Assignments for Chapter 1

Homework (Week 2)

- **1.20, 1.21 (c) (f), 1.24 (a), 1.26, 1.27 (a) (f), 1.41**

Tutorial Problems (Week 3)

- **Basic Problems with Answers 1.15, 1.18**
- **Basic Problems 1.29, 1.31**
- **Advanced Problems 1.33, 1.42**

Tutorial session

- **Teaching Building 1-111, 9pm-10pm, Monday-Thursday, from Week 3.**

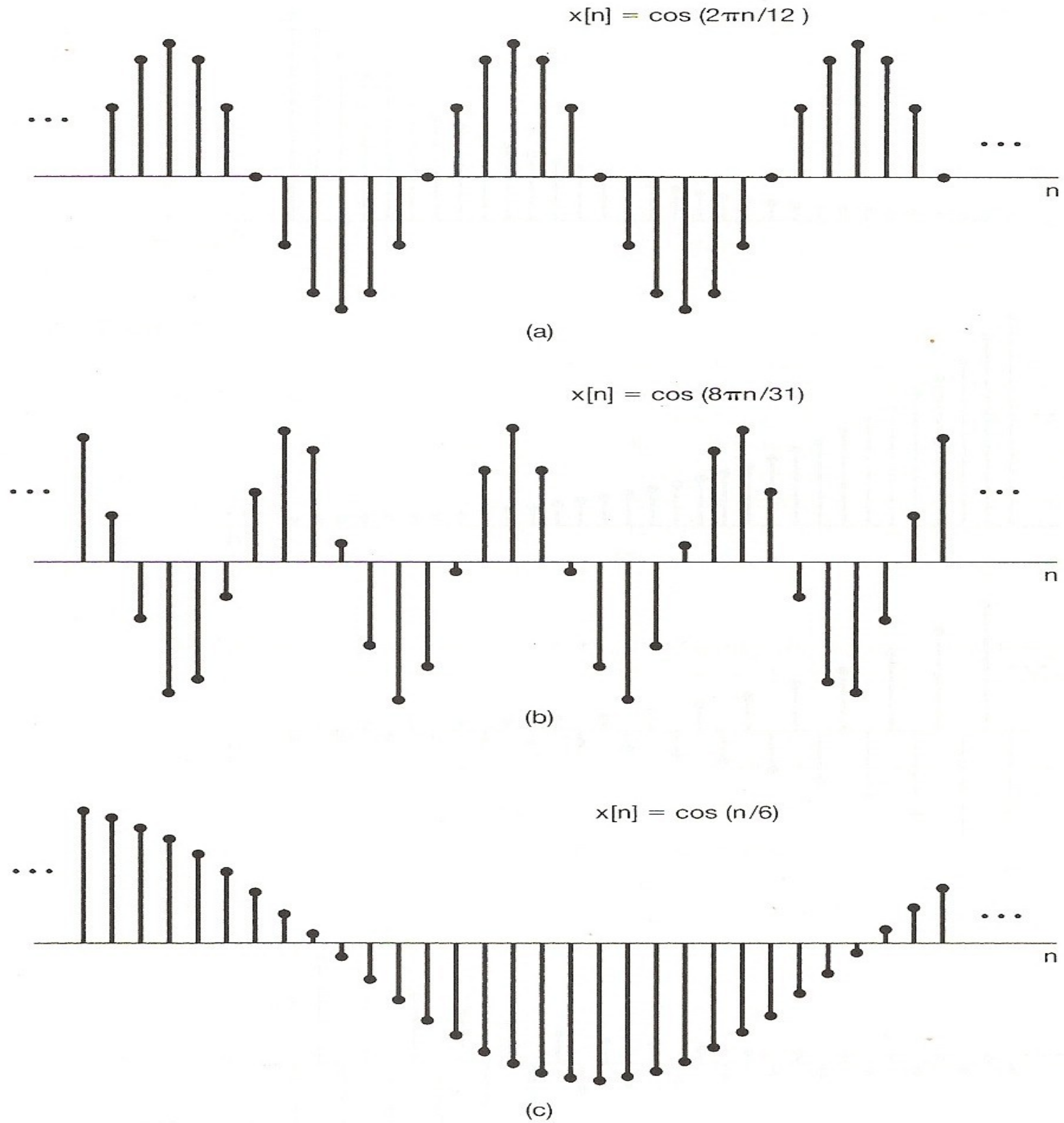
Periodicity of DT Complex Exponentials

Important difference between $e^{j\omega_0 n}$ and $e^{j\omega_0 t}$:

- $e^{j\omega_0 n}$ is a periodic signal only when $\frac{\omega_0}{2\pi}$ is a rational number

$$e^{j\omega_0 n} = e^{j\omega_0 (n+N)} \rightarrow e^{j\omega_0 N} = 1 \rightarrow \omega_0 N = 2\pi m$$

Hence, $$\frac{\omega_0}{2\pi} = \frac{m}{N}$$



$$\frac{\omega_0}{2\pi} = \frac{2\pi/12}{2\pi} = \frac{1}{12}$$

$$\frac{\omega_0}{2\pi} = \frac{8\pi/31}{2\pi} = \frac{4}{31}$$

$$\frac{\omega_0}{2\pi} = \frac{1/6}{2\pi} = \frac{1}{12\pi}$$

Figure 1.25 Discrete-time sinusoidal signals.

How to determine the fundamental period of $e^{j\omega_0 n}$?

Solution:

- Let N be the fundamental period, then

$$e^{j\omega_0(n+N)} = e^{j\omega_0 n} \rightarrow e^{j\omega_0 N} = 1.$$

- $\exists\ integer\ m,\ \ \omega_0 N = 2\pi m.$
- Therefore,

$$N = \frac{2\pi}{\omega_0} m.$$

- Hence, N is the minimum positive integer in the set $\{\frac{2\pi}{\omega_0} m | \forall\ integer\ m\}$.

