Multi-Agent Systems

Homework Assignment 3

MSc AI, VU

E.J. Pauwels

Version: November 19 — Deadline: Friday November 26, 2021 (23h59)

3 Sequential Games with Perfect Information

3.1 Reduced centipede game

Consider a sequential 2-player game with the following game-tree: at each decision node the associated player needs to decide whether to continue (c) or stop (s). The tree (including utilities) and the players' rationality are common knowledge.

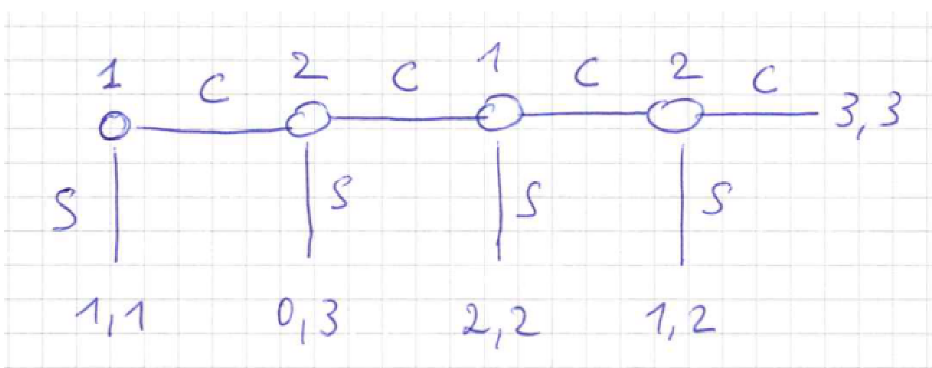


Figure 1: Shortened version of the centipede game.

Questions

- Write the normal form for this game and find all Nash equilibria in pure strategies (PNEs).
- List all subgames and determine which of these PNEs are also subgame-perfect?
- Can you solve this game using backward induction? Discuss.

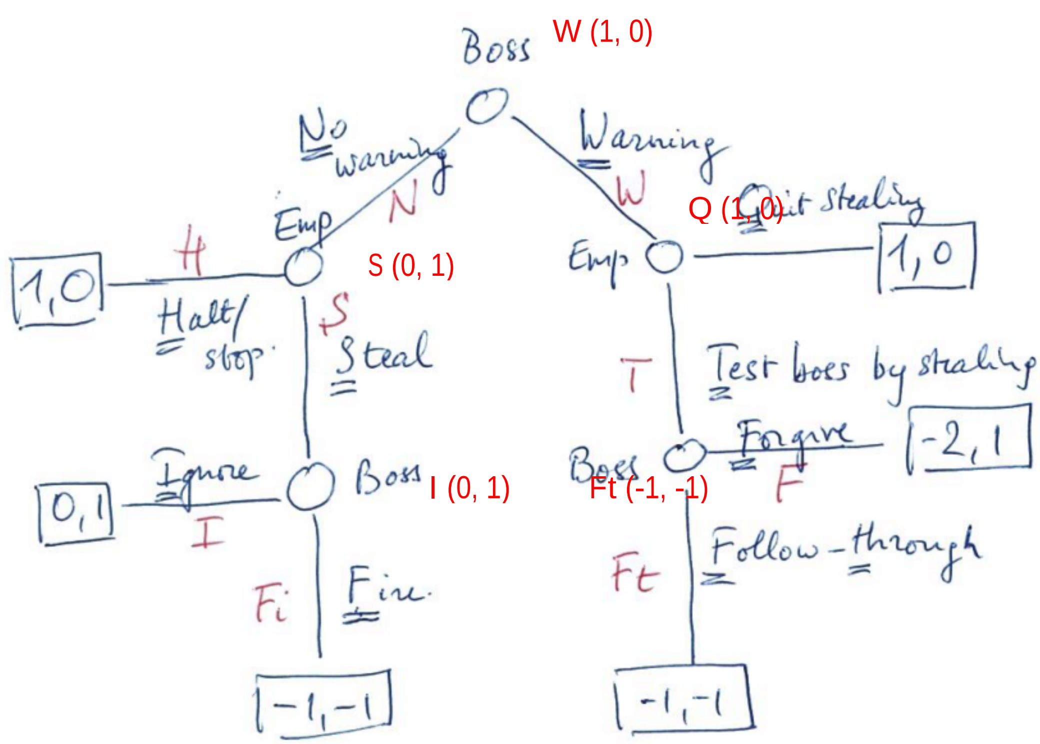
fa

3.2 Boss and stealing employee

A boss notices that one of her employers has been stealing company material lately. The material was not all that valuable, so she is inclined to let it pass, preferring to keep the employee around rather than firing him and having to hire and retrain a replacement. Nevertheless she wants the stealing to stop.

