5 Eigenvalues and Eigenvectors
（特征值与特征向量）

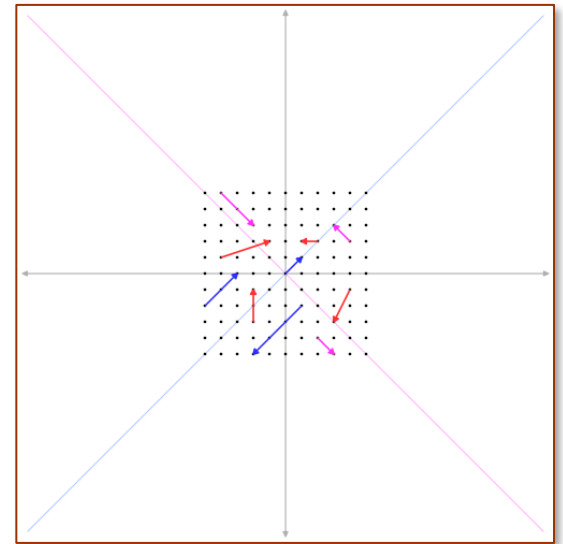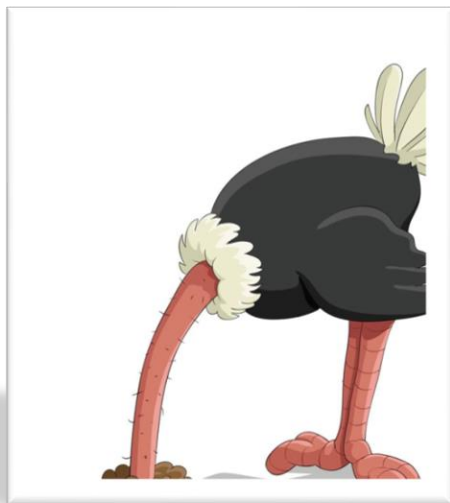## 5.5

COMPLEX MATRICES

Operations in the Complex Case

Hermitian Matrices

Unitary Matrices

Since $|A - \lambda I|$ is a polynomial of degree n, the equation always has exactly n roots, counting multiplicities, *provided that possibly complex roots are included.*

A real matrix has real coefficients in $|A - \lambda I|$, but the eigenvalues (as in rotations) may be *complex*.

We cannot avoid complex numbers and vectors any more.

The key is to let A act on the space $\mathbf{C}^n$.

The new definitions coincide with the old when the vectors and matrices are *real*.

Main results:

1. *Every symmetric matrix (and Hermitian matrix) has real eigenvalues.*

2. *Its eigenvectors can be chosen to be orthonormal.*

I. Some Definitions in the Complex Case

(1) Take a complex number $z = a + ib$, where $i = \sqrt{-1}$.

- Conjugate (共轭) $\bar{z} = a - ib$.
- Absolute value $r = |z| = \sqrt{a^2 + b^2}$.
- Polar form: $a + ib = r(\cos\theta + i\sin\theta) = r\mathbf{e}^{i\theta}$.
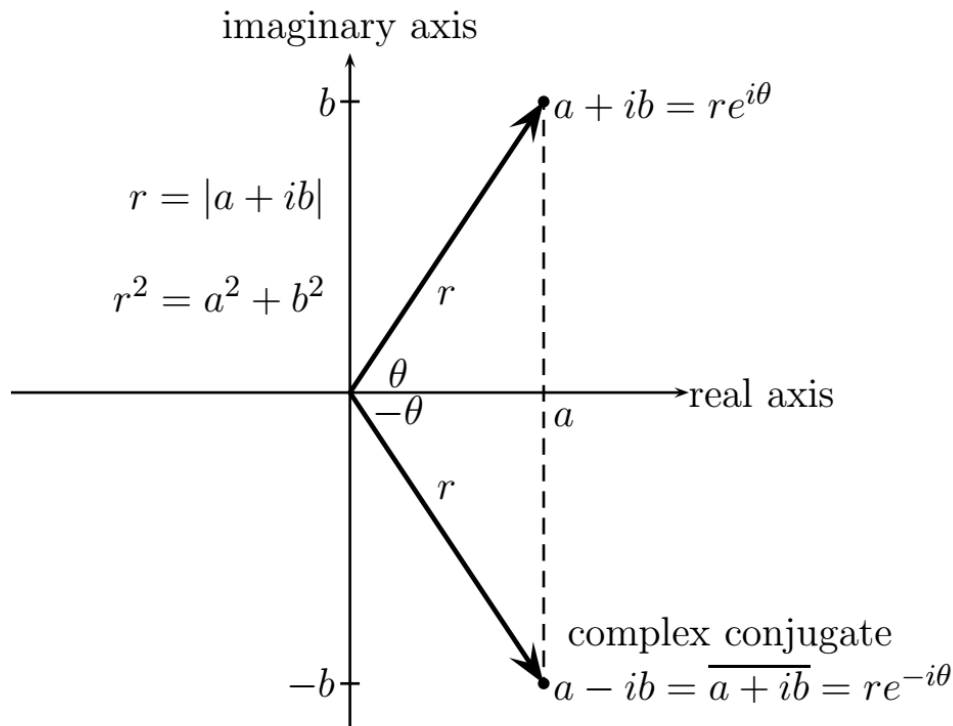
Complex addition
$$(a + ib) + (c + id)$$
$$= (a + c) + i(b + d).$$

