Introduction to Game Theory

Basic Concepts

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Reading

- **Recommended**
  - Shoham and Leyton-Brown: Chapter 3, sections 3.1-3.3
- **Optional**
  - William Spaniel: Game Theory 101: The Complete Textbook (paperback): *Very accessible and clear, teaching through examples. Accompanying YouTube channel.*
  - N. Nisan, T. Roughgarden, E. Tardos, V. Vazirani: Algorithmic Game Theory. Cambridge UP. *Solid, mathematical. Advanced.*
  - A. Dixit, B. Nalebuff: Thinking Strategically. Norton. *Lots of context and background. Interesting and non-technical.*

Overview

Table of Contents

Games people play ...

Games as entertainment or challenge

- Board games: chess, backgammon, GO, etc.
- Card games
- Rock-paper-scissors, etc.
- ...

Games as models

- War games, simulations, etc.
- More generally: many **interactions in society or nature** share the same ingredients!
- *All models are wrong, but some are useful!* (G. Box)

Game Theory: Science of Strategic Thinking

- Originally, tool in **economics**
  - 1944: von Neumann & Morgenstern, *Theory of Games and Economic Behavior,*
  - Nobel Prize in Economics was awared for GT-work in the years: 1994 (J. Harsanyi, J.Nash and R. Selten), 2005 (R. Aumann and T. Schelling), 2007 (L. Hurwicz, E. Maskin and R. Myerson), 2012 (A. Roth and L. Shapley), 2014 (J. Tirole), 2020 (P.R. Milgrom and R.B. Wilson)
- Game theory provides a level of abstraction appropriate to study a wide range of **socio-economic, political** and even **biological** phenomena.

Ingredients of interesting games

- Players: You against one or more opponent(s)
  - Opponent: other agents, other version of yourself, nature, lady luck, etc.
- Rules determine which actions can be taken, and what the corresponding pay-offs or utilities are;
- Maximize your pay-off: Everyone wants to win!
- **Competition and collaboration:** individuals or teams (non-cooperative and cooperative GT)

Cooperative versus Non-Cooperative Games

- **Non-Cooperative**
  - Selfish individuals, only consider their own interest;
  - Do **not coordinate** their actions in groups
    - Coordination might happen as "accident" of selfish behaviour (*emergent property*)
  - Agreements need to be self-enforcing (no contracts!)

- **Cooperative:** Binding commitments (*"contracts"*) allow groups of players to coordinate their actions
  - Non-transferable utility: pay-off of each individual increases!
    E.g. Stable marriage problem;
  - Transferable utility: need to find a fair way to divide the additional value (e.g. money) generated by collaboration:
    E.g. Shapley value

Simultaneous vs. Sequential Games

- **Simultaneous games:** players make their moves simultaneously, i.e. without knowing what the other players will do!
  - Rock-paper-scissors
  - Sealed bid auctions
- **Sequential games:** Sequence of successive moves by players who can see each other's moves:
  - Chess
  - Card games
  - Open cry auctions

Encoding utilities of actions: Matrix form

Simultaneous game: two players, finite number of actions

	Player 2 chooses *Left*	Player 2 chooses *Right*
Player 1 chooses *Up*	4, 3	−1, −1
Player 1 chooses *Down*	0, 0	3, 4

Normal form or payoff matrix of a 2-player, 2-strategy game

Encoding utilities of actions: Extensive Form

Sequential game: Decision tree

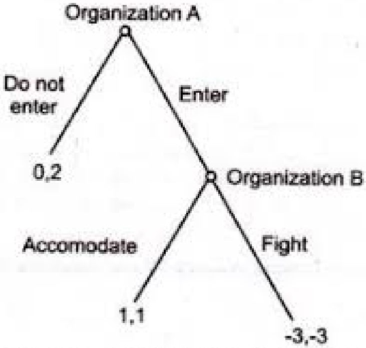


Figure-2: Extensive Form Games

Encoding utilities of actions: General case

- Utility is a function of the joint action of players:
- Ultimatum game (one shot)
  - Player A can choose any fraction $0 \leq x \leq 1$ for himself, and offer the rest $(1 - x)$ to player B;
  - If player B accepts this offer, then that is the outcome. If he rejects it, then both get zero.

$$u_A(x) = \begin{cases} x \\ 0 \end{cases} \quad \text{and} \quad u_B(x) = \begin{cases} 1 - x & \text{if B accepts} \\ 0 & \text{if B rejects} \end{cases}$$

Important applicability issue: Rationality vs Emotionality

Table of Contents

Economics: Cournot's Duopoly Model

- Two companies make an interchangeable product (e.g. bottled water).
- Without knowing the other company's strategy (i.e. simultaneously), both need to determine the quantity they will produce, say q_1 and q_2 respectively.
- The unit price p of the product in the market depends on the total produced quantity $q_1 + q_2$; specifically

$$p(q_1, q_2) = \alpha - \beta(q_1 + q_2) \quad (\alpha, \beta > 0).$$

- Firm $i = 1, 2$ can produce the item at a unit-cost c_i.
- What quantity q_i should each company produce in order to maximise its profit?

Children dividing pie

Illustrates various GT concepts!

- Selfish agents, but rational (cutter knows that chooser will pick largest piece)
- Satisfactory solution, notwithstanding selfish behaviour
- Solution: minimax and maximin
  - Cutter: mitigating the worst result that chooser can enforce;
  - Chooser: maximizing his pay-off
- **Mechanism design:**
  - How to set up **rules of game** so that **selfish individual behaviour** will lead to **social welfare**?

Prisoners' dilemma (RAND, 1950s)

Two suspects in a crime are put into separate cells. The police officer tells them: *Currently you're charged with trespassing which carries a jail sentence of one month. I know you were planning a robbery though, but cannot prove it – I need your testimony. If you confess and cooperate, I will drop the charges against you, but your partner will be charged to the fullest extent of the law: 12 months in jail. I'm offering the same deal to him. If you both confess, your individual testimony is less valuable, and you will get 8 months each.*

	quiet	confess
quiet	$-1, -1$	$-12, 0$
confess	$0, -12$	$-8, -8$

What should the suspects do?

Penalty kicks

- Penalty-kick game
  - Soccer penalties have been studied extensively

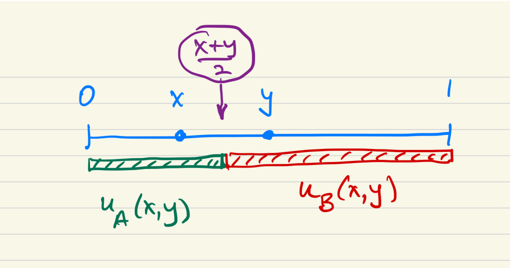
	Defend left	Defend right
Left	0.58, -0.58	0.95, -0.95
Right	0.93, -0.93	0.70, -0.70

Ice cream time!

Ice cream time! (Hotelling's game)

- **Two players** but **continuous (infinite) action space:**
  - each player can choose any position between 0 and 1.
  - Player A chooses x, player B chooses y $(0 \leq x < y \leq 1)$;
- **Utility**

$$
\begin{aligned}
u_1(x, y) &= x + \frac{y-x}{2} = \frac{x+y}{2} \\
u_2(x, y) &= 1 - y + \frac{y-x}{2} = 1 - \frac{x+y}{2}
\end{aligned}
$$



=> x+y=1 => 0.5 for each person

Partnership game

- Two students work on project, and divide profits evenly (50-50).
- Each student must decide how much effort (e.g. hours) he's contributing to the project. Hence, student i chooses action s_i = amount of effort. (assume $0 \leq s_1, s_2 \leq 4$);
- Project generates reward: $4(s_1 + s_2 + bs_1 s_2)$ (with $0 < b < 1$)
- The cost of the work to student i equals s_i^2
- Pay-off for individual student:

$$u_i(s_1, s_2) = \frac{4}{2}(s_1 + s_2 + bs_1 s_2) - s_i^2$$

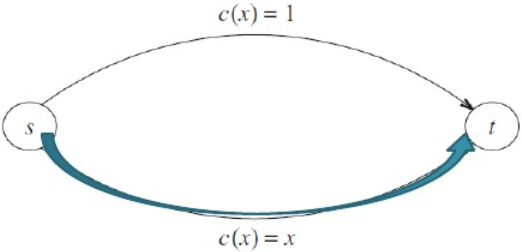
Selfish routing in congestion games

Congestion games:

- **Context:** routing in network
- Single shot, n-player game
- Player chooses some resource (route) from **set of resources**;
- **Congestion:** Cost of resource depends on number of agents selecting this resource;

Selfish routing in congestion games

- Pigou's example
  - Selfish players
  - Each player wants to minimize cost



- Cost at equilibrium (1) exceeds optimal cost (3/4)
- **Price of anarchy:** 4/3

无政府状态

Tragedy of the Commons

悲剧

- n players sharing some common resource (of total size 1)
  - E.g., village green, bandwidth in network, etc.
- Each player i would like to have a big share ($0 \leq x_i \leq 1$)!
- However, each player's utility (pay-off) depends on what the others do:

$$u_i(x_i, x_{-i}) = \begin{cases} x_i \left(1 - \sum_{j=1}^{n} x_j\right) & \text{if } \sum_j x_j < 1 \\ 0 & \text{otherwise} \end{cases}$$

- Is there a stable strategy?

