An Application of the Trigonometric Interpolation on Second Order Linear ODE

Xiaorong Zou (`xiaorong.zou@bofa.com`)
Global Market Risk Analytic, Bank of America

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Abstract. In this paper, we propose a trigonometric-interpolation based approach (TIBA) to approximate solutions of second-order linear ODEs with mixed boundary conditions. TIBA is expected to achieve high accuracy with moderate number of grid points and outperform significantly difference-based algorithm, especially in the case where target solutions are highly oscillating. In addition, TIBA can be applied to solve singular linear ODEs of second-order with the same degree of accuracy as in its application on normal ODEs.

Keywords: Trigonometric Interpolation, Ordinary Differential Equation, Singular Ordinary Differential Equation

MSC2000: Primary 65T40, Secondary 65T50

1. Introduction

In this paper, we develop an algorithm to solve the boundary value problems of the form:

$$w(x)y''(x) = p(x)y'(x) + q(x)y(x) + r(x), \quad x \in [s, e] \tag{1}$$

$$a_{11}y(s) + a_{12}y'(s) + a_{13}y(e) + a_{14}y'(e) = \alpha, \tag{2}$$
$$a_{21}y(s) + a_{22}y'(s) + a_{23}y(e) + a_{24}y'(e) = \beta, \tag{3}$$

where $p(x), q(x), r(x), w(x)$ is continuously differential on the range $[s, e]$, $(a_{ij})_{1 \le i \le 2, 1 \le j \le 4}$ and α, β are constant real numbers.

The second-order linear differential equations are used in wide scientific areas and relevant researches in literature are generally limited to the standard form with $w(x) \equiv 1$. Theoretic solutions are not available in general and many numerical algorithms on standard ODEs have been proposed in the literature. The algorithms are often boundary-condition specific. For instance, the classic Runge-Kutta is popular with convergence order 4 for Neumann type with $y(s) = \alpha, y'(s) = \beta$. A comprehensive discussion on Dirichlet type $y(s) = \alpha, y(e) = \beta$ is refereed to [2]. The general mixed boundary condition 2 and 3 was studied in [3] and [5]. In [3], the solution is represented by multi-point Taylor expansion. In [5], a multiscale orthogonal basis of reproducing

kernel space is constructed and used to approximate a solution by certain linear combination of finite basis function. The coefficients of the approximation can be identified by solving a linear equation system.

A new trigonometric interpolation algorithm was recently introduced in [7]. It converges at an optimal speed aligned with smoothness of underlying function and can be used to approximate non-periodic functions defined on bounded intervals. In this study, we propose a novel trigonometric interpolation based algorithm, named TIBA hereafter, to solve ODE (1-3). The analysis in this paper has been be used to solve second-order integro-differential equations in [9]-[10].

In a nutshell, TIBA converts ODE (1-3) into a linear algebraic system whose solution can be solved directly. The boundary conditions is captured by two parameters of a close-form approximation of the target solution, similar to how initial condition is captured in Adomian decomposition method [4].

Eq (1) is singular if $w(x)$ can reach to 0. Researches for the numerical method of singular ODEs is motivated by numerous applications from physics, chemistry, mechanics or ecology [6]. To the author's knowledge, there is no numerical algorithm to solve singular ODE (1-3) in the literature. A difference-based method is likely not effective to cope with a singular ODE since its estimation around a singular point can generate significant errors, which can deteriorate the overall performance because of error compounding disadvantage. TIBA approximates the solution at all grid points simultaneously and is effective to handle isolated singular points. As shown Section 6, the algorithm converges almost equally well as in the treatment of normal ODE with $|w(x)| > 0$.

The rest of paper is organized as follows. In Section 2, we summarize the relevant results of trigonometric interpolation algorithm developed in [7]. Section 3 is devoted to develop TIBA, summarized in Algorithm 3.1, for a second-order nonlinear ODE (4) with the mixed boundary condition (2-3). Section 4 is used to enhance Algorithm 3.1 to solve linear second-order ODE (1-3) as stated in Algorithm 4.1, and some details of derivation is shown in Appendix A. In Section 5, we show how Algorithm 4.1 can be used to solve singular linear ODEs of second-order by a certain transformation. Numerical tests are conducted in Section 6 to assess the performance on convergence and accuracy. The results are presented in two subsections, Subsection 6.1 and 6.2 for the application in normal ODEs and singular ODEs respectively. The summary is made on Section 7.

2. Trigonometric Interpolation on Non-Periodic Functions

In this section, we review relevant results of trigonometric interpolation algorithm developed in [7] starting with following interpolation algorithm on periodic functions.

Theorem 2.1. *Let $f(x)$ be an odd periodic function [1] with period $2b$ and $N = 2M = 2^{q+1}$ for some integer $q \geq 1$ and x_j, y_j are defined by*

$$x_j := -b + j\lambda, \quad \lambda = \frac{2b}{N}, \quad 0 \leq j < N,$$
$$y_j := f(x_j),$$

then there is a unique $M - 1$ degree trigonometric polynomial

$$f_M(x) = \sum_{0 < j < M} a_j \sin \frac{j\pi x}{b},$$
$$a_j = \frac{2}{N} \sum_{0 < k < N} (-1)^j y_k \sin \frac{2\pi jk}{N}, \quad 0 < j < M$$

such that it fits to all grid points, i.e.

