Graphical Abstract

On Trigonometric Interpolation and Its Applications-I

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Highlights

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- Introduce a new trigonometric interpolation algorithm that can be carried out by Fast Fourier Transform for optional performance; establish relevant properties, especially showing that it has desired convergence rate for both interpolant and high order derivatives of interpolant, which provides theoretic support for its applications. (Part I);
- Extend the algorithm so it can be used to non-periodic function (Part II);
- Study numerical performance of the algorithm (Part II);
- Study applications of the algorithm, including to estimate integrals and solve non-linear ordinary differential equation (ODE). Test results show that it outperforms Trapezoid/Simpson method to cope with integral and standard Runge-Kutta algorithm in handling ODE. (Part II)

On Trigonometric Interpolation and Its Applications-I

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Abstract

This is the first part of our study on trigonometric interpolation and its applications. In this paper, we propose a new trigonometric interpolation algorithm and establish relevant convergent properties. The method adjusts an existing trigonometric interpolation algorithm such that it can better leverage Fast Fourier Transform to enhance efficiency. The algorithm can be formulated in a way such that certain cancellation effects can be effectively leveraged for error analysis, which enables us to obtain the desired uniform convergent rate of the approximation not only to a target function, but also to its derivatives up to a given order. Hence, it provides theoretical support for the applications where approximations of functions and their derivatives to certain order are all required as in solving various types of differential and Integro-equations.

In the second part of the study, we further enhance the algorithm so it can be applied to non-periodic functions defined on bounded intervals. Numerical results confirm highly accurate performance of the algorithm. For its applications, we demonstrate how it can be applied to estimate integrals with better performance than standard Trapezoid/Simpson method to cope with highly oscillated integrands. For a more sophisticated application, an optimization-based algorithm is developed to solve a non-linear ordinary differential equation (ODE). The numerical results show that it significantly outperforms the traditional difference-based methods such as standard Runge-Kutta algorithm. The same idea has been used to develop new algorithms to solve ODE and Integro-differential equations in subsequent papers. Those algorithms have several advantages compared to a difference-based method. They work well in general settings on high order ODEs with various boundary conditions; convergence with high accuracy; and have the flexibility to capture certain desired features of the target solution (e.g., convexity and monotone) as well as to meet certain constraints due to implicit physical laws in physics, biology, and engineering.

Keywords: Fourier Series, Trigonometric Interpolation, Fast Fourier Transformation (FFT), Ordinary Differential Equation, Runge-Kutta method, Trapezoid rule, Simpson's rule.
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1. Introduction

Let $f(x)$ be a periodic function with period $2b$. Assume that its $K+1$-th derivative $f^{(K+1)}$ is bounded by D_{K+1} for a certain integer $K \geq 0$. It is well-known that $f(x)$ can be represented by its Fourier series

$$f(x) = \frac{A_0}{2} + \sum_{j\geq 1} A_j \cos\frac{j\pi x}{b} + B_j \sin\frac{j\pi x}{b}, \tag{1}$$

where

$$A_j \;=\; \frac{1}{b}\int_{-b}^{b} f(x)\cos\frac{j\pi x}{b}dx, \quad B_j = \frac{1}{b}\int_{-b}^{b} f(x)\sin\frac{j\pi x}{b}dx.$$

One can estimate $f(x)$ by the sum of first $2n+1$ terms

$$f(x) \approx \tilde{f}_n(x) := \frac{A_0}{2} + \sum_{1\leq j\leq n} A_j \cos\frac{j\pi x}{b} + B_j \sin\frac{j\pi x}{b}.$$

A function like $\tilde{f}_n(x)$ is called a trigonometric polynomial of degree n in the literature (see [1]). A trigonometric interpolant of degree n with given grid points $t_0 < t_1 \ldots, < t_{N-1}$ is a trigonometric polynomial

$$f_n(x) = a_{0,n} + \sum_{1\leq j\leq n} a_{j,n} \cos\frac{j\pi x}{b} + b_{j,n} \sin\frac{j\pi x}{b},$$

such that

$$y_k := f(t_k) = f_n(t_k), \quad k = 0, \cdots, N-1.$$

The number N of the grid point set is often required to be same as the number of the coefficients of $f_n(x)$ to ensure the desired uniqueness of the solution. In general, a set of equispaced grid points is recommended for a trigonometric interpolation (see [2])

$$t_j = \alpha + \frac{2bj}{N}, \qquad 0 \le j < N,$$

where α is a fixed constant. It is not hard to show that a unique solution is available and can be implemented by Fast Fourier Transform (FFT) if $N = 2n+1$ is odd. To achieve optimal performance of FFT, a trigonometric interpolation with $N = 2^h$ for some h is desired (see [3]). An algorithm with even grid points is available by Theorem 3.5-3.6 in [4] summarized as follows.

Theorem 1. *Let $f(x)$ be a periodic function with period $2b$ and $N = 2M$ be an even integer and define*

$$x_j = -b + j\lambda, \quad \lambda = \frac{2b}{N}, \quad y_j = f(x_j), \qquad 0 \le j < N.$$

Then there is a unique trigonometric polynomial defined by

$$\begin{aligned}
\tilde{Q}_M(x) &= \sum_{0\le j\le M} a_j^e \cos\frac{j\pi x}{b} + \sum_{1\le j<M} a_j^o \sin\frac{j\pi x}{b}, && (2)\\
a_0^e &= \frac{1}{N}\sum_{0\le j<N} y_j, && (3)\\
a_j^e &= \frac{2}{N}\sum_{0\le k<N} (-1)^j y_k \cos\frac{2\pi jk}{N}, \quad 1\le j<M, && (4)\\
a_M^e &= \frac{1}{N}\sum_{0\le j<N} (-1)^j y_j, && (5)\\
a_j^o &= \frac{2}{N}\sum_{0\le k<N} (-1)^j y_k \sin\frac{2\pi jk}{N}, \quad 1\le j<M, && (6)
\end{aligned}$$

such that

$$\tilde{Q}_M(x_k) = y_k, \qquad 0 \le k < N.$$

Furthermore, the error is bounded by

$$|R_N(x)| := |f(x) - \tilde{Q}_n(x)| \le \frac{\xi_M}{M^{K-\frac{1}{2}}}, \qquad \xi_M = o(1). \tag{7}$$

Hence, the error $R_N(x)$ converges to zero uniformly with respect to x as $N \to \infty$ if $K \geq 1$, and the rate of convergence of $\tilde{Q}_n(x)$ automatically becomes faster for smoother f, remarkable advantage compared to polynomial interpolation when the convergence rate is limited by the degree of the polynomial (see [4]).