Ultimatum game with impatient players
最后通牒

- Kids get a box of icecream and can divide it between themselves;
- They can have all of it as long as they agree on the division;
- If they fail to agree, their parents take away all the ice cream (conflict deal);
- It's a hot day and the icecream is melting! The longer they argue, the less icecream there is left!

Overview of Topics in Game Theory Course

- Non-cooperative games
  - Matrix games (2 players, finite action sets)
  - Sequential games, e.g. bargaining
- Cooperative (coalitional) games; 联合的
  - Shapley value;
- Mechanism design (inverse game theory)
  - Vickrey auction

Table of Contents

Game theory and strategic agents

- **Game theory** studies **multiagent decision problems**, that is, problems in which **independent decision-makers interact**.
- What each agent does has an effect on the other agents in the group (through utility);
- **Assumptions:**
  - agents have **preferences** encoded in **utility function (pay-off)**
  - **self-interest:** agents strive to maximize their own pay-off;
  - **rational behaviour:** agents reason about the actions of other agents and decide rationally.

Games: Normal-form vs. extensive-form

Models for games for which actions are:

- **Sequential:** player moves after observing action of other players;
- **Repeated:** special case of sequential game with (possibly) many iterations;

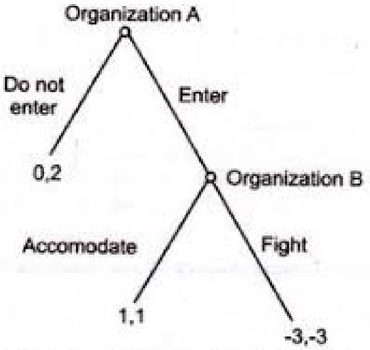


Figure-2: Extensive Form Games

Normal-form Games (Matrix Games)

- **Players:**
  - make **simultaneous moves** and receive **immediate payoffs**;
  - **payoffs** are specified for the combinations of actions played.
- **Payoff matrix:**
  - Specifies for given action combination $\mathrm{a} = (a_1, a_2, \ldots, a_n)$ the corresponding utility (pay-off) $u_i(\mathrm{a})$ for player $i = 1 \ldots n$

	Player 2 chooses *Left*	Player 2 chooses *Right*
Player 1 chooses *Up*	4, 3	−1, −1
Player 1 chooses *Down*	0, 0	3, 4

Normal form or payoff matrix of a 2-player, 2-strategy game

A graphical representation: matrix games

In the special case of *two* agents, a strategic game can be graphically represented by a **payoff matrix**, for example:

	left	centre	right
up	$1,0$	$1,2$	$0,1$
down	$0,3$	$0,1$	$2,0$

- Rows correspond to actions of agent 1 and columns to actions of agent 2. Here: $A_1 = \{\text{up}, \text{down}\}$, $A_2 = \{\text{left}, \text{centre}, \text{right}\}$.
- Each entry contains the payoffs (u_1, u_2) of the two agents for each possible joint action. For example, $a = (\text{down}, \text{centre})$ gives $(u_1, u_2) = (0, 1)$.

Formal definition of normal-form game

Norm-form, also konwn as strategic form, is the most familiar representation of strategic interaction in game theory, in which the state of the world only dependes on the players' combined actions - Multiagent System Book.

A n-person **normal-form game** is a tuple (N, A, u):

- N is a set of n **players (agents)**
- **Actions or Strategies** $A = A_1 \times A_2 \times \ldots \times A_n$ where each A_i is the set of actions available to agent i, i.e. set of allowable moves player i can make.
  An A-element a $= (a_1, a_2, \ldots, a_n)$ is called an **action profile**.
- **Pay-off or utility function:** u : $A \longrightarrow \mathbb{R}^n$ where u $= (u_1, u_2, \ldots, u_n)$ and each $u_i : A \longrightarrow \mathbb{R}$ is the corresponding utility function for player i. Notice, payoff u_i(a) for *each* agent depends on the *joint actions* of all agents.

Utility functions

von Neumann and Morgenstern, 1944

If there exists a preference relation $\succcurlyeq$ on the outcomes of a game that satisfies a number of "natural conditions" (completeness, transitivity, substituability, decomposability, monotonicity and continuity), then there exists a function $u : \mathcal{O} \longrightarrow \mathbb{R}$ such that:

- $u(o_1) \geq u(o_2) \quad \text{iff} \quad o_1 \succcurlyeq o_2$
- $u(\{(o_1 : p_1), (o_2 : p_2), \ldots, (o_n : p_n)\}) = \sum_{i=1}^{n} p_i u(o_i)$

Examples of competitive and cooperative (matrix) games

A strategic game can model a variety of situations where agents interact. These are two well-known cases:

Matching Pennies

	head	tail
head	$1, -1$	$-1, \ 1$
tail	$-1, \ 1$	$1, -1$

In a **strictly competitive** or **zero-sum** game, $\sum_i u_i(\text{a}) = 0$ for all a (anti-coordination game).

Going to the Movies

	action	comedy
action	$1, 1$	$0, 0$
comedy	$0, 0$	$1, 1$

In a **strictly cooperative** game, a type of **coordination** game, $u_i(\text{a}) = u_j(\text{a})$ for all i, j, a.

More examples

Chicken

	swerve	straight
swerve	$0, 0$	$-1, 1$
straight	$1, -1$	$-5, -5$

Stag Hunt

	stag	hare
stag	$2, 2$	$0, 1$
hare	$1, 0$	$1, 1$

Battles of the Sexes 1

	action	comedy
action	$3, 2$	$0, 0$
comedy	$0, 0$	$2, 3$

Battle of the Sexes 2

	action	comedy
action	$3, 2$	$2, 1$
comedy	$0, 0$	$2, 3$

Continuous action space

Hotelling's Game (ice-cream time):

- **Two players**
- **Continuous (infinite) action space:**
  - each player can choose any position between 0 and 1.
  - Assume first player choses x while second player chooses y where for simplicity: $0 \leq x < y \leq 1$;
- **Utility**

$$
\begin{aligned}
u_1(x, y) &= x + \frac{y - x}{2} = \frac{x + y}{2} \\
u_2(x, y) &= 1 - y + \frac{y - x}{2} = 1 - \frac{x + y}{2}
\end{aligned}
$$

Strategies

- A player's **strategy** is the **algorithm** that determines the action the player will take at **any stage of the game.**
- **Pure strategy:** Select single action and play it.
- **Mixed strategy:** Select single action according to probability distribution and play it. Rationale? Think of *matching pennies*:

	Heads	Tails
Heads	1, -1	-1 , 1
Tails	-1, 1	1, -1

Mixed strategy: using **randomness** not to be *outsmart-ed* by opponent.

- **Strategy profile:** $s = (s_1, s_2, \ldots, s_n)$, i.e. one specified strategy for each agent.

Expected Utility for Mixed Strategies

- **Pure strategy:** (Expected) utility u_i for agent i selecting action a_i equals $u_i(a_i, a_{-i})$.
- **Mixed strategy:** Agent i plays strategy s_i which is a probability distribution over k possible actions:

$$s_i = \{(a_{i1}, p_{i1}), (a_{i2}, p_{i2}), \ldots, (a_{ik}, p_{ik})\} \qquad \text{(where } p_k = P(a_k))$$

- **Expected utility** for mixed strategies:
  - agent i playing mixed strategy $s_i = \{(a_{i1}, p_{i1}) \ldots (a_{in}, p_{in})\}$
  - agent j playing mixed strategy $s_j = \{(a_{j1}, p_{j1}) \ldots (a_{jm}, p_{jm})\}$

$$EU_i(s_i, s_j) = \sum_{k=1}^{n} \sum_{\ell=1}^{m} u_i(a_{ik}, a_{j\ell}) p_{ik} p_{j\ell}$$

Expected Utility for Mixed Strategies

B

		b_1 (q)	b_2 $(1-q)$
A	a_1 (p)	α, α' $\boxed{pq}$	β, β' $\boxed{p(1-q)}$
	a_2 $(1-p)$	γ, γ' $\boxed{(1-p)q}$	δ, δ' $\boxed{(1-p)(1-q)}$

Strategies

$S_A = \{(a_1, p), (a_2, 1-p)\}$

$S_B = \{(b_1, q), (b_2, 1-q)\}$

$$EU_A = \alpha pq + \beta p(1-q) + \gamma(1-p)q + \delta(1-p)(1-q)$$

$$EU_B = \alpha' pq + \beta' p(1-q) + \gamma'(1-p)q + \delta'(1-p)(1-q)$$

Expected utility for mixed strategies

Expected utility

		2	
		L	R
1	U	1, 2	0, 2
	D	2, 4	3, 1

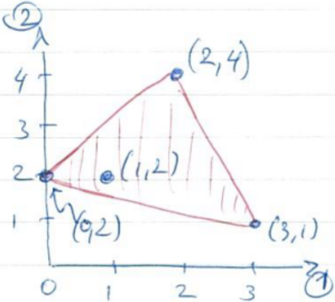


Table of Contents

Analyzing games: Solution concepts for games

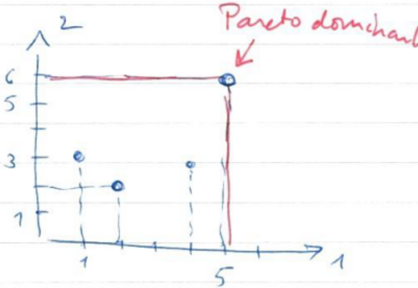
Consider point of view of a **single (self-interested) agent:**