Multiplication
$$(a + ib)(c + id)$$
$$= (ac - bd) + i(bc + ad).$$

The complex plane, with $a + ib = re^{i\theta}$ and its conjugate $a - ib = re^{-i\theta}$.

Three important properties:

1. The conjugate of a product equals the product of the conjugates:
$$\overline{(a + ib)(c + id)} = (ac - bd) - i(bc + ad)$$
$$= \overline{(a + ib)}\,\overline{(c + id)}.$$

2. The conjugate of a sum equals the sum of the conjugates:
$$\overline{(a + c) + i(b + d)} = (a + c) - i(b + d)$$
$$= \overline{(a + ib)} + \overline{(c + id)}.$$

3. Multiplying any $a + ib$ by its conjugate $a - ib$ produces a real number $a^2 + b^2$:

$$(a + ib)(a - ib) = a^2 + b^2 = r^2.$$

This distance r is the ***absolute value*** $|a + ib| = \sqrt{a^2 + b^2}.$

For example,

$x = 3 + 4i$ times its conjugate $\bar{x} = 3 - 4i$ is the absolute value squared:

$$x\bar{x} = (3 + 4i)(3 - 4i) = 25 = |x|^2,$$

so $r = |x| = 5$.

To divide by $3 + 4i$, multiply numerator and denominator by its conjugate $3 - 4i$:

$$\frac{2+i}{3+4i} = \frac{2+i}{3+4i}\,\frac{3-4i}{3-4i} = \frac{10-5i}{25}.$$

In *polar coordinates*(极坐标), multiplication and division are easy:
$re^{i\theta}$ times $Re^{i\alpha}$ has absolute value rR and angle $\theta + \alpha$.
$re^{i\theta}$ divided by $Re^{i\alpha}$ has absolute value r/R and angle $\theta - \alpha$.

(2) Pick a complex vector $x = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} \in \mathbf{C}^n$ (*the complex vector space*

containing all vectors with n complex components),   where $x_j = a_j + ib_j$.

- Vector addition: $x + y = (x_1 + y_1, x_2 + y_2, \ldots, x_n + y_n)^{\mathrm{T}}$.

- Scalar multiplication: $cx,\ c \in \mathbf{C}$.

- The vectors $v_1, v_2, \ldots, v_k$ are linearly *dependent* if some *nontrivial* combination gives $c_1 v_1 + \cdots + c_k v_k = 0$; the c_j may now be complex.

- $\mathbf{C}^n$ is a complex vector space of dimension n. (The unit coordinate vectors are still in $\mathbf{C}^n$; they are still independent; and they still form a basis.)

For $x, y \in \mathbf{C}^n$,

— The length squared $\|x\|^2 = |x_1|^2 + |x_2|^2 + \cdots + |x_n|^2$.

— The conjugate: $\bar{x} = (\bar{x}_1, \bar{x}_2, \ldots, \bar{x}_n)^{\mathrm{T}}$.

— Inner product: $\overline{x}^{\mathrm{T}} y = \bar{x}_1 y_1 + \cdots + \bar{x}_n y_n$.

In particular, $\quad \overline{x}^{\mathrm{T}} x = \bar{x}_1 x_1 + \cdots + \bar{x}_n x_n = \|x\|^2$.

Attention: Length is computed differently.
The inner product, the definitions of symmetric and orthogonal matrices, all need to be modified for complex numbers.

For example,

$x = \begin{bmatrix} 1 \\ i \end{bmatrix}$, then $\|x\|^2 = 2$.

$y = \begin{bmatrix} 2 + i \\ 2 - 4i \end{bmatrix}$, then $\|y\|^2 = 25$.

