Optimization for Data Science
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Lecture 10: Mirror Descent, Smoothing, Proximal Algorithms

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https://www.ti.inf.ethz.ch/ew/courses/ODS23/index.html

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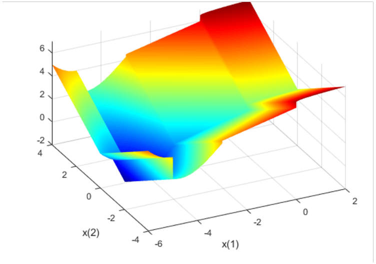
Recap: Convex Nonsmooth Optimization

$$\begin{array}{ll} \text{minimize} & f(\mathbf{x}) \\ \text{subject to} & \mathbf{x} \in X \end{array}$$



- For convex functions, subgradients always exist in the interior.
- Subgradients share lots of similar properties as gradients.
- Subgradient methods can be slow.

NB: For nonconvex nonsmooth functions, finding an approximately stationary point with first-order methods is intractable in general [Zha20].

Recap: Subgradient Descent

Subgradient Descent

$$\mathbf{x}_{t+1} = \Pi_X(\mathbf{x}_t - \gamma_t \mathbf{g}_t) = \underset{\mathbf{x}\in X}{\operatorname{argmin}} \left\{ \frac{1}{2}\|\mathbf{x}-\mathbf{x}_t\|_2^2 + \langle \gamma_t \mathbf{g}_t, \mathbf{x}\rangle \right\}, \quad \mathbf{g}_t \in \partial f(\mathbf{x}_t).$$

- **Convergence rate**: $O\left(\frac{B\cdot R}{\sqrt{t}}\right)$ for convex objectives
- **Subgradient complexity**: $O\left(\frac{B\cdot R}{\epsilon^2}\right)$ for convex objectives
- From information-theoretic viewpoint, the rate of subgradient descent cannot really be improved, despite being slow.

$$B := \sup_{\mathbf{x}\in X} \frac{|f(\mathbf{x})-f(\mathbf{y})|}{\|\mathbf{x}-\mathbf{y}\|_2}, R := \max_{\mathbf{x},\mathbf{y}\in X} \|\mathbf{x}-\mathbf{y}\|_2, BR = \|\cdot\|_2\text{-variation of } f \text{ on } X$$

Clicker Question (EduApp)

Consider the example:

$$f(\mathbf{x}) = \sum_{i=1}^{d} |x_i - a_i|,$$

$$X = \Delta_d := \{\mathbf{x} \in \mathbb{R}_+^d : \sum_{i=1}^{d} x_i = 1\}.$$

What's the order of the convergence rate when applying subgradient descent?

- $O\left(\frac{1}{\sqrt{t}}\right)$
- $O\left(\frac{\sqrt{d}}{\sqrt{t}}\right)$
- $O\left(\frac{d}{\sqrt{t}}\right)$
- None of the above

Motivation

In practice, we often have extra information about set X and nonsmooth function f.

- Can we exploit non-Euclidean geometry of convex set X? (instead of Euclidean geometry)
  $\Rightarrow$ **Mirror Descent!**

- Can we exploit additional structure of nonsmooth objective f? (instead of treating it as black box)
  $\Rightarrow$ **Smoothing and Proximal Algorithms!**

General Norms and Dual Norms

- **Norm:** A function $\|\cdot\|: \mathbb{R}^d \to \mathbb{R}_+$ is a <u>norm</u> if
  (a) $\|\mathbf{x}\| = 0$ if and only if $\mathbf{x} = 0$;
  (b) $\|\alpha\mathbf{x}\| = |\alpha| \cdot \|\mathbf{x}\|$;
  (c) $\|\mathbf{x}+\mathbf{y}\| \le \|\mathbf{x}\| + \|\mathbf{y}\|$.

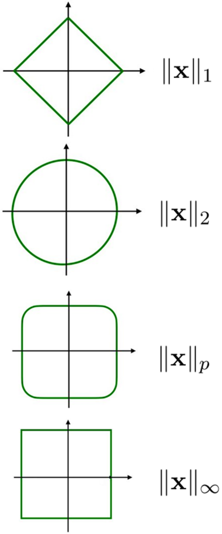
- **Dual norm:**

$$\|\mathbf{y}\|_* := \max_{\|\mathbf{x}\|\le 1} \langle \mathbf{x}, \mathbf{y}\rangle.$$

- Example: for $p \ge 1$ and $1/p + 1/q = 1$,

$$\|\mathbf{x}\|_p := \left(\sum_{i=1}^d |x_i|^p\right)^{1/p}, \|\cdot\|_{p,*} = \|\cdot\|_q$$

- Inequality: $\frac{1}{\sqrt{d}}\|\mathbf{x}\|_2 \le \|\mathbf{x}\|_\infty \le \|\mathbf{x}\|_2 \le \|\mathbf{x}\|_1 \le \sqrt{d}\|\mathbf{x}\|_2$



General Smoothness and Strong Convexity

Smoothness: $f(\mathbf{x})$ is L-smooth on X if $f(\mathbf{x})$ is differentiable and

$$f(\mathbf{x}) \leq f(\mathbf{y}) + \nabla f(\mathbf{y})^T(\mathbf{x}-\mathbf{y}) + \frac{L}{2}\|\mathbf{x}-\mathbf{y}\|^2, \forall \mathbf{x}, \mathbf{y} \in X.$$

Lipschitz continuity: $f(\mathbf{x})$ is B-Lipschitz continuous on X if

$$|f(\mathbf{x}) - f(\mathbf{y})| \leq B\|\mathbf{x}-\mathbf{y}\|, \forall \mathbf{x}, \mathbf{y} \in X.$$

Strong convexity: $f(\mathbf{x})$ is μ-strongly convex on X if for any $\mathbf{g} \in \partial f(\mathbf{x})$,

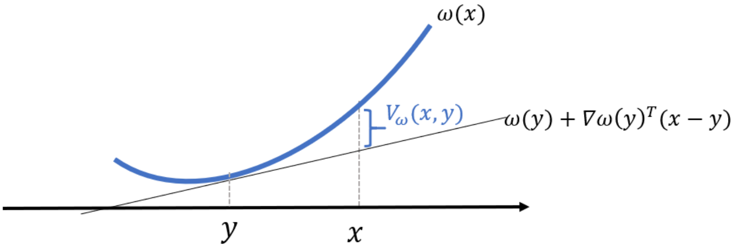
$$f(\mathbf{x}) \geq f(\mathbf{y}) + \mathbf{g}^T(\mathbf{x}-\mathbf{y}) + \frac{\mu}{2}\|\mathbf{x}-\mathbf{y}\|^2, \forall \mathbf{x}, \mathbf{y} \in X.$$

Bregman Divergence

Let $\omega(\cdot) : \Omega \to \mathbb{R}$ be continuously differentiable on Ω and 1-strongly convex w.r.t. some norm $\|\cdot\|$: $\omega(\mathbf{x}) \geq \omega(\mathbf{y}) + \nabla\omega(\mathbf{y})^T(\mathbf{x}-\mathbf{y}) + \frac{1}{2}\|\mathbf{x}-\mathbf{y}\|^2, \forall x, y \in \Omega$.

