COMP6212: Computational Finance

Mahesan Niranjan

School of Electronics and Computer Science
University of Southampton

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This module is in two parts:

- Part I — Financial data analysis, taught by Prof. M. Niranjan
- Part II — Crypto-currencies and blockchain technology, taught by Dr Jie Zhang

Spring Semester 2017/2018


jamesnichollsillustration.blogspot.co.uk

Financial Markets


www.investors411.com

- Generate products and services
- In need of
  - stability against fluctuations (e.g. demand, exchange rate)
  - capital investment (e.g. to modernise, grow)

- Pocess wealth & capital
- Driven by gambling instinct and greed

The Setting

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- What are the sources of computational problems?
  - Time - present value of money.
  - Uncertainty - of the future.

Overview of the Module

Topics in Part I: Financial Data Analysis

- Portfolio Optimization
- Derivatives Pricing

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Keywords:

Mean-Variance optimization, Linear and quadratic programming, Multivariate Gaussian distribution, Constrained optimization, Value at risk and Conditional value at risk, Sharpe ratio, Present value, Stochastic differential equations, Ito's Lemma, Black-Scholes model, Options pricing, Stochastic Simulations and Monte Carlo methods.

Resources


PAOLO BRANDIMARTE
Numerical Methods in
Finance and Economics
A MATLAB®-Based Introduction
Second Edition
WILEY
STATISTICS IN PRACTICE

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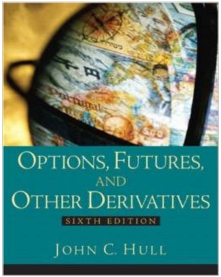
OPTIONS, FUTURES, AND OTHER DERIVATIVES
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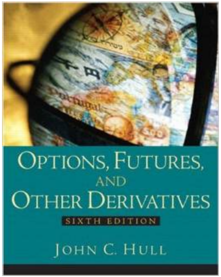
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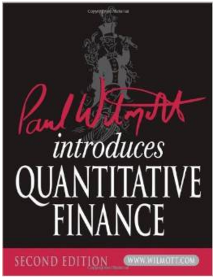
Paul Wilmott
introduces
QUANTITATIVE FINANCE
SECOND EDITION
WWW.WILMOTT.COM

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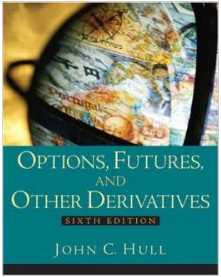

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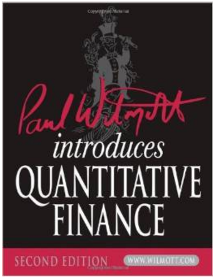

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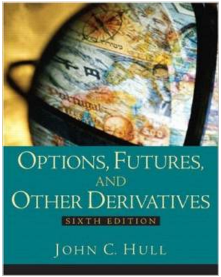
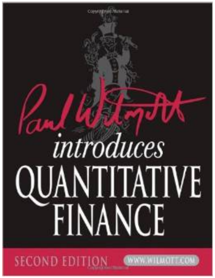
Time Series Analysis
James D. Hamilton

Resources





- plus several academic papers.

Financial Instruments (broad classes)

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- Derivatives
  - Contracts written on the basis of a future value of a a stock, currency etc. Usually there is a time of *maturity* and a promised *payoff* in the contract. Variations in style of *exercising* the contract.

Time: Present Value

- Wealth W_0 deposit in bank and get W_1 after one year
- $W_1 \;=\; (1+r)\,W_0, \quad r$ interest rate
- Compound interest over n years: $W_n \;=\; (1+r)^n\,W_0$

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- Continuous compounding $m \rightarrow \infty$

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- Present value of your promise to give me cash C in time t is

$$\exp(-rt)\, C$$

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  - Inducing sparsity – l_1 or *lasso* regularization
  - Convex optimization using `CVX` toolbox

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$$dG = \left(\mu S \frac{\partial G}{\partial S} + \frac{\sigma^2 S^2}{2} \frac{\partial^2 G}{\partial S^2} + \frac{\partial G}{\partial t} \right) dt + \sigma S \frac{\partial G}{\partial S} \, dZ$$

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- Black-Scholes: options pricing under specific assumptions
- Monte Carlo / Stochastic simulations: general cases

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- Returns on the portfolio has a multivariate Gaussian distribution

$$\boldsymbol{r} = \begin{pmatrix} r_1 \\ r_2 \\ \vdots \\ r_N \end{pmatrix} \quad \boldsymbol{\mu} \begin{pmatrix} \mu_1 \\ \mu_2 \\ \vdots \\ \mu_N \end{pmatrix} \quad \boldsymbol{\Sigma} = \begin{pmatrix} \sigma_1^2 & \sigma_{12}^2 & \dots & \sigma_{1N}^2 \\ \sigma_{12}^2 & \sigma_2^2 & \dots & \sigma_{2N}^2 \\ \vdots & \vdots & \vdots & \vdots \\ \sigma_{1N}^2 & \sigma_{2N}^2 & \dots & \sigma_N^2 \end{pmatrix}$$

Gaussian Density

$$p(x) = \frac{1}{\sigma\sqrt{2\pi}} \exp\left\{-\frac{(x-m)^2}{2\sigma^2}\right\}$$

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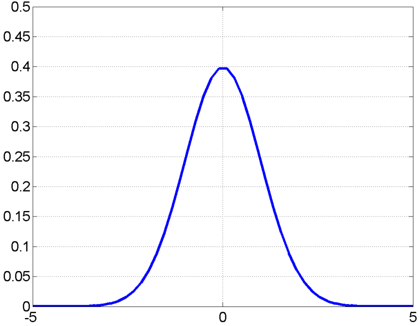


```
x = linspace(-5,5,50);
m = 0;
s = 1;
y = normpdf(x,m,s);
figure(1), clf
plot(x,y,'LineWidth',3);

grid on
```

Multivariate Gaussian Distribution

$$p(\boldsymbol{x}) = \frac{1}{(2\pi)^{d/2} |\boldsymbol{\Sigma}|^{1/2}} \exp \left\{ -\frac{1}{2} (\boldsymbol{x} - \boldsymbol{\mu})^T \boldsymbol{\Sigma}^{-1} (\boldsymbol{x} - \boldsymbol{\mu}) \right\}$$

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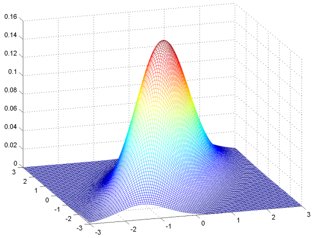
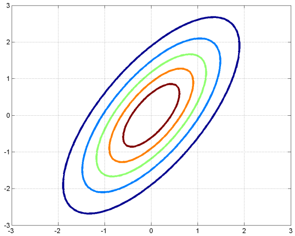
- Parameters: mean vector $\boldsymbol{\mu}$ covariance matrix $\boldsymbol{\Sigma}$





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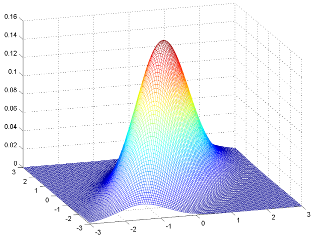

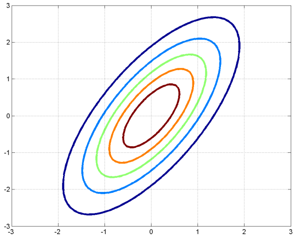

- How do these shapes change with $\boldsymbol{\mu}$ and $\boldsymbol{\Sigma}$?

Portfolio Return

- Linear transform of multivariate Gaussian

$$\boldsymbol{x} \sim \mathcal{N}(\boldsymbol{\mu}, \; \boldsymbol{\Sigma}); \;\; \boldsymbol{y} = \boldsymbol{A}\,\boldsymbol{x} \quad \Longrightarrow \quad \boldsymbol{y} \sim \mathcal{N}\left(\boldsymbol{A}\,\boldsymbol{\mu}, \; \boldsymbol{A}\Sigma\boldsymbol{A}^t\right)$$

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- Mean return $M = \boldsymbol{\pi}^T\,\boldsymbol{\mu}$ and the variance on it $V = \boldsymbol{\pi}^T\,\boldsymbol{\Sigma}\,\boldsymbol{\pi}$
- When π changes, M and V change — how?