Also $\overline{y}^{\mathrm{T}} y = \overline{(2 + i)}(2 + i) + \overline{(2 - 4i)}(2 - 4i) = 5 + 20 = 25$.

(3) Let $\boldsymbol{A} = \left[a_{ij}\right]_{m \times n}$ be a complex matrix.

– The conjugate: $\overline{\boldsymbol{A}} = \left[\bar{a}_{ij}\right]_{m \times n}$.

– The conjugate transpose (共轭转置) : $\overline{\boldsymbol{A}}^{\mathrm{T}} = \left[\bar{a}_{ji}\right]_{n \times m}$,

called '$\boldsymbol{A}$ **Hermitian**' ($\boldsymbol{A}$的厄米特矩阵), denoted by $\boldsymbol{A}^H$.
(Instead of a bar for the conjugate and a T for the transpose, a superscript H combines both operations)

For example, $\begin{bmatrix} 2+i & 3i \\ 4-i & 5 \\ 0 & 0 \end{bmatrix}^H = \begin{bmatrix} 2-i & 4+i & 0 \\ -3i & 5 & 0 \end{bmatrix}$.

- For $\boldsymbol{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$, $\boldsymbol{x}^H = (\bar{x}_1, \bar{x}_2, \dots, \bar{x}_n)$.

- Inner product $\overline{\boldsymbol{x}}^{\mathrm{T}} \boldsymbol{y}$ can also be written as $\boldsymbol{x}^H \boldsymbol{y}$.
- Orthogonal vectors have $\boldsymbol{x}^H \boldsymbol{y} = 0$.
- The squared length of $\boldsymbol{x}$ is $\boldsymbol{x}^H \boldsymbol{x}$.

Remark We note that
$(\boldsymbol{AB})^H = \boldsymbol{B}^H \boldsymbol{A}^H$,
and $(\boldsymbol{A}^H)^H = \boldsymbol{A}$.

II. Hermitian Matrices and Properties

Real cases: Symmetric matrices: $A = A^{\mathrm{T}}$.

With complex entries, this idea of symmetry has to be extended.

Generalization: ***matrices that equal their conjugate transpose***.

Definition 1 A matrix A is called a **Hermitian matrix** (*A*是厄米特矩阵) if $A^H = A$. (即满足：A的共轭转置矩阵等于它本身)

For example,

$A = \begin{bmatrix} 2 & 3 - 3i \\ 3 + 3i & 5 \end{bmatrix} = A^H$, so A is a Hermitian matrix.

The diagonal entries must be real; Each off-diagonal entry is matched with its mirror image across the main diagonal .

Remark *A real symmetric matrix is certainly Hermitian*. (For real matrices there is no difference between A^{T} and A^H.)

Property 1 *If $A = A^H$, then for all complex vectors x, the number $x^H A x$ is real.*

Proof. Notice that $x^H A x$ is a number, and
$$(x^H A x)^H = x^H A^H (x^H)^H = x^H A x.$$
That is to say, $x^H A x$ is a number which is equal to its conjugate.
So $x^H A x$ is a real number.

Property 2 *If $A = A^H$, then every eigenvalue is a real number.*

Proof. Let A be a Hermitian matrix, and assume $A x = \lambda x$.

Then $x^H A x = x^H \lambda x = \lambda x^H x = \lambda \|x\|^2$.

By Property 1, $x^H A x$ is real.

And since $x \neq 0$, $\|x\|^2$ is real and positive,

thus $\lambda = \dfrac{x^H A x}{\|x\|^2}$, and so λ is a real number.

Property 2 *If $A = A^H$, then every eigenvalue is a real number.*

A Second Proof. (without using Property 1)

$$\bar{\lambda} x^H x = (\lambda x)^H x = (Ax)^H x = x^H A^H x = x^H A x = \lambda x^H x.$$

So $\bar{\lambda} = \lambda$, and λ is a real number.

For example,

$A = \begin{bmatrix} 2 & 3 - 3i \\ 3 + 3i & 5 \end{bmatrix} = A^H$, so A is a Hermitian matrix.

Then

$$|A - \lambda I| = \begin{vmatrix} 2 - \lambda & 3 - 3i \\ 3 + 3i & 5 - \lambda \end{vmatrix}$$

$$= \lambda^2 - 7\lambda + 10 - |3 - 3i|^2$$

$$= \lambda^2 - 7\lambda - 8 = (\lambda - 8)(\lambda + 1).$$

The eigenvalues of A are 8 and -1.

