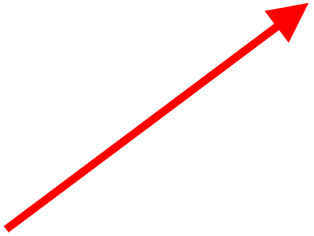
Vector Algebra

What are vectors?

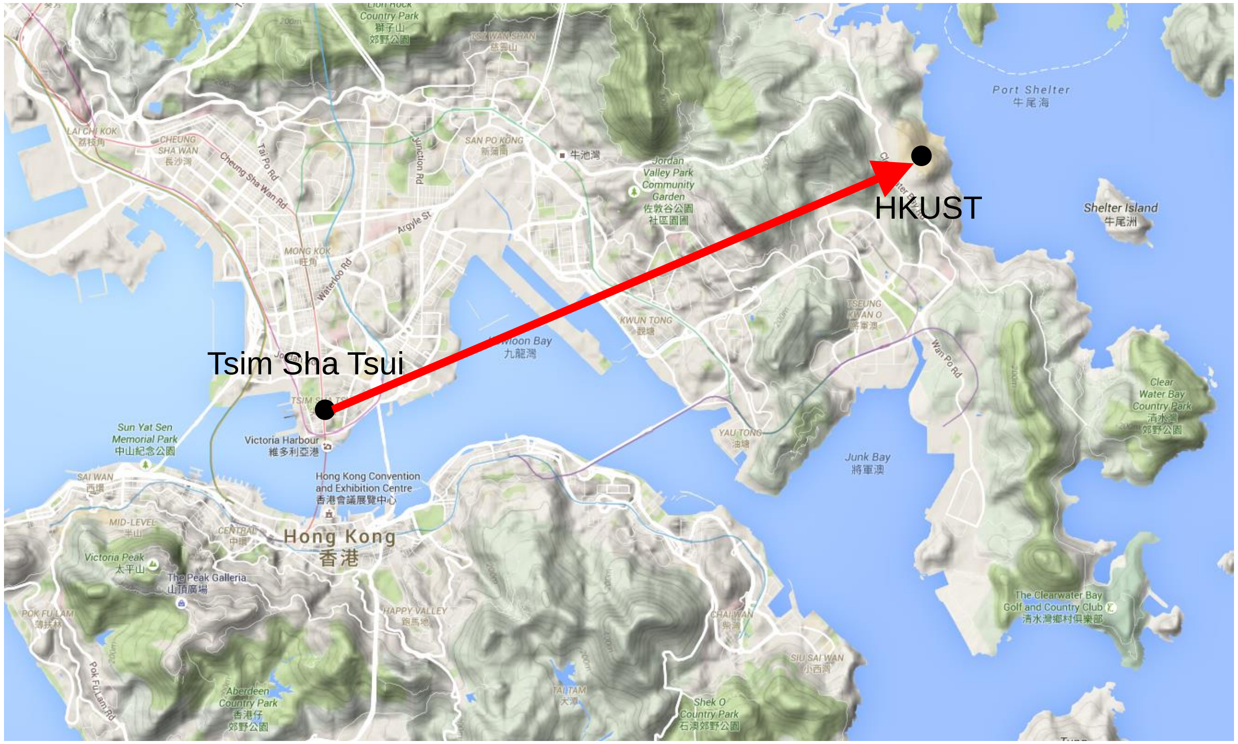
- Intuitively, a vector is a quantity with both magnitude and direction. Graphically, it can be represented by an arrow.



- Two vectors are the same iff they have the same direction and magnitude → represented by the same arrow
- Examples of vectors in physics: displacement, velocity, acceleration, force, momentum, angular momentum,….
- In contrast, a quantity that is just a number is called a scalar
  - Examples of scalars in physics: mass, energy, temperature,…

Example: Displacement vector

- The displacement vector from point A to point B is defined as the arrow pointing from A to B
- The magnitude is the straight-line distance from A to B

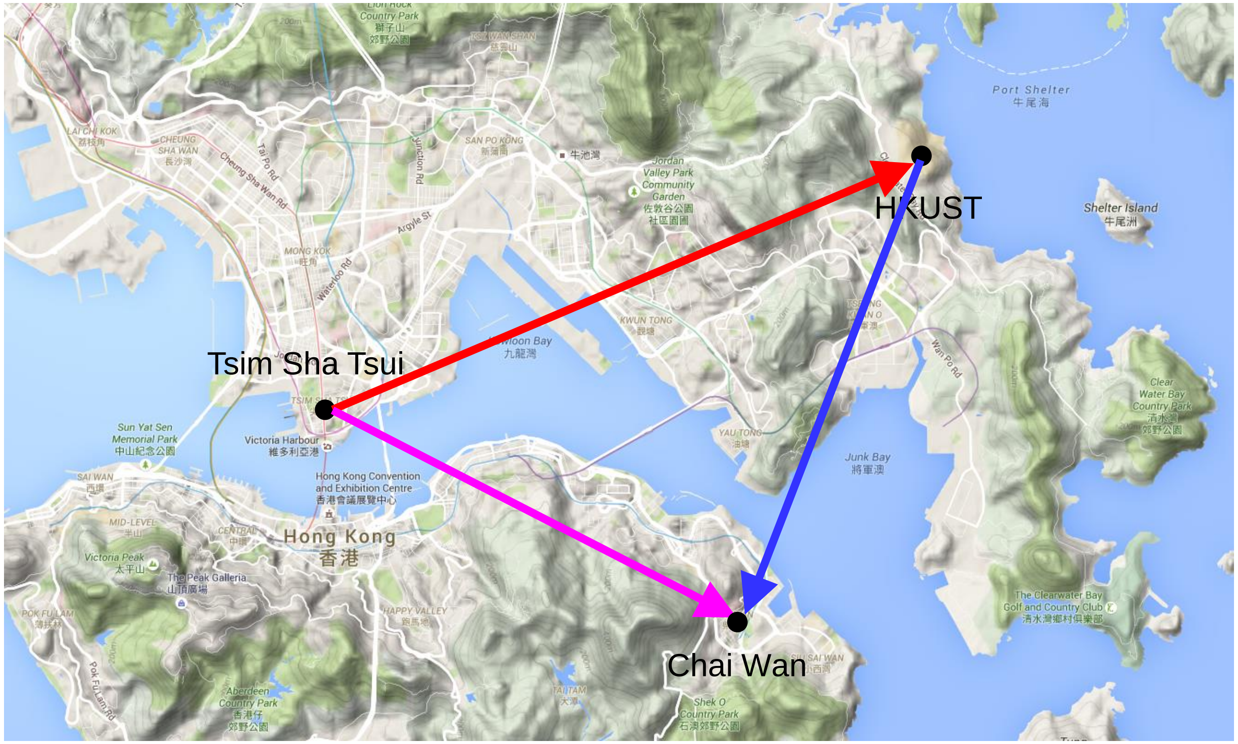


The displacement vector from Tsim Sha Tsui to HKUST

Vector Addition and Scalar Multiplication

Addition of Vectors

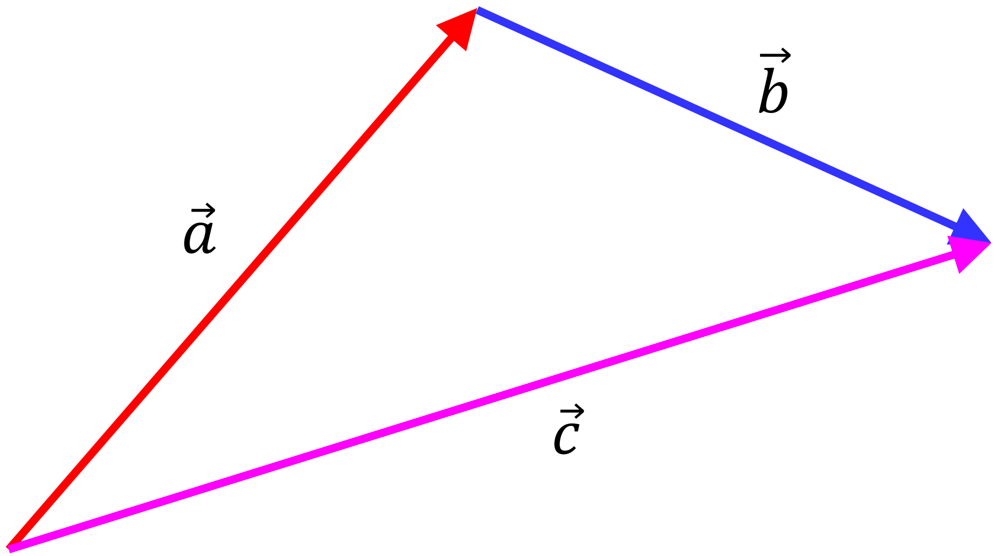
- Red arrow: Displacement vector from Tsim Sha Tsui to HKUST
- Blue arrow: Displacement vector from HKUST to Chai Wan
- If you go from Tsim Sha Tsui to HKUST, then from HKUST to Chai Wan, the final displacement will be from Tsim Sha Tsui to Chai Wan: The purple arrow