The Bregman divergence is defined as

$$V_\omega(\mathbf{x},\mathbf{y}) = \omega(\mathbf{x}) - \omega(\mathbf{y}) - \nabla\omega(\mathbf{y})^T(\mathbf{x}-\mathbf{y}), \forall \mathbf{x}, \mathbf{y} \in \Omega.$$



Examples

- Euclidean distance: $\Omega = \mathbb{R}^d$, $\omega(\mathbf{x}) = \frac{1}{2}\|\mathbf{x}\|_2^2$, $\|\cdot\| = \|\cdot\|_2$

$$V_\omega(\mathbf{x}, \mathbf{y}) = \frac{1}{2}\|\mathbf{x} - \mathbf{y}\|_2^2.$$

- Mahalanobis distance: $\Omega = \mathbb{R}^d$, $\omega(\mathbf{x}) = \frac{1}{2}\mathbf{x}^T Q \mathbf{x}$ (where $Q \succeq I$), $\|\cdot\| = \|\cdot\|_2$,

$$V_\omega(\mathbf{x}, \mathbf{y}) = \frac{1}{2}(\mathbf{x} - \mathbf{y})^T Q(\mathbf{x} - \mathbf{y}).$$

- Kullback-Leibler divergence: $\Omega = \Delta_d$, $\omega(\mathbf{x}) = \sum_{i=1}^d x_i \log x_i$, $\|\cdot\| = \|\cdot\|_1$,

$$V_\omega(\mathbf{x}, \mathbf{y}) = \mathrm{KL}(\mathbf{x}|\mathbf{y}) := \sum_{i=1}^d x_i \log \frac{x_i}{y_i}.$$

Clicker Question (EduApp)

Recall the definition of Bregman divergence:

$$V_\omega(\mathbf{x}, \mathbf{y}) = \omega(\mathbf{x}) - \omega(\mathbf{y}) - \nabla\omega(\mathbf{y})^T(\mathbf{x} - \mathbf{y}), \forall \mathbf{x}, \mathbf{y} \in \Omega.$$

Which one of the following statements does not always hold?

A. Nonnegativity: $V_\omega(\mathbf{x}, \mathbf{y}) \geq 0$.

B. Symmetry: $V_\omega(\mathbf{x}, \mathbf{y}) = V_\omega(\mathbf{y}, \mathbf{x})$.

C. Convexity: $V_\omega(\mathbf{x}, \mathbf{y})$ is convex in $\mathbf{x}$.

D. $V_\omega(\mathbf{x}, \mathbf{y}) \geq \frac{1}{2}\|\mathbf{x} - \mathbf{y}\|^2$.

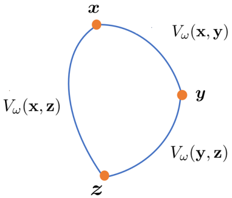
Key Property of Bregman Divergence

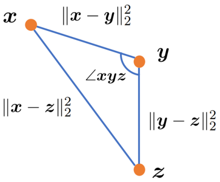
Lemma 10.1 (Three Point Identity)

$$V_\omega(\mathbf{x},\mathbf{z}) = V_\omega(\mathbf{x},\mathbf{y}) + V_\omega(\mathbf{y},\mathbf{z}) - \langle \nabla\omega(\mathbf{z}) - \nabla\omega(\mathbf{y}), \mathbf{x}-\mathbf{y}\rangle, \forall \mathbf{x},\mathbf{y},\mathbf{z} \in \Omega$$

- Special case: $\omega(\mathbf{x}) = \frac{1}{2}\|\mathbf{x}\|_2^2$, this is the law of cosine:

$$\|\mathbf{x}-\mathbf{z}\|_2^2 = \|\mathbf{x}-\mathbf{y}\|_2^2 + \|\mathbf{y}-\mathbf{z}\|_2^2 - 2\langle \mathbf{z}-\mathbf{y}, \mathbf{x}-\mathbf{y}\rangle.$$





- Proof follows by the definition of Bregman divergence (see supplementary).

Mirror Descent

Mirror Descent Algorithm: (Nemirovski & Yudin, 1983)

$$\mathbf{x}_{t+1} = \operatorname*{argmin}_{\mathbf{x}\in X}\{V_\omega(\mathbf{x},\mathbf{x}_t) + \langle \gamma_t \mathbf{g}_t, \mathbf{x}\rangle\}, \text{ where } \mathbf{g}_t \in \partial f(\mathbf{x}_t).$$

Example:

- Subgradient descent: $\omega(\mathbf{x}) = \frac{1}{2}\|\mathbf{x}\|_2^2$, $V_\omega(\mathbf{x},\mathbf{x}_t) = \frac{1}{2}\|\mathbf{x}-\mathbf{x}_t\|_2^2$.

$$\mathbf{x}_{t+1} = \Pi_X(\mathbf{x}_t - \gamma_t \mathbf{g}_t).$$

- Entropic descent: $X = \Delta_d$, $\omega(\mathbf{x}) = \sum_{i=1}^d x_i \log x_i$, $V_\omega(\mathbf{x},\mathbf{x}_t) = \mathrm{KL}(\mathbf{x}|\mathbf{x}_t)$.

$$\mathbf{x}_{t+1} \propto \mathbf{x}_t \odot \exp(-\gamma_t \mathbf{g}_t).$$

Here $\odot$ is element-wise multiplication.

Remarks

Mirror Descent is closely related to many classical algorithms in other fields:

- AdaBoost algorithm in machine learning (Freund & Schapire, 1995)
- Winnow algorithm in learning theory (Littlestone, 1988)
- Exponentiated gradient in online learning (Kivinen & Warmuth, 1997)
- Multiplicative update algorithm in game theory in 1950s
- Richardson-Lucy algorithm in imaging processing in 1970s
- Follow-the-regularized-leader (FTRL) in online learning
- Relative Entropy Policy Search in reinforcement learning
- Natural policy gradient (NPG) in reinforcement Learning
- ...

Convergence of Mirror Descent

Let f be convex and $\omega(\cdot)$ be 1-strongly convex on X w.r.t. norm $\|\cdot\|$.

Lemma 10.2

$$\gamma_t(f(\mathbf{x}_t) - f^*) \leq V_\omega(\mathbf{x}^*, \mathbf{x}_t) - V_\omega(\mathbf{x}^*, \mathbf{x}_{t+1}) + \frac{\gamma_t^2}{2}\|\mathbf{g}_t\|_*^2.$$

Theorem 10.3

$$\min_{1\leq t\leq T} f(\mathbf{x}_t) - f^* \leq \frac{V_\omega(\mathbf{x}^*, \mathbf{x}_1) + \frac{1}{2}\sum_{t=1}^T \gamma_t^2\|\mathbf{g}_t\|_*^2}{\sum_{t=1}^T \gamma_t}.$$

- Generalizes the previous results for subgradient descent.

