南方科技大学
SOUTHERN UNIVERSITY OF SCIENCE AND TECHNOLOGY

考试科目：概率论与数理统计　　　　开课单位：数学系

考试时长：2 小时　　　　命题教师：杨丽丽

<table>
<tr><td rowspan="2">题号</td><td rowspan="2">Part 1</td><td rowspan="2">Part 2</td><td colspan="6">Part 3</td></tr>
<tr><td>1</td><td>2</td><td>3</td><td>4</td><td>5</td><td>6</td></tr>
<tr><td>分值</td><td></td><td></td><td></td><td></td><td></td><td></td><td></td><td></td></tr>
</table>

本试卷共三大部分，满分 100 分（考试结束后请将试卷、答题本、草稿纸一起交给监考老师）

第一部分　选择题（每题 4 分，总共 20 分）

Part One　Select one from the given four options (4 marks each question, in total 20 marks)

1. 设 A、B 为不相容事件，且 $P(A)>0$，$P(B)>0$，下面四个结论中正确的是（ ）:

   (A) $P(B\mid A)>0$

   (B) $P(A\mid B)=P(A)$

   (C) $P(A\mid B)=0$

   (D) $P(AB)=P(A)P(B)$

   Assume A and B are disjoint events, $P(A)>0$，$P(B)>0$, which conclusion is correct?

   (A) $P(B\mid A)>0$

   (B) $P(A\mid B)=P(A)$

   (C) $P(A\mid B)=0$

   (D) $P(AB)=P(A)P(B)$

2. 设 $F(x,y)$ 是二维随机变量 (X,Y) 的分布函数，下面四个结论中错误的是（ ）:

   (A) $F(+\infty,+\infty)=1$

   (B) $F(-\infty,-\infty)=0$

   (C) $F(+\infty,y)=1$

   (D) $F(x,-\infty)=0$

   Assume $F(x,y)$ is the distribution function of two dimensional r.v. (X,Y), which conclusion is wrong?

   (A) $F(+\infty,+\infty)=1$

   (B) $F(-\infty,-\infty)=0$

   (C) $F(+\infty,y)=1$

   (D) $F(x,-\infty)=0$

3. 一种零件的加工由两道工序组成. 第一道工序的废品率为 p_1，第二道工序的废品率为 p_2，则该零件加工的成品率为（　）.

第二部分 填空题（每空2分，总共20分）

Part Two　Fill in the boxes for each Question (2 marks each box, in total 20 marks)

1. 已知事件A、B仅发生一个的概率为 0.3（即$P(A\bar{B})+P(\bar{A}B)=0.3$），且$P(A)+P(B)=0.5$, 则$A$、$B$至少有一个不发生的概率为_________.
   If the probability for exactly one of the events A or B to happen is 0.3 (i.e., $P(A\bar{B})+P(\bar{A}B)=0.3$), and $P(A)+P(B)=0.5$, then the probability for at least one of A and B not to happen is _________.

2. 设A,B,C是两两独立且三事件不能同时发生的随机事件，且它们发生的概率相等，则$P(A\cup B\cup C)$的最大值为_________.
   Suppose A,B,C are pairwise independent events while the three events cannot happen at the same time. If each of the events happens with the same probability, then the maximum of $P(A\cup B\cup C)$ is _________.

3. 将一枚硬币重复掷五次，则正、反面都至少出现二次的概率为_________.
   A coin is tossed five times, then the probability of getting at least two heads and at least two tails is _________.

4. 假设随机变量X与Y相互独立且都服从参数为λ的指数分布$EXP(\lambda)$，即$f_X(x)=\begin{cases}\lambda e^{-\lambda x}, x>0\\ 0,\ x\le 0\end{cases}$，则$\min(X,Y)$服从的分布为_________.（写出分布类型及参数）
   Suppose the random variables X and Y are independent and they both follow the exponential distribution $EXP(\lambda)$, $f_X(x)=\begin{cases}\lambda e^{-\lambda x}, x>0\\ 0,\ x\le 0\end{cases}$, then the distribution of $\min(X,Y)$ is_________. (Please write down the distribution and its parameter.)

5. 设随机变量(X,Y)的联合频率函数为

X \ Y	0	1
0	1/4	1/4
1	0	1/2

   记(X,Y)的分布函数为$F(x,y)$，则$F\left(\frac{1}{2},1\right)=$_________.