$$f_M(x_k) = y_k, \quad 0 \leq k < N.$$

Theorem 2.1 has been enhanced so it can be applied to a nonperiodic function f whose $K + 1$-th derivative $f^{(K+1)}(x)$ exists over a bounded interval $[s, e]$. To seek for a periodic extension with same smoothness, we assume that f can be extended smoothly such that $f^{(K+1)}$ exists and is bounded over $[s - \delta, e + \delta]$ for certain $\delta > 0$. A periodic extension of f can then be achieved by a cut-off smooth function $h(x)$ with following property:

$$h(x) = \begin{cases} 1 & x \in [s, e], \\ 0 & x < s - \delta \text{ or } x > e + \delta. \end{cases}$$

A cut-off function with closed-form analytic expression is proposed in [7]. Let

$$o = s - \delta, \quad b = e + \delta - o,$$

and define $F(x) := h(x + o)f(x + o)$ for $x \in [0, b]$. One can treat $F(x)$ as an odd periodic function with period $2b$. Apply Theorem 2.1 to generate the trigonometric interpolation of degree $M - 1$ with N evenly-spaced grid points over $[-b, b]$

$$F_M(x) = \sum_{0 < j < M} a_j \sin \frac{j\pi x}{b},$$

---

[1] Similar results for even periodic function is also available in [7].

and let

$$\hat{f}_M(x) = F_M(x - o) = \sum_{0 < j < M} a_j \sin \frac{j\pi(x - o)}{b}.$$

$\hat{f}_M(x)|_{[s,e]}$ can be treated as an trigonometric interpolation of f since $\hat{f}_M(x_k) = f(x_k)$ for all grid points $x_k \in [s, e]$. Numerical tests on certain functions demonstrate that $\hat{f}$ approaches to f with decent accuracy [7].

3. TIBA for Second-Order Nonlinear ODE

In this section, we aim to develop an algorithm for the solution of following second-order nonlinear ODE with the boundary conditions (2-3)

$$w(x)y''(x) = f(x, y, y'), \quad x \in [s, e]. \tag{4}$$

We shall convert the nonlinear ODE to a nonlinear algebraic system through the trigonometric interpolation on the target solution and its derivatives. Following the notations in previous sections, we assume that w, f are continuously differentiable on $[s - \delta, e + \delta]$ and $[s - \delta, e + \delta] \times R^2$ respectively for certain $\delta > 0$. By a parallel shifting if needed, we assume $s = \delta$ without loss of generality. Let h be a cut-off function specified in Section 2 and construct $F(x, v, u)$ as follows

$$F(x, v, u) = f(x, v, u)h(x), \quad (x, v, u) \in [0, b] \times R^2.$$

Consider a solution $v(x)$ of the following ODE system

$$
\begin{aligned}
w(x)v''(x) &= F(x, v, v'), \quad x \in [0, b], & (5) \\
\alpha &= a_{11}v(s) + a_{12}v'(s) + a_{13}v(e) + a_{14}v'(e), & (6) \\
\beta &= a_{21}v(s) + a_{22}v'(s) + a_{23}v(e) + a_{24}v'(e). & (7)
\end{aligned}
$$

It is clear that $v(x)|_{[s,e]}$ solves ODE (4) with boundary condition (3). Define $u(x) := v'(x)$ and $z(x) := v''(x)$. By Eq (5), $z(x)$ and its derivatives vanish at boundary points $\{0, b\}$, hence it can be smoothly extended as an odd periodic function with period $2b$ and be approximated by a trigonometric polynomial. Assume that

$$\tilde{z}_M(x) = \sum_{0 \le j < M} b_j \sin \frac{j\pi x}{b} \tag{8}$$

is an interpolant of $z(x)$ with N equispaced grid points over $[-b, b]$ by Theorem 2.1. u and v can be approximated accordingly by

$$\tilde{u}_M(x) = a_0 - \frac{b}{\pi} \sum_{1 \leq j < M} \frac{b_j}{j} \cos \frac{j\pi x}{b}, \tag{9}$$

$$\tilde{v}_M(x) = a_1 + a_0 x - (\frac{b}{\pi})^2 \sum_{1 \leq j < M} \frac{b_j}{j^2} \sin \frac{j\pi x}{b}, \tag{10}$$

where a_0, a_1 are two constant and can be determined by boundary conditions as shown in Eq (15) below.

The following notations and conventions will be adopted in the rest of this paper. A k-dim vector is considered as $(k, 1)$ dimensional matrix unless specified otherwise. Define

$$x_k = k\lambda, \quad \lambda = \frac{b}{M}, \quad X = (x_k)_{0 \leq k \leq M},$$
$$u_k = \tilde{u}_M(x_k), \quad v_k = \tilde{v}_M(x_k), \quad z_k = \tilde{z}_M(x_k), \quad f_k = F(x_k, v_k, u_k)$$
$$U = (u_k)_{0 \leq k \leq M}, V = (v_k)_{0 \leq k \leq M}, Z = (z_k)_{0 \leq k \leq M}, W = (w_k)_{0 \leq k \leq M}$$
$$\hat{U} = (u_k)_{0 < k < M}, \hat{V} = (v_k)_{0 < k < M}, \hat{Z} = (z_k)_{0 < k < M}, \hat{W} = (w_k)_{0 < k < M}$$
$$F = (f_k)_{0 \leq k \leq M}, \quad K = (1, 2, \cdots, M-1)^T, \quad B = (b_i)_{1 \leq i < M}$$
$$I = (1, 1, \cdots, 1)^T_{M-1}, \quad I_a = (-1, 1, -1, \cdots, -1)^T_{M-1}.$$

