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**New Form of the Region-to-Factor Message**

1. **New Form of the Region-to-Factor Message**

Previously, we proposed the following form of the region-to-factor message,



However, this gives rise to recursion relations with terms difficult to interpret. We need to explore more convenient forms of the region-to-factor messages.

Some insights can be obtained from the Edwards-Anderson model [1]. In that model, a factor-to-node message is a single-spin message contributing to the magnetic field of a node, and a region-to-factor message consists of two single-spin messages passed to the two nodes belonging to the factor, and a two-spin message passed to the factor. The single-spin messages contribute to the magnetic fields of the nodes, and the two-spin message contributes to the coupling between the nodes. The region-to-factor message then corresponds to a spin chain with modified coupling strengths, with each spin subject to magnetic fields due to different contributions.

Extending to the network flow model, a factor-to-node message corresponds to a current supply through a potential and a conductance. For the region-to-factor message, it is natural to propose that there are two single-potential messages that supply current through a potential and a conductance to the two nodes belonging to the factor, and a double-potential message that contributes to the conductance of the link between the nodes. The region-to-factor message then corresponds to a chain of nodes, with modified conductance and each node fed by a few current supplies. Hence we propose



Note that this region-to-factor message consists of 5 parameters. This is the same number of parameters as in the previous form. Hence there is a one-to-one correspondence between them.

1. **Factor-to-Node Messages**

Consider generalized belief-propagation. A factor collects factor-to-node messages from factors connected from nodes 2, 3 and 4 neighboring node *j* and region-to-factor messages from regions *p* and *q* feeding the factor. The factor then sends a message from node *j* to node *i*.



***i***

**2**

**4**

***j***

***p***

***q***

**3**

The generalized belief-propagation equation becomes



*g* is inserted for checking. It will be set to *g* = 1 at the end. Assume that the messages have the forms





Minimization equation:



Introduce the conductances and the potential

, , 

Then the equation is simplified to





The factor-to-node message becomes







Quadratic term in *Fj*(*μi*)

Linear term in *Fj*(*μi*) 

Further introduce the conductance and potential

, 

The recursion relation becomes

Factor-to-node message from node *j* to *i*, no base stations

, , , ,



 and



1. **Region-to-Factor Messages**

Consider a region *p* consisting of factors *a*, *b*, *c* and *d*. It collects 4 factor-to-node messages from factors connected from nodes 3 and 4, 7 and 8 to nodes 5 and 6 respectively, and 3 region-to-factor messages feeding respectively nodes 2 and 5, 5 and 6, 6 and 1. The region then sends a message to factor 1 and 2.

**7**

**8**

**4**

**61**

**25**

**56**

**12**



**1**

**2**

**5**

**3**

**6**

Then the generalized belief-propagation equation becomes







Minimization equation:









Introduce the conductances and the potentials















In matrix form,



Solution:



The region-to-factor message becomes, after neglecting constant terms,









Coefficient of .

Coefficient of .

Coefficient of .

Coefficient of  

Coefficient of  

Hence the set of messages consists of:

Conductance between nodes 1 and 2 = coefficient of .

Conductance to node 1 from its voltage source = 2(coefficient of ) − coefficient of 

.

Conductance to node 2 from its voltage source = 2(coefficient of ) − coefficient of 

.

Current from the voltage source of node 1 = coefficient of 



Current from the voltage source of node 2 = coefficient of 



The recursion relation becomes, immediately after updating 

Region-to-factor message from nodes 5 and 6 in the direction of nodes 1 and 2, no base stations

, ,

, ,

, , ,

,



.

.





1. **Messages from Base Stations**

The recursion relations are modified when a node is a base station, where the potential is fixed at 0. Consider the modified factor-to-node message supposing that node *j* is a base station.



***i***

**2**

**4**

***j***

***p***

***q***

**3**

The generalized belief-propagation equation becomes



The recursion relation becomes





Introduce the conductances and the potential

, , 

Then the recursion relations become





Factor-to-node message from base station *j*

, , 

 and



Consider the region-to-factor message supposing node 5 is a base station.

**7**

**8**

**4**

**61**

**25**

**56**

**12**



**1**

**2**

**5**

**3**

**6**

Then the generalized belief-propagation equation becomes







Minimization equation:





Introduce the conductances and the potentials











This simplifies the equation to



The region-to-factor message becomes, after neglecting constant terms,





Coefficient of 

Coefficient of 

Coefficient of 

Coefficient of  

Coefficient of  

The recursion relation becomes, immediately after updating 

Region-to-factor message from nodes 5 and 6 in the direction of 1 and 2, 5 being a base station

,

, ,

, , ,



,

,





1. **The Full Potential**

After obtaining the messages, the full potential can be calculated according to



The optimal equation is given by





1. **Example**

Consider a square lattice with nodes being sources, destinations and relays, denoted as *s*, *d* and *r* respectively.



