Direct Runge-Kutta Discretization Achieves Acceleration

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1 Problem setup and backgrounds

The article [1] studies gradient-based optimization methods obtained by directly discretizinga second-order ordinary differential equation (ODE) related to the continuous limit of Nesterov's accelerated gradient method. It introduces some conditions under which the sequence of iterates generated by discretizing the proposed second-order ODE converges to the optimal solution at a certain rate.

Firstly let's focus the essential target problem:

$$\min_{x\in\mathbb{R}^d} f(x), \tag{1}$$

where f is convex and sufficiently smooth. For solving (1), there are several methods.

- Classical method: gradient decent, which displays a sub-optimal convergence rate of $\mathcal{O}(N^{-1})$.
- Nesterov's seminal accelerated gradient method, matches the oracle lower bound of $\mathcal{O}(N^{-2})$.

Several articles have pursued approaches to NAG (and accelerated methods in general) via a continuous-time perspective. However, they fail to provide a general discretization procedure that generates provably convergent accelerated methods. This article takes Runge-Kutta integrater as tool and introduces a second-order ODE that generates an accelerated first-order method for smooth functions if using Runge-Kutta method.

2 Main results

To build up a iterative method, letting x_0 be the initial point, firstly we consider the sublevel set

$$\mathcal{S} := \{x \in \mathbb{R}^d | f(x) \leq \exp(1)(f(x_0) - f(x^*) + \|x_0 - x^*\|^2) + 1\}, \tag{2}$$

where x^* is the minimum of (1). The introduction of set $\mathcal{S}$ actually gives a restriction of the sequence of iterates obtained from discretizing a suitable ODE (would be proved later). Denote subset

$$\mathcal{A} := \{x \in \mathbb{R}^d | \exists x' \in \mathcal{S}, \|x - x'\| \leq 1\}, \tag{3}$$

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Then all assumptions we may require can be considered just in $\mathcal{A}$.

Assumption 1. *There exists an integer $p \geq 2$ and a positive constant L such that for any point $x \in \mathcal{A}$, and for all indices $i \in \{1, \ldots, p-1\}$, we have the lower-bound*

$$f(x) - f(x^*) \geq \frac{1}{L} \|\nabla^{(i)} f(x)\|^{\frac{p}{p-i}}, \tag{4}$$

where x^* minimizes f and $\|\nabla^{(i)} f(x)\|$ denotes the operator norm of the tensor $\nabla^{(i)} f(x)$.

Assumption 2. *There exists an integer $s \geq p$ and a constant $M \geq 0$, such that $f(x)$ is order $(s+2)$ differentiable. Furthermore, for any $x \in \mathcal{A}$, the following operator norm bounds hold:*

$$\|\nabla^{(i)} f(x)\| \leq M, \text{ for } i = p, p+1, \ldots, s, s+1, s+2. \tag{5}$$

When the sublevel sets of f are compact and hence the set $\mathcal{A}$ is also compact; as a result, the bound (5) on high order derivatives is implied by continuity.

2.1 Runge-Kutta integrators

The explicit Runge-Kutta integrators used in the article appears in the form below.

Definition 1. *Given a dynamical system $\dot{y} = F(y)$, let the current point be y_0 and the step size be h. An explicit S stage Runge-Kutta method generates the next step via the following update:*

$$g_i = y_0 + h \sum_{j=1}^{i-1} a_{ij} F(g_j), \quad \Phi_h(y_0) = y_0 + h \sum_{i=1}^{S} b_i F(g_i), \tag{6}$$

where a_{ij} and b_i are suitable coefficients defined by the integrator; $\Phi_h(y_0)$ is the estimation of the state after time step h, while g_i (for $i = 1, \ldots, S$) are a few neighboring points where the gradient information $F(g_i)$ is evaluated.

In general, Runge-Kutta methods offer a powerful class of numerical integrators, and with the knowledge of its concergence behaviour when discretizing for solutions, the article gets to use it to discretize ODE with controlment of its convergence rates.