- Given all game information: **what strategy** should he adopt?
- Complicated: depends on **actions of other agents**!
  Make the most of it . . .
- From **optimality** to **best response** (equilibrium)
- **From (weak) optimality . . .**
  - Pareto Optimality 帕累托最优
  - Best Response (BR) given the actions of the other agents;
  - Iterated elimination of strictly dominated strategies (IESDS)
- **. . . to Equilibrium**
  - Maximin and minimax strategyies
  - Nash equilibrium (John Nash, 1950): Next section

Pareto optimality

Pareto Dominance

① \ ②	L	R
U	2, 2	4, 3
D	1, 3	5, 6



Pareto optimality

Pareto optimality

Pareto optimality is a **solution property** (not solution concept itself)

A joint action/strategy profile a is **Pareto dominated** by another joint action a′ if $u_i(\mathrm{a}') \geq u_i(\mathrm{a})$ for all agents i and $u_j(\mathrm{a}') > u_j(\mathrm{a})$ for some j.

A joint action/strategy profile a is **Pareto optimal** if there is no other joint action a′ that Pareto dominates it.

Pareto dominance defines a **partial ordering** over strategy profiles.

Best response (1)

- **Best response** from agent *i*'s point of view:
- Let's assume that we know the strategies of all the other agents, i.e. s_{-i} is known;
- Agent *i*'s **best response** to strategy profile s_{-i} is (a possibly mixed) strategy $s_i^* \in S_i$ such that

$$\forall s_i \in S_i : \quad u_i(s_i^*, \mathrm{s}_{-i}) \geq u_i(s_i, \mathrm{s}_{-i}).$$

- Of course, . . . in a realistic setting we don't know the strategies of the other agents!
  Not solution concept, but **essential ingredient** for Nash eq.!

Best response (2)

Chicken

	swerve	straight
swerve	$0, 0$	$-1, 1$
straight	$1, -1$	$-5, -5$

Stag Hunt

	stag	hare
stag	$2, 2$	$0, 1$
hare	$1, 0$	$1, 1$

- Chicken:

$$BR_1(a_2 = \text{swerve}) = \text{straight} \qquad BR_1(a_2 = \text{straight}) = \text{swerve}$$

- Stag hunt:

$$BR_1(a_2 = \text{stag}) = \text{stag} \qquad BR_1(a_2 = \text{hare}) = \text{hare}$$

Best response (3)

- Best response **not necessarily unique**!
- When the best response includes two (or more) actions, then the agent must be **indifferent** among them!
- In fact: **any mixture** of these actions would also be a best response (mixed) strategy.
- Indeed,
  - If a_{i1} and a_{i2} are best both best response actions to s_{-i}, then $u_i(a_{i1}, \mathsf{s}_{-i}) = u_i(a_{i2}, \mathsf{s}_{-i}) =: u_i^*$.
  - Then, for any mixed strategy $s_i = \{(a_{i1}, p_1), (a_{i2}, p_2)\}$:

$$u_i(s_i, \mathsf{s}_{-i}) = p_1 u_i(a_{i1}, \mathsf{s}_{-i}) + p_2 u_i(a_{i2}, \mathsf{s}_{-i}) \equiv u_i^*.$$

Best response(4): Continuous state space

Partnership game (two players)

- **Actions:** choice of individual contributions to joint project

$$0 \leq x, y \leq 4$$

- **Utilities:** *utility* = *profit* − *cost*

$$\begin{cases} u_1(x,y) = 2(x+y+bxy) - x^2 & (0 \leq b < 1) \\ u_2(x,y) = 2(x+y+bxy) - y^2 \end{cases}$$

- Utility is quadratic (high input is very costly);

Best response(5): Continuous state space

Partnership game (two players): continued

- **Best response:** For a given input x of player 1, what input y of player 2 maximizes the latter's utility ($u_2(x, y)$)?
- **Finding the maximum utility $u_2(x, y)$ for given x:**

$$\frac{\partial u_2}{\partial y} = 2(1 + bx) - 2y = 0$$

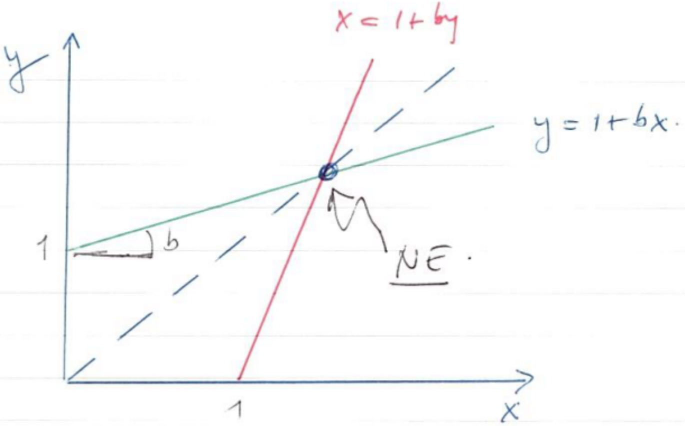
- **Best response solution:**

$$y^* \equiv BR_2(x) = 1 + bx$$

- **Similarly:**

$$x^* \equiv BR_1(y) = 1 + by$$

Best response (6): Continuous state space



Domination for strategies

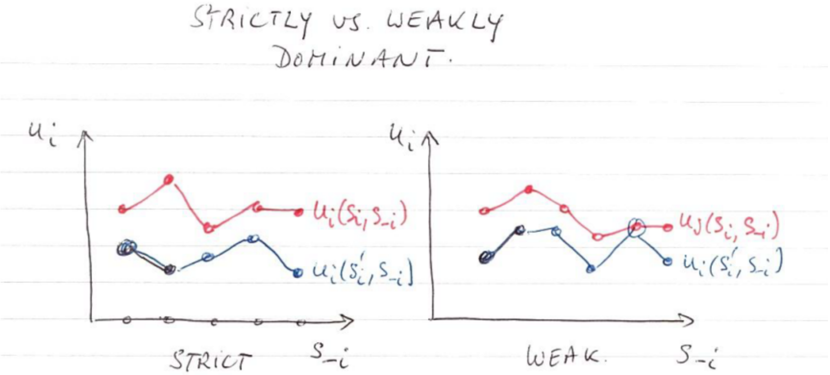
Let s_i and s'_i be two strategies for player i, and S_{-i} set of all strategy profiles for the other players:

- s_i **strictly dominates** s'_i if

$$u_i(s_i, s_{-i}) > u_i(s'_i, s_{-i}) \qquad \forall s_{-i} \in S_{-i}$$

- s_i **weakly dominates** s'_i if
  1. $u_i(s_i, s_{-i}) \geq u_i(s'_i, s_{-i}) \qquad \forall s_{-i} \in S_{-i}$, and
  2. $u_i(s_i, s_j) > u_i(s'_i, s_j)$ for at least one $s_j \in S_{-i}$

Strict vs. weak dominance



Dominant and dominated strategies

- **(Strictly/Weakly) Dominant Strategy:** (strictly/weakly) dominates every other strategy of the agent;
- **Strictly/Weakly) Dominanted Strategy:** is (strictly/weakly) dominated by at least one of the agent;
- A strictly dominated strategy will never be the best response to anything!
- For a dominating strategy, we don't have to worry what the opponents are going to do!
- Dominance plays important role in **mechanism design.**

Example Dominated Strategies: Prisoner's Dilemma

	Quiet	Confess
Quiet	$-1, -1$	$-12, 0$
Confess	$0, -12$	$-8, -8$

Prisoner's Dilemma

- *Quiet* is a strictly dominated strategy for both players, hence can be **eliminated**.
- Players will therefore both play confess, yielding pay-off $(-8, -8)$.

Iterated elimination of strictly dominated strategies (IESDS)

IESDS (a.k.a. What NOT to do?) is based on the following assumptions:

- It is **common knowledge** that all agents are rational.
- Rational agents **never** play strictly dominated actions.
- Hence, strictly dominated actions can be **eliminated.**

	left	centre	right
up	13, 3	1, 4	7, 3
middle	4, 1	3, 3	6, 2
down	−1, 9	2, 8	8, −1

What would IESDS predict in this game?

