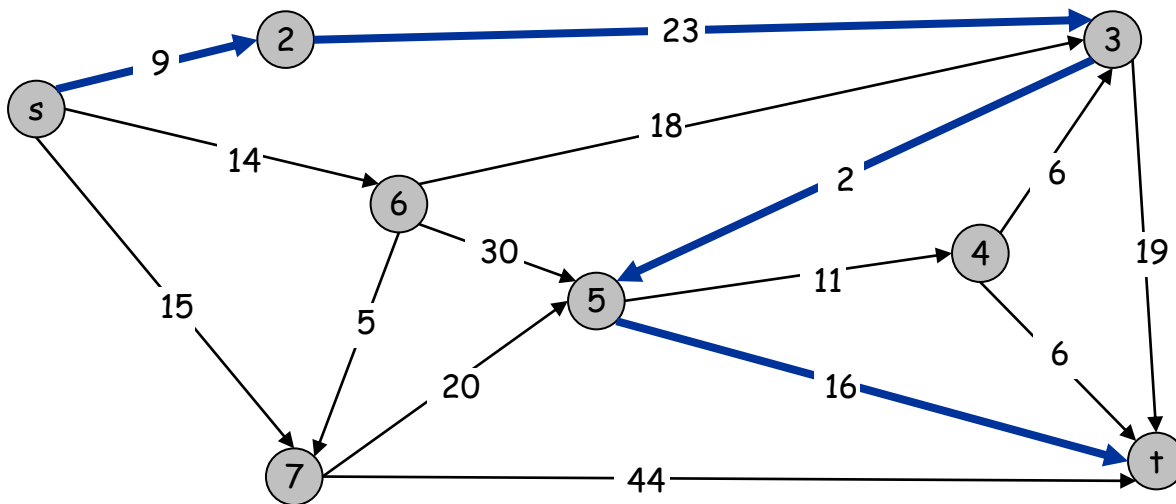
Source s , destination t .

Length $\ell_e \geq 0$, length of edge e .

Shortest path problem: find shortest directed path from s to t .



cost of path = sum of edge costs in path



Cost of path $s-2-3-5-t$
 $= 9 + 23 + 2 + 16$
 $= 48.$

Dijkstra's Algorithm

Dijkstra's algorithm.

Maintain a set of **explored nodes** S for which we have determined the shortest path distance $d(u)$ from s to u .

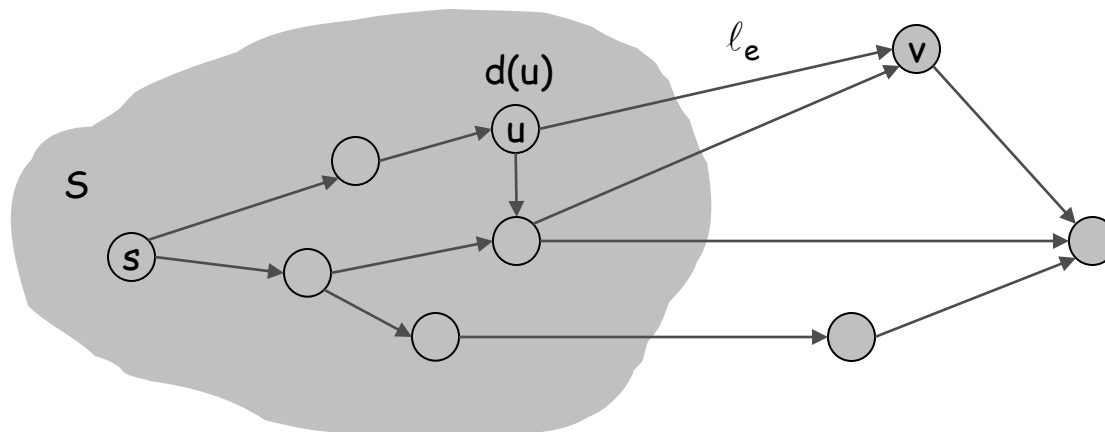
Initialize $S = \{s\}$, $d(s) = 0$.

Repeatedly choose unexplored node v which minimizes

$$\pi(v) = \min_{e = (u,v) : u \in S} d(u) + \ell_e,$$

add v to S , and set $d(v) = \pi(v)$.

← shortest path to some u in explored part, followed by a single edge (u, v)



Dijkstra's Algorithm

Dijkstra's algorithm.

Maintain a set of **explored nodes** S for which we have determined the shortest path distance $d(u)$ from s to u .