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- Other constraints possible: $\sum_{i=1}^{N} \pi_i = 1, \ \pi_i \geq 0, \ \alpha \leq \pi_i \leq \beta$

Estimation

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- Estimation:

$$
\begin{aligned}
\widehat{\boldsymbol{\mu}} &= \frac{1}{T} \sum_{t=1}^{T} \boldsymbol{r}(t) \\
\widehat{\boldsymbol{\Sigma}} &= \frac{1}{T} \sum_{t=1}^{T} \left(\boldsymbol{r}(t) - \widehat{\boldsymbol{\mu}}\right) \left(\boldsymbol{r}(t) - \widehat{\boldsymbol{\mu}}\right)^{T}
\end{aligned}
$$

Solving quadratic program in MATLAB

$$\min_{\boldsymbol{x}} \ \boldsymbol{x}^T \boldsymbol{H} \boldsymbol{x} + \boldsymbol{f}^T \boldsymbol{x} \text{ such that } \begin{cases} \boldsymbol{A} \boldsymbol{x} \leq 0 \\ \boldsymbol{A}_{eq} \boldsymbol{x} = b_{eq} \\ lb \leq \boldsymbol{x} \leq ub. \end{cases}$$

```
x = quadprog(H ,f, A, b, Aeq, beq ,lb, ub, x0)
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For the portfolio optimization problem, we might have:

```
pi = quadprog(Sigma, [], [], [], mu', rMax, 0, 1, [])
```

Solving quadratic program in MATLAB

$$\min_{\boldsymbol{x}} \; \boldsymbol{x}^T \boldsymbol{H} \boldsymbol{x} + \boldsymbol{f}^T \boldsymbol{x} \text{ such that } \begin{cases} \boldsymbol{A} \boldsymbol{x} \leq 0 \\ \boldsymbol{A}_{eq} \boldsymbol{x} = b_{eq} \\ lb \leq \boldsymbol{x} \leq ub. \end{cases}$$

```
x = quadprog(H ,f, A, b, Aeq, beq ,lb, ub, x0)
```

Do `>doc quadprog` in `MATLAB` and read more.

For the portfolio optimization problem, we might have:

```
pi = quadprog(Sigma, [], [], [], mu', rMax, 0, 1, [])
```

Map the problem variables to the function in the tool.

Efficient Frontier

- Given $\boldsymbol{\mu}$ and $\boldsymbol{\Sigma}$
- What portfolio has the highest return, unconstrained by risk?

$$\max \boldsymbol{\pi}^T \boldsymbol{\mu} \quad \text{subject to} \sum_{i=1}^{N} \pi_i = 1, \quad \text{and} \quad \pi_i \geq 0$$

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Linear Programming:

$$\min_{\boldsymbol{x}} \boldsymbol{f}^T \boldsymbol{x} \quad \text{such that} \quad \begin{cases} \boldsymbol{A}\,\boldsymbol{x} \leq \boldsymbol{b} \\ \boldsymbol{A}_{eq}\,\boldsymbol{x} = \boldsymbol{b}_{eq} \\ lb \leq \boldsymbol{x} \leq ub \end{cases}$$

```
x = linprog( f, A, b, Aeq, beq, lb, ub )
w1 = linprog(-mu, [], [], ones(1,N), 1, 0, 0);
r1 = w1 * mu;
```

Efficient Frontier (cont'd)

- What portfolio has lowest variance (unconstrained by expectation)?

$$\min \boldsymbol{\pi}^T \boldsymbol{\Sigma} \boldsymbol{\pi} \quad \text{subject to} \quad \sum_{i=1}^{N} \pi_i = 1$$

Efficient Frontier (cont'd)

- What portfolio has lowest variance (unconstrained by expectation)?

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```
w2=quadprog(mu,zeros(N,1),[],[],ones(1,N),1,zeros(N,1),[],[]);
r2 = w2' * mu;
```

- Portfolios on the efficient frontier will have returns in range `r1` to `r2`

Efficient Frontier (cont'd)

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```
M = linspace(r1, r2, p)
for j=1:p
    ret = M(j);
    w = quadprog(...);
    V(j) = w' * Sigma * w;
end
```

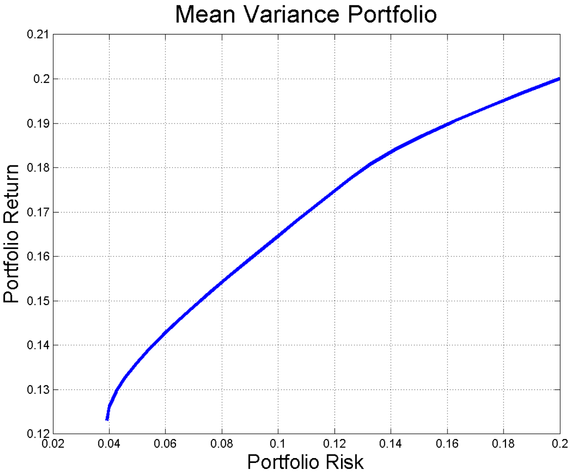
Complete Function to Compute the Efficient Frontier

```
function [PRisk, PRoR, PWts] = NaiveMV(ERet, ECov, NPts)
ERet = ERet(:);       % makes sure it is a column vector
NAssets = length(ERet);  % get number of assets
% vector of lower bounds on weights
V0 = zeros(NAssets, 1);
% row vector of ones
V1 = ones(1, NAssets);
% set medium scale option
options = optimset('LargeScale', 'off');
% Find the maximum expected return
MaxReturnWeights = linprog(-ERet, [], [], V1, 1, V0);
MaxReturn = MaxReturnWeights' * ERet;
% Find the minimum variance return
MinVarWeights = quadprog(ECov,V0,[],[],V1,1,V0,[],[],options);
MinVarReturn = MinVarWeights' * ERet;
MinVarStd = sqrt(MinVarWeights' * ECov * MinVarWeights);
% check if there is only one efficient portfolio
if MaxReturn > MinVarReturn
   RTarget = linspace(MinVarReturn, MaxReturn, NPts);
   NumFrontPoints = NPts;
else
      RTarget = MaxReturn;
      NumFrontPoints = 1;
end
```

Complete Function (cont'd)

```
...
% Store first portfolio
PRoR = zeros(NumFrontPoints, 1);
PRisk = zeros(NumFrontPoints, 1);
PWts = zeros(NumFrontPoints, NAssets);
PRoR(1) = MinVarReturn;
PRisk(1) = MinVarStd;
PWts(1,:) = MinVarWeights(:)';
% trace frontier by changing target return
VConstr = ERet';
A = [V1 ; VConstr ];
B = [1 ; 0];
for point = 2:NumFrontPoints
B(2) = RTarget(point);
Weights = quadprog(ECov,V0,[],[],A,B,V0,[],[],options);
PRoR(point) = dot(Weights, ERet);
PRisk(point) = sqrt(Weights'*ECov*Weights);
PWts(point, :) = Weights(:)';
end
```

Summary: what have we achieved?



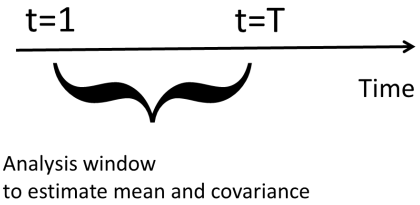
Homework

- Three assets with the following properties:

$$m = \begin{bmatrix} 0.1 \\ 0.2 \\ 0.15 \end{bmatrix}; \quad C = 100 * \begin{bmatrix} 0.005 & -0.010 & 0.004 \\ -0.010 & 0.040 & -0.002 \\ 0.004 & -0.002 & 0.023 \end{bmatrix};$$

- Study the code of function `NaiveMV` and draw the efficient frontier.
- Use the function `frontcon` in `MATLAB` and draw the efficient frontier.

Estimation of Parameters



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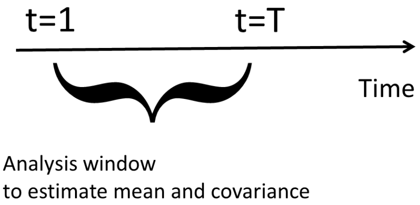

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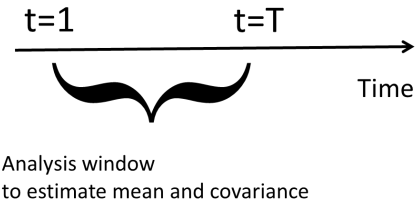

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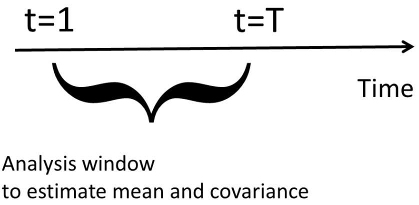

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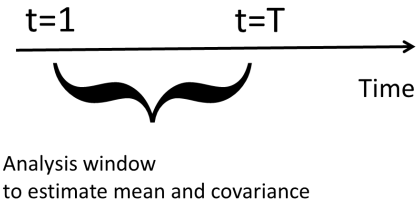

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- Shrinkage in covariance estimates

Advances on the Mean-Variance Portfolio

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- Optimizing the execution of trade

> TBC

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- cVAR: Conditional Value at Risk (later)

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  - Combination of portfolios (model averaging / mixing)
- No method *consistently* beats the naive strategy!

- N assets; $\boldsymbol{r}_t$, return vector at time t
- Expected return and covariance:

$$\boldsymbol{r}_t = \begin{pmatrix} r_{1,t} \\ r_{2,t} \\ \vdots \\ r_{N,t} \end{pmatrix} \quad \boldsymbol{E}\left[\boldsymbol{r}_t\right] = \boldsymbol{\mu} \quad \boldsymbol{E}\left[\left(\boldsymbol{r}_t - \boldsymbol{\mu}\right)\left(\boldsymbol{r}_t - \boldsymbol{\mu}\right)^T\right] = \boldsymbol{C}$$

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- Brodie *et al.* suggest l_1 regularizer

$$\begin{cases} \widehat{\boldsymbol{w}} = \min_{\boldsymbol{w}} \left[\|\rho \mathbf{1}_T - \boldsymbol{R}\,\boldsymbol{w}\|_2^2 + \tau \|\boldsymbol{w}\|_1 \right] \\ \text{subject to } \boldsymbol{w}^T \widehat{\boldsymbol{\mu}} = \rho, \ \boldsymbol{w}^T \mathbf{1}_N = 1 \end{cases}$$

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Brodie *et al.* (2007) (cont'd)

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Brodie *et al.* (2007) (cont'd)

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- Transaction costs:
  - Usually have fixed (overhead) part and transaction-dependent part
  - Institutional investors fixed part negligible
  - Small investors can assume fixed cost only

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- We want to make an adjustment $\boldsymbol{\Delta_w}$, new portfolio $\boldsymbol{w} + \boldsymbol{\Delta_w}$
- Transaction costs only on the adjustments

$$\begin{cases} \boldsymbol{\Delta_w} = \min_{\boldsymbol{\Delta_w}} \left[|| \rho \mathbf{1}_T - \boldsymbol{R} \left(\boldsymbol{w} + \boldsymbol{\Delta_w} \right) ||_2^2 + \tau || \boldsymbol{\Delta_w} ||_1 \right] \\ \text{subject to } \boldsymbol{\Delta_w}^T \widehat{\boldsymbol{\mu}} = 0 \text{ and } \boldsymbol{\Delta_w}^T \mathbf{1}_N = 1 \end{cases}$$

Brodie *et al.* (2007) (cont'd)

Homework:

Coursework 1 will involve confirming some claims in Brodie *et al.*'s paper. Please download the paper and start reading.

Convex Optimization: CVX

- We will use the `CVX` toolbox within `MATLAB` to implement optimization
- `http://cvxr.com/`

Convex Optimization: CVX

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- `http://cvxr.com/`
- Download, uncompress, set `MATLAB` to the `cvx` directory and do `cvx_setup`
- Take `MATLAB` back into your working directory

Example of using CVX