For any two matrices A, B with same shape, $A \circ B$ denotes the Hadamard product, which applies the element-wise product to two matrices. $A \cdot B$ denote the standard matrix multiplication when applicable. $\sum(W)$ denotes the sum of all elements in a vector W. $A(i, :)$ and $A(:, j)$ is used to denote the i-th row and j-th column of A respectively. $W(k : l)$ denote the $l - k + 1$-th vector $(w_k, \cdots, w_l)^T$. In addition, $diag(W)$ is the diagonal matrix constructed by W. Note we have

$$s = x_m, \qquad e = x_{m+n}.$$

Eq. (5) always holds at $x_0 = 0$ and $x_M = b$; at other grid points, it is equivalent to

$$diag(\hat{W}) \cdot \hat{Z} = \hat{F}. \tag{11}$$

Z, U, V can be calculated based on Eq. (8-10):

$$z_k = \sum_{0 \leq j < M} b_j \sin \frac{2\pi jk}{N}, \tag{12}$$

$$u_k = a_0 - \frac{b}{\pi} \sum_{1 \leq j < M} \frac{b_j}{j} \cos \frac{2\pi jk}{N}, \tag{13}$$

$$v_k = a_1 + a_0 x_k - (\frac{b}{\pi})^2 \sum_{1 \leq j < M} \frac{b_j}{j^2} \sin \frac{2\pi jk}{N}. \tag{14}$$

Eq (14) can be used to solve a_0 and a_1:

$$a_0 = \frac{v_M - v_0}{b}, \quad a_1 = v_0. \tag{15}$$

Define

$$S = (\sin \frac{2\pi jk}{N})_{1 \leq j,k < M}, \qquad C = (\cos \frac{2\pi jk}{N})_{1 \leq j,k < M}. \tag{16}$$

It is easy to check $SS = \frac{M}{2}E$, where E is the $M - 1$ identity matrix, and therefore $O := \sqrt{\frac{2}{M}}S$ is a symmetric orthogonal matrix. Define

$$\Theta = (\theta_{ij})_{1 \leq i,j < M} = O \cdot diag(1/K^2) \cdot O.$$

We need represent B, U in term of V. First, rewrite (14) in vector format:

$$\hat{V} = a_1 I + a_0 \frac{b}{M} K - (\frac{b}{\pi})^2 S \cdot diag(1/K^2) \cdot B,$$

which implies

$$B = diag(K^2) \cdot S \cdot (\frac{2a_1 \pi^2}{Mb^2} I + \frac{2a_0 \pi^2}{bM^2} K - \frac{2\pi^2}{Mb^2} \hat{V}). \tag{17}$$

Applying Eqs. (15, 17) to Eq. (12), we obtain

$$\Theta \cdot \hat{Z} = \frac{v_0 \pi^2}{Mb^2}(MI - K) + \frac{v_M \pi^2}{Mb^2} K - \frac{\pi^2}{b^2} \hat{V},$$

and a discretization of Eq (11)

$$diag(\hat{W}) \cdot \Theta^{-1} \cdot (\frac{v_0 \pi^2}{Mb^2}(MI - K) + \frac{v_M \pi^2}{Mb^2} K - \frac{\pi^2}{b^2} \hat{V}) = \hat{F}. \tag{18}$$

By Eq (13), we can represent U by a linear combination of V

$$U = AV. \tag{19}$$

We leave details of derivation of $A = (a_{ij})_{0 \le i,j \le M}$ in Appendix A and present the results as follows.

$$
\begin{aligned}
a_{0,0} &= \frac{\pi}{b} sum(I_a \circ \cot(\pi K/N)) \\
&\quad - \frac{\pi}{bM} sum(I_a \circ K \circ \cot(\pi K/N)) - \frac{1}{b} \qquad (20) \\
a_{0,1:M-1} &= -\frac{\pi}{b} I_a' \circ \cot(\pi K'/N) \\
a_{0,M} &= \frac{\pi}{bM} sum(I_a \circ K \circ \cot(\pi K/N)) + \frac{1}{b} \\
a_{i,0} &= \frac{\pi}{2b} sum((-1)^i \cot(i,:)I_a) \\
&\quad - \frac{\pi}{2bM} sum((-1)^i I_a \circ \cot(i,:) \circ K) - 1/b, \qquad (21) \\
a_{i,1:M-1} &= \frac{\pi}{2b}(-1)^{i+1} I_a' \circ \cot(i,:), \\
a_{i,M} &= \frac{\pi}{2bM} sum((-1)^i I_a \circ \cot(i,:) \circ K) + 1/b. \\
a_{M,0} &= -\frac{\pi}{b} sum(I_a \circ \tan(\pi K/N)) \\
&\quad - \frac{\pi}{bM} sum((K \circ \cot(\pi K/N)) - \frac{1}{b}, \qquad (22) \\
a_{M,1:M-1} &= \frac{\pi}{b} I_a' \circ \tan(\pi K'/N), \\
a_{M,M} &= \frac{\pi}{bM} sum((K \circ \cot(\pi K/N)) + \frac{1}{b},
\end{aligned}
$$

where $0 < i < M, \cot(k,i) := cot\frac{k+i}{N}\pi + cot\frac{k-i}{N}\pi$, and $cot(x)$ is defined to be 0 if x/π is an integer. Note that A is singular since $\sum_j a_{i,j} = 0$ for all $0 \le i \le M$, and V can not be recovered by U, which is expected. Let

$$
\hat{A} = (a_{ij})_{1 \le i,j \le M-1}, \quad \hat{A}_0 = (a_{i,0})_{1 \le i \le M-1}, \quad \hat{A}_M = (a_{i,M})_{1 \le i \le M-1}
$$