2.2 Formal work and inspiration

Then focus on the NAG method that is defined according to the updates

$$x_k = y_{k-1} - h\nabla f(y_{k-1}), \quad y_k = x_k + \frac{k-1}{k+2}(x_k - x_{k-1}). \tag{7}$$

Su, Boyd, and Candès [2] showed that the iteration (7) in the limit is equivalent to the following ODE

$$\ddot{x}(t) + \frac{3}{t}\dot{x}(t) + \nabla f(x(t)) = 0, \quad \text{where } \dot{x} = \frac{dx}{dt} \tag{8}$$

when one drives the step size h to zero. It can be further shown that in the continuous domain the function value $f(x(t))$ decreases at the rate of $\mathcal{O}(1/t2)$ along the trajectories of the ODE. This convergence rate can be accelerated to an arbitrary rate in continuous time via time dilation as in [Wibisono et al., 2016]. In particular, the solution to

$$\ddot{x}(t) + \frac{p+1}{t}\dot{x}(t) + p^2 t^{p-2} \nabla f(x(t)) = 0 \tag{9}$$

has a convergence rate $\mathcal{O}(1/t^p)$. When $p > 2$, Wibisono et al. [2016] proposed rate matching algorithms via utilizing higher order derivatives. However, this article focuses purely on first-order methods and study the stability of discretizing the ODE directly when $p \geq 2$.

According to some related work, deriving the ODE from the algorithm is now a solved problem. Nevertheless, to derive the update of NAG or any other accelerated method by directly discretizing an ODE is not. Some work points out that explicit Euler discretization of the ODE in (8) may not lead to a stable algorithm. Betancourt, Jordan, and Wilson [3] observed empirically that Verlet integration is stable and suggested that the stability relates to the symplectic property of the Verlet integration, but for dissipative systems such as (9), this doesn't hold. This article offers a different approach: it augments the state with time in (9), and focuses the following ODE

$$\ddot{x}(t) + \frac{2p+1}{t}\dot{x}(t) + p^2 t^{p-2} \nabla f(x(t)) = 0. \tag{10}$$

This actually turns the non-autonomous dynamical system into an autonomous one.

2.3 Main conclusion

The ODE in (10) can also be written as the dynamical system

$$\dot{y} = F(y) = \begin{bmatrix} -\frac{2p+1}{t}v - p^2 t^{p-2}\nabla f(x) \\ v \\ 1 \end{bmatrix}, \quad \text{where } y = [v; x; t]. \tag{11}$$

Algorithm 1: Input(f, x_0, p, L, M, s, N) ▷ Constants p, L, M are the same as in Assumptions
1. Set the initial state $y_0 = [\overrightarrow{0}; x_0; 1] \in \mathbb{R}^{2d+1}$
2. Set step size $h = C/N^{(1/s+1)}$ ▷ C is determined by p, L, M, s, x_0
3. $x_N \leftarrow$ Oreder-s-Runge-Kutta-Integrater(F, y_0, N, h) ▷ F is defined in equation (11)
4. **return** x_N

Since the article has augmented the state with time to obtain an autonomous system, it can be readily solved numerically with a Runge-Kutta integrator as in **Algorithm 1**.

Theorem 1. *Consider the second-order ODE in* (10)*. Suppose that the function f is convex and Assumptions 1 and 2 are satisfied. Further, let s be the order of the Runge-Kutta integrator used in **Algorithm 1**, N be the total number of iterations, and x_0 be the initial point. Also, let $\mathcal{E}_0 := f(x_0) - f(x^*) + \|x_0 - x^*\|^2 + 1$. Then, there exists a constant C_1 such that if we set the step size as $h = C_1 N^{-1/(s+1)}(L + M + 1)^{-1}\mathcal{E}_0^{-1}$, the iterate x_N generated after running **Algorithm 1** for N iterations satisfies the inequality*

$$f(x_N) - f(x^*) \leq C_2 \mathcal{E}_0 \left[\frac{(L + M + 1)\mathcal{E}_0}{N^{\frac{s}{s+1}}}\right]^p = \mathcal{O}(N^{-p\frac{s}{s+1}}), \tag{12}$$

where the constants C1 and C2 only depend on s, p, and the Runge-Kutta integrator. Since each iteration consumes S gradient, $f(x_N) - f(x^*)$ will converge as $\mathcal{O}(S^{\frac{ps}{s+1}} N^{-\frac{ps}{s+1}})$ with respect to the number of gradient evaluations. Note that for commonly used Runge-Kutta integrators, $S \leq 8$.

