Homework & Tutorial

Homework: 5.2, 5.5, 5.15, 5.21(a-f, h) Tutorial Problems: 5.1, 5.3, 5.4, 5.41



Chapter 5: The Discrete-Time Fourier Transform

Department of Electrical & Electronic Engineering

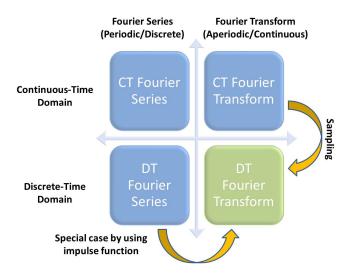
2020-Fall

Last Update on: Sunday 8th November, 2020





A Big Picture



Outline

- Definition of discrete time Fourier transform (DTFT)
 - From Fourier series to Fourier transform
 - Convergence issue
- Periodic signals' DTFT
- Properties of DTFT





Discrete-Time Fourier Transform

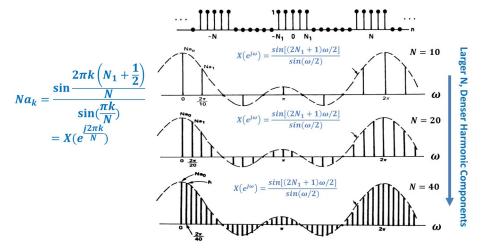
Discrete-time Fourier series (for periodic signals)

Synthesis Equation:
$$x[n] = \sum_{k=< N>} a_k e^{jk(2\pi/N)n}$$

Analysis Equation: $a_k = \frac{1}{N} \sum_{n=< N>} x[n] e^{-jk(2\pi/N)n}$

- ullet Aperiodic signals can be treated as periodic signals with period ${\it N}
 ightarrow \infty$
 - \blacktriangleright x[n] must be like $\sum_k b_k e^{jk(2\pi/N)n}$ or $\int_{\omega} b(\omega)e^{j\omega n}d\omega$
 - b_k or $b(\omega)$ can be calculated from $x[\tilde{n}]$

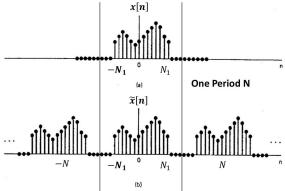
Example: From Periodic To Aperiodic







Derivation (1/3)



- Original signal: x[n]
- Define new periodic signal with period $N: \widetilde{x}[n]$, such that

$$\tilde{x}[n] = x[n], \ n = -N/2, ..., N/2 - 1$$

• Notice: when $N \to \infty$, $\widetilde{x}[n]$ becomes x[n]





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Derivation (2/3)

• Look at the Fourier series of $\widetilde{x}[n]$:

$$a_k = \frac{1}{N} \sum_{n=-N/2}^{N/2+1} \widetilde{x}[n] e^{-jk(2\pi/N)n} = \frac{1}{N} \sum_{n=-\infty}^{\infty} x[n] e^{-jk(2\pi/N)n}$$

- Define $X(e^{j\omega}) = \sum_{-\infty}^{\infty} x[n]e^{-j\omega n}$, so that $a_k = X(e^{jk\omega_0})/N$ ($\omega_0 = 2\pi/N$)
- Look at the synthesis equation:

$$\widetilde{X}[n] = \sum_{k=< N>} a_k e^{jk(2\pi/N)n} = \frac{1}{N} \sum_{k=< N>} X(e^{jk\omega_0}) e^{jk\omega_0 n}$$
$$= \frac{1}{2\pi} \sum_{k=< N>} X(e^{jk\omega_0}) e^{jk\omega_0 n} \omega_0$$

• When $N \to \infty$, $\omega_0 \to d\omega \Sigma \to \int k\omega_0 \to \omega \widetilde{x}[n] \to x[n]$, thus,

$$x[n] = \frac{1}{2\pi} \int_{2\pi} X(e^{j\omega}) e^{j\omega n} d\omega$$





Derivation (3/3)