She is therefore thinking to issue a warning at the next company meeting: the next person caught stealing company property will be fired immediately. She envisages the following game tree with pay-offs (see fig below).

1. Analyse this game using backward induction.
2. What are the pure actions for the two players (boss and employee)? Construct the normal form matrix.
3. Use this matrix to identify all the pure Nash equilibria of the normal form game.
4. Determine the subgame-perfect equilibrium (equilibria?) by eliminating all the Nash equilibria that fail to induce a NE in subgames.
5. Compare to the solution based on backward induction.



3.3 Stackelberg's Duopoly Model

Stackelberg's duopoly model is a sequential version of Cournot's duopoly model. There are two firms that produce some bland product (e.g. fertilizer). Firm 1 moves first and decides to produce a total quantity q_1 (think of this as a continuous variable). Firm 2 observes this move and then decides to produce a quantity q_2. The market price (per unit) depends (linearly) on the total amount produced:

$$P(q_1, q_2) = \alpha - \beta(q_1 + q_2)$$

where $\alpha, \beta > 0$ are known (positive) constants. Assume that both firms can produce the product at a fixed unit cost c. Hence the pay-off for each firm equals:

$$u_i(q_1, q_2) = P(q_1, q_2)q_i - cq_i.$$

1. Use backward induction to determine the optimal quantities for both firms.

2. Compare your results to the once obtained for the Cournot (simultaneous) model. Is there a "first mover" advantage?

SOLUTIONS

3.1 Reduced centipede game

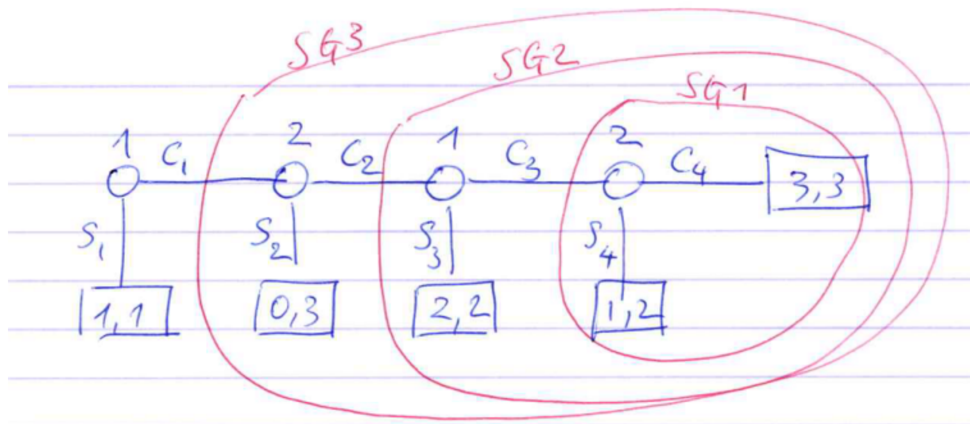


Figure 2: Centipede game with non-trivial subgames for future reference.

Each player has two decision nodes yielding the following pure strategies:

- player 1 : $\{s_1, c_1\} \times \{s_3, c_3\} = \{s_1s_3, s_1c_3, c_1s_3, c_1c_3\}$
- player 2 : $\{s_2, c_2\} \times \{s_4, c_4\} = \{s_2s_4, s_2c_4, c_2s_4, c_2c_4\}$

Conversion to normal form yields the following matrix, with five Nash equilibria as indicated.

	s_2s_4	s_2c_4	c_2s_4	c_2c_4
s_1s_3	1) 1,1	2) 1,1	1,1	1,1
s_1c_3	3) 1,1	4) 1,1	1,1	1,1
c_1s_3	0,3	0,3	2,2	2,2
c_1c_3	0,3	0,3	1,2	5) 3,3

Figure 3: Centipede, full game in normal form, with 5 pure NE (numbered for future reference).

Next, we determine the pure NE's for the three non-trivial subgames:

SG1: Player 2 chooses: C_4. (Nash Eq).

SG2

player 2

		S_4	C_4
player 1	S_3	2, 2	2, 2
	C_3	1, 2	3, 3

$$PNE(SG_2) = \{(S_3, S_4), (C_3, C_4)\}$$

Figure 4: PNEs for subgames SG1 and SG2

SG3: player 1 $<$ S_3, C_3

player 2: $\{S_2, C_2\} \times \{S_4, C_4\}$

$= \{S_2S_4, S_2C_4, C_2S_4, C_2C_4\}$.