   The joint probability mass function (PMF) of the random variable (X,Y) is listed as follows:

X \ Y	0	1
0	1/4	1/4
1	0	1/2

   Let $F(x,y)$ be the joint CDF of (X,Y), then $F\left(\frac{1}{2},1\right)=$_________.

第三部分 大题（每题 10 分，总共 60 分）

Part Three Questions and Answers (10 marks each question, in total 60 marks)

1. 某射击小组有 20 名射手，其中一级射手 4 人，二级 8 人，三级 7 人，四级 1 人。各级射手能通过选拔进入比赛的概率依次为 0.9、0.7、0.5、0.2。求任选一名射手能通过选拔进入比赛的概率。

   A shooting team has 20 shooters, of whom 4 are in the first level, 8 are in the second level, 7 are in the third level, and 1 is in the fourth level. The probability of each level of the shooters entering the competition through selection is 0.9, 0.7, 0.5, 0.2. Compute the probability that a randomly selected shooter could enter the competition.

2. 设猎人在猎物 100m 处对猎物打第一枪，命中猎物的概率为 0.5。若第一枪未命中，则猎人继续打第二枪，此时猎物与猎人已相距 150m。若第二枪仍未命中，则猎人继续打第三枪，此时猎物与猎人已相距 200m。若第三枪还未命中，则猎物逃逸。假如该猎人命中猎物的概率与距离成反比 $P(X=x)=\frac{k}{x}$（x是距离，k是待求的常数)，试求该猎物被击中的概率。

   A hunter shoots at the first time in 100m from the prey, and the probability of hitting the prey is 0.5. If the first shooting misses, the hunter continues to shoot at the second time, and now the prey is in 150m away from the hunter. If the second shooting still misses, the hunter continues to shoot at the third time, and right now the prey is in 200m away from the hunter. If the third shooting has not hit, the prey escapes. If the probability of the hunter hitting the prey is inversely proportional to the distance $P(X=x)=\frac{k}{x}$ (x is the distance and k is a constant), find the probability of the prey having been hit.

3. 设随机变量X的密度函数满足：

$$f(x)=\begin{cases} kx(1-x), & 0<x<1, \\ 0, & \text{其他} \end{cases}$$

   (1) 求常数k;

   (2) 求$Y=-3X+3$的取值范围和密度函数.

   Suppose a random variable X has the density function:

$$f(x)=\begin{cases} kx(1-x), & 0<x<1, \\ 0, & other \end{cases}$$

   (1) Compute the constant k;

   (2) Find the value range and the density function of $Y=-3X+3$.

Answers:

Part One

CCCAB

Part Two

1. 0.9
2. $3/4$
3. $5/8$
4. $EXP(2\lambda)$
5. $1/2$
6. -1
7. $1-1/e$
8. $1/2$
9. $b(m+n,p)$
10. 3/4

Part Three

1. 某射击小组有 20 名射手, 其中一级射手 4 人, 二级 8 人, 三级 7 人, 四级 1 人, 各级射手能通过选拔进入比赛的概率依次为 0.9, 0.7, 0.5, 0.2. 求任选一名射手能通过选拔进入比赛的概率.

解 记事件 B = {所选射手能进入比赛}, A_i = {所选射手为第 i 级}, $i=1,2,3,4$. 已知

$$P(A_1)=\frac{4}{20},\quad P(A_2)=\frac{8}{20},\quad P(A_3)=\frac{7}{20},\quad P(A_4)=\frac{1}{20},$$

(2)

$$P(B|A_1)=0.9,\ P(B|A_2)=0.7,\quad P(B|A_3)=0.5,\quad P(B|A_4)=0.2.$$

(4)

用全概率公式, 则所求概率为

$$P(B)=\sum_{i=1}^{4}P(A_i)\cdot P(B|A_i)=\frac{4}{20}\times 0.9+\frac{8}{20}\times 0.7+\frac{7}{20}\times 0.5+\frac{1}{20}\times 0.2=0.645\ .$$

(4)

2 设猎人在猎物 100m 处对猎物打第一枪，命中猎物的概率为 0.5。若第一枪未命中，则猎人继续打第二枪，此时猎物与猎人已相距 150m。若第二枪仍未命中，则猎人继续打第三枪，此时猎物与猎人已相距 200m。若第三枪还未命中，则猎物逃逸。假如该猎人命中猎物的概率与距离成反比 $P(X=x)=\frac{k}{x}$ (x 是距离，k是待求的常数)，试求该猎物被击中的概率。

