

Homework: 7.21, 7.23
Tutorial Problems: 7.25, 7.37, 7.40



Chapter 7: Sampling

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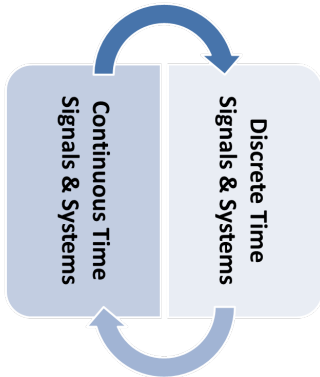
2020 - Fall

Last Update on: Saturday 21st November, 2020



Introduction

Sampling: to facilitate digital processing via computers or chips



Any lossless conversion?

Process CT signals with DT systems?

Interpolation: to present the output of digital processing

Video Recording

- **Signal to be sampled:** real scene (continuous-time signals)
- **Sampling:** record by camera with a rate of 24, 25 or 30 frames per second
- **Sampled signal:** video tapes, mp4 files, avi files and etc. (discrete-time signals)
- **Reconstruction:** watch via eyes and interpret in the brain
- In our consciousness, the real scene can be reconstructed without information loss

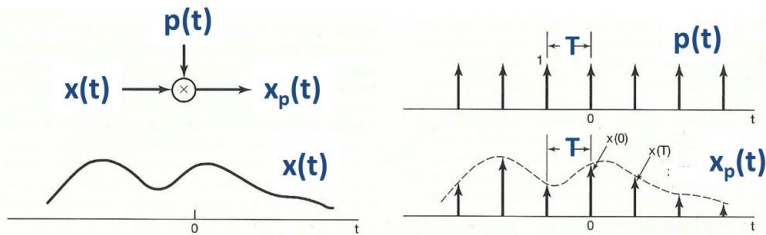


Outline

- Sampling is a general procedure to generate DT signals from CT signals, where information of the original signals can be kept
- Core sampling theory:
 - ▶ Impulse train, zero-order hold, first-order hold and etc
 - ▶ Analysis in frequency domain
 - ▶ Nyquist rate
- Undersampling: Aliasing
- Application: process continuous-time signals discretely
- More sampling techniques: decimation, downsampling and upsampling

Impulse-Train Sampling

- Mathematically, sampling can be represented by multiplication



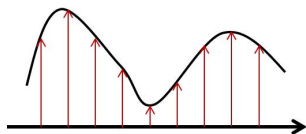
- Sampling function: $p(t) = \sum_{n=-\infty}^{\infty} \delta(t - nT)$
- Sampling period: T
- Sampling:

$$x_p(t) = x(t) \times p(t) = x(t) \times \sum_{n=-\infty}^{\infty} \delta(t - nT) = \sum_{n=-\infty}^{\infty} x(nT) \delta(t - nT)$$

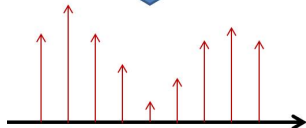
Sampling discards most of points in the original signals.
Is there any information loss in sampling?



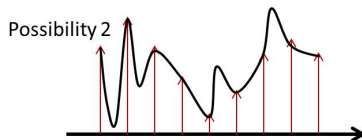
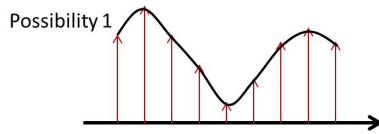
Observation (1/2)



Sampling

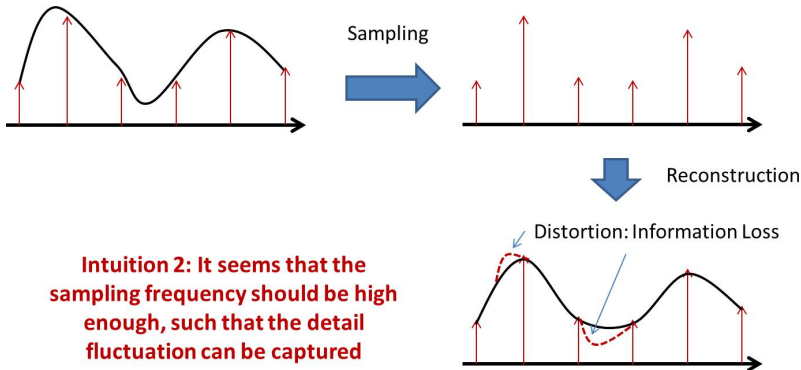


Reconstruction



Intuition 1: It seems that we need a smooth interpolation

Observation (2/2)