Example

- **What is the fundamental period of $e^{j\frac{6}{5}\pi n}$?**

$$\left\{\frac{2\pi}{\omega_0}m \middle| \forall\ integer\ m\right\} = \left\{\frac{5}{3}m \middle| \forall\ integer\ m\right\}$$

$$= \{\dots, 0, \frac{5}{3}, \frac{10}{3}, 5, \frac{20}{3}, \dots\}$$

Hence, the fundamental period is 5 and fundamental frequency is $\frac{2\pi}{5}$.

Periodicity Properties of DT Complex Exponentials

Important difference between $e^{j\omega_0 n}$ and $e^{j\omega_0 t}$:

- $e^{j\omega_0 n}$ is periodic w.r.t. ω_0

$$\boxed{e^{j(\omega_0+m\cdot 2\pi)n} = e^{j\omega_0 n} \cdot e^{jm\cdot 2\pi n} = e^{j\omega_0 n}}$$

- However, $e^{j\omega_0 t}$ *is aperiodic w.r.t.* ω_0

$$\boxed{\forall x \neq 0, e^{j(\omega_0+x)t} = e^{j\omega_0 t} e^{jxt} \neq e^{j\omega_0 t}}$$

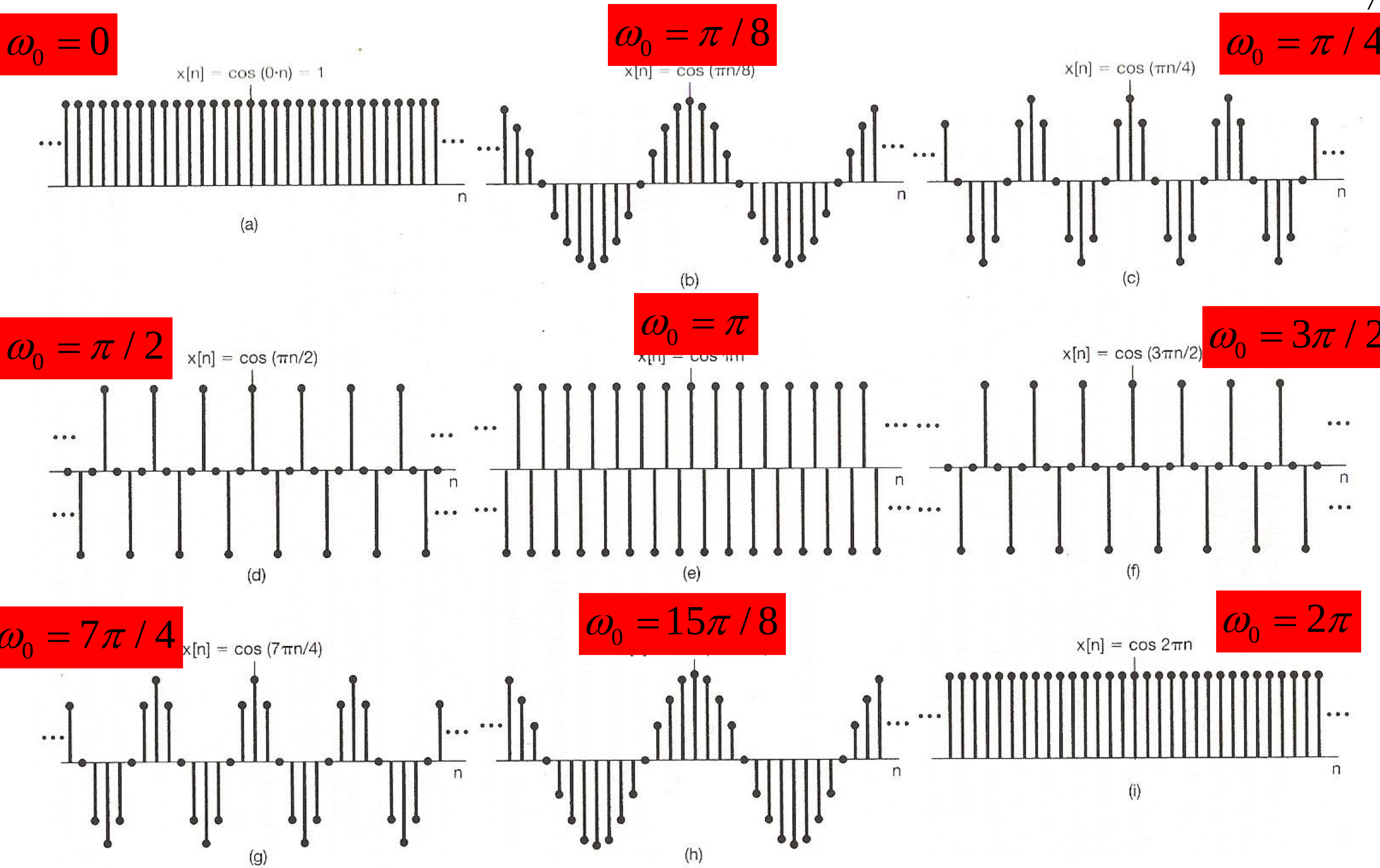


Figure 1.27 Discrete-time sinusoidal sequences for several different frequencies.

- We need only consider an ω interval of length 2π, and on most cases, we use the interval: $0 \le \omega_0 < 2\pi$, or $-\pi \le \omega_0 < \pi$

- $e^{j\omega_0 n}$ does **not** have a continually increasing rate of oscillation as ω_0 is increased.

  lowest-frequency (slowly varying): ω_0 near 0, 2π, …, or 2k • π

  highest-frequency (rapid variation): ω_0 near ± π, …, or (2k+1) • π

$$e^{j(2k+1)\pi n} = e^{j\pi n} = (e^{j\pi})^n = (-1)^n$$

$$e^{j2\pi n} = (e^{j2\pi})^n = (1)^n = 1$$

CT Harmonically Related Sets

- A set of periodic exponentials which have a common period.

$$\{\phi_k(t) = e^{jk\omega_0 t}, k = 0, \pm 1, \pm 2, \ldots.\}$$

Fundamental (Angular) Frequency : $|k\omega_0|$

Fundamental Period: $\dfrac{2\pi}{|k\omega_0|}$

Common Period: $\dfrac{2\pi}{|\omega_0|}$

DT Harmonically Related Set

Harmonically related discrete-time signal sets

$$\{\phi_k[n] = e^{jk(\frac{2\pi}{N})n} \quad , \qquad \mathrm{k} = 0, \pm 1, \pm 2, \ldots\ldots\}$$

all with common period N

There are only N elements in the above set.