Convergence Rate of Mirror Descent

- Suppose f is B-Lipschitz continuous such that $|f(\mathbf{x}) - f(\mathbf{y})| \leq B\|\mathbf{x} - \mathbf{y}\|$, namely, $\|\mathbf{g}\|_* \leq B$ for any $\mathbf{g} \in \partial f(\mathbf{x})$.

- Define $R^2 := \sup_{\mathbf{x} \in X} V_\omega(\mathbf{x}, \mathbf{x}_1)$, where $R \geq 0$ and set $\gamma_t = \frac{\sqrt{2}R}{B\sqrt{T}}$.

$$\min_{1 \leq t \leq T} f(\mathbf{x}_t) - f^* \leq O\left(\frac{BR}{\sqrt{T}}\right).$$

- Similar results can be obtained when $\gamma_t = \frac{\sqrt{2}R}{B\sqrt{t}}$ or using weighted average.

Convergence of Mirror Descent for Convex Problems

- Generalizes the previous results for subgradient descent.

$$\min_{1\le t\le T} f(\mathbf{x}_t) - f^* = O\left(\frac{BR}{\sqrt{T}}\right),$$

where $R = \sqrt{\max_{\mathbf{x}\in X} V_\omega(\mathbf{x}, \mathbf{x}_1)}$ and $B := \sup_{\mathbf{x}\in X} \frac{|f(\mathbf{x})-f(\mathbf{y})|}{\|\mathbf{x}-\mathbf{y}\|}$.

- Subgradient descent: special case with $\|\cdot\| = \|\cdot\|_2$ and $\omega(\cdot) = \frac{1}{2}\|\cdot\|_2^2$.

Proof of Lemma 10.2

- Since $\mathbf{x}_{t+1} = \operatorname{argmin}_{\mathbf{x}\in X}\{V_\omega(\mathbf{x},\mathbf{x}_t) + \langle \gamma_t \mathbf{g}_t, \mathbf{x}\rangle\}$, by the optimality condition,

$$\langle \nabla\omega(\mathbf{x}_{t+1}) + \gamma_t \mathbf{g}_t - \nabla\omega(\mathbf{x}_t), \mathbf{x} - \mathbf{x}_{t+1}\rangle \geq 0, \forall \mathbf{x} \in X.$$

- From three point identity, we have for $\forall \mathbf{x} \in X$:

$$\langle \gamma_t \mathbf{g}_t, \mathbf{x}_{t+1} - \mathbf{x}\rangle \leq \langle \nabla\omega(\mathbf{x}_{t+1}) - \nabla\omega(\mathbf{x}_t), \mathbf{x} - \mathbf{x}_{t+1}\rangle = V_\omega(\mathbf{x},\mathbf{x}_t) - V_\omega(\mathbf{x},\mathbf{x}_{t+1}) - \underbrace{V_\omega(\mathbf{x}_{t+1},\mathbf{x}_t)}_{\geq \frac{1}{2}\|\mathbf{x}_t - \mathbf{x}_{t+1}\|^2}$$

- As a result,

$$\begin{aligned}
\langle \gamma_t \mathbf{g}_t, \mathbf{x}_t - \mathbf{x}^*\rangle &\leq \langle \gamma_t \mathbf{g}_t, \mathbf{x}_{t+1} - \mathbf{x}^*\rangle + \langle \gamma_t \mathbf{g}_t, \mathbf{x}_t - \mathbf{x}_{t+1}\rangle \\
&\leq V_\omega(\mathbf{x}^*,\mathbf{x}_t) - V_\omega(\mathbf{x}^*,\mathbf{x}_{t+1}) - \frac{1}{2}\|\mathbf{x}_t - \mathbf{x}_{t+1}\|^2 + \langle \gamma_t \mathbf{g}_t, \mathbf{x}_t - \mathbf{x}_{t+1}\rangle \\
&\leq V_\omega(\mathbf{x}^*,\mathbf{x}_t) - V_\omega(\mathbf{x}^*,\mathbf{x}_{t+1}) + \frac{\gamma_t^2}{2}\|\mathbf{g}_t\|_*^2
\end{aligned}$$

- By convexity of f, we further have the key lemma. $\square$

Subgradient Descent vs. Mirror Descent

$$\min_{1\le t\le T} f(\mathbf{x}_t) - f^* = O\left(\frac{BR}{\sqrt{T}}\right),$$

where $R = \sqrt{\max_{\mathbf{x}\in X} V_\omega(\mathbf{x}, \mathbf{x}_1)}$ and $B := \sup_{\mathbf{x}\in X} \frac{|f(\mathbf{x})-f(\mathbf{y})|}{\|\mathbf{x}-\mathbf{y}\|}$.

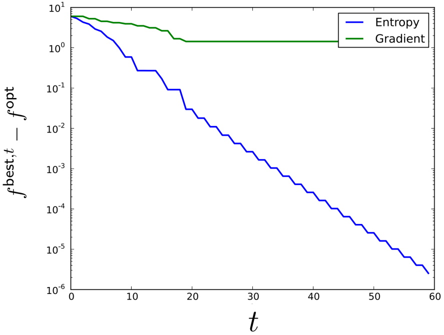
Optimization over simplex:

Assume $\|\mathbf{g}\|_\infty \le 1, \forall \mathbf{g} \in \partial f(\mathbf{x})$ and $X = \Delta_d$. Set $\mathbf{x}_1 = [1/d; \dots; 1/d]$.

- Subgradient Descent: $O\left(\frac{\sqrt{d}}{\sqrt{T}}\right)$,where $B = O(\sqrt{d}), R = O(1)$.
- Mirror Descent: $O\left(\frac{\sqrt{\log d}}{\sqrt{T}}\right)$, where $B = O(1), R = O(\sqrt{\log d})$.

Numerical Illustration: Robust Regression

$$\min_{\mathbf{x}\in\Delta} \; f(\mathbf{x}) = \|Ax - b\|_1 \qquad (A \in \mathbb{R}^{20\times 3000})$$



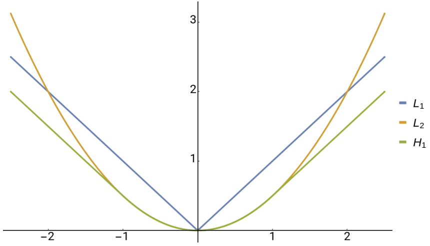
From Boyd's ECE364B lecture

Motivation: absolute value function

Consider the simplest non-smooth and convex function: $f(x) = |x|$.

- Huber function is a smooth approximation of the absolute value function.

$$f_\mu(x) = \begin{cases} \frac{x^2}{2\mu}, |x| \leq \mu \\ |x| - \frac{\mu}{2}, |x| > \mu \end{cases}.$$



- $f_\mu(x) \to f(x)$ as $\mu \to 0$.

$$f(x) - \frac{\mu}{2} \leq f_\mu(x) \leq f(x).$$

- $\nabla f_\mu(x)$ is $\frac{1}{\mu}$-Lipschitz continuous.