• Therefore, we get the discrete-time Fourier transform pair

Discrete-Time Fourier Transform

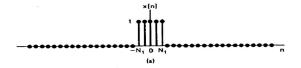
Synthesis Equation:
$$x[n] = \frac{1}{2\pi} \int_{2\pi} X(e^{j\omega}) e^{j\omega n} d\omega$$

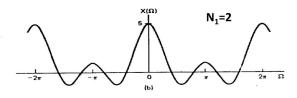
Analysis Equation:
$$X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n]e^{-j\omega n}$$

- Observations
 - Continuous spectrum: Similar to CTFT
 - ▶ Periodic with period 2π : Different from CTFT
 - ▶ Low frequency: close to 0 and 2π ; high frequency: close to π



Fourier Transform Examples (1/2)





- $x[n] = 1 (n = -N_1, ..., 0, ..., N_1)$
- $X(e^{j\omega}) = \frac{\sin\omega(N_1+1/2)}{\sin(\omega/2)}$
- Width of x[n]: $W_t=2N_1+1$; width of $X(e^{j\omega})$: $W_f=\frac{4\pi}{2N_1+1}$
- $W_t \times W_f = 4\pi$, which is a constant





Fourier Transform Examples (2/2)

$$x[n] = a^{n}u[n] \quad 0 < a < 1 \iff X(e^{j\omega}) = \frac{1}{1 - ae^{-j\omega}}$$

$$|X(e^{j\omega})|$$

$$\frac{1}{(1-a)}$$

$$\frac{1}{(1+a)}$$

$$\frac{1}{($$

- See textbook, Example 5.1
- What's the shape of magnitude when $a \to 1$ or $a \to 0$?





Convergence Issue of Analysis Equation

Sufficient Condition of Convergence

The analysis equation $X(e^{j\omega}) = \sum_{-\infty}^{\infty} x[n]e^{-j\omega n}$ will converge either if x[n] is absolutely summable or if the sequence has finite energy, thus,

$$\sum_{-\infty}^{\infty} |x[n]| < \infty \text{ or } \sum_{-\infty}^{\infty} |x[n]|^2 < \infty$$

- Do the following signals have Fourier transform:
 - $\rightarrow a^n u[n] (0 < a < 1)$
 - δ[n]
 - ► *u*[*n*]

 - $a^n u[n] (a > 1)$



Can Periodic Signals Have DTFT?

Definition of DTFT:

$$X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n]e^{-j\omega n}$$

- Justification of divergence:
 - Let $\omega = 2k\pi$, we have $X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n]$
 - Since x[n] is periodic, the summation $\sum_{n=-\infty}^{\infty} x[n]$ will never converge unless x[n]=0
- Conclusion: Most of periodic signals do NOT have DTFT according to the definition
- However, it's of significant engineering importance to extend Fourier transform to periodic signals



DTFT with Periodic Signals (1/2)

Fourier Transform of $e^{j\omega n}$

The following transform pair is actually NOT rigorously defined:

$$x[n] = e^{j\omega_0 n} \longleftrightarrow X(e^{j\omega}) = \sum_{i=1}^{\infty} 2\pi \delta(\omega - \omega_0 - 2\pi I)$$

Synthesis:

$$\frac{1}{2\pi}\int\limits_{\Omega_{-}}X(e^{j\omega})e^{j\omega n}d\omega=\frac{1}{2\pi}\int\limits_{\Omega_{-}}\sum_{l=-\infty}^{\infty}2\pi\delta(\omega-\omega_{0}-2\pi l)e^{j\omega n}d\omega=e^{j\omega_{0}n}$$

Analysis:

$$\sum_{n=0}^{\infty} x[n]e^{-j\omega n} = \sum_{n=0}^{\infty} e^{j(\omega_0 - \omega)n}$$
 converge??