Proof:

$$\phi_{k+N}[n] = e^{j(k+N)(\frac{2\pi}{N})n} = e^{jk(\frac{2\pi}{N})n} \cdot e^{j2\pi n} = e^{jk(\frac{2\pi}{N})n} = \phi_k[n]$$

This is different from continuous case. Only N distinct signals in this set.

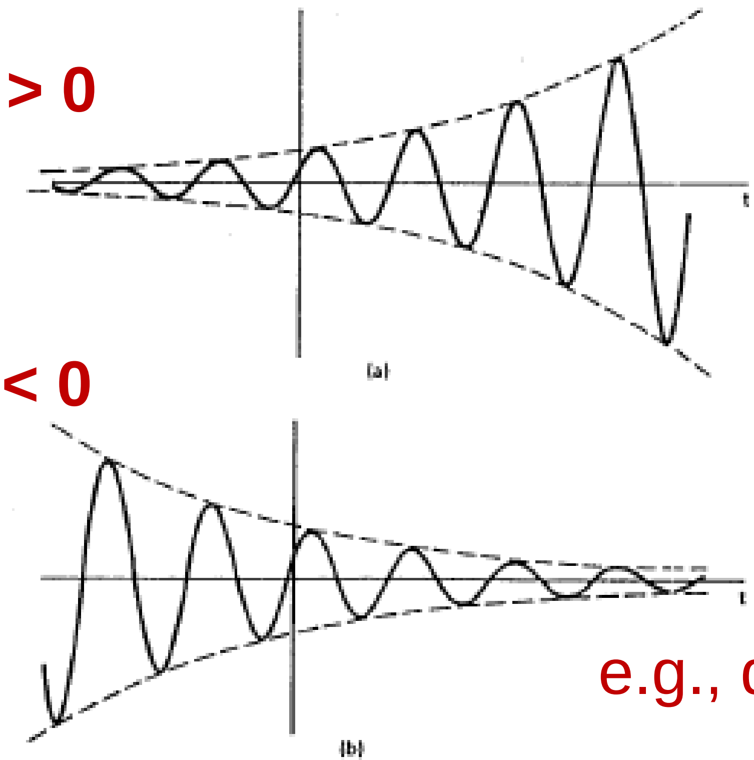
General Complex Exponential Signals - CT

- General format (C and a are complex numbers)

$$x(t) = Ce^{at} = |C|e^{j\theta} \cdot e^{(r+j\omega_0)t} = |C|e^{rt} \cdot e^{j(\omega_0 t+\theta)}$$

$r > 0$

$r < 0$



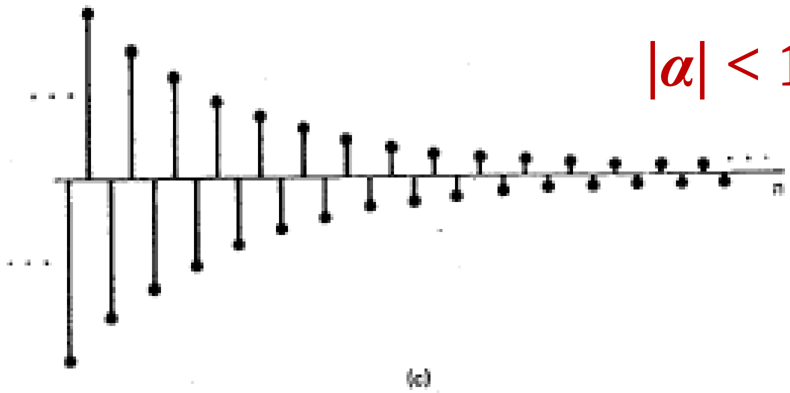
e.g., damped sinusoids

General Complex Exponential Signals - DT

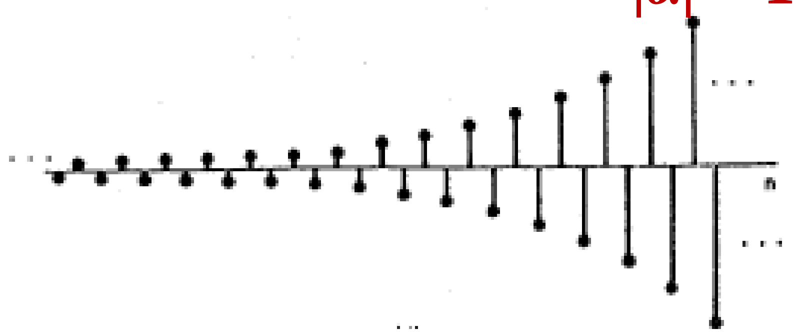
- General format (C and α are complex numbers)

$$x[n] = C\alpha^n = |C|e^{j\vartheta} \cdot |\alpha|^n e^{j\omega_0 n} = |C||\alpha|^n e^{j(\omega_0 n + \vartheta)}$$

