Multi-Agent Systems

Homework Assignment 1

MSc AI, VU

E.J. Pauwels

Deadline: Thursday Nov 11, 2021 (23h59)

1 Game Theory: Concepts and Nash Equilibrium

1.1 Iterated Best response dynamics for a simple matrix game

Consider the following simultaneous two player game: Player 1 (row player) can play U, M or D, while player 2 (column player) can pick L, C or R. They pay-off is given below:

	L	C	R
U	$4,3$	$2,0$	$8,2$
M	$8,2$	$7,6$	$-1,1$
D	$6,-3$	$0,0$	$1,-1$

- Are there any dominated strategies? Player1: U, D
- The concept of *Nash equilibrium* plays a central role in game theory and will be discussed at length in the next assignment(s). But some basic intuition can be gleaned from the following approach (known as: best response dynamics). One way to visualise Nash equilibria is to interpret the simultaneous game as a repeated sequential one: One player is allowed to make a move, then the other player makes a move, after which the first player can make another move and so on. What happens when you do this for this game?
- Can you think of simple games where the outcome of this dynamics would be qualitatively different?

1.2 Elimination of dominated strategies

A game is played with a group of say 50 players (e.g. all students sitting in a large auditorium). Upon request, they all pick an (integer) number between 1 and 100 (inclusive), and write it down on a card without communicating with any of the other players. These cards are collected and the average (say m) of all the numbers is computed. The winner is the person(s) whose chosen number was closest to $2/3$ of the mean m.

- What would be a rational strategy?
- If you had to play this game, would you play the above rational strategy? Explain.

1.3 Cournot's Duopoly (continuous version)

Cournot's duopoly is a game that models economic competition with strategic substitutes. Strategies are called *strategic substitutes* if an increase in your strategy will cause the competitor to decrease the use of his strategy (*"if you increase your part, I will reduce mine"*). This contrasts with *strategic complements* when an increase in the strategy of one player causes the other player to follow suit (*"if you increase your part, I will do the same."*.)

Two companies make an interchangeable product (e.g. bottled water). Both need to determine (simultaneously!) the quantity they will produce (say for next week). Call these quantities q_1 and q_2, respectively. The unit price p (price of each unit, e.g. one bottle) of the product in the market depends on the total produced quantity $q_1 + q_2$. Specifically

$$p(q_1, q_2) = \alpha - \beta(q_1 + q_2) \qquad (\alpha, \beta > 0).$$

Firm 1 can produce each unit at a unit-cost c_1, whereas the unit-cost for firm 2 equals c_2.

- What is the best response for each company given the quantity the other company will produce? For firm 1, get revenue function R(q1) -> get argmax_{q1}R(q1)
- Suppose the companies need **not** decide on their quantity at the same time, but can react to one another (an unlimited number of times). What will be the outcome? (Provide a diagram.)

1.4 Ice cream time!

Three competing ice-cream vendors (Alice, Bob and Charlize) are trying to sell their refreshments to tourists on the beach. We are making the following assumptions:

- the beach has total length of 1, while its width is uniform and much smaller than its length. So the beach can be represented as a line segment of length 1, and each position on the beach can be represented by the position parameter $0 \leq x \leq 1$.
- Tourists are uniformly distributed along the total length of the beach and will buy their ice-cream at the stall that is closest to their location;

Questions:

1. On a beautiful summer morning Charlize makes her way to the beach and upon arrival finds that her two competitors have already set up their stalls: Alice at location $a = 0.1$ and Bob at location $b = 0.8$. Discuss what Charlize's best response is: i.e. what location should she choose, given $a = 0.1$ and $b = 0.8$? any location between 0.1 and 0.8
2. Same question as above, but now assume that all we know is that $a = 0.1$ and $a < b \leq 1$.
3. Earlier that morning, Bob arrived and discovered that Alice had already set up her stall at $a = 0.1$, while Charlize hadn't shown up yet. But Bob knows that Charlize will arrive before too long, and that she will try to position herself in such a way as to maximize her revenue. What location should Bob pick in order to maximize his own expected revenue?
4. At sunrise that morning, Alice arrived before both Bob and Charlize, and set up her stall at location $a = 0.1$. However she knows for sure that the other two vendors will show up later that morning. Where should she set up her stall in order to maximise her expected revenue?

SOLUTIONS

1 Game Theory: Concepts and Nash Equilibrium

1.1 Iterated best response

1. Iterated best response

1. No Dominated Strategies.

2. Iteration → yields equilibrium position in which both players won't deviate.

	L	C	R
U	4,3	2,0	8,2
M	8,2	7,6	1,1
D	6,−3	0,0	1,−1

— : first player.