Smoothing Idea

Nonsmooth Optimization

$$\begin{array}{ll} \text{minimize} & f(\mathbf{x}) \\ \text{subject to} & \mathbf{x} \in X \end{array}$$

$$\Longrightarrow$$

Smooth Optimization

$$\begin{array}{ll} \text{minimize} & f_\mu(\mathbf{x}) \\ \text{subject to} & \mathbf{x} \in X \end{array}$$

- Solving smooth approximation allows for richer and faster algorithms
- Can deal with nonsmooth nonconvex problems
- Desiderata: approximation accuracy, smoothness, computational efficiency

Smoothing Techniques

- Nesterov smoothing (only for convex objectives)
  [Nesterov 2005]
- Moreau-Yosida smoothing/regularization (only for convex objectives)
  [Bauschke et al., 2011]
- Lasry-Lions regularization
  [Lasry and Lions, 1986, Attouch and Aze, 1993]
- Randomized smoothing
  [Duchi et al., 2012]
- ...

A Quick Tour of Convex Conjugate Theory

Definition 10.4

The conjugate function of f is

$$f^{\star}(\mathbf{y}) = \sup_{\mathbf{x}\in\mathbf{dom}(f)} \left\{\mathbf{y}^T\mathbf{x} - f(\mathbf{x})\right\},$$

also called Legendre-Fenchel transformation.

Fenchel's inequality:

$$f(\mathbf{x}) + f^*(\mathbf{y}) \geq \mathbf{x}^T\mathbf{y}, \forall \mathbf{x}, \mathbf{y}$$

A. Legendre

(1752-1833)

Werner Fenchel

(1905-1988)

A Quick Tour of Convex Conjugate Theory

Lemma 10.5 (Chapter C.6, [Nem01])

1. *(Duality) If f is lower semi-continuous (l.s.c.)*[1] *and convex, then $f^{\star\star} = f$.*
2. *(Fenchel's inequality): $\mathbf{x}^T\mathbf{y} = f(\mathbf{x}) + f^\star(\mathbf{y}) \Leftrightarrow \mathbf{y} \in \partial f(\mathbf{x}) \Leftrightarrow \mathbf{x} \in \partial f^\star(\mathbf{y})$.*
3. *If f and g are l.s.c. and convex, then $(f+g)^\star(\mathbf{x}) = \inf_{\mathbf{y}} \{f^\star(\mathbf{y}) + g^\star(\mathbf{x}-\mathbf{y})\}$.*
4. *If f is μ-strongly convex, then $f^\star$ is differentiable and $\frac{1}{\mu}$-smooth.*

[1] Function f is l.s.c. if $f(\mathbf{x}) \leq \liminf_{t\to\infty} f(\mathbf{x}_t)$ for $\mathbf{x}_t \to \mathbf{x}$.

Examples

1. Quadratic: $f(\mathbf{x}) = \frac{1}{2}\mathbf{x}^T Q \mathbf{x}$ where $Q \succ 0$, $f^\star(\mathbf{y}) = \frac{1}{2}\mathbf{y}^T Q^{-1}\mathbf{y}$.
2. Negative entropy: $f(\mathbf{x}) = \sum_{i=1}^n x_i \log(x_i)$, $f^\star(\mathbf{y}) = \sum_{i=1}^n e^{y_i - 1}$.
3. Negative logarithm: $f(\mathbf{x}) = -\sum_{i=1}^n \log(x_i)$, $f^\star(\mathbf{y}) = -\sum_{i=1}^n \log(-y_i) - n$.
4. Norm: $f(\mathbf{x}) = \|\mathbf{x}\|$, $f^\star(\mathbf{y}) = \begin{cases} 0, & \|\mathbf{y}\|_* \le 1 \\ +\infty, & \|\mathbf{y}\|_* > 1 \end{cases}$.

Smoothing Techniques I: Nesterov's smoothing

$$f_\mu(\mathbf{x}) = \max_{\mathbf{y}\in\mathbf{dom}(f^\star)} \left\{\mathbf{x}^T\mathbf{y} - f^\star(\mathbf{y}) - \mu \cdot d(\mathbf{y})\right\}$$

- Here $f^\star(\mathbf{y})$ is the convex conjugate of f.
- Proximity function: $d(\mathbf{y})$ is 1-strongly convex and nonnegative everywhere.
  - $d(\mathbf{y}) = \frac{1}{2}||\mathbf{y} - \mathbf{y}_0||_2^2$;
  - $d(\mathbf{y}) = \frac{1}{2}\sum w_i(y_i - y_{0,i})^2$ with $w_i \geq 1$;
  - $d(\mathbf{y}) = \omega(\mathbf{y}) - \omega(\mathbf{y}_0) - \nabla\omega(\mathbf{y}_0)^T(\mathbf{y} - \mathbf{y}_0)$ with $\omega(\mathbf{x})$ being 1-strongly convex.

Smoothing Techniques I: Nesterov's smoothing

$$f_\mu(\mathbf{x}) = \max_{\mathbf{y}\in\mathbf{dom}(f^\star)} \left\{\mathbf{x}^T\mathbf{y} - f^\star(\mathbf{y}) - \mu \cdot d(\mathbf{y})\right\}$$

- Smoothness: Function $f_\mu(\mathbf{x})$ is $\frac{1}{\mu}$-smooth.
- Approximation: For convex f with bounded $\mathbf{dom}(f^\star)$, we have

$$f(\mathbf{x}) - \mu D^2 \leq f_\mu(\mathbf{x}) \leq f(\mathbf{x}), \text{ where } D^2 = \max_{\mathbf{y}\in\mathbf{dom}(f^\star)} d(\mathbf{y}).$$

- Tradeoff between approximation error and optimization efficiency:

$$f(\mathbf{x}) - f^* \leq \underbrace{f(\mathbf{x}) - f_\mu(\mathbf{x})}_{\text{approximation error}} + \underbrace{f_\mu(\mathbf{x}) - \min_{\mathbf{x}} f_\mu(\mathbf{x})}_{\text{optimization error}}$$

Smoothing Techniques I: Nesterov's smoothing

- If we apply Accelerated Gradient Descent to solve the smoothed problem:

$$f(\mathbf{x}_t) - f^* \leq O\left(\mu D^2 + \frac{R^2}{\mu t^2}\right).$$

- To achieve accuracy $\epsilon > 0$, need $\mu = O(\frac{\epsilon}{D^2})$.
- The number of AGD iterations is at most $T_\epsilon = O(\frac{R}{\sqrt{\epsilon\mu}}) = O(\frac{RD}{\epsilon})$.
- This is faster than directly applying subgradient methods.

