Pricing American options on exponential Levy processes

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August, 10, 2010

Abstract. to be added

1. Introduction and Motivation

To be added.

2. the formulation of the algorithm and implementation

We are assuming that the dynamics of the prices of the underlying risky security $\{S_t\}_{0\leq t\leq T}$ follows a process of the form

$$S_t = e^{X_t}$$

where X_t follows a Levy process with $X_0 = \ln S_0$.

Let us denote the characteristic function of X_t by $\phi_t(v) = E[e^{iv\cdot X_t}], x \in R$.

We shall focus on the Bermuda put option with M periods. The the value for call option can be derived by the parity equation. The algorithm can be used for the other popular options where exercise payoff has a simple expression in term of underlying asset.

The exercise value at any time t before maturity is

$$G(S_t) = \begin{cases} (\alpha K - \beta S_t)^+ & S_t \leq K \\ 0 & S_t > K \end{cases}$$

where K is the strike price and α, β are non negative parameters. Let $\Delta = T/M$, $k = \ln K$. Let

$$\phi(v) = E[e^{ivX_\Delta}].$$

be the characteristic function. We are going to write $S_{j\Delta}$ and $X_{j\Delta}$ as S_j and X_j for convenience. So

$$S_{j+1} = e^{X_{j+1}} = e^{X_j} e^{X_{j+1}-X_j} \sim S_j e^Z$$

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where $Z = X_{j+1} - X_j$ has the density $q(z)$ (without depending on j). We shall scale the X_j by

$$X_j = \sigma Y_j + \mu$$

where

$$\sigma = var(X_j), \quad \mu = E[X_j]$$

and

$$Z = X_{j+1} - X_j = \sigma(Y_{j+1} - Y_j) = \sigma W$$

where the distribution of $W = Y_{j+1} - Y_j$ is independent on j.

Define

$$C_j^s(y) = C_j(e^{\sigma y+\mu}), \quad V_j^s(y) = V_j(e^{\sigma y+\mu}), \quad G^s(y) = G(e^{\sigma y+\mu})$$

Let $f_W(w)$ be the density function of W and $f_Z(z)$ be the density of Z, we have

$$f_W(w) = \sigma f_Z(\sigma w)$$

so

$$\begin{aligned} e^{r\delta} C_j^s(y) &= e^{r\delta} C_j(e^{\sigma y+\mu}) = E[V_{j+1}(e^{\sigma y+\mu} e^{\sigma W})] \\ &= \int_{-\infty}^{\infty} V_{j+1}(e^{\sigma y+\mu+\sigma w}) f_W(w) dw \\ &= \int_{-\infty}^{\infty} V_{j+1}^s(y+w) f_W(w) dw \end{aligned}$$

Assume that

$$f_W(w)|_{[L,R]} \approx F_0/2 + \sum_{k=1}^{N-1} F_k cos(k\pi \frac{x-L}{R-L})$$

where

$$\begin{aligned} F_k &= \frac{2}{R-L} Re(\int_L^R f_W(w) e^{ik\pi(w-L)/(R-L)} dw) \\ &\approx \frac{2}{R-L} Re(\phi_W(\frac{k\pi}{R-L}) e^{\pi k L/(R-L)}) \end{aligned}$$

where

$$\phi_W(t) = \int_{-\infty}^{\infty} e^{itw} f_W(w) dw$$

Let

$$b_0 = -\infty < b_1 < \ldots b_k < cdots < b_{M+1} = R < \infty$$

be a partition of $(-\infty, \infty)$ and the V_{j+1} is equal to the payoff at

$$V_{j+1}^s(y) = G^s(y), \quad -\infty < y \leq b_1$$

and V_{j+1} can be approximated by d-degree polynomial on each interval $[b_k, b_{k+1})$ for $(1 \leq k \leq M)$:

$$V_{j+1}^s(y) = \sum_{h=0}^{d} c_{k,h}(y - b_k)^h, \quad b_k \leq y < b_{k+1},$$

so

$$\begin{aligned} e^{r\delta} C_j^s(y) &= \int_{-\infty}^{b_1 - y} G^s(w+y) f_W(w) dw \\ &+ \sum_{k=1}^{M} \int_{b_k - y}^{b_{k+1} - y} V_{j+1}^s(y+w) f_W(w) dw \\ &+ \int_{b_{M+1} - y}^{\infty} V_{j+1}^s(w+y) f_W(w) dw \\ &:= I + II + III \end{aligned}$$

To estimate I,let

$$y_1 = \min(L, b_1 - y), \quad y_2 = \min(\max(L, b - y_1), R), y_3 = \max(R, b_1 - y)$$

we discuss I in three cases. We assume that $G^s(y)$ is an decreasing function,i.e. a put-style option.

1. $b_1 - y \leq L$. $y_1 = b_1 - y$ and $y_2 = L$. It is clear that we have

$$0 \leq I - \int_{L}^{y_2} G^s(y+w) f_W(w) dw \leq G^s(-\infty) F_W(L) \tag{1}$$

2. $L \leq b_1 - y \leq R$, $y_1 = L$ and $y_2 = b_1 - y$,

$$I = \int_{L}^{y_2} G^s(y+w) f_W(w) dw + \int_{-\infty}^{L} G^s(y+w) f_W(w) dw$$

   It is clear

$$0 \leq \int_{-\infty}^{L} G^s(y+w) f_W(w) dw \leq G^s(-\infty) F_W(L)$$

   so the equation (1) holds as in the case 1.

3. $b_1 - y > R$, $y_1 = L$ and $y_2 = R$, and

$$I = (\int_{-\infty}^{L} + \int_{L}^{y_2} + \int_{R}^{b_1-y})G^s(y+w)f_W(w)dw$$

It is straightforward to show

$$0 \leq \int_{R}^{b_1-y} G^s(y+w)f_W(w)dw \leq G^s(-\infty)(1-F_W(R))$$

So we have

$$0 \leq I - \int_{L}^{y_2} G^s(y+w)f_W(w)dw \leq G^s(-\infty)(1-P(L \leq W \leq R)) \quad (2)$$

and

$$0 \leq III \leq V_{j+1}^s(R)P(W \geq R-y)$$

Let

$$low(y,k) = \min(\max(b_k - y, L), R), \quad up(y,k) = \min(\max(b_{k+1} - y, L), R),$$

then

$$\begin{aligned} 0 \ &\leq \ \int_{b_k-y}^{b_{k+1}-y} V_{j+1}^s(y+w)f_W(w)dw - \int_{low(y,k)}^{up(y,k)} V_{j+1}^s(y+w)f_W(w)dw \\ &\leq \ V_{j+1}^s(b_k)(1-P(L \leq W \leq R)) \end{aligned}$$

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