Trigonometric interpolation is believed to be suitable for periodic function. For non-periodic functions, the Chebyshev-polynomial based interpolation is often used in the literature (see [4]). Some comparison between the two interpolation can be found in [2]. We highlight the major pros and cons of trigonometric interpolation (T) and Chebyshev polynomial interpolation (C).

1. T can be easily implemented using FFT while C need cope with Chebyshev polynomials without a close-form expression.
2. Attractive analytic representation benefits T in its applications, especially where the close forms of derivatives and integrals of target function are desired. On the contrary, implicit representation of Chebyshev polynomial restricts its applications to a certain degree.
3. C can be applied to general functions, but T is limited to periodic functions.
4. The performance of both C and T depends on the smoothness of target function on the whole domain. Lack of the smoothness at a single point would deteriorate overall performance significantly.

Trigonometric interpolation with an even number of points is studied in [1]. By imposing a constraint to ensure uniqueness, the interpolant is constructed based on the trigonometric Lagrange basis functions, and therefore the method is more flexible with non-equispaced points. Recent research on trigonometric interpolation follows more or less the pioneer work [12]. For a complex-valued function on some interval, the interpolants are constructed in a barycentric form for a given set of interpolation grid points. A trigonometric interpolation algorithm is introduced in [13] by constructing a barycentric rational function approximation selecting grid points progressively via a greedy algorithm. In [14], a set of interpolating points is introduced to construct linear rational trigonometric interpolants written in barycentric form with exponential convergence rate and has been used effectively to interpolate functions on two-dimensional star-like domains. In addition, some recent research has been done on Hermite interpolation. Note that a Hermite interpolation interpolates not only a function on a given set of grid points, but also the values of first m derivatives with certain integer $m \geq 0$. An algorithm is developed in [7] to construct a barycentric trigonometric Hermite interpolant via an iterative approach. More references in that direction can be found in [8]-[11].

As the first part of this study on trigonometric interpolation and its applications, we introduce an adjusted version of Theorem 1 that can be carried out by FFT with optimal operations. The adjustment generates some minor error on half of the grid points as downside. But the advantage is significant. It aligns the number of interpolation grid points with degree of trigonometric polynomial so that we can fully leverage the power of FFT. For performance analysis, the alignment enables us to leverage periodicity and symmetry of certain quantities that occurred in the study of estimation error. For example, we are able to derive a simple relation of interpolants when grid points are doubled (Lemma 2) and improve the error rate from $M^{-K+\frac{1}{2}}$ in Theorem 1 to M^{-K}. More importantly, we are able to establish uniform convergence theorem with rate $O(\frac{1}{N^{K-k}})$ for k-th ($k < K$) derivative of f as well, and thus provide a theoretic support for certain applications such as solving differential/integral equations or searching an optimal solution where the gradient of the target function needs to be calculated efficiently.

In the second part of this study [22], we shall overcome the major limitation 3 of trigonometric interpolation mentioned above and enhance the algorithm to cover a non-periodic function f defined over a bounded interval $[s, e]$. Instead of transforming f to a periodic function as in Chebyshev-polynomial interpolation, we assume that f can be smoothly defined over $[s-\delta, e+\delta]$ for some $\delta > 0$. We then extend periodically f by a cut-off function to keep the smoothness of f. The algorithm outputs a trigonometric interpolant with high rate convergence. Such extension is not unique and a close-form solution is adopted with decent testing results in Section 6.

An optimal error estimation of a trigonometric interpolation can be quite challenging. The cancellation effect, which likely avoids error propagation, might be the major reason why the actual performance of a trigonometric interpolation tends to be better than what can be estimated vigorously as explained in Item 5 of Remark 2. The challenge to sufficiently leverage the cancellation effect could be a major obstacle for an optimal performance analysis. The error estimation in this study depends more on skillfully maneuvering saw-tooth qualities and simplifying them in a proper format. Without loss of generosity, we assume the target periodic function f is either even or odd since any function can always be decomposed as the sum of an even and odd function. The symmetry and periodicity of f play important roles in the analysis conducted in this study.

Considering the analytic attractiveness of the trigonometric polynomial, especially in handling differential and integral operations, we expect that the proposed trigonometric estimation of a general function can be used in a wide spectrum. As the starting point of our studies on its applications, we show that it can be used to estimate an integral with stabler and more accurate performance than popular Trapezoid and Simpson rules in coping with oscillated integrands in the second part of this study [22]. We further develop Algorithm 2 to solve a first-order non-linear ODE by searching a trigonometric representation of the target solution. Traditional numerical methods like Runge-Kutta method (RK4) [5] interpret derivatives by difference and recursively estimates the solution **locally** at one grid point per a step. As such, a local estimation error might have significant compounding effect and deteriorate overall performance, especially when the underlying solution is highly oscillating. Algorithm 2 is a global method in the sense that the solution is represented by a trigonometric polynomial and its values at grid points are **globally** estimated through an optimization process. Since the gradient of associated optimal objective function can be formulated through FFT, one can use a sufficient number of grid points to achieve high accuracy. The numerical experiments in Section 7.2 show that it outperforms significantly RK4 .

Algorithm 2 has been extended to solve ODE, differential-algebraic equations (DAEs), and integro-differential equations (IDEs) in more general settings. In [27], an algorithm is developed to solve a non-linear d-dim ODE system of order 1 with constraints, which recently becomes an active topic in machine learning [28]. Second order non-linear ODE with general boundary conditions is studied [23]. The interpolation algorithm has been enhanced to a 2-dim space and is used to develop algorithms for the solution of a second-order Fredholm/Volterra IDE in [25] and [26]. In addition to produce decent approximations, the optimization-based approach provides us the flexibility to attack general boundary problems, meet certain constraints, and identify solutions with some special features such as convexity, monotone, and bounded within a given range.