Smoothing Techniques II: Moreau-Yosida Regularization

$$f_\mu(\mathbf{x}) = \min_{\mathbf{y}} \left\{ f(\mathbf{y}) + \frac{1}{2\mu} ||\mathbf{x} - \mathbf{y}||_2^2 \right\}$$

- Here $\mu > 0$ and $f_\mu(\mathbf{x})$ is called the Moreau envelope of $f(\mathbf{x})$.
- Example: Huber function is the Moreau envelope of $f(x) = |x|$:

$$f_\mu(x) = \begin{cases} \frac{x^2}{2\mu}, |x| \leq \mu \\ |x| - \frac{\mu}{2}, |x| > \mu \end{cases} .$$

Smoothing Techniques II: Moreau-Yosida Regularization

$$f_\mu(\mathbf{x}) = \min_{\mathbf{y}} \left\{ f(\mathbf{y}) + \frac{1}{2\mu} ||\mathbf{x} - \mathbf{y}||_2^2 \right\}$$

- Special case of Nesterov's smoothing with $d(\mathbf{y}) = \frac{1}{2}\|\mathbf{y}\|^2$.

$$\begin{aligned} f_\mu(\mathbf{x}) &= \max_{\mathbf{y}} \left\{ \mathbf{x}^T\mathbf{y} - f^\star(\mathbf{y}) - \frac{\mu}{2}\|\mathbf{y}\|_2^2 \right\} \\ &= (f^\star + \frac{\mu}{2}\| \cdot \|_2^2)^\star(\mathbf{x}) \\ &= \inf_{\mathbf{y}} \left\{ f(\mathbf{y}) + \frac{1}{2\mu}\|\mathbf{x} - \mathbf{y}\|_2^2 \right\} \end{aligned}$$

- Smoothness: Function $f_\mu(\mathbf{x})$ is $\frac{1}{\mu}$-smooth.
- Exact Minimization: We have $\min_{\mathbf{x}} f(\mathbf{x}) = \min_{\mathbf{x}} f_\mu(\mathbf{x})$.

Smoothing Techniques II: Moreau-Yosida Regularization

$$
\begin{aligned}
f_\mu(\mathbf{x}) &= \min_{\mathbf{y}} \left\{ f(\mathbf{y}) + \frac{1}{2\mu}||\mathbf{x}-\mathbf{y}||_2^2 \right\} \\
\mathbf{prox}_{\mu f}(\mathbf{x}) &:= \operatorname*{argmin}_{\mathbf{y}} \left\{ f(\mathbf{y}) + \frac{1}{2\mu}||\mathbf{x}-\mathbf{y}||_2^2 \right\}
\end{aligned}
$$

- Gradient of smooth function: (based on Danskin's theorem or Fenchel duality)

$$
\nabla f_\mu(\mathbf{x}) = \frac{1}{\mu}(\mathbf{x} - \mathbf{prox}_{\mu f}(\mathbf{x}))
$$

- GD on smooth $f_\mu(\mathbf{x})$ reduces to proximal minimization on $f(\mathbf{x})$:

$$
\mathbf{x}_{t+1} = \mathbf{x}_t - \mu \nabla f_\mu(\mathbf{x}_t) \Longleftrightarrow \mathbf{x}_{t+1} = \mathbf{prox}_{\mu f}(\mathbf{x}_t).
$$

Proximal Operators

Definition 10.6

The **proximal operator** of convex function g at $\mathbf{x}$ is defined as

$$\mathbf{prox}_f(\mathbf{x}) = \underset{\mathbf{y}}{\operatorname{argmin}} \left\{ f(\mathbf{y}) + \frac{1}{2} \|\mathbf{x} - \mathbf{y}\|_2^2 \right\}$$

- For continuous convex function f, $\mathbf{prox}_f(\mathbf{x})$ exists and is unique.
- For many nonsmooth functions, proximal operators can be computed efficiently (*closed form solution, low-cost computation, polynomial time*).

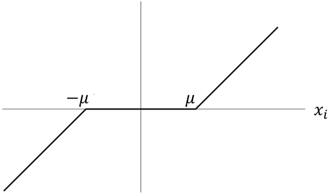
Proximal Operators

Examples:

- If $f(\mathbf{x}) = \delta_X(\mathbf{x}) = \begin{cases} 0, & \mathbf{x} \in X \\ +\infty, & \mathbf{x} \notin X \end{cases}$, then $\mathbf{prox}_f(\mathbf{x}) = \Pi_X(\mathbf{x})$ is the projection.

- If $f(\mathbf{x}) = \mu\|\mathbf{x}\|_1$, then $\mathbf{prox}_f(\mathbf{x})$ is the soft thresholding operator.

$$\mathbf{prox}_{\mu|\cdot|}(x_i) = \begin{cases} x_i - \mu & \text{if } x_i > \mu \\ 0 & \text{if } |x_i| \leq \mu \\ x_i + \mu & \text{if } x_i < -\mu \end{cases}.$$



Equivalently, $\mathbf{prox}_{\mu\|\cdot\|_1}(\mathbf{x}) = \mathrm{sign}(\mathbf{x}) \odot \max\{|\mathbf{x}| - \mu, 0\}$.

A non-exhaustive list of proximal operators

Name	Function	Proximal operator	Complexity
ℓ_1-norm	$f(\mathbf{x}) := \|\mathbf{x}\|_1$	$\mathrm{prox}_{\lambda f}(\mathbf{x}) = \mathrm{sign}(\mathbf{x}) \otimes [\lvert\mathbf{x}\rvert - \lambda]_+$	$\mathcal{O}(p)$
ℓ_2-norm	$f(\mathbf{x}) := \|\mathbf{x}\|_2$	$\mathrm{prox}_{\lambda f}(\mathbf{x}) = [1 - \lambda/\|\mathbf{x}\|_2]_+ \mathbf{x}$	$\mathcal{O}(p)$
Support function	$f(\mathbf{x}) := \max_{\mathbf{y}\in\mathcal{C}} \mathbf{x}^T\mathbf{y}$	$\mathrm{prox}_{\lambda f}(\mathbf{x}) = \mathbf{x} - \lambda \pi_{\mathcal{C}}(\mathbf{x})$	
Box indicator	$f(\mathbf{x}) := \delta_{[\mathbf{a},\mathbf{b}]}(\mathbf{x})$	$\mathrm{prox}_{\lambda f}(\mathbf{x}) = \pi_{[\mathbf{a},\mathbf{b}]}(\mathbf{x})$	$\mathcal{O}(p)$
Positive semidefinite cone indicator	$f(\mathbf{X}) := \delta_{\mathbb{S}^p_+}(\mathbf{X})$	$\mathrm{prox}_{\lambda f}(\mathbf{X}) = \mathbf{U}[\Sigma]_+\mathbf{U}^T$, where $\mathbf{X} = \mathbf{U}\Sigma\mathbf{U}^T$	$\mathcal{O}(p^3)$
Hyperplane indicator	$f(\mathbf{x}) := \delta_{\mathcal{X}}(\mathbf{x})$, $\mathcal{X} := \{\mathbf{x} : \mathbf{a}^T\mathbf{x} = b\}$	$\mathrm{prox}_{\lambda f}(\mathbf{x}) = \pi_{\mathcal{X}}(\mathbf{x}) = \mathbf{x} + \left(\frac{b - \mathbf{a}^T\mathbf{x}}{\|\mathbf{a}\|_2}\right)\mathbf{a}$	$\mathcal{O}(p)$
Simplex indicator	$f(\mathbf{x}) = \delta_{\mathcal{X}}(\mathbf{x})$, $\mathcal{X} := \{\mathbf{x} : \mathbf{x} \geq 0,\ \mathbf{1}^T\mathbf{x} = 1\}$	$\mathrm{prox}_{\lambda f}(\mathbf{x}) = (\mathbf{x} - \nu\mathbf{1})$ for some $\nu \in \mathbb{R}$, which can be efficiently calculated	$\tilde{\mathcal{O}}(p)$
Convex quadratic	$f(\mathbf{x}) := (1/2)\mathbf{x}^T\mathbf{Q}\mathbf{x} + \mathbf{q}^T\mathbf{x}$	$\mathrm{prox}_{\lambda f}(\mathbf{x}) = (\lambda\mathbb{I} + \mathbf{Q})^{-1}\mathbf{x}$	$\mathcal{O}(p\log p) \to \mathcal{O}(p^3)$
Square ℓ_2-norm	$f(\mathbf{x}) := (1/2)\|\mathbf{x}\|_2^2$	$\mathrm{prox}_{\lambda f}(\mathbf{x}) = (1/(1+\lambda))\mathbf{x}$	$\mathcal{O}(p)$
log-function	$f(\mathbf{x}) := -\log(x)$	$\mathrm{prox}_{\lambda f}(x) = ((x^2 + 4\lambda)^{1/2} + x)/2$	$\mathcal{O}(1)$
$\log\det$-function	$f(\mathbf{x}) := -\log\det(\mathbf{X})$	$\mathrm{prox}_{\lambda f}(\mathbf{X})$ is the log-function prox applied to the individual eigenvalues of $\mathbf{X}$	$\mathcal{O}(p^3)$