解　记 X 为猎人与猎物的距离，因为该猎人命中猎物的概率与距离成反比，所以有 $P(X=x)=k/x$. 又因为在 100 m 处命中猎物的概率为 0.5，所以 $0.5=P(X=100)=k/100$，从中解得 $k=50$. 若以事件 A,B,C 依次记“猎人在 100 m、150 m、200 m 处击中猎物”，则 $P(A)=1/2, P(B)=1/3, P(C)=1/4$. 因为各次射击是独立的，所以

(2)

(3)

$$\underbrace{P(\text{命中猎物})=P(A)+P(\bar{A}B)+P(\bar{A}\,\bar{B}C)}_{(3)}=\underbrace{P(A)+P(\bar{A})P(B)+P(\bar{A})P(\bar{B})P(C)}_{(1)}$$

$$=\underbrace{\frac{1}{2}+\frac{1}{2}\times\frac{1}{3}+\frac{1}{2}\times\frac{2}{3}\times\frac{1}{4}=\frac{3}{4}}_{(1)}.$$

3 设随机变量 X 的密度函数满足：

$$f(x)=\begin{cases}kx(1-x), & 0<x<1,\\ 0, & 其他\end{cases}$$

（1）求常数 k; (3)

（2）求 $Y=-3X+3$ 的取值范围和密度函数；(7)

解

(1) **Solution:** A p.d.f must integrate to 1, so

$$\begin{aligned}\underline{1=\int_{-\infty}^{+\infty}f_X(x)\mathrm{d}x}_{(1)} &= \int_{-\infty}^{0}f_X(x)\mathrm{d}x+\int_0^1 f_X(x)\mathrm{d}x+\int_1^{+\infty}f_X(x)\mathrm{d}x\\ &=\int_{-\infty}^{0}0\mathrm{d}x+\int_0^1 kx(1-x)\mathrm{d}x+\int_1^{+\infty}0\mathrm{d}x\\ &=\int_0^1 kx(1-x)\mathrm{d}x=\int_0^1 k\left(x-x^2\right)\mathrm{d}x\\ &=\left[k\left(\frac{x^2}{2}-\frac{x^3}{3}\right)\right]_0^1=\frac{k}{6}. \quad (1)\end{aligned}$$

Hence, we get $k=6$. (1)

(2) If $Y=-3X+3$. Then,

$$\begin{aligned}F_Y(y)&=\mathrm{P}\{\mathrm{Y}\le \mathrm{y}\}=\mathrm{P}\{-3\mathrm{X}+3\le \mathrm{y}\}=\mathrm{P}\{-3\mathrm{X}\le \mathrm{y}-3\}\\ &=\mathrm{P}\{\mathrm{X}\ge\frac{\mathrm{y}-3}{-3}\}\\ &=1-\mathrm{P}\{\mathrm{X}<\frac{\mathrm{y}-3}{-3}\}=1-\mathrm{P}\{\mathrm{X}\le\frac{\mathrm{y}-3}{-3}\}\\ &=1-F_X(\frac{y-3}{-3}). \quad (2)\end{aligned}$$

Hence,

$$\begin{aligned}\underline{f_Y(y)=\frac{\mathrm{d}F_Y(y)}{\mathrm{d}y}} \quad (1)\\ &=\frac{\mathrm{d}}{\mathrm{d}y}\left(1-F_X(\frac{y-3}{-3})\right)\\ &=-\frac{\mathrm{d}}{\mathrm{d}y}F_X(\frac{y-3}{-3})\end{aligned}$$

$$= -\frac{d}{du}F_X(u)\Big|_{u=\frac{y-3}{-3}} \cdot \frac{d}{dy}\left(\frac{y-3}{-3}\right)$$
$$= -f_X(u)\Big|_{u=\frac{y-3}{-3}} \cdot \left(\frac{1}{-3}\right)$$
$$= -f_X\left(\frac{y-3}{-3}\right)\cdot\left(\frac{1}{-3}\right) = \frac{1}{3}f_X\left(\frac{y-3}{-3}\right).$$