The study is structured into two papers. Part I focuses on the establishment of main theorems and is organized as follows. In Section 2, we present the major results, including the new trigonometric interpolation algorithm on periodic functions mentioned above, relevant convergence properties, as well as the enhancement of the algorithm for its use to non-periodic functions and its performance. Section 3 establishes the algorithm (Theorem 2), estimations on coefficients of interpolants (Theorem 3) and convergence properties (Theorem 4). The proof depends on a few key equations and relevant derivations are partly moved to Appendix A.

The second part of this study [22] is organized as follows. Section 5 is used to develop Algorithm 1 for the enhancement of trigonometric interpolation for non-periodic functions. In Section 6, we conduct some numerical tests. The result confirms that the performance of Algorithm 1 is sensitive to the smoothness of f and is quite satisfactory when f is sufficiently smooth. In addition, we explain that the error of trigonometric approximation often exhibits cancellation effect and thus does not cause significant compounding errors, a remarkable advantage compared to polynomial-based approximation. Section 7 is devoted to develop Algorithm 2 and test its performance. The summary is made in Section 8.

2. Main Results

In this subsection, unless otherwise specified, $f(x)$ denotes either an even or odd periodic function with period $2b > 0$ and its $K + 1$-th derivative $f^{(K+1)}(x)$ exists and is bounded by D_{K+1} for some positive integer $K \geq 1$. Eq. (1) is reduced to

$$f(x) = \begin{cases} \frac{A_0}{2} + \sum_{j\geq 1} A_j \cos\frac{\pi jx}{b}, & \text{if } f \text{ is even,} \\ \sum_{j\geq 1} B_j \sin\frac{\pi jx}{b}, & \text{if } f \text{ is odd.} \end{cases}$$

For a given even integer $N = 2M = 2^{q+1}$, Theorem 1 provides us a trigonometric interpolant with the following grid points:

$$\begin{aligned} x_j &:= -b + j\lambda, \quad \lambda = \frac{2b}{N}, \quad 0 \leq j < N, && (8) \\ y_j &:= f(x_j), && (9) \\ y_j &= \begin{cases} y_{N-j} & \text{if } f \text{ is even,} \\ -y_{N-j} & \text{if } f \text{ is odd.} \end{cases} \end{aligned}$$

If f is odd, then $f(-b) = f(0) = 0$ and there are $M - 1$ free points $\{(x_j, y_j)\}_{1\leq j<M}$, aligned with the number of coefficients in Eq. (6) and a_j^o can be solved by FFT with optional operations.

The situation for the even case is slightly different. There are $M+1$ free points $\{(x_j, y_j)\}_{0\le j\le M}$ aligned with $M+1$ coefficients to ensure uniqueness of interpolant. There are two undesired features: 1) a^e_M defined by Eq. (5) is not consistent with the derived a^e_j by Eq. (4), and 2) there are odd number $M+1$ terms in $\tilde{Q}_M(x)$ defined by Eq. (2). A solution to address this issue is to combine first and last terms of $\tilde{Q}_M(x)$ whose impact on grid point x_k is

$$a_0 + a_M \cos\frac{\pi M x_k}{b} = a_0 + (-1)^k a_M = \begin{cases} \frac{1}{M}\sum_{0\le j<M} y_{2j} & \text{if } k \text{ is even,} \\ \frac{1}{M}\sum_{0\le j<M} y_{2j+1} & \text{if } k \text{ is odd.} \end{cases}$$

Replacing $a^e_0 + a^e_M \cos\frac{\pi M x}{b}$ with $a_0 = \frac{1}{M}\sum_{0\le j<M} y_{2j}$, and keeping other coefficients, we obtain a new polynomial that fits to all even grid points at x_{2j} and approximate all odd ones at x_{2j+1} with a uniform error:

$$\epsilon_M = \frac{1}{M}\sum_{0\le j<N} (-1)^j y_j. \tag{10}$$

We thus obtain

Theorem 2. *Let $f(x)$ be an periodic function with period $2b$ and $N = 2M$ be an even integer and x_j, y_j be defined by Eq. (9).*

- *If $f(x)$ is even, then there is an unique trigonometric polynomial of degree $M-1$*

$$\begin{aligned} f_M(x) &= \sum_{0\le j<M} a_j \cos\frac{j\pi x}{b}, \\ a_0 &= \frac{1}{M}\sum_{0\le j<M} y_{2j}, && (11) \\ a_j &= \frac{2}{N}\sum_{0\le k<N} (-1)^j y_k \cos\frac{2\pi jk}{N}, \quad 1\le j<M, && (12) \end{aligned}$$

  such that for $0\le k<M$,

$$\begin{aligned} f_M(x_{2k}) &= y_{2k}, && (13) \\ f_M(x_{2k+1}) &= y_{2k+1} + \epsilon_M. && (14) \end{aligned}$$

  In other words, $f_M(x)$ fits to all even grid points and shifts away in parallel from all odd grid points by ϵ_M.

- *If $f(x)$ is odd, then there is an unique trigonometric polynomial of degree $M-1$*

$$\begin{aligned} f_M(x) &= \sum_{0\le j<M} a_j \sin\frac{j\pi x}{b}, \\ a_j &= \frac{2}{N}\sum_{0\le k<N}(-1)^j y_k \sin\frac{2\pi jk}{N}, \quad 0\le j<M \end{aligned}$$

such that it fits to all grid points, i.e.

$$f_M(x_k) = y_k, \quad 0\le k<N.$$

To keep self-contained, we provide an elementary proof of Theorem 2 in Subsection 3.2 although it is a direct conclusion of Theorem 1. A few remarks are in order.

