# Introduction to Game Theory 3

Eric Pauwels (CWI & VU)

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# Reading

- **Recommended**
    - Shoham and Leyton-Brown: Chapter 5, sections 5.1
- **Optional**
    - William Spaniel: Game Theory 101: The Complete Textbook (paperback): *Very accessible and clear, teaching through examples. Accompanying YouTube channel.*
    - N. Nisan, T. Roughgarden, E. Tardos, V. Vazirani: Algorithmic Game Theory. Cambridge UP. *Solid, mathematical. Advanced.*
    - A. Dixit, B. Nalebuff: Thinking Strategically. Norton. *Lots of context and background. Interesting and non-technical.*

# Overview

Bargaining as example of sequential game

Repeated Games

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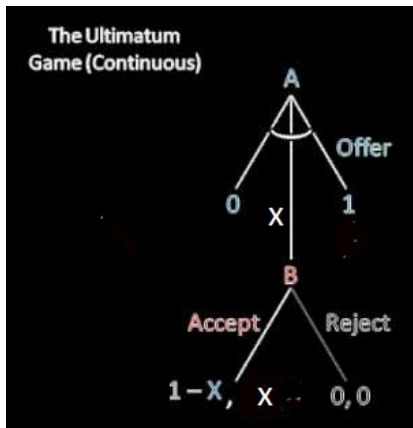
# The Ultimatum Game

**Ultimatum Game (UG): baseline** (simplest non-trivial) model for **bargaining:** *take-it-or-leave-it!*

**Assumptions**

- **Surplus** can be divided continuously ($0 \leq x \leq 1$)
- **Two agents:**
  - A **proposes** split $x$ versus $1 - x$, (**proposal power**)
  - B accepts or rejects;
- **No deal (conflict deal)** is considered **worst outcome**;
- Both agents aim to **maximize their utility**;

# One round ultimatum game
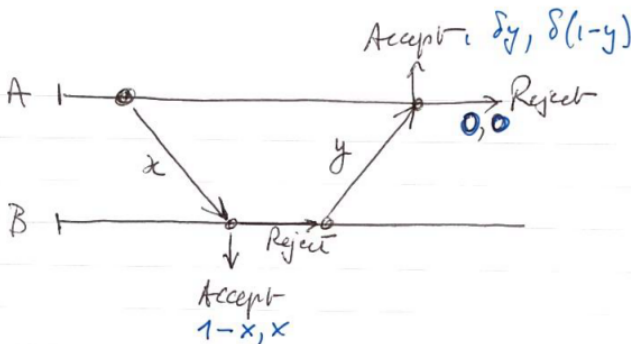
- A makes single offer, B either accepts or rejects!

# Two rounds ultimatum game with impatient players

**Power of counter-offer:**

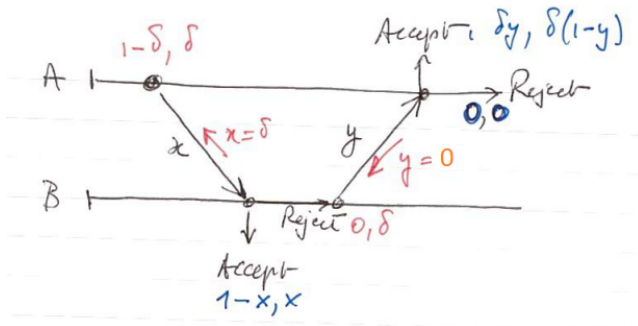- A makes offer;
- B either accepts, or makes counter-offer.
- However, in each round the total is reduced by factor $\delta < 1$ (the icecream is melting!)

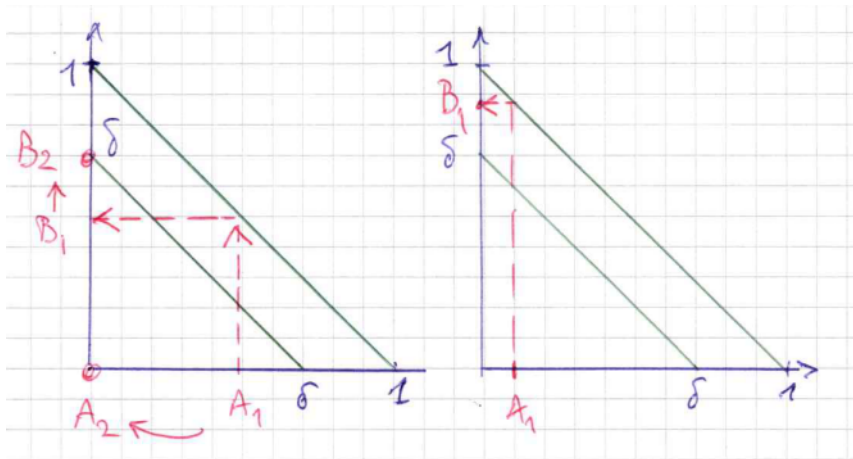A makes offer, but B can make counter-offer!