3 Proof

In this section we're going to explore the proof of Theorem 1, the details of the proof is included in Appendix A.

3.1 Summary

- The proof begins by defining a Lyapunov function $\mathcal{E}$ that quantifies progress and shows that it is monotonically non-increasing along the continuous trajectory of the ODE. This means that the value of the Lyapunov function is always decreasing or remaining the same along the trajectory, which implies that the suboptimality of the solution $f(x) - f(x^*)$ and the norm of the gradient v are bounded above by some constants.
- The proof then focuses on bounding the distance between the points in the discretized and continuous trajectories. This is done by showing that the difference between the two points is bounded by the step size of the numerical integrator. Specifically, it is shown that there exists a constant C_1 such that the distance between the points is bounded by $C_1 h^{s+1}$, where h is the step size and s is the order of the Runge-Kutta integrator.
- Using the bound on the distance between the points in the discretized and continuous trajectories, the proof then shows that the Lyapunov function is also monotonically non-increasing along the discretized trajectory. This means that the value of the Lyapunov function is always decreasing or remaining the same along the discretized trajectory, which implies that the suboptimality of the solution is also decreasing or remaining the same.
- Finally, the proof uses the continuity of the Lyapunov function and the bound on the distance between the points in the discretized and continuous trajectories to show that the suboptimality of the discretized sequence of points also converges to zero quickly. Specifically, it is shown that there exists a constant C_2 such that the suboptimality of the discretized sequence of points is bounded by $C_2\mathcal{E}_0[(L + M + 1)\mathcal{E}_0/N^{s/(s+1)}]^p$, where $\mathcal{E}_0$ is the initial value of the Lyapunov function, N is the total number of iterations, and p is

a positive constant that depends on the order of differentiability of the objective function. This result implies that the suboptimality of the discretized sequence of points converges to zero at a rate that is close to $O(N^{-p})$ with respect to the number of gradient evaluations.

3.2 Comments

- The proof relies on the assumption that the objective function f is convex and satisfies certain conditions on its derivatives, which are stated as Assumptions 1 and 2 in the theorem. These assumptions are necessary for the Lyapunov function to be monotonically non-increasing along the continuous trajectory of the ODE and for the suboptimality of the solution to converge to zero.
- The bound on the distance between the points in the discretized and continuous trajectories depends on the order of the Runge-Kutta integrator used. Higher-order integrators can provide a tighter bound, which leads to a faster convergence rate for the suboptimality of the discretized sequence of points.
- The theorem states that the Direct Runge-Kutta Discretization method can achieve a convergence rate that is close to $O(N^{-p})$ with respect to the number of gradient evaluations, where N is the total number of iterations and p is a positive constant that depends on the order of differentiability of the objective function. This convergence rate is faster than the $O(N^{-1})$ rate which is typically achieved by first-order methods such as gradient descent.
- One possible remark is that the constants C_1 and C_2 in the proof depend on the order of the Runge-Kutta integrator and the positive constant p that depends on the order of differentiability of the objective function. These constants can affect the convergence rate of the algorithm and the step size h that needs to be chosen. In particular, choosing a higher order integrator or a higher order of differentiability may lead to better convergence rates, but also requires a smaller step size to achieve these rates. On the other hand, choosing a smaller step size may increase the computational cost of the algorithm. It is important to carefully balance these trade-offs in practice to achieve good performance.
- Another remark is that the proof assumes the existence of a solution x^* that minimizes the objective function $f(x)$. This assumption is necessary for the Lyapunov function $\mathcal{E}$ to be well-defined and for the convergence result to hold. In practice, it may be challenging to verify the existence of such a solution, especially when the objective function is nonconvex. In these cases, it is important to carefully choose the initial point x_0 and the step size h to ensure that the algorithm converges to a good approximate solution.
- Finally, it is worth noting that the proof of Theorem 1 relies on several technical assumptions, such as convexity of the objective function, Lipschitz continuity of the gradient, and boundedness of high order derivatives. These assumptions are typically required to establish convergence results for optimization algorithms, but may not always hold in practice. It is important to carefully verify these assumptions before applying the algorithm and to choose appropriate algorithms or methods if these assumptions are not satisfied.