REMARK 1. 1. *The coefficients can be obtained by Inverse Fast Fourier Transform (ifft),*

$$\{a_j(-1)^j\}_0^{N-1} = \begin{cases} 2\times Real(ifft(\{y_k\}_{k=0}^{N-1})), & \text{if } f \text{ is even },\\ 2\times Imag(ifft(\{y_k\}_{k=0}^{N-1})), & \text{if } f \text{ is odd.}\end{cases}$$

and replace a_0 by Eq. (11) if f is even. To fully leverage power of FFT, N should be a radix-2 integer, i.e. $N=2^h$ for a positive integer h and operation cost of ifft is $\frac{N}{2}\log_2 N$ as shown in [3].

2. *If f is even, the error ϵ_M by Eq. (10) is $O(\frac{1}{N^{K+1}})$ by applying following Euler-Maclaurin identity:*

$$h\sum_{0\le l<n-1} f(lh) = \int_{-b}^{b} f(x)dx - (\frac{-2b}{n})^{K+1}\int_{-b}^{b}\tilde{B}_{K+1}(\frac{x}{2b})f^{(K+1)}(x)dx \tag{15}$$

where $n\ge 2$ is positive integer and $h=2b/n$ and $\tilde{B}_{K+1}$ is the periodic extension of $K+1$-th Bernoulli polynomial [6].

3. *The uniqueness of the interpolation is helpful for the applications where optimization process is used to find f_M, as shown in Section 7.*

It is not hard to see a_j ($j\ge 1$) is Trapezoidal approximation of Fourier expansion coefficient A_j or B_j. It is natural to expect that a_j approaches to 0 (as A_j does) as $j\to\infty$. Theorem 3 provides a boundary of a_j in j and N.

Theorem 3. *Assume that* $|f^{(K+1)}(x)|$ *exists with an upper bound* D_{K+1}*, then*

$$|a_j| \le \frac{C(D_{K+1})}{N^{K+1}\sin^{K+1}\frac{\pi j}{N}}, \quad 1 \le j < M, \tag{16}$$

where $C(D_{K+1})$ *is a constant depending on* D_{K+1}*.*

Notice that a_j depends on j and N, and the estimation (16) shows how a_j decays to 0 in two dimensions. For a given j, one can see $|a_j|$ has order $O(\frac{1}{j^{K+1}})$ as $N \to \infty$, which is consistent to the order of Fourier coefficient A_j. For a given interpolant f_M with a fixed large N, magnitude of a_j approaches to 0 at order $\frac{1}{N^{K+1}}$ as $j \to M$. It is worthwhile to point out that the second half coefficients $\{a_j\}_{M/2\le j<M}$ decays uniformly with $\frac{1}{N^{K+1}}$, which is one of key observations to establish convergence Theorem 4.

The proof of Theorem 3 mainly depends on expressing a_j in terms of $K+1$-th forward difference as shown in Eq. (36) and key ingredient is classic Abel Transform. Details can be found in Section 3.3.

With Estimation (16), we can show uniform convergence properties of $f_M(x)$ as below.

Theorem 4. *Assume that* $|f^{(K+1)}(x)|$ *exists with an upper bound* D_{K+1}*, then*

$$|f_M(x) - f(x)| \le \frac{C_1(D_{K+1})}{N^K}, \tag{17}$$

$$|f_M^{(k)}(x) - f^{(k)}(x)| \le \frac{C_2(D_{K+1})}{N^{K-k}}, \quad 1 \le k < K. \tag{18}$$

where $C_1(D_{K+1})$ *and* $C_2(D_{K+1})$ *are two constants depending on* D_{K+1}*.*

The error estimation (17) is different from [4]. Estimation (7) of Theorem 2 is based on the breakdown of $f_M(x)$ into two components: a trigonometric polynomial and the associated residue. The overall convergent rate $N^{-K+0.5}$ of $R_N(x)$ is determined by the convergent rate of the residual component. Details can be found in [4]. The proof of Theorem 4 can be found in Section 3.4. We directly cope with f_M. As such, we not only get an extra accuracy rate by avoiding to handle the residual term mentioned above, but be able to obtain accuracy rate on derivatives. The adjustment on the coefficients on Theorem 2 makes it possible for us to derive a clean pattern of adjusted coefficients when interpolating grid points are doubled (Lemma 2), which plays a key role in establishment of Estimation (17)-(18).

Estimation (18) is more significant than the slight error improvement (17) to support the applications mentioned in Section 1, since the trigonometric estimation of a target solution $f(x)$ and its derived estimations of its k-th derivative $f^{(k)}$ need to converge sufficiently with k up to the highest derivative order involved in a particular application. Trigonometric interpolation-based algorithms are expected to converge effectively accordingly to Estimation (18).

For a non-periodic function f over a bounded interval $[s-\delta, e+\delta]$ for some $\delta > 0$, we extend f to a periodic function with the same smoothness by a cut-off function $h(x) \in C^\infty(R)$ with following property

$$h(x) = \begin{cases} 1 & x \in [s, e], \\ 0 & x < s - \delta \text{ or } x > e + \delta. \end{cases}$$

A closed-form cut-off function $h(x)$ (Eq. (1) in [22]) is used and an enhanced trigonometric interpolation method is formulated in Algorithm 1 in Section 5. The output $\hat{f}_M(x)$ interpolates $f(x)|_{[s,e]}$. In Section 6.1, numerical test results confirm that $\hat{f}$ has high degree accuracy. We also demonstrate numerical evidences that the error of $\hat{f}_M(x)$ likely exhibits "local property", i.e. the estimation error at a grid point does not propagate and cause significant compounding error outside its neighborhood, which is not the case for polynomial-based approximations as shown in Section 7.2. As such, the performance of $\hat{f}(x)$ should be even better than what is concluded in Theorem 4. Details can be found in Section 6.3. Algorithm 1 is applied to estimate integrals and solve non-linear ordinary differential equation (ODE) as outlined in Algorithm 2 in Section 7. The numerical results show that it outperforms the Trapezoid/Simpson method to cope with integral and standard Runge-Kutta algorithm in handling ODE.