Remark

• We just believe the following equation is true:

$$\sum_{n=-\infty}^{\infty} e^{j\omega n} = \sum_{l=-\infty}^{\infty} 2\pi \delta(\omega - 2\pi l)$$

• When $\omega = 2k\pi$,

$$\sum_{n=-\infty}^{\infty} 1 = 2\pi \delta(0)$$



DTFT with Periodic Signals (2/2)

• According to the Fourier series, for a periodic signal with period N:

$$x[n] = a_0 + a_1 e^{j(2\pi/N)n} + ... + a_k e^{jk(2\pi/N)n} + ... + a_{N-1} e^{j(N-1)(2\pi/N)n}$$

- $e^{jk(2\pi/N)n} \longleftrightarrow \sum_{n=0}^{\infty} 2\pi\delta(\omega k(2\pi/N) 2\pi I)$
- Then, due to the linearity of Fourier transform

$$\mathcal{F}\left\{x[n]\right\} = \sum_{k=0}^{N-1} a_k \mathcal{F}\left\{e^{jk(2\pi/N)n}\right\} = \sum_{k=0}^{N-1} a_k \sum_{l=-\infty}^{\infty} 2\pi\delta(\omega - k(2\pi/N) - 2\pi l)$$
$$= \sum_{l=-\infty}^{\infty} \sum_{k=0}^{N-1} a_k 2\pi\delta(\omega - k(2\pi/N) - 2\pi l)$$

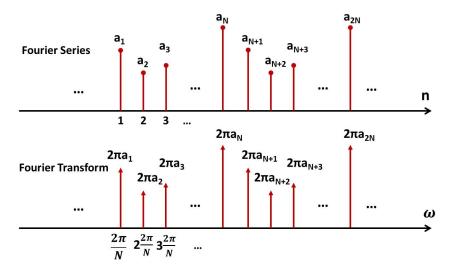
- Fourier transform of a periodic signal is a periodic sequence of impulses
 - What's the period? How many impulses within one period?





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Fourier Series v.s. Fourier Transform



Example: Discrete-Time Impulse Chain

- What's the Fourier transform of $x[n] = \sum_{k=0}^{+\infty} \delta[n kN]$?
- First of all, we calculate the Fourier series:

$$a_k = \frac{1}{N} \sum_{n=} x[n] e^{-jk\frac{2\pi}{N}n}$$

$$= \frac{1}{N} \sum_{n=} \sum_{k=-\infty}^{+\infty} \delta[n-kN] e^{-jk\frac{2\pi}{N}n}$$

$$= \frac{1}{N} \sum_{n=} \delta[n] e^{-jk\frac{2\pi}{N}n} = \frac{1}{N}$$

• Then, we have

$$X(e^{j\omega}) = \sum_{l=-\infty}^{\infty} \sum_{k=0}^{N-1} \frac{2\pi}{N} \delta(\omega - k(2\pi/N) - 2\pi I) = \frac{2\pi}{N} \sum_{k=-\infty}^{\infty} \delta(\omega - \frac{2\pi k}{N})$$

• Time domain period × Frequency domain period = ?



Periodicity, Linearity and Shifting

Periodicity

$$X(e^{j(\omega+2\pi)})=X(e^{j\omega})$$

- ► How about CTFT? Why?
- Linearity

$$ax_1[n] + bx_2[n] \longleftrightarrow aX_1(e^{j\omega}) + bX_2(e^{j\omega})$$

Time Shifting and Frequency Shifting

$$x[n-n_0] \longleftrightarrow e^{-j\omega n_0}X(e^{j\omega})$$

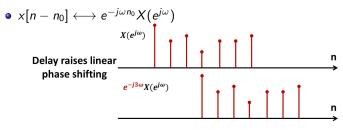
 $e^{j\omega_0 n}x[n] \longleftrightarrow X(e^{j(\omega-\omega_0)})$

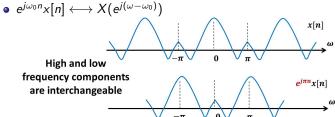
What's the physical meaning?