3.3 Further improvements

- One additional point that may be helpful in understanding the proof is that the Lyapunov function is used to quantify the progress of the algorithm in terms of the suboptimality of the solution and the norm of the gradient. The monotonicity of the Lyapunov function along the continuous and discretized trajectories implies that these quantities are non-increasing, which in turn implies that the algorithm is making progress towards the optimal solution.
- Another point to consider is that the proof relies on the existence of a constant C_1 such that the distance between the points in the discretized and continuous trajectories is bounded by $C_1 h^{s+1}$. This constant is not explicitly calculated in the proof, but it is stated that replacing C_1 with any smaller positive constant leads to the same polynomial rate of convergence. This means that the specific value of C_1 is not important for achieving the desired convergence rate, as long as it is small enough.

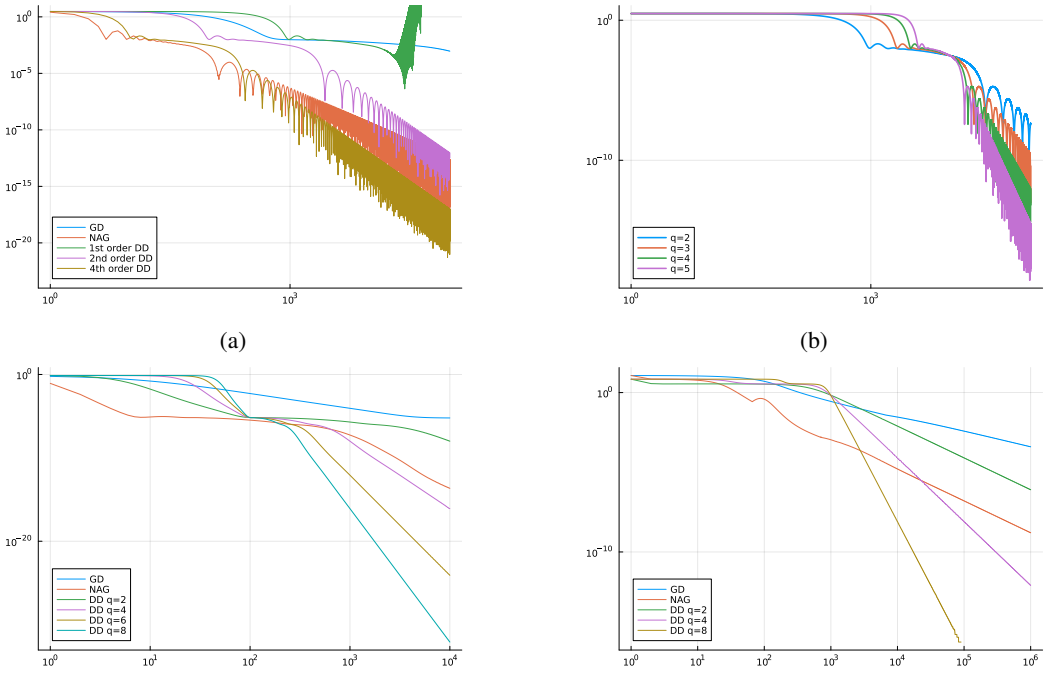
4 Numerical Experiments

In this section, we implement the algorithms in the original article with `Julia` and its package `DifferentialEquations.jl` [4]. By comparing ODE direct discretizating (DD) methods described in the article against gradient descent (GD) and Nesterov's accelerated gradient (NAG) methods, we con verify the main results in the theoretical part. The code of these experiments can be found here: `https://github.com/xiaoxuan-yu/Direct-Runge-Kutta-Discretization-Achieves-Acceleration-PKU`.

Inspired by the numerical results by Wilson, Mackey, and Wibisono [5], we generate normal distributed separable dataset and fit a linear model $Ax = b$. Then, we minimize three different kinds of loss functions:

$$\begin{aligned} f_1(x) &= \|Ax - b\|_2^2 \\ f_2(x) &= \sum_i \log(1 + e^{-w_i^{\mathrm{T}} x y_i}) \\ f_3(x) &= \frac{1}{4}\|Ax - b\|_4^4 \end{aligned} \tag{13}$$

where $f_1(\cdot), f_2(\cdot), f_3(\cdot)$ are L_2 loss, logistic loss and L_4 loss, respectively. For each test case and optimization algorithm, we empirically select the learning rate as the largest step length among $\{10^{-k} | k \in \mathbb{Z}\}$ that the method remains stable during the optimization process. Main results are shown in Figure 1 where all figures are on log-log scale.



