

2

Vector Spaces (向量空间)

2.4

THE FOUR FUNDAMENTAL SUBSPACES (四个基本子空间)

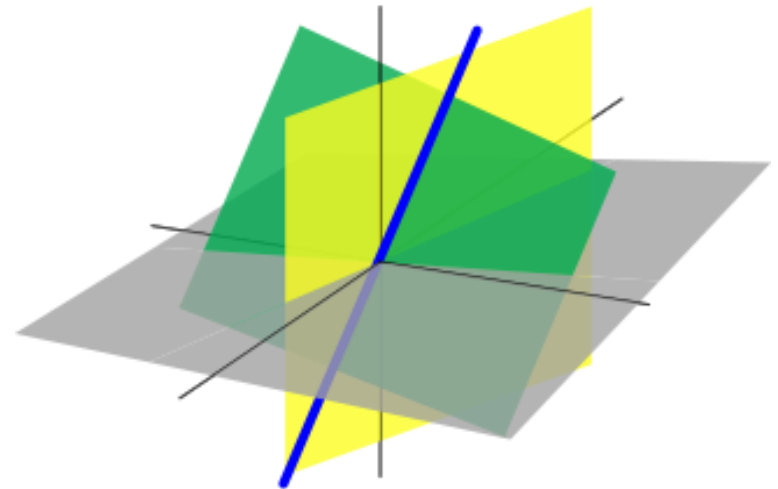
Column space

Row space

Nullspace

Left nullspace

Full Rank and Rank 1





DEFINITION

vector spaces

subspaces

spanned subspaces

spanning set

column spaces

nullspaces

free variables

basic / pivot variables

special solutions

particular solution

complete solution

rank

linearly dependent

linearly independent

basis

dimension

coordinate

The previous sections dealt with *definitions*.



Introduction

Subspaces can be described in *two ways*:

1. We may be given a set of vectors that span the space.
 - *Example*: The columns span the column space $C(A)$.
 - may include useless vectors (dependent columns)
2. We may be told which conditions the vectors in the space must satisfy.
 - *Example*: The nullspace $N(A)$ consists of all vectors that satisfy $A\mathbf{x} = \mathbf{0}$.
 - may include repeated conditions (dependent rows)

We know what a basis is, but *how to find one*?

We need a systematic procedure to compute **an explicit basis (基的显示表达)**.

THE FOUR FUNDAMENTAL SUBSPACES

Let A be an $(m \times n)$ -matrix.

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}$$

There are 4 vector spaces related to A :

$C(A)$: *column space* of A
列空间

$C(A^T)$: *row space* of A
行空间

$N(A)$: *nullspace* of A
零空间

$N(A^T)$: *left nullspace* of A
左零空间

Some equivalent notations and definitions:

$C(A)$ or $\text{Col}(A)$: column space of A

the subspace spanned by the columns of A

$C(A^T)$ or $R(A)$: row space of A

the subspace spanned by the rows of A

It is the column space of A^T .

$N(A)$ or $\text{null}(A)$: nullspace of A

i.e., $\{\mathbf{x}: \mathbf{x} \in \mathbf{R}^n, A\mathbf{x} = \mathbf{0}\}$.

$N(A^T)$ or $\text{Lnull}(A)$: left nullspace of A

It contains all vectors $\mathbf{y} \in \mathbf{R}^m$ such that $A^T \mathbf{y} = \mathbf{0}$, so it is the *nullspace* of A^T , i.e., $\{\mathbf{y}: \mathbf{y} \in \mathbf{R}^m, A^T \mathbf{y} = \mathbf{0}\}$.

It can also be written as $\mathbf{y}^T A = \mathbf{0}$, so it is called the *left nullspace* of A .

The nullspace $N(A)$ and row space $C(A^T)$ are subspaces of \mathbf{R}^n .

The left nullspace $N(A^T)$ and column space $C(A)$ are subspaces of \mathbf{R}^m .

Example 1 For a simple matrix like

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix},$$

The column space $C(A)$ is the line through $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$, and it is in \mathbf{R}^2 .