## Two rounds ultimatum game: BI solution

- A makes offer, but B can make counter-offer!
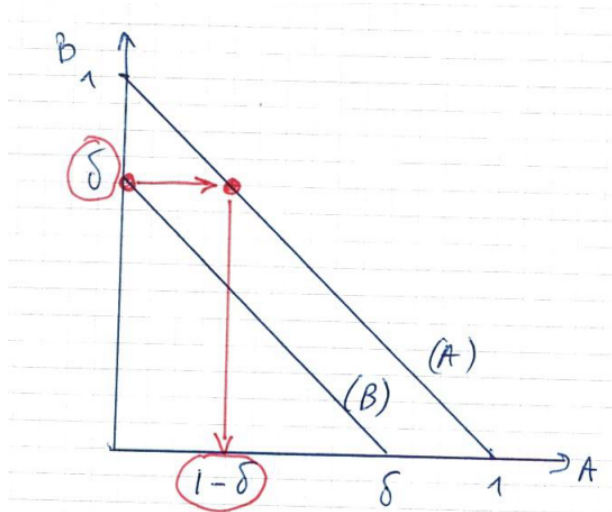- Use **backward induction** to find optimal solution.



**Conclusion:** A offers split $(1 - \delta, \delta)$ which B accepts.
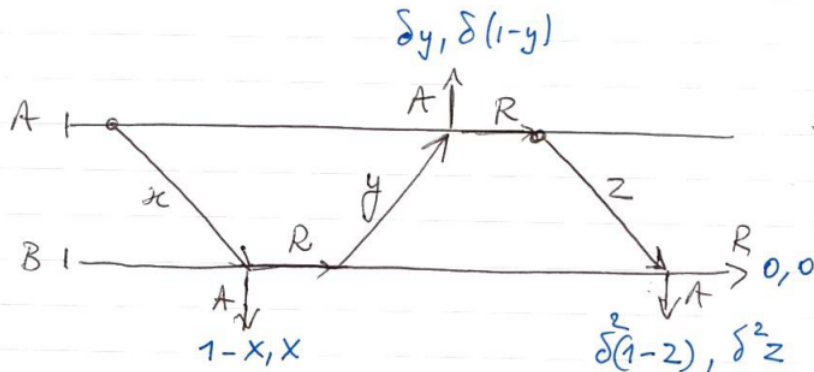
# Alternative interpretation

# Two rounds ultimatum game: "Pareto Shuffle"

# Three rounds ultimatum game

Room for two counter-offers!