	S_2S_4	S_2C_4	C_2S_4	C_2C_4
S_3	0, 3	0, 3	2, 2	2, 2
C_3	0, 3	0, 3	1, 2	3, 3

Figure 5: PNEs for subgame SG3

NE of original game induce following strategies in subgames:

Original NE		SG1	SG2	SG3
(1) $s_1 s_3, s_2, s_4$	$\rightarrow$	$\cancel{s_4}$ (NOK)	s_3, s_4 OK	$s_3, s_2 s_4$ OK
(2) $s_1 s_3, s_2 c_4$	$\rightarrow$	c_4 OK	$\cancel{s_3 c_4}$	$s_3, s_2 c_4$ OK
(3) $s_1 c_3, s_2 s_4$	$\rightarrow$	$\cancel{s_4}$ (NOK)	$c_3 s_4$	$c_3, s_2 s_4$
(4) $\boxed{s_1 c_3, s_2 c_4}$	$\rightarrow$	c_4 ✓	$c_3 c_4$ ✓	$c_3, s_2 c_4$ ✓
(5) $\boxed{c_1 c_3, c_2 c_4}$	$\rightarrow$	c_4 ✓	$c_3 c_4$ ✓	$c_3, c_2 c_4$ ✓

Figure 6: Only NE 4 and 5 induce Nash equilibria in all three subgames.

Induced Nash eq. in subgames Conclusion Only $NE_4 = (s_1c_3, s_2c_4)$ and $NE_5 = (c_1c_3, c_2c_4)$ induce Nash eq. in all subgames and are therefore subgame-perfect.

Backward induction Backward induction shows that player 2 is indifferent between his two options at his first decision point (both yield 3). He could therefore play any mixture of the two pure strategies. Hence it is impossible for player 1 to decide what his best option is. His choice will depend on other factors (not part of the game setup). E.g. if he's really risk-averse then he will go for s_1 (i.e. stop at node 1) since this guarantees a pay-off of 1.

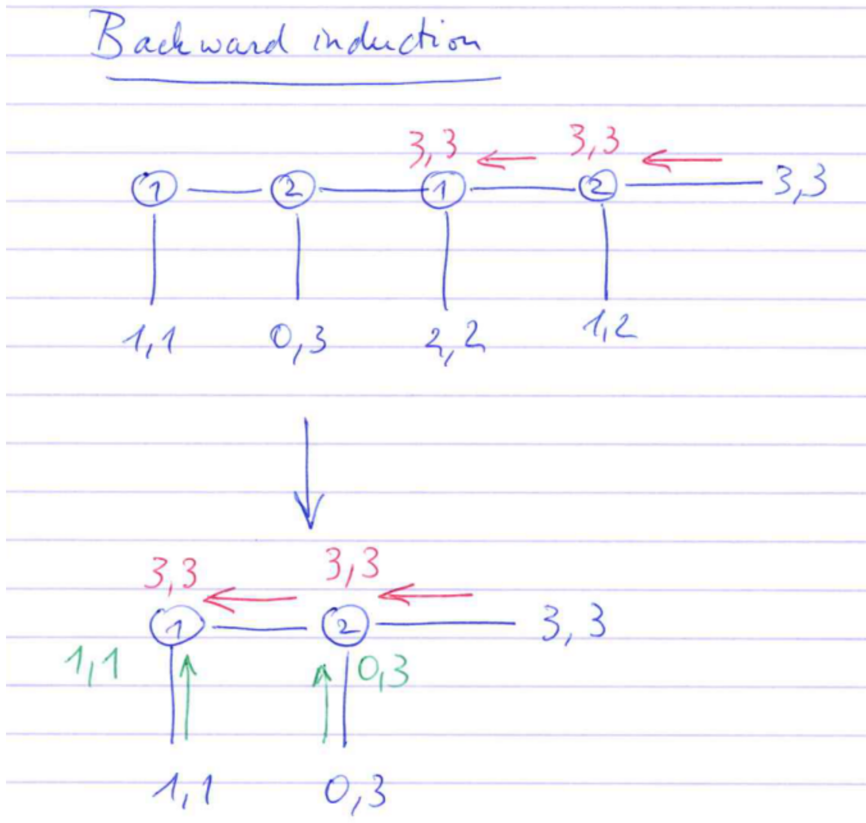
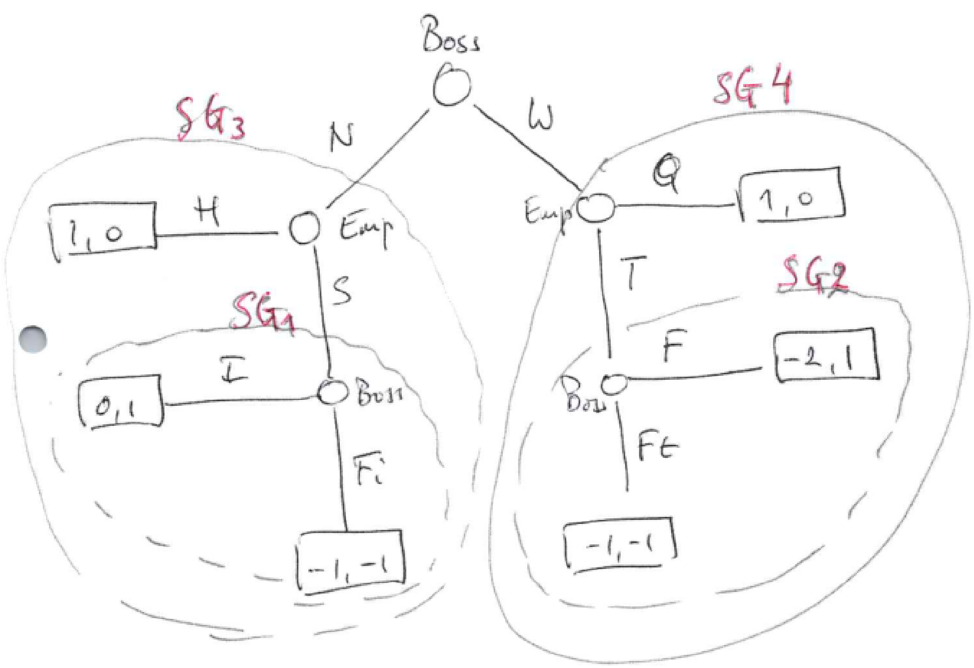