The row space $C(A^T)$ is the line through $[1 \ 0 \ 0]^T$, and it is in \mathbf{R}^3 .

The nullspace is a plane in \mathbf{R}^3 , $N(A)$ contains $\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$.

The left nullspace is a line in \mathbf{R}^2 , $N(A^T)$ contains $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$.

Remark: Note that all vectors here are expressed as **column vectors**. (Even the rows are transposed, and the row space of A is the *column* space of A^T .)

Example 2 For a non-zero 2×4 matrix

$$\mathbf{A} = \begin{bmatrix} 0 & 1 & 2 & 3 \\ 1 & 2 & 3 & 4 \end{bmatrix},$$

Column space: $C(\mathbf{A}) = \text{Span} \left\{ \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 3 \\ 4 \end{bmatrix} \right\} \subseteq \mathbf{R}^2,$

Row space: $C(\mathbf{A}^T) = \text{Span} \left\{ (0, 1, 2, 3)^T, (1, 2, 3, 4)^T \right\} \subseteq \mathbf{R}^4.$

Obviously, both subspaces are of dimension equal to 2.

To find a basis of $N(\mathbf{A})$, we convert \mathbf{A} into reduced row echelon form

$$\begin{bmatrix} 1 & 0 & -1 & -2 \\ 0 & 1 & 2 & 3 \end{bmatrix}, \quad \text{So } \mathbf{A}\mathbf{x} = \mathbf{0} \text{ has two free variables, say } x_3 = s \text{ and } x_4 = t.$$

Then a general solution is of the form $(s + 2t, -2s - 3t, s, t)^T$, where $s, t \in \mathbf{R}$.

Thus $(1, -2, 1, 0)^T$ and $(2, -3, 0, 1)^T$ form a basis of $N(\mathbf{A})$.

Example 2 For a non-zero 2×4 matrix

$$\mathbf{A} = \begin{bmatrix} 0 & 1 & 2 & 3 \\ 1 & 2 & 3 & 4 \end{bmatrix},$$

(Continued)

To obtain a basis of $\mathbf{N}(\mathbf{A}^T)$, we convert \mathbf{A}^T into row echelon form

$$\mathbf{A}^T = \begin{bmatrix} 0 & 1 \\ 1 & 2 \\ 2 & 3 \\ 3 & 4 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 2 \\ 0 & 1 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$$

It follows that the left nullspace $\mathbf{N}(\mathbf{A}^T)$ only contains $\mathbf{0}$ (zero vector).

I. Row Spaces

Let A be a matrix, and U the row echelon form of A . Then

the row spaces of A and U are the same, i.e., $R(A) = R(U)$.

This is because each row of U is a linear combination of the rows of A .

On the other hand, since each elementary row operation is reversible, each row of A can be written as a linear combination of the rows of U .

Thus the rows of A and U span the same vector space.

This leads to a method for finding a basis of the row space of a matrix A , i.e., converting A into row echelon form U so that

the non-zero rows of U form a basis of $R(A)$, and the row space has dimension r (rank of A and U).

For example,

$$A = \begin{bmatrix} 1 & 3 & 3 & 2 \\ 2 & 6 & 9 & 7 \\ -1 & -3 & 3 & 4 \end{bmatrix} \rightarrow U = \begin{bmatrix} 1 & 3 & 3 & 2 \\ 0 & 0 & 3 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

II. Nullspaces

Given an $(m \times n)$ -matrix A with row echelon form U and reduced row echelon form R , the system of linear equations $Ax = 0$ has the same solutions with the system $Ux = 0$ and $Rx = 0$.

Thus **the nullspace of A is the same as the nullspace of U (and R).**

Let r be the rank of A , i.e., the number of non-zero rows of U .