By (19), we have

$$
\hat{U} = \hat{A}\hat{V} + \hat{A}_0 v_0 + \hat{A}_M v_M \qquad (23)
$$

By Eq (19), we obtain a nonlinear algebraic system on V by combining boundary conditions (6-7) and the discretization (18) of the

differential equation (5):

$$
\begin{aligned}
h_0(V) &:= a_{11}v_m + a_{13}v_{n+m} \\
&+ \sum_{0 \le i \le M} (a_{12}a_{m,k} + a_{14}a_{m+n,k})v_k - \alpha = 0 \qquad (24)
\end{aligned}
$$

$$
\begin{aligned}
h_i(V) &:= \frac{v_0\pi^2}{Mb^2}(M-i) + \frac{v_M\pi^2}{Mb^2}i - \frac{\pi^2}{b^2}v_i \\
&- \sum_{1 \le j < M} \theta_{ij}F_j = 0, \quad 0 < i < M, \qquad (25)
\end{aligned}
$$

$$
\begin{aligned}
h_M(W) &:= a_{21}v_m + a_{23}v_{n+m} \\
&+ \sum_{0 \le i \le M} (a_{22}a_{m,k} + a_{24}a_{m+n,k})v_k - \beta = 0, \qquad (26)
\end{aligned}
$$

with

$$
u_j = \sum_{0 \le k \le M} a_{jk}v_k.
$$

TIBA for general nonlinear ODE (4, 2-3) can be summarized as follows

Algorithm 3.1. *(TIBA with second-order nonlinear ODE)*

1. *Select proper N, δ and follow Section 2 to generate cut-off function h;*

2. *Apply Eq (20-22) to generate matrix A;*

3. *Solve $M+1$-dim system that consists of $M+1$ equations (24-26) with variables $\{v_j\}_{0 \le j \le M}$.*

4. *Apply Eq (15) and (17) to calculate parameters $a_0, a_1, b_1, \ldots, b_{M-1}$ of $\tilde{v}_M$ defined by Eq. (10);*

5. *Restrict $\tilde{v}_M$ to $[s, e]$ as approximation of target solution.*

In general, the performance of Algorithm 3.1 depends on how effective the nonlinear system (24-26) can be solved. Note that close form of Jacobian $\frac{\partial h_i}{\partial v_j}$ is available and the classic Newton method can be applied. TIBA becomes particularly attractive when ODE (4) is reduced to a linear system as discussed in Section 4.

4. TIBA for Second Order Linear ODEs

In this section, we focus on second-order linear ODE and assume $f(x, v, u)$ in Eq. (4) is a linear function in u, v as follows

$$f(x, v, u) = p(x)u + q(x)v(x) + r(x),$$

and p, q, r is continuous differential on $[s - \delta, e + \delta]$. Eq. (4) is reduced to Eq. (1). We follow same conventions and notations as in Section 3. In addition, define

$$p_h(x) = p(x)h(x), \quad q_h(x) = q(x)h(x), \quad r_h(x) = r(x)h(x), \qquad x \in [0, b]$$

and

$$\begin{aligned} p_k &= p_h(x_k), \quad q_k = q_h(x_k), \quad r_k = r_h(x_k), \quad 0 < k < M \\ \hat{P} &= (p_k)_{0<k<M}, \quad \hat{Q} = (q_k)_{0<k<M}, \quad \hat{R} = (r_k)_{0<k<M}. \end{aligned}$$

Replacing $\hat{F}$ by $\hat{P} \circ \hat{U} + \hat{Q} \circ \hat{V} + \hat{R}$ in Eq (18) and applying (19), we obtain

$$diag(\hat{W})\Theta^{-1}\left(\frac{v_0\pi^2}{Mb^2}(MI - K) + \frac{v_M\pi^2}{Mb^2}K - \frac{\pi^2}{b^2}\hat{V}\right) = \hat{R} + \hat{Q}\circ\hat{V} + \hat{P}\circ\hat{U}. \quad (27)$$

Let

$$\begin{aligned} \hat{C} &= \frac{\pi^2}{b^2}diag(\hat{W})\Theta^{-1} + diag(\hat{Q}) + diag(\hat{P}) \cdot \hat{A} \\ C_0 &= diag(\hat{P})\hat{A}_0 - \frac{\pi^2}{Mb^2} \cdot diag(\hat{W}) \cdot \Theta^{-1}(MI - K) \\ C_M &= diag(\hat{P})\hat{A}_M - \frac{\pi^2}{Mb^2} \cdot diag(\hat{W}) \cdot \Theta^{-1}K \end{aligned}$$

By Eq (23), Eq (27) is equivalent to

$$\hat{C}\hat{V} + C_0 v_0 + C_M v_M = -\hat{R} \qquad (28)$$

Eqs. (6-7) can be written as two linear equations in V. Insert Eq. (6) in the front and attach Eq. 7 to the end of the linear system (28), we obtain a $M + 1$-dim linear system

$$\Phi V = \Psi, \qquad (29)$$

where $\Psi = (\psi_i)_{0 \le i \le M}$ is determined by

$$\psi_0 = \alpha, \quad \psi_M = \beta, \quad \Psi(1 : M - 1) = -\hat{R}; \qquad (30)$$

$\Phi = (\phi_{ij})_{0 \le i,k \le M}$ is determined by $(0 \le k \le M)$,

$$
\begin{aligned}
\phi_{0,k} &= a_{12} \cdot A(m, k) + a_{14} \cdot A(m + n, k) \\
&+ \delta_{m,k} a_{11} + \delta_{m+n,k} a_{13}, \quad (31) \\
\phi_{M,k} &= a_{22} \cdot A(m, k) + a_{24} \cdot A(m + n, k) \\
&+ \delta_{m,k} a_{21} + \delta_{m+n,k} a_{23}, \quad (32)
\end{aligned}
$$

and

$$
(\phi_{i,0})_{0<i<M} = C_0, \quad (\phi_{i,M})_{0<i<M} = C_M, \quad (\phi(i,j))_{0<i,j<M} = \hat{C}. \quad (33)
$$

We have the following Algorithm to solve ODE (1-3) as follows.