$|\alpha| < 1$



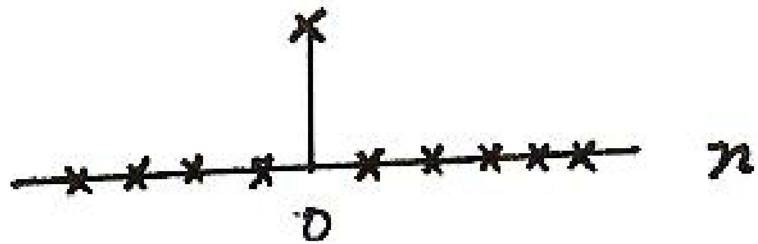
$|\alpha| > 1$



DT Unit Impulse Function

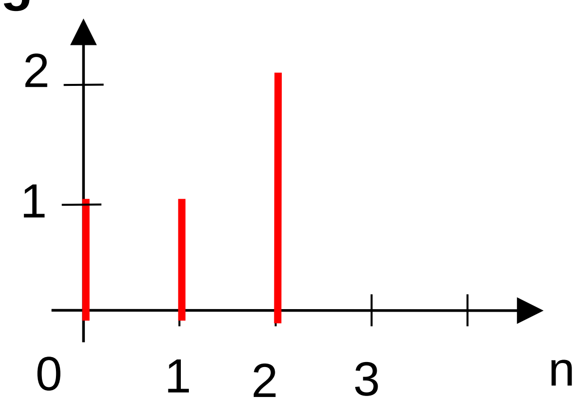
Discrete-time

$$\delta[n] = \begin{cases} 0, & n \neq 0 \\ 1, & n = 0 \end{cases}$$



- As a basic building function, we can use unit impulse function to represent other different signals.

e.g.1



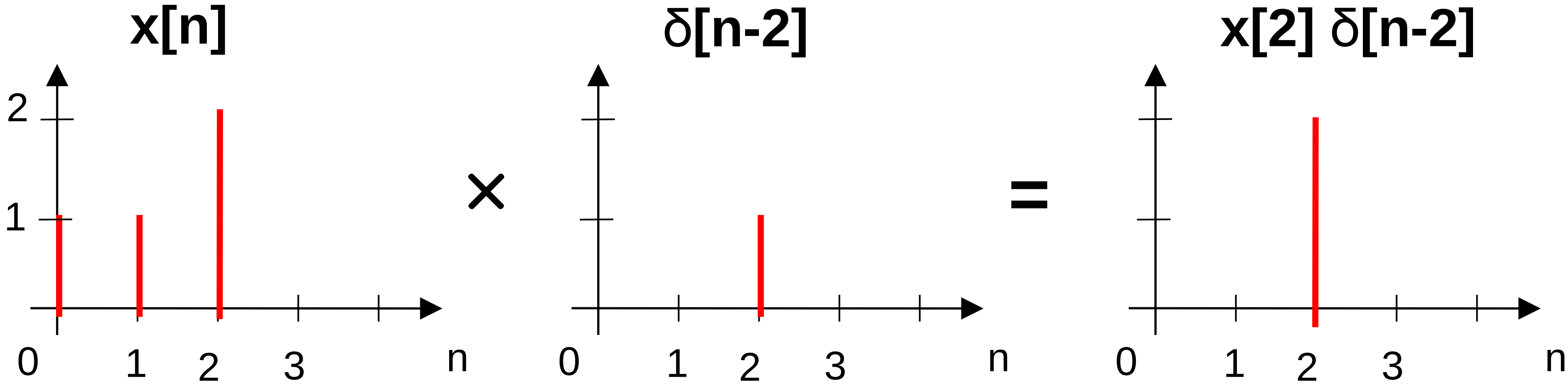
$$=\delta[n] + \delta[n-1] + 2\delta[n-2]$$

DT Unit Impulse Function (cont.)

- Sampling property

$$x[n]\ \delta[n] = x[0]\ \delta[n]$$

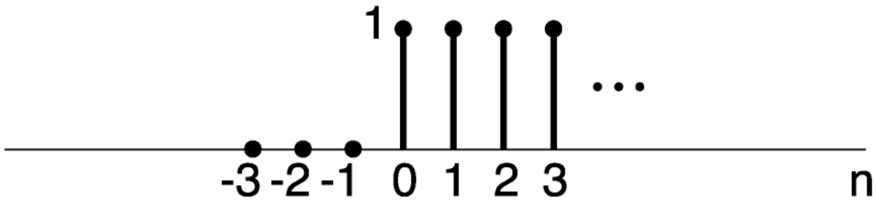
$$x[n]\ \delta[n-n_0] = x[n_0]\ \delta[n-n_0]$$