— : 2nd player

Conclusion: we always end up in (M, C) with pay-off (7, 6). = Nash Eq.

3. This type of equilibrium position is NOT always possible.

Eg: matching pennies. → CYCLE

	H	T
H	1,−1	−1,1
T	−1,1	1,−1

1.2 Elimination of dominated strategies

1. **Rational strategy:** Every player can pick one of 100 numbers (actions, options). Suppose all players choose 100, then we have a mean $m = 100$ and therefore the target value $(2/3)m \approx 67$ at most. So choosing anything in excess of 67 would be non-rational, because you can always do better by choosing 67. But since we assume that all players are rational, all of them will pick a number lower than 67. Hence the target value is at most $(2/3) \cdot 67 \approx 44$. Continuing this argument, we conclude that purely rational players would all pick the number 1.

2. In a real game it would be unwise to pick the theoretical optimum (1) as most players would fail to make the above deduction in limited time.

1.3 Cournot's Duopoly

Each company needs to decide which quantity to produce, and this means that each of the two players has an infinite number of possible actions to choose from. Let's introduce some notation: Company i produces quantity q_i at unit costs c_i $(i = 1, 2)$. The price setting (per unit) depends linearly on the total quantity produced (i.e. $q_1 + q_2$):

$$\text{unit-price:} \quad p = \alpha - \beta(q_1 + q_2) \qquad \text{where } \alpha, \beta > 0.$$

The expected utility (profig) for company 1 assuming that quanities q_1 and q_2 are produced, equals:

$$u_1(q_1, q_2) = pq_1 - c_1 q_1 = (p - c_1)q_1 = ((\alpha - c_1) - \beta(q_1 + q_2))q_1$$

Similarly:

$$u_2(q_1, q_2) = pq_2 - c_2 q_2 = (p - c_2)q_2 = ((\alpha - c_2) - \beta(q_1 + q_2))q_2.$$

Best response Assuming that company 2 produces quantity q_2, the best response for company 1 by maximising utility u_1. To this end we compute the (partial) derivative and set it to zero:

$$\frac{\partial u_1}{\partial q_1} = -2\beta q_1 + (\alpha - c_1 - \beta q_2) = 0$$

From this it follows that the best response $BR_1(q_2)$ for company 1 (given company 2 produces q_2) equals:

$$q_1^* := BR_1(q_2) = \frac{\alpha - c_1 - \beta q_2}{2\beta} = \frac{\alpha - c_1}{2\beta} - \frac{q_2}{2}.$$

Notice how q_1^* is a decreasing function in q_2 as expected for strategic substitutes.

Applying the same logic for company 2 yields:

$$q_2^* := BR_2(q_1) = \frac{\alpha - c_2 - \beta q_1}{2\beta} = \frac{\alpha - c_2}{2\beta} - \frac{q_1}{2}.$$

The corresponding dynamics is illustrated in Fig 1.