Algorithm 4.1. *(TIBA with second-order linear ODE)*

1. *Select proper N, δ and follow Section 2 to generate cut-off function h;*

2. *Apply Eq (20-22) to generate matrix A;*

3. *Calculate Φ, Ψ by Eq (30) and (31-33) and then obtain V by Eq. (29);*

4. *Apply Eq (15) and (17) to calculate parameters $a_0, a_1, b_1, \ldots, b_{M-1}$ of $\tilde{v}_M$ defined by Eq. (10);*

5. *Restrict $\tilde{v}_M$ to $[s, e]$ as approximation of target solution.*

Algorithm 4.1 works under the condition that $rank(\Phi) = M + 1$ and the solution is unique in this case. The condition can be failed under certain circumstances. A study of existence and uniqueness of the linear ODE with $w(x) \equiv 1$ is refereed to [1].

Note that $rank(\Phi) < M + 1$ is always true for a singular ODE with $w(s) = 0$ under Neumann condition. In this case, we have $f(s) = 0$, which implies the initial conditions (2-3) are not independent and thus not sufficient to determine a unique solution of ODE (1-3). In fact, one can obtain $\phi(m, :) = q_m \phi(0, :) + p_m \phi(M, :)$ by Eqs. (31-33).

5. TIBA for Second order Singular ODEs

In this section, we show how Algorithm 4.1 can be used to solve ODEs with singularities from $p(x), q(x), r(x)$. As a demonstrative example, it is easy to see that $y = \frac{cos(\theta x)}{(x-e)}, x \in [s, e]$ solve the following singular ODE

$$
y''(x) = \frac{-y'}{(x - e)} + \frac{y}{(x - e)^2} + \frac{\theta \sin(\theta x)}{(x - e)^2} - \frac{\theta^2 \cos(\theta x)}{(x - e)} \quad (34)
$$

with *Neumann* type of boundary condition

$$y(s) = \frac{\cos(\theta s)}{(s - e)}, \quad y'(s) = -\frac{\cos(\theta e)}{(s - e)^2} - \frac{\theta \sin(\theta e)}{(s - e)}, \tag{35}$$

or *Dirichlet* type of boundary condition

$$y(s) = \frac{\cos(\theta s)}{(s - e)}, \quad \lim_{x \to e} y(x)(x - e) = \cos(\theta e). \tag{36}$$

We shall focus on a numerical solution of following singular ODE

$$y''(x) = p(x)y'(x) + q(x)y(x) + r(x), \quad x \in [s, e] \tag{37}$$

with *Neumann* (38) or *Dirichlet* (39) type of boundary conditions

$$y(s) = \alpha, \quad y'(s) = \beta, \tag{38}$$

$$y(s) = \alpha, \quad \lim_{x \to e}(x - e)^d y(x) = \beta, \tag{39}$$

where $p(x), q(x), r(x)$ is continuous differentiable over $[s, e)$ and the order of singularity of $p(x), q(x), r(x)$ at e, if exists, is no larger than certain integer $d \geq 0$.

Let $v(x) = t(x) \cdot y(x)$ with $t(x) = (x - e)^d$, Eq. (37) is transformed to

$$t^2 v'' = (pt^2 + 2tt')v'$$
$$+ (qt^2 - ptt' + tt'' - 2(t')^2)v + rt^3. \tag{40}$$

and the boundary conditions (38-39) are transformed to

$$v(s) = \alpha t(s), \quad v'(s) = t(s)\beta + t'(s)\alpha, \tag{41}$$

$$v(s) = \alpha t(s), \quad v(e) = \beta. \tag{42}$$

The transformed ODE (40) with boundary condition (41) or (42) can be solved by Algorithm 4.1.

Applying transformation $v(x) = (x - e)y(x)$, Eqs. (34-36) are reduced to

$$(x - e)v'' = v' + \theta \sin(\theta x) - \theta^2 \cos(\theta x)(x - e), \tag{43}$$

$$v(s) = \cos(\theta s), \quad v'(s) = \theta \cos(\theta s), \tag{44}$$

$$v(s) = \cos(\theta s), \quad v(e) = \cos(\theta e). \tag{45}$$

We provide relevant numerical results in Section 6 on this sample.

6. Performance

In this section, we discuss the performance of Algorithm 4.1. In Subsection 6.1, We construct normal ODEs with closed-form solutions under four types of boundary conditions in Table I.The numerical results show that the algorithm converges stably and provides decent accuracy with moderate number of grid points. In the case of Neumann initial condition, the test results shows that Algorithm 4.1 significantly outperforms the classic RK4, especially when the target solution becomes more oscillated.

In Subsection 6.2, we apply Algorithm 4.1 to solve ODE (34) under both Neumann initial condition (35) and Dirichlet boundary condition (36). The algorithm converges similarly as observed in Subsection 6.1.