Then there are $n - r$ free variables in the system $Ux = 0$, and **the nullspace of U is of dimension $n - r$.** We therefore obtain

Theorem 1 (*rank-nullity theorem*) *Let A be an $(m \times n)$ -matrix. Then*

$$\text{rank}(A) + \text{nullity}(A) = n.$$

Similarly,

$$\text{rank}(A^T) + \text{nullity}(A^T) = m.$$

The nullspace is also called the *kernel* (核) of A , and its dimension $n - r$ is the *nullity* (零度) of A .
 $\text{nullity}(A) = \text{dimension}(\mathcal{N}(A)).$

For example

$$A = \begin{bmatrix} 1 & 3 & 3 & 2 \\ 2 & 6 & 9 & 7 \\ -1 & -3 & 3 & 4 \end{bmatrix} \rightarrow U = \begin{bmatrix} 1 & 3 & 3 & 2 \\ 0 & 0 & 3 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\rightarrow R = \begin{bmatrix} 1 & 3 & 0 & -1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

The “special solutions” are a basis—each free variable is given the value 1, while the other free variables are 0.

Free variables: x_2, x_4

Special solutions: $\begin{bmatrix} -3 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ -1 \\ 1 \end{bmatrix}$

are definitely independent.

$$n = 4$$

$$r = 2$$

The dimension of $N(A) = n - r = 2$
(nullity)

III. Column Spaces

We notice that, in Example 2, $A = \begin{bmatrix} 0 & 1 & 2 & 3 \\ 1 & 2 & 3 & 4 \end{bmatrix}$,

$C(A^T)$ and $C(A)$ have the same dimension.



Is it always true that row space and column space have the same dimension?

Example 3 A non-zero $2 \times n$ matrix $A = \begin{bmatrix} a_1 & a_2 & \cdots & a_n \\ b_1 & b_2 & \cdots & b_n \end{bmatrix}$,

has row space and column space:

$$C(A^T) = \text{span}\{(a_1, a_2, \dots, a_n)^T, (b_1, b_2, \dots, b_n)^T\} \subseteq \mathbf{R}^n,$$

$$C(A) = \text{span}\{(a_1, b_1)^T, (a_2, b_2)^T, \dots, (a_n, b_n)^T\} \subseteq \mathbf{R}^2.$$

Obviously, both are of dimension at most 2.

If $C(A^T)$ is of dimension 1, then

$$(b_1, b_2, \dots, b_n) = k(a_1, a_2, \dots, a_n),$$

so $b_i = ka_i$ for all i .

Suppose $a_1 \neq 0$. Then for $a_i \neq 0$, we have

$$\frac{b_i}{a_i} = k = \frac{b_1}{a_1} \quad \text{and} \quad \frac{b_i}{b_1} = \frac{a_i}{a_1} = k_i$$

Thus, $(a_i, b_i) = k_i(a_1, b_1)$ for all i ,

and hence $C(A)$ is of dimension 1.

Similarly, if $C(A)$ is of dimension 1, then $C(A^T)$ is also of dimension 1.

So the row space and the column space of A have the same dimension.

This is actually true in the general case.

Theorem 2 *For any matrix, its row space and column space have the same dimension, which equals the rank of the matrix.*

For example,

$$A = \begin{bmatrix} 1 & 3 & 3 & 2 \\ 2 & 6 & 9 & 7 \\ -1 & -3 & 3 & 4 \end{bmatrix} \rightarrow U = \begin{bmatrix} 1 & 3 & 3 & 2 \\ 0 & 0 & 3 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix} \rightarrow R = \begin{bmatrix} 1 & 3 & 0 & -1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Note: the **column spaces** of U and A *are different* (just look at the matrices!)

but their **dimensions are the same**.

The first and third columns of U are a basis for its column space.

They are the columns with pivots.

Furthermore, the same is true of the original A —even though its columns are different.

The pivot columns of A are a basis for its column space.



Every linear dependence $A\mathbf{x} = \mathbf{0}$ among the columns of A is matched by a dependence $U\mathbf{x} = \mathbf{0}$ ($R\mathbf{x} = \mathbf{0}$) among the columns of U (R), with exactly the same coefficients. (行变换不改变列之间的相关性)
If a set of columns of A is independent, then so are the corresponding columns of U (and R), and vice versa.

