Introduction to Game Theory 3

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Reading

- **Recommended**
  - Shoham and Leyton-Brown: Chapter 5, sections 5.1
- **Optional**
  - William Spaniel: Game Theory 101: The Complete Textbook (paperback): *Very accessible and clear, teaching through examples. Accompanying YouTube channel.*
  - N. Nisan, T. Roughgarden, E. Tardos, V. Vazirani: Algorithmic Game Theory. Cambridge UP. *Solid, mathematical. Advanced.*
  - A. Dixit, B. Nalebuff: Thinking Strategically. Norton. *Lots of context and background. Interesting and non-technical.*

Overview

Bargaining as example of sequential game

Repeated Games

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Bargaining as example of sequential game

Repeated Games

The Ultimatum Game

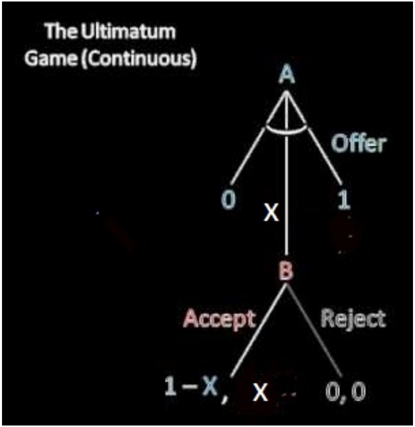
Ultimatum Game (UG): baseline (simplest non-trivial) model for **bargaining:** *take-it-or-leave-it!*

Assumptions

- **Surplus** can be divided continuously ($0 \leq x \leq 1$)
- **Two agents:**
  - A **proposes** split x versus $1 - x$, (**proposal power**)
  - B accepts or rejects;
- **No deal (conflict deal)** is considered **worst outcome**;
- Both agents aim to **maximize their utility**;

One round ultimatum game

- A makes single offer, B either accepts or rejects!

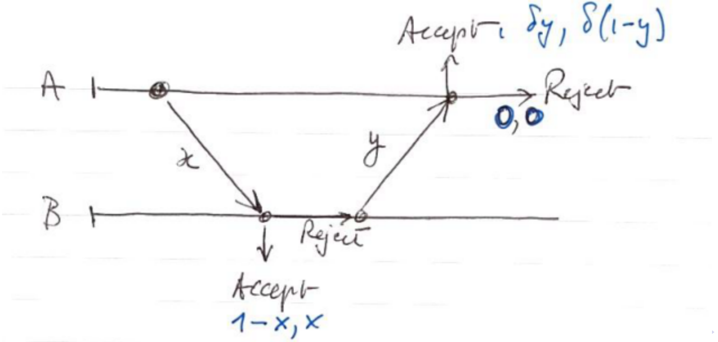


Two rounds ultimatum game with impatient players

Power of counter-offer:

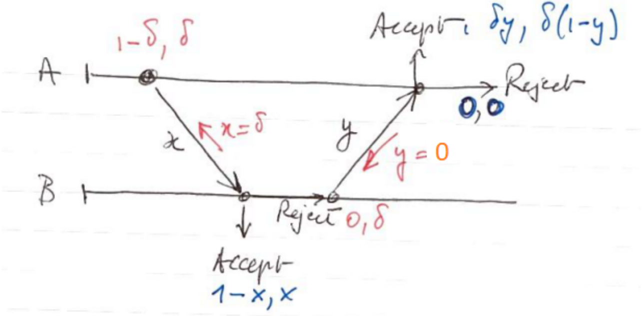
- A makes offer;
- B either accepts, or makes counter-offer.
- However, in each round the total is reduced by factor $\delta < 1$ (the icecream is melting!)

A makes offer, but B can make counter-offer!