- Sampling: the frequency should be high enough
- Reconstruction: the interpolation should be smooth enough

Frequency Analysis (1/2)

- Theoretical tool: continuous-time Fourier transform
- Principle:

$$x(t) \times p(t) \Leftrightarrow \frac{1}{2\pi} X(j\omega) * P(j\omega)$$

- Fourier series of $p(t)$:

$$\begin{aligned} a_k &= \frac{1}{T} \int_{-T/2}^{T/2} \sum_{n=-\infty}^{\infty} \delta(t - nT) e^{-jk\omega_s t} dt \quad \text{where} \quad \omega_s = \frac{2\pi}{T} \\ &= \frac{1}{T} \int_{-T/2}^{T/2} \delta(t) e^{-jk\omega_s t} dt = \frac{1}{T} \end{aligned}$$

- Fourier Transform of $p(t)$:

$$P(j\omega) = 2\pi a_k \sum_{k=-\infty}^{+\infty} \delta(\omega - k\omega_s) = \frac{2\pi}{T} \sum_{k=-\infty}^{+\infty} \delta(\omega - k\omega_s)$$

Navigation icons: back, forward, search, etc.

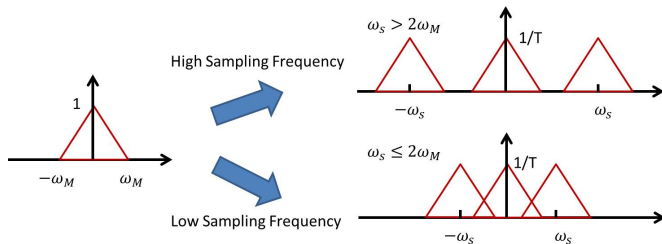


Frequency Analysis (2/2)

- Fourier transform of sampled signal $x_p(t)$:

$$\begin{aligned}X_p(j\omega) &= \frac{1}{2\pi} \int_{-\infty}^{\infty} X(j\theta) P(j(\omega - \theta)) d\theta \\&= \frac{1}{T} \sum_{k=-\infty}^{\infty} X(j(\omega - k\omega_s))\end{aligned}$$

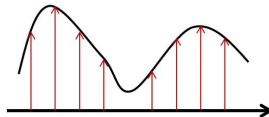
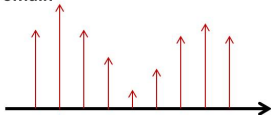
- Sampling: the Fourier transform of input signal is repeated with period ω_s



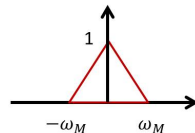
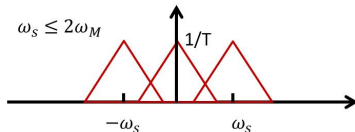
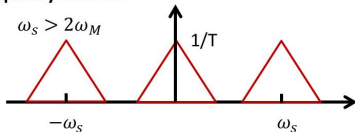
Reconstruction Problem

- Given the sampled signal, can we perfectly reconstruct the signal before sampling?