Table I. The types of boundary conditions. $\{a_{ij}\}_{1 \le i,j \le 4}$ are parameters in Eq (2-3).

type	a_{11}	a_{12}	a_{13}	a_{14}	a_{21}	a_{22}	a_{23}	a_{24}	condition on
Neumann	1	0	0	0	0	1	0	0	y_s, y'_s
Dirichlet	1	0	0	0	0	0	1	0	y_s, y_e
Mix$_1$	1	0	0	0	0	0	0	1	y_s, y'_e
Mix$_2$	1	1	0	0	0	0	1	1	$y_s + y'_s, y_e + y'_e$

6.1. PERFORMANCE ON THE ODEs WITH BOUNDED $p(x), q(x), r(x)$

In this subsection, we apply TIBA to a second-order linear ODE with continuously differentiable $p(x), q(x), r(x)$. For a given $\hat{y}(x)$ over $[s, e]$, we set $r(x)$ in Eq. 1 as follows

$$r(x) := w(x)\hat{y}''(x) - p(x)\hat{y}'(x) - q(x)\hat{y}(x).$$

It is clear that $\hat{y}(x)$ solves ODE (1-3) if α, β are defined by

$$\alpha = a_{11}\hat{y}(s) + a_{12}\hat{y}'(s) + a_{13}\hat{y}(e) + a_{14}\hat{y}'(e)$$
$$\beta = a_{21}\hat{y}(s) + A_{22}\hat{y}'(s) + a_{23}\hat{y}(e) + a_{24}\hat{y}'(e).$$

We use the following setting for all the tests conducted in this subsection

$$w(x) = (x - e)^{\eta}, \quad p(x) \equiv 0.1, \quad q(x) \equiv 1, \quad \hat{y}(x; \theta) = x^2 \cos\theta x,$$
$$\theta \in \{\pi/2, \pi, 3\pi/2, 2\pi, 7\pi/2, 4\pi\}, \quad \eta \in \{0, 1, 2\}, \quad [s, e] = [1, 3],$$

Figure 1 compares the numerical solution y_{tiba} to the target solution y_{true} under the setting $\theta = \pi, 4\pi$ respectively with $q = 7, \eta = 0$ under
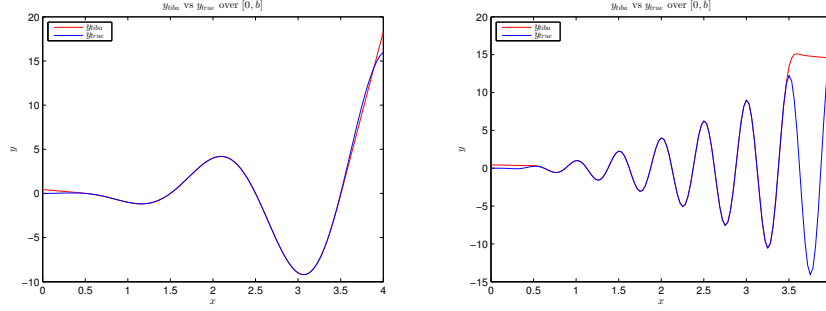
Figure 1. The comparison of TIBA's solution y_{tiba} (red) and the target function y_{true} (blue) with $\theta = \pi$ (left) and $\theta = 4\pi$ (right) under Dirichlet boundary condition and $q = 7, \eta = 0$.

Dirichlet boundary condition. Note that two curves are almost identical over the range $[s, e]$ as expected.

6.1.1. *Convergence*

We test the convergence performance by three sets of grid points with $q = 6, 7, 8$. Under each of scenarios, we report two sets of results, labeled as *tiba* and *trin* respectively. *tiba* follows Algorithm 4.1; *trin* is same as Algorithm 4.1 but without multiplying p, q, r by the cut-off function h to extend them as smooth periodic functions with period $2b$. With *trin*, the second derivative $y''(x)$ of the solution y can't be treated as a smooth periodic function with period $2b$ in general. Table II shows the maximum errors of an estimation $\tilde{v}(x)$ on a target function $v(x)$ under those scenarios, i.e.

$$max_{error} = \max_{x \in [s,e]} |\tilde{v}(x) - v(x)|.$$

One can observe that TIBA converges quickly with decent accuracy at

Table II. The maximum errors of the test scenarios described in above paragraph with $\theta = \pi/2, \eta = 0$

q	Neumann		Dirichlet		mix₁		mix₂	
	tiba	*trin*	*tiba*	*trin*	*tiba*	*trin*	*tiba*	*trin*
6	7.3E-06	4.4E-04	1.1E-08	1.2E-06	3.8E-06	4.5E-04	2.8E-05	1.1E-03
7	7.9E-09	5.5E-05	7.6E-12	4.2E-08	4.5E-09	5.7E-05	2.9E-08	1.3E-04
8	2.5E-11	6.9E-06	1.7E-12	1.3E-09	1.8E-11	7.1E-06	9.0E-11	1.7E-05

$q = 8$ and it significantly outperforms *trim* for all cases.

6.1.2. *Impact on θ*

Trigonometrical interpolation is expected to outperform difference-based approximation in the treatment of highly oscillated functions as shown in [7]. Table III shows the impact of θ on the performance of TIBA under the setting $\eta = 0, q = 7$. In addition to the four testing scenarios, we include the maximum errors of the classic Runge-Kutta method under *Neumann* initial condition, labeled as $RK4$ in the table. The

Table III. The maximum errors of TIBA with the setting $\eta = 0, q = 7$. The results include all covered boundary conditions together with the classic RK4 under Neumann condition in column $RK4$.

