CHAPTER 1
Rosenblatt's Perceptron

Problem 1.1

(1) If $\mathbf{w}^T(n)\mathbf{x}(n) > 0$, then $y(n) = +1$.
If also $\mathbf{x}(n)$ belongs to C_1, then $d(n) = +1$.
Under these conditions, the error signal is
$$e(n) = d(n) - y(n) = 0$$
and from Eq. (1.22) of the text:
$$\mathbf{w}(n + 1) = \mathbf{w}(n) + \eta e(n)\mathbf{x}(n) = \mathbf{w}(n)$$
This result is the same as line 1 of Eq. (1.5) of the text.

(2) If $\mathbf{w}^T(n)\mathbf{x}(n) < 0$, then $y(n) = -1$.
If also $\mathbf{x}(n)$ belongs to C_2, then $d(n) = -1$.
Under these conditions, the error signal $e(n)$ remains zero, and so from Eq. (1.22) we have
$$\mathbf{w}(n + 1) = \mathbf{w}(n)$$
This result is the same as line 2 of Eq. (1.5).

(3) If $\mathbf{w}^T(n)\mathbf{x}(n) > 0$ and $\mathbf{x}(n)$ belongs to C_2 we have
$$y(n) = +1$$
$$d(n) = -1$$
The error signal $e(n)$ is -2, and so Eq. (1.22) yields
$$\mathbf{w}(n + 1) = \mathbf{w}(n) -2\eta\mathbf{x}(n)$$
which has the same form as the first line of Eq. (1.6), except for the scaling factor 2.

(4) Finally if $\mathbf{w}^T(n)\mathbf{x}(n) < 0$ and $\mathbf{x}(n)$ belongs to C_1, then
$$y(n) = -1$$
$$d(n) = +1$$
In this case, the use of Eq. (1.22) yields
$$\mathbf{w}(n + 1) = \mathbf{w}(n) +2\eta\mathbf{x}(n)$$
which has the same mathematical form as line 2 of Eq. (1.6), except for the scaling factor 2.

Problem 1.2

The output signal is defined by

$$\begin{aligned} y &= \tanh\left(\frac{v}{2}\right) \\ &= \tanh\left(\frac{b}{2} + \frac{1}{2}\sum_i w_i x_i\right) \end{aligned}$$

Equivalently, we may write

$$b + \sum_i w_i x_i = y' \tag{1}$$

where

$$y' = 2\tanh^{-1}(y)$$

Equation (1) is the equation of a hyperplane.

Problem 1.3

(a) AND operation: Truth Table 1

Inputs		Output
x_1	x_2	y
1	1	1
0	1	0
1	0	0
0	0	0

This operation may be realized using the perceptron of Fig. 1

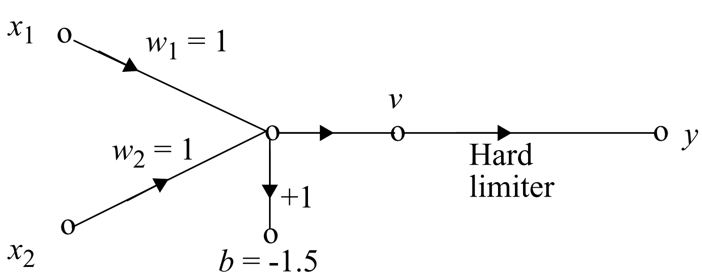

Figure 1: Problem 1.3

The hard limiter input is

$$\begin{aligned} v &= w_1 x_1 + w_2 x_2 + b \\ &= x_1 + x_2 - 1.5 \end{aligned}$$

If $x_1 = x_2 = 1$, then $v = 0.5$, and $y = 1$
If $x_1 = 0$, and $x_2 = 1$, then $v = -0.5$, and $y = 0$
If $x_1 = 1$, and $x_2 = 0$, then $v = -0.5$, and $y = 0$
If $x_1 = x_2 = 0$, then $v = -1.5$, and $y = 0$

These conditions agree with truth table 1.

OR operation: Truth Table 2

Inputs		Output
x_1	x_2	y
1	1	1
0	1	1
1	0	1
0	0	0

The OR operation may be realized using the perceptron of Fig. 2:

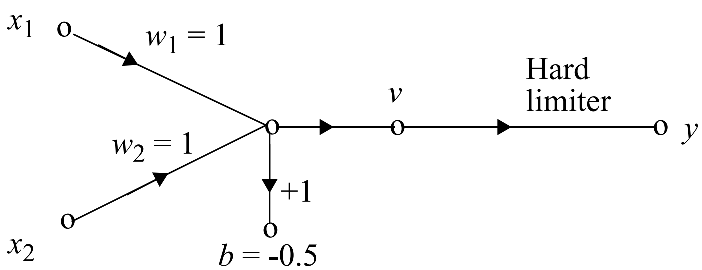

Figure 2: Problem 1.3

In this case, the hard limiter input is

$$v = x_1 + x_2 - 0.5$$

If $x_1 = x_2 = 1$, then $v = 1.5$, and $y = 1$
If $x_1 = 0$, and $x_2 = 1$, then $v = 0.5$, and $y = 1$
If $x_1 = 1$, and $x_2 = 0$, then $v = 0.5$, and $y = 1$
If $x_1 = x_2 = 0$, then $v = -0.5$, and $y = -1$

These conditions agree with truth table 2.

COMPLEMENT operation: Truth Table 3

Input x,	Output, y
1	0
0	1

The COMPLEMENT operation may be realized as in Figure 3::

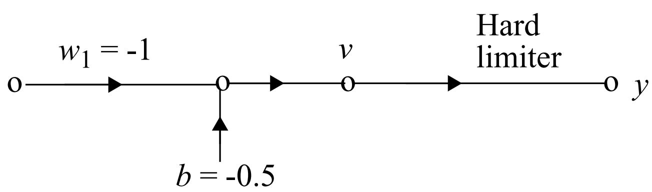

Figure 3: Problem 1.3

The hard limiter input is

$$v = wx + b = -x + 0.5$$

If $x = 1$, then $v = -0.5$, and $y = 0$
If $x = 0$, then $v = 0.5$, and $y = 1$

These conditions agree with truth table 3.

(b) EXCLUSIVE OR operation: Truth table 4

Inputs		Output
x_1	x_2	y
1	1	0
0	1	1
1	0	1
0	0	0

This operation is nonlinearly separable, which cannot be solved by the perceptron.

Problem 1.4

The Gaussian classifier consists of a single unit with a single weight and zero bias, determined in accordance with Eqs. (1.37) and (1.38) of the textbook, respectively, as follows:

$$\begin{aligned} w &= \frac{1}{\sigma^2}(\mu_1 - \mu_2) \\ &= -20 \end{aligned}$$

$$
\begin{aligned}
b &= \frac{1}{2\sigma^2}(\mu_2^2 - \mu_1^2) \\
&= 0
\end{aligned}
$$

Problem 1.5

Using the condition

$$\mathbf{C} = \sigma^2 \mathbf{I}$$

in Eqs. (1.37) and (1.38) of the textbook, we get the following formulas for the weight vector and bias of the Bayes classifier:

$$
\begin{aligned}
\mathbf{w} &= \frac{1}{\sigma^2}(\mu_1 - \mu_2) \\
\mathbf{b} &= \frac{1}{2\sigma^2}(\|\mu_1\|^2 - \|\mu_2\|^2)
\end{aligned}
$$