Addition of Vectors

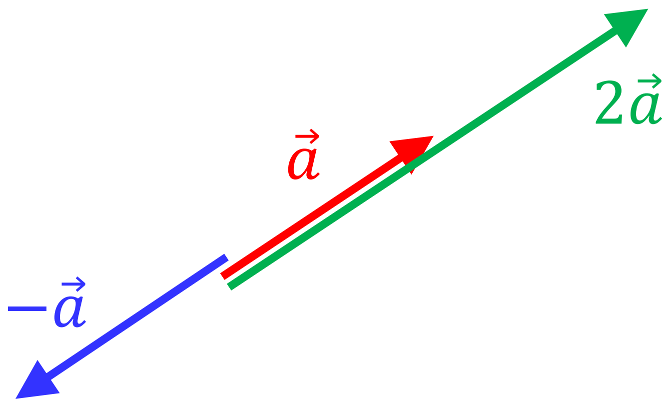
Vectors are added by *triangle rule* -- "Tip to tail"

$$\vec{c} = \vec{a} + \vec{b}$$



Scalar Multiplication

$\alpha\vec{a}$ $(\alpha > 0)$ changes the magnitude but not the direction of $\vec{a}$



$-\vec{a}$ flips the direction of $\vec{a}$ but keeps its magnitude

Null (Zero) Vector

A vector with zero length, called the null vector $\vec{0}$

It satisfies $\vec{a} + \vec{0} = \vec{0} + \vec{a} = \vec{a}$ for all vector **a**

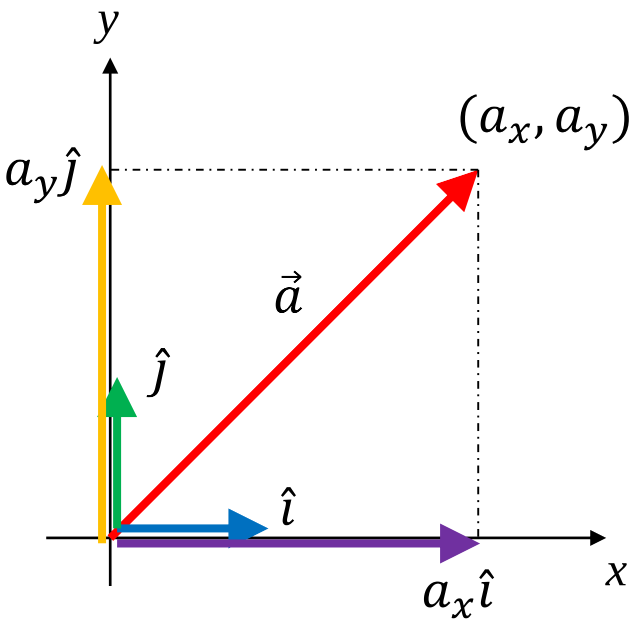
The null vector is the only vector with zero length

$$\left|\vec{0}\right| = 0$$

It is also the only vector that has no direction

Two Dimensions

In two dimensions, we can define an x-y coordinate system. With its tail at the origin, a vector can identified by the coordinates (a_x, a_y) of its tip

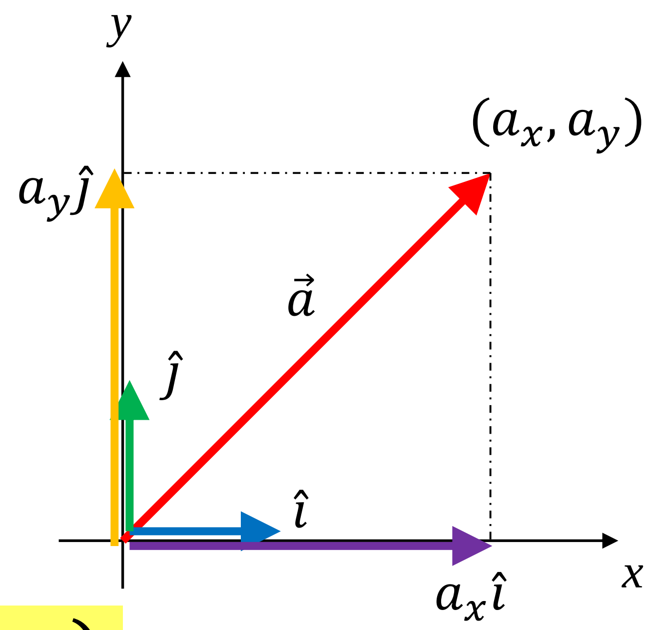


By the vector addition law:

$$\vec{a} = a_x\hat{\imath} + a_y\hat{\jmath}$$

Two Dimensions

$$\vec{a} = a_x\hat{\imath} + a_y\hat{\jmath}$$



$a_x\hat{\imath}$, $a_y\hat{\jmath}$: components (component vectors) of $\vec{a}$

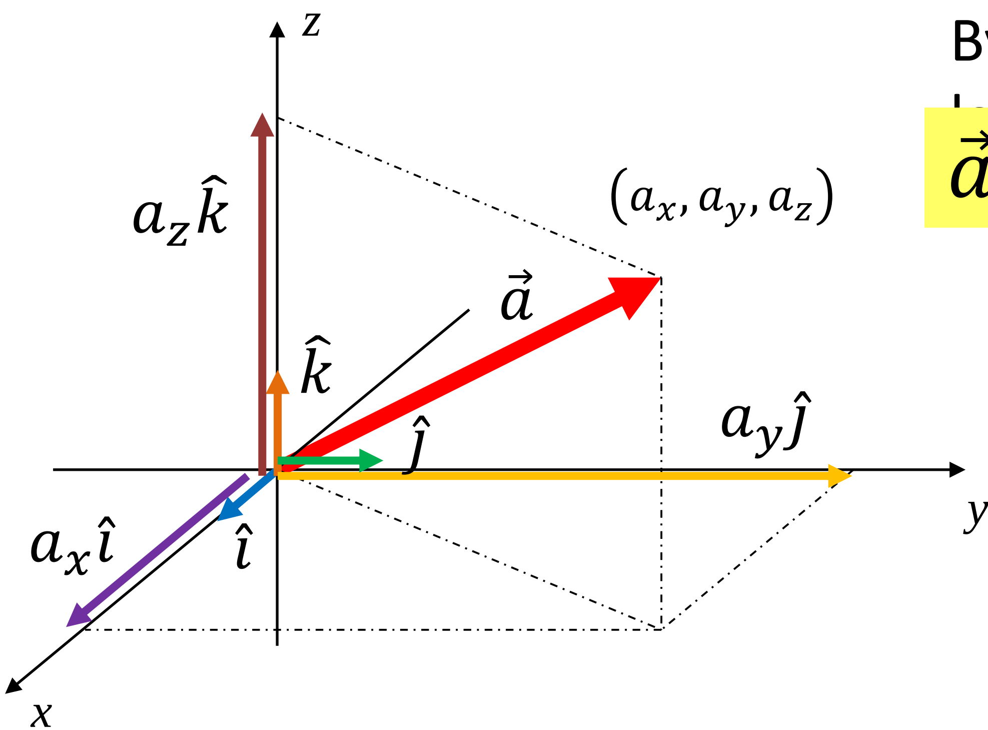
a_x, a_y: components of $\vec{a}$

We can also represent a vector by its components: $\vec{a} = (a_x, a_y)$

By Pythagoras' theorem: $|\vec{a}| = \sqrt{{a_x}^2 + {a_y}^2}$

Three Dimensions

Similarly, in three dimensions, we can define an x-y-z coordinate system. Then, with its tail at the origin, a vector can be identified by the coordinates (a_x, a_y, a_z) of its tip