DT Unit Step Function

Discrete-time

$$u[n] = \begin{cases} 1, & n \geq 0 \\ 0, & n < 0 \end{cases}$$



Relation between unit impulse and unit step functions

– First difference

$$\delta[n] = u[n] - u[n-1]$$

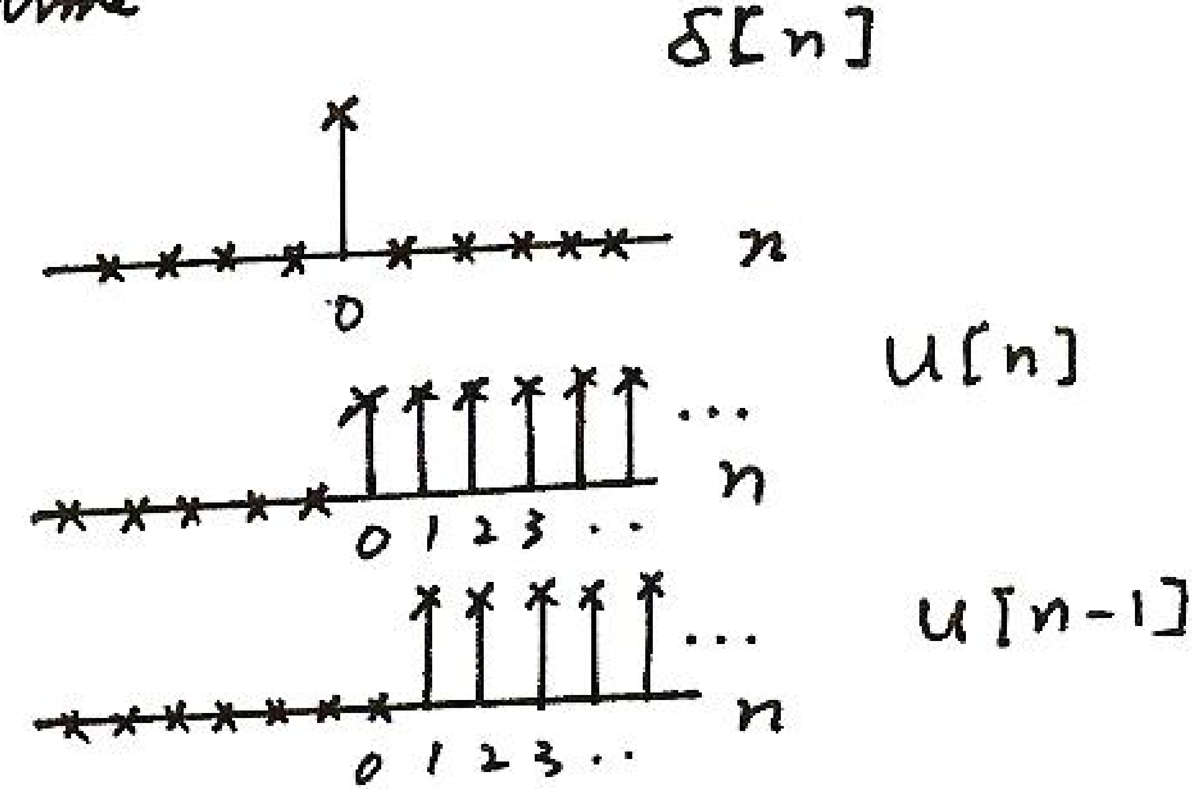
– Running Sum

$$u[n] = \sum_{m=-\infty}^{n} \delta[m] \begin{cases} =0, & n<0 \\ =1, & n \geq 0 \end{cases}$$

$$u[n] = \sum_{k=0}^{\infty} \delta[n-k]$$

DT Unit Step Function: First Difference

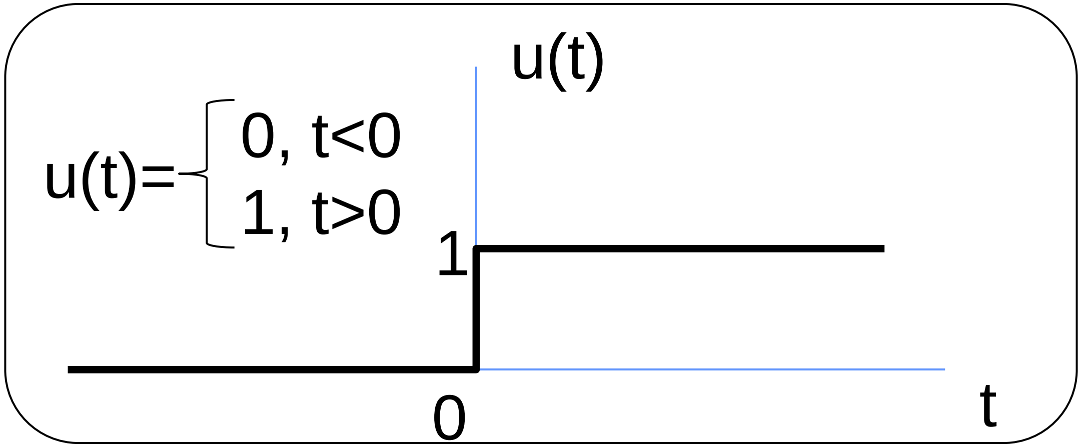
Discrete-Time

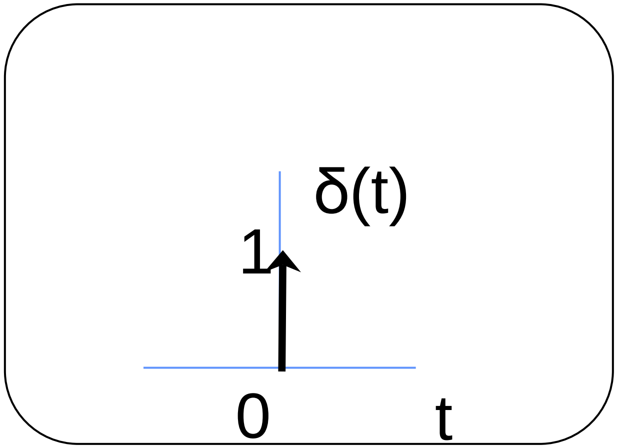


$$\delta[n] = u[n] - u[n-1]$$

CT Unit Impulse and Unit Step Functions

Continuous-time





Relation between unit impulse and unit step functions

– First Derivative

$$\delta(t) = \frac{du(t)}{dt}$$

– Running Integral

$$u(t) = \int_{-\infty}^{t} \delta(\tau)\, d\tau$$

CT Unit Impulse and Unit Step Functions: Asymptotic View

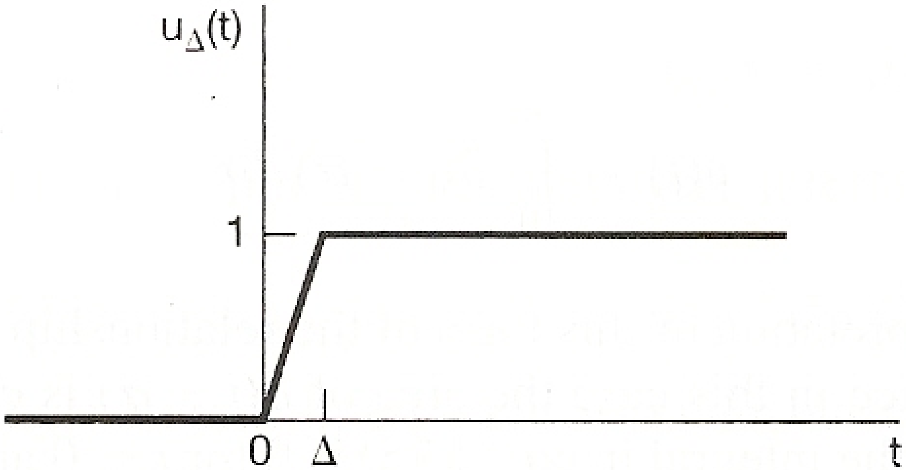


Figure 1.33 Continuous approximation to the unit step, $u_\Delta(t)$.