Final result: middle <-> centre

Iterated elimination of strictly dominated strategies (2)

Centre **strictly dominates** *right*. Row player knows that column player will never play the dominated action *right*. Hence he can eliminate that action and only needs to consider the simpler game:

	left	centre
up	13, 3	1, 4
middle	4, 1	3, 3
down	-1, 9	2, 8

For the row player, action *middle* strictly dominates *down*; hence eliminate! We are left with the simpler game where *centre* dominates *left*:

	left	centre
up	13, 3	1, 4
middle	4, 1	**3, 3**

Cournot Duopoly (discrete version)

Unit production cost: $c = 1$;
$Q_A(Q_B) =$ quantity produced by A (B)
P= Market price (per unit) : $P = 12 - 2(Q_A + Q_B)$
Pay-off:

$$u_A(A2, B3) = Q_A(P(Q_A, Q_B) - c) = 2(P(2,3) - c) = 2(12 - 2 \cdot 5 - 1) = 2$$

	$B0$	$B1$	$B2$	$B3$	$B4$	$B5$
$A0$	$0,0$	$0,9$	$0,14$	$0,15$	$0,12$	$0,5$
$A1$	$9,0$	$7,7$	$5,10$	$3,9$	$1,4$	$-1,-5$
$A2$	$14,0$	$10,5$	$6,6$	$2,3$	$-2,-4$	$-2,-5$
$A3$	$15,0$	$9,3$	$3,2$	$-3,-3$	$-3,-4$	$-3,-5$
$A4$	$12,0$	$4,1$	$-4,-2$	$-4,-3$	$-4,-4$	$-4,-5$
$A5$	$5,0$	$-5,-1$	$-5,-2$	$-5,-3$	$-5,-4$	$-5,-5$

Cournot Duopoly (discrete version)

1. A3 (B3) strictly dominates A5 (B5), eliminate A5/B5
2. A3 (B3) strictly dominates A4 (B4), eliminate A4/B4
3. A1 (B1) strictly dominates A0 (B0), eliminate A0/B0

	$B1$	$B2$	$B3$
$A1$	$7, 7$	$5, 10$	$3, 9$
$A2$	$10, 5$	$6, 6$	$2, 3$
$A3$	$9, 3$	$3, 2$	$-3, -3$

4. A2(B2) strictly dominates A3 (B3), eliminate A3/B3
5. A2(B2) strictly dominates A1 (B1), resulting in strategy profile (A2,B2) with utility $(6, 6)$;
6. Notice: **not Pareto-optimal!** (dominated by (A1,B1), with utilitity $(7, 7)$

Iterated elimination of strictly dominated actions (3)

More challenging example:

- Rules of the game:
  - Game played in large group (e.g. auditorium)
  - Each player picks number between 1 and 100.
  - Collect all numbers and compute the mean.
  - Winner is player whose number was closest to 1/2 of mean.
- What strategy should you use when picking your number?
- **Bounded rationality** vs. "homo economicus"! Rationality is bounded by limits to our resources (Simon, 1982):
  - cognitive capacity, available information, time, etc.

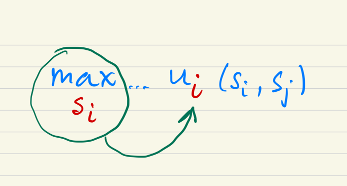
有限理性 VS 经济人

Best response and IESDS

- Strictly dominated strategies are **never a best response;**
- **Church-Rosser property:** Order of elimination does not matter for IESDS (strict dominance)!
- Eliminating **weakly dominated** strategies might be too drastic!

Maximin and Minimax Strategies

- Actual solution concepts: allow an agent to choose a strategy with specific guarantees/properties;
- Focus on 2-player game;
- Most natural interpretation for (2-player) zero-sum games : *maximising own payoff = minimising opponent's payoff*
- General sum games: assume opponent is malicious 恶意的
  - . . . or threatening (e.g. in case of repeated game)
- We consider player i's point of view: (i.e. maximise over s_i)



Minimax Value against Punishment Strategy

Helpful interpretation: Dealing with a spy

- Player i knows that player j is vindictive and malicious ...
- ... but player i has spy in player j camp who informs him (JIT) of j's action (s_j);
- Player i will then play his best response yielding utility

$$BR_i(s_j) \longrightarrow \max_{s_i} u_i(s_i, s_j)$$

- However, player j suspects he has a mole in his camp and plays the action that minimises the pay-off of i's best response,
- This yields the minimax value for player i:

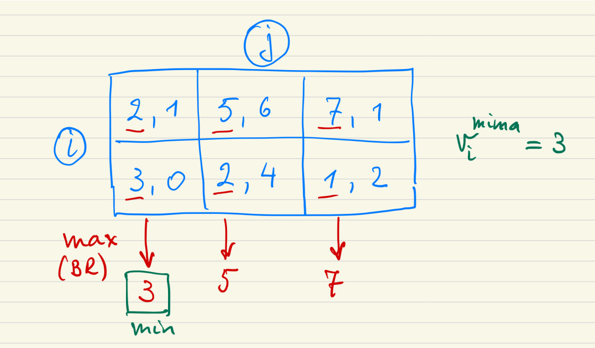
$$v_i^{mima} := \min_{s_j} \max_{s_i} u_i(s_i, s_j)$$

- This is worst pay-off player j can **force** on player i.

Minimax value for player i: Algorithm

1. Player i computes best response for each of j's strategies s_j: $BR_i(s_j) = \max_{s_i} u_i(s_i, s_j)$;
2. Then player j picks action to minimise player i best pay-off. This results in the minimax value for player i:

$$v_i^{mima} := \min_{s_j} \max_{s_i} u_i(s_i, s_j)$$



Recap: Minimax Value and Punishment Strategy

- **Minimax value** for agent i:
  - Given the strategy s_j of his opponent, agent i will play its **best response**, resulting in a pay-off:

$$\max_{s_i} u_i(s_i, s_j)$$

  - The opponent is aware of this and wants to *"punish"* i by *minimizing* this pay-off, yielding the minimax value for player i:

$$v_i^{mima} := \min_{s_j} \max_{s_i} u_i(s_i, s_j)$$

- The corresponding minimising strategy s_j^{mima} is called the minimax strategy for player j.
- If j plays his minimax strategy s_j^{mima}, then i cannot do better than v_i^{mima} (even if i plays best response $BR_i(s_j^{mima})$).

Maximin Value and Strategy (Safety Strategy)

Helpful interpretation: Dealing with a spy (2)

- Player i knows that player j is vindictive and malicious . . .
- Furthermore, player j has spy in player i's camp who informs him (JIT) of i's action (s_i);
- Player j will then play strategy to minimise i's pay-off resulting in pay-off for i who is playing s_i:

$$\min_{s_j} u_i(s_i, s_j)$$

- However, player i suspects he has a mole in his camp and plays the action that maximises his worst pay-off.
- This yields the maximin value for player i:

$$v_i{}^{mami} := \max_{s_i} \min_{s_j} u_i(s_i, s_j)$$

- This is the best pay-off player i can **guarantee** himself.

Maximin strategy (safety strategy): Algorithm

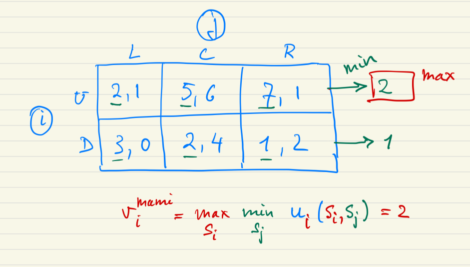
1. Player i computes for each of his actions the worst possible outcome:

$$s_i \longrightarrow \min_{s_j} u_i(s_i, s_j)$$

2. Next player i chooses action s_i to maximise his minimal (worst) pay-off:

$$v_i^{mami} := \max_{s_i} \min_{s_j} u_i(s_i, s_j)$$

Agent tries to maximise pay-off of worst possible outcome



Maximin Value and Strategy

Context: 2-player, general sum game

- **Maximin value** (or security level for WORST CASE) is the guaranteed minimal pay-off for agent i playing strategies in S_i:

$$v_i^{mami} := \max_{s_i \in S_i} \min_{s_j \in S_j} u_i(s_i, s_j)$$

- **Maximin strategy** for agent i maximizes his worst case pay-off:

$$s_i^{mami} = \arg \underbrace{\max_{s_i} \min_{s_j} u_i(s_i, s_j)}_{\text{security level}}$$

Maximin (safety) value and strategy

- **Why play maximin strategy?**
  - Pay-off guarantee! Highest pay-off agent i can guarantee himself irrespective of the actions taken by other agent(s).
  - Worst case analysis: assume that opponent is malicious!
  -

Example: both agents play maximin-strategy

Both players play maximin.

	L	CL	CR	R	min
U	3, 1	4, 6	8, 1	5, 2	$\boxed{3}$ max.
M	4, 2	7, 4	1, 5	4, 5	1
D	6, 2	1, 3	7, 0	0, 4	0
min	1	$\boxed{3}$ max	0	2	

$v_1^{mami} = 3$, $s_1^{mami} = U$

$v_2^{mami} = 3$, $s_2^{mami} = CL$

$\rightarrow$ $\boxed{u_1(U, CL) = 4 \quad u_2(U, CL) = 6}$

Minimax and Maximin Value for Player i

- In general:

$$v_i^{mami} = \max_{s_i} \min_{s_j} u_i(s_i, s_j) \quad \leq \quad \min_{s_j} \max_{s_i} u_i(s_i, s_j) = v_i^{mima}$$

"Your guaranteed pay-off is a lower bound for the worst your opponent can force onto you!"

Safety and Punishment Strategy

- **Player i's maximin strategy is safety strategy**
  - player i concerned about his own safety
  - strategy yields highest guaranteed outcome for player i
  - Viable solution algorithm.