```
T = 150; N =  50;
R = randn(T, N);
rho = 0.02;
tau = 1;
mu = rand(N,1);

cvx_begin quiet
variable w(N)
   minimize( norm(rho*ones(T,1)-R*w) + tau*norm(w,1) )
   subject to
         w'*ones(N,1) == 1;
         w'*mu == rho;
         w > 0;
cvx_end

figure(1), clf, bar(w); grid on
```

Note: Data random - probably won't work all the time

- Portfolio weights: $\boldsymbol{w} = [w_1 \; w_2 \; ... \; w_n]^T$
- Returns: $\boldsymbol{a}$; $\quad E[\boldsymbol{a}] = \overline{\boldsymbol{a}}$; $\quad E\left[(\boldsymbol{a} - \overline{\boldsymbol{a}})(\boldsymbol{a} - \overline{\boldsymbol{a}})^T\right] = \boldsymbol{\Sigma}$

Portfolio Optimization with Transaction Costs

Lobo *et al.* (2007) *Ann Oper Res* **152**:341

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$$\begin{aligned} E[W] &= \overline{\boldsymbol{a}}^T(\boldsymbol{w} + \boldsymbol{x}) \\ E\left[(W - E[W])^2\right] &= (\boldsymbol{w} + \boldsymbol{x})^T \boldsymbol{\Sigma} (\boldsymbol{w} + \boldsymbol{x}) \end{aligned}$$

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$$

- Transaction cost: $\phi(\boldsymbol{x})$
- Budget Constraint:

$$
\mathbf{1}^T \boldsymbol{x} + \phi(\boldsymbol{x}) \le 0
$$

Lobo *et al.* (2007) (cont'd)

Possible Optimizations:

Lobo *et al.* (2007) (cont'd)

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$\mathcal{S}$ some feasible set (with other constraints)

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$$\begin{array}{ll} \underset{\boldsymbol{x}}{\text{minimize}} & \phi(\boldsymbol{x}) \\ \text{subject to} & \bar{\boldsymbol{a}}^T (\boldsymbol{w} + \boldsymbol{x}) \ge r_{\min} \\ & \boldsymbol{w} + \boldsymbol{x} \in \mathcal{S} \end{array}$$

Costs are separable (usual assumption):

$$\phi(\boldsymbol{x}) = \sum_{i=1}^{n} \phi_i(x_i)$$

Lobo *et al.* (2007) (cont'd)

Modeling Transaction Costs

Costs are separable (usual assumption):

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$$\phi_i(x_i) \;=\; \begin{cases} \alpha_i^{+}\, x_i, & x_i \geq 0 \\ -\,\alpha_i^{-}\, x_i, & x_i \leq 0 \end{cases}$$

- α_i^{+} and α_i^{-} cost rates for buying and selling asset i

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Fixed plus linear transaction costs:

$$\phi_i(x_i) = \begin{cases} 0, & x_i = 0 \\ \beta_i^+ + \alpha_i^+ x_i, & x_i \geq 0 \\ \beta_i^- - \alpha_i^- x_i, & x_i \leq 0 \end{cases}$$

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- This is non-convex

Diversification Constraints

- Limit the amount of investment in any asset

$$w_i + x_i \leq p_i, \quad i = 1, 2, ..., n$$

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- Limit the fraction of total wealth held in each asset

$$w_i + x_i \le \mathbf{1}^T \, (\boldsymbol{w} + \boldsymbol{x})$$

- Limit exposure in any small group of assets (say in a sector)

$$\sum_{i=1}^{r} (w_i + x_i)_{[i]} \le \mathbf{1}^T \, (\boldsymbol{w} + \boldsymbol{x})$$

(Tricky, but can show this is convex – see Eqn (11) in paper)

- Constraints on short-selling

  ... on individual asset

$$w_i + x_i \geq -s_i, \quad i = 1, ..., n$$

Lobo *et al.* (2007) (cont'd)

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Lobo *et al.* (2007) (cont'd)

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- Collateralization:

$$\sum_{i=1}^{n} (w_i + x_i)_- \leq \gamma \sum_{i=1}^{n} (w_i + x_i)_+$$

What I have borrowed to sell is smaller than a fraction of what I own

Variance:

$$(\boldsymbol{w} + \boldsymbol{x})^T \, \Sigma \, (\boldsymbol{w} + \boldsymbol{x}) \leq \sigma_{\max}$$

Lobo *et al.* (2007) (cont'd)

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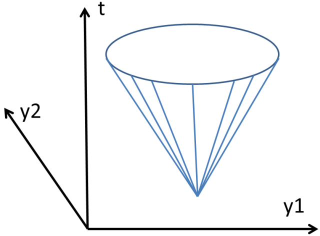
- This is *Second Order Cone* constraint.



Lobo *et al.* (2007) (cont'd)

Shortfall Risk Constraint:

$$P\left(W \geq W^{\text{low}}\right) \;\geq\; \eta$$

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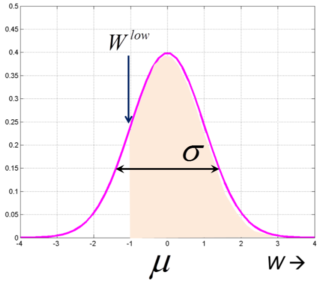

Shortfall Risk Constraint:

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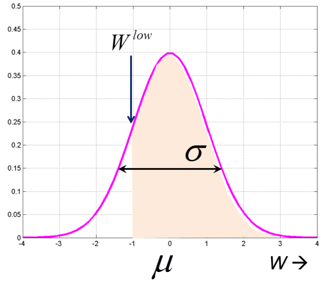

$$P\left(\frac{W - \mu}{\sigma} \leq \frac{W^{\text{low}} - \mu}{\sigma}\right) \leq 1 - \eta$$

Lobo *et al.* (2007) (cont'd)

- But $(W - \mu)/\sigma \;\sim\; \mathcal{N}(0,\, 1)$

Lobo *et al.* (2007) (cont'd)

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- Hence

$$P\left(\frac{W - \mu}{\sigma} \leq \frac{W^{\mathrm{low}} - \mu}{\sigma}\right) = \Phi\left(\left(W^{\mathrm{low}} - \mu\right) / \sigma\right)$$

$$\Phi(z) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{z} \exp\left\{-\frac{t^2}{2}\right\} dt$$

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$$\Phi(z) \;=\; \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{z} \exp\left\{-\frac{t^2}{2}\right\}\,dt$$

- $\frac{W^{\text{low}} - \mu}{\sigma} \;\leq\; \Phi^{-1}(1-\eta)$

Lobo *et al.* (2007) (cont'd)

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- $\frac{W^{\text{low}} - \mu}{\sigma} \leq \Phi^{-1}(1 - \eta)$
- $\Phi^{-1}(1 - \eta) = -\Phi^{-1}(\eta)$

$$\mu - W^{\text{low}} \geq \Phi^{-1}(\eta)\, \sigma$$

Lobo *et al.* (2007) (cont'd)

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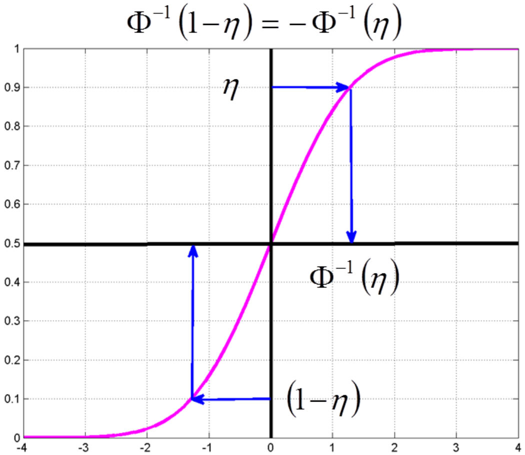
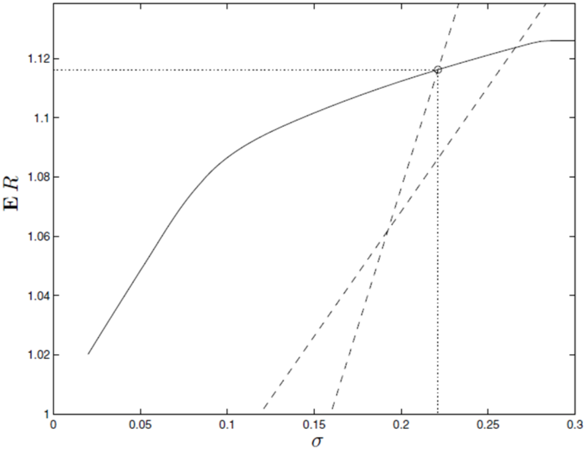
- $\Phi^{-1}(1-\eta) \;=\; -\Phi^{-1}(\eta)$

$$\mu \;-\; W^{\rm low} \;\geq\; \Phi^{-1}(\eta)\,\sigma$$

- Using $\mu = \overline{\boldsymbol{a}}^T(\boldsymbol{w}+\boldsymbol{x})$ and $\sigma^2 = (\boldsymbol{w}+\boldsymbol{x})^T\boldsymbol{\Sigma}(\boldsymbol{w}+\boldsymbol{x})$

$$\Phi^{-1}(\eta)\,||\boldsymbol{\Sigma}^{1/2}(\boldsymbol{w}+\boldsymbol{x})|| \;\leq\; \overline{\boldsymbol{a}}^T(\boldsymbol{w}+\boldsymbol{x}) \;-\; W^{\rm low}$$


$\Phi^{-1}(1-\eta) = -\Phi^{-1}(\eta)$
η
$\Phi^{-1}(\eta)$
$(1-\eta)$
1
0.9
0.8
0.7
0.6
0.5
0.4
0.3
0.2
0.1
0
-4
-3
-2
-1
0
1
2
3
4


1.12
1.1
1.08
1.06
1.04
1.02
1
E R
0
0.05
0.1
0.15
0.2
0.25
0.3
σ

$$\text{maximize} \quad \bar{\boldsymbol{a}}^T \left(\boldsymbol{w} + \boldsymbol{x}^+ - \boldsymbol{x}^- \right)$$

$$\text{subject to} \quad \mathbf{1}^T \left(\boldsymbol{x}^+ - \boldsymbol{x}^- \right) + \sum_{i=1}^{n} \left(\alpha_i^+ x_i^+ + \alpha_i^- x_i^- \right) \leq 0$$

$$x_i^+ \geq 0, \; x_i^- \geq 0, \; i = 1, 2, ..., n$$

$$w_i + x_i^+ - x_i^- \geq s_i, \; i = 1, 2, ..., n$$

$$\Phi^- 1(\eta_j) \, || \boldsymbol{\Sigma}^{1/2} \left(\boldsymbol{w} + \boldsymbol{x}^+ - \boldsymbol{x}^- \right) || \leq \bar{\boldsymbol{a}}^T \left(\boldsymbol{w} + \boldsymbol{x}^+ - \boldsymbol{x}^- \right) - W_j^{\text{low}}, j = 1, 2$$

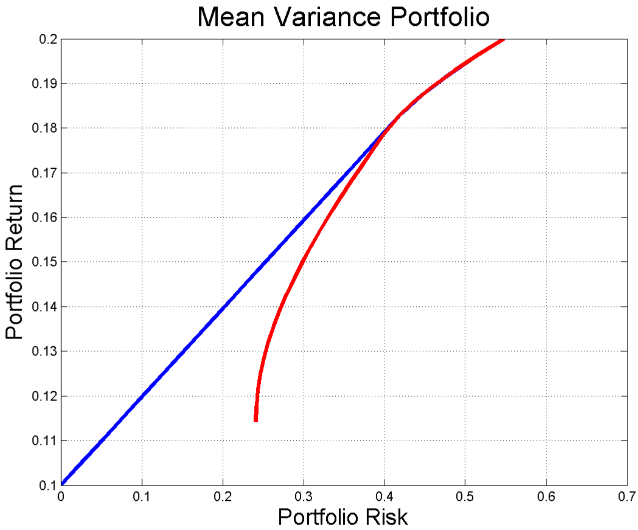
Puzzle