By the vector addition

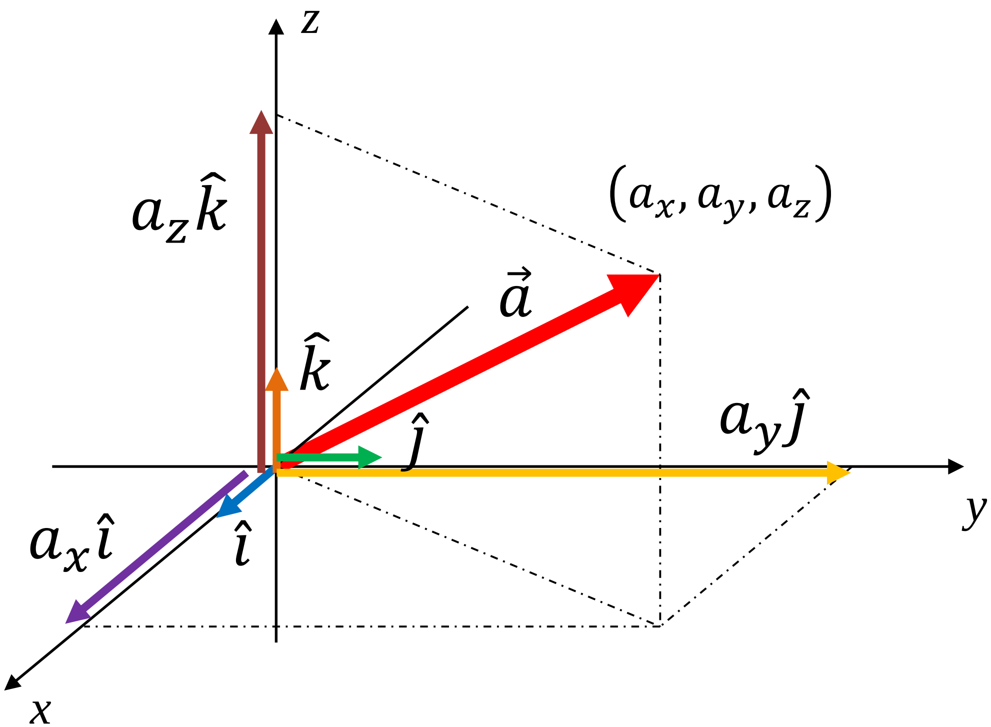
$$\vec{a} = a_x\hat{\imath} + a_y\hat{\jmath} + a_z\hat{k}$$

Three Dimensions

$$\vec{a} = a_x\hat{\imath} + a_y\hat{\jmath} + a_z\hat{k}$$

$a_x\hat{\imath},\ a_y\hat{\jmath},\ a_z\hat{k}$: components (component vectors) of $\vec{a}$

$a_x,\ a_y,\ a_z$: components of $\vec{a}$



We can also represent a vector by its components:

$$\vec{a} = (a_x, a_y, a_z)$$

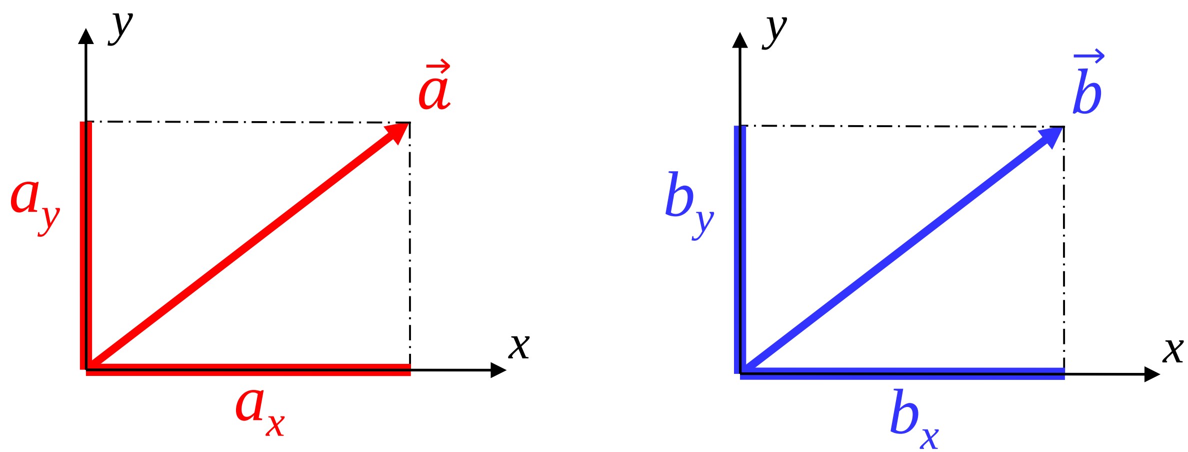
By Pythagoras' theorem:

$$|\vec{a}| = \sqrt{{a_x}^2 + {a_y}^2 + {a_z}^2}$$

Equality of Vectors

Two vectors are equal iff all their corresponding components are the same

For example,



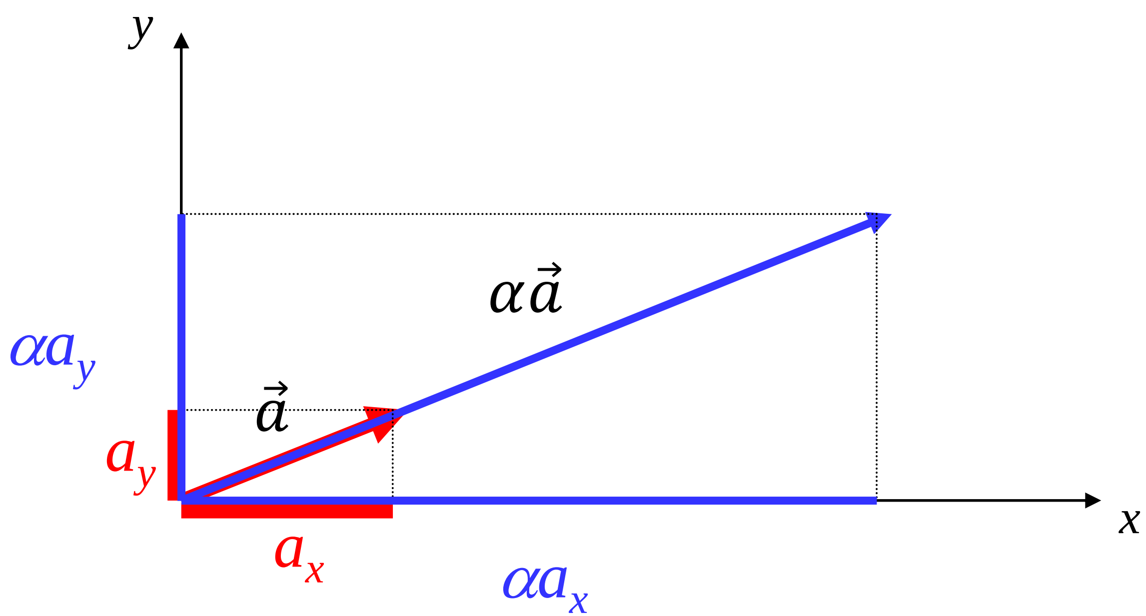
$$\vec{a} = \vec{b} \quad \text{iff} \quad \begin{cases} a_x = b_x \\ a_y = b_y \end{cases}$$

Scalar Multiplication in Component Form

If you multiply vector by α, its components are all multiplied by α. E.g., in two dimensions

$$\alpha\vec{a} = \alpha\left(a_x\hat{\imath} + a_y\hat{\jmath}\right) = \alpha a_x\hat{\imath} + \alpha a_y\hat{\jmath}$$