Figure 7: Reduction of game tree by using backward induction. At this stage, player 2 is indifferent between both actions.

3.2 Boss and stealing employee

BSE/1

Boss & Stealing employee



Normal form

	HQ	HT	SQ	ST	NOT PERFECT for
N I F	1,0	1,0	0,1	$\boxed{0,1}$	→ SG2
N I Ft	1,0	1,0	0,1	$\boxed{0,1}$	→ SG4 (*
N Fi F	$\boxed{1,0}$	$\boxed{1,0}$	-1,-1	-1,-1	→ SG1
N Fi Ft	$\boxed{1,0}$	$\boxed{1,0}$	-1,-1	-1,-1	→ SG1
W I F	1,0	-2,1	1,0	-2,1	
W I Ft	$\boxed{1,0}$	-1,-1	$\boxed{1,0}$	-1,-1	→ SG3 (**)
W Fi F	1 0	-2,1	1,0	-2,1	
W Fi Ft	$\boxed{1,0}$	-1,-1	$\boxed{1,0}$	-1,-1	→ SG1

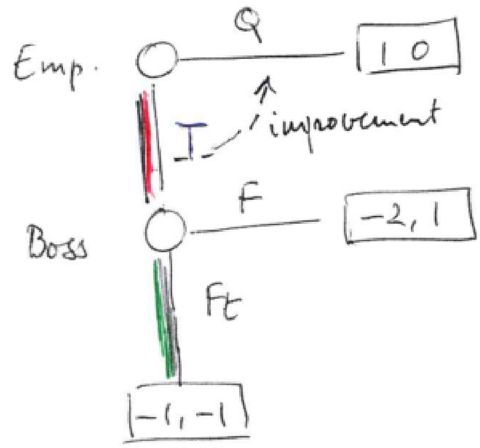
Figure 8: TOP: Game tree with subgames. Bottom: The normal form game with all Nash equilibria (black boxes) and the SPNE (boxed in green). Last column indicates in which subgame the NE fails to induce a subgame NE.

BSE/2

(*)

Boss Em.

Consider $(NI\,F_t, ST)$ for SG4 : induces the strategy below.

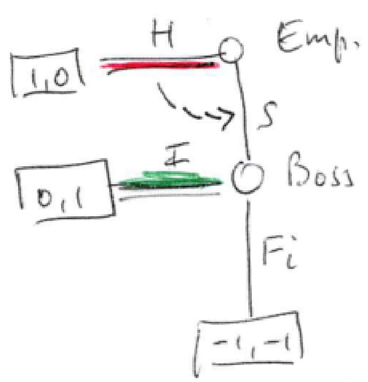


Emp. wants to change $T \longrightarrow Q$

(payoff : $-1 \longrightarrow 0$)

Hence : NOT NE!