Property 3 *Let $\boldsymbol{A}$ be a Hermitian matrix (i.e., $\boldsymbol{A} = \boldsymbol{A}^H$), and λ_1, λ_2 be two different eigenvalues of $\boldsymbol{A}$. Then the eigenvectors corresponding to λ_1, λ_2 are orthogonal to each other.*

In particular, this is true for real symmetric matrices.

Proof. Let $\boldsymbol{x}_1, \boldsymbol{x}_2$ be the eigenvectors of $\boldsymbol{A}$ corresponding to λ_1, λ_2, respectively. Then

$$\boldsymbol{A}\boldsymbol{x}_1 = \lambda_1\boldsymbol{x}_1, \quad \text{and} \quad \boldsymbol{A}\boldsymbol{x}_2 = \lambda_2\boldsymbol{x}_2.$$

Hence

$$\lambda_1\boldsymbol{x}_1^H\boldsymbol{x}_2 = (\lambda_1\boldsymbol{x}_1)^H\boldsymbol{x}_2 = (\boldsymbol{A}\boldsymbol{x}_1)^H\boldsymbol{x}_2 = \boldsymbol{x}_1^H\boldsymbol{A}^H\boldsymbol{x}_2$$
$$= \boldsymbol{x}_1^H\boldsymbol{A}\boldsymbol{x}_2 = \boldsymbol{x}_1^H\lambda_2\boldsymbol{x}_2 = \lambda_2\boldsymbol{x}_1^H\boldsymbol{x}_2.$$

Since $\lambda_1 \neq \lambda_2$, we conclude that $\boldsymbol{x}_1^H\boldsymbol{x}_2 = 0$, and $\boldsymbol{x}_1, \boldsymbol{x}_2$ are orthogonal.

For example,

$$A = \begin{bmatrix} 2 & 3 - 3i \\ 3 + 3i & 5 \end{bmatrix} = A^H, \text{ so } A \text{ is a Hermitian matrix.}$$

The eigenvalues of A are 8 and -1.

$$(A - 8I)x = \begin{bmatrix} -6 & 3 - 3i \\ 3 + 3i & -3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \quad x = \begin{bmatrix} 1 \\ 1 + i \end{bmatrix}.$$

$$(A + I)y = \begin{bmatrix} 3 & 3 - 3i \\ 3 + 3i & 6 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \quad y = \begin{bmatrix} 1 - i \\ -1 \end{bmatrix}.$$

The two eigenvectors are *orthogonal :*

$$x^H y = \begin{bmatrix} 1 & 1 - i \end{bmatrix} \begin{bmatrix} 1 - i \\ -1 \end{bmatrix} = 0.$$

The next is one of the great theorems in linear algebra.

Theorem 1 (*Spectral Theorem*, part I) *A real symmetric matrix* (实对称矩阵) *A can be factored into*

$$A = Q\Lambda Q^T .$$

Its orthonormal eigenvectors are in the orthogonal matrix Q and its eigenvalues are in Λ.

Proof. (We only prove this for A with distinct eigenvalues.)

Let Q be the matrix with columns being n eigenvectors of A which are orthonormal. Then

$$Q^{-1}AQ = \Lambda = \mathrm{diag}(\lambda_1, \lambda_2, \ldots, \lambda_n),$$

and so $A = Q\Lambda Q^{-1} = Q\Lambda Q^{\mathrm{T}}$.

(*Even with repeated eigenvalues, a symmetric matrix still has a complete set of orthonormal eigenvectors.* – Next section)

Remark 1

$$A = Q\Lambda Q^T = [x_1 \quad \dots \quad x_n] \begin{bmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{bmatrix} \begin{bmatrix} x_1^T \\ \vdots \\ x_n^T \end{bmatrix}$$

$$= \lambda_1 x_1 x_1^T + \lambda_2 x_2 x_2^T + \cdots + \lambda_n x_n x_n^T.$$

So A becomes a combination of one-dimensional projections—which are the special matrices $x_i x_i^T$ of rank 1, multiplied by λ_i.