- **Player i's minimax strategy is punishment strategy**
  - i's strategy is directed against player j
  - player i tries to minimize best (i.e. maximum) pay-off for j
  - Useful as threat (e.g. in repeated games);
  - i's maximin **strategy** gives rise to j's maximin **value**

Minimax and Maximin for Zero-Sum Games

- **Zero-sum game:** *"your loss is my gain!"*

$$u_i(s_i, s_j) = -u_j(s_i, s_j)$$

- For (finite) **zero-sum games**: **minimax equals maximin**.
  - But beware! Could be result of mixed strategies!
- **Maximin strategy**
  - maximize minimal gain for myself
- **Minimax strategy**
  - minimize maximal gain for opponent
  - minimize maximal loss for myself (zero-sum game)
- **Value of zero-sum game:** Common value of minimax and maximin

Minimax Theorem

Let A be the $n \times m$ pay-off matrix for the first player in a **2-player, finite, zero-sum game** (i.e. pay-off for player 2 equals $-A$). Furthermore, let $\mathrm{p} = (p_1, p_2, \ldots, p_n)$ and $\mathrm{q} = (q_1, q_2, \ldots, q_n)$ be probability distributions representing mixed strategies for player 1 and 2, respectively. Notice that the corresponding pay-off for player 1 is then given by:

$$V(\mathrm{p}, \mathrm{q}) = \mathrm{p}^T A \mathrm{q}$$

Minimax Theorem, Von Neumann, 1928 ???

Under the assumptions above, there exists probability vectors p^* and q^* such that

$$V(\mathrm{p}^*, \mathrm{q}^*) = \max_{\mathrm{p}} \min_{\mathrm{q}} V(\mathrm{p}, \mathrm{q}) = \min_{\mathrm{q}} \max_{\mathrm{p}} V(\mathrm{p}, \mathrm{q})$$

Intuition for Minimax Thm for 2p-zerosum game

Expected utility $u(p,q) = p^T V q$ is bi-linear function of p and q.

	q_1	q_2	q_3
p_1	3, −3	2, −2	6, −6
p_2	−1, 1	1, −1	−7, 7
p_3	0, 0	−5, 5	0, 0

$$\longrightarrow \quad V = \begin{pmatrix} 3 & 2 & 6 \\ -1 & 1 & -7 \\ 0 & -5 & 0 \end{pmatrix}$$

$$\boxed{EU_1 = p^T V q}$$

where $p = \begin{pmatrix} p_1 \\ p_2 \\ p_3 \end{pmatrix}$, $q = \begin{pmatrix} q_1 \\ q_2 \\ q_3 \end{pmatrix}$

Figure: Caption

Intuition for Minimax Thm for 2p-zerosum game

Equilibrium in simultaneous game, but ...

1. Suppose player 1 goes first:
   - If he picks p_0 then player 2 will try to minimise $u(p_0, q)$ resulting in $\min_q u(p_0, q)$;
   - Pl 1 knows this and will therefore pick p to maximise result:

$$\max_p \min_q u(p, q)$$

2. Suppose player 2 goes first:
   - If he picks q_0 then player 1 will try to maximise $u(p, q_0)$ resulting in $\max_p u(p, q_0)$;
   - Pl 2 knows this and will therefore pick q to minimise result:

$$\min_q \max_p u(p, q)$$

Minimax Theorem

Let A be the $n \times m$ pay-off matrix for the first player in a **2-player, finite, zero-sum game** (i.e. pay-off for player 2 equals $-A$). Furthermore, let $\mathrm{p} = (p_1, p_2, \ldots, p_n)$ and $\mathrm{q} = (q_1, q_2, \ldots, q_n)$ be probability distributions representing mixed strategies for player 1 and 2, respectively. Notice that the corresponding pay-off for player 1 is then given by:

$$V(\mathrm{p}, \mathrm{q}) = \mathrm{p}^T A \mathrm{q}$$

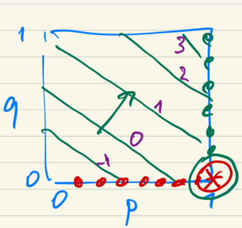
Minimax Theorem, Von Neumann, 1928

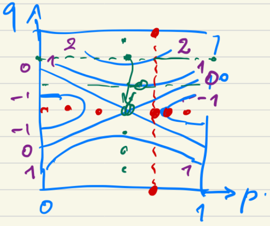
Under the assumptions above, there exists probability vectors p^* and q^* such that

$$V(\mathrm{p}^*, \mathrm{q}^*) = \max_{\mathrm{p}} \min_{\mathrm{q}} V(\mathrm{p}, \mathrm{q}) = \min_{\mathrm{q}} \max_{\mathrm{p}} V(\mathrm{p}, \mathrm{q})$$

Intuition for Minimax Thm for 2p-zerosum game

In 2-dim the $u(p,q)$ level curves look like:





$$\underbrace{\max_p \underbrace{\min_q u(p,q)}}$$

$$\underbrace{\max_p \min_q u(p,q)}$$

$$\underbrace{\min_q \max_p u(p,q)}$$

$$\underbrace{\min_q \max_p u(p,q)}$$

Regret minimisation

Regret for player 1

2

1	L	R	
T	10, ? R = 0	1, ? R = 3	max → $\boxed{3}$ min
B	2, ? R = 8	4, ? R = 0	→ 8

Note: $v_1^{mami} = 2$ (safety level)

Regret minimisation

- **Regret** (for agent i) is the difference between the actual and maximal pay-off for a given action (this depends on action s_j taken by opponent)

$$R_i(s_i, s_j) = \max_{s_i'} u_i(s_i', s_j) - u_i(s_i, s_j)$$

- For each action agent i can take, there is maximum regret (depending on what opponent j does):

$$R_i^{max}(s_i) = \max_{s_j} R_i(s_i, s_j)$$

- **Regret minimisation** (minimax regret): agent i picks action s_i that mimimises max regret:

$$s_i^{rm} := \arg\min_{s_i} R_i^{max}(s_i) = \arg\min_{s_i} \max_{s_j} R_i(s_i, s_j).$$

Table of Contents

Nash equilibrium (1)

- A **Nash equilibrium** (NE, 1950) is a solution concept based on *conditions* instead of an *algorithm*.
- A NE is a joint strategy profile s^* such that **for each agent** i the strategy s_i^* is a **best response** to s_{-i}^*;
- **Formally:** A strategy profile $s^* = (s_1^*, s_2^*, \ldots, s_n^*)$ is a **strict NE** if:

$$\forall \text{agents } i, \forall s_i' \neq s_i^* : u(s_i^*, s_{-i}^*) > u(s_i', s_{-i}^*).$$

- **Strict** ($>$) versus **weak** ($\geq$) NE
- **No Regret/Self-enforcing:** a (strict) NE is a stable strategy profile for which no agent has an incentive to **unilaterally deviate**;

Nash equilibrium: Computation of NE (pure strategy)

Find mutual best responses:

- **Battle of the Sexes** (two pure strategy NEs)

	action	comedy
action	$\underline{2}, \underline{1}$	$0, 0$
comedy	$0, 0$	$\underline{1}, \underline{2}$

- **Prisoner's dilemma** (single NE, not Pareto-optimal!)

	hush	confess
hush	$-1, -1$	$-12, \underline{0}$
confess	$\underline{0}, -12$	$\underline{-8}, \underline{-8}$

Nash's Theorem

Existence of Nash Equilibrium (Nash, 1950)

A **finite strategic game** (i.e. finite number of players and actions) always has **at least one Nash equilibrium** (allowing mixed strategies).

- A **pure** Nash equilibrium can be strict or weak;
- A **mixed** Nash equilibrium is necessarily weak;

Nash equilibrium: Nash's Theorem

- A **finite strategic game** is a game with a finite number of agents and a finite number of actions;
- A game may have **zero**, **one**, or **more pure-strategy NE**.
- If there's a **single** NE: natural solution concept, but might be sub-optimal!
- If there are **multiple NEs**: there might be no compelling reasons to pick a particular one; but ...
  - Schelling's focal points

Computation of NE

- **Pure NE** **for each agent** i the strategy s_i^* is a **best response** to s_{-i}^* (mutual best response);
  - Matrix games (discrete state/action) space;
  - Continuous action space
- **Mixed NE**: make opponent indifferent (matrix games only);

Given the player's strategy, the second player must be indifferent among all the actions, otherwise, the second player would be better off switching to a pure strategy action.