(a) (b) (c) (d)

Figure 1: Experimental results comparing DD with GD and NAG. (a) Convergence path of GD, NAG and DD with different Runge-Kutta integrators of degree $s = 1, 2, 4$ on L_2 loss. (b) The optimization of L_2 loss by DD with different choices of q values with 4-th order Runge-Kutta integrator RK4. (c) Minimization of L_4 loss by GD, NAG and DD with different q vaues with a 2-nd order Runge-Kutta integrator. (d) Minimization of logistic loss by GD, NAG and DD with different q vaues with a 4-th order Runge-Kutta integrator.

First, we explore the optimization path of a quadratic function, the L_2 loss, w.r.t. iteration. In particular, we labeled half of the generated data by 0 and the rest by 1. In Figure 1a, the ODE is discretized for $p = 2$ with different Runge-Kutta integrators with $s \in \{1, 2, 4\}$ and compared against GD and NAG algorithm. We can find that except the integrator with $s = 1$ can not converge due to

the unstability of the differential format itself, the DD methods shows superiority over GD. By using higher order iterator, the local acceleration is achieved and 4th order DD even converges faster than NAG (although for each iteration, it is obviously more costly than NAG). In Figure 1b, we explore the effect of q is the ODE. Since in the article p keeps the same as the one in Assumption 1, thus we denotes q the true parameter used in the ODE as below

$$\ddot{x}(t) + \frac{2q+1}{t}\dot{x}(t) + q^2 t^{q-2}\nabla f(x(t)) = 0.$$

We optimize the same L_2 loss with different values of q. By selecting smaller learning rates and increasing the numerical precision by using longer floats, the phenomenon that DD method diverges when $q > 2$ is not observed. Instead, we found that for $q \in \{2, 3, 4, 5\}$, larger q will give out faster convergence.

Then the minimization of L_4 loss (Figure 1c) and logistic loss (Figure 1d) is studied. We use 2-nd order Runge-Kutta integrator SSPRK22 for logistic loss optimization and 4-th order Runge-Kutta integrator RK4 for L_4 loss. As shown in Figure 1c and 1d, the loss decrease faster for larger q, as we can observed in above experiment about L_2 loss.

5 Discussion

5.1 Intuitive knowledge

Roughly speaking, this article allows for the design of optimization methods via direct discretization using Runge-Kutta integrators. However, the two assumptions required would be essential. **Assumption 1** quantifies the local flatness of convex functions in a way, and it actually contradicts our normal impression that gradient descent converges fast when the objective is not flat. This innovative discovery may inspire people to hold a more modern opinion towards the connection between convergence and local flatness. Also, the article claims that with careful analysis, discretizing ODE can preserve some of its trajectories properties. As a result, making further research on continuous ODE or appling the KR method to more general ODE cases can be valuable.

5.2 Potential research directions

To make further steps, there are quite some choices to take. The article uses conditions of higher-order differentiability to finally achieve an algorithm involving only first-order differential. We can see if allowing second and higher-order differential in the final algorithm will make things different, though in that case NAG method would be useless so we have to find another acceleration method to start with. Furthermore, as discussed above, the influence of local flatness to the convergence behaviour in discretized integraters is worth digging. How does the process of integration approaching actually work? What's the instinctive impact of local differentials and higher-order differentials? With techniques we know, some new results might be discovered.

To make a bold move, adding some random part to the conditions might leads to some interesting facts.

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A Proof Details

The proof of Theorem 1 consists of three main steps:

1. Showing that the suboptimality of the continuous trajectory of the ODE (10) converges to zero sufficiently fast.
2. Bounding the distance between points in the discretized and continuous trajectories, which measures the error introduced by using a numerical integrator.
3. Using continuity of the Lyapunov function and the bound on the distance between the points in the discretized and continuous trajectories to show that the suboptimality of the discretized sequence of points also converges to zero quickly.