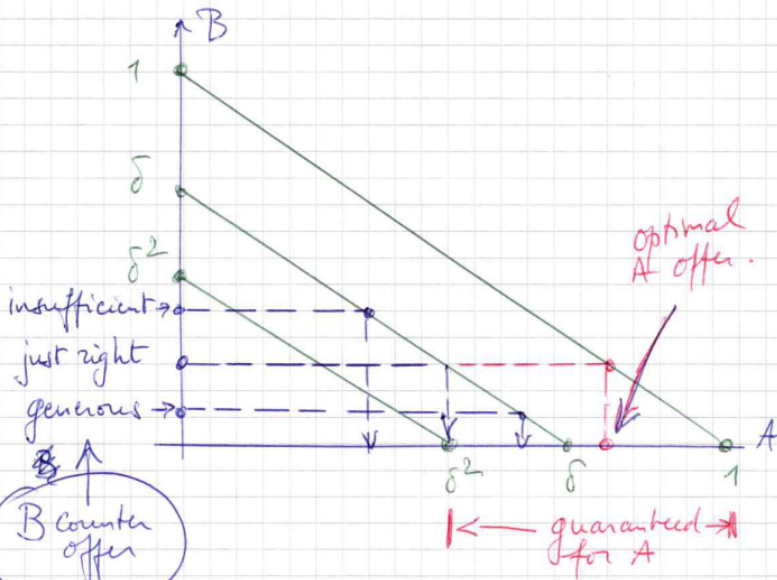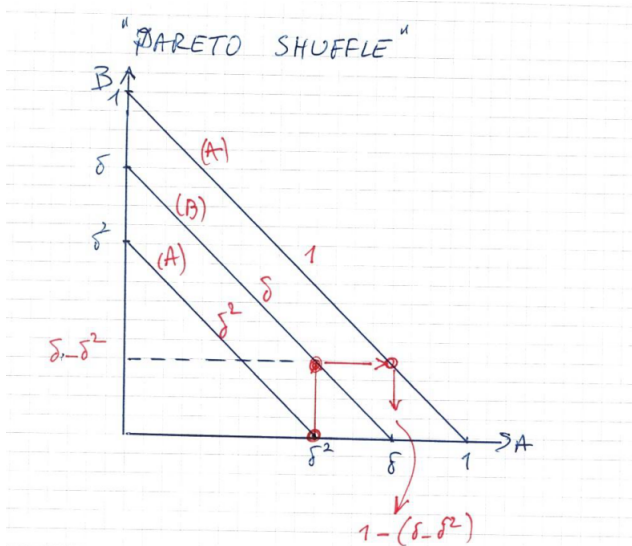# Three rounds ultimatum game: Backwards induction
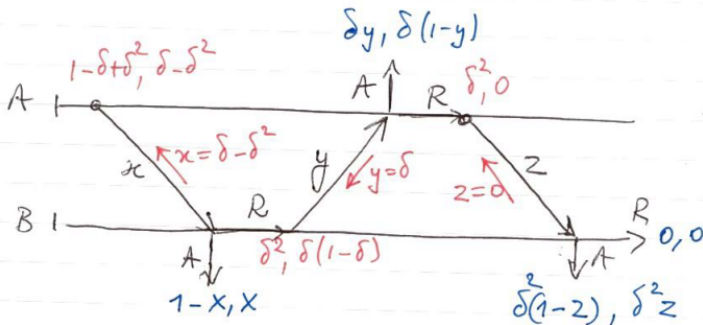


**Conclusion:** A proposes split $(1 - \delta + \delta^2, \delta - \delta^2)$ which B accepts.

# Pareto shuffle forward

## Three rounds ultimatum game: "Pareto Shuffle"

# Four rounds ultimatum game: backward induction



B indifferent
A improves

A indifferent
B improves

B indifferent
A improves
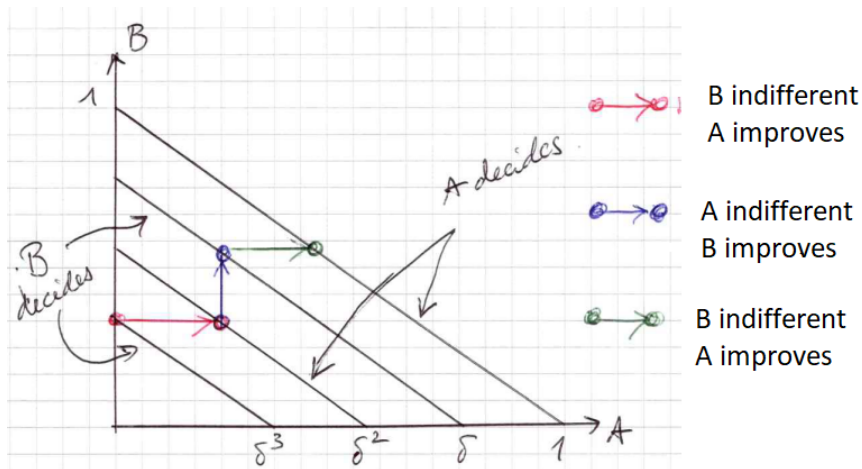
# Four rounds ultimatum game: "Pareto Shuffle"
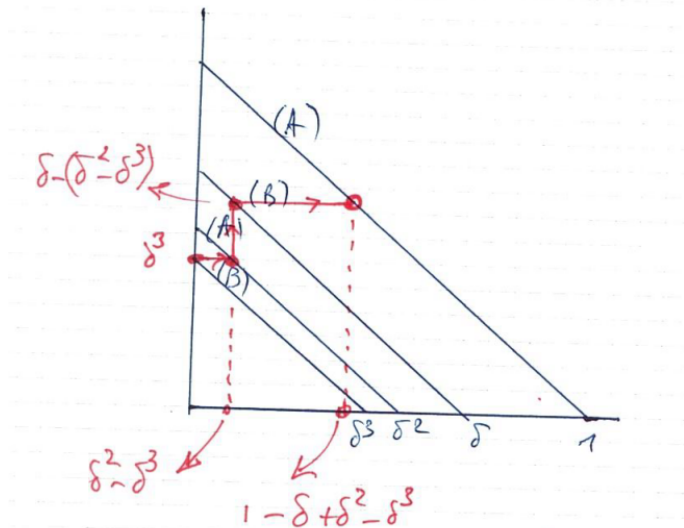
# Alternating offers bargaining