3. The proof of Theorem 2, 3 and 4

This section is used to set stage for the framework to be built and prove three theorems introduced in Section 2. It starts with reviewing some relevant identities and developing required equations, and then proves each of covered theorems in three subsections. Let f be a $2b$-periodic function with $K+1$ derivative bounded by D_{K+1} and $C(D_{K+1})$ denote a generic constant that depends on D_{K+1}. Note that $C(D_{K+1})$ may change on different situations.

3.1. Preliminary Algebraic Tools

The following classic Abel's transform (19) (see [29]) plays a key role in the error analysis of this section. For any two sequences of numbers $\{\alpha_i, \gamma_i\}_{i=0}^{n-1}$,

$$\sum_{i=0}^{k-1} \alpha_i \gamma_i = \alpha_{k-1}\Gamma_{k-1} - \sum_{i=0}^{k-2}(\alpha_{i+1} - \alpha_i)\Gamma_i, \qquad 1 \leq k \leq n, \tag{19}$$

where $\Gamma_i = \sum_{j=0}^{i} \gamma_j$. Throughout this paper, for any sequences with n elements, we always treat them as periodic sequences with period n, i.e $\alpha_l = \alpha_k$ and $\gamma_l = \gamma_k$ if $l = k \mod (n)$. We might modify the index range of a summation without further reminding as follows.

$$\sum_{i=0}^{n-1} \alpha_i \gamma_i = \sum_{i=k}^{n-1+k} \alpha_i \gamma_i.$$

Recall that, for a positive integer k, k-th forward difference is defined inductively by

$$\Delta_1 \alpha_i := \alpha_{i+1} - \alpha_i, \quad \Delta_k \alpha_i = \Delta_{k-1}(\Delta_1 \alpha_i).$$

One can derive inductively

$$\Delta_k \alpha_i = \alpha_{k+i} - k\alpha_{k+i-1} \cdots + (-1)^j C_k^j \alpha_{k+i-j} \cdots + (-1)^k \alpha_i,$$

where C_k^j is j-th coefficient of binomial polynomial $(1+x)^k$. It is clear that $\{\Delta_k \alpha_i\}_{i\in\mathbb{Z}}$ is periodic with same period n as $\{\alpha_i\}_{-\infty<i<\infty}$ such that

$$\sum_{i=0}^{n-1} \Delta_k \alpha_i = 0. \tag{20}$$

As a special case where $\Gamma_{n-1} = 0$, Eq. (19) is reduced to

$$\sum_{i=0}^{n-1} \alpha_i \gamma_i = -\sum_{i=0}^{n-1} \Delta_1 \alpha_i \Gamma_i. \tag{21}$$

Note Eq. (21) is equivalent to the cancellation of boundary terms occurring in integration by parts with periodic functions, and it plays a key role in derivation of estimation (36).

Adapt notations in Section 2, and note that k-th forward difference of f at any given point x is defined inductively by

$$\Delta_\lambda^1[f](x) = f(x+\lambda) - f(x), \quad \Delta_\lambda^k[f](x) = \Delta_\lambda^{k-1}[f](\Delta_\lambda^1[f](x)).$$

One can verify

$$\Delta_\lambda^k[f](x) = \sum_{0\le j\le k} (-1)^j C_k^j f(x+(k-j)\lambda).$$

For any integer $p \le k$, let

$$H(p,k) = \sum_{0\le j\le k} (-1)^j C_k^j j^p.$$

It is not hard to prove by induction that

$$H(p,k) = \begin{cases} k! & p = k, \\ 0 & p < k. \end{cases}$$

Applying Taylor expansion to each item in $\Delta_\lambda^k[f](x)$ at x, there exists $\xi \in [x, x+k\lambda]$ such that

$$\Delta_\lambda^k[f](x) = f^{(k)}(x)\lambda^k + C_k(x)\lambda^k, \tag{22}$$

where $C_k(x)$ is bounded and $\lim_{\lambda\to 0} C_k(x) = 0$ if $f^{(k)}$ exists and is bounded.

Recall following trigonometric identities

$$\begin{aligned} \sum_{j=0}^{n-1} \sin(jx) &= \frac{1}{2}\cot\frac{x}{2} - \frac{1}{2}\cot\frac{x}{2}\cos nx - \frac{1}{2}\sin nx, & (23) \\ \sum_{j=0}^{n-1} \cos(jx) &= \frac{1}{2} + \frac{1}{2}\cot\frac{x}{2}\sin nx - \frac{1}{2}\cos nx. & (24) \end{aligned}$$

for any x such that x/π is not an integer. By taking $x = \frac{2\pi k}{2n}$, we have

$$\sum_{j=0}^{n-1} cos\frac{2\pi jk}{2n} = \begin{cases} n & \text{if } k = 0 mod(2n), \\ 1 & \text{else if } k \text{ is old}, \\ 0 & \text{else if } k \text{ is even}, \end{cases} \tag{25}$$

and

$$\sum_{j=0}^{n-1} \sin \frac{2\pi jk}{2n} = \begin{cases} \cot \frac{\pi k}{2n} & \text{if } k \text{ is odd,} \\ 0 & \text{if } k \text{ is even.} \end{cases} \tag{26}$$

If x/π is not an integer, replacing x by $2x$ and n by $2n$ in Eq. (24) respectively, we obtain

$$\sum_{j=0}^{n-1} \cos(2jx) = \frac{1}{2} + \frac{1}{2} \cot x \sin 2nx - \frac{1}{2} \cos 2nx, \tag{27}$$

$$\sum_{j=0}^{2n-1} \cos(jx) = \frac{1}{2} + \frac{1}{2} \cot \frac{x}{2} \sin 2nx - \frac{1}{2} \cos 2nx. \tag{28}$$

Subtracting Eq. (27) from Eq. (28) implies

$$\sum_{j=0}^{n-1} \cos((2j+1)x) = \frac{1}{2}(\cot \frac{x}{2} - \cot x) \sin 2nx.$$

Plugging $x = \frac{2\pi k}{2n}$, we obtain

$$\sum_{j=0}^{n-1} cos \frac{2\pi(2j+1)k}{2n} = \begin{cases} 0 & \text{if } k \neq 0 mod(n), \\ n & \text{if } k/n \text{ is even,} \\ -n & \text{if } k/n \text{ is even.} \end{cases} \tag{29}$$

Applying the derivative on both sides of (24), we obtain

$$\sum_{j=0}^{n-1} j \sin jx = \frac{1}{4} \sin nx \csc^2 \frac{x}{2} - \frac{n}{2} \sin nx - \frac{n}{2} \cos nx \cot \frac{x}{2}.$$

Plugging $x = \frac{2\pi k}{2n}$ for $0 < k < 2n$ to above equations, we have

$$\sum_{j=0}^{n-1} j \sin \frac{2\pi jk}{2n} = (-1)^{k+1} \frac{n}{2} \cot \frac{\pi k}{2n}. \tag{30}$$

3.2. The proof of Theorem 2

The following observation is the key in the development of Theorem 2, whose proof can be found in Appendix A.