```
m1 = [0.15 0.2 0.08 0.1]';
C1 = [ 0.2    0.05    -0.01    0.0
       0.05   0.30     0.015   0.0
      -0.01   0.015    0.10    0.0
       0.0    0.0      0.0     0.0
       ];

m2 = [0.15 0.2 0.08]';
C2 = [ 0.2    0.05    -0.01
       0.05   0.30     0.015
      -0.01   0.015    0.10
       ];

[V1, M1, PWts1] = NaiveMV(m1, C1, 25);
[V2, M2, PWts2] = NaiveMV(m2, C2, 25);

figure(2), clf,
plot(V1, M1, 'b', V2, M2, 'r', 'LineWidth', 3),
title('Mean Variance Portfolio', 'FontSize', 22)
xlabel('Portfolio Risk', 'FontSize',18)
ylabel('Portfolio Return', 'FontSize', 18);
```


Mean Variance Portfolio
Portfolio Return
Portfolio Risk
0.2
0.19
0.18
0.17
0.16
0.15
0.14
0.13
0.12
0.11
0.1
0
0.1
0.2
0.3
0.4
0.5
0.6
0.7

Derivatives Pricing

- Efficiency, no-arbitrage and fair price

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  - At time T, A owes the bank $S(0)\exp(rT)$ and has the asset to sell to B.

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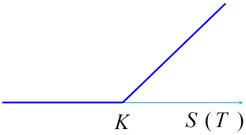
  - At time T, A owes the bank $S(0)\exp(rT)$ and has the asset to sell to B.
  - $F = S(0)\exp(rT)$, else arbitrage opportunity

Options

- Call: right to buy at price K at time T



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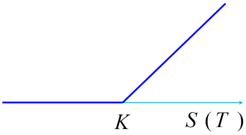
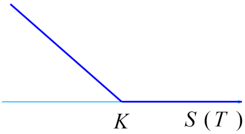
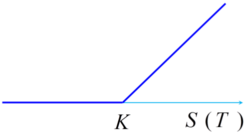


- Put: right to sell at price K at time T

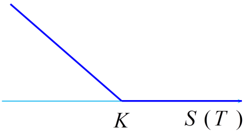


Options

- Call: right to buy at price K at time T



- Put: right to sell at price K at time T



- Exercise of contract
  - European style: only at time T
  - American style: any time in $0 \rightarrow T$

- Example: Put-Call Parity
  - Portfolio P_1: European Call + cash $K \exp(-rT)$

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  - Portfolio P_1: European Call + cash $K \exp(-rT)$
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  - Values at time $t = 0$

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$$P_1 \quad C \; + \; K \exp -rT$$

- Example: Put-Call Parity

  - Portfolio P_1: European Call + cash $K\exp(-rT)$
  - Portfolio P_2: European Put + one share of underlying stock
  - Values at time $t=0$

$$
\begin{array}{ll}
P_1 & C + K\exp{-rT} \\
P_2 & P + S(0)
\end{array}
$$

- Example: Put-Call Parity
  - Portfolio P_1: European Call + cash $K\exp(-rT)$
  - Portfolio P_2: European Put + one share of underlying stock
  - Values at time $t = 0$

$$\begin{array}{ll} P_1 & C \;+\; K\exp -rT \\ P_2 & P \;+\; S(0) \end{array}$$

  - Value of portfolios at time $t = T$

$$S(T) > K \quad P_1 \quad [S(T) - K] + K \;=\; S(T)$$

- Example: Put-Call Parity

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  - Portfolio P_2: European Put + one share of underlying stock
  - Values at time $t = 0$

$$\begin{array}{ll} P_1 & C \;+\; K\exp{-rT} \\ P_2 & P \;+\; S(0) \end{array}$$

  - Value of portfolios at time $t = T$

$$\begin{array}{lll} S(T) > K & P_1 & [S(T) - K] + K \;=\; S(T) \\ & P_2 & 0 \;+\; S(T) \;=\; S(T) \end{array}$$

- Example: Put-Call Parity

  - Portfolio P_1: European Call + cash $K\exp(-rT)$
  - Portfolio P_2: European Put + one share of underlying stock
  - Values at time $t = 0$

$$\begin{array}{ll} P_1 & C \,+\, K\exp -rT \\ P_2 & P \,+\, S(0) \end{array}$$

  - Value of portfolios at time $t = T$

$$\begin{array}{lll} S(T) > K & P_1 & [S(T) - K] + K \,=\, S(T) \\ & P_2 & 0 \,+\, S(T) \,=\, S(T) \\ S(T) < K & P_1 & 0 \,+\, K \,=\, K \end{array}$$

- Example: Put-Call Parity
  - Portfolio P_1: European Call + cash $K\exp(-rT)$
  - Portfolio P_2: European Put + one share of underlying stock
  - Values at time $t = 0$

$$\begin{array}{ll} P_1 & C + K\exp -rT \\ P_2 & P + S(0) \end{array}$$

  - Value of portfolios at time $t = T$

$$\begin{array}{lll} S(T) > K & P_1 & [S(T) - K] + K = S(T) \\ & P_2 & 0 + S(T) = S(T) \\ S(T) < K & P_1 & 0 + K = K \\ & P_2 & [K - S(T)] + S(T) = K \end{array}$$

- Example: Put-Call Parity
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  - Both portfolios having the same value at time $t = T$ should also have the same value at $t = 0$.

$$C \,+\, K\exp(-rT) \,=\, P \,+\, S(0)$$

- Geometric Brownian motion for stock price

$$
\begin{aligned}
dS(t) &= \mu S(t)dt + \sigma S(t)dW(t) \\
\frac{dS(t)}{S(t)} &= \mu dt + \sigma dW(t)
\end{aligned}
$$

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- Stochastic differential equation for the log of the process

$$F(S,t) = \log S(t)$$

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\end{aligned}
$$

- Stochastic differential equation for the log of the process

$$
F(S, t) = \log S(t)
$$

- Ito's lemma tells us about increments dF
- Terms needed to apply Ito's lemma

$$
\boxed{
\begin{aligned}
\frac{\partial F}{\partial t} &= 0 \\
\frac{\partial F}{\partial S} &= \frac{1}{S} \\
\frac{\partial^2 F}{\partial S^2} &= -\frac{1}{S^2}
\end{aligned}
}
$$

$$dF = \left(\frac{\partial F}{\partial t} + \mu S \frac{\partial F}{\partial S} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 F}{\partial S^2} \right) dt + \sigma S \frac{\partial F}{\partial S} dW$$

$$
\begin{aligned}
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&= \left(\mu - \frac{\sigma^2}{2}\right) dt + \sigma dW
\end{aligned}
$$

$$
\log S(t) = \log S(0) + \left(\mu - \frac{\sigma^2}{2}\right) t + \sigma dW(t)
$$

$$
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$$

$$
\log S(t) = \log S(0) + \left(\mu - \frac{\sigma^2}{2}\right) t + \sigma dW(t)
$$

- $dW(t) = \epsilon\sqrt{t}$ where $\epsilon \sim \mathcal{N}(0,1)$

$$
\log S(t) \sim \mathcal{N}\left[\log S(0) + \left(\mu - \frac{\sigma^2}{2}\right) t, \;\; \sigma^2 t\right]
$$

- Log of asset price has a normal distribution

$$
\begin{aligned}
dF &= \left(\frac{\partial F}{\partial t} + \mu S \frac{\partial F}{\partial S} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 F}{\partial S^2} \right) dt + \sigma S \frac{\partial F}{\partial S} dW \\