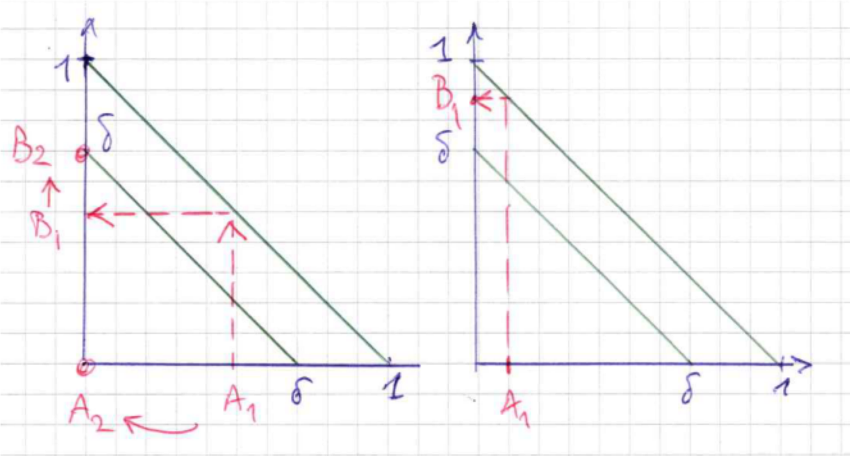
Two rounds ultimatum game: BI solution

- A makes offer, but B can make counter-offer!
- Use **backward induction** to find optimal solution.

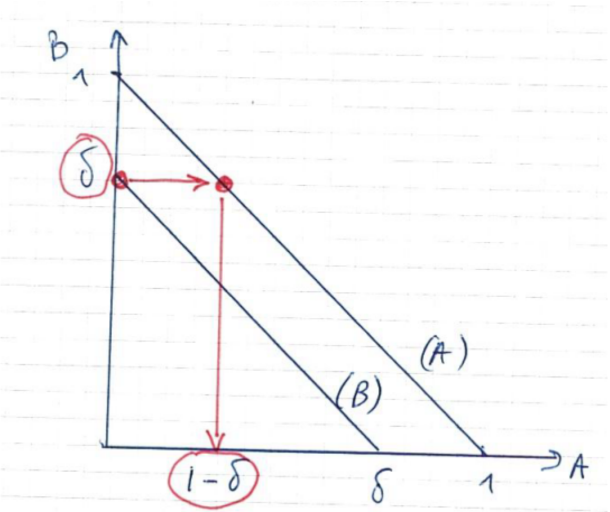


Conclusion: A offers split $(1-\delta, \delta)$ which B accepts.

Alternative interpretation

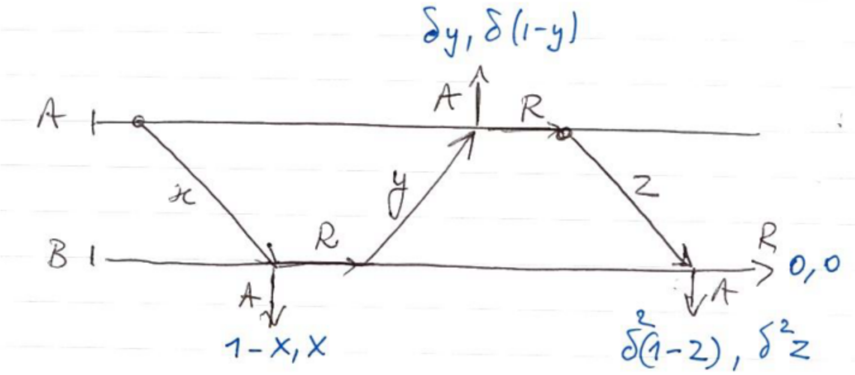


Two rounds ultimatum game: "Pareto Shuffle"

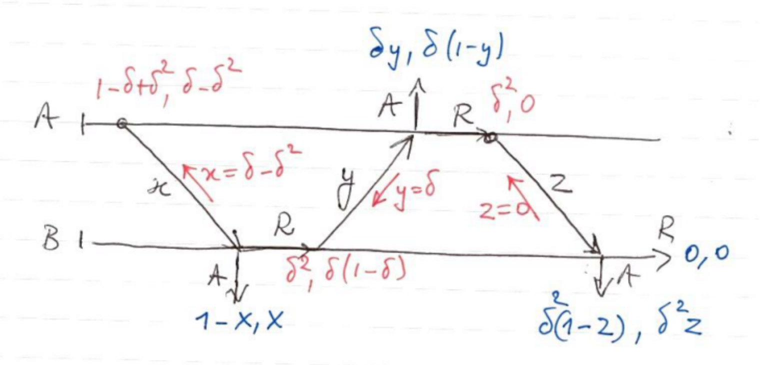


Three rounds ultimatum game

Room for two counter-offers!