Source from Volkan Cevher's EE-556 lecture notes. More examples can be found in Parikh & Boyd (2013).

Proximal Point Algorithm

$$\textbf{PPA}: \qquad \mathbf{x}_{t+1} = \mathbf{prox}_{\lambda_t f}(\mathbf{x}_t)$$

Theorem 10.7 (Convergence of PPA)

If f is convex, then for any $T \geq 1$,

$$f(\mathbf{x}_{T+1}) - f^* \leq \frac{\|\mathbf{x}_1 - \mathbf{x}^*\|_2^2}{2\sum_{t=1}^{T} \lambda_t}.$$

- Setting $\lambda_t = \lambda$, this implies a $O(1/t)$ convergence rate.

Convergence Proof of Proximal Point Algorithm

Proof.

- First we can prove the following recursion based on optimality of $\mathbf{x}_{t+1}$ (following similar argument as the analysis of Mirror Descent).

$$\lambda_t[f(\mathbf{x}_{t+1}) - f(\mathbf{x})] \leq \frac{1}{2}\|\mathbf{x} - \mathbf{x}_t\|_2^2 - \frac{1}{2}\|\mathbf{x} - \mathbf{x}_{t+1}\|_2^2 - \frac{1}{2}\|\mathbf{x}_t - \mathbf{x}_{t+1}\|_2^2, \forall \mathbf{x}.$$

- Note that $f(\mathbf{x}_{t+1}) \leq f(\mathbf{x}_t)$.
- Combining these two results leads to the desired result.

$\square$

Smoothing Techniques III: Randomized Smoothing

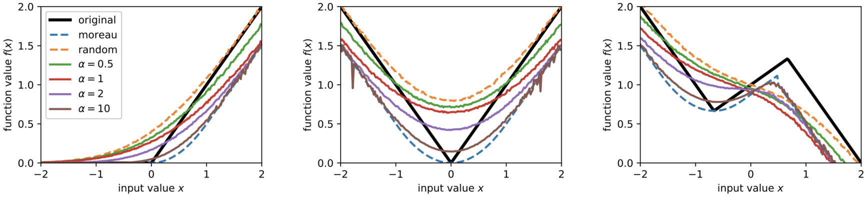
$$f_\mu(\mathbf{x}) = \mathbb{E}_{\mathbf{Z}}[f(\mathbf{x} + \mu\mathbf{Z})]$$

where $\mathbf{Z}$ is an isotopic Gaussian or uniform random variable.

- Choosing $\mu = O(\epsilon)$ guarantees ϵ approximation error [Duc12].
- $f_\mu(\mathbf{x})$ is $O(\frac{\sqrt{d}}{\epsilon})$-smooth (dimension dependent) [Duc12].
- Can compute stochastic gradient very efficiently through sampling.

Other Techniques

BMR: Combination of randomized smoothing and Moreau-Yosida smoothing [Sca20]



(a) $f(x) = \max\{0, x\}$. (b) $f(x) = |x|$. (c) f non-convex.

Figure 1: Effect of the parameter α on BMR smoothing (with $\gamma = \min\{1, \alpha^{-1/2}\}$). When $\alpha \to 0$ (resp. $\alpha \to +\infty$), BMR tends to randomized smoothing (resp. Moreau envelope).

Convex Composite Optimization

$$\min_{\mathbf{x}\in\mathbb{R}^d} \quad f(\mathbf{x}) + g(\mathbf{x})$$

Assume both f and g are convex.

- $f(\mathbf{x})$ is smooth, $g(\mathbf{x}) = 0$
- $f(\mathbf{x})$ is nonsmooth, $g(\mathbf{x}) = \delta_X(\cdot)$ is indicator function
- $f(\mathbf{x})$ is smooth, $g(\mathbf{x})$ is a "simple" nonsmooth regularizer
- $f(\mathbf{x})$ and $g(\mathbf{x})$ are both "simple" nonsmooth functions
- ....

Application I: Supervised Learning

Most supervised learning problems can be cast into the form:

$$\min_{\theta} \frac{1}{n} \sum_{i=1}^{n} \ell(h_\theta(\mathbf{x}_i), y_i) + g(\theta)$$

- $\ell(\cdot, \cdot)$ is the loss function, e.g., square loss, hinge loss, logistic loss, etc.
- $h_\theta(\cdot)$ is the predictor, e.g., linear predictor, neural networks, etc.
- $g(\theta)$ is some regularizer, e.g., ℓ_2-norm, ℓ_1-norm, elastic net, etc.
- $(\mathbf{x}_1, y_1), \ldots, (\mathbf{x}_n, y_n)$ are the input data.

Application II: Image Processing

The goal is to recover a clean image $\mathbf{x} \in \mathbb{R}^{n\times m}$ given observation $\mathbf{b} = \mathcal{A}(\mathbf{x}) + \boldsymbol{\epsilon}$.

$$\min_{\mathbf{x}} \ \|\mathcal{A}(\mathbf{x}) - \mathbf{b}\|_2^2 + \lambda\|\mathbf{x}\|_{TV} \qquad \text{(Gaussian noise)}$$

$$\min_{\mathbf{x}} \ \sum_i [\langle \mathbf{a}_i, \mathbf{x}\rangle - b_i \log(\langle \mathbf{a}_i, \mathbf{x}\rangle)] + \lambda\|\mathbf{x}\|_{TV} \qquad \text{(Poisson noise)}$$

- $\mathcal{A}(\mathbf{x}) = A\mathbf{x}$ is some linear operator that captures image blur or subsampling.
- Here $\|\mathbf{x}\|_{TV} := \sum_{i,j} |\mathbf{x}_{i,j+1} - \mathbf{x}_{i,j}| + |\mathbf{x}_{i+1,j} - \mathbf{x}_{i,j}|$ is the total variation norm.