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$$

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$$
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$$

- Log of asset price has a normal distribution
- Also

$$
S(t) = S(0) \exp \left((\mu - \sigma^2/2) t + \sigma \sqrt{t} \epsilon \right)
$$

Black-Scholes Model

- Model

$$dS(t) = \mu S(t)dt + \sigma S(t)dW(t)$$

Black-Scholes Model

- Model

$$dS(t) = \mu S(t)dt + \sigma S(t)dW(t)$$

- Change in option price

$$df = \frac{\partial f}{\partial t}dt + \frac{\partial f}{\partial S}dS + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 f}{\partial S^2}dt$$

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- At maturity

$$f(S(T), T) = \max\{S(T) - K, 0\}$$

Black-Scholes Model

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- At maturity

$$f(S(T), T) = \max\{S(T) - K, 0\}$$

- Consider a portfolio
  - Own Δ stocks (long)
  - One call option sold

$$\Pi = \Delta S - f(S, t)$$

$$
\begin{aligned}
d\Pi &= \Delta dS - df \\
&= \left(\Delta - \frac{\partial f}{\partial S}\right) dS - \left(\frac{\partial f}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 f}{\partial S^2}\right) dt
\end{aligned}
$$

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- Term in dS (stochastic) can be eliminated by choosing Δ

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- With this choice of Δ (balance between short and long), the portfolio is riskless.
- $d\Pi = r\,\Pi\,dt$

$$
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- With this choice of Δ (balance between short and long), the portfolio is riskless.
- $d\Pi = r\,\Pi\,dt$
- Eliminating $d\Pi$

$$
\left(\frac{\partial f}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 f}{\partial S^2}\right) dt = r\left(f - S\frac{\partial f}{\partial S}\right) dt
$$

$$
\frac{\partial f}{\partial t} + rS\frac{\partial f}{\partial S} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 f}{\partial S^2} - rf = 0
$$

- Partial differential equation

$$\frac{\partial f}{\partial t} + rS\frac{\partial f}{\partial S} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 f}{\partial S^2} - rf = 0$$

- Boundary condition
    - European Call: $f(S, T) = \max\{S - K, 0\}$
    - European Put: $f(S, T) = \max\{K - S, 0\}$

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  - European Call: $f(S, T) = \max\{S - K, 0\}$
  - European Put: $f(S, T) = \max\{K - S, 0\}$
- Black-Scholes

$$C = S_0 \mathcal{N}(d_1) - K \exp(-rT)\mathcal{N}(d_2)$$

$$\begin{aligned} d_1 &= \frac{\log(S_0/K) + (r + \sigma^2/2)T}{\sigma\sqrt{T}} \\ d_2 &= \frac{\log(S_0/K) + (r - \sigma^2/2)T}{\sigma\sqrt{T}} \\ &= d_1 - \sigma\sqrt{T} \\ \mathcal{N}(x) &= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{x} \exp(-y^2/2)dy \end{aligned}$$

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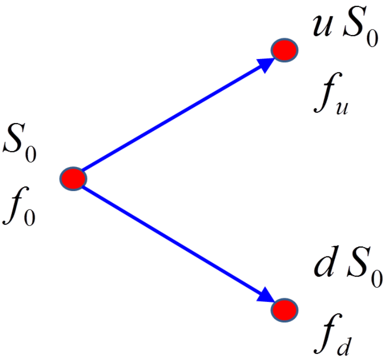
- Put-Call parity

$$P = K \exp(-rT)\mathcal{N}(-d_2) - S_0\mathcal{N}(-d_1)$$

Binomial Lattice



Options Pricing on a Binomial Model

- Construct a portfolio:
  - A riskless bond, initial price $B_0 = 1$ and future value $B_1 = \exp(r\delta t)$
  - Underlying asset, initial value S_0
  - Number of stocks Δ, number of bonds Ψ

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$$\Pi_0 = \Delta S_0 + \Psi$$

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$$\Pi_0 = \Delta S_0 + \Psi$$

- Future value depends on price movement up or down

$$\begin{cases} \Pi_u = \Delta S_0 u + \Psi \exp(r\delta t) \\ \Pi_d = \Delta S_0 d + \Psi \exp(r\delta t) \end{cases}$$

Options Pricing on a Binomial Model

- Construct a portfolio:
  - A riskless bond, initial price $B_0 = 1$ and future value $B_1 = \exp(r\delta t)$
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$$\begin{cases} \Pi_u \; = \; \Delta S_0 u \, + \, \Psi \exp(r\delta t) \\ \Pi_d \; = \; \Delta S_0 d \, + \, \Psi \exp(r\delta t) \end{cases}$$

- We can solve for a portfolio that will replicate option payoff

$$\begin{aligned} \Delta S_0 u \, + \, \Psi \exp(r\delta t) \;\; &= \;\; f_u \\ \Delta S_0 d \, + \, \Psi \exp(r\delta t) \;\; &= \;\; f_d \end{aligned}$$

and solve for Δ and Ψ

- ... solving

$$\Delta = \frac{f_u - f_d}{S_0(u-d)}$$

$$\Psi = \exp(-r\delta t)\frac{uf_d - df_u}{u-d}$$

- ... solving

$$\begin{aligned}\Delta &= \frac{f_u - f_d}{S_0(u-d)} \\ \Psi &= \exp(-r\delta t)\frac{uf_d - df_u}{u-d}\end{aligned}$$

- No arbitrage $\implies$ initial value of this portfolio should be f_0

$$\begin{aligned}f_0 &= \Delta S_0 + \Psi \\ &= \frac{f_u - f_d}{(u-d)} + \exp(-r\delta t)\frac{uf_d - df_u}{u-d} \\ &= \exp(-r\delta t)\left\{\frac{--}{u-d} f_u + \frac{--}{u-d} f_d /\right\}\end{aligned}$$

- ... solving

$$\Delta = \frac{f_u - f_d}{S_0(u-d)}$$
$$\Psi = \exp(-r\delta t)\frac{uf_d - df_u}{u-d}$$

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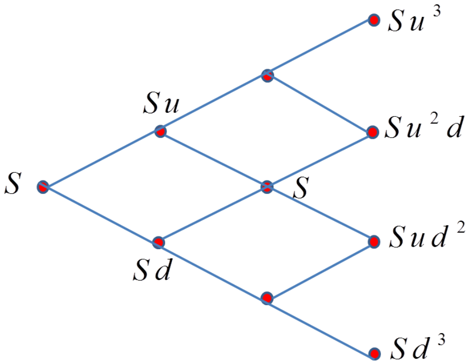
- Defining probabilities

$$\pi_u = \frac{\exp(r\delta t) - d}{u-d} \text{ and } \pi_d = \frac{u - \exp(r\delta t)}{u-d}$$

option price interpreted as discounted expected value

$$f_0 = \exp(-r\delta t)\left(\pi_u f_u + \pi_d f_d\right)$$

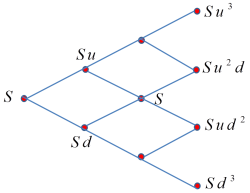
Binomial Lattice



Calibrating a Binomial Lattice

- When are these equivalent?

$$dS = r\,S\,dt + \sigma\,S\,dW$$



Calibrating a Binomial Lattice

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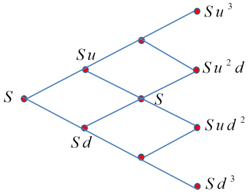

- Log normal distribution

$$\log\left(S_{t+\delta t}\right) \sim \mathcal{N}\left(\left(r - \sigma^2/2\right),\ \sigma^2\,\delta t\right)$$

Calibrating a Binomial Lattice

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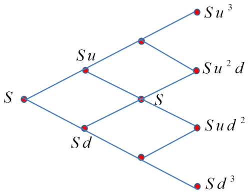

- Log normal distribution

$$\log\left(S_{t+\delta t}\right) \sim \mathcal{N}\left(\left(r - \sigma^2/2\right),\ \sigma^2\,\delta t\right)$$