Let's go through each of these steps in more detail:

1. To show that the suboptimality of the continuous trajectory of the ODE (10) converges to zero sufficiently fast, the proof uses a Lyapunov function $\mathcal{E}$ defined as:

$$\mathcal{E}([v;x;t]) := \frac{t^2}{4p^2}|v|^2 + \left|x + \frac{t}{2p}v - x^*\right|^2 + t^p\left(f(x) - f\left(x^*\right)\right).$$

   The Lyapunov function is a measure of the suboptimality of the solution and the norm of the gradient, and it is chosen such that it is monotonically non-increasing along the continuous trajectory of the ODE. This property is established in Proposition 5, which shows that the time derivative of the Lyapunov function is non-positive and bounded above:

$$\dot{\mathcal{E}}(y) \leq -\frac{t}{p}|v|^2.$$

   This monotonicity of the Lyapunov function implies that the suboptimality of the solution and the norm of the gradient are non-increasing along the continuous trajectory of the ODE, which in turn implies that the algorithm is making progress towards the optimal solution.

   In the first step of the proof, the goal is to show that the function $\mathcal{E}$ defined as:

$$\mathcal{E}([v;x;t]) := \frac{t^2}{4p^2}|v|^2 + |x + \frac{t}{2p}v - x^*|^2 + t^p(f(x) - f(x^*))$$

   is non-increasing with time, i.e., $\dot{\mathcal{E}}(y) \leq 0$.

   To prove this, the proof first computes the time derivative of $\mathcal{E}$:

$$\dot{\mathcal{E}}(y) = \left\langle \frac{\partial \mathcal{E}}{\partial y}, F(y) \right\rangle$$

   where $y = [v;x;t] \in \mathbb{R}^{2d+1}$ and $F(y) = [v;x;t]$ is the vector field defined in the ODE (10).

   Then, using the definition of $\mathcal{E}$ and the expression for $F(y)$, the proof obtains:

$$\dot{\mathcal{E}}(y) = \frac{t^2}{p^2}\left\langle v, \frac{\nabla f(x)}{|\nabla f(x)|}\right\rangle - \frac{t}{p}\left\langle x + \frac{t}{2p}v - x^*, \frac{\nabla f(x)}{|\nabla f(x)|}\right\rangle$$

   The proof then uses the convexity of the function f and the Cauchy-Schwarz inequality to bound this expression and obtain:

$$\dot{\mathcal{E}}(y) \leq -\frac{t}{p}|v|^2$$

   This inequality shows that the function $\mathcal{E}$ is non-increasing with time.

2. To bound the distance between points in the discretized and continuous trajectories, the proof uses a standard result from the theory of numerical integration that states that for a given ODE $\dot{y} = F(y)$, the error between the true solution $\varphi_h(y_0)$ and the numerical solution $\Phi_h(y_0)$ generated by a numerical integrator with step size h is bounded by $C_1 h^{s+1}$, where C_1 is a constant and s is the order of the numerical integrator. In the second step of the proof, the goal is to bound the distance between the points in the discretized and continuous trajectories. To do this, the proof defines a sequence of points y_k in the continuous trajectory such that $y_k = \varphi_h(y_{k-1})$, where y_0 is the initial point. Then, the proof defines the discretized sequence of points z_k as $z_k = \Phi_h(y_{k-1})$.

   The proof proceeds by using a Taylor expansion to express the difference between the continuous and discretized points as:

$$|z_k - y_k| \leq \frac{h^{s+1}}{(s+1)!} |F^{(s+1)}(\xi_k)|$$

   where ξ_k is some point on the segment connecting y_{k-1} and y_k.

   The proof then uses Assumption 2, which states that the $(s+1)$th derivative of the vector field F is bounded, to obtain:

$$|z_k - y_k| \leq C_1 h^{s+1}$$

   where C_1 is a constant that depends on the bound on the $(s+1)$th derivative of F. This inequality provides a bound on the distance between the points in the discretized and continuous trajectories.