$$\alpha\left(a_x, a_y\right) = \left(\alpha a_x, \alpha a_y\right)$$

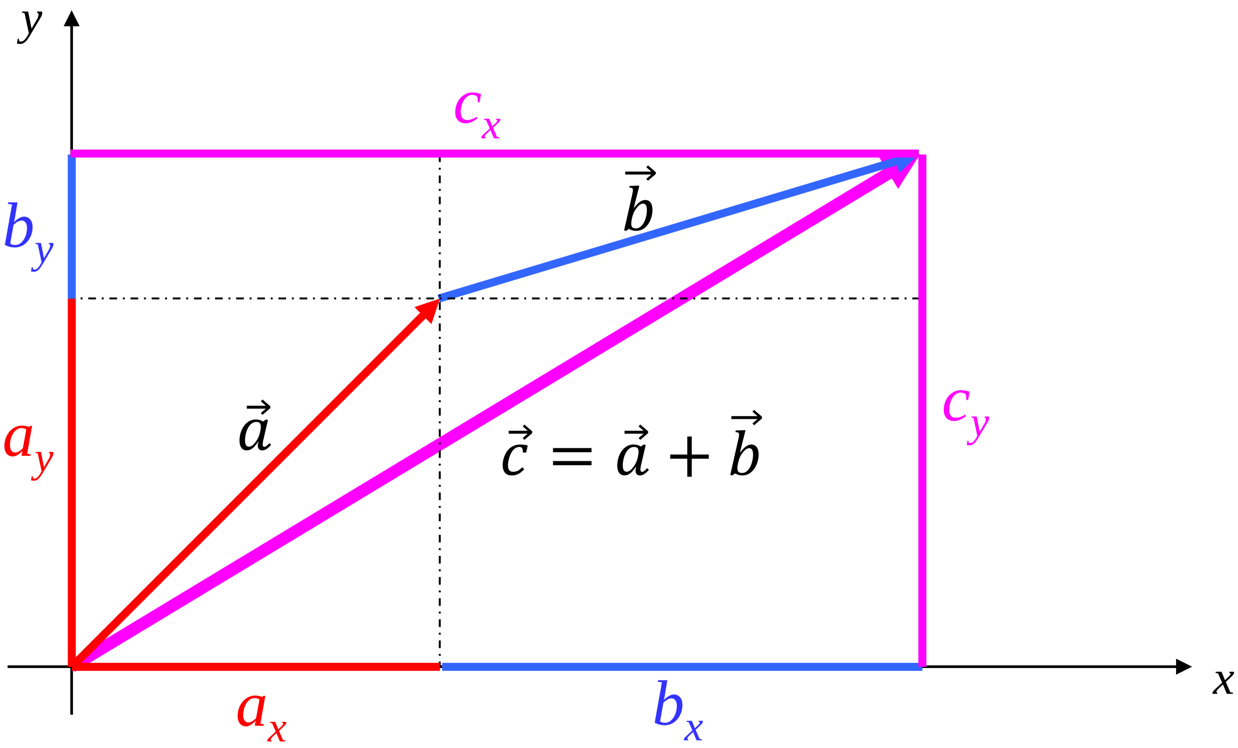


Vector Addition in Component Form

If you multiply vector, their components add

E.g., in two dimensions: $\vec{a}+\vec{b}=(a_x+b_x)\hat{\imath}+(a_y+b_y)\hat{\jmath}$

$$(a_x,a_y)+(b_x,b_y)=(a_x+b_x,a_y+b_y)$$



Vector Operations in Component Form

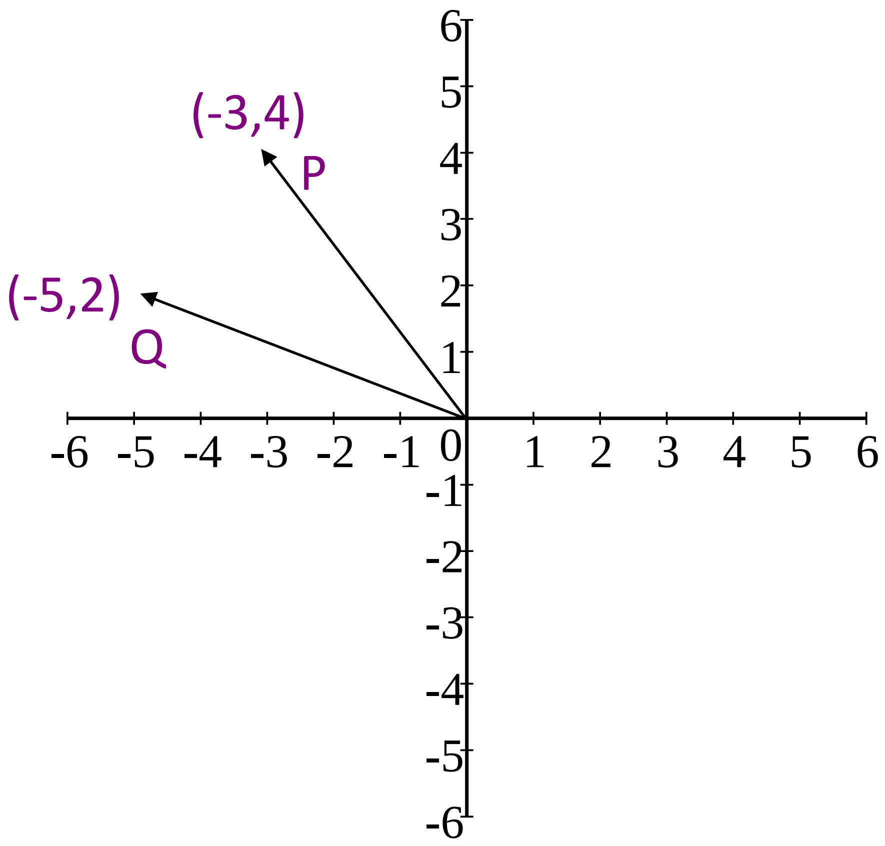
$$\alpha\vec{a} = \alpha(a_x, a_y\ ) = (\alpha a_x, \alpha a_y\ )$$

$$\vec{a} + \vec{b} = (a_x, a_y) + (b_x, b_y) = (a_x + b_x, a_y + b_y)$$

$$-\vec{a} = -1(a_x, a_y) = (-a_x, -a_y)$$

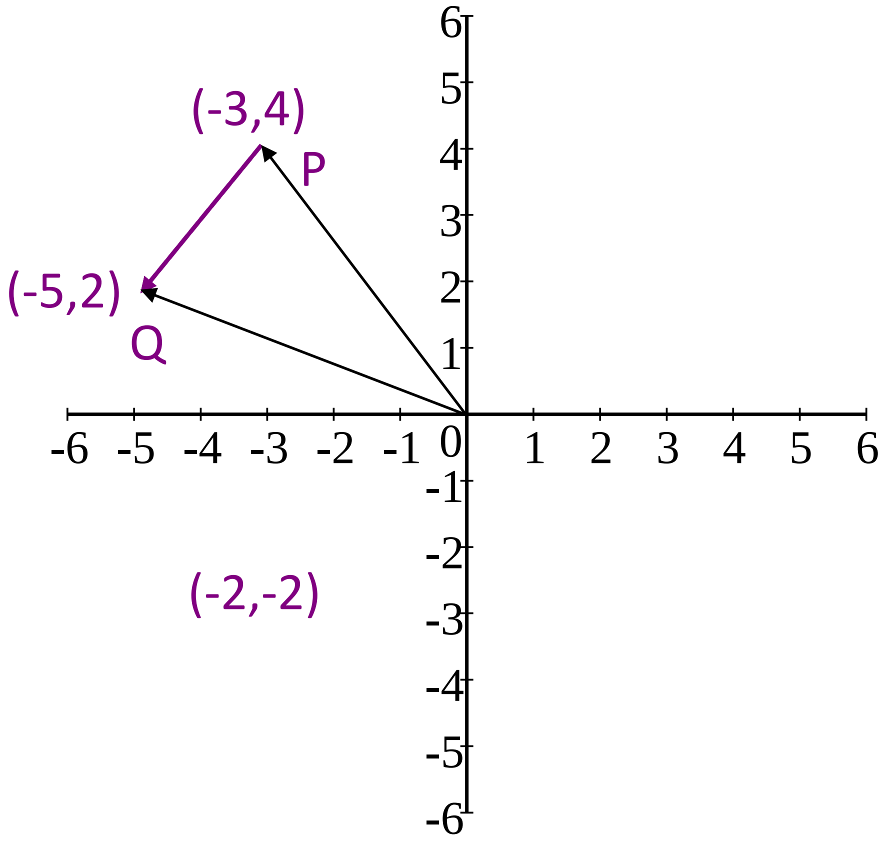
$$\vec{a} - \vec{b} = \vec{a} + (-\vec{b}) = (a_x - b_x, a_y - b_y)$$

Example



$$\left|\overrightarrow{PQ}\right| = ?$$

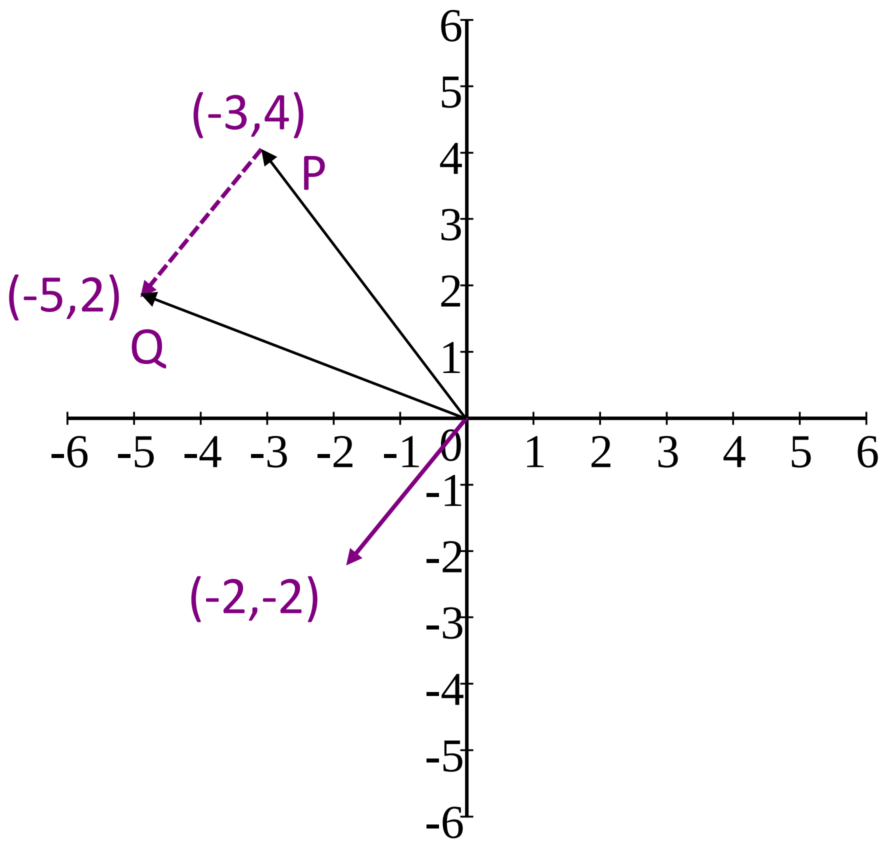
Example



$$\overrightarrow{PQ} = \overrightarrow{OQ} - \overrightarrow{OP}$$
$$= (-5,2) - (-3,4)$$
$$= (-2,-2)$$