- Generalizing to negotiation with $n$ rounds;
- When number of negotiation rounds is fixed at $n$, the equilibrium split is given in table below:

| payoff for $n$ rounds | $n = 1$ | $n = 2$ | $n = 3$ | $n = 4$ |
|:---:|:---:|:---:|:---:|:---:|
| for A | 1 | $1 - \delta$ | $1 - \delta + \delta^2$ | $1 - \delta + \delta^2 - \delta^3$ |
| for B | 0 | $\delta$ | $\delta - \delta^2$ | $\delta - \delta^2 + \delta^3$ |

# Alternating offers bargaining (mildly impatient, $\delta = 0.9$)



Payoff for A (red) and B (blue) for $\delta$=0.9

(Fixed) Number of negotiation rounds
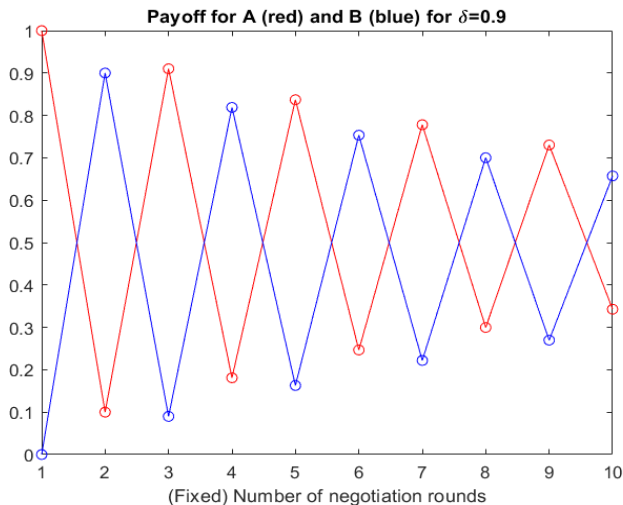
# Alternating offers bargaining (seriously impatient, $\delta = 0.5$)



Payoff for A (red) and B (blue) for $\delta$=0.5

(Fixed) Number of negotiation rounds
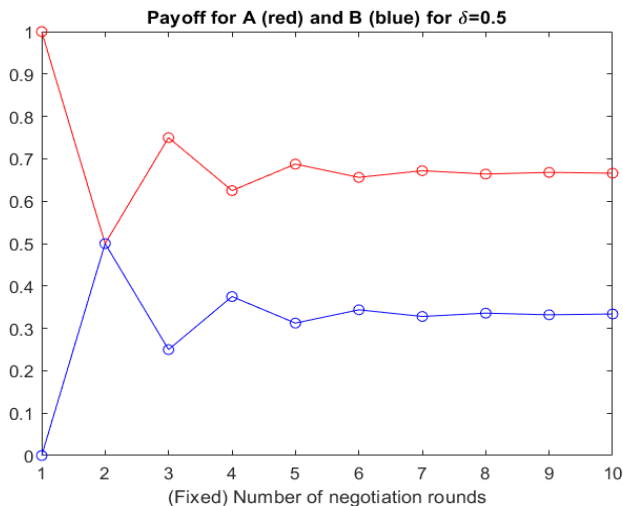
# Alternating offers bargaining (very impatient, $\delta = 0.2$)