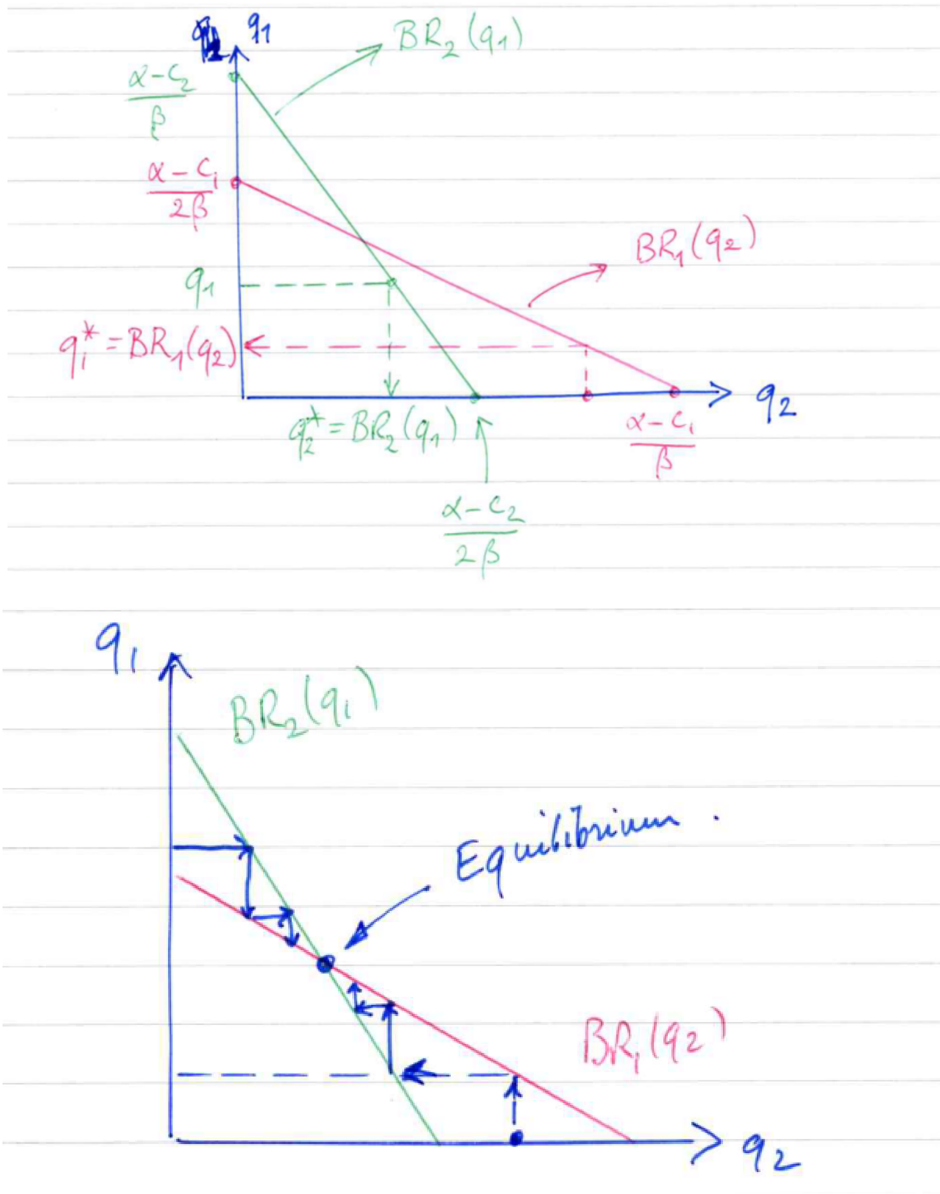

Figure 1: TOP: Best response functions for both companies. BOTTOM: Best response dynamics: one company starts with initial random quantity, the other company reacts, and this is iterated until the (Nash) equilibrium is reached.

1.4 Ice cream time!

1.4.1 Expected utiity (as function of position x) for Charlize $(EU_C(x))$

The expected utility of each player is equal to the number of customers the vendor will attract, or equivalently: what fraction of the x-positions is closes to the vendor. This result is graphed in Fig. 2.

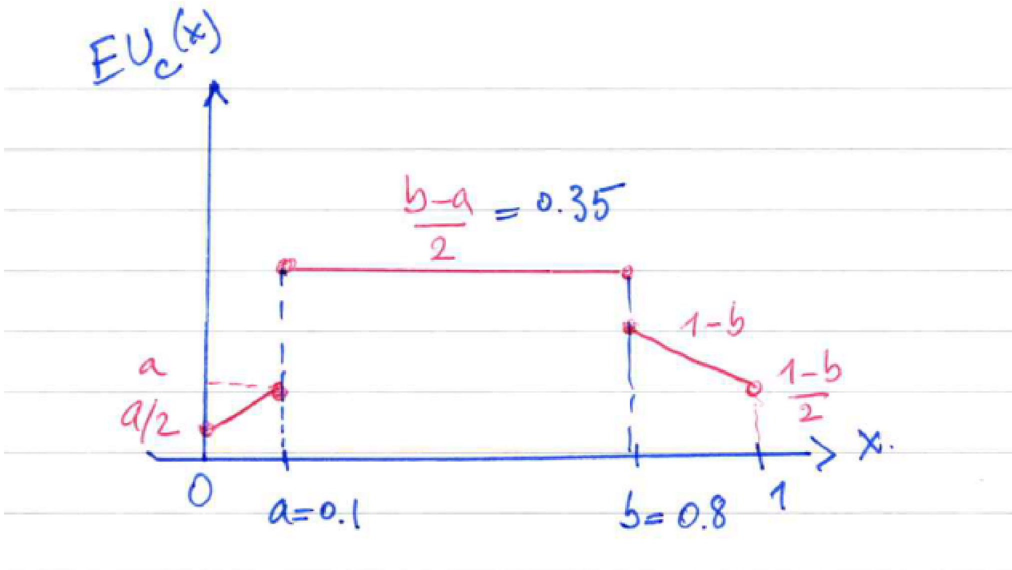

Figure 2: Expected utility EU_C for Charlize for given positions of Alice and Bob.