Proximal Gradient Method

Convex composite optimization: $\min_{\mathbf{x}} \; F(\mathbf{x}) := f(\mathbf{x}) + g(\mathbf{x})$

- f is convex and L-smooth;
- g is convex and proximal-friendly.

Proximal Gradient Method: choose $\mathbf{x}_0 \in \mathbb{R}^d$.

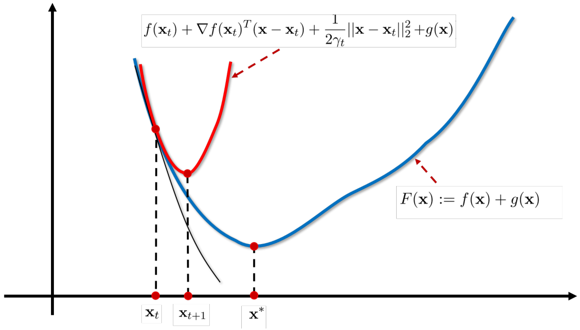
$$\mathbf{x}_{t+1} = \mathbf{prox}_{\gamma_t g}(\mathbf{x}_t - \gamma_t \nabla f(\mathbf{x}_t)).$$

- Alternates between gradient update and proximal operator.
- Update can be computed efficiently.

Interpretation

Proximal gradient update ≈ majorization-minimization

$$\mathbf{x}_{t+1} = \arg\min_{\mathbf{x}} \Big\{ \underbrace{f(\mathbf{x}_t) + \nabla f(\mathbf{x}_t)^T(\mathbf{x}-\mathbf{x}_t) + \frac{1}{2\gamma_t}\|\mathbf{x}-\mathbf{x}_t\|_2^2}_{\geq f(\mathbf{x}) \quad (\text{if } \gamma_t \leq \frac{1}{L})} + g(\mathbf{x}) \Big\}.$$



Convergence of PGM for Convex Problems

Theorem 10.8

Assume $f(\mathbf{x})$ is convex and L-smooth, $g(\mathbf{x})$ is convex and possibly nonsmooth. Proximal gradient method with fixed step size $\gamma_t = \frac{1}{L}$ satisfies:

$$F(\mathbf{x}_t) - F(\mathbf{x}^*) \leq \frac{L||\mathbf{x}_0 - \mathbf{x}^*||_2^2}{2t}.$$

- Behaves as if there is no nonsmooth term $g(\mathbf{x})$.
- Faster than directly applying subgradient method.
- Can be further accelerated with $O(1/t^2)$ rate.

Accelerated Proximal Gradient Method

Accelerated Proximal Gradient: Initialize $\mathbf{x}_0 \in \mathbb{R}^d$ and $\mathbf{y}_0 = \mathbf{x}_0$.

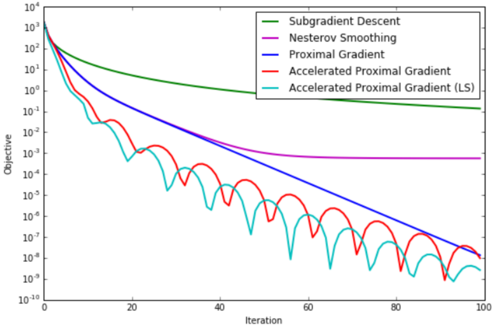
$$\begin{aligned} \mathbf{x}_{t+1} &= \mathbf{prox}_{\gamma_t g}(\mathbf{y}_t - \gamma_t \nabla f(\mathbf{y}_t)) \\ \mathbf{y}_{t+1} &= \mathbf{x}_{t+1} + \frac{t}{t+3}(\mathbf{x}_{t+1} - \mathbf{x}_t) \end{aligned}$$

- There exist several acceleration schemes, e.g., Nesterov (1983, 2004), Beck and Teboulle (2009), Tseng (2008)
- $O\left(\sqrt{\frac{LR^2}{\epsilon}}\right)$ for convex problems

Example: Lasso

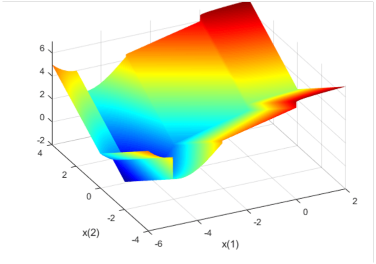
$$\min_{\mathbf{x}} \underbrace{\frac{1}{2}||A\mathbf{x} - b||_2^2}_{f(\mathbf{x})} + \underbrace{\mu||\mathbf{x}||_1}_{g(\mathbf{x})}.$$

Proximal Gradient (a.k.a. ISTA) : $\mathbf{x}_{t+1} = \mathbf{prox}_{\mu\gamma_t\|\cdot\|_1}(\mathbf{x}_t - \gamma_t A^T(A\mathbf{x}_t - b))$.



Summary: Convex Nonsmooth Optimization

- Subgradient Method
- Exploiting non-Euclidean geometry
  - Mirror Descent
- Exploiting nonsmooth structure:
  - Smoothing techniques
  - Proximal point algorithm
  - Proximal gradient methods
  - ....



Additional resources:
Neal Parikh and Stephen Boyd. "Proximal algorithms". Foundations and trends in Optimization 1.3 (2014): 127-239.

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Supplementary Material

Proof of Lemma 10.1

Proof.

This can be easily derived from the definition. We have

$$
\begin{aligned}
V_\omega(\mathbf{x},\mathbf{y}) + V_\omega(\mathbf{y},\mathbf{z}) &= \omega(\mathbf{x}) - \omega(\mathbf{y}) + \omega(\mathbf{y}) - \omega(\mathbf{z}) - \langle \nabla\omega(\mathbf{y}), \mathbf{x}-\mathbf{y}\rangle - \langle \nabla\omega(\mathbf{z}), \mathbf{y}-\mathbf{z}\rangle \\
&= V_\omega(\mathbf{x},\mathbf{z}) + \langle \nabla\omega(\mathbf{z}), \mathbf{x}-\mathbf{z}\rangle - \langle \nabla\omega(\mathbf{y}), \mathbf{x}-\mathbf{y}\rangle - \langle \nabla\omega(\mathbf{z}), \mathbf{y}-\mathbf{z}\rangle \\
&= V_\omega(\mathbf{x},\mathbf{z}) + \langle \nabla\omega(\mathbf{z}) - \nabla\omega(\mathbf{y}), \mathbf{x}-\mathbf{y}\rangle.
\end{aligned}
$$

□

Basic Properties of Proximal Operators

Lemma 10.9

Let g be a convex function, we have

(a) *(Subgradient characterization)*

$$\mathbf{y} = \mathbf{prox}_g(\mathbf{x}) \Longleftrightarrow \mathbf{x} - \mathbf{y} \in \partial g(\mathbf{y}).$$

(b) *(Fixed Point) A point $\mathbf{x}^*$ minimizes $g(\mathbf{x}) \Longleftrightarrow \mathbf{x}^* = \mathbf{prox}_g(\mathbf{x}^*)$.*

(c) *(Non-expansiveness) $\left\|\mathbf{prox}_g(\mathbf{x}) - \mathbf{prox}_g(\mathbf{y})\right\|_2 \leq \|\mathbf{x} - \mathbf{y}\|_2$.*

Proof follows definition and monotonicity of subgradient.