θ	$Neumann$	$RK4$	$Dirichlet$	Mix_1	Mix_2
π	9.3E-09	1.6E-06	6.4E-11	3.2E-08	3.9E-08
2π	3.4E-08	2.8E-05	1.6E-10	1.0E-07	9.2E-08
4π	2.8E-07	4.5E-04	1.2E-09	3.6E-07	2.4E-06

numerical results shows that TIBA keeps at a decent performance when θ increase from π to 4π, especially under *Dirichlet* boundary condition, and significantly outperforms RK4 in a consistent way.

6.1.3. *Impact on η*

Difference-based methods are not effective to cope with the singular ODE with $\eta > 0$ as mentioned in Section 1. TIBA is not restricted by the request $|w(x)| > 0$ as long as $rank(\Phi) = M + 1$ holds. We compare TIBA performance at three levels of $\eta = 0, 1, 2$ and the results are shown in Table IV. One can observe consistent performance of TIBA

Table IV. The performance of TIBA with three levels of η under the setting $\theta = \pi/2, q = 7$

η	$Neumann$	$Dirichlet$	mix_1	mix_2
0	7.9E-09	7.6E-12	4.5E-09	2.9E-08
1	2.0E-09	4.3E-11	4.3E-11	1.7E-09
2	2.9E-09	1.1E-10	1.1E-10	8.7E-09

under different degrees of roots of $w(x)$.

6.2. Treatment of singular solutions

We applied the transformation proposed in Section 5 to solve the ODE (34-36) with $[s, e] = [1, 3]$ in this subsection. *tibo* can be used to solve the transformed ODE (43-45) and the solution can be approximated by $y(x) = v(x)/(x - e)$. Figure 2 compares the numerical solution y_{tiba} to the target solution y_{true} under the setting $\theta = 2\pi, 3\pi$ respectively with $q = 7$ under Dirichlet boundary condition over the range $[s, e)$. Note that two curves approach to ∞ as $x \to e$.
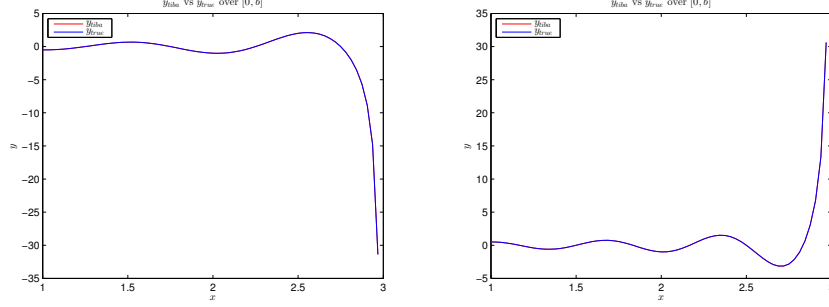

Figure 2. The comparison between the target solution y_{true} (blue) and the estimation y_{tiba} with $\theta = 2\pi$ (left) and $\theta = 3\pi$ (right) under $q = 7$

Table V shows the maximum error of TIBA defined in Subsection 6.1.1. We can see TIBA converges quickly with decent accuracy at $q = 8$ and its performance is stable on the oscillating parameter θ.

Table V. The maximum errors of TIBA for 6 testing scenarios with $\theta = \pi, 3\pi$ and $q = 6, 7, 8$

θ q	π $Neumann$	3π $Neumann$	π $Dirichlet$	3π $Dirichlet$
6	2.3E-06	3.1E-05	1.2E-08	2.4E-07
7	2.8E-09	8.9E-09	7.5E-12	1.7E-11
8	3.6E-12	4.4E-12	3.8E-13	5.5E-13

7. Summary

In this paper, we propose a trigonometric-interpolation based approach (TIBA) to approximate solutions of second-order linear ODEs with mixed boundary conditions. TIBA has the flexibility to cope with a singular second-order linear ODEs and can be used to estimate singular solutions by a transformation. The algorithm is expected to coverage at an optimal rate aligned to the order of smoothness of ODE. The numerical experiments in Section 6 show that TIBA can achieve decent accuracy by moderate number of grid points under various boundary conditions and outperforms significantly the classic Runge-Kutta under Neumann condition.

Appendix

A. The derivation of Eq (20-22)

In this Appendix, we derive Eq (20-22). Let S, C be defined by Eq. (16). Recall the following identity established in [7].

$$\sum_{j=0}^{n-1} j \sin \frac{2\pi jk}{2n} = (-1)^{k+1} \frac{n}{2} \cot \frac{\pi k}{2n}, \qquad 0 < k < 2n. \qquad (46)$$

We start with two special cases u_0 and u_M. By Eq (13),

$$u_0 = a_0 I - \frac{b}{\pi} I' \times diag(1/K)B$$

$$= v_0 \left(\frac{2\pi}{bM^2} I' \cdot diag(K) \cdot SK - \frac{2\pi}{Mb} I' \cdot diag(K)SI - \frac{1}{b} \right)$$

$$+ v_M \left(\frac{1}{b} - \frac{2\pi}{bM^2} I' \cdot diag(K)SK \right) + \frac{2\pi}{bM} I' \cdot diag(K)SV$$

$$= v_0 \left(\frac{2\pi}{bM^2} \sum_{1 \le i,j < M} ij \sin \frac{2\pi ij}{N} - \frac{2\pi}{Mb} \sum_{1 \le i,j < M} i \sin \frac{2\pi ij}{N} - \frac{1}{b} \right)$$

$$+ v_M \left(\frac{1}{b} - \frac{2\pi}{bM^2} \sum_{1 \le i,j < M} ij \sin \frac{2\pi ij}{N} \right) + \frac{2\pi}{bM} \sum_{1 \le i,j < M} v_i j \sin \frac{2\pi ij}{N}$$

$$= v_0 \left(\frac{\pi}{bM} \sum_{1 \le i < M} i(-1)^{i+1} \cot \frac{\pi i}{N} - \frac{\pi}{b} \sum_{1 \le j < M} (-1)^{j+1} cot \frac{j\pi}{N} - \frac{1}{b} \right) \quad (47)$$

$$+ v_M \left(\frac{1}{b} - \frac{\pi}{bM} \sum_{1 \le i < M} i(-1)^{i+1} cot \frac{\pi i}{N} \right) + \frac{\pi}{b} \sum_{1 \le i < M} v_i (-1)^{i+1} \cot \frac{i\pi}{N}.$$