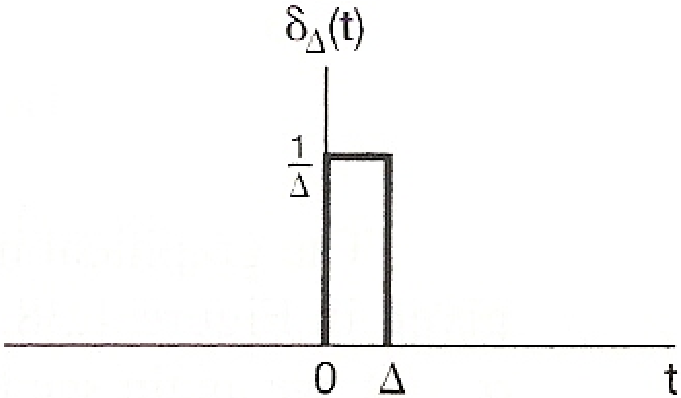


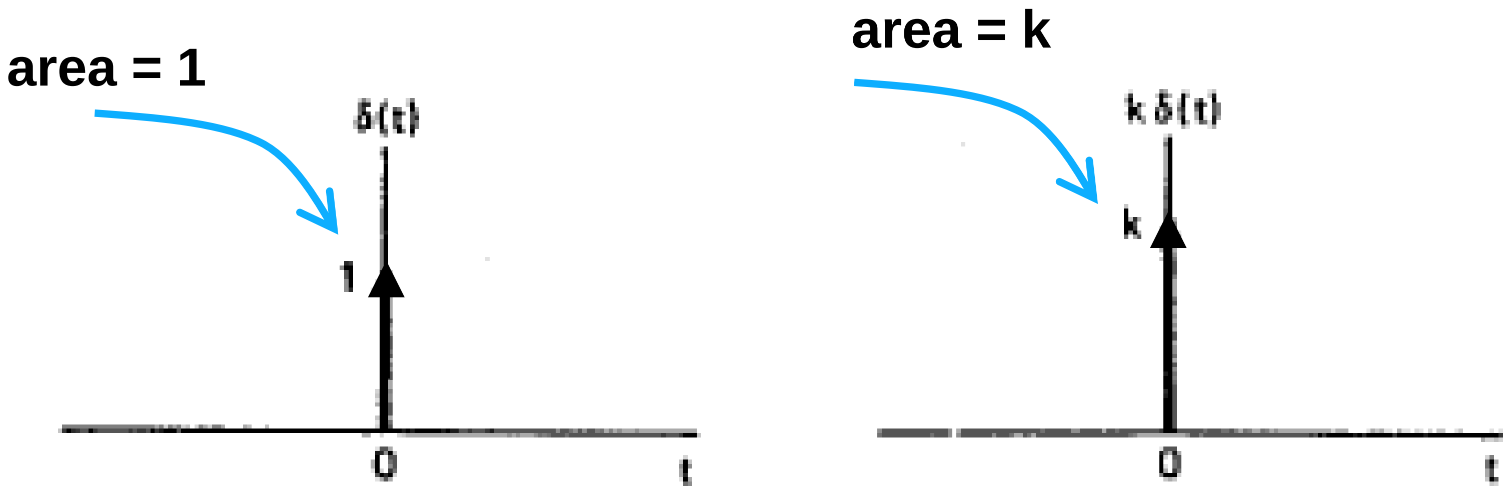
Figure 1.34 Derivative of $u_\Delta(t)$.

$$u(t) = \lim_{\Delta \to 0} u_\Delta(t)$$

$$\delta(t) = \lim_{\Delta \to 0} \delta_\Delta(t)$$

More on CT unit impulse function:

- **δ(t) has in effect no duration, but unit area.**



CT unit impulse

Scaled impulse

- **Or the integration of CT unit impulse function is unit.** $\int_{-\infty}^{\infty} \delta(t)dt = 1$

Sampling Property

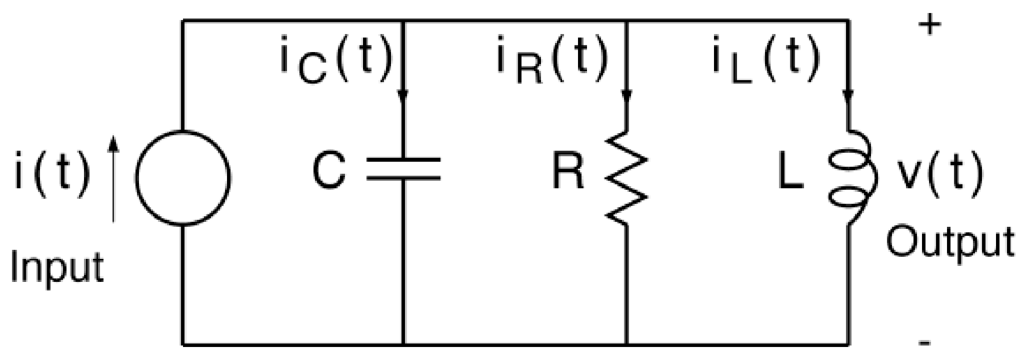
– Sampling property

$$x(t)\,\delta(t-t_0)=x(t_0)\,\delta(t-t_0)$$

System Examples

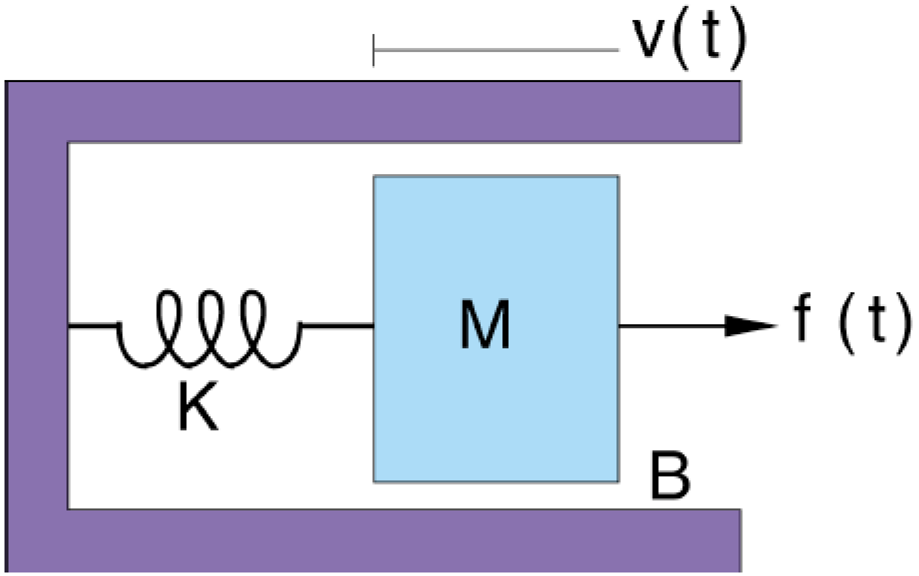
$x(t) \longrightarrow \boxed{\text{CT System}} \longrightarrow y(t)$ $\quad$ $x[n] \longrightarrow \boxed{\text{DT System}} \longrightarrow y[n]$