~~Now since the continuous random variable X has p.d.f~~

$$\cancel{f_X(x) = \begin{cases} 6x(1-x) & 0 < x < 1, \\ 0 & \text{otherwise.} \end{cases}}$$

Therefore, for $0 < \frac{y-3}{-3} < 1$, i.e. $0 < y < 3$,

$$f_Y(y) = \frac{1}{3}f_X\left(\frac{y-3}{-3}\right) = \frac{1}{3}\times 6\times\left(\frac{y-3}{-3}\right)\left(1-\frac{y-3}{-3}\right) = \frac{2}{9}y(3-y).$$ (2)

otherwise, $f_Y(y) = \frac{1}{3}f_X(\frac{y-3}{-3}) = \frac{1}{3}\times 0 = 0$. i.e.,

$$f_Y(y) = \begin{cases} \frac{2}{9}y(3-y) & 0 < y < 3, \\ 0 & \text{otherwise.} \end{cases}$$ (2)

Hence, the non-zero range of f_Y is $(0, 3)$.

4 假设 $X \sim P(\lambda_1)$， $Y \sim P(\lambda_2)$. 进一步地，假设 X 和 Y 独立。

（1）求 X 和 Y 的联合频率函数； (3)

（2）求条件概率 $P(X=k \mid X+Y=n)$，其中 $n \geq k$. 是非负整数 (6)

(提示： $P\{X+Y=n\}=\sum_{k=0}^{n} P\{X=k,\, Y=n-k\}$)

解

Solution:

(a) Since X and Y are independent, then for any non-negative integers n and m, we have

$$p_{(X,Y)}(m,n) = p_X(m)p_Y(n) = \frac{\lambda_1^m}{m!}e^{-\lambda_1} \cdot \frac{\lambda_2^n}{n!}e^{-\lambda_2} = \frac{\lambda_1^m \lambda_2^n}{m!n!}e^{-\lambda_1-\lambda_2}$$ ~~(4)~~ (3)

(b)

$$\begin{aligned} P\{X=k|X+Y=n\} &= \frac{P\{X=k,\, X+Y=n\}}{P\{X+Y=n\}} \\ &= \frac{P\{X=k,\, Y=n-k\}}{P\{X+Y=n\}} \\ &= \frac{P\{X=k\}P\{Y=n-k\}}{P\{X+Y=n\}} \end{aligned}$$ (2)

$$\begin{aligned} P\{X+Y=n\} &= \sum_{k=0}^{n} P\{X=k,\, Y=n-k\} \\ &= \sum_{k=0}^{n} P\{X=k\}P\{Y=n-k\} \\ &= \sum_{k=0}^{n} \frac{\lambda_1^k \lambda_2^{n-k}}{k!(n-k)!}e^{-\lambda_1-\lambda_2} \quad (2) \\ &= \frac{e^{-(\lambda_1+\lambda_2)}}{n!}\sum_{k=1}^{n} \binom{n}{k}\lambda_1^k\lambda_2^{n-k} \\ &= e^{-\lambda_1-\lambda_2}\frac{(\lambda_1+\lambda_2)^n}{n!} \text{(by the Binomial Theorem)} \quad (1) \end{aligned}$$

It follows that

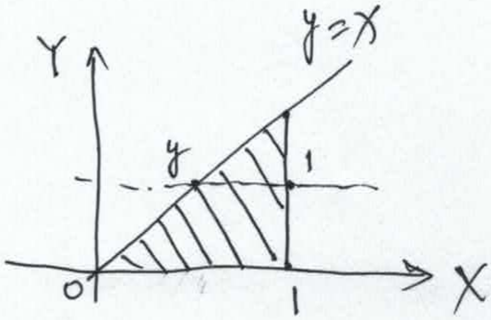
$$\begin{aligned} P\{X=k|X+Y=n\} &= \frac{\frac{\lambda_1^k\lambda_2^{n-k}}{k!(n-k)!}e^{-\lambda_1-\lambda_2}}{e^{-\lambda_1-\lambda_2}\frac{(\lambda_1+\lambda_2)^n}{n!}} \quad (2) \\ &= \frac{n!}{k!(n-k)!}\left(\frac{\lambda_1}{\lambda_1+\lambda_2}\right)^k\left(\frac{\lambda_2}{\lambda_1+\lambda_2}\right)^{n-k}. \end{aligned}$$