Time Domain



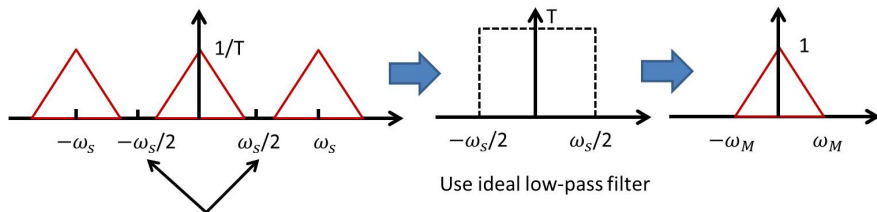
Frequency Domain



It seems that the frequency domain approach makes sense

Reconstruction (1/2)

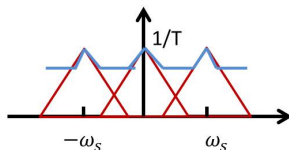
- Scenario of $\omega_s > 2\omega_M$



The spectrum of desired signal is within
 $\left(-\frac{\omega_s}{2}, \frac{\omega_s}{2}\right) \rightarrow$ No overlapping

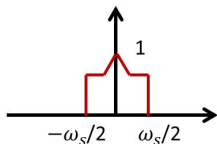
Reconstruction (2/2)

- Scenario of $\omega_s \leq 2\omega_M$

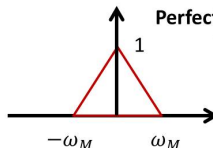


Since $\omega_s \leq 2\omega_M$, we don't know the frequency range of the desired signal

Sampling on the following signals can generate the same result:



OR



Perfect reconstruction is impossible

Observation: the original signal $x(t)$ can be Uniquely and perfectly reconstructed from $x(nT)$ only when $\omega_s > 2\omega_M$

Sampling Theorem

Let $x(t)$ be a band-limited signal with

$$X(j\omega) = 0 \text{ for } |\omega| > \omega_M.$$

Then, $x(t)$ is uniquely determined by its samples $x(nT)$ or $x_p(t)$ if

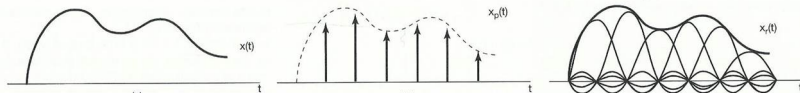
$$\omega_s = \frac{2\pi}{T} > 2\omega_M,$$

where $2\omega_M$ is referred to as the *Nyquist rate*.

- Questions:
 - ▶ How about $\omega_s = 2\omega_M$?
 - ▶ Sampling on band-pass signals

Signal Reconstruction: Interpolation

- If $\omega_s > 2\omega_M$, original signal can be perfectly reconstructed by ideal low-pass filter.
- Time domain interpretation of lowpass filtering



$$\begin{aligned}x_r(t) &= x_p(t) * h(t) = \sum_{n=-\infty}^{+\infty} x(nT)h(t - nT) \\&= \sum_{n=-\infty}^{+\infty} x(nT) \frac{\sin \frac{\omega_s}{2}(t - nT)}{\frac{\omega_s}{2}(t - nT)} = \sum_{n=-\infty}^{+\infty} x(nT) \text{sinc}\left(\frac{t - nT}{T}\right)\end{aligned}$$

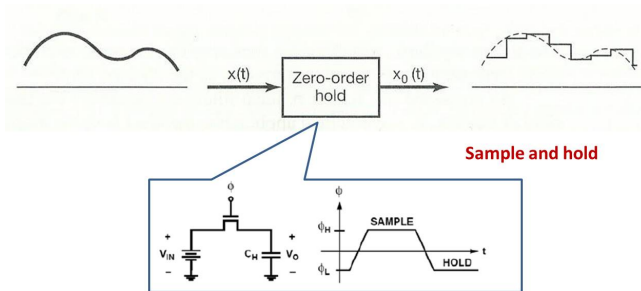
- Ideal lowpass filtering: interpolation with sinc function

Zero-Order Hold

- It's difficult to generate ideal impulse chain in practical implementation.