Payoff for A (red) and B (blue) for $\delta$=0.2
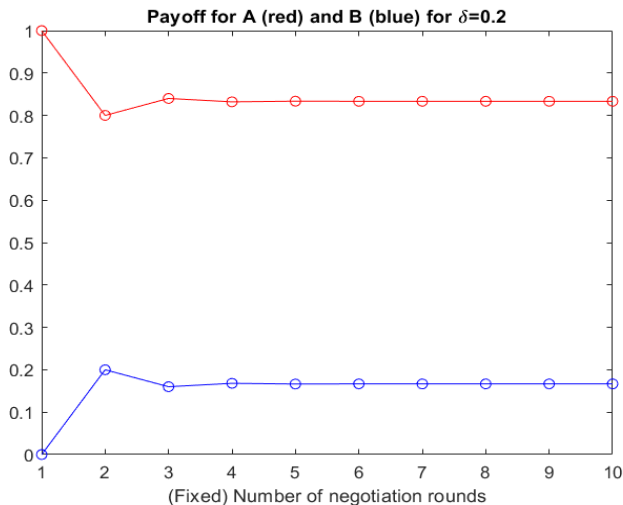
## General conclusions

- Limit behaviour for pay-offs:

$$A(n) = \frac{1 - (-\delta)^n}{1 - (-\delta)} \quad \implies \quad \lim_{n \to \infty} A(n) = \frac{1}{1 + \delta}$$

$$B(n) = 1 - A(n) \quad \implies \quad \lim_{n \to \infty} B(n) = \frac{\delta}{1 + \delta}$$

- **First offer advantage:** $\quad \lim A(n) \geq \lim B(n)$
- First offer advantage disappears for very patient negotiators:

$$\delta \longrightarrow 1 \quad \implies \quad A(n) \searrow 1/2, \quad B(n) \nearrow 1/2;$$

# Rubinstein's Model: Infinite Horizon Bargaining

- 2 agents, infinite horizon (# rounds **not fixed in advance**);
- **Time is valuable:** discount factors for both agents $(\delta_1, \delta_2)$;
- **Optimal split:**

$$u_1 = \frac{1 - \delta_2}{1 - \delta_1 \delta_2} \qquad \text{and} \qquad u_2 = \frac{\delta_2(1 - \delta_1)}{1 - \delta_1 \delta_2}$$

- **Tragedy of bargaining:**
  The more time matters, the lower the share!
  E.g. if $\delta_2 \approx 0$, then $u_2 \approx 0$, etc.

- Notice **first mover's advantage** for $\delta_1 = \delta_2 = \delta$:

$$u_1 = \frac{1 - \delta}{1 - \delta^2} = \frac{1}{1 + \delta} \qquad \text{and} \qquad u_2 = \frac{\delta(1 - \delta)}{1 - \delta^2} = \frac{\delta}{1 + \delta}$$

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Bargaining as example of sequential game

Repeated Games

# Repeated Games

- Repeat the **same** one-shot (stage) game (special case of sequential game);
- At each stage, information about preceding games is known;
- **Finite number** $n$ (known!) of repetitions;
  - Maximize total reward:

$$R_n = \sum_{t=1}^{n} r_t$$

- **Infinite (unlimited) number** of repetitions
  - Maximize **discounted** total reward:

$$R = \sum_{t=1}^{\infty} \delta^{t-1} r_t \qquad \text{discount factor} \quad 0 < \delta < 1$$

  - equivalently: ending after **random number** of repetitions
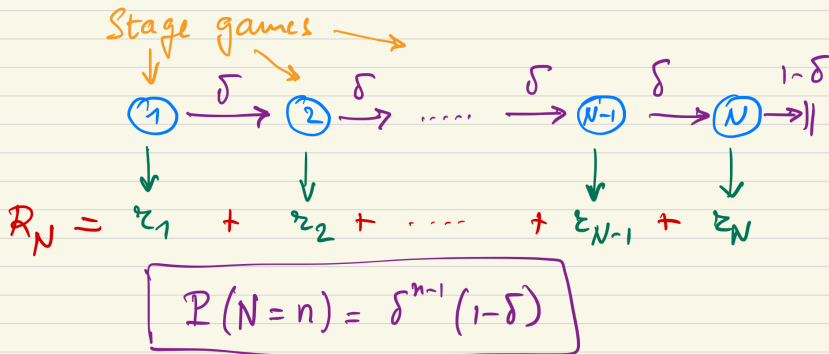
## Mathematical aside

- Recall:

$$\sum_{k=0}^{\infty} x^k = 1 + x + x^2 + x^3 + \ldots = \frac{1}{1-x} \qquad (\text{for } |x| < 1)$$

$$\sum_{k=1}^{\infty} x^k = x + x^2 + x^3 + x^4 \ldots = \frac{x}{1-x} \qquad (\text{for } |x| < 1)$$

# Interpretation of discount factor

## Repeated games: interpretation of discount factor

- After every stage-game there is a
  - probability $1 - \delta$ that game will be ended;
  - probability $\delta$ that game will proceed to next round

- Reward $R$ now becomes random variable:

$$R_N = \sum_{t=1}^{N} r_t \qquad \text{where} \quad P(N = n) = \delta^{n-1}(1 - \delta)$$

  (Assuming at least one stage game is played, hence $N \geq 1$).