3. Using this bound on the distance between the points in the discretized and continuous trajectories and the continuity of the Lyapunov function, the proof shows that the suboptimality of the discretized sequence of points also converges to zero quickly. More specifically, the proof defines a sequence of points y_k in the continuous trajectory such that $y_k = \varphi_h(y_{k-1})$, where y_0 is the initial point. Then, the proof defines the discretized sequence of points z_k as $z_k = \Phi_h(y_{k-1})$. Using the bound on the distance between the points in the discretized and continuous trajectories, the proof shows that:

$$|z_k - y_k| \leq C_1 h^{s+1}$$

   Then, using the continuity of the Lyapunov function, the proof shows that:

$$|\mathcal{E}(z_k) - \mathcal{E}(y_k)| \leq L|z_k - y_k|$$

   where L is the Lipschitz constant of the Lyapunov function. Combining these two inequalities, the proof obtains:

$$|\mathcal{E}(z_k) - \mathcal{E}(y_k)| \leq LC_1 h^{s+1}$$

   Then, by the monotonicity of the Lyapunov function, the proof has:

$$\mathcal{E}(z_k) \leq \mathcal{E}(y_k) \leq \mathcal{E}(y_0)$$

   Substituting this inequality back into the previous one, the proof obtains:

$$\mathcal{E}(y_0) - \mathcal{E}(z_k) \leq LC_1 h^{s+1}$$

   Finally, the proof sets the step size $h = C_1 N^{-1/(s+1)} (L + M + 1)^{-1} \mathcal{E}_0^{-1}$ and shows that the suboptimality of the discretized sequence of points $f(x_N) - f(x^*)$ is bounded by:

$$f(x_N) - f(x^*) \leq C_2 \mathcal{E}_0 \left[\frac{(L + M + 1)\mathcal{E}_0}{N^{s/(s+1)}} \right]^p$$

   where the constants C_1 and C_2 depend on the order s of the numerical integrator and the parameter p.

The objective function f is convex and satisfies certain conditions, then using a high-order numerical integrator to discretize the ODE (10) results in an algorithm that converges to the optimal solution at a rate close to $\mathcal{O}(N^{-p})$.

In the third step of the proof, the goal is to use the bound on the distance between the points in the discretized and continuous trajectories, together with the continuity of the Lyapunov function $\mathcal{E}$, to show that the suboptimality of the discretized sequence of points also converges to zero quickly.

To do this, the proof first defines a sequence of points z_k in the discretized trajectory such that $z_k = \Phi_h(y_{k-1})$, where $y_k = \varphi_h(y_{k-1})$ is the corresponding point in the continuous trajectory.

Then, the proof uses the bound on the distance between the points in the discretized and continuous trajectories, together with the continuity of $\mathcal{E}$, to obtain:

$$|\mathcal{E}(y_k) - \mathcal{E}(z_k)| \leq C_2 |y_k - z_k|$$

where C_2 is a constant that depends on the Lipschitz constant of $\mathcal{E}$.

Finally, the proof uses the bound on the distance between the points in the discretized and continuous trajectories to bound the right-hand side of this inequality and obtain:

$$|\mathcal{E}(y_k) - \mathcal{E}(z_k)| \leq C_3 h^{s+1}$$

where C_3 is a constant that depends on C_1 and C_2. This inequality shows that the suboptimality of the discretized sequence of points also converges to zero quickly.

the suboptimality of the continuous and discretized sequences of points converges to zero by using a Lyapunov function $\mathcal{E}$ defined as:

$$\mathcal{E}([v; x; t]) := \frac{t^2}{4p^2}|v|^2 + \left| x + \frac{t}{2p}v - x \right|^2 + t^p \left(f(x) - f\left(x\right) \right).$$

The Lyapunov function is a measure of the suboptimality of the solution and the norm of the gradient, and it is chosen such that it is monotonically non-increasing along the continuous and discretized trajectories of the ODE. This property shows that the time derivative of the Lyapunov function is non-positive and bounded above:

$$\dot{\mathcal{E}}(y) \leq -\frac{t}{p}|v|^2.$$

This monotonicity implies that both the suboptimality of the solution $f(x) - f(x^*)$ and the norm of the gradient $|v|$ are bounded above by some constants.

To bound the distance between points in the discretized and continuous trajectories, the proof shows that there exists a constant C_1 such that the distance between the points is bounded by $C_1 h^{s+1}$, where h is the step size and s is the order of the Runge-Kutta integrator. This bound on the distance between the points is used to show that the Lyapunov function is also monotonically non-increasing along the discretized trajectory.

This completes the proof of Theorem 1.