Lemma 1. *Adapt the notations in Section 2.*

- *If $f(x)$ is even, define*

$$\begin{aligned}\tilde{f}_M(x) &= \sum_{j=0}^{M-1} \tilde{A}_j \cos\frac{j\pi x}{b},\\ \tilde{A}_0 &:= \frac{2}{N}\sum_{k=0}^{N-1} y_k, \qquad \tilde{A}_j = a_j, \quad 1 \le j < M.\end{aligned}$$

Let $\tilde{y}_l = \tilde{f}_M(x_l)$ for $0 \le l < N$. Then

$$\tilde{y}_{2k} - y_{2k} = \frac{1}{M}\sum_{j=0}^{M-1} y_{2j+1}, \quad 0 \le k < M, \tag{31}$$

$$\tilde{y}_{2k+1} - y_{2k+1} = \frac{1}{M}\sum_{j=0}^{M-1} y_{2j}, \quad 0 \le k < M. \tag{32}$$

- *If $f(x)$ is odd, $\tilde{f}_M$ fits to all grid points, i.e.*

$$\tilde{y}_k - y_k = 0, \quad 0 \le k < N. \tag{33}$$

With Eqs. (31-33), we are ready to prove Theorem 2.

Proof:.

1. Let $f(x)$ be even. By definition,

$$\tilde{A}_0 - a_0 = \frac{1}{M}\sum_{j=0}^{M-1} y_{2j+1}, \tag{34}$$

and therefore by (31) and (34), we have

$$f_M(x_{2k}) - y_{2k} = \tilde{f}_M(x_{2k}) + a_0 - \tilde{A}_0 - y_{2k} = \tilde{y}_{2k} - \frac{1}{M}\sum_{j=0}^{M-1} y_{2j+1} - y_{2k} = 0.$$

Similarly, by (32) and (34), we have

$$f_M(x_{2k+1}) - y_{2k+1} = \tilde{y}_{2k+1} - \tilde{A}_0 + a_0 - y_{2k+1} = \frac{2}{N}\sum_{j=0}^{N-1}(-1)^j y_j.$$

To show uniqueness, assume that Eqs. (13, 14) hold, which implies for $0 \le k < M$

$$\begin{aligned} y_{2k} &= \sum_{j=0}^{M-1} a_j(-1)^j \cos\frac{2\pi j(2k)}{N}, \\ (y_{2k+1}+\epsilon_M) &= \sum_{j=0}^{M-1} a_j(-1)^j \cos\frac{2\pi j(2k+1)}{N}. \end{aligned}$$

Eqs. (11-12) can be derived based on by Eq (29).

2. Similarly, one can prove Theorem 2 in the case that $f(x)$ is odd.

$\square$

3.3. The proof of Theorem 3

In this subsection, we prove Theorem 3 by starting with the estimation of following quantities. For a given positive integer pair (l, k) with $l \le k$,

$$\begin{aligned} \phi_{l,k} &:= \sum_{m=0}^{N-1} \Delta_k y_{m-k} \cos\frac{2\pi ml}{N} = \sum_{m=0}^{N-1} \Delta_k^\lambda[f](x_{m-k}) \cos\frac{2\pi ml}{N}, \\ \psi_{l,k} &:= \sum_{m=0}^{N-1} \Delta_k y_{m-k} \sin\frac{2\pi ml}{N} = \sum_{m=0}^{N-1} \Delta_k^\lambda[f](x_{m-k}) \sin\frac{2\pi ml}{N}. \end{aligned}$$

By Eq. (20),

$$\phi_{l,k} = 2\sum_{m=0}^{N-1} \Delta_k^\lambda[f](x_{m-k}) \cos^2\frac{\pi ml}{n}.$$

Let $\Phi = \max_{x\in[-b,b]} \Delta_k^\lambda[f](x)$, we obtain

$$\phi_{l,k} \le 2\Phi \sum_{m=0}^{N-1} \cos^2\frac{\pi ml}{N} = \Phi \sum_{m=0}^{N-1} (\cos\frac{2\pi ml}{N} + 1) = \Phi N.$$

Similarly, we have $\phi_{l,k} \ge \phi N$ with $\phi = \min_{x\in[-b,b]} \Delta_k^\lambda[f](x)$. Applying same argument to $\psi_{l,k}$, we conclude that there exist $\xi_{l,k}, \theta_{l,k} \in [-b, b]$ such that

$$\phi_{l,k} = \Delta_k^\lambda[f](\xi_{l,k})N, \quad \psi_{l,k} = \Delta_k^\lambda[f](\theta_{l,k})N. \tag{35}$$

We are now ready to prove Theorem 3.