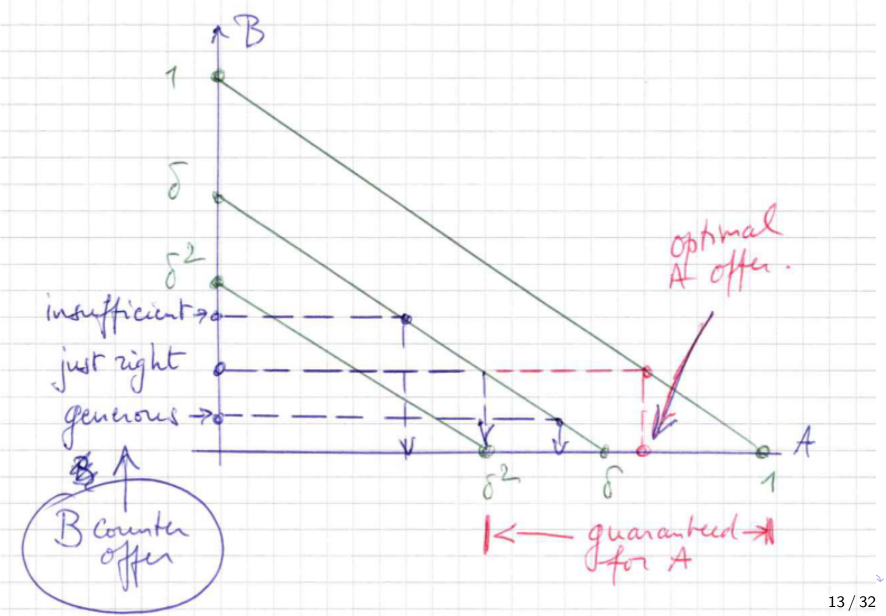


Three rounds ultimatum game: Backwards induction

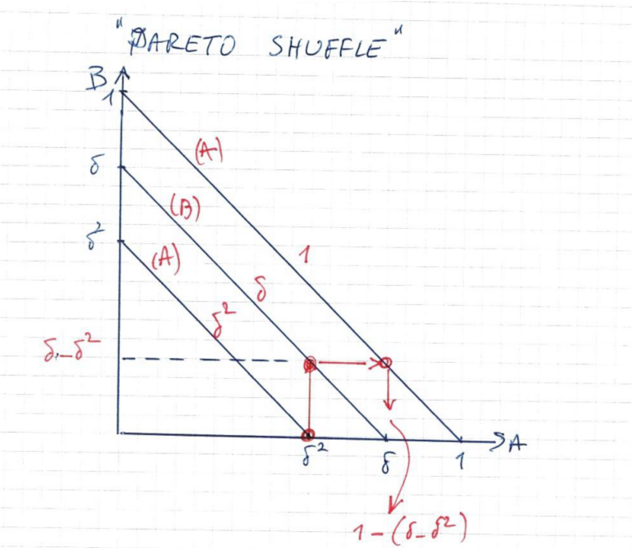


Conclusion: A proposes split $(1-\delta+\delta^2, \delta-\delta^2)$ which B accepts.

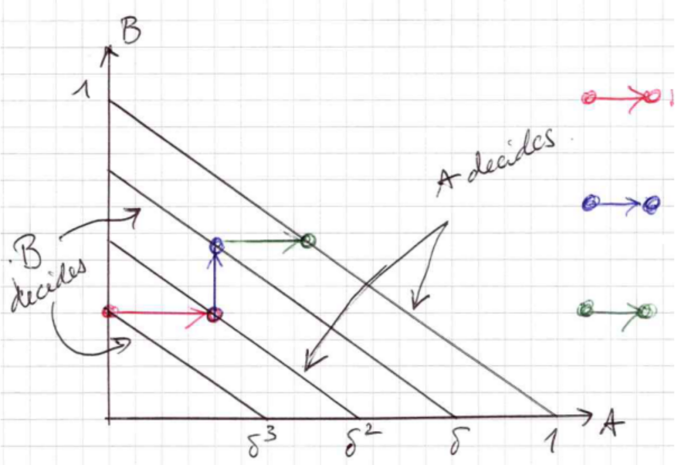
Pareto shuffle forward



Three rounds ultimatum game: "Pareto Shuffle"