Example 1 $A = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$ has eigenvalues $\lambda_1 = 1$ and $\lambda_2 = 3$.
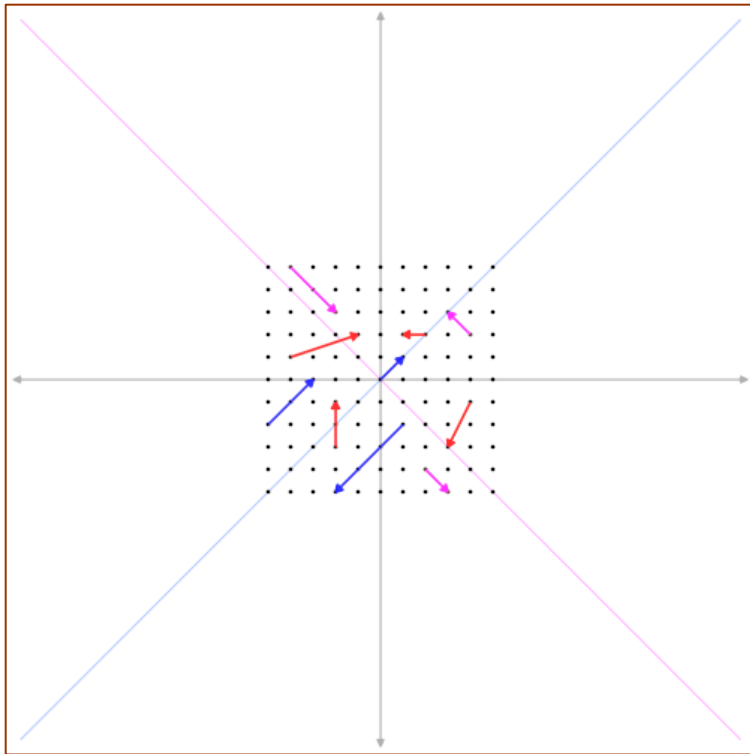
The eigenvectors, with length scaled to 1, are

$$x_1 = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ -1 \end{bmatrix} \quad \text{and} \quad x_2 = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 1 \end{bmatrix}.$$

$$A = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} \frac{1}{2} & -\frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} \end{bmatrix} + 3 \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix} = \lambda_1 x_1 x_1^T + \lambda_2 x_2 x_2^T.$$

— combination of two one-dimensional projections.

For example,

$$A = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$$

The Eigenvectors

$$k_1 \begin{bmatrix} 1 \\ -1 \end{bmatrix} \qquad k_2 \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$(k_1 \neq 0) \qquad (k_2 \neq 0)$$

Corresponding respectively to the Eigenvalues:

$$\lambda_1 = 1 \qquad \lambda_2 = 3$$

http://en.wikipedia.org/wiki/File:Eigenvectors-extended.gif

Remark 2 If A is *real* and its eigenvalues *happen to be real*, then its eigenvectors are also real.

(solve $(A - \lambda I)x = 0$ and compute by elimination.)

But they will not be orthogonal unless A is symmetric:

$A = Q\Lambda Q^{\mathrm{T}}$ leads to $A^{\mathrm{T}} = A$.

Remark 3 If A is *real*, all complex eigenvalues come in conjugate pairs: $Ax = \lambda x$ and $A\overline{x} = \overline{\lambda}\overline{x}$.

(This is true because $A\overline{x} = \overline{Ax} = \overline{\lambda x} = \overline{\lambda}\overline{x}$)

Hence $\overline{\lambda}$ is also an eigenvalue of A, with $\overline{x}$ a corresponding eigenvector.

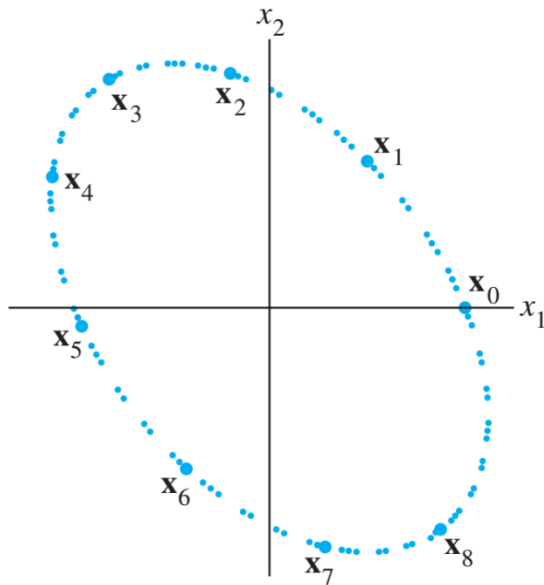*If $a + ib$ is an eigenvalue of a real matrix, so is $a - ib$.*

(If $A = A^{\mathrm{T}}$ then $b = 0$. 实对称矩阵的特征值都是实数)

For example, $A = \begin{bmatrix} 0.5 & -0.6 \\ 0.75 & 1.1 \end{bmatrix}$

The eigenvalues are $\lambda_{1,2} = 0.8 \pm 0.6i$.