Illustration on Shifting







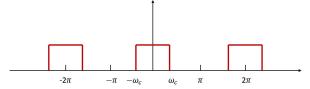


Example: Shifting

 Similar to CT, the DTFT of impulse response of a DT LTI system is frequency response.

$$H(e^{j\omega}) = \sum_{n=-\infty}^{+\infty} h[n]e^{-j\omega n}$$

Given an ideal LPF with the following frequency response



• Suppose its impulse response is $h_{lpf}[n]$, please predict the behavior of another LTI system with impulse response $(-1)^n h_{lpf}[n]$.

Conjugation, Differencing and Accumulation

Conjugation

$$x^*[n] \longleftrightarrow X^*(e^{-j\omega})$$

- $X(e^{j\omega}) = X^*(e^{-j\omega}) \Leftrightarrow x[n]$ is real
- ▶ If x[n] is real, then $\Re\{X(e^{j\omega})\}$ is even, $\Im\{X(e^{j\omega})\}$ is odd
- Differencing

$$x[n] - x[n-1] \longleftrightarrow (1 - e^{-j\omega})X(e^{j\omega})$$

- ► High-pass or low-pass?
- Accumulation

$$\sum_{m=-\infty}^{n} x[m] \longleftrightarrow \frac{1}{1-e^{-j\omega}} X(e^{j\omega}) + \pi X(e^{j0}) \sum_{k=-\infty}^{\infty} \delta(\omega - 2\pi k)$$

How to derive it via differencing?





Problem 5.37

Problem

Let $X(e^{j\omega})$ be the Fourier transform of x[n]. Derive expressions in terms of $X(e^{j\omega})$ for the Fourier transforms of the following signals.

- $Re\{x[n]\}$
- $x^*[-n]$
- **Ev**{x[n]}

Effect of Differencing





•
$$J(i,j) = |M(i,j) - M(i+1,j+1)| + |M(i+1,j) - M(i,j+1)|$$

Fourier Transform of u[n]

- How to derive the Fourier transform of u[n]?
- Option 1: From definition of Fourier transform

$$\mathcal{F}\{u[n]\} = \sum_{n=-\infty}^{+\infty} u[n]e^{-j\omega n} = \sum_{n=0}^{+\infty} e^{-j\omega n} \quad \text{Converge??}$$

• Option 2: Since $u[n] = \sum_{m=-\infty}^{n} \delta[m]$, according to the property of accumulation

$$\mathcal{F}\{u[n]\} = \frac{1}{1 - e^{-j\omega}} + \pi \sum_{k = -\infty}^{\infty} \delta(\omega - 2\pi k)$$

• Observation: Fourier transform of u[n] does not exist according to the definition; however, it can be expressed in terms of $\delta(\cdot)$





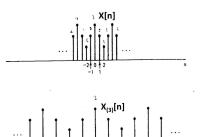
Time Reversal and Expansion

Time Reversal

$$x[-n]\longleftrightarrow X(e^{-j\omega})$$

- Time Expansion
 - ▶ Define $x_{(k)}[n] = \begin{cases} x[n/k], & \text{if } n \text{ is a multiple of k} \\ 0, & \text{Otherwise} \end{cases}$, then

$$x_{(k)}[n] \longleftrightarrow X(e^{jk\omega})$$
 (1)







Differentiation and Parseval

Differentiation in Frequency

$$nx[n] \longleftrightarrow j\frac{dX(e^{j\omega})}{d\omega}$$

Parseval's Relation

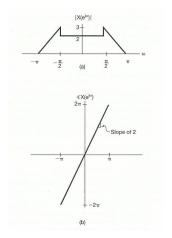
$$\sum_{n=-\infty}^{+\infty} |x[n]|^2 = \frac{1}{2\pi} \int_{2\pi} |X(e^{j\omega})|^2 d\omega$$

► Energy density spectrum — $\frac{|X(e^{j\omega})|^2}{2\pi}$





Example



- See textbook, Example 5.10
- \bullet Spectrum within $[-\pi,\pi]$
- Is it periodic, real, even, of finite energy?