Ex. #1 RLC circuit — an electrical system



$$i(t) = \underbrace{C\frac{dv(t)}{dt}}_{\text{capacitance}} + \underbrace{\frac{v(t)}{R}}_{\text{resistance}} + \underbrace{\frac{1}{L}\int_{-\infty}^{t} v(\tau)d\tau}_{\text{inductance}} .$$

Ex. #2 A shock absorber – a mechanical system

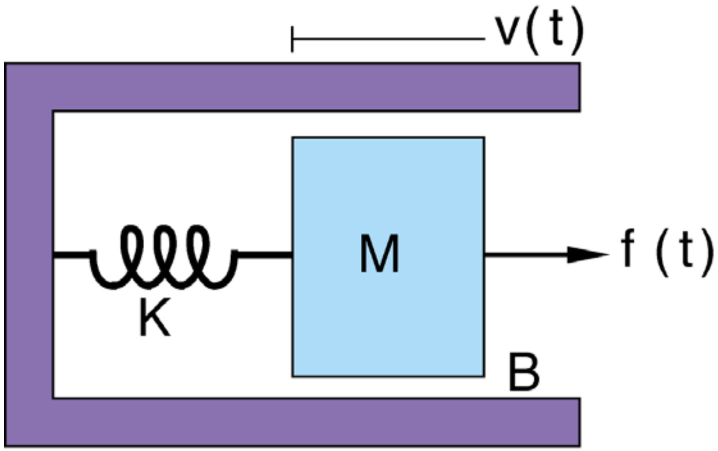


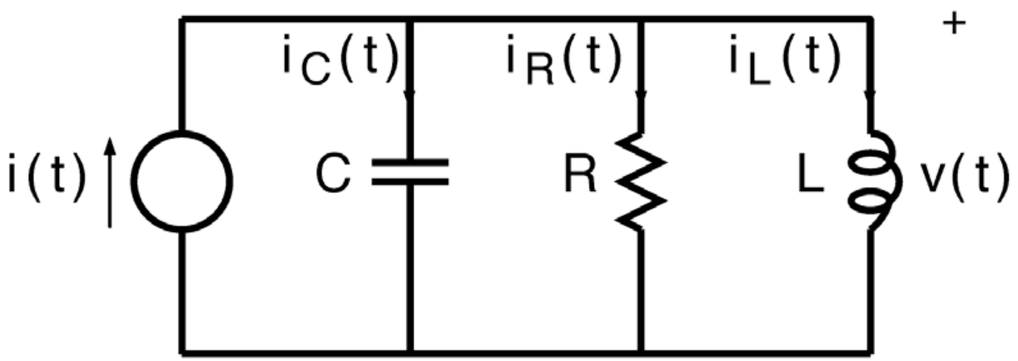
Force Balance:

$$f(t) = \underbrace{M\frac{dv(t)}{dt}}_{inertial\ force} + \underbrace{Bv(t)}_{friction} + \underbrace{K\int_{-\infty}^{t} v(\tau)d\tau}_{spring\ force}.$$

This equation looks quite familiar, we just saw it earlier.

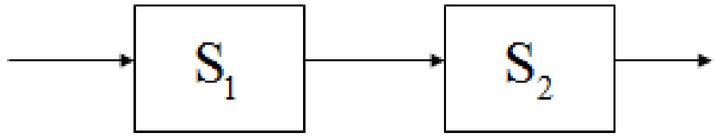
- **Observation: different systems could be described by the same input/output relations**
- **In this course, we focus on the mathematical relation between input and output**



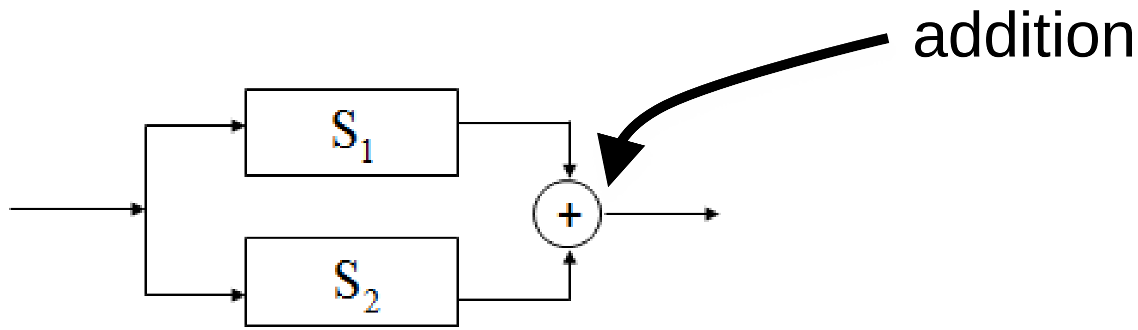


Interconnection of Systems

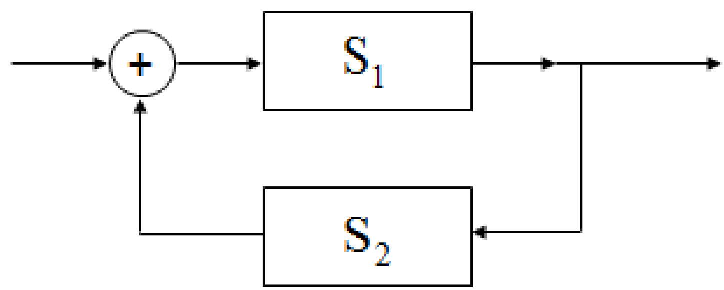
- **Series (cascade)**



- **Parallel**



- **Feedback**



System Properties :
1) Memoryless or With Memory

Memoryless : output at a given time depends only on the input at the same time

eg.