The basis for the eigenspace corresponding to λ_1 and λ_2 are

$\boldsymbol{v}_1 = \begin{bmatrix} -2 + 4i \\ 5 \end{bmatrix}$ and $\boldsymbol{v}_2 = \begin{bmatrix} -2 - 4i \\ 5 \end{bmatrix}$.

Iterates of a point $\boldsymbol{x}_0$ under the action of a matrix with a complex eigenvalue

One way to see how multiplication by A affects points is to plot an arbitrary initial point – say, $\boldsymbol{x}_0 = (2,0)^{\mathrm{T}}$ – and then to plot

$$\boldsymbol{x}_1 = A\boldsymbol{x}_0, \ \boldsymbol{x}_2 = A\boldsymbol{x}_1, \ \boldsymbol{x}_3 = A\boldsymbol{x}_2, \ldots$$

Example 2 Let $A = \begin{bmatrix} 2 & 2 & -2 \\ 2 & 5 & -4 \\ -2 & -4 & 5 \end{bmatrix}$.

Find an orthogonal matrix Q such that $Q^{-1}AQ$ is a diagonal matrix.

Solution

$$|A - \lambda I| = \begin{vmatrix} 2 - \lambda & 2 & -2 \\ 2 & 5 - \lambda & -4 \\ -2 & -4 & 5 - \lambda \end{vmatrix} = \begin{vmatrix} 2 - \lambda & 2 & -2 \\ 0 & 1 - \lambda & 1 - \lambda \\ -2 & -4 & 5 - \lambda \end{vmatrix}$$

$$= \begin{vmatrix} 2 - \lambda & 2 & -4 \\ 0 & 1 - \lambda & 0 \\ -2 & -4 & 9 - \lambda \end{vmatrix} = (1 - \lambda) \begin{vmatrix} 2 - \lambda & -4 \\ -2 & 9 - \lambda \end{vmatrix}$$

$$= (1 - \lambda)^2(10 - \lambda).$$

So the eigenvalues of A are $\lambda_1 = 1$ (Algebraic multiplicity is 2) and $\lambda_2 = 10$ (Algebraic multiplicity is 1).

For the eigenvalue $\lambda_1=1$, by $(A-\lambda_1 I)x = 0$, i.e.,

$$\begin{bmatrix} 1 & 2 & -2 \\ 2 & 4 & -4 \\ -2 & -4 & 4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

The basis for the eigenspace of λ_1: $\quad x_1 =(-2, 1, 0)^T, \quad x_2 =(2, 0, 1)^T.$

By Gram-Schmidt orthogonalization, let

$$\beta_1 = x_1 = (-2, 1, 0)^T,$$

$$\beta_2 = x_2 - \frac{x_2^T \beta_1}{\beta_1^T \beta_1} \beta_1 = \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix} - \frac{-4}{5} \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} \frac{2}{5} \\ \frac{4}{5} \\ 1 \end{bmatrix} = \frac{1}{5} \begin{bmatrix} 2 \\ 4 \\ 5 \end{bmatrix},$$

and normalize β_1, β_2 into:

$$\gamma_1 = \frac{\beta_1}{\|\beta_1\|} = \frac{\sqrt{5}}{5}\begin{bmatrix} -2 & 1 & 0 \end{bmatrix}^T, \qquad \gamma_2 = \frac{\beta_2}{\|\beta_2\|} = \frac{\sqrt{5}}{15}\begin{bmatrix} 2 & 4 & 5 \end{bmatrix}^T.$$

For the eigenvalue $\lambda_2=10$, by $(A-\lambda_2 I)x=0$, i.e.,
$$\begin{bmatrix} -8 & 2 & -2 \\ 2 & -5 & -4 \\ -2 & -4 & -5 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

We can get $x_3 =(1, 2, -2)^{\mathrm{T}}$ and the corresponding unit vector:
$$\gamma_3 = \tfrac{1}{3}\begin{bmatrix} 1, & 2, & -2 \end{bmatrix}^{\mathrm{T}}.$$

Take the orthogonal matrix
$$Q = \begin{bmatrix} \gamma_1, \gamma_2, \gamma_3 \end{bmatrix} = \begin{bmatrix} -2\sqrt{5}\big/5 & 2\sqrt{5}\big/15 & 1\big/3 \\ \sqrt{5}\big/5 & 4\sqrt{5}\big/15 & 2\big/3 \\ 0 & \sqrt{5}\big/3 & -2\big/3 \end{bmatrix}$$
which will make
$$Q^{-1}AQ = \mathrm{diag}(\lambda_1, \lambda_2, \lambda_3) = \mathrm{diag}(1, 1, 10).$$

III. Unitary Matrices

A *real* orthogonal matrix— $Q^{\mathrm{T}}Q = I$.