- Mean and variance of log normal distribution

(log of the variable is normal, what is mean and variance of the variable?)

$$\begin{aligned} E\left[\,S_{t+\delta t}\,\right] &= \exp(r\,\delta t) \\ \mathrm{Var}\left[\,S_{t+\delta t}\,\right] &= \exp(2r\delta t)\left(\exp(\sigma^2\delta t) - 1\right) \end{aligned}$$

Calibrating binomial lattice (cont'd)

Calibrating binomial lattice (cont'd)

- Mean for the lattice

$$E\left[S_{t+\delta t}\right] \;=\; p\,u\,S_t \;+\; (1-p)\,d\,S_t$$

Calibrating binomial lattice (cont'd)

- Mean for the lattice

$$E\left[S_{t+\delta t}\right] \;=\; p\,u\,S_t \;+\; (1-p)\,d\,S_t$$

- Equating the means...

$$puS_t \;+\; (1-p)dS_t \;=\; \exp(r\,\delta t)\,S_t$$

$$p \;=\; \frac{\exp(r\,\delta t) \;-\; d}{u \;-\; d}$$

Calibrating binomial lattice (cont'd)

- Mean for the lattice

$$E\left[S_{t+\delta t}\right] \;=\; p\,u\,S_t \;+\; (1-p)\,d\,S_t$$

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$$puS_t \;+\; (1-p)dS_t \;=\; \exp(r\,\delta t)\,S_t$$

$$p \;=\; \frac{\exp(r\,\delta t) \;-\; d}{u \;-\; d}$$

- Variance on the lattice

$$\begin{aligned}\mathrm{Var}\left[S_{t+\delta t}\right] &= E\left[S^2_{t+\delta t}\right] - E^2\left[S_{t+\delta t}\right] \\ &= S_t^2\left(pu^2 + (1-p)d^2\right) - S_t^2\exp(2r\delta t)\end{aligned}$$

Calibrating binomial lattice (cont'd)

- Mean for the lattice

$$E\left[S_{t+\delta t}\right] \;=\; p\, u\, S_t \;+\; (1-p)\, d\, S_t$$

- Equating the means...

$$puS_t \;+\; (1-p)dS_t \;=\; \exp(r\,\delta t)\, S_t$$

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- Variance on the lattice

$$\begin{aligned} \mathrm{Var}\left[S_{t+\delta t}\right] \;&=\; E\left[S^2_{t+\delta t}\right] \;-\; E^2\left[S_{t+\delta t}\right] \\ &=\; S_t^2\left(pu^2 + (1-p)d^2\right) - S_t^2\exp(2r\delta t) \end{aligned}$$

... which from the dynamical model is...

$$\mathrm{Var}\left[S_{t+\delta t}\right] \;=\; S_t^2\exp(2r\delta t)\left(\exp(\sigma^2\delta t) - 1\right)$$

(cont'd)

- Equating the two variances

$$S_t^2 \exp(2r\delta t)\left(\exp(\sigma^2\delta t) - 1\right) \;=\; S_t^2\left(pu^2 + (1-p)d^2\right) - S_t^2 \exp(2r\delta t)$$

(cont'd)

- Equating the two variances

$$S_t^2 \exp(2r\delta t)\left(\exp(\sigma^2\delta t) - 1\right) \; = \; S_t^2\left(pu^2 + (1-p)d^2\right) - S_t^2 \exp(2r\delta t)$$

Which reduces to

$$\exp(2r\delta t + \sigma^2\delta t) \; = \; pu^2 \; + \; (1-p)d^2$$

(cont'd)

- Equating the two variances

$$S_t^2 \exp(2r\delta t) \left(\exp(\sigma^2 \delta t) - 1\right) \;=\; S_t^2 \left(pu^2 + (1-p)d^2\right) - S_t^2 \exp(2r\delta t)$$

Which reduces to

$$\exp(2r\delta t + \sigma^2 \delta t) \;=\; pu^2 \;+\; (1-p)d^2$$

Substitute for p and simplify

$$\exp(2r\delta t + \sigma^2 \delta t) \;=\; (u+d)\exp(r\delta t) \;-1$$

...and because $u = 1/d$,

$$u^2 \exp(r\delta t) \;-\; u \;\left(1 + \exp(2r\delta t + \sigma^2 \delta t)\right) \;+\; \exp(r\delta t) = \;0$$

... a quadratic equation in u.

$$u \;=\; \frac{\left(1+\exp(2r\delta t+\sigma^2\delta t)\right) \;+\; \sqrt{(1+\exp(2r\delta t+\sigma^2\delta t)^2 - 4\exp(2r\delta t)}}{2\exp(r\delta t)}$$

Taylor series expansion of $\exp(x)$

$$\left(1+\exp(2r\delta t+\sigma^2\delta t)\right)^2 - 4\exp(2r\delta t) \;\approx\; (2+(2r+\sigma^2)\delta t)^2 - 4(1+2r\delta t) \;\approx\; 4\sigma^2\delta t$$

$$\begin{aligned}
u &\approx \frac{2+(2r+\sigma^2)\delta t \;+\; 2\sigma\sqrt{\delta t}}{2\exp(r\delta t)} \\
&\approx \left(1+r\delta t+\frac{\sigma^2}{2}\delta t+\sigma\sqrt{\delta t}\right)(1-r\delta t) \\
&\approx 1+r\delta t+\frac{\sigma^2}{2}\delta t+\sigma\sqrt{\delta t}-r\delta t \\
&= 1+\sigma\sqrt{\delta t}+\frac{\sigma^2}{2}\delta t
\end{aligned}$$

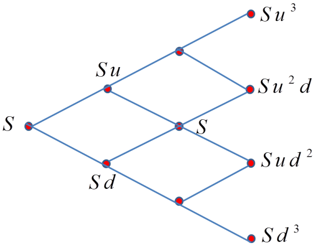
Calibrating the Binomial Lattice (cont'd)

$$u = \exp(\sigma\sqrt{\delta t})$$

$$d = \exp(-\sigma\sqrt{\delta t})$$

$$p = \frac{\exp(r\delta t) - d}{u - d}$$

$$dS = r\,S\,dt + \sigma\,S\,dW$$



Example

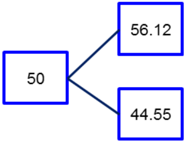
- European call option; $S_0 = K = 50$; $r = 0.1$; $\sigma = 0.4$; maturity in five months.

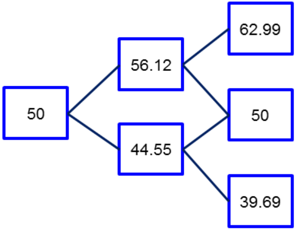
```
>> call = blsprice(50, 50, 0.1, 5/12, 0.4)
call =
      6.1165
```

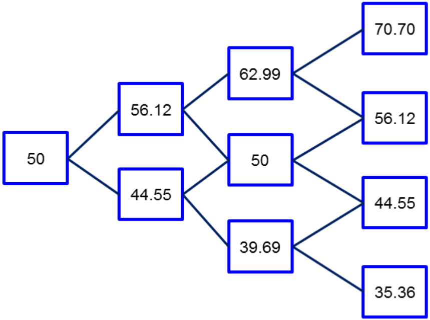
We can now build the lattice

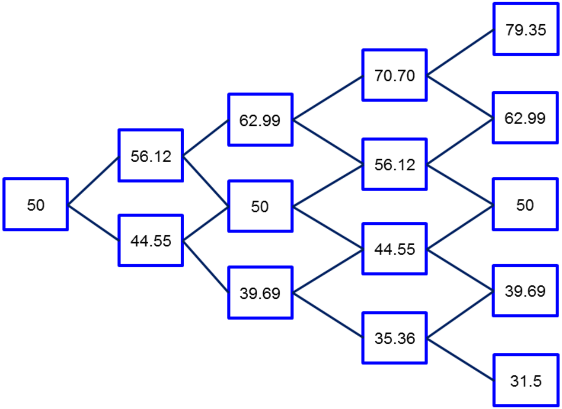
$$
\begin{array}{ccc}
\delta t & 1/12 & 0.0833 \\
u & \exp(\sigma\sqrt{t}) & 1.1224 \\
d & 1\,/\,u & 0.8909 \\
p & (exp(r\delta t) - d)\,/\,(u - d) & 0.5073
\end{array}
$$

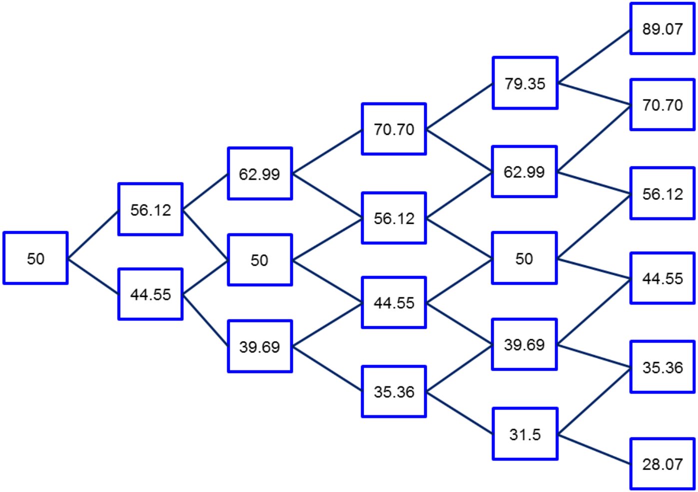
50


50
56.12
44.55


50
56.12
44.55
62.99
50
39.69


50
56.12
44.55
62.99
50
39.69
70.70
56.12
44.55
35.36


50
56.12
44.55
62.99
50
39.69
70.70
56.12
44.55
35.36
79.35
62.99
50
39.69
31.5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44.55
62.99
50
39.69
70.70
56.12
44.55
35.36
79.35
62.99
50
39.69
31.5
89.07
70.70
56.12
44.55
35.36
28.07

39.07

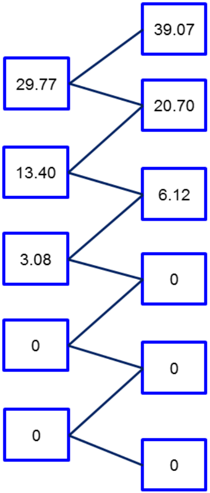
20.70

6.12

0

0

0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29.77
20.70
13.40
6.12
3.08
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0
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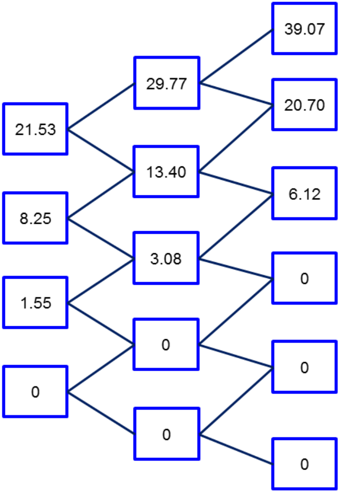
39.07
29.77
20.70
21.53
13.40
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8.25
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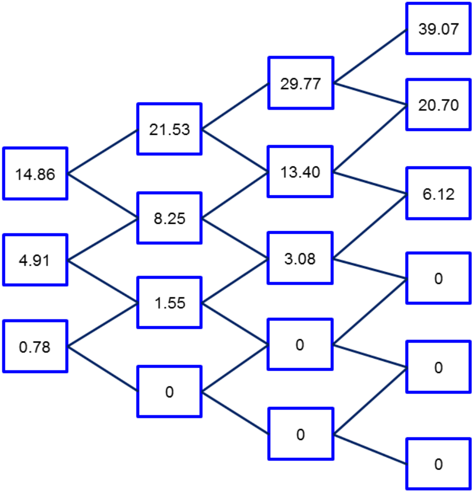
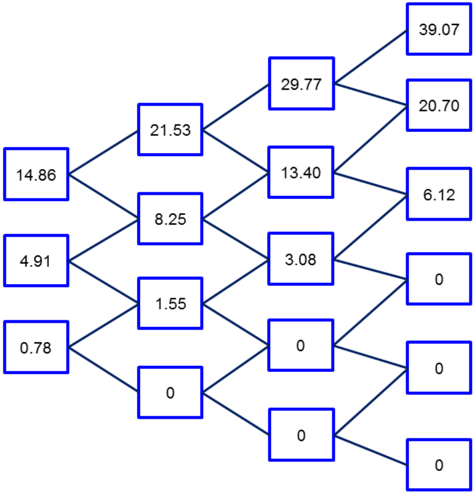

39.07
29.77
20.70
21.53
13.40
14.86
6.12
8.25
3.08
4.91
0
1.55
0
0
0.78
0
0
0


39.07
29.77
20.70
21.53
13.40
14.86
6.12
8.25
3.08
4.91
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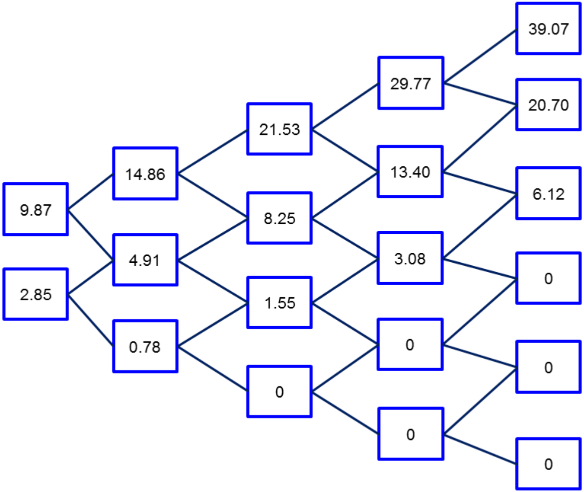
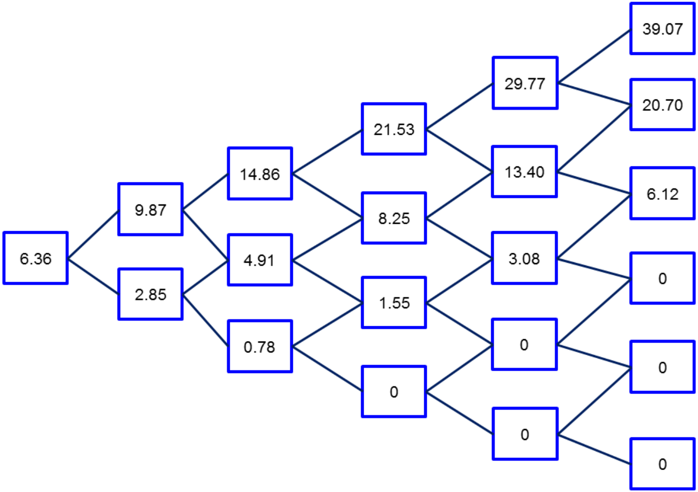

39.07
29.77
20.70
21.53
13.40
14.86
6.12
9.87
8.25
4.91
3.08
0
2.85
1.55
0.78
0
0
0
0
0


6.36
9.87
2.85
14.86
4.91
0.78
21.53
8.25
1.55
0
29.77
13.40
3.08
0
0
39.07
20.70
6.12
0
0
0

Pricing European Call Option by Binomial Lattice

