

# Estimate density function through fast Fourier transformation

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**Abstract.** This is the specification for estimating a function and its derivatives/antiderivatives of any order through fast Fourier transformation(FFT).

## 1. cos expansion of even functions

For an even continuous periodic function  $h$  on  $[-\pi, \pi]$ , we have by the standard Fourier analysis

$$h(x) = \sum_{j \geq 0} A'_j \cos(jx)$$

where

$$A_j = \frac{2}{\pi} \int_0^\pi h(x) \cos(jx) dx. \quad (1)$$

and

$$A'_0 = A_0/2, \quad A'_j = A_j, \quad j \geq 1. \quad (2)$$

REMARK 1. We shall use similar notation  $\{c_j\}'_{j \geq 0}$  for any sequence  $\{c_j\}_{j \geq 0}$ , i.e.

$$c'_0 = c_0/2, \quad c'_j = c_j, \quad j \geq 1. \quad (3)$$

Assume a function  $f(x)$  is defined over the interval  $[L, R]$ . Extend it to an even periodic function over the real number space  $\mathbb{R}$ , and denote the extended function also by  $f$ . Transform the interval  $[L, R]$  to  $[0, \pi]$  by

$$y = \frac{\pi(x-L)}{l}, \quad x \in [L, R] \quad (4)$$

where  $l = R - L$ . We have

$$f(x) = \frac{A_0}{2} + \sum_{j=1}^{\infty} A_j \cos \frac{j\pi(x-L)}{l} = \sum_{j=0}^{\infty} A'_j \cos \frac{j\pi(x-L)}{l}, \quad x \in [L, R]$$

where

$$A_j = \frac{2}{l} \int_L^R f(x) \cos \frac{j\pi(x-L)}{l} dx. \quad (5)$$

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Using first  $N$  terms in above series to Approximate  $f$ ,

$$\tilde{f}_N(x) = \sum_{j=0}^{N-1} (-1)^j A'_j \cos(\pi j \frac{x - (2L - R)}{l}). \quad (6)$$

Let  $F(x, 0) = \tilde{f}_N(x)$  and

$$\begin{aligned} F(x, k) &:= F(x, k+1)', \quad k = -1, -2, -3, \dots \\ F(x, k) &:= \int_L^x F(u, k-1) du, \quad k = 1, 2, 3, \dots, \end{aligned}$$

denote the derivatives and anti derivatives of order  $k$  of  $\tilde{f}_N$  respectively. The explicit expression (6) of  $\tilde{f}_N$  make it possible for us to compute effectively  $F(x, k)$  at the points  $\{x_k\}_{k=0}^N$ :

$$x_k = (2L - R) + k \times \lambda, \quad \lambda = \frac{2(R - L)}{N}, \quad 0 \leq k \leq N \quad (7)$$

Note that

$$x_0 = 2L - R, \quad x_{N/2} = L, \quad x_N = R.$$

By (6),

$$F(x_k, 0) = \sum_{j=0}^{N-1} A'_j (-1)^j \cos(2kj\pi/N) = \Re \sum_{j=0}^{N-1} A'_j (-1)^j \omega_N^{kj} \quad (8)$$

where  $\omega_N = e^{2\pi i/N}$ . For any complex number  $z$ ,  $\Re(z)$  and  $\Im(z)$  denote the real part and imaginary part of  $z$  respectively. of the complex number  $z$ . The equation (8) can be obtained by inverse FFT (*ifft*):

$$\{F(x_k, 0)\}_{0 \leq k \leq N-1} = N \times \Re \{ifft(\{(-1)^j A'_j\}_{0 \leq j \leq N-1})\}. \quad (9)$$

One can apply (6) to derive  $s$ -th order derivatives ( $s = -1, -2, \dots$ )

$$\begin{aligned} F(x, -(2h-1)) &= (-1)^h \sum_{j=0}^{N-1} (-1)^j (j\pi/l)^{2h-1} A'_j \sin(\pi j \frac{x - (2L - R)}{l}) \\ F(x, -2h) &= (-1)^h \sum_{j=0}^{N-1} (-1)^j (j\pi/l)^{2h} A'_j \cos(\pi j \frac{x - (2L - R)}{l}). \end{aligned}$$

and apply inverse FFT to estimate the values at  $\{x_k\}_{0 \leq k \leq N-1}$

$$\{F(x_k, -(2h-1))\} = (-1)^h N \times \Im(ifft\{(-1)^j (j\pi/l)^{2h-1} A'_j\}) \quad (10)$$

$$\{F(x_k, -2h)\} = (-1)^h N \times \Re(ifft\{(-1)^j (j\pi/l)^{2h} A'_j\}) \quad (11)$$