For *complex* matrices, the transpose will be replaced by the conjugate transpose. The condition will become $U^H U = I$.

The new letter U reflects the new name: *A complex matrix with orthonormal columns is called a unitary matrix*.

Definition 2   A matrix U is called a **unitary matrix** (酉矩阵) if $U^H = U^{-1}$.

Equivalently, $U^H U = I$, and $UU^H = I$.

Unitary matrices have many nice properties.

Theorem 2 Let U be a unitary matrix. Then the following hold.

1. *Inner products and lengths are preserved by U.*

Proof. $(Ux)^H(Uy) = x^H U^H U y = x^H y$, and $\|Ux\|^2 = \|x\|^2$.

2. *Every eigenvalue of U has absolute value $|\lambda| = 1$.*

Proof. This follows directly from $Ux = \lambda x$, by comparing the lengths of the two sides: $\|Ux\| = \|x\|$, and always $\|\lambda x\| = |\lambda| \cdot \|x\|$.

Therefore $|\lambda| = 1$.

3. *Eigenvectors of U corresponding to different eigenvalues are orthogonal (and can be scaled to orthonormal).*

Proof. Start with $Ux = \lambda_1 x$ and $Uy = \lambda_2 y$, and take inner products:
$$x^H y = (Ux)^H(Uy) = (\lambda_1 x)^H(\lambda_2 y) = \overline{\lambda_1}\lambda_2 x^H y.$$

Comparing the left to the right, $\overline{\lambda_1}\lambda_2 = 1$ or $x^H y = 0$. But Property 2 is $\overline{\lambda_1}\lambda_1 = 1$, so we cannot also have $\overline{\lambda_1}\lambda_2 = 1$. Thus $x^H y = 0$ and the eigenvectors are orthogonal (*and can be scaled to unit length*).

Check the properties by working on the following matrices.

Example 3 $U = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$ has eigenvalues $\mathbf{e}^{i\theta}$ and $\mathbf{e}^{-i\theta}$.

The orthonormal eigenvectors are $\boldsymbol{x}_1 = \dfrac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ -i \end{bmatrix}$ and $\boldsymbol{x}_2 = \dfrac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ i \end{bmatrix}$.

Example 4 $P = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{bmatrix}$ has eigenvalues $-1, i, -i, 1$.

The orthonormal eigenvectors are $\begin{bmatrix} -\frac{1}{2} \\ \frac{1}{2} \\ -\frac{1}{2} \\ \frac{1}{2} \end{bmatrix}, \begin{bmatrix} \frac{1}{2} \\ \frac{i}{2} \\ -\frac{1}{2} \\ -\frac{i}{2} \end{bmatrix}, \begin{bmatrix} \frac{1}{2} \\ -\frac{i}{2} \\ -\frac{1}{2} \\ \frac{i}{2} \end{bmatrix}, \begin{bmatrix} -\frac{1}{2} \\ -\frac{1}{2} \\ -\frac{1}{2} \\ -\frac{1}{2} \end{bmatrix}$.

Let A be a matrix of degree n. Let λ be an eigenvalue of A.

The *eigenspace* V_λ is the subspace spanned by the eigenvectors of A corresponding to λ.

By Gram-Schmidt procedure, an eigenspace has an orthonormal basis.

Lemma 1 *A Hermitian matrix has a complete set of orthonormal eigenvectors.* (more discussions in Section 5.6)

Remark Assume that A is Hermitian. From each eigenspace of A, choose an orthonormal basis by Gram-Schmidt process.

Since any two vectors corresponding to different eigenvalues are orthogonal, the eigenvectors in these orthonormal bases are orthonormal, i.e., there are n eigenvectors of A which are orthonormal.

Let A be a Hermitian matrix of degree n, and let $v_1, v_2, \ldots, v_n$ be a complete set of orthonormal eigenvectors, corresponding to eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_n$ respectively.

Let $U = [v_1, v_2, \ldots, v_n]$, then U is a unitary matrix, and
$$AU = A[v_1 \ v_2 \ldots v_n] = [Av_1 \ Av_2 \ldots Av_n]$$
$$= [\lambda_1 v_1 \ \lambda_2 v_2 \ \ldots \ \lambda_n v_n] = U \text{diag}(\lambda_1, \lambda_2, \ldots, \lambda_n).$$

Thus U diagonalizes A: $U^{-1}AU = \text{diag}(\lambda_1, \lambda_2, \ldots, \lambda_n).$

This gives the following important theorem.