```
function [price, lattice] = LatticeEurCall(S0,K,r,T,sigma,N)

deltaT = T/N;
u=exp(sigma * sqrt(deltaT));
d=1/u;
p=(exp(r*deltaT) - d)/(u-d);
lattice = zeros(N+1,N+1);

for i=0:N
   lattice(i+1,N+1)=max(0 , S0*(u^i)*(d^(N-i)) - K);
end

for j=N-1:-1:0
   for i=0:j
      lattice(i+1,j+1) = exp(-r*deltaT) * ...
         (p * lattice(i+2,j+2) + (1-p) * lattice(i+1,j+2));
   end
end

price = lattice(1,1);
```

Pricing American Style Put Option

```
function price = AmPutLattice(S0,K,r,T,sigma,N)
deltaT = T/N;
u=exp(sigma * sqrt(deltaT));
d=1/u;
p=(exp(r*deltaT) - d)/(u-d);
discount = exp(-r*deltaT);
p_u = discount*p;
p_d = discount*(1-p);
SVals = zeros(2*N+1,1);
SVals(N+1) = S0;

[...]
```

```
function price = AmPutLattice(S0,K,r,T,sigma,N)



[...]

for i=1:N
    SVals(N+1+i) = u*SVals(N+i);
    SVals(N+1-i) = d*SVals(N+2-i);
end
PVals = zeros(2*N+1,1);
for i=1:2:2*N+1
    PVals(i) = max(K-SVals(i),0);
end

[...]
```

Pricing American Style Put Option (cont'd

```
function price = AmPutLattice(S0,K,r,T,sigma,N)



[...]
for tau=1:N
    for i= (tau+1):2:(2*N+1-tau)
        hold = p_u*PVals(i+1) + p_d*PVals(i-1);
        PVals(i) = max(hold, K-SVals(i));
    end
end
price = PVals(N+1);
```

- Decisions at every point during backtracking

$$f_{i,j} \;=\; \max\left\{K - S_{i,j},\;\; \exp(-r\delta t)\left(p\, f_{i+1,j+1} \,+\, (1-p)\, f_{i,j+1}\right)\right\}$$

- We will look at inference as expectations...

$$E\left[g(X)\right] = \int_{-\infty}^{+\infty} g(x) f_X(x)\, dx$$

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(of a function of a uniform random variable):

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$$E\left[g(U)\right], \quad \text{where} \quad U \sim (0,1)$$

- We approximate the integral by

$$\widehat{I}_m \;=\; \frac{1}{m} \sum_{i=1}^{m} g(U_i)$$

- Where will we use this?

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- European call option

$$f = \exp(-rT)\, E\left[f_T\right]$$

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```
% BlsMC1.m
function Price = BlsMC1(S0,K,r,T,sigma,NRepl)
nuT = (r - 0.5*sigma^2)*T;
siT = sigma * sqrt(T);
DiscPayoff = exp(-r*T)*max(0, S0*exp(nuT+siT*randn(NRepl,1))-K);
Price = mean(DiscPayoff);
```

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```

```
> S0=50; K=60; r=0.05; T=1; sigma=0.2;
> randn('state', 0);
> BlsMC1(S0, K, r, T, sigma, 1000)
ans =
   1.2562
```

Is this a good approach?