Nash equilibrium: Computation of **mixed** NE

Matching pennies (zero-sum game)

- **No pure strategy NE . . .**

	heads	tails
heads	$\underline{1}, -1$	$-1, \underline{1}$
tails	$-1, \underline{1}$	$\underline{1}, -1$

- . . . hence, at least one **mixed strategy NE!**
- **Intuitively** this is obvious; play each action with prob $= 1/2$:

$$s_1 = s_2 = \{(H, 1/2), (T, 1/2)\}$$

- **Expected utility (pay-off)**:

$$u_1(s_1, s_2) = \frac{1}{4}u_1(H, H) + \frac{1}{4}u_1(T, H) + \frac{1}{4}u_1(H, T) + \frac{1}{4}u_1(T, T) = 0.$$

Matching pennies: Computation of mixed NE

$$
\begin{array}{cc|c|c|}
 & & q & 1-q \\
 & & H & T \\
\hline
p & H & 1,-1 & -1,1 \\
\hline
1-p & T & -1,1 & 1,-1 \\
\hline
\end{array}
$$

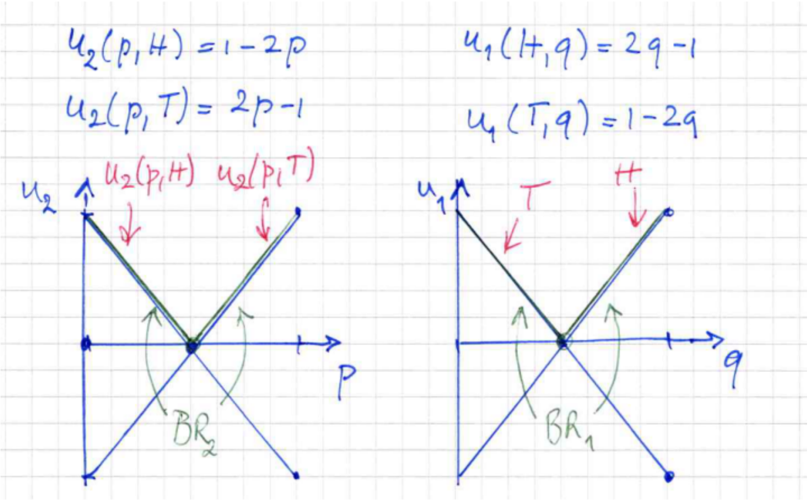
$$u_2(p,H) = (-1)p + 1\cdot(1-p) = 1-2p.$$

$$u_2(p,T) = p + (-1)(1-p) = 2p-1$$

$$u_1(H,q) = q-(1-q) = 2q-1$$

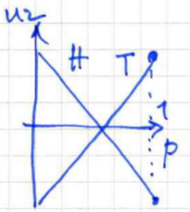
$$u_1(T,q) = -1\cdot q + (1-q) = 1-2q$$

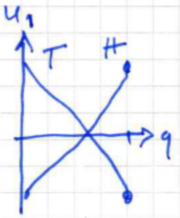
Matching pennies: Computation of best response

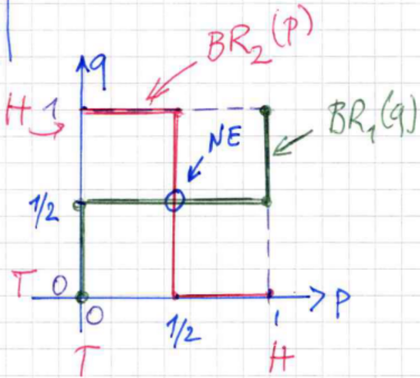


Matching pennies: Best response graph

		q H	$1-q$ T
p	H	1, −1	−1, 1
$1-p$	T	−1, 1	1, −1







Matching pennies: Mixed Nash Equilibrium

Nash equilibrium characteristics:

- At the intersection point ($p = 1/2, q = 1/2$), players are simultaneously playing best response to each other;
- No player can do strictly better by unilaterally deviating:
  - If player 2 keeps playing $q = 1/2$ then player 1's utility $u_1(p, q = 1/2)$ can be computed as follows:

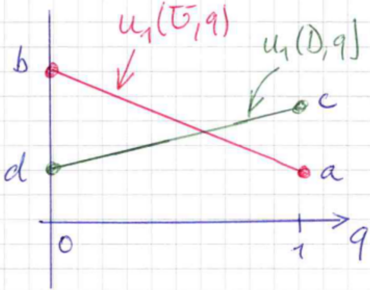
$$\begin{aligned} u_1(p, 1/2) &= (1 \cdot p + (-1) \cdot (1-p) + (-1) \cdot p + 1 \cdot (1-p)) \cdot \frac{1}{2} \\ &= 0 \end{aligned}$$

    Player 1 therefore has no incentive to change his strategy (change p) ? ? ?

  - Same consideration for player 2.

Computation of mixed Nash equilibrium

		q	$1-q$
		L	R
p	U	a, A	b, B
$1-p$	D	c, C	d, D



(Expected) utility for player 1:

$$u_1(p,q) = apq + bp(1-q) + c(1-p)q + d(1-p)(1-q)$$

Linear in p (for fixed q) and vice versa.

Computation of mixed Nash equilibrium

$$u_1(p,q) = apq + bp(1-q) + c(1-p)q + d(1-p)(1-q)$$

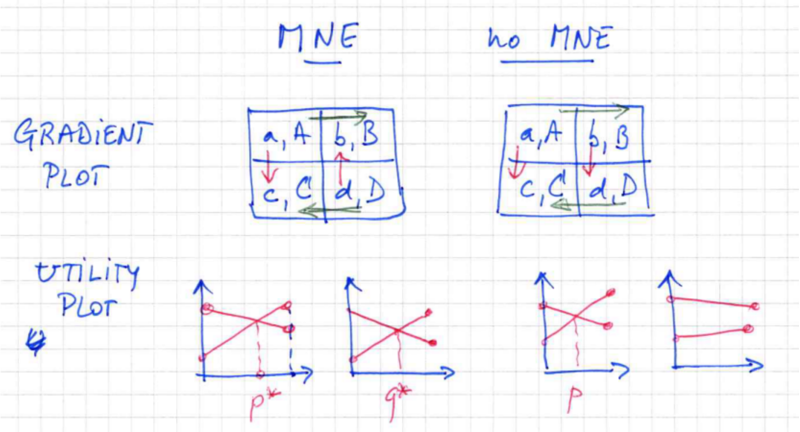
Player 1: no incentive to change.

$$0 = \frac{\partial u_1}{\partial p} = \underbrace{\left(aq + b(1-q)\right)}_{= u_1(U,q)} - \underbrace{\left(cq + d(1-q)\right)}_{= u_1(D,q)}$$

$$u_1(U,q) - u_1(D,q) = 0$$

Player 2 needs to pick q such that player 1 is indifferent btw U and D.

Some useful graphical representations



Nash equilibrium: Computation of mixed equilibrium

- **Battle of the Sexes** (two pure strategy NEs)
- Additional NE by **mixing pure strategies**?

$$s_1 = \{(A, p), (C, 1-p)\} \quad \text{and} \quad s_2 = \{(A, q), (C, 1-q)\}.$$

	Action (q)	Comedy ($1-q$)
Action (p)	2, 1	0, 0
Comedy ($1-p$)	0, 0	1, 2

- Determining the mixture parameters p and q
  - Ag 1 chooses p such that Ag 2 is **indifferent** btw actions A and C; if not Ag 2 would focus on the most lucrative option.

Condition for Ag 1: $u_2(s_1(p), A) = u_2(s_1(p), C)$

Nash equilibrium: Computation of mixed equilibrium

- **Battle of the Sexes** (mixed strategy NEs)

	Action (q)	Comedy ($1-q$)
Action (p)	2, 1	0, 0
Comedy ($1-p$)	0, 0	1, 2

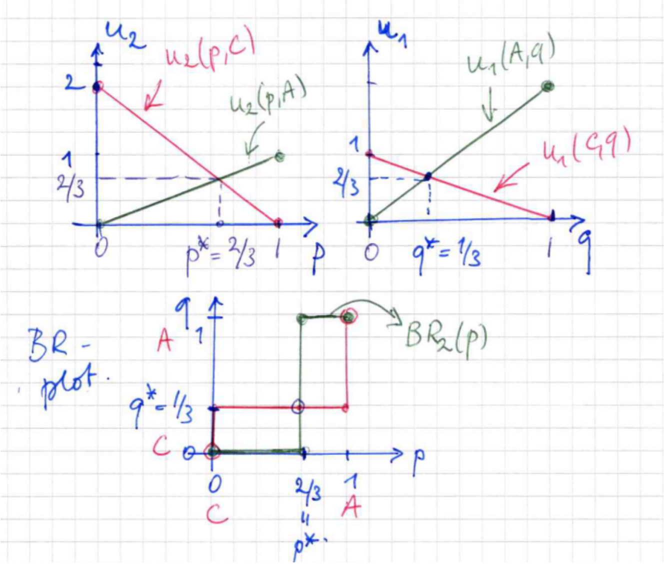
- Determining the mixture parameters p and q

Ag 1: $EU_2(s_1(p), A) = EU_2(s_1(p), C) \implies p = 2(1-p)$

Ag 2: $EU_1(A, s_2(q)) = EU_1(C, s_2(q)) \implies 2q = (1-q)$

Conclusion: $p = \frac{2}{3},\ q = \frac{1}{3}$ $\quad EU_1(s_1, s_2) = EU_2(s_1, s_2) = 2/3.$

Battle of the sexes: Mixed Nash Equilibrium



Nash equilibrium: Computation of mixed equilibrium

- **Prisoner's Dilemma** (does mixed strategy NE exist?)