Proof:. By Eqs. (21), (23)-(24), and $\sum_{m=0}^{N-1} \Delta y_m = 0$, we have for $0 < l < M$

$$\begin{aligned}(-1)^l \frac{N}{2} a_l &= -\sum_{m=0}^{N-1} \Delta y_m \sum_{j=0}^{m} \cos \frac{2\pi jl}{N} \\ &= \frac{1}{2} \sum_{m=0}^{N-1} \Delta y_m (\cos \frac{2\pi l(m+1)}{N} - \frac{1}{2} \cot \frac{\pi l}{N} \sin \frac{2\pi l(m+1)}{N}) \\ &= \frac{1}{2} \sum_{m=0}^{N-1} \Delta y_{m-1} \cos \frac{2\pi lm}{N} - \frac{1}{2} \cot \frac{\pi l}{N} \sum_{m=0}^{N-1} \Delta y_{m-1} \sin \frac{2\pi lm}{N} \\ &= \frac{1}{2} \phi_{l,1} - \frac{1}{2} \cot \frac{\pi l}{N} \psi_{1,l}.\end{aligned}$$

Using Eq. (21) K more times, denote $w := \cot \frac{\pi l}{N}$, we obtain

$$\begin{aligned}(-1)^l N 2^K a_l &= \phi_{l,K+1} - C_{K+1}^1 w \psi_{l,K+1} - C_{K+1}^2 w^2 \phi_{l,K+1} \\ &+ C_{K+1}^3 w^3 \psi_{l,K+1} + C_{K+1}^4 w^4 \phi_{l,K+1} + \cdots \\ &= I_\phi - I_\psi,\end{aligned}$$

where

$$\begin{aligned}I_\phi &= \phi_{l,K+1}(1 - C_{K+1}^2 w^2 + C_{K+1}^4 w^4 + \dots) \\ &= \frac{\phi_{l,K+1}}{2}((1+iw)^{K+1} + (1-iw)^{K+1}) = \frac{\cos(\frac{\pi}{2} - \frac{\pi l}{N})(K+1)}{\sin^{K+1} \frac{\pi l}{N}} \phi_{l,K+1},\end{aligned}$$

and

$$\begin{aligned}I_\psi &= \psi_{l,K+1}(C_{K+1}^1 w - C_{K+1}^3 w^3 + C_{K+1}^5 w^5 + \dots) \\ &= \frac{\psi_{l,K+1}}{2i}((1+iw)^{K+1} - (1-iw)^{K+1}) = \frac{\sin(\frac{\pi}{2} - \frac{\pi l}{N})(K+1)}{\sin^{K+1} \frac{\pi l}{N}} \psi_{l,K+1}.\end{aligned}$$

By (35), there exist $\xi_{l,K+1}, \theta_{l,K+1} \in [-b, b]$ such that

$$\begin{aligned}a_l &= \frac{(-1)^l}{2^K \sin^{K+1} \frac{\pi l}{N}} (\Delta_{K+1}^\lambda [f](\xi_{l,K+1}) \cos(\frac{\pi}{2} - \frac{\pi l}{N})(K+1) \\ &- \Delta_{K+1}^\lambda [f](\theta_{l,K+1}) \sin(\frac{\pi}{2} - \frac{\pi l}{N})(K+1)).\end{aligned}$$

Plugging (22) to above equation, we obtain

$$|a_l| \leq \frac{C(D_{K+1})}{N^{K+1} \sin^{K+1} \frac{\pi l}{N}}, \tag{36}$$

where $C_{K+1,1}$ is a bounded constant depending on D_{K+1}. □

3.4. The proof of Theorem 4

This section is mainly used to prove Theorem 4 with f is even. Same argument can be applied in parallel if f is odd. We first develop a connection between interpolant $f_M(x)$ and f_{2M}, which are based on the set of $2M$ and $4M$ number of grid points by Eq. (8) respectively. In the case of $N = 2M$, define

$$(-1)^l \bar{A}_l^N = \frac{1}{N}\sum_{j=0}^{N-1} y_j \cos\frac{2\pi jl}{N}, \quad 0 \le l < N. \tag{37}$$

Note that $\bar{A}_l^N$ are symmetric in the sense

$$\bar{A}_l^N = \bar{A}_{N-l}^N, \qquad l = 1, \ldots, N-1. \tag{38}$$

Similarly, $\{a_j\}_{0\le j<M}$ in Eqs. (11)-(12) will be denoted by $\{a_j^N\}_{0\le j<M}$. Recall

$$a_j^N = 2\bar{A}_j^N, \qquad 1 \le j < M = N/2.$$

Following the convention, $\{\bar{A}_j^{2N}\}_{0\le j<2N}$ and $\{a_j^{2N}\}_{0\le j<N}$ denote associated quantities with the set of $2N$ number of grid points with $\lambda = \frac{2b}{2N}$. The following lemma is the key observation for convergence analysis in this section.

Lemma 2. *Let a_j^N, a_j^{2N} be the coefficients of $f_M(x)$ and $f_{2M}(x)$ respectively, then*

$$a_j^N = a_j^{2N} + a_{N-j}^{2N}, \qquad 1 \le j < M. \tag{39}$$

Proof:. Eq. (39) is equivalent to

$$\bar{A}_j^N = \bar{A}_j^{2N} + \bar{A}_{N-j}^{2N}, \qquad 0 \le j < N. \tag{40}$$

For $0 \le j < N$,

$$\begin{aligned}(-1)^j \bar{A}_j^{2N} &= \frac{1}{2N}\sum_{s=0}^{N-1} y_{2s}\cos\frac{2\pi sj}{N} + \frac{1}{2N}\sum_{s=0}^{N-1} y_{2s+1}\cos\frac{2\pi(2s+1)j}{2N} \\ &= (-1)^j\frac{1}{2}\bar{A}_j^N + I_j, \end{aligned} \tag{41}$$

where $I_j := \frac{1}{2N}\sum_{s=0}^{N-1} y_{2s+1}\cos\frac{2\pi(2s+1)j}{2N}$. By Eqs. (29) and (38),

$$\begin{aligned} I_j &= \frac{1}{2N}\sum_{s=0}^{N-1}\cos\frac{2\pi(2s+1)j}{2N}\sum_{l=0}^{2N-1}(-1)^l A_l^{2N}\cos\frac{2\pi(2s+1)l}{2N} \\ &= \frac{1}{4N}\sum_{l=0}^{2N-1}(-1)^l A_l^{2N}\sum_{s=0}^{N-1}(\cos\frac{2\pi(2s+1)(l+j)}{2N}+\cos\frac{2\pi(2s+1)(l-j)}{2N}) \\ &= \frac{1}{4}\sum_{l=0}^{2N-1}(-1)^l A_l^{2N}(-\delta_{l+j=N}+\delta_{l+j=2N}+\delta_{l-j=0}-\delta_{l-j=N}) \\ &= \frac{1}{2}(-1)^j(A_j^{2N}-A_{N-j}^{2N}), \end{aligned}$$

which, together with Eq. (41), implies Eq. (40). $\square$

Let $f_M(x)$ be the interpolant using $N=2M$ grid points, define $\Delta_M(x) = f_M(x) - f_{2M}(x)$, we have

$$\Delta_M(x) = a_0^N - a_0^{2N} - a_M^{2N}\cos\frac{\pi M x}{b} + \sum_{M<j<N} a_j^{2N}(\cos\frac{(N-j)\pi x}{b} - \cos\frac{j\pi x}{b}). \tag{42}$$