Theorem 2 is one of the most important theorems in linear algebra. It is often abbreviated as

“row rank = column rank.” (行秩=列秩)

For example,

$$U = \begin{bmatrix} d_1 & * & * & * & * & * \\ 0 & 0 & 0 & d_2 & * & * \\ 0 & 0 & 0 & 0 & 0 & d_3 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

- Given an $m \times n$ matrix A with row echelon form U ,

$$\begin{array}{l} \text{Dimension of } C(U) \\ \text{ERO's do} \\ \text{not change} \\ \text{the linear} \\ \text{dependence} \\ \text{relations} \\ \text{among} \\ \text{the columns.} \end{array} \quad \begin{array}{c} \text{=} \\ \\ \\ \\ \\ \\ \\ \end{array} \quad \begin{array}{l} \text{Dimension of } C(U^T) \\ C(U^T) \\ \parallel \\ C(A^T) \end{array} \quad \begin{array}{c} \text{=} \\ \\ \\ \\ \\ \\ \\ \end{array} \quad \begin{array}{l} \text{rank}(U) \\ \\ \\ \\ \\ \\ \\ \end{array} \quad \begin{array}{c} \text{By definition} \end{array}$$

$$\text{Dimension of } C(A) = \text{Dimension of } C(A^T) = \text{rank}(A)$$

Summary

- For any matrix, its row space and column space have the same dimension, which equals the rank of the matrix.

row rank = column rank

- For any matrix A , $\text{rank}(A) = \text{rank}(A^T)$.
- For **square matrices**: *If the rows of a square matrix are linearly independent, then so are the columns (and vice versa) .*

IV. Left Nullspaces

In [Theorem 1](#) Let A be an $(m \times n)$ -matrix. Then

$$\text{rank}(A) + \text{nullity}(A) = n.$$

Similarly, $\text{rank}(A^T) + \text{nullity}(A^T) = m.$

Now that $\text{rank}(A) = \text{rank}(A^T)$ ([Theorem 2](#)), we have

$$\text{nullity}(A^T) = \text{dimension}(N(A^T)) = m - r.$$

$C(A)$ = column space of A ; dimension r .

$N(A)$ = nullspace of A ; dimension $n - r$.

$C(A^T)$ = row space of A ; dimension r .

$N(A^T)$ = left nullspace of A ; dimension $m - r$.

Fundamental Theorem of Linear Algebra (Part I)

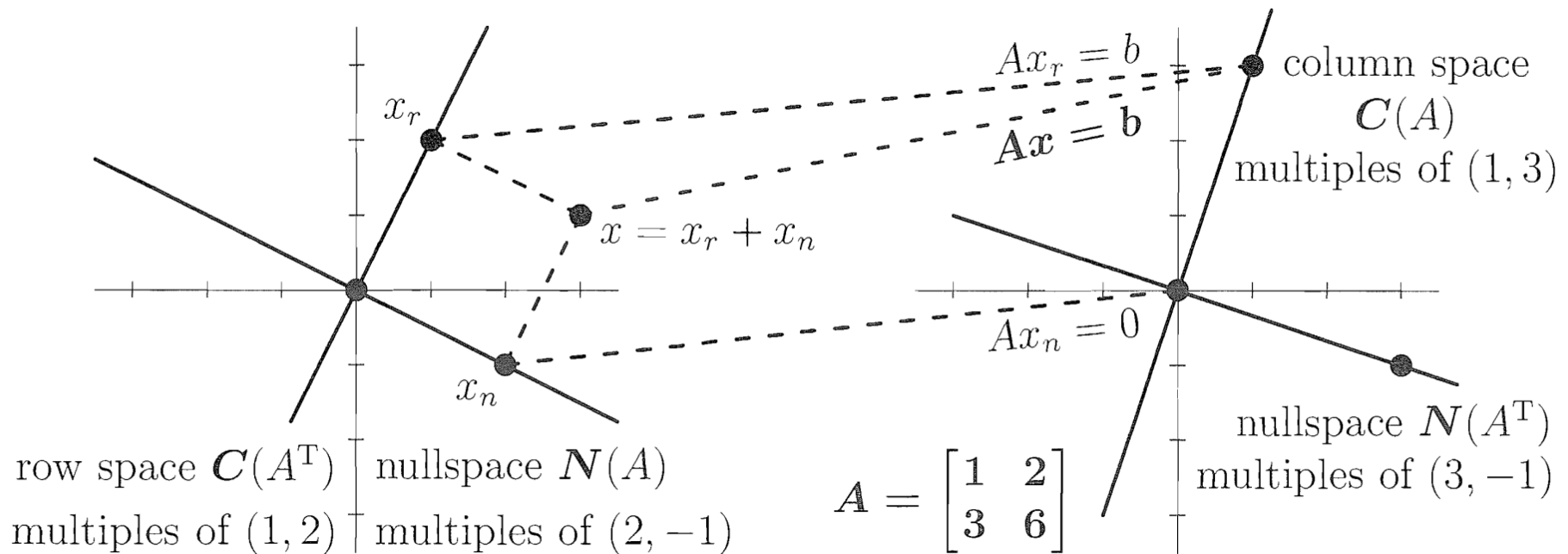
Example 4 $A = \begin{bmatrix} 1 & 2 \\ 3 & 6 \end{bmatrix}$, $m = n = 2$, **Singular matrix**
 $r = 1$.