	Quiet (q)	Confess ($1-q$)
Quiet (p)	$-1, -1$	$-12, 0$
Confess ($1-p$)	$0, -12$	$-8, -8$

- Determining the mixture parameters p and q

$$\text{Ag 1: } EU_2(s_1, Q) = EU_2(s_1, C) \Longrightarrow \quad -p - 12(1-p) = -8(1-p)$$

$$\text{Ag 2: } EU_1(Q, s_2) = EU_1(C, s_2) \Longrightarrow \quad -q - 12(1-q) = -8(1-q)$$

Conclusion: $p = q = \dfrac{4}{3}$ **impossible!**

Mixed NE for game with three actions

Rock-Paper-Scissors: no PURE NE!

		v	w	$1-(v+w)$
		R	P	S
p	R	$0,0$	$-1,1$	$1,-1$
q	P	$1,-1$	$0,0$	$-1,1$
$1-(p+q)$	S	$-1,1$	$1,-1$	$0,0$

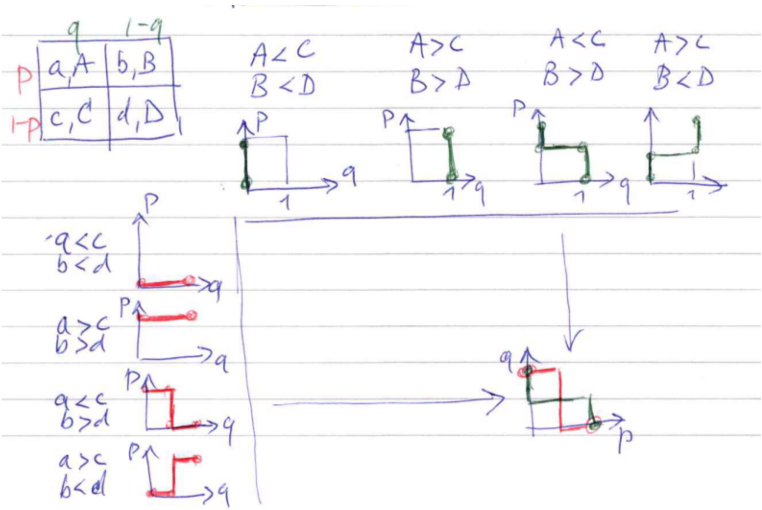
Determine p,q by insisting that player 2 is indifferent btw actions:

$$\begin{aligned} 0\cdot p+(-1)\cdot q+1\cdot(1-p-q) &= 1\cdot p+0\cdot q+(-1)\cdot(1-p-q) \\ 0\cdot p+(-1)\cdot q+1\cdot(1-p-q) &= -1\cdot p+1\cdot q+0\cdot(1-p-q) \end{aligned}$$

Result: $p=q=1/3$. Determine mixing parameters u,v similarly.

Table of Contents

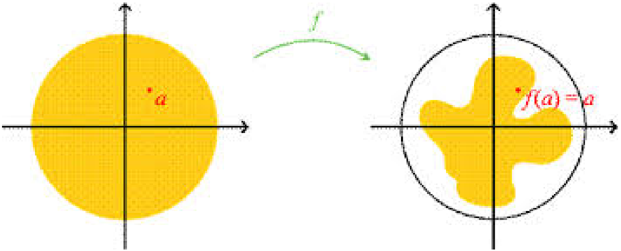
Nash equilibrium for 2 × 2 matrix game



Nash Theorem is based on Fixed-Point Theorem

Brouwer's Fixed Point Thm

Let $K \subset \mathbb{R}^n$ be a compact and convex, and $f : K \longrightarrow K$ continuous. Then f has a fix-point in K, i.e. there exists a $x_0 \in K : f(x_0) = x_0$.

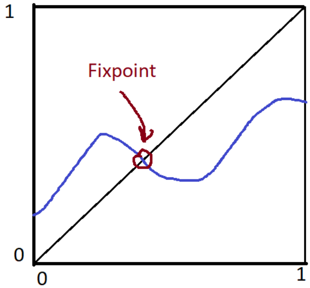


: **Non-constructive existence proof!**

Aside: Sperner's Lemma (1928) and fix-point theorems

Defintion: Fix-point

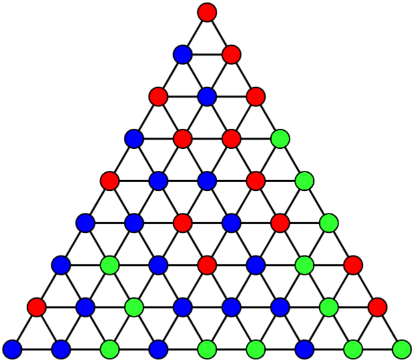
Point $a \in A$ is a **fix-point** for function $f : A \longrightarrow A$, iff $f(a) = a$.



f: [0,1] ---> [0,1] continuous

Sometimes(!), fixpoints can be computed using **function iteration.**

Aside: Sperner's Lemma (1928) and fix-point theorems

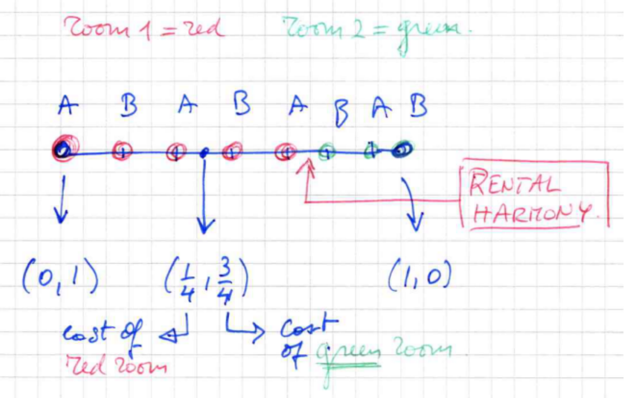


If in triangle:

- corner nodes have different colors
- nodes on outer edges have two possible colors (determined by corners)

Then there is a subtriangle with 3 differently colored corners.

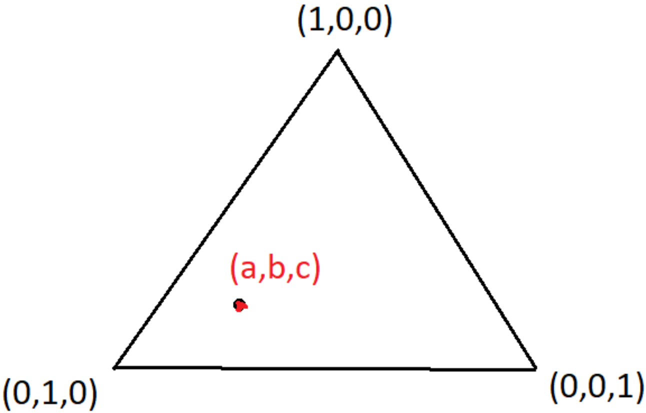
Sperner's lemma: Rental Harmony: Fair division of rent



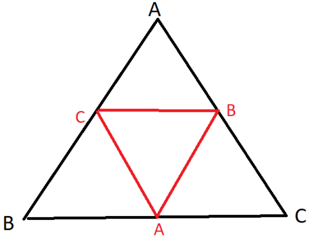
- Divide interval in segments by adding points
- Assign alternating decision makers (A and B)
- Decision maker decides which room he picks (for given rental division)

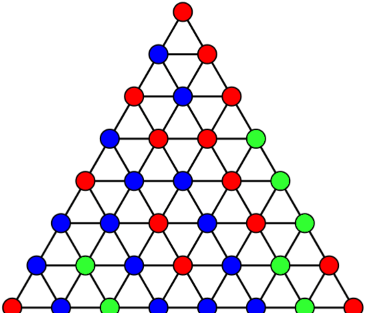
Sperner's lemma – Application 1: Rental Harmony

Flat sharing: three bedroom flat shared by tree friends but rooms are not of equal quality: how much should each contribute to rent?

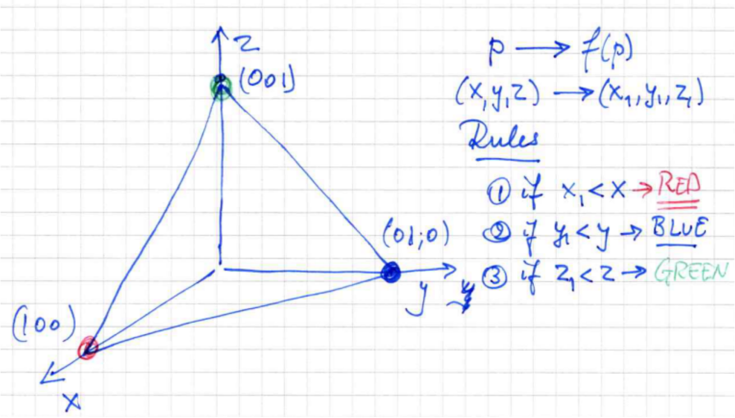


Sperner's lemma – Application 1: Rental Harmony





Sperner's lemma – Application 2: Proof of Brouwer's fixpoint thm



Youtube: Trefor Bazzett: A beautiful combinatorical proof of the Brouwer Fixed Point Theorem - Via Sperner's Lemma