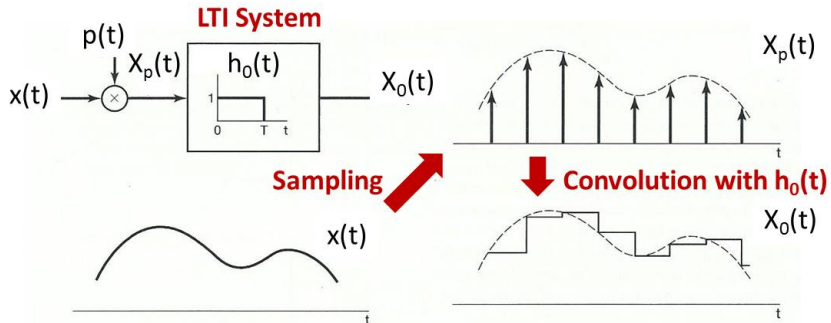
$$x_p(t) = \sum_{n=-\infty}^{\infty} x(nT)\delta(t - nT)$$

- Alternative approach: zero-order hold



- How to interpret the system of "zero-order hold" mathematically?

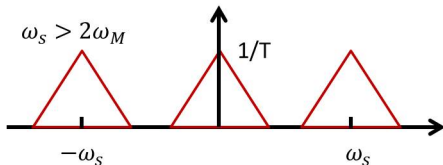
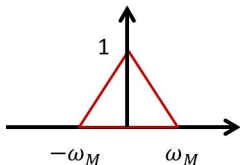
Interpretation of Zero-Order Hold



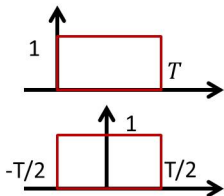
- Zero-order hold: sampling + interpolation with rectangular impulse response
- An approximation of the signal to be sampled.

Frequency Analysis (1/2)

- Step 1: Impulse-train sampling



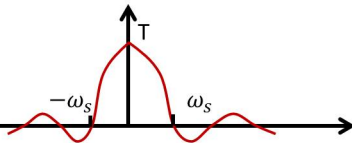
- Step 2: Frequency response of $h_0(t)$



Fourier Transform



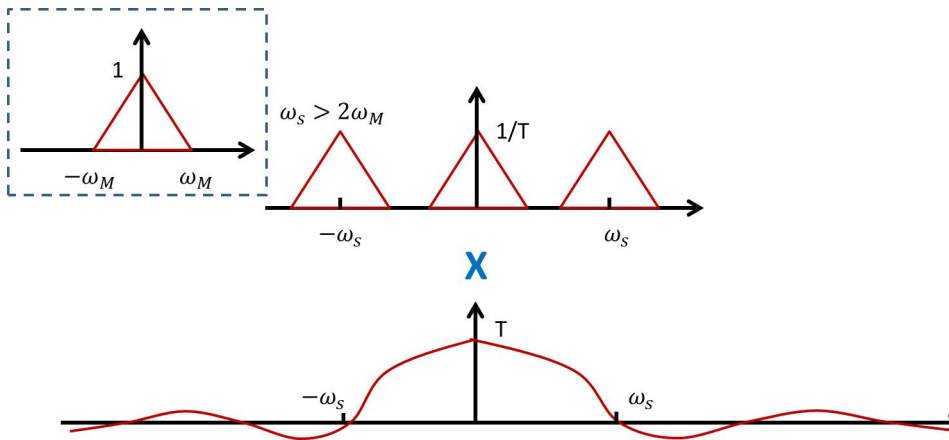
Same Amplitude



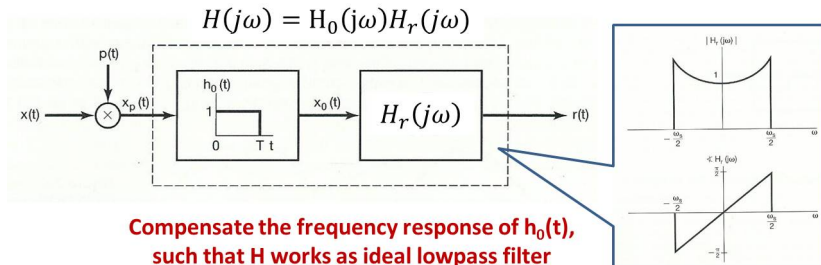
$$H(j\omega) = 2e^{-j\omega T/2} \frac{\sin \omega T/2}{\omega}$$



Frequency Analysis (2/2)



