Stochastic Process

A rigorous but painless introduction

shuqi

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xiaosq2000 xiaosq2000@gmail.com

Prerequistes & Gains

Readers are assumed to have an undergraduate-level basic understanding of

- Multivariate Probability Theory
- Linear Algebra
- Matrix Caculus
- Fourier Analysis

You are expected to learn diverse related knowledge, such as measure theory, Lebesgue integration, power spectral density, and white noise.

Outline

1. Basic concepts of measure and probability
2. Basic concepts of stochastic process

Basic concepts of measure and probability

Remark (motivation of measure)

A measure is a generalization and formalization of geometrical measures (length, area, volume) and other common notions, such as magnitude, mass, and probability of events. It is fundamental in many mathematical fields, such as probability theory and integration theory.

Definition (measure)

Let X be a set and $\mathcal{F}$ a σ-algebra over X. A function $\mu : \mathcal{F} \mapsto \mathbb{R}^1_\infty$, where $\mathbb{R}^1_\infty$ is the extended real number field, is called a measure if the following three conditions hold:

1. empty-is-zero: $\mu(\emptyset) = 0$
2. non-negativity: $\forall E \in \mathcal{F} \, (\mu(E) \geq 0)$
3. countable-additivity: $\mu(\bigcup_{k=1}^{\infty} E_k) = \sum_{k=1}^{\infty} \mu(E_k)$, where $\{E_k\}_{k=1}^{\infty}$ is all countable collections of pairwise disjoint sets in $\mathcal{F}$

Remark (σ-algebra)

The "σ-algebra" and "countable" (actually, closely related to σ-algebra) make the rigorous definition of measure (def. 4) peculiar.

You can check out the definition of σ-algebra and motivations about it in measure theory on Wikipedia [1] , which is quite enlightening. In summary,

- Introducing the set algebra to deliver the addition-preserving property of a measure is natural, and σ-algebra is a set algebra with countable-additivity, alias σ-additivity. But why countable? [2] is a good explanation.
- ZFC (precisely, axiom of choice) entails non-measurable set of $\mathbb{R}^n$, i.e., it is actually impossible to assign a length to all subsets of $\mathbb{R}$ in a way that preserves some natural additivity and translation invariance properties. The Vitali set and the Banach–Tarski paradox are famous examples.

Definition (measurable space and measure space)

measurable space

$$\overbrace{\left(\underbrace{X}_{\text{a set}}, \underbrace{\mathcal{F}}_{\text{a } \sigma\text{-algebra on } X}\right)}^{\text{tuple}} \tag{1}$$

measure space

$$\overbrace{\left(\underbrace{X}_{\text{a set}}, \underbrace{\mathcal{F}}_{\text{a } \sigma\text{-algebra on } X}, \underbrace{\mu}_{\text{a measure on } (X, \mathcal{F})}\right)}^{\text{triple}} \tag{2}$$

Definition (measurable function)

Let (X, Σ) and (Y, T) be measurable spaces. A function $f : X \mapsto Y$ is measurable if and only if

$$\forall E \in T \left(f^{-1}(E) \in \Sigma\right) \tag{3}$$

Corollary

$$f \text{ is measurable} \Leftrightarrow \sigma(f) \subset \Sigma, \tag{4}$$

where $\sigma(f)$ is the σ-algebra generated by f.

Definition (probability)

Kolmogorov Axioms

$$\overbrace{\left(\underbrace{\Omega}_{\text{sample space}}, \underbrace{\mathcal{F}}_{\text{event space}}, \underbrace{P}_{\text{probability measure}}\right)}^{\text{probability space}} \tag{5}$$

the probability is a measure with two additional properties,

1. finiteness: $\forall E \in \mathcal{F}\,(P(E) \in \mathbb{R})$
2. unitarity: $P(\Omega) = 1$

Definition (random variable)

$$\underbrace{X}_{\text{measurable function}} : \underbrace{\Omega}_{\text{sample space}} \mapsto \underbrace{S}_{\text{state space}} \tag{6}$$

Remark (random variable)

It is a function named "variable" to refer to the state space(codomain), usually subsets of $\mathbb{R}^n$ or $\mathbb{Z}^n$, which is more convenient for manipulation than the abstract sample space. For example, the event $E := \{\omega \in \Omega : u < X(\omega) \leq v\}$ is usually denoted by $u < X \leq v$, since $\omega \in X^{-1}((u, v]) \Leftrightarrow u < X(\omega) \leq v$.

Basic concepts of stochastic process

Definition (stochastic process)

A stochastic process is collection of indexed random variables, denoted by

$$\{X(t) : t \in T\}\,, \tag{7}$$

where T is the index/parameter set.

Remark (index set)

t usually has a physical meaning of time(continuous) or timestamp(discrete).