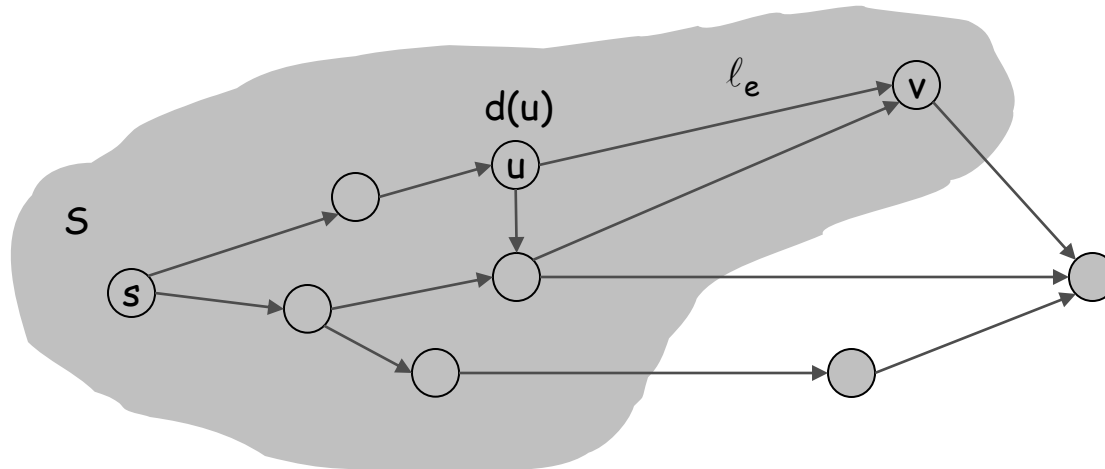
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← shortest path to some u in explored part, followed by a single edge (u, v)



Dijkstra's Algorithm: Proof of Correctness

Invariant. For each node $u \in S$, $d(u)$ is the length of the shortest s - u path.

Pf. (by induction on $|S|$)

Base case: $|S| = 1$ is trivial.

Inductive hypothesis: Assume true for $|S| = k \geq 1$.

Let v be next node added to S , and let u - v be the chosen edge.

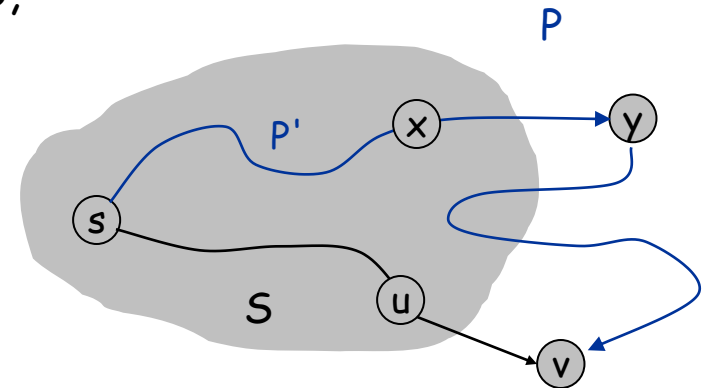
The shortest s - u path plus (u, v) is an s - v path of length $\pi(v)$.

Consider any s - v path P . We'll see that it's no shorter than $\pi(v)$.

Let x - y be the first edge in P that leaves S ,

and let P' be the subpath to x .

P is already too long as soon as it leaves S .



$$\begin{array}{ccccccc} \ell(P) & \geq & \ell(P') + \ell(x, y) & \geq & d(x) + \ell(x, y) & \geq & \pi(y) \geq \pi(v) \\ \uparrow & & \uparrow & & \uparrow & & \uparrow \\ \text{nonnegative} & & \text{inductive} & & \text{defn of } \pi(y) & & \text{Dijkstra chose } v \\ \text{weights} & & \text{hypothesis} & & & & \text{instead of } y \end{array}$$

Dijkstra's Algorithm

Dijkstra's Algorithm(G, l)

- 1: Let S be the set of explored nodes.
- 2: For each $u \in S$, we store a distance $d(u)$.
- 3: Initially $S \leftarrow \{s\}$ and $d(s) \leftarrow 0$.
- 4: **while** $S \neq V$ **do**
- 5: Select a node $v \notin S$ with at least one edge from S for which $d'(v) = \min_{e=(u,v): u \in S} d(u) + l_e$ is as small as possible.
- 6: Add v to S and define $d(v) \leftarrow d'(v)$
- 7: **end while**
- 8: **return** S .

Dijkstra's Algorithm: Implementation

For each unexplored node, explicitly maintain $\pi(v) = \min_{e=(u,v): u \in S} d(u) + \ell_e$.

Next node to explore = node with minimum $\pi(v)$.

When adding v , for each incident edge $e = (v, w)$, update

$$\pi(w) = \min \{ \pi(w), \pi(v) + \ell_e \}.$$

Efficient implementation. Dijkstra's algorithm can find the shortest path in $O(n^2)$ time.

Homework

.Read Chapter 4 of the textbook.

.Exercises 4, 6, 7 & 13 in Chapter 4.