Solution: , 

Now consider the belief propagation solution.

 and 

Conductances:



⇒ 

Flows:





⇒ 

⇒ ⇒ 

Note that this set of equations have solutions only if Λ*s* + 2Λ*r* + Λ*d* = 0.

⇒ 

,





 as expected.

Now consider the generalized belief propagation solution.

Conductances:

,

, , ,

,

.

Dividing the last two equations, .

We arrive at an equation for : .

Substituting into the equation for :

⇒ , .

.

Substituting into the first equation:

,

⇒ ⇒

⇒

**The Presence of Base Stations**

Since this set of equations have solutions only if Λ*s* + 2Λ*r* + Λ*d* = 0, it is useful to introduce base stations. The solution should be consistent with the assumption that all destination nodes are base stations. Thus and .



⇒ 

⇒ , , ,







In matrix form,













 as expected.

Next consider the generalized belief propagation solution. We first work out the conductances. The 10 variables are represented in the following figure, namely, from left to right, *grd*, *gdr*, *grs*, *gsr*, *gsrs*, *hsr*, *gsrr*, *gdrd*, *hdr*, and *gdrr*.



***s***

***r***

***d***

***r***

***r***

***d***

***d***

***r***

***r***

***s***

***s***

***r***



***d***

***r***

***s***

***r***

, (1)

, (2)

, (3)

, (4)

, (5)

, (6)

, (7)

, (8)

, (9)

. (10)

At the steady state solution,

(1) allows us to eliminate : . (11)

(7) – (1): (12)

This will be useful in simplifying the equations.

(12) – (11) allows us to eliminate : . (13)

(4) – (6) and using (12) and (13): . (14)

(3) – 2(13) and using (14): . (15)

This will be useful in simplifying the equations, and allows us to eliminate :

. (16)

(4) and using (13), (14) and (15) allows us to calculate : .

⇒ . (17)

(16): . (18)

(8) and using (12), (14) and (15) allows us to calculate :

. (19)

(9) and using (8), (14), (15), (18) allows us to calculate :

. (20)

(2) allows us to calculate :

. (21)

(10): ⇒ (22)

Note that Eqs. (18) an (22) are dependent. Hence the solution space is a one-dimensional subspace. We need to find the stable solution in this subspace.

Numerical results show that the solution is stable for .

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Comparison of BP and GBP

|  |  |  |  |  |  |  |
| --- | --- | --- | --- | --- | --- | --- |
|  |  | *grs* | *grd* | *gsr* | *gdr* |  |
|  | BP | 0.6861 | 1 | 0.7287 | 0.7035 |  |
|  | GBP | 0.7287 | 0.9574 | 0.7287 | 0.6483 |  |

|  |  |  |  |  |  |  |
| --- | --- | --- | --- | --- | --- | --- |
|  | *gsrs* | *gsrr* | *hsr* | *gdrd* | *gdrr* | *hdr* |
| BP | 0 | 0 | 0 | 0 | 0 | 0 |
| GBP | 0 | 0.0213 | 0 | −0.0603 | −0.1144 | 0.0931 |

When the steady state solution is set to be the initial condition of the GBP, the solution remains a fixed point (Fig. 1). When noises are added to the initial condition, the solution converges after some iteration (Fig. 2).

Fig. 1: Results of GBP when the initial condition is set to be the steady state solution.

Fig. 2: Results of GBP when the initial condition is set to be the steady state solution plus noise.

Cavity potentials in GBP:

, (1)

, (2)

, (3)

, (4)

, (5)

, (6)

, (7)

. (8)

(1): . (9)

(1) and (6): . (10)

Substituting (9) and (10) into (4): . (11)

Substituting (9) and (10) into (5): . (12)

(11) or (12): . (13)

(3): . (14)

Substituting (14) into (13): . (15)

Substituting (15) into (14): . (16)

(7): . (17)

(2): . (18)

(8): , consistent with (16). (19)

**Reference**

[1] E. Dominguez, A. Lage-Castellanos, R. Mulet, F. Ricci-Tersenghi, T. Rizzo, “Characterizing and Improving Generalized Belief Propagation Algorithms on the 2D Edwards-Anderson Model”, J. Stat. Mech. P12007 (2011). arXiv:1110.1259