$$\left|\overrightarrow{PQ}\right| = ?$$

Example



$$\overrightarrow{PQ} = \overrightarrow{OQ} - \overrightarrow{OP}$$
$$= (-5,2) - (-3,4)$$
$$= (-2,-2)$$

$$\left|\overrightarrow{PQ}\right| = \sqrt{(-2)^2 + (-2)^2}$$
$$= \sqrt{8}$$
$$= 2\sqrt{2}$$

Your Turn!

(Vector algebra practices)

Calculus

Differential Calculus

Let $y = f(x)$. Conceptually, the derivative of y with respect to (w.r.t.) x is the instantaneous rate of change of y w.r.t. x

Average rate of change = $\dfrac{f(x+\Delta x) - f(x)}{\Delta x}$

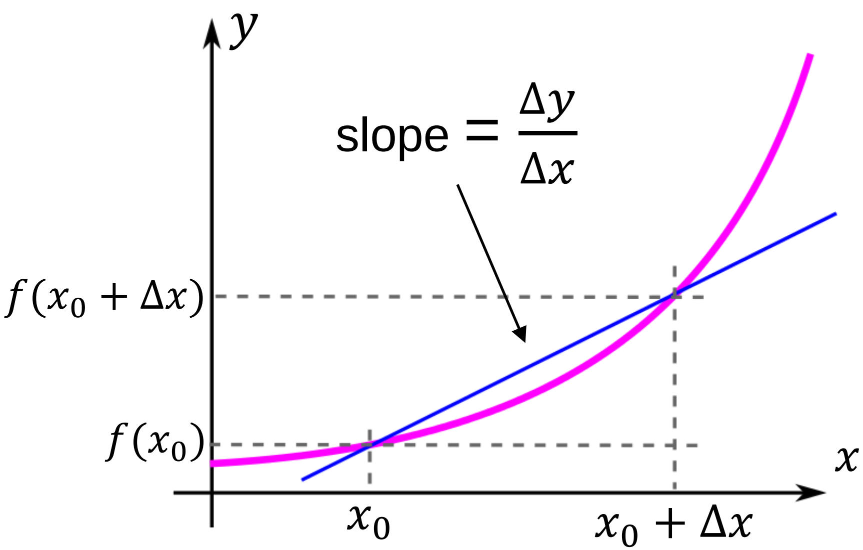
Instantaneous rate of change = $\lim\limits_{\Delta x \to 0} \dfrac{f(x+\Delta x) - f(x)}{\Delta x}$

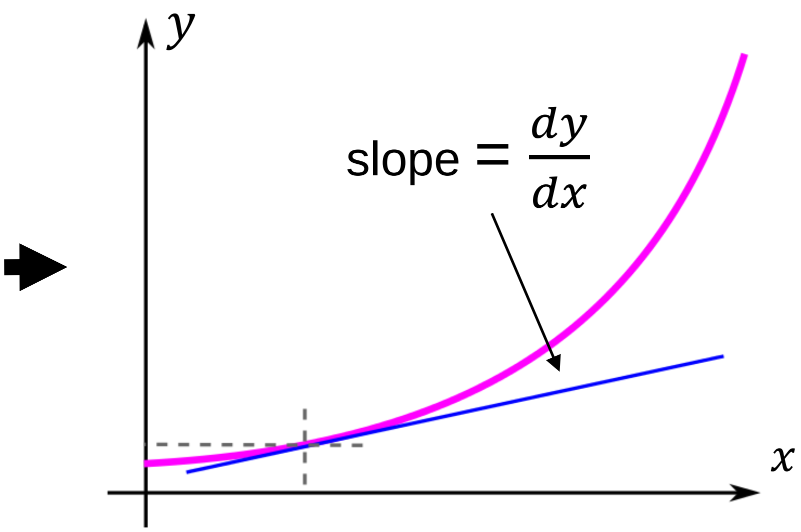
The derivative of $y = f(x)$ w.r.t. x may be denoted as $f'(x)$, y', $\dot{y}$, or $\dfrac{dy}{dx}$.

In general, y can be any dependent variable and x can be any independent variable.

Graphical Interpretation

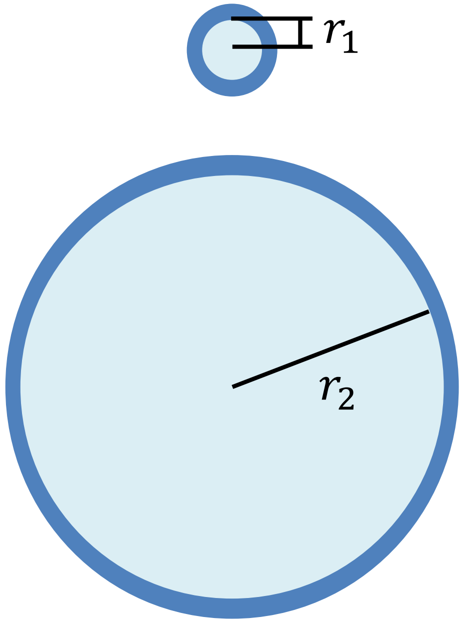
Graphically, $\frac{dy}{dx}$ evaluated at $x = x_0$ can be understood as the slope of tangent of the graph $y = f(x)$ at $x = x_0$





Example:

The area of a circle is given by $A = \pi\, r^2$



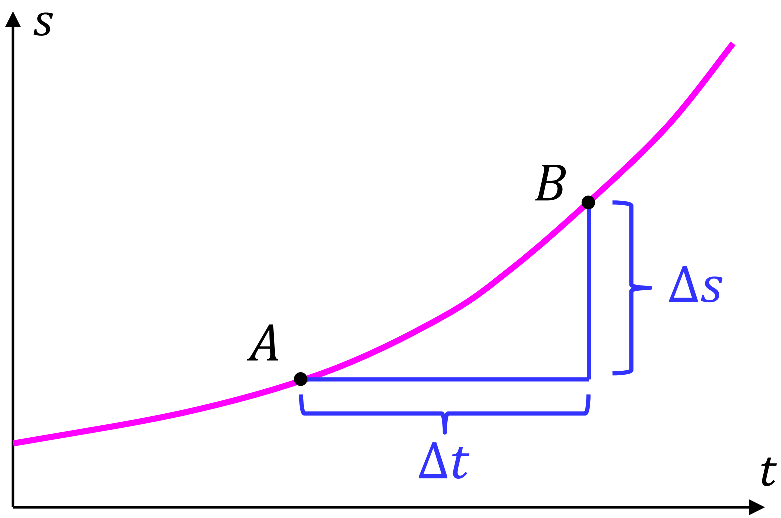
The rate of change of A w.r.t. r is

$$\frac{dA}{dr} = 2\pi r$$

which is the circumference of the circle

Rate of change increases with r
→ For a larger circle, the area changes faster with respect to the same amount of change in r

Velocity in one dimension



Average velocity can be found by taking:

$$\frac{\text{change in position}}{\text{change in time}} = \frac{\Delta s}{\Delta t}$$

$$v_{\text{avg}} = \frac{\Delta s}{\Delta t} = \frac{s(t + \Delta t) - s(t)}{\Delta t}$$

The instantaneous velocity at t is the derivative obtained when $B \rightarrow A$

$$v(t) = \frac{ds}{dt} = \lim_{\Delta t \to 0} \frac{s(t + \Delta t) - s(t)}{\Delta t}$$