Four rounds ultimatum game: backward induction

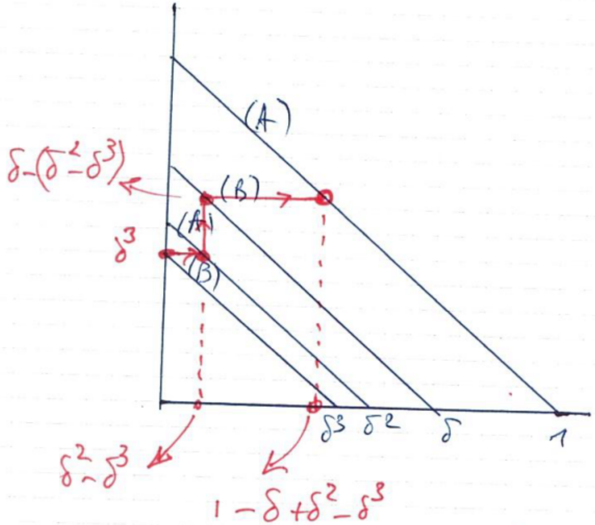


B indifferent
A improves

A indifferent
B improves

B indifferent
A improves

Four rounds ultimatum game: "Pareto Shuffle"

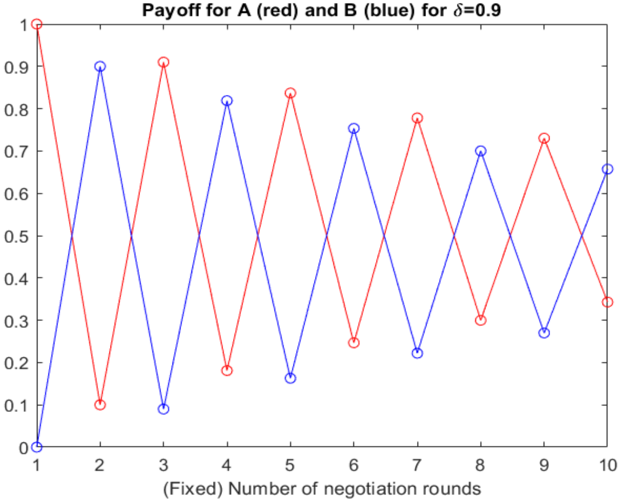


Alternating offers bargaining

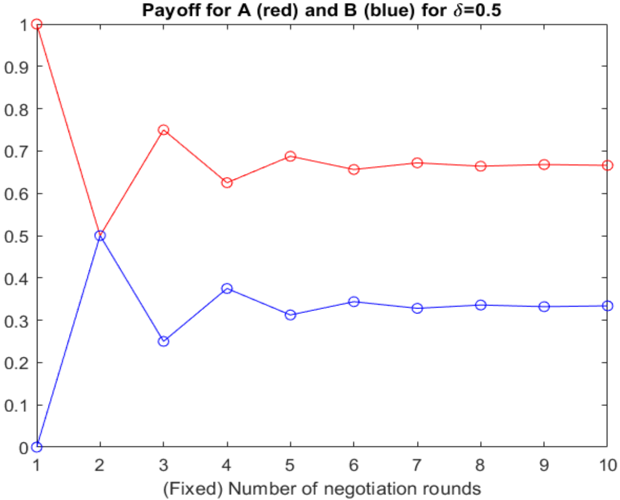
- Generalizing to negotiation with n rounds;
- When number of negotiation rounds is fixed at n, the equilibrium split is given in table below:

payoff for n rounds	$n=1$	$n=2$	$n=3$	$n=4$
for A	1	$1-\delta$	$1-\delta+\delta^2$	$1-\delta+\delta^2-\delta^3$
for B	0	δ	$\delta-\delta^2$	$\delta-\delta^2+\delta^3$