5 设二维随机变量(X,Y)的联合密度函数为

$$f(x,y)=\begin{cases}2(x+y), & 0<x<1,0<y<x\\ 0, & \text{其他}\end{cases},$$

(1) 求边缘分布函数$F_Y(y)$；(6)

(2) 判断X与Y的独立性，并给出理由。(4)



解

(1) 根据定义

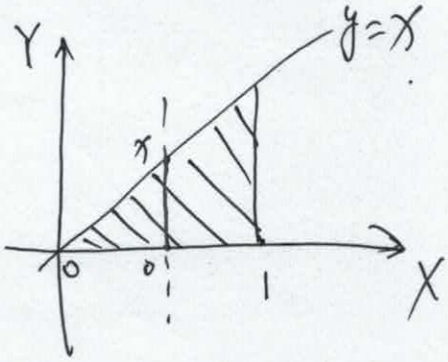
$$f_Y(y)=\int_{-\infty}^{\infty}f(x,y)dx=\begin{cases}\displaystyle\int_y^1 2(x+y)dx=1+2y-3y^2, & 0<y<1,\\ 0, & \text{其他.}\end{cases}$$ (3)

于是

$$F_Y(y)=\int_{-\infty}^{y}f_Y(u)du=\begin{cases}0, & y<0\\ y+y^2-y^3, & 0\leqslant y<1,\\ 1, & y\geqslant 1.\end{cases}$$ (2)

(2) 可求得

$$f_X(x)=\int_{-\infty}^{\infty}f(x,y)dy=\begin{cases}\displaystyle\int_0^x 2(x+y)dy=3x^2, & 0<x<1,\\ 0, & \text{其他.}\end{cases}$$ (3)



由于存在面积不为0的区域，于是

$$f(x,y)\neq f_X(x)f_Y(y),$$

故X与Y不独立。(2)

6 设随机变量 $X \sim U(0,1)$，当给定 $X = x$ 时，随机变量 Y 的条件密度函数为

$$f_{Y|X}(y|x)=\begin{cases} x, & 0<y<\dfrac{1}{x}, \\ 0, & \text{其他}. \end{cases}$$

(1) 求 X 和 Y 的联合密度函数 $f(x,y)$；③

(2) 求边缘密度函数 $f_Y(y)$；④

(3) 求 $P\{X \le Y\}$ 的值. ③

解

(1) 因 $f(x,y) = \underset{(1)}{f_{Y|X}(y|x)} f_X(x)$，其中

$$f_X(x)=\begin{cases} 1, & 0<x<1, \\ 0, & \text{其他}, \end{cases} \quad (1)$$

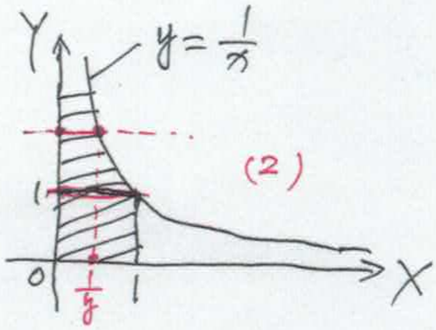
故有

$$f(x,y)=\begin{cases} x, & 0<y<\frac{1}{x}, 0<x<1, \\ 0, & \text{其他}. \end{cases} \quad (1)$$

可知 $f(x,y)$ 仅在区域 $D: \{(x,y) | 0<x<1, 0<y<\frac{1}{x}\}$ 上不等于零.

(2) 由定义有

$$f_Y(y)=\int_{-\infty}^{\infty} f(x,y)dx=\begin{cases} \int_0^1 x dx=\frac{1}{2}, & 0<y<1, \\ \int_0^{1/y} x dx=\frac{1}{2y^2}, & 1\leqslant y<\infty, \\ 0, & \text{其他}. \end{cases} \quad (2)$$



即

$$f_Y(y)=\int_{-\infty}^{\infty} f(x,y)dx=\begin{cases} \frac{1}{2}, & 0<y<1, \\ \frac{1}{2y^2}, & 1\leqslant y<\infty, \\ 0, & \text{其他}. \end{cases}$$

(3) 有

$$P\{X \leqslant Y\}=\int_0^1 \int_x^{1/x} x dy dx=\int_0^1 (1-x^2) dx=\frac{2}{3}.$$

(1)