1.4.2 Same question as above, but now for variable b-value

When the b-position is no longer fixed but can vary ($a < b < 1$) we can express the expected utility for Charlize as follows:

$$EU_C(x;b) = \begin{cases} \dfrac{x+a}{2} & \text{if} \quad 0 < x < a \\ \dfrac{b-a}{2} & \text{if} \quad a < x < b \\ (1-x) + \dfrac{x-b}{2} = 1 - \dfrac{x+b}{2} & \text{if} \quad b < x < 1 \end{cases} \tag{1}$$

1.4.3 Optimal positioning for Bob

When Charlize arrives she will select a position $x = c$ that maximizes her expected utility. Referring to Fig 3 we observe the following:

- CENTRE panel: when the expected utility EU_C is continuous at $x = b$, Bob is indifferent about C's decision. Even if Charlize positions herself immediately to the left $c = b^-$ or to the right $c = b^+$ of Bob, Bob's utility will be the same. Continuity requires that

$$\frac{b-a}{2} = 1 - b \quad \Longrightarrow \quad b = \frac{a+2}{3}$$

with corresponding utility:

$$1 - b = 1 - \frac{(a+2)}{3} = \frac{1-a}{3}$$

- LEFT panel: if $(b-a)/2 > 1-b$ or equivalently $b > 0.7$, Charlize will take a position to the left of Bob, and in the worst case this could be right next to him ($c = b^-$). In that case Bob would be left with expected utility $1 - b < 0.3$.

- RIGHT panel: if $(b-a)/2 < 1-b$, or $b < 0.7$ Charlize will position herself just to the right of Bob (i.e. $c = b^+$) and Bob will get utility $(b-a)/2 < 0.3$

So from the above we conclude that the optimal choice (best guaranteed minimum outcome) for Bob is (assuming $b^* > a$):

$$b^* = \frac{a+2}{3} \quad \text{with corresponding guaranteed minimal utility } u_B^* = \frac{1-a}{3}.$$

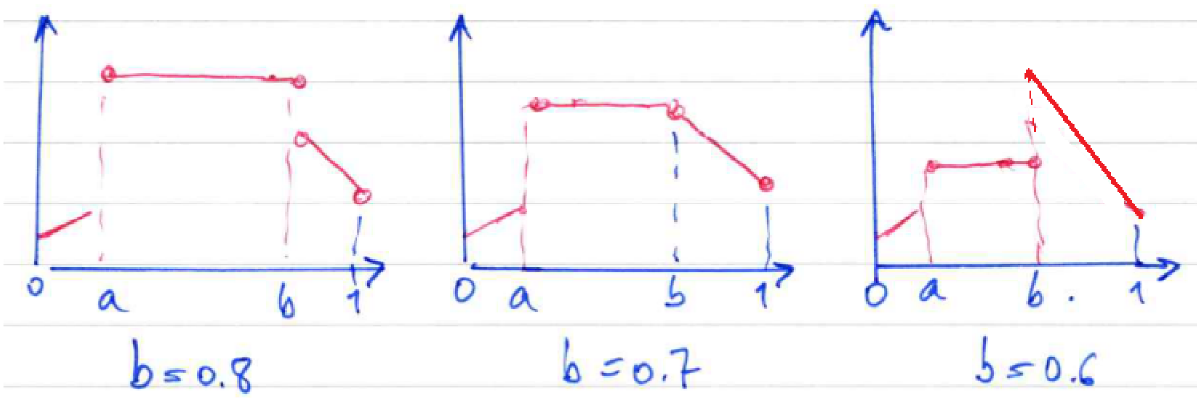

Figure 3: Expected utility $EU_C(x)$ for different choices of Bob's position b.

1.4.4 Optimal positioning for Alice

Alice knows that whatever her position, Bob will pick a position that maximizes his utility. This is either immediately to the left of Alice (i.e. $b = a$, with a utility a) or to the right of a at $b^* = (a+2)/3$ with utility $(1-a)/3$. Alice will be indifferent to Bob's choice if these two utilities agree: So the optimal position needs to satisfy the following equation:

$$a^* = \frac{1-a^*}{3} \quad \Longrightarrow \quad a^* = 1/4$$