Theorem 3 (*Spectral Theorem*)

(1) *Each real symmetric matrix A can be diagonalized by an orthogonal matrix Q.*

(2) *Every Hermitian matrix A can be diagonalized by a unitary matrix U.*

The columns of Q (or U) consist of orthonormal eigenvectors of A.

Skew-Hermitian matrices

- Skew-symmetric matrices satisfy $K^{\mathrm{T}} = -K$.

- Skew-Hermitian matrices (反厄米特矩阵) satisfy $K^{\mathrm{H}} = -K$.

Property   If A is Hermitian then $K = iA$ is skew-Hermitian.

(i.e., If $A = A^{\mathrm{H}}$, and $K = iA$, then $K^{\mathrm{H}} = -K$.)

Remark The eigenvalues of K are purely imaginary instead of purely real (反厄米特矩阵的特征值是纯虚数). For example,

$A = \begin{bmatrix} 2 & 3 - 3i \\ 3 + 3i & 5 \end{bmatrix} = A^{\mathrm{H}}$, so A is a Hermitian matrix.

$K = iA = \begin{bmatrix} 2i & 3 + 3i \\ -3 + 3i & 5i \end{bmatrix} = -K^{\mathrm{H}}$.

The diagonal entries are multiples of i (allowing zero).

The eigenvalues are $8i$ and $-i$.

The eigenvectors are still orthogonal, and we still have $K = U\Lambda U^{\mathrm{H}}$ — with a unitary U instead of a real orthogonal Q, and with $8i$ and $-i$ on the diagonal of Λ.

Real versus Complex

$\mathbf{R}^n$ (n real components)	$\leftrightarrow$	$\mathbf{C}^n$ (n complex components)
length: $\|x\|^2 = x_1^2 + \cdots + x_n^2$	$\leftrightarrow$	length: $\|x\|^2 = |x_1|^2 + \cdots + |x_n|^2$
transpose: $A_{ij}^{\mathrm{T}} = A_{ji}$	$\leftrightarrow$	Hermitian transpose: $A_{ij}^{\mathrm{H}} = \overline{A_{ji}}$
$(AB)^{\mathrm{T}} = B^{\mathrm{T}} A^{\mathrm{T}}$	$\leftrightarrow$	$(AB)^{\mathrm{H}} = B^{\mathrm{H}} A^{\mathrm{H}}$
inner product: $x^{\mathrm{T}} y = x_1 y_1 + \cdots + x_n y_n$	$\leftrightarrow$	inner product: $x^{\mathrm{H}} y = \bar{x}_1 y_1 + \cdots + \bar{x}_n y_n$
$(Ax)^{\mathrm{T}} y = x^{\mathrm{T}} (A^{\mathrm{T}} y)$	$\leftrightarrow$	$(Ax)^{\mathrm{H}} y = x^{\mathrm{H}} (A^{\mathrm{H}} y)$
orthogonality: $x^{\mathrm{T}} y = 0$	$\leftrightarrow$	orthogonality: $x^{\mathrm{H}} y = 0$
symmetric matrices: $A^{\mathrm{T}} = A$	$\leftrightarrow$	Hermitian matrices: $A^{\mathrm{H}} = A$
$A = Q\Lambda Q^{-1} = Q\Lambda Q^{\mathrm{T}}$ (real Λ)	$\leftrightarrow$	$A = U\Lambda U^{-1} = U\Lambda U^{\mathrm{H}}$ (real Λ)
skew-symmetric $K^{\mathrm{T}} = -K$	$\leftrightarrow$	skew-Hermitian $K^{\mathrm{H}} = -K$
orthogonal $Q^{\mathrm{T}} Q = I$ or $Q^{\mathrm{T}} = Q^{-1}$	$\leftrightarrow$	unitary $U^{\mathrm{H}} U = I$ or $U^{\mathrm{H}} = U^{-1}$
$(Qx)^{\mathrm{T}}(Qy) = x^{\mathrm{T}} y$ and $\|Qx\| = \|x\|$	$\leftrightarrow$	$(Ux)^{\mathrm{H}}(Uy) = x^{\mathrm{H}} y$ and $\|Ux\| = \|x\|$

The columns, rows, and eigenvectors of Q and U are orthonormal, and every $|\lambda| = 1$

Key words:

Real Hermitian matrices are symmetric; real unitary matrices are orthogonal.

Spectral Theorem:
*(1) Each real symmetric matrix **A** can be diagonalized by an orthogonal matrix **Q**.*
*(2) Every Hermitian matrix **A** can be diagonalized by a unitary matrix **U**.*

Homework

See Blackboard