Autocorrelation and autocovariance

Let $\mathsf{x}(\omega, t) : \Omega \times \mathbb{R} \mapsto \mathbb{R}^n$ be a continuous-time multivariate real-valued stochastic process[1],

Definition (autocorrelation)

$$\mathsf{R}_{\mathsf{xx}}(t_1, t_2) = \mathsf{E}\left(\mathsf{x}(t_1)\mathsf{x}(t_2)^{\mathrm{T}}\right) \tag{8}$$

Definition (autocovariance)

$$\mathsf{K}_{\mathsf{xx}}(t_1, t_2) = \mathsf{Cov}\left(\mathsf{x}(t_1), \mathsf{x}(t_2)\right) = \mathsf{E}\left(\left(\mathsf{x}(t_1) - \mathsf{E}\left(\mathsf{x}(t_1)\right)\right)\left(\mathsf{x}(t_2) - \mathsf{E}\left(\mathsf{x}(t_2)\right)\right)^{\mathrm{T}}\right) \tag{9}$$

[1] Due to the author's background, continuous-time multivariate real-valued stochastic processes are more common, on which the following definitions, remarks, etc., are based if not mentioned.

Definition (strict stationary process)

Let $F_X\left(X_{t_1+\tau}, \cdots, X_{t_n+\tau}\right)$ be the cumulative distribution function(CDF) of the unconditional joint distribution of the stochastic process $\{X_t\}$ at times $t_1 + \tau, \cdots, t_n + \tau$.

$\{X_t\}$ is a (strict(ly)/strong(ly)) stationary process, if the unconditional joint CDF does not change when shifted in time, i.e.

$$(\forall\, \tau, t_1, \cdots, t_n \in \mathbb{R})\,(\forall\, n \in \mathbb{N}_+)\,\left(F_X\left(x_{t_1+\tau}, \cdots, x_{t_n+\tau}\right)\right) \tag{10}$$

Definition (wide stationary process)

A wide/weak stationary process loosens the constraints on CDF to the following first two conditions, with an additional ”finite second-moment” condition.

$$\mathsf{E}\left(\mathsf{x}(t+\tau)\right) = \mathsf{E}\left(\mathsf{x}(t)\right), \qquad \forall\, t, \tau \in \mathbb{R} \tag{11}$$

$$\mathsf{K}_{\mathsf{xx}}\left(t_1, t_2\right) = \mathsf{K}_{\mathsf{xx}}\left(t_1 - t_2, 0\right), \qquad \forall\, t_1, t_2 \in \mathbb{R} \tag{12}$$

$$\mathsf{E}\left(|\mathsf{x}_t|^2\right) < \infty, \qquad \forall\, t \in \mathbb{R} \tag{13}$$

Stationary process iii

Corollary (wide stationary process)

- *The expectation is always a constant.*
- *The autocovariance and autocorrelation are better indexed by one variable instead of two.*
- *Any strictly stationary process which has a finite mean and a covariance is also a wide-sense stationary process.*

Remark (motivation of wide-sense stationarity, WSS)

If you have learned some advanced linear algebra or functional analysis and are good at associating, the ”finite second-moment” condition may remind you about Hilbert space.

The Wikipedia [3] has a wonderful explanation of its mathematical motivation and the reason why the WSS assumption is widely employed in signal processing algorithms. In summary,

- Under the zero-mean assumption, the autocovariance, also the autocorrelation is an inner product in the Hilbert space, to be precise, L^2 Lebesgue space.

Spectral density i

Definition (energy)

the energy E of a signal $x(t) : \mathbb{R} \mapsto \mathbb{R}$ is

$$E := \int_{-\infty}^{+\infty} |x(t)|^2 \, \mathrm{d}t \tag{14}$$

Theorem (Parseval's theorem)

$$\int_{-\infty}^{+\infty} |x(t)|^2 \, \mathrm{d}t = \int_{-\infty}^{+\infty} |\hat{x}(f)|^2 \, \mathrm{d}f, \tag{15}$$

where $\hat{x}(f)$ is the Fourier transform of $x(t)$, i.e.,

$$\hat{x}(f) = \int_{-\infty}^{+\infty} \mathrm{e}^{-\mathrm{i}2\pi ft} x(t) \, \mathrm{d}t \tag{16}$$

Spectral density ii

Definition (energy spectral density)

$$\bar{S}_{xx} := |\hat{x}(f)|^2 \tag{17}$$

Theorem (Wiener–Khinchin theorem)

References

[1] Wikipedia contributors. *σ-algebra — Wikipedia, The Free Encyclopedia.* URL: https://en.wikipedia.org/w/index.php?title=%CE%A3-algebra&oldid=1173641705 (cit. on p. 6).

[2] Carl Mummert. *Why do we want probabilities to be countably additive?* Mathematics Stack Exchange. URL: https://math.stackexchange.com/q/566154 (cit. on p. 6).

[3] Wikipedia contributors. *Stationary process — Wikipedia, The Free Encyclopedia.* motivation of weak-sense stationarity. URL: https://en.wikipedia.org/wiki/Stationary_process#Motivation (cit. on p. 17).