For anti derivatives  $F(x, s)$  ( $s = 1, 2, \dots$ ), one can use induction to show for  $h \geq 1$

$$\begin{aligned}
F(x, 2h-1) &= (-1)^{h-1} \sum_{j=0}^{N-1} (-1)^j \frac{\hat{A}_j l^{2h-1}}{(j\pi)^{2h-1}} \sin(j\pi \frac{x - (2L - R)}{l}) \\
&+ \sum_{m=1}^{h-1} \frac{(-1)^{h-m-1}}{(2m-1)!} (x-L)^{2m-1} \sum_{j=0}^{N-1} \frac{\hat{A}_j l^{2(h-m+1)}}{(\pi j)^{2(h-m+1)}} \\
&+ \frac{A'_0 (x-L)^{2h-1}}{(2h-1)!}
\end{aligned} \tag{12}$$

and

$$\begin{aligned}
F(x, 2h) &= (-1)^h \sum_{j=0}^{N-1} (-1)^j \frac{\hat{A}_j l^{2h}}{(j\pi)^{2h}} \cos(j\pi \frac{x - (2L - R)}{l}) \\
&+ \sum_{m=1}^h \frac{(-1)^{h-m}}{(2m-2)!} (x-L)^{2m-2} \sum_{j=0}^{N-1} \frac{\hat{A}_j l^{2(h-m+1)}}{(\pi j)^{2(h-m+1)}} \\
&+ \frac{A'_0 (x-L)^{2h}}{(2h)!}
\end{aligned} \tag{13}$$

where

$$\hat{A}_0 = 0, \quad \hat{A}_j = A'_j = A_j, \quad j \geq 1$$

For  $h \geq 1$ ,

$$\begin{aligned}
\{Z(h, k)\}_{0 \leq k \leq N-1} &= \sum_{j=0}^{N-1} (-1)^j \frac{\hat{A}_j l^{2h-1}}{(j\pi)^{2h-1}} \sin(j\pi \frac{x_k - (2L - R)}{l}) \\
&= N \times \Im(\text{fft}\{(-1)^j \frac{\hat{A}_j l^{2h-1}}{(j\pi)^{2h-1}}\})
\end{aligned}$$

and

$$\begin{aligned}
\{Y(h, k)\}_{0 \leq k \leq N-1} &= \sum_{j=0}^{N-1} (-1)^j \frac{\hat{A}_j l^{2h}}{(j\pi)^{2h}} \cos(j\pi \frac{x_k - (2L - R)}{l}) \\
&= N \times \Re(\text{fft}\{(-1)^j \frac{\hat{A}_j l^{2h}}{(j\pi)^{2h}}\})
\end{aligned}$$

Notice that

$$Y(h, \frac{N}{2}) = \sum_{j=0}^{N-1} \frac{\hat{A}_j l^{2h}}{(j\pi)^{2h}}$$

One can rewrite (12) as

$$\begin{aligned} \{F(x_k, 2h-1)\}_k &= (-1)^{h-1} \{Z(j, k)\}_k + \left\{ \frac{A'_0(x_k - L)^{2h-1}}{(2h-1)!} \right\}_k \\ &\quad + \left\{ \sum_{m=1}^{h-1} \frac{(-1)^{h-m-1} Y(\frac{N}{2}, h-m+1)}{(2m-1)!} (x_k - L)^{2m-1} \right\}_k \end{aligned} \quad (14)$$

and rewrite (13) as

$$\begin{aligned} \{F(x_k, 2h)\}_k &= (-1)^h \{Y(j, k)\}_k + \left\{ \frac{A'_0(x_k - L)^{2h}}{(2h)!} \right\}_k \\ &\quad + \left\{ \sum_{m=1}^h \frac{(-1)^{h-m} Y(\frac{N}{2}, h-m+1)}{(2m-2)!} (x_k - L)^{2m-2} \right\}_k \end{aligned} \quad (15)$$

Since we assume that  $f_X = 0$  outside  $[L, R]$ , one should only take the second parts of  $F$ :

$$F(i, h) = F(N/2 : (N-1), h), \quad h = 0, 1, \dots, \frac{N}{2} - 1$$

## 2. Estimate certain relevant integrations

FFT method can also be used to compute the following integration, which is required for our purpose.

$$E(x, t) = \int_L^x e^{tu} \tilde{f}_N(u) du, \quad t \geq 0, \quad x < R \quad (16)$$

In fact,

$$\begin{aligned} E(x, t) &= e^{tx} \sum_{j=0}^{N-1} \frac{\frac{1}{t} \cos(j\pi(x-L)/l) + \frac{j\pi}{lt^2} \sin(j\pi(x-L)/l)}{1 + (\frac{j\pi}{tl})^2} A'_j \\ &\quad - \sum_{j=0}^{N-1} \frac{e^{tL}/t}{1 + (\frac{j\pi}{tl})^2} A'_j \\ &= e^{tx} \sum_{j=0}^{N-1} \frac{\frac{1}{t} (-1)^j \cos(j\pi(x - (2L - R))/l)}{1 + (\frac{j\pi}{tl})^2} A'_j \\ &\quad + e^{tx} \sum_{j=0}^{N-1} \frac{\frac{j\pi}{lt^2} (-1)^j \sin(j\pi(x - (2L - R))/l)}{1 + (\frac{j\pi}{tl})^2} A'_j \end{aligned} \quad (17)$$

$$- \sum_{j=0}^{N-1} \frac{e^{tL}/t}{1 + (\frac{j\pi}{tl})^2} A'_j \quad (18)$$

We can use FFT inverse transformation to obtain  $\{E(x_k, t)\}_{k=0}^{N-1}$ . Finally, using integration by parts, we can calculate the following integration for any nonnegative integer  $j$

$$P(a, b, c, j) = \int_a^b (x - c)^j \tilde{f}_X(x) dx, \quad [a, b] \subseteq [L, R]$$

In fact,

$$P(a, b, c, j) = \sum_{k=0}^j \frac{(-1)^k j!}{(j-k)!} \{F(b, k)(b - y)^{j-k} - F(a, k)(a - y)^{j-k}\} \quad (19)$$