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- Different answers on different runs

```
> S0=50; K=60; r=0.05; T=1; sigma=0.2;
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ans =
   1.2562
> BlsMC1(S0, K, r, T, sigma, 1000)
ans =
   1.8783
> BlsMC1(S0, K, r, T, sigma, 1000)
ans =
   1.7864
```

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- Different answers on different runs

```
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> randn('state', 0);
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   1.2562
> BlsMC1(S0, K, r, T, sigma, 1000)
ans =
   1.8783
> BlsMC1(S0, K, r, T, sigma, 1000)
ans =
   1.7864
```

- What if we had large number of samples?

```
> BlsMC1(S0, K, r, T, sigma, 1000000)
ans =
   1.6295
> BlsMC1(S0, K, r, T, sigma, 1000000)
ans =
   1.6164
> BlsMC1(S0, K, r, T, sigma, 1000000)
ans =
   1.6141
```

Sampling: Inverse Transform

- Sample X from $f(x)$; Cumulative distribution $F(x)$

  - Draw $U \sim U(0.1)$
  - Return $X = F^{-1}(U)$

Sampling: Inverse Transform

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$$\boxed{\begin{array}{l} \bullet \text{ Draw } U \sim U(0.1) \\ \bullet \text{ Return } X = F^{-1}(U) \end{array}}$$

$$\begin{aligned} P\{X \leq x\} &= P\left\{F^{-1}(U) \leq x\right\} \\ &= P\{U \leq F(x)\} \\ &= F(x) \end{aligned}$$

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&= P\{U \leq F(x)\} \\
&= F(x)
\end{aligned}
$$

- Example: Exponential distribution $X \sim \exp(\mu)$
- Cumulative

$$F(x) = 1 - \exp(-\mu x)$$

- Inverse

$$x = -\frac{1}{\mu} \log(1 - U)$$

- Distributions of U and $(1 - U)$ are the same
  Hence return: $-\log(U)/\mu$

Sampling: Acceptance-Rejection Method

- Probability density function: $f(x)$

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$$r(x) \;=\; t(x)/c \qquad c = \int_{\mathcal{I}} t(x)\, dx$$

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1. Generate $Y \sim r$
2. Generate $U \sim U(0, 1)$
3. If $U \leq f(Y)/t(Y)$ return
   $X = Y$
   Else go to 1

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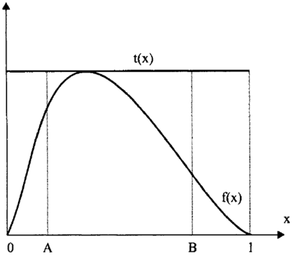

Homework
Page 235, Brandimarte

- $f(x) = 30(x^2 - 2x^3 + x^4), \qquad x \in [0, 1]$

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- Algorithm

1. Draw U_1 and U_2
2. If $U2 \leq 16(U_1^2 - 2U_1^3 + U_1^4)$
   accept $X = U_1$
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Homework

Page 235, Brandimarte

- $f(x) = 30(x^2 - 2x^3 + x^4), \quad x \in [0, 1]$
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     accept $X = U_1$
     Else
     go to 1
- Exercise:
  - Draw the graph of $f(x)$
  - Simulate 1000 samples using above algorithm
  - Draw a histogram to the same scale as $f(x)$ – do they match? Is it better with 100000 samples?
  - On average, how many trials were needed through the accept-reject loop for each sample?

Variance Reduction

- Independent samples X_i
- Sample mean (estimates mean $\mu = E[X_i]$ from n samples)

$$\overline{X}(n) = \frac{1}{n}\sum_{i=1}^{n} X_i$$

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$$S^2(n) = \frac{1}{(n-1)}\sum_{i=1}^{n}\left[X_i - \overline{X}(n)\right]^2$$

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- Error of the estimator

$$\begin{aligned} E\left[\left(\overline{X}(n) - \mu\right)^2\right] &= \mathrm{Var}\left[\overline{X}(n)\right] \\ &= \mathrm{Var}\left[\frac{1}{n} \sum_{i=1}^{n} X_i\right] \\ &= \frac{1}{n^2} \times n \times \sigma^2 \;=\; \frac{\sigma^2}{n} \end{aligned}$$

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- Two points:
  - More samples n reduces the variance in estimation
  - Variance reduction schemes can control σ^2

Variance reduction: Antithetic Sampling

- Pair of sequences

$$\left\{ \begin{array}{cccc} X_1^{(1)} & X_1^{(2)} & \dots & X_1^{n} \\ X_2^{(1)} & X_2^{(2)} & \dots & X_2^{n} \end{array} \right\}$$

- Columns (horizontally) are independent
- $X_1^{(i)}$ and $X_2^{(i)}$ are dependent.

Variance reduction: Antithetic Sampling

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- Sample is a function of each pair: $X^{(i)} = \left(X_1^{(i)} + X_2^{(i)} \right) / 2$

Variance reduction: Antithetic Sampling

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- $X_1^{(i)}$ and $X_2^{(i)}$ are dependent.
- Sample is a function of each pair: $X^{(i)} = \left(X_1^{(i)} + X_2^{(i)} \right) /2$
- Variance

$$\begin{aligned} \mathrm{Var}\left[\overline{X}(n) \right] &= \frac{1}{n} \mathrm{Var}\left[X^{(i)} \right] \\ &= \frac{1}{4n} \left\{ \mathrm{Var}(X_1^{(i)}) + \mathrm{Var}(X_2^{(i)}) + 2\,\mathrm{Cov}(X_1^{(i)}, X_2^{(i)}) \right\} \\ &= \frac{1}{2n} \mathrm{Var}(X)\,(1 + \rho) \end{aligned}$$

- Uniform random number $\{U_k\}$ and $\{1 - U_k\}$ as sequences.

Application

```
function [Price, CI] = BlsMC2(S0,K,r,T,sigma,NRepl)
nuT = (r - 0.5*sigma^2)*T;
siT = sigma * sqrt(T);
DiscPayoff = exp(-r*T)*max(0, S0*exp(nuT+siT*randn(NRepl,1))-K);
[Price, VarPrice, CI] = normfit(DiscPayoff);
```

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function [Price, CI] = BlsMC2(S0,K,r,T,sigma,NRepl)
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```

```
function [Price, CI] = BlsMCAV(S0,K,r,T,sigma,NPairs)
nuT = (r - 0.5*sigma^2)*T;
siT = sigma * sqrt(T);
Veps = randn(NPairs,1);
Payoff1 = max( 0 , S0*exp(nuT+siT*Veps) - K);
Payoff2 = max( 0 , S0*exp(nuT+siT*(-Veps)) - K);
DiscPayoff = exp(-r*T) * 0.5 * (Payoff1+Payoff2);
[Price, VarPrice, CI] = normfit(DiscPayoff);
```

Homework

Test the two functions: `BlsMC` and `BlsMCAV`

(Brandimarte, p248)

```
> randn('state', 0)
> [Price, CI] = BlsMC2(50,50,0.05,1,0.4,200000)
Price=
          9.0843
CI =
          9.0154
          9.1532
\pause
> (CI(2)-CI(1))/Price
ans =
        0.0152
\pause
> randn('state', 0)
> [Price, CI] = BlsMCAV(50,50,0.05,1,0.4,200000)
Price=
          9.0553
CI =
           8.9987
           9.1118
\pause
> (CI(2)-CI(1))/Price
ans =
        0.0125
```

Approximating option prices with a neural network

- We have seen three tools for pricing options
  - Closed form Black-Scholes
  - Binomial lattice
  - Monte Carlo

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$$\boldsymbol{x} = [\, S/X \quad (T-t)\,]^T$$

$$c = \sum_{j=1}^{J} \lambda_j \, \phi_j(\boldsymbol{x}) + \boldsymbol{w}^T \boldsymbol{x} + w_0$$

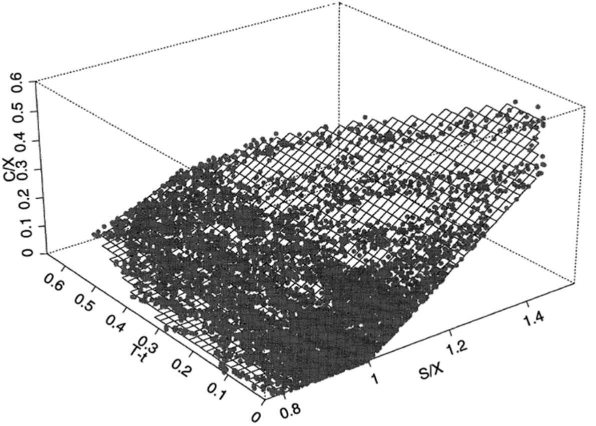

Figure 4. Simulated call option prices normalized by strike price and plotted versus

$$
\begin{aligned}
\widehat{C/X} = {} & -0.06\sqrt{\begin{bmatrix} S/X - 1.35 \\ T - t - 0.45 \end{bmatrix}' \begin{bmatrix} 59.79 & -0.03 \\ -0.03 & 10.24 \end{bmatrix} \begin{bmatrix} S/X - 1.35 \\ T - t - 0.45 \end{bmatrix} + 2.55} \\
& - 0.03\sqrt{\begin{bmatrix} S/X - 1.18 \\ T - t - 0.24 \end{bmatrix}' \begin{bmatrix} 59.79 & -0.03 \\ -0.03 & 10.24 \end{bmatrix} \begin{bmatrix} S/X - 1.18 \\ T - t - 0.24 \end{bmatrix} + 1.97} \\
& + 0.03\sqrt{\begin{bmatrix} S/X - 0.98 \\ T - t + 0.20 \end{bmatrix}' \begin{bmatrix} 59.79 & -0.03 \\ -0.03 & 10.24 \end{bmatrix} \begin{bmatrix} S/X - 0.98 \\ T - t + 0.20 \end{bmatrix} + 0.00} \\
& + 0.10\sqrt{\begin{bmatrix} S/X - 1.05 \\ T - t + 0.10 \end{bmatrix}' \begin{bmatrix} 59.79 & -0.03 \\ -0.03 & 10.24 \end{bmatrix} \begin{bmatrix} S/X - 1.05 \\ T - t + 0.10 \end{bmatrix} + 1.62} \\
& + 0.14S/X - 0.24(T - t) - 0.01. \qquad (9)
\end{aligned}
$$