Games with NO Nash Equilibrium

- Sealed bid auction: highest bid gets item, 2nd highest pays this amount!
- More generally: One can construct games without NE by making sure that either
  - state space is **not compact**
  - utility function is **not continuous**
- **Ex. for 2-player games with continuous state spaces:**
  - Non-compact: $u_i(x, y) = xy \qquad \textit{for} \quad 0 \leq x, y < 1$
  - Non-continuous: $0 \leq x, y \leq 1$ and

$$u_i(x, y) = \begin{cases} xy & \textit{if} \quad x, y < 1 \\ 0 & \textit{if} \quad x, y = 1 \end{cases}$$

Nash equilibrium: Computational aspects

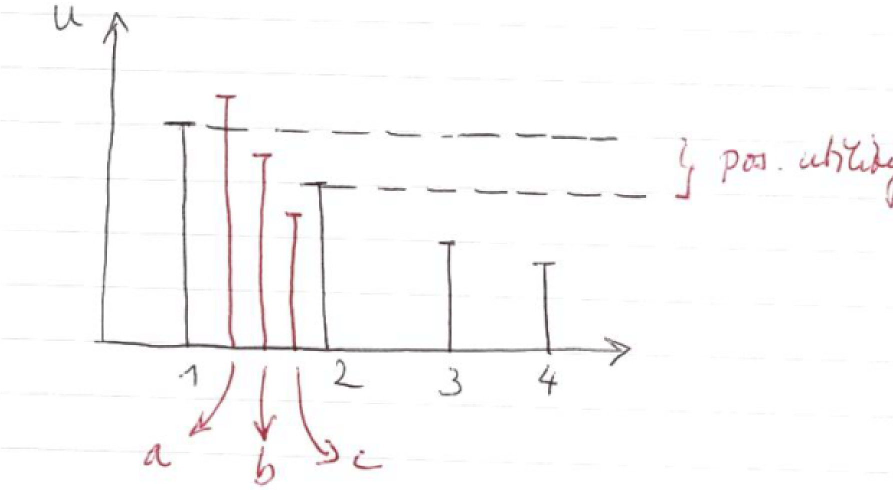
- Finding a Nash equilibrium for 2-player zero-sum games can be done efficiently by formulating a linear program. Notice that in this case: NE = minimax = maximin.
- Finding a Nash equilibrium is not known to be NP-complete because it is not a decision problem
- PPAD (polynomial parity argument, directed version) is a class describing problems for which a solution always exists
- Daskalakis, Goldberg, and Papadimitriou showed that finding a sample Nash equilibrium of a general-sum finite game with two or more players is PPAD-complete (i.e. "difficult!").

Table of Contents

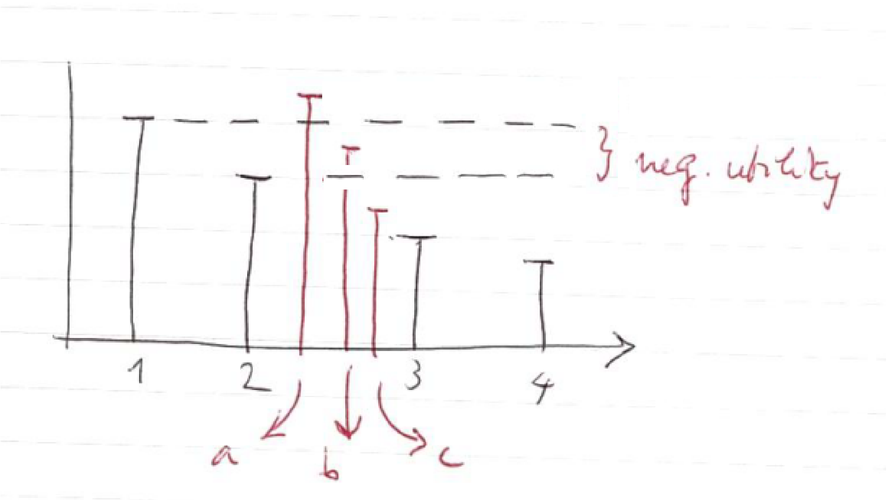
Vickrey auction: Second Price Auction

- **Vickrey Auction:**
  - n sealed bid auction for single item;
  - highest bid wins, but pays 2nd highest price;
- **Truth-telling** is (weakly) **dominant strategy;**
- NE: Neither winner nor loser(s) have incentive to deviate:
- **Winner:**
  - Higher: still winner, same price;
  - Lower: might lose, but if still winner, still paying 2nd price;
- **Loser:**
  - Higher: possibly winner, but at higher price;
  - Lower: still loser;
- Example of **mechanism design**.

Vickrey Auction: Alternatives for winner



Vickrey Auction: Alternatives for runner-up



Hawk or Dove

- Equilibria as a function of **exogenous pay-off variables**;
- **Exogenous variables** are imposed on the game (not by players);
- **Strategy: Hawk or dove:**
  - two parties are in conflict over some good of value $v > 0$;
  - Fighting over it comes at cost c
  - **Pay-off matrix:**

	Hawk	*Dove*
Hawk	$\frac{v}{2} - c, \frac{v}{2} - c$	$v, 0$
Dove	$0, v$	$\frac{v}{2}, \frac{v}{2}$

- Different outcomes depending on $c < v/2, c = v/2, c > v/2$

Investment game

- Many agents (n) but only two strategies;
- **Strategies:** Each agent can either not invest (N) or invest 10 euro (I).
- **Pay-offs:**
  - N: No investment: pay-off $= 0$;
  - I: Invest 10 Euro:

$$\text{net pay-off (in Euro)} = \begin{cases} 5 & \text{if at least 90\% of agents invest} \\ -10 & \text{otherwise} \end{cases}$$

- **Coordination game!**

Strategic Effects

- **Symmetric penalty kick game:**

		goal keeper	
		left	right
kicker	left	$0, 0$	$1, -1$
	right	$1, -1$	$0, 0$

- Suppose kicker has weak left kick: **direct and indirect effect!**

		goal keeper	
		left	right
kicker	left	$0, 0$	$a, -a$
	right	$1, -1$	$0, 0$

$0 < a < 1$

Strategic effect

goalie

		(q) L	(1-q) R
Striker	(p) L	0, 0	a, -a
	(1-p) R	1, -1	0, 0

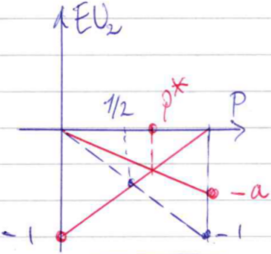
$0 < a < 1.$

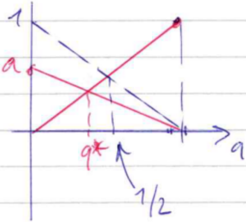
$$\left.\begin{aligned} EU_2(p, L) &= 0.p + (-1)(1-p) = p-1 \\ EU_2(p, R) &= -ap + 0(1-p) = -ap. \end{aligned}\right\} \text{goalie.}$$

$$\left.\begin{aligned} EU_1(L, q) &= 0.q + a(1-q) = a(1-q) \\ EU_1(R, q) &= q + 0 = q \end{aligned}\right\} \text{striker}$$

Strategic effect

		q L	$1-q$ R
Striker	(p) L	$0,0$	$a,-a$
	$(1-p)$ R	$1,-1$	$0,0$





$\boxed{a\downarrow \Rightarrow p^*\uparrow}$

$\boxed{a\downarrow \Rightarrow q^*\downarrow}$

Nash equilibrium: NE and IEWDS

Eliminating **weakly dominated** strategies might erase NEs!

	L	R
U	$2,3$	$4,3$
D	$3,3$	$1,1$

- Two NEs: (4,3) and (3,3)
- L weakly dominates R;
- Eliminating R would result in single solution (3,3);
- Notice that the Pareto-optimal NE would be eliminated.

Focal points (Schelling)

Focal points

- Some equilibria are more "natural" than others;
- Allow coordination without communication;

Example: allow two persons to split 100 Euro: if they match, they can keep the money. Majority proposes 50=50.

Other solution concepts

- **Minimax equilibrium**: zero-sum special case of Nash
- **Trembling-hand perfect equilibrium**: each player's action is a best-response even if other players make small mistakes
- ϵ-**Nash equilibrium**: deviating benefits no agent more than ϵ
- **Correlated equilibrium**: agents can condition strategy on external signal: "If there is intelligent life on other planets, in a majority of them, they would have discovered correlated equilibrium before Nash equilibrium." -Myerson
- **Evolutionary stable state**: "A population is said to be in an evolutionarily stable state if its genetic composition is restored by selection after a disturbance, provided the disturbance is not too large." -Maynard Smith

Summary

- Game theory studies utility-based multiagent decision making.
- Solving a game means trying to predict its outcome.
- Rational agents never play strictly dominated actions.
- No agent has an incentive to **unilaterally deviate** from a Nash equilibrium.
  单方面
- There is always an NE in mixed strategies. ？？？
- Nash equilibria may not be Pareto optimal.

Some additional literature

- Presh Talwalkar: The Joy of Game Theory: An introduction to strategic thinking.
- Avinash K. Dixit, Barry J. Nalebuff: The Art of Strategy: A Game Theorist's Guide to Success in Business and Life
- William Poundstone: Prisoner's Dilemma: John von Neumann, Game Theory, and the Puzzle of the Bomb