$$y[n] = (ax[n] - x^2[n])^2$$

With Memory

eg.

$$y[n] = \sum_{k=-\infty}^{n} x[k]$$

summer or accumulator

2) Invertability

invertible : distinct inputs lead to distinct outputs, i.e. an inverse system exits

No information loss

eg.

$$y[n] = \sum_{k=-\infty}^{n} x[k]$$

$$z[n] = y[n] - y[n-1] \; = x[n]$$

3) Causality

- <u>**Causality**</u>: A system is causal if the output does **not** anticipate future values of the input, i.e., if the output at any time depends only on values of the input up to that time.
- All real-time physical systems are **causal**, because time only moves forward, and effect occurs after cause. (Imagine if you own a noncausal system whose output depends on tomorrow's stock price.)
    - Do **not** apply to spatially varying signals. (We can move both left and right, up and down.)
    - Do **not** apply to systems processing *recorded (or non-realtime)* signals, e.g. taped sports games vs. live broadcast.

Causal or Non-causal?

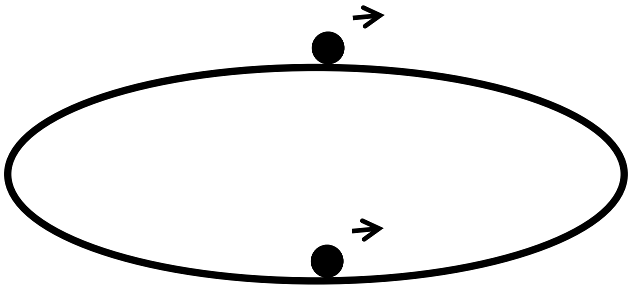
- $y(t) = x^2(t-1)$
- $y(t)=x(t+1)$
- $y(t)=x(t) \cos(t+1)$
- $y[n]=x[-n]$
- $y[n]=(1/2)^{n+1} x^3[n-1]$

4) Stability

- If the input to a stable system is bounded, the output must also be bounded.

- e.g.: S_1: $y(t) = t\,x(t)$

  S_2: $y(t) = e^{x(t)}$



5) Time Invariance (TI)

- DT: A system $x[n] \rightarrow y[n]$ is TI if for *any* input $x[n]$ and *any* time shift n_0

$$x[n - n_0] \rightarrow y[n - n_0]$$

- Similarly for CT time-invariant system

$$x(t - t_0) \rightarrow y(t - t_0)$$

Time-invariant or Time-varying?

- **Steps:**

1) **Calculate $y_1(t) \leftarrow x_1(t)$**
2) **Calculate $y_2(t) \leftarrow x_2(t)=x_1(t-t_0)$**
3) **Does $y_1(t-t_0)$ equal $y_2(t)$?**

e.g.: $y[n] = \left(\frac{1}{2}\right)^{n+1} x^3[n-1]$

① $y_1[n] = \left(\frac{1}{2}\right)^{n+1} x_1^3[n-1]$

② $x_2[n] = x_1[n-n_0]$

$y_2[n] = \left(\frac{1}{2}\right)^{n+1} x_2^3[n-1]$

$= \left(\frac{1}{2}\right)^{n+1} x_1^3[n-n_0-1]$

③ $y_1[n-n_0] = \left(\frac{1}{2}\right)^{n-n_0+1} \cdot x_1^3[n-n_0-1]$

$\therefore$ $y_1[n-n_0] \neq y_2[n]$

$\therefore$ Time-varying

Now we can deduce something:

- If the input to a TI system is periodic, then the output is also periodic with the same period (Problem 1.43 (a)).

Proof: Suppose $x(t+T) = x(t)$

and $x(t) \quad \longrightarrow \quad y(t)$

Then by TI

$$\underset{\substack{\uparrow \\ \text{But these are} \\ \text{the same input!}}}{x(t+T)} \quad \longrightarrow \quad \underset{\substack{\uparrow \\ \text{So these must be} \\ \text{the same output,}}}{y(t+T)}$$

i.e., $y(t) = y(t+T)$

Linearity

Suppose $x_1(t) \rightarrow y_1(t)$ and $x_2(t) \rightarrow y_2(t)$, such system is linear, if

1) **<u>Additivity</u> property:** $x_1(t) + x_2(t) \rightarrow y_1(t) + y_2(t)$

2) **<u>Scaling (or homogeneity)</u> property:**

$$a\, x_1(t) \rightarrow a\, y_1(t)$$

where *a* is a complex number

e.g.: $y(t) = 2\,x(t)$ $\qquad$ $y(t) = x^2(t)$

Linear system or not?

- **Steps**

1) Have $y_1(t)$ and $y_2(t)$ as output signals to $x_1(t)$ and $x_2(t)$

2) Have $y_3(t)$ as output signal to $x_3(t) = a\ x_1(t) + b\ x_2(t)$

3) Does $y_3(t)$ equal "$a\ y_1(t) + b\ y_2(t)$"?

More examples on textbook

Read Example 1.17 ~ 1.20

Linearity (cont.)

- Superposition

If $x_k[n] \xrightarrow{\text{Linear System}} y_k[n]$ k=1,2,3,…

Then $$\boxed{\sum_k a_k x_k[n] \xrightarrow{\text{Linear System}} \sum_k a_k y_k[n]}$$

- This property seems to be almost trivial now, but it is one of the most important ones

Linear Time-invariant (LTI) Systems

- LTI: Linear + Time invariant
- A basic fact: If we know the response of an LTI system to some inputs, we actually know the response to many inputs.

Example: DT LTI System