Column space: $\text{Span}\left\{\begin{bmatrix} 1 \\ 3 \end{bmatrix}\right\}$

Nullspace: $\text{Span}\left\{\begin{bmatrix} 2 \\ -1 \end{bmatrix}\right\}$

Row space: $\text{Span}\left\{\begin{bmatrix} 1 \\ 2 \end{bmatrix}\right\}$

Left nullspace: $\text{Span}\left\{\begin{bmatrix} 3 \\ -1 \end{bmatrix}\right\}$



Example 4
What if --

$$\mathbf{B} = \begin{bmatrix} 1 & 2 \\ 3 & \textcolor{red}{7} \end{bmatrix},$$

$$m = n = 2, \\ r = 2.$$

Invertible matrix

Column space: $\text{Span}\left\{\begin{bmatrix} 1 \\ 3 \end{bmatrix}, \begin{bmatrix} 2 \\ 7 \end{bmatrix}\right\}$
 $= \mathbf{R}^2$

Nullspace: $\left\{\begin{bmatrix} 0 \\ 0 \end{bmatrix}\right\}$

Row space: $\text{Span}\left\{\begin{bmatrix} 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 3 \\ 7 \end{bmatrix}\right\}$
 $= \mathbf{R}^2$

Left nullspace: $\left\{\begin{bmatrix} 0 \\ 0 \end{bmatrix}\right\}$

V. Full Rank Matrices (满秩矩阵: extreme case)

We now consider some extremal cases. Let A be an $(m \times n)$ -matrix.

- If $\text{rank}(A) = m$, then A is said to have **full row rank** (行满秩).
- If $\text{rank}(A) = n$, then A is said to have **full column rank** (列满秩).

For example,

The following matrix A
has full row rank,
and B has full column
rank.

$$A = \begin{bmatrix} 1 & 0 & 1 & 2 \\ 0 & 2 & 2 & 5 \end{bmatrix}, \quad B = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 6 \\ 2 & 7 & 3 \\ 1 & 2 & 3 \end{bmatrix}.$$

Definition 1 Let A be an $(m \times n)$ -matrix.

- If there exists a matrix C such that $AC = I_m$, then we say A has **right-inverse** (右逆) C .
- If there exists a matrix B such that $BA = I_n$, then we say A has **left-inverse** (左逆) B .

- If A has a left-inverse ($BA = I$) and a right-inverse ($AC = I$), then the two inverses are equal: $B = B(AC) = (BA)C = C$.
- The rank always satisfies $r \leq \min(m, n)$.
- Existence and Uniqueness of solution to $Ax=b$
- **EXISTENCE:** Full row rank $r = m$. (possible only if $m \leq n$)
 $Ax = b$ has *at least* one solution x for every b if and only if the columns span \mathbf{R}^m .