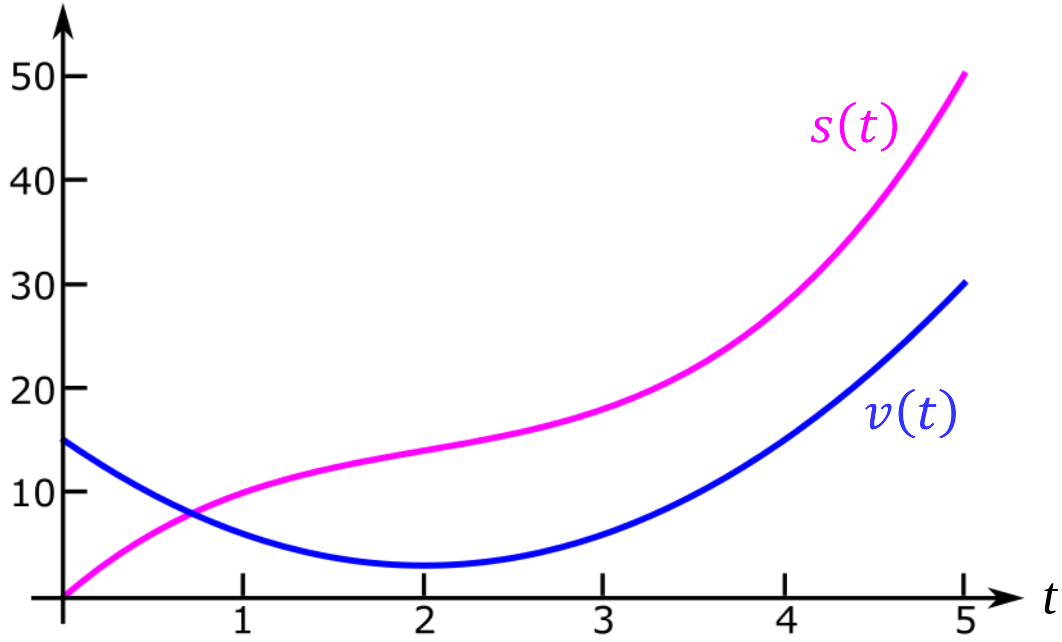
The velocity at one moment in time

Example:

Suppose the position of a particle is given by

$$s(t) = t^3 - 6t^2 + 15t$$

Then the velocity is $v(t) = \dfrac{ds}{dt} = 3t^2 - 12t + 15$



Velocity in higher dimensions

Velocity is the *rate of change* of displacement with respect to time

$$\vec{v}(t) = \frac{d\vec{s}}{dt}$$

Velocity has both magnitude and direction

Speed is the *magnitude* of velocity

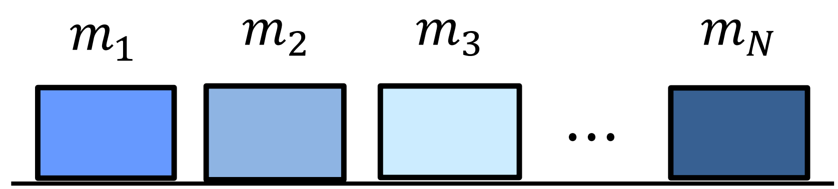
$$\text{Speed} = |\vec{v}| \; (= v)$$

Speed has no direction

Integral calculus

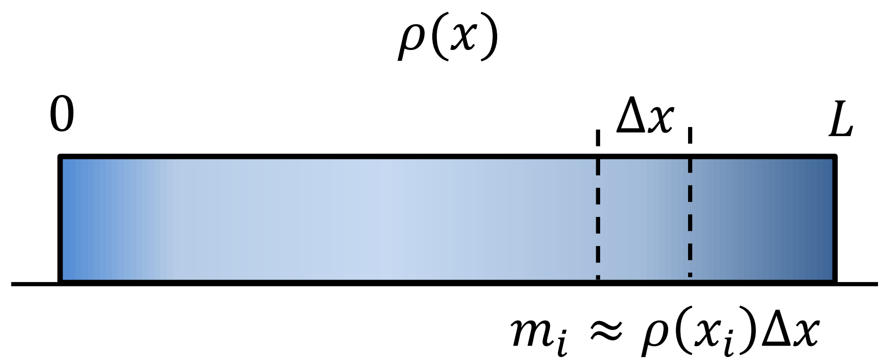
Conceptually, the definite integral $\int_a^b f(x)dx$ is a big sum of little pieces

$$M = \sum_{i=1}^{N} m_i$$



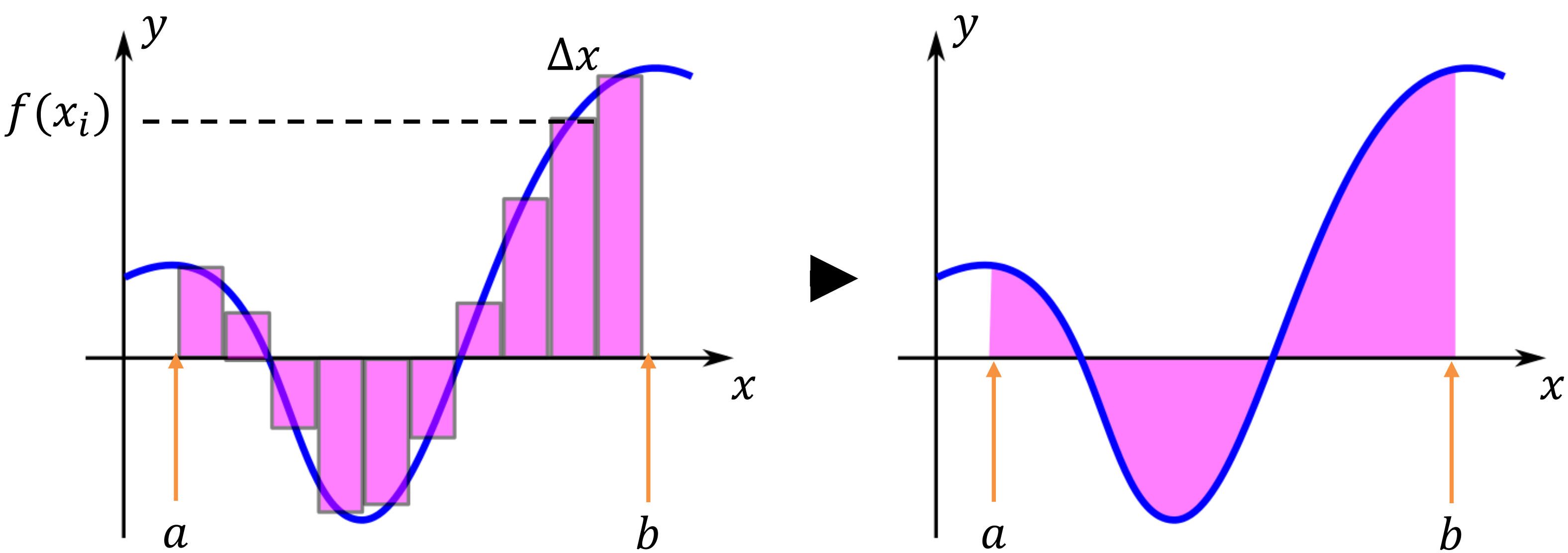
$$M = \lim_{\Delta x \to 0} \sum_i \rho(x_i)\Delta x$$

$$= \int_0^L \rho(x)dx$$



Graphical Interpretation

Graphically, $\int_a^b f(x)dx$ is the signed area under $f(x)$ between the two points a and b



$$\sum_i f(x_i)\Delta x \longrightarrow \int_a^b f(x)dx$$

Fundamental Theorem of Calculus

$$\frac{d}{dx}\int_a^x f(\xi)d\xi = f(x)$$

Since $\frac{dF}{dx} = \frac{dG}{dx}$ implies $F(x) = G(x) + C$, if we know a particular antiderivative $F(x)$ of $f(x)$, then:

$$\int_a^x f(\xi)d\xi = F(x) + C$$

To find C, note that $0 = \int_a^a f(x)dx = F(a) + C$. Thus,

$$\int_a^b f(x)dx = F(b) - F(a)$$

Examples:

$$\frac{d}{dx}x = 1 \quad \Rightarrow \quad \int dx = x + C$$

$$\frac{d}{dx}\frac{x^2}{2} = x \quad \Rightarrow \quad \int x\,dx = \frac{x^2}{2} + C$$

$$\frac{d}{dx}\left(-\frac{1}{x}\right) = \frac{1}{x^2} \quad \Rightarrow \quad \int \frac{1}{x^2}dx = -\frac{1}{x} + C$$

$$\frac{d}{dx}\frac{x^n}{n} = x^{n-1} \quad \Rightarrow \quad \int x^{n-1}\,dx = \frac{x^n}{n} + C \qquad \text{Integer } n \neq 0$$