(**) $(W, I, F_t\,;\, HQ)$ in SG3



Emp. wants to change $H \longrightarrow S$

(payoff : $0 \longrightarrow 1$)

Hence not NE!

Conclusion : (WIF_t, SQ) is SGPE.

Figure 9: Amplification: induced NE

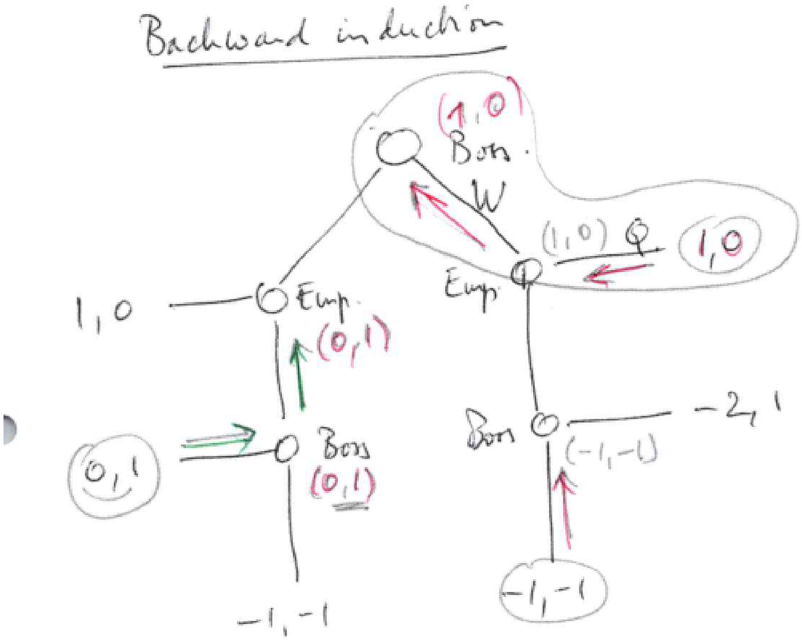


Figure 10: Backward induction solution: boss issues a warning, and employee quits stealing.

3.3 Stackelberg's Duopoly

Sequential game: firm 1 (leader) moves first and produces quantity q_1. Subsequently, firm 2 responds by producing quantity q_2. We solve this game by backward induction.

Given q_1, what is the optimal value q_2^* for firm 2? To this end we compute:

$$\frac{\partial u_2}{\partial q_2} = \frac{\partial}{\partial q_2}\{(P(q_1,q_2)-c)q_2\} = (\alpha-c)-\beta(q_1+2q_2)$$

Hence:

$$\frac{\partial u_2}{\partial q_2} = 0 \qquad \Longrightarrow \qquad q_2^* = \frac{1}{2}(A-q_1) \quad \text{where} \quad A = (\alpha-c)/\beta.$$

Given the anticipated optimal response q_2^* of firm 2, what is the best action (q_1^*) for firm 1? The total produced quanity $q_T = q_1 + q_2^* = (A+q_1)/2$, hence:

$$P(q_1,q_2^*) = \alpha - \frac{\beta}{2}(A+q_1)$$

whence

$$u_1(q_1,q_2^*) = \frac{1}{2}((\alpha-c)-\beta q_1)q_1$$

For optimality, we need to insist on:

$$0 = \frac{\partial u_1}{\partial q_1} = \beta(A/2 - q_1)$$

from which we conclude:

$$q_1^* = A/2 \qquad \text{whence} \qquad q_2^* = A/4.$$

Corresponding utilities The optimal total quantity is given by $q_T^* = q_1^* + q_2^* = (3/4)A$. This implies that $P(q_1^*,q_2^*) - c = (\alpha-c) - \beta q_T^* = (\beta/2)A$. From this we find

$$u_1^* = \frac{1}{8}A^2\beta \qquad \text{and} \qquad u_2^* = \frac{1}{16}A^2\beta$$

clearly showing the *first mover's advantage* for the leader (firm 1).

Comparison with Cournot's duopoly In the case of Cournot we found (previous homework):

$$q_1^* = (A-q_2)/2 \qquad \text{and} \qquad q_2^* = (A-q_1)/2.$$

Substituting the 2nd one in the first, yields for the optimal quantities in the Cournot case:

$$q_1^* = q_2^* = \frac{1}{3}A \qquad \text{with corresponding optimal utilities} \quad u_i^* = \frac{1}{9}A^2\beta.$$