Then A has an n by m *right-inverse* C such that $AC = I_m$ (m by m).

- **UNIQUENESS:** Full column rank $r = n$. (possible only if $m \geq n$)
 $Ax = b$ has *at most* one solution x for every b if and only if the columns are linearly independent.

Then A has an n by m *left-inverse* B such that $BA = I_n$ (n by n).

- Only a square matrix can have both $r = m$ and $r = n$, and therefore--
 Only a square matrix can achieve both existence and uniqueness of solution to $Ax=b$. Only a square matrix has a two-sided inverse.

How can we find a left-inverse, or a right-inverse for a matrix?

Lemma

Let \mathbf{A} be an $(m \times n)$ -matrix.

(1) If \mathbf{A} has full row rank, and the first m columns are linearly independent, then, writing

$$\mathbf{A} = \left[\mathbf{A}_0 \mid \mathbf{X} \right],$$

the matrix

$$\mathbf{C} = \begin{bmatrix} \mathbf{A}_0^{-1} \\ \mathbf{0} \end{bmatrix}$$

is a right-inverse of \mathbf{A} .

(2) If \mathbf{A} has full column rank, and the first n rows are linearly independent, then, writing

$$\mathbf{A} = \begin{bmatrix} \mathbf{A}_0 \\ \mathbf{Y} \end{bmatrix},$$

the matrix

$$\mathbf{B} = \left[\mathbf{A}_0^{-1} \mid \mathbf{0} \right]$$

is a left-inverse of \mathbf{A} .

Example 5 Using this lemma to the matrices A , B , it is easy to find a right-inverse of A , and a left-inverse of B .

$$A = \begin{bmatrix} 1 & 0 & 1 & 2 \\ 0 & 2 & 2 & 5 \end{bmatrix}, \quad B = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 6 \\ 2 & 7 & 3 \\ 1 & 2 & 3 \end{bmatrix}.$$

Remark: Generally, if AA^T is invertible, then a right-inverse of A is

$$C = A^T(AA^T)^{-1},$$

while if A^TA is invertible, then a left-inverse of A is

$$B = (A^TA)^{-1}A^T.$$

We shall study this stuff further later.

VI. Matrices of Rank 1 (秩为1的矩阵: simplest case)

Finally, we make a simple observation about matrices of rank 1.
(The rank is as small as possible, except for the zero matrix.)

Let A be an $(m \times n)$ -matrix of rank 1.

For example,

$$A = \begin{bmatrix} 2 & 1 & 1 \\ 4 & 2 & 2 \\ 8 & 4 & 4 \\ -2 & -1 & -1 \end{bmatrix}, \quad \text{has rank}(A)=1.$$

Then

$$A = \mathbf{u}\mathbf{v}^T,$$

where \mathbf{u} is a column vector (i.e., $(m \times 1)$ -matrix), and \mathbf{v}^T is a row vector $((1 \times n)$ -matrix).

This is because

- the rows are all multiples of the same row \mathbf{v}^T , and
- the columns are all multiples of the same column \mathbf{u} .

$$A = \begin{bmatrix} 2 & 1 & 1 \\ 4 & 2 & 2 \\ 8 & 4 & 4 \\ -2 & -1 & -1 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 4 \\ -1 \end{bmatrix} \begin{bmatrix} 2 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 2 \\ 4 \\ 8 \\ -2 \end{bmatrix} \begin{bmatrix} 1 & \frac{1}{2} & \frac{1}{2} \end{bmatrix}.$$

\uparrow
 u
 \uparrow
 v^T

*Every matrix of **rank 1** has the simple form*

$$A = uv^T = \text{column times row}.$$

The row space and column space are lines—the easiest case.

We remark that if $A = uv^T$, then $A = (cu)(c^{-1}v^T)$, where c is a non-zero number.

Key words:

*row space, column space, nullspace, left nullspace,
rank, full rank*

Homework

See Blackboard

Note: In Ex. 35, u, v, w, z are column vectors.