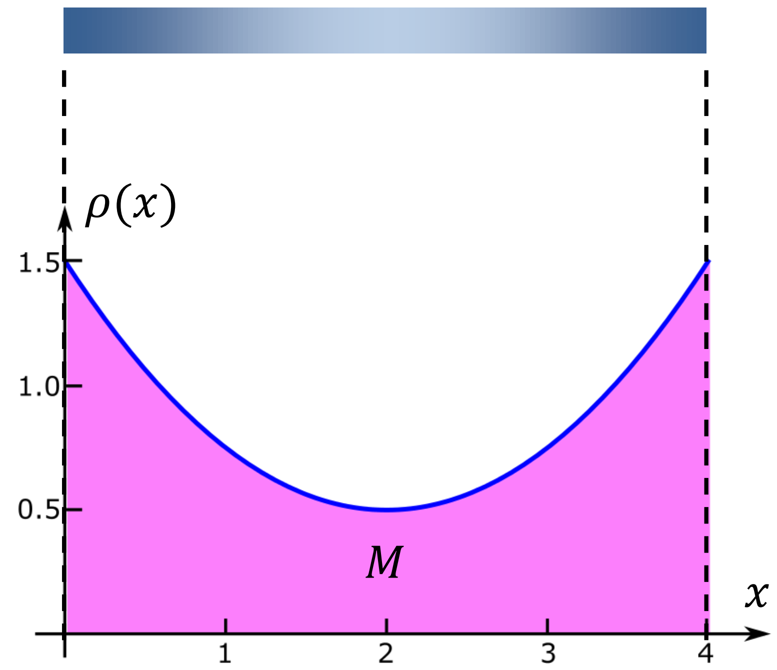
$$\frac{d}{dx}\sin x = \cos x \quad \Rightarrow \quad \int \cos x\,dx = \sin x + C$$

$$\frac{d}{dx}\cos x = -\sin x \quad \Rightarrow \quad \int \sin x\,dx = -\cos x + C$$

$$\frac{d}{dx}e^{kx} = ke^{kx} \quad \Rightarrow \quad \int e^{kx}\,dx = \frac{1}{k}e^{kx} + C$$

Example:

The density (mass per length) of a rod spanning from $x = 0$ to $x = 4$ is given by $\rho(x) = \frac{1}{4}(x^2 - 4x + 6)$



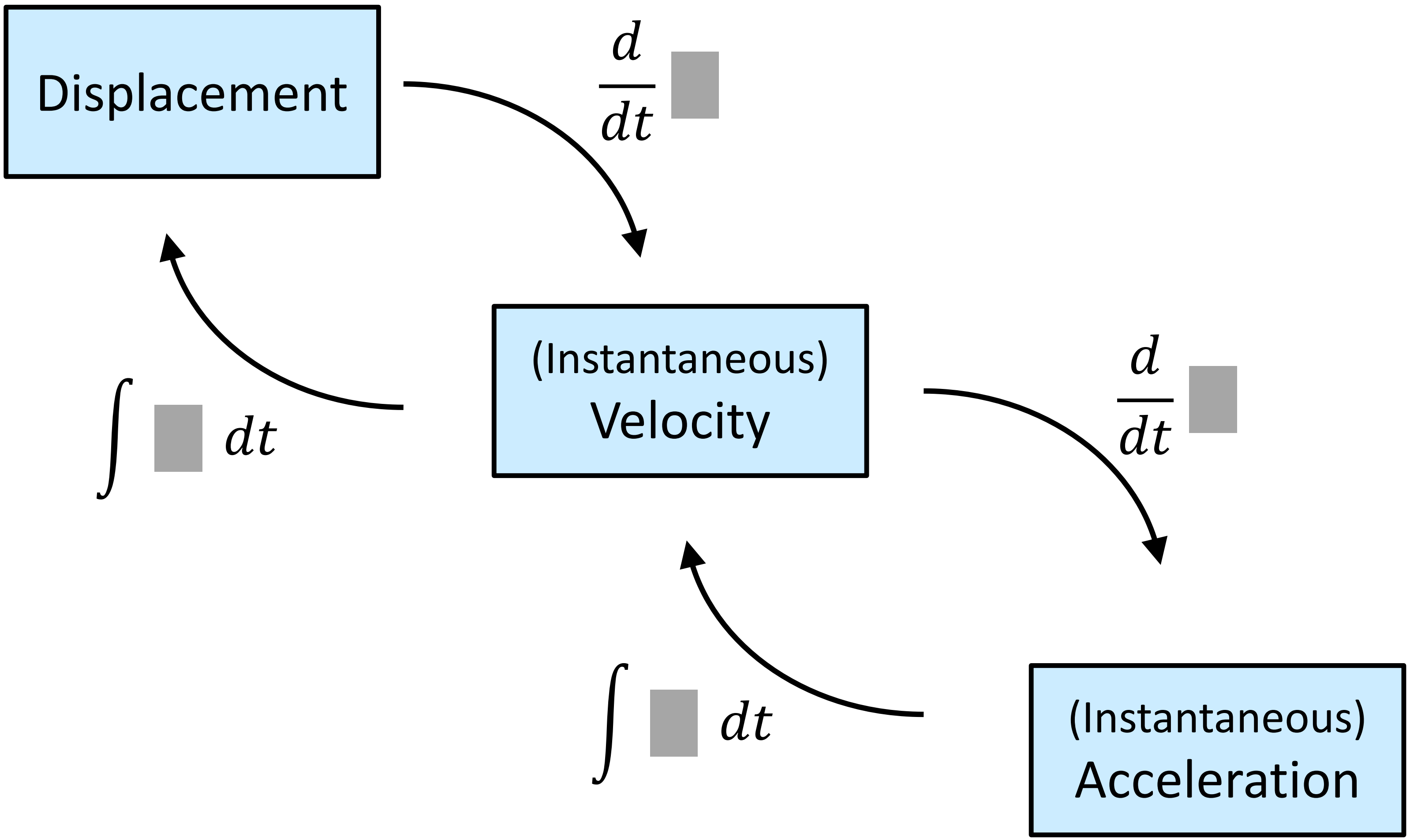
The total mass of the rod is then:

$$M = \int_0^4 \rho(x)dx$$

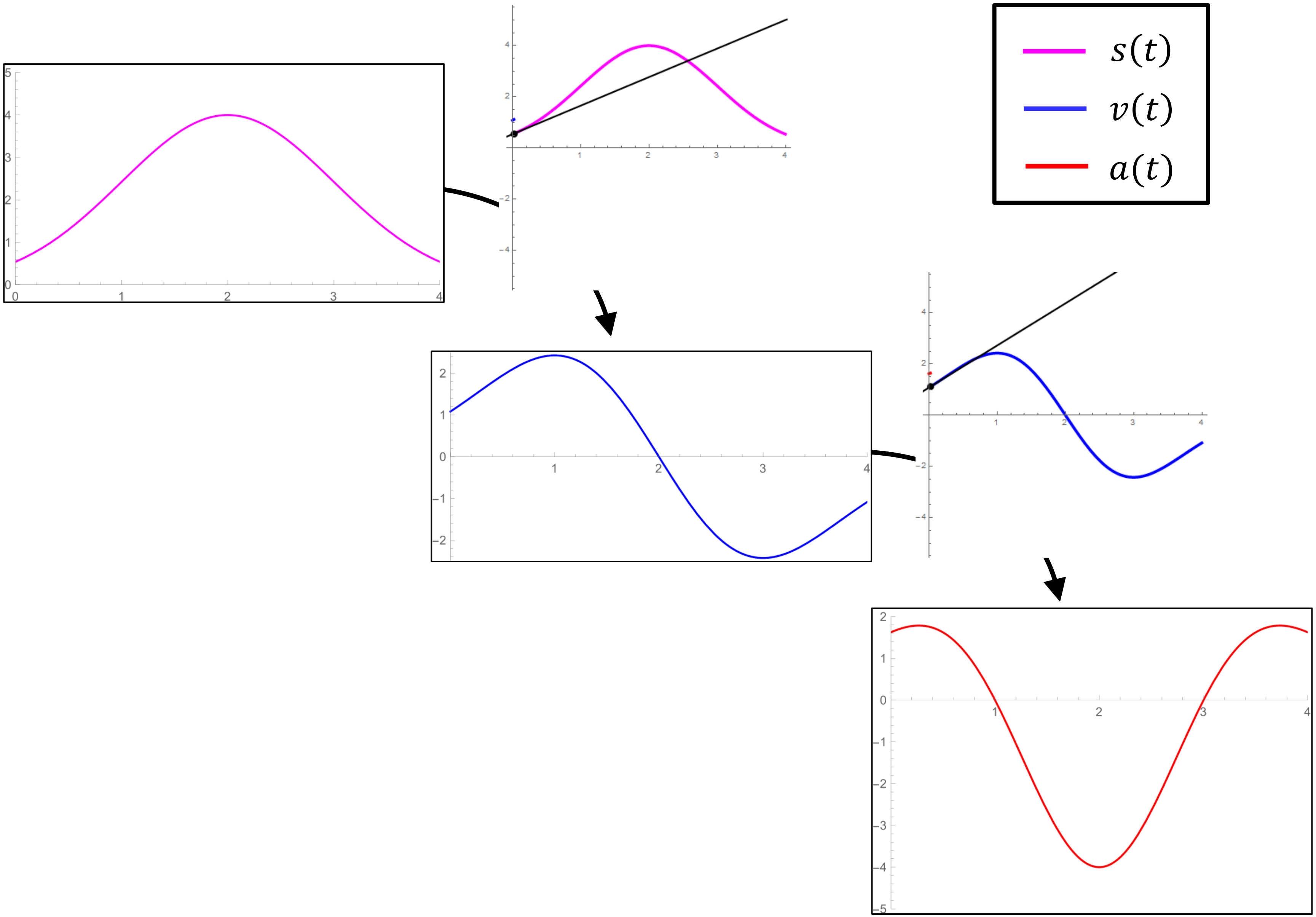
$$= \left[\frac{1}{4}\left(\frac{x^3}{3} - 2x^2 + 6x\right)\right]_0^4$$