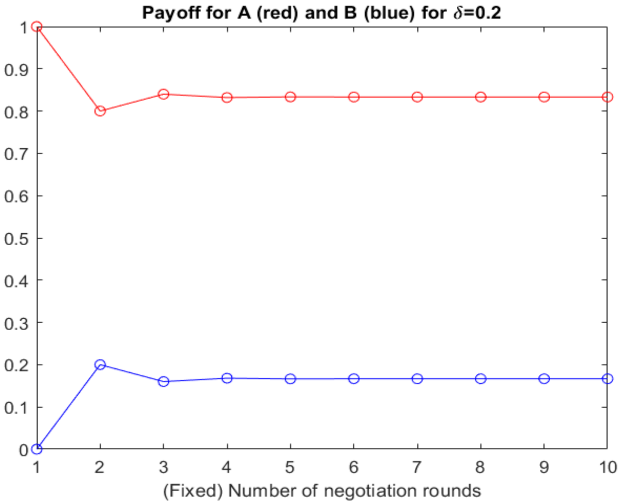
Alternating offers bargaining (mildly impatient, $\delta = 0.9$)



Alternating offers bargaining (seriously impatient, $\delta = 0.5$)



Alternating offers bargaining (very impatient, $\delta = 0.2$)



General conclusions

- Limit behaviour for pay-offs:

$$A(n) = \frac{1-(-\delta)^n}{1-(-\delta)} \quad \Longrightarrow \quad \lim_{n\to\infty} A(n) = \frac{1}{1+\delta}$$

$$B(n) = 1 - A(n) \quad \Longrightarrow \quad \lim_{n\to\infty} B(n) = \frac{\delta}{1+\delta}$$

- **First offer advantage:** $\lim A(n) \geq \lim B(n)$
- First offer advantage disappears for very patient negotiators:

$$\delta \longrightarrow 1 \quad \Longrightarrow \quad A(n) \searrow 1/2, \quad B(n) \nearrow 1/2;$$

Rubinstein's Model: Infinite Horizon Bargaining

- 2 agents, infinite horizon (# rounds **not fixed in advance**);
- **Time is valuable:** discount factors for both agents (δ_1, δ_2);
- **Optimal split:**

$$u_1 = \frac{1-\delta_2}{1-\delta_1\delta_2} \qquad \text{and} \qquad u_2 = \frac{\delta_2(1-\delta_1)}{1-\delta_1\delta_2}$$

- **Tragedy of bargaining:**
  The more time matters, the lower the share!
  E.g. if $\delta_2 \approx 0$, then $u_2 \approx 0$, etc.
- Notice **first mover's advantage** for $\delta_1 = \delta_2 = \delta$:

$$u_1 = \frac{1-\delta}{1-\delta^2} = \frac{1}{1+\delta} \qquad \text{and} \qquad u_2 = \frac{\delta(1-\delta)}{1-\delta^2} = \frac{\delta}{1+\delta}$$

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Bargaining as example of sequential game

Repeated Games

Repeated Games

- Repeat the **same** one-shot (stage) game (special case of sequential game);
- At each stage, information about preceding games is known;
- **Finite number** n (known!) of repetitions;
  - Maximize total reward:

$$R_n = \sum_{t=1}^{n} r_t$$

- **Infinite (unlimited) number** of repetitions
  - Maximize **discounted** total reward:

$$R = \sum_{t=1}^{\infty} \delta^{t-1} r_t \qquad \text{discount factor} \quad 0 < \delta < 1$$

  - equivalently: ending after **random number** of repetitions

Mathematical aside

- Recall:

$$\sum_{k=0}^{\infty} x^k = 1 + x + x^2 + x^3 + \ldots = \frac{1}{1-x} \qquad \text{(for } |x| < 1\text{)}$$

$$\sum_{k=1}^{\infty} x^k = x + x^2 + x^3 + x^4 \ldots = \frac{x}{1-x} \qquad \text{(for } |x| < 1\text{)}$$

Interpretation of discount factor

Stage games

$$1 \xrightarrow{\delta} 2 \xrightarrow{\delta} \cdots \cdots \xrightarrow{\delta} N-1 \xrightarrow{\delta} N \xrightarrow{1-\delta} \|$$

$$R_N = r_1 + r_2 + \cdots + r_{N-1} + r_N$$

$$\boxed{\mathbb{P}(N=n) = \delta^{n-1}(1-\delta)}$$

Repeated games: interpretation of discount factor

- After every stage-game there is a
  - probability $1-\delta$ that game will be ended;
  - probability δ that game will proceed to next round
- Reward R now becomes random variable:

$$R_N = \sum_{t=1}^{N} r_t \qquad \text{where} \quad P(N=n) = \delta^{n-1}(1-\delta)$$

(Assuming at least one stage game is played, hence $N \geq 1$).