Interpretation II of Proximal Gradient Methods

Proximal gradient update $\approx$ fixed point iteration

Lemma 10.10

$\mathbf{x}^*$ *is optimal if and only if* $\forall \gamma > 0$: $\mathbf{x}^* = \mathbf{prox}_{\gamma g}(\mathbf{x}^* - \gamma \nabla f(\mathbf{x}^*))$.

Proof.

$$\begin{aligned} & \mathbf{x}^* = \mathbf{prox}_{\gamma g}(\mathbf{x}^* - \gamma \nabla f(\mathbf{x}^*)) \\ \Leftrightarrow \quad & 0 \in \frac{1}{\gamma}(\mathbf{x}^* - (\mathbf{x}^* - \gamma \nabla f(\mathbf{x}^*))) + \partial g(\mathbf{x}^*) \\ \Leftrightarrow \quad & 0 \in \nabla f(\mathbf{x}^*) + \partial g(\mathbf{x}^*). \end{aligned}$$

$\square$

Interpretation III of Proximal Gradient Methods

Proximal gradient update $\approx$ forward-backward operator

$$\mathbf{x}_{t+1} = (I + \gamma_t \partial g)^{-1}(I - \gamma_t \nabla f)(\mathbf{x}_t)$$

- $(I - \gamma_t \nabla f)$ is the 'forward' operator;
- $(I + \gamma_t \partial g)^{-1}$, called the resolvent of operator ∂g, is the 'backward' operator.

$$\mathbf{y} = \mathbf{prox}_g(\mathbf{x}) \iff \mathbf{x} \in (I + \partial g)(\mathbf{y}) \iff \mathbf{y} = (I + \partial g)^{-1}(\mathbf{x}).$$

- Also called forward-backward algorithm.

Interpretation IV of Proximal Gradient Methods

Proximal gradient update $\approx$ generalized gradient update

$$\mathbf{x}_{t+1} = \mathbf{x}_t - \gamma_t G_{\gamma_t}(\mathbf{x}_t)$$

where

$$G_\gamma(\mathbf{x}) := \frac{1}{\gamma}(\mathbf{x} - \mathbf{prox}_{\gamma g}(\mathbf{x} - \gamma \nabla f(\mathbf{x})))$$

- $G_\gamma(\mathbf{x})$ is called the generalized gradient.
- $G_\gamma(\mathbf{x}) = 0$ if and only if $\mathbf{x}$ is optimal.
- $G_\gamma(\mathbf{x}) \in \nabla f(\mathbf{x}) + \partial g(\mathbf{x} - \gamma G_\gamma(\mathbf{x}))$.
  (Easy to check based on the subgradient characterization of proximal operators)

Proof of Theorem 10.8

Lemma 10.11

$$F(\mathbf{x}-\gamma_t G_\gamma(\mathbf{x})) \le F(\mathbf{y})+G_\gamma(\mathbf{x})^T(\mathbf{x}-\mathbf{y})-\frac{\gamma}{2}||G_\gamma(\mathbf{x})||_2^2, \text{ for } \gamma \le 1/L.$$

Applying the inequality at $\mathbf{x} = \mathbf{x}_t$ an $\mathbf{y} = \mathbf{x}^*$, we have:

$$\begin{aligned}
F(\mathbf{x}_{t+1}) - F(\mathbf{x}^*) &\le G_\gamma(\mathbf{x}_t)^T(\mathbf{x}_t - \mathbf{x}^*) - \frac{\gamma_t}{2}||G_{\gamma_t}(\mathbf{x}_t)||_2^2 \\
&= \frac{1}{2\gamma_t}\left[||\mathbf{x}_t - \mathbf{x}^*||_2^2 - ||\mathbf{x}_t - \mathbf{x}^* - \gamma_t G_{\gamma_t}(\mathbf{x}_t)||_2^2\right] \\
&= \frac{1}{2\gamma_t}\left[||\mathbf{x}_t - \mathbf{x}^*||_2^2 - ||\mathbf{x}_{t+1} - \mathbf{x}^*||_2^2\right].
\end{aligned}$$

- $F(\mathbf{x}_t)$ is non-increasing (applying $\mathbf{y} = \mathbf{x}_t$).
- $\|\mathbf{x}_t - \mathbf{x}^*\|_2$ is non-increasing ($F(\mathbf{x}_t) \ge F(\mathbf{x}^*)$).
- Taking sums of both sides over all t and setting $\gamma_t = \frac{1}{L}$ leads to desired result.

Proof of Lemma 10.11

- By smoothness of f, we have

$$f(\mathbf{x} - \gamma G_\gamma(\mathbf{x})) \leq f(\mathbf{x}) - \gamma \nabla f(\mathbf{x})^T G_\gamma(\mathbf{x}) + \frac{L\gamma^2}{2} ||G_\gamma(\mathbf{x})||_2^2.$$

- By convexity of f, we have

$$f(\mathbf{x}) \leq f(\mathbf{y}) + \nabla f(\mathbf{x})^T (\mathbf{x} - \mathbf{y}).$$

- By convexity of g and the fact that $G_\gamma(\mathbf{x}) - \nabla f(\mathbf{x}) \in \partial g(\mathbf{x} - \gamma G_\gamma(\mathbf{x}))$ we have

$$g(\mathbf{x} - \gamma G_\gamma(\mathbf{x})) \leq g(\mathbf{y}) + (G_\gamma(\mathbf{x}) - \nabla f(\mathbf{x}))^T (\mathbf{x} - \mathbf{y} - \gamma G_\gamma(\mathbf{x})).$$

Combing the above three inequalities lead to the desired result in $(\star)$. $\square$

Proximal Gradient with Backtracking Line-search

In practice, we often do not know L a priori. How to choose stepsize?

We can use backtracking line-search to find the local Lipschitz constant.

- Initialize $L_0 = 1$ and some $\alpha > 1$.
- At each iteration t, we find the smallest integer i such that $L = \alpha^i L_{t-1}$ satisfies the Lipschitz condition:

$$f(\mathbf{x}^+) \leq f(\mathbf{x}_t) + \nabla f(\mathbf{x}_t)(\mathbf{x}^+ - \mathbf{x}_t) + \frac{L}{2}||\mathbf{x}^+ - \mathbf{x}_t||_2^2$$

  where $\mathbf{x}^+ = \mathbf{prox}_{\frac{g}{L}}(\mathbf{x}_t - \frac{1}{L}\nabla f(\mathbf{x}_t))$.
- Then update $L_t = L$ and $\mathbf{x}_{t+1} = \mathbf{x}^+$.