For the treatment of u_M, applying Eq (46) to Eq (13), we have

$$u_M = a_0 I - \frac{b}{\pi} I_a \cdot diag(1/K)B$$

$$= v_0(\frac{2\pi}{bM^2} I_a \cdot diag(K)SK - \frac{2\pi}{Mb} I_a \cdot diag(K)SI - \frac{1}{b})$$

$$+ v_M(\frac{1}{b} - \frac{2\pi}{bM^2} I_a \cdot diag(K)SK) + \frac{2\pi}{bM} I_a \cdot diag(K)SV$$

$$= v_0(\frac{2\pi}{bM^2} \sum_{1 \le i,j < M} (-1)^i ij \sin\frac{2\pi ij}{N} - \frac{2\pi}{Mb} \sum_{1 \le i,j < M} (-1)^i i \sin\frac{2\pi ij}{N} - \frac{1}{b})$$

$$+ v_M(\frac{1}{b} - \frac{2\pi}{bM^2} \sum_{1 \le i,j < M} (-1)^i ij \sin\frac{2\pi ij}{N}) + \frac{2\pi}{bM} \sum_{1 \le i,j < M} (-1)^j v_i j \sin\frac{2\pi ij}{N}$$

$$= v_0(-\frac{\pi}{bM} \sum_{1 \le i < M} i \cot\frac{\pi i}{N} + \frac{2\pi}{Mb} \sum_{1 \le i,j < M} i \sin\frac{2\pi i(M-j)}{N} - \frac{1}{b})$$

$$+ v_M(\frac{1}{b} + \frac{\pi}{bM} \sum_{1 \le i < M} i \cot\frac{\pi i}{N}) - \frac{2\pi}{bM} \sum_{1 \le i,j < M} v_i j \sin\frac{2\pi(M-i)j}{N}$$

$$= v_0(-\frac{\pi}{bM} \sum_{1 \le i < M} i \cot\frac{\pi i}{N} + \frac{\pi}{b} \sum_{1 \le j < M} (-1)^{M-j+1} \cot\frac{(M-j)\pi}{N} - \frac{1}{b})$$

$$+ v_M(\frac{1}{b} + \frac{\pi}{bM} \sum_{1 \le i < M} i \cot\frac{\pi i}{N}) - \frac{\pi}{b} \sum_{1 \le i < M} (-1)^{M-i+1} v_i \cot\frac{(M-i)\pi}{N}$$

$$= v_0(-\frac{\pi}{bM} \sum_{1 \le i < M} i \cot\frac{\pi i}{N} + \frac{\pi}{b} \sum_{1 \le j < M} (-1)^{j+1} \tan\frac{j\pi}{N} - \frac{1}{b})$$

$$+ v_M(\frac{1}{b} + \frac{\pi}{bM} \sum_{1 \le i < M} i \cot\frac{\pi i}{N}) - \frac{\pi}{b} \sum_{1 \le i < M} (-1)^{i+1} v_i \tan\frac{i\pi}{N}. \tag{48}$$

For other u_i with $0 < i < M$, we have by (13)

$$U(1 : M - 1) = a_0 I - \frac{b}{\pi} C \cdot diag(1/K)B$$

$$= a_0 I - \frac{2a_1\pi}{Mb} C \cdot diag(K)SI$$

$$- \frac{2a_0\pi}{M^2} C \cdot diag(K)SK + \frac{2\pi}{bM} C \cdot diag(K)SV$$

$$= v_0(\frac{2\pi}{bM^2} C \cdot diag(K)SK - \frac{2\pi}{Mb} C diag(K)SI - \frac{1}{b}I) \tag{49}$$

$$+ v_M(\frac{1}{b}I - \frac{2\pi}{bM^2} C \cdot diag(K)SK) + \frac{2\pi}{bM} C \cdot diag(K)SV(1 : M - 1).$$

Let $c_i = C(i, :)$ and for $0 < i < M$, by (49),

$$u_i = (v_M - v_0)(\frac{1}{b} - \frac{2\pi}{M^2 b} c_i \cdot diag(K) \cdot S \cdot K)$$

$$- v_0 \frac{2\pi}{bM} c_i \cdot diag(K) \cdot S \cdot I + \frac{2\pi}{bM} c_i \cdot diag(K) \cdot S \cdot V(1 : M - 1)$$

$$= v_0(\frac{\pi}{2b} \sum_{1 \le k < M} (-1)^{i+k} \cot(k, i) - \frac{\pi}{2bM} \sum_{1 \le k < M} (-1)^{i+k} k \cdot \cot(k, i) - \frac{1}{b})(50)$$

$$+ v_M(\frac{\pi}{2bM} \sum_{1 \le k < M} (-1)^{i+k} k \cdot \cot(k, i) + \frac{1}{b}) - \frac{\pi}{2b} \sum_{1 \le k < M} (-1)^{i+k} \cot(k, i) v_k.$$

One can see that Eq (19) represent the system 47), (48), and (50).

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