$$= \frac{10}{3}$$

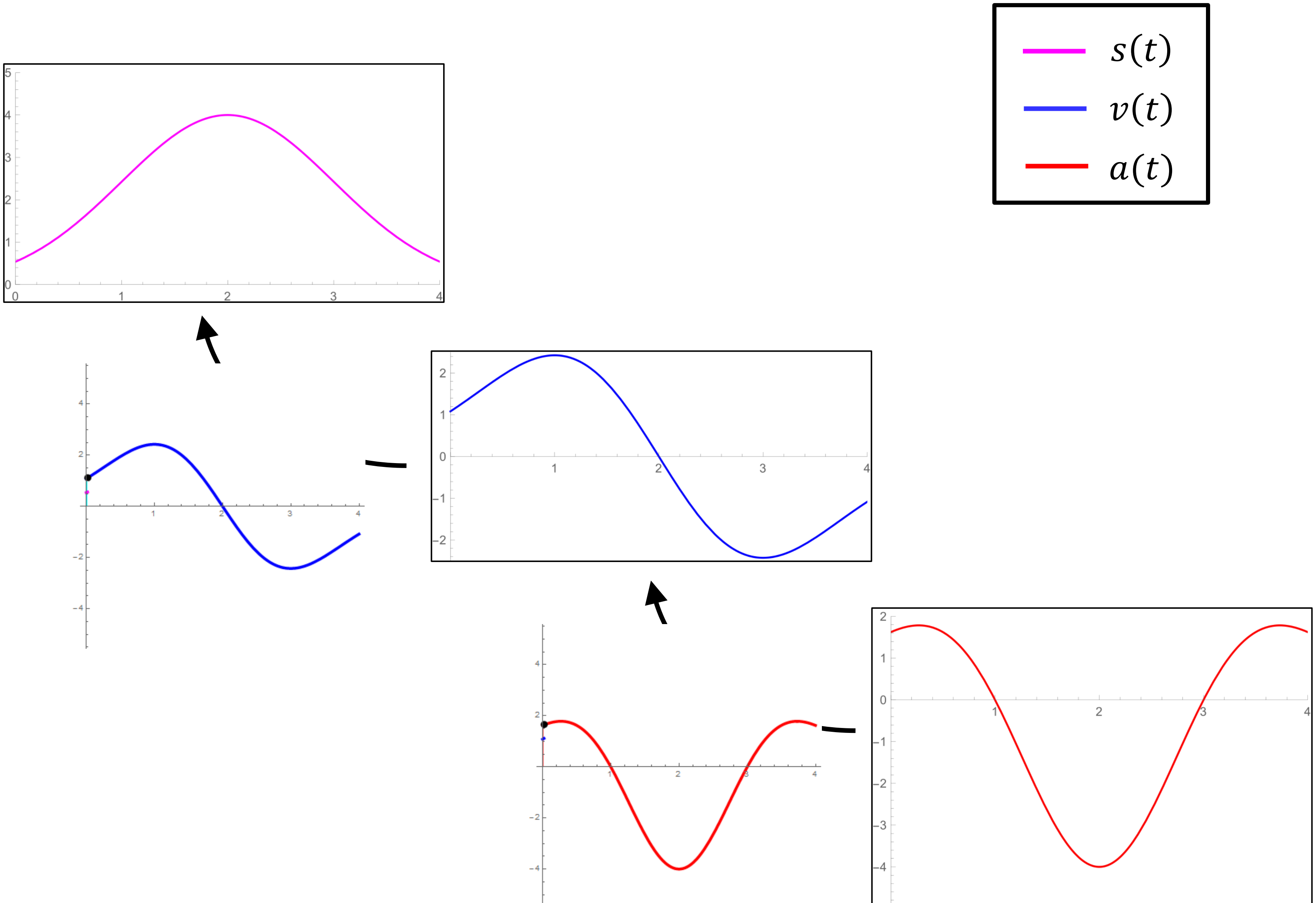
Displacement, Velocity and Acceleration



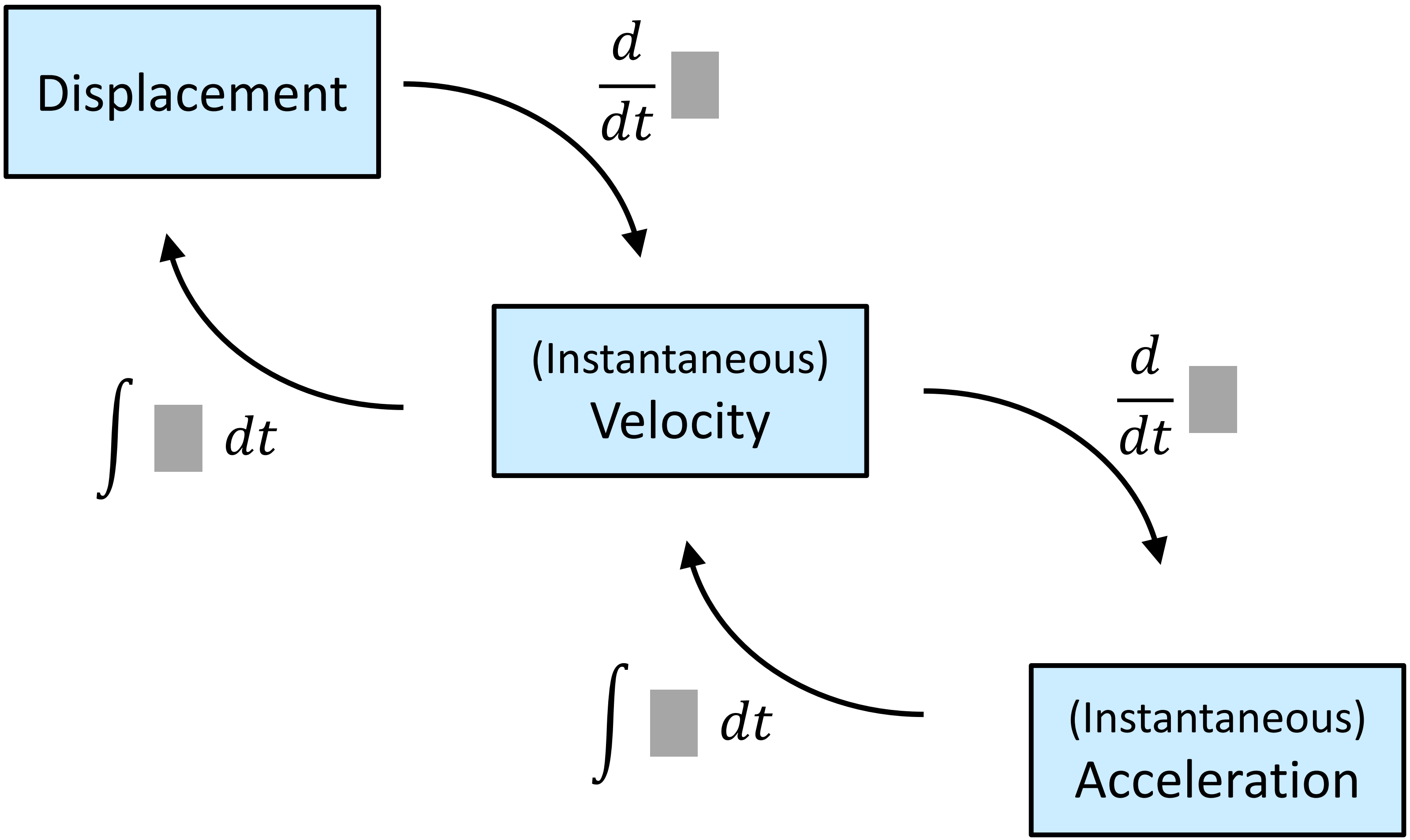
Displacement, Velocity and Acceleration



Displacement, Velocity and Acceleration



Displacement, Velocity and Acceleration



Your Turn!

(Calculus practices)