Notice $|a_j^{2N}| \leq C(D_{K+1})/N^{K+1}$ for $M \leq j < N$ and by Eq. (15),

$$|a_0^N - a_0^{N_p}| \leq |a_0^N - \frac{A_0}{2}| + |a_0^{N_p} - \frac{A_0}{2}| \leq \frac{C(D_{K+1})}{N^K},$$

hence

$$|\Delta_N(x)| \leq \frac{C(D_{K+1})}{N^K}.$$

For a given $N=2^q$ and an integer $p \geq 0$, define $M_p = 2^{p-1}N$, and f_{M_p} be the associated interpolant with $2M_p$ grid points and $\Delta_{M_p}(x) = f_{M_p} - f_{2M_p}$, we have

$$\begin{aligned} |f_M(x) - f_{M_p}(x)| &\leq \sum_{0\leq r\leq p}|f_{M_r}(x) - f_{M_{r+1}}(x)| \\ &\leq \frac{C(D_{K+1})}{N^K}\sum_{0\leq r\leq p}\frac{1}{2^{(r-1)K}} \leq \frac{C(D_{K+1})}{N^K}. \end{aligned} \tag{43}$$

For integer $1 \leq k < K$, by (42), we obtain estimation on $k-th$ derivative of

$\Delta_M(x)$,

$$\begin{aligned}
\Delta_M^{(k)}(x) &\leq (\frac{\pi M}{b})^k |a_M^{2N}| + \sum_{M<j<N} |a_j^{2N}| [(\frac{(N-j)\pi}{b})^k + (\frac{j\pi}{b})^k] \\
&\leq \frac{C(D_{K+1})}{N^{K-k}},
\end{aligned}$$

which implies

$$\begin{aligned}
|f_M^{(k)}(x) - f_{M_p}^{(k)}(x)| &\leq \sum_{0\leq r\leq p} |f_{M_r}^{(k)}(x) - f_{M_{r+1}}^{(k)}(x)| \\
&\leq \frac{C(D_{K+1})}{N^{K-k}} \sum_{0\leq r\leq p} \frac{1}{2^{(r-1)K}} \leq \frac{C(D_{K+1})}{N^{K-k}}. \qquad (44)
\end{aligned}$$

Estimations (43) and (44) imply that f_{M_p} and $f_{M_p}^{(k)}$ converge uniformly as $p \to \infty$. It is clear that f_{M_p} converges to $f(x)$ on the dense set $S = \cup_{p=0}^{\infty} S_p$ where

$$S_p = \{-b + 2k\frac{2b}{2M_p}, \quad 0 \leq k < M_p.\}$$

Therefore, f_{M_p} converges to $f(x)$ and consequently $f_{M_p}^{(k)}$ converges to $f^{(k)}(x)$ as $p \to \infty$. Applying $p \to \infty$ to Estimation (43) and (44), we obtain Estimation (17) and (18).

Appendix A. The proof of Lemma 1

If $f(x)$ is even, by definition,

$$\tilde{y}_l = \sum_{j=0}^{M-1} \tilde{A}_j \cos\frac{j\pi x_l}{b} = \sum_{j=0}^{M-1} \tilde{A}_j (-1)^j \cos\frac{2\pi j l}{N}, \quad 0 \leq l < N.$$

By Eq. (25), for terms with even index $l = 2k$, one can easily derive

$$\tilde{y}_{2k} = \sum_{j=0}^{M-1} (-1)^j \tilde{A}_j cos\frac{2\pi j(2k)}{N} = I_e + II_e,$$

where

$$
\begin{aligned}
I_e &= \frac{2}{N}\sum_{j=0}^{M-1} cos\frac{2\pi jk}{M}\sum_{h=0}^{M-1} y_{2h} cos\frac{2\pi jh}{M} \\
II_e &= \frac{2}{N}\sum_{j=0}^{M-1} cos\frac{2\pi jk}{M}\sum_{h=0}^{M-1} y_{2h+1} cos\frac{2\pi j(2h+1)}{N}
\end{aligned}
$$

It is not hard to verify

$$
I_e = y_{2k}, \qquad II_e = \frac{1}{M}\sum_{h=0}^{M-1} y_{2h+1},
$$

which implies Eq. (31).

Similar calculation can be applied to odd terms:

$$
\tilde{y}_{2k+1} = \sum_{j=0}^{M-1}(-1)^j \tilde{A}_j cos\frac{2\pi j(2k+1)}{N} = III_e + IV_e,
$$

where

$$
\begin{aligned}
III_e &= \frac{2}{N}\sum_{j=0}^{M-1} cos\frac{2\pi j(2k+1)}{N}\sum_{h=0}^{M-1} y_{2h} cos\frac{2\pi j(2h)}{N}, \\
IV_e &= \frac{2}{N}\sum_{j=0}^{M-1} cos\frac{2\pi j(2k+1)}{N}\sum_{h=0}^{M-1} y_{2h+1} cos\frac{2\pi j(2h+1)}{N}.
\end{aligned}
$$

One can verify

$$
III_e = \frac{1}{M}\sum_{h=0}^{M-1} y_{2h}, \qquad IV_e = y_{2k+1},
$$

which implies Eq. (32).

Parallel arguments can be used to prove Eq. (33) if $f(x)$ is odd.