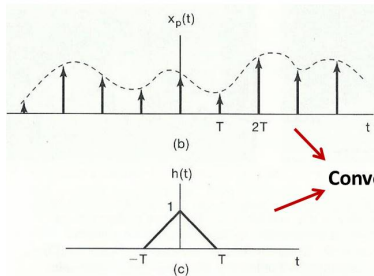
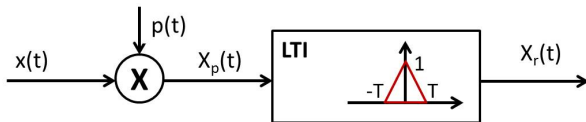
Reconstruction



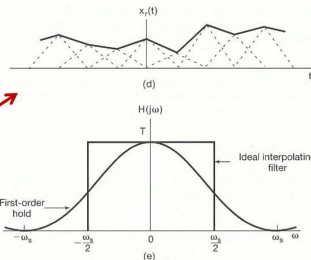
$$H_r(j\omega) = e^{j\omega T/2} \frac{\omega T}{2 \sin \omega T/2} \quad -\frac{\omega_s}{2} \leq \omega \leq \frac{\omega_s}{2}$$

- $H(j\omega)$ should be an ideal low-pass filter from $-\omega_s/2$ to ω_s

First-Order Hold

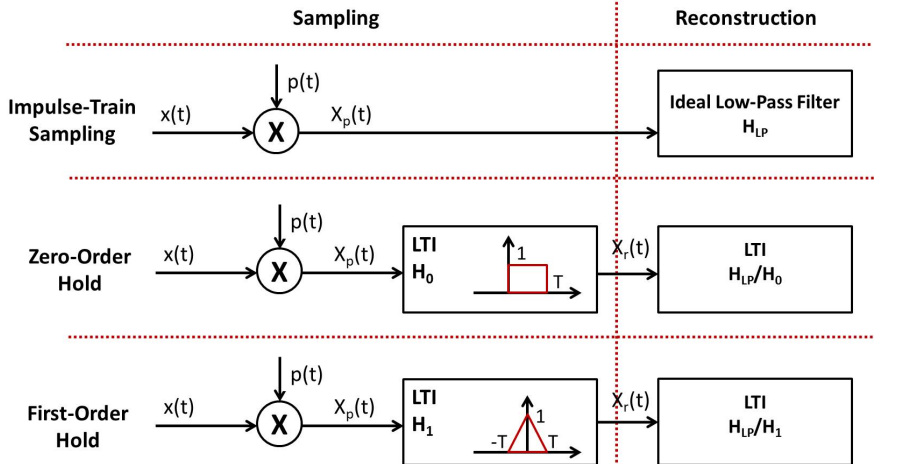


Convolution



- First-order hold: sampling + interpolation with triangular wave
- How to reconstruct?

Summary: Sampling Approaches



Problem 1

Problem (7.5)

Let $x(t)$ be a signal with Nyquist rate ω_0 . Also, let

$$y(t) = x(t)p(t - 1),$$

where

$$p(t) = \sum_{n=-\infty}^{\infty} \delta(t - nT), \quad \text{and} \quad T < \frac{2\pi}{\omega_0}.$$

Specify the constraints on the magnitude and phase of the frequency response of a filter that gives $x(t)$ as its output when $y(t)$ is the input.

Problem 2

Problem (7.7)

A signal $x(t)$ undergoes a zero-order hold operation with an effective sampling period T to produce a signal $x_0(t)$. Let $x_1(t)$ denote the result of a first-order hold operation on $x(t)$. Specify the frequency response of a filter that produces $x_1(t)$ as its output when $x_0(t)$ is the input.

Problem 3

Problem (7.36)

Let $x(t)$ be a band-limited signal such that $X(j\omega) = 0$ for $|\omega| \geq \pi/T$.

(a) If $x(t)$ is sampled using a sampling period T , determine an interpolating function $g(t)$ such that

$$\frac{dx(t)}{dt} = \sum_{n=-\infty}^{\infty} x(nT)g(t - nT).$$

(b) Is the function $g(t)$ unique?