凸优化及其在信号处理中的应用-01

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- https://www.se.cuhk.edu.hk/~manchoso/2122/x2te2109
- 刻画优化问题的结构，分类->针对性求解
- **P**: $V^* = \inf_{x \in X} f(x)$，**inf**表示极限的最大值，当确定存在最优值时，可以用min代替
  - 目标函数（objective function）
  - 可行域（fesaible region）
  - 优化值（optimal value of **P**）
  - $$(P) \quad V^* = \underset{x \in X}{\overset{\text{infimum}\ \text{"minimize"}}{\inf}} f(x)$$
    - $f: \mathbb{R}^n \to \mathbb{R}$ objective function
    - $X \subseteq \mathbb{R}^n$ feasible region ; $x \in \mathbb{R}^n$ decision variable
    - V^* : optimal value of (P)
- Optimal value与optimal solution不同，前者是值后者是集
  - (P)
    - infeasible : $X = \emptyset$; $V^* = +\infty$ by convention
    - feasible $X \neq \emptyset$
      - $V^* = -\infty$ e.g. $n=1$, $\inf\ x$ s.t. $x \le 0$
      - $V^* > -\infty$

    Def: optimal solution $x^* \in X$ s.t. $V^* = f(x^*)$
  - 求解优化问题，最优解可能不存在

    Def: optimal solution $x^* \in X$ s.t. $V^* = f(x^*)$
    - Optimal solution exists

      e.g. $n=1$

      $\inf\ x$ s.t. $x \ge 0$

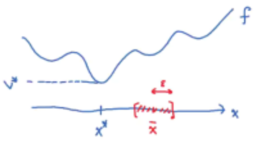
      $V^* = 0$, attained by $x^* = 0$

      optimal Solution does not exist

      e.g. $n=1$

      $\inf\ 1/x$ s.t. $x \ge 0$

      $V^* = 0$; $\exists\, x^* \ge 0$ s.t. $1/x^* = 0$?

      No.
- 局部最优解定义，local minimizer
  - 

    Def. local minimizer

    $\bar{x}$ is local minimizer if $\bar{x} \in X$ and

    $\exists\, \varepsilon > 0$ s.t. $\forall x \in X \cap B(\bar{x}, \varepsilon)$, we have

    $f(\bar{x}) \le f(x)$

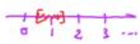
    $(B(\bar{x}, \varepsilon) = \{ y \in \mathbb{R}^n : \| y - \bar{x} \|_2 \le \varepsilon \})$

    center radius
- 举例，Simple Examples of **P**
  - (1) Unconstrained : $X = \mathbb{R}^n$ $\quad \inf_{x \in \mathbb{R}^n} f(x)$

    If f is differentiable, then $\nabla f(x) = 0$ is necessary for optimality.

    Gradient
    $$\nabla f(x) = \begin{bmatrix} \frac{\partial f}{\partial x_1} \\ \vdots \\ \frac{\partial f}{\partial x_n} \end{bmatrix}$$
    - 如果不连续/不可导，怎么处理？另一种梯度的形式
  - (2) Discrete : X is a discrete set; i.e.,

    $\forall x \in X$, $\exists\, \varepsilon > 0$ s.t. $X \cap B(x, \varepsilon) = \{x\}$

    e.g. $X = \mathbb{Z}_+$, $X = \{0,1\}^n$

    $X = [0,1]$ not discrete

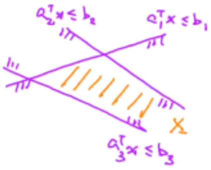
    Note: For discrete optimization problems, local minimality is meaningless!
  - (3) Linear Programming (LP)

    $f(x) = c_1 x_1 + c_2 x_2 + \cdots + c_n x_n = c^T x$ (linear function)

    $c = (c_1, \cdots, c_n)$; $x = (x_1, \cdots, x_n)$

X : defined by a finite number of linear inequalities

$$X = \{ x \in \mathbb{R}^n : \underbrace{a_i^T x \le b_i}_{\text{linear function}} ; \ i = 1, \dots, \underbrace{m}_{\text{finite}} \} \qquad a_i \in \mathbb{R}^n ; \ b_i \in \mathbb{R}$$



$$= \{ x \in \mathbb{R}^n : Ax \le b \} \qquad A = \begin{bmatrix} - a_1^T - \\ \vdots \\ - a_m^T - \end{bmatrix} \in \mathbb{R}^{m \times n}$$

$$b = (b_1, \dots, b_m) \in \mathbb{R}^m$$

Note: $u \le v$

$\Leftrightarrow u_i \le v_i \ \forall i$

- What if we want $a_i^T x = b_i$? Simply consider

$$a_i^T x = b_i \iff \begin{cases} a_i^T x \le b_i \\ -a_i^T x \le -b_i \end{cases}$$

  - 有限且线性的交集：无穷个多边形交集是圆，非线性

(4) Quadratic Programming (QP)

- $$f(x) = \sum_{i,j=1}^{n} Q_{ij} x_i x_j = x^T Q x \qquad Q = [Q_{ij}] \in \mathbb{R}^{n \times n}$$

homogeneous

$f(\alpha x) = \alpha^2 f(x)$

$$x^T Q x = x^T \left(\frac{Q + Q^T}{2} \right) x$$

$$= x^T \frac{Q}{2} x + x^T \frac{Q^T}{2} x$$

$$= \frac{1}{2} x^T Q x + \frac{1}{2} (x^T Q^T x)^T$$

$$= x^T Q x$$

X : Same as LP

Note: In the above definition, Q need not be symmetric.

However, observe that

- $$x^T Q x = x^T \underbrace{\left(\frac{Q + Q^T}{2} \right)}_{\text{Symmetric}} x$$

Hence, we can assume without loss that Q is symmetric.

  - 可行域可以由二次不等式构成