Then:

$$E(R_N) := E\left(\sum_{t=1}^{N} r_t\right) = r_1 + \delta r_2 + \delta^2 r_3 + \ldots = \sum_{t=1}^{\infty} \delta^{t-1} r_t.$$

Interpretation of discount factor

$$E(R_N) = E\left(\sum_{t=1}^{N} r_t\right)$$

$$= \sum_{n=1}^{\infty} \underbrace{E\left(\sum_{t=1}^{N} r_t \,\middle|\, N=n\right)} P(N=n)$$

$$= \sum_{n=1}^{\infty} \left(\sum_{t=1}^{n} r_t\right) \underbrace{P(N=n)}_{\delta^{n-1}(1-\delta)}$$

$$= (1-\delta) \sum_{n=1}^{\infty} \left(\sum_{t=1}^{n} r_t\right) \delta^{n-1}$$

Interpretation of discount factor

$$E(R_N) = (1-\delta) \underbrace{\sum_{n=1}^{\infty} \left(\sum_{t=1}^{n} r_t \right) \delta^{n-1}}$$

$$\begin{aligned}
&= r_1 \\
&+ \delta r_1 + \delta r_2 \\
&+ \delta^2 r_1 + \delta^2 r_2 + \delta r_3
\end{aligned}$$

$$\underbrace{\left(\sum_{k=0}^{\infty} \delta^k \right)} r_1 + \delta \underbrace{\left(\sum_{k=0}^{\infty} \delta^k \right)} r_2 + \dots$$

$$= (1-\delta) \left[\frac{1}{1-\delta} r_1 + \delta \left(\frac{1}{1-\delta} \right) r_2 + \dots \right]$$

$$= r_1 + \delta r_2 + \delta^2 r_3 + \dots = \sum_{t=1}^{\infty} \delta^{t-1} r_t . \quad \square$$

Example: Repeated Prisoner's dilemma

- Pay-off matrix for **stage game:**

	$C(oop)$	$D(efect)$
C	$3,3$	$1,4$
D	$4,1$	$2,2$

- For **finite number** of repetitions:
  - Single Nash eq.: Play D-D for each repetition;
  - Can be proved using **backward induction**
- For **infinite number** of repetitions:
  - More interesting strategies, including cooperation
  - Examples: Tit-for-Tat, Grim Trigger

Grim Trigger Strategy for Repeated Prisoner's Dilemma

- **Grim Trigger and Tit-for-Tat strategy**
  Start by cooperating . . .
  - GT: continue cooperating, until someone defects; from then onwards, always defect!
  - TfT: from then onwards, copy last move of opponent.
- If both parties play GT, is it rational to defect?
  Consider player 1:
  - Utility for continued **cooperation:**

$$u_1(C) = 3 + 3\delta + 3\delta^2 + \ldots = \frac{3}{1-\delta}$$

  - Utility for **defection**:

$$u_1(D) = 4 + 2\delta + 2\delta^2 + \ldots = 4 + 2\frac{\delta}{1-\delta}$$

- Continued **cooperation is rational** if $u_1(C) > u_1(D)$.

Grim Trigger Strategy for Repeated Prisoner's Dilemma

- Continued **cooperation is rational** if $u_1(C) > u_1(D)$:

$$u_1(C) > u_1(D) \qquad \iff \qquad \frac{3}{1-\delta} = 4 + 2\frac{\delta}{1-\delta}$$

$$\iff \qquad \delta > 1/2.$$

- **Interpretation:** If the players are sufficiently patient (i.e. future rewards are sufficiently valuable) then it's rational to cooperate.