  *Then:*

$$E(R_N) := E\left(\sum_{t=1}^{N} r_t\right) = r_1 + \delta r_2 + \delta^2 r_3 + \ldots = \sum_{t=1}^{\infty} \delta^{t-1} r_t.$$

## Interpretation of discount factor

$$E(R_N) = E\left(\sum_{t=1}^{N} z_t\right)$$

$$= \sum_{n=1}^{\infty} \underbrace{E\left(\sum_{t=1}^{N} z_t \mid N=n\right) P(N=n)}$$

$$= \sum_{n=1}^{\infty} \left(\sum_{t=1}^{n} z_t\right) \underbrace{P(N=n)}_{\delta^{n-1}(1-\delta)}$$

$$= (1-\delta) \sum_{n=1}^{\infty} \left(\sum_{t=1}^{n} z_t\right) \delta^{n-1}$$

# Interpretation of discount factor

$$E(R_N) = (1-\delta) \sum_{n=1}^{\infty} \left( \sum_{t=1}^{n} r_t \right) \delta^{n-1}$$

$$= r_1$$
$$+ \delta r_1 + \delta r_2$$
$$+ \delta^2 r_1 + \delta^2 r_2 + \delta r_3$$
$$+ \ \cdots$$

$$\left( \sum_{k=0}^{\infty} \delta^k \right) r_1 + \delta \left( \sum_{k=0}^{\infty} \delta^k \right) r_2 + \cdots$$

$$= (1-\delta) \left[ \frac{1}{1-\delta} r_1 + \delta \left( \frac{1}{1-\delta} \right) r_2 + \cdots \right]$$

$$= r_1 + \delta r_2 + \delta^2 r_3 + \cdots = \sum_{t=1}^{\infty} \delta^{t-1} r_t.$$

# Example: Repeated Prisoner's dilemma

- Pay-off matrix for **stage game:**

|   | $C(oop)$ | $D(efect)$ |
|---|---|---|
| $C$ | 3, 3 | 1, 4 |
| $D$ | 4, 1 | 2, 2 |

- For **finite number** of repetitions:
  - <span style="color:red">Single Nash eq.:</span>   Play D-D for each repetition;
  - Can be proved using **backward induction**
- For **infinite number** of repetitions:
  - More interesting strategies, including <span style="color:blue">cooperation</span>
  - Examples: Tit-for-Tat, Grim Trigger

# Grim Trigger Strategy for Repeated Prisoner's Dilemma

- **Grim Trigger and Tit-for-Tat strategy**
  Start by cooperating . . .
    - GT: continue cooperating, until someone defects; from then onwards, always defect!
    - TfT: from then onwards, copy last move of opponent.

- If both parties play GT, is it rational to defect?
  Consider player 1:
    - Utility for continued **cooperation:**

    $$u_1(C) = 3 + 3\delta + 3\delta^2 + \ldots = \frac{3}{1-\delta}$$

    - Utility for **defection**:

    $$u_1(D) = 4 + 2\delta + 2\delta^2 + \ldots = 4 + 2\frac{\delta}{1-\delta}$$

- Continued **cooperation is rational** if $u_1(C) > u_1(D)$.

# Grim Trigger Strategy for Repeated Prisoner's Dilemma

- Continued **cooperation is rational** if $u_1(C) > u_1(D)$:

$$u_1(C) > u_1(D) \qquad \Longleftrightarrow \qquad \frac{3}{1-\delta} = 4 + 2\frac{\delta}{1-\delta}$$

$$\Longleftrightarrow \qquad \delta > 1/2.$$

<span style="color:red">sufficiently patient??</span>

- **Interpretation:** <mark>If the players are sufficiently patient (i.e. future rewards are sufficiently valuable) then it's rational to cooperate.</mark>