Introduction to Game Theory: Coalitional Games

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Reading

- **Recommended**
  - Shoham and Leyton-Brown: Chapter 12, sections 12.1-12.2

Overview

Coalitional Game Theory

- Basic modelling unit is **group** rather than indvidual agent.
- Transferable vs. non-transferable utility

transferable utility assumption that the payoffs to a coalition may be freely redistributed among its members.

- **Coalitional game with transferable utility (N, v):**
  - N finite set of players:
  - $v : 2^N \longrightarrow \mathbb{R}$ pay-off function ($v(\emptyset) = 0$)
- Fundamental questions:
  - Which coalitions will form?
  - How should coalition divide its pay-off among its members?

Classes of coalitional games

- **Super-additive game** Game (N, v) is super-additive iff

$$\forall S, T \subset N : S \cap T = \emptyset \Longrightarrow v(S \cup T) \geq v(S) + v(T).$$

  In particular: $v(S \cup i) \geq v(S) + v(i)$ for any $S \subset N \setminus \{i\}$.

- As a consequence, for super-additive game, the grand coalition has the highest pay-off of all coalitional structures:

$$v(N) = v(S \cup S^c) \geq v(S) + v(S^c) \geq v(S).$$

- Therefore focus on a fair redistribution of total pay-off among the members of the grand coalition.

Ways to allocate common benefits

Motivating example:

Both Bob and Cedric are *very* fond of Alice, who unfortunately lives far away. They invite her to visit and will gladly pay for her long-haul trip. The fares for roundtrips are as follows:

- $A \leftrightarrow B$: round trip visiting B: 900 Euros;
- $A \leftrightarrow C$: round trip visiting C: 1100 Euros;
- $A \leftrightarrow B \leftrightarrow C$: round trip visiting both B and C: 1600 Euros;

Combining the trips saves 400 Euros. How should that be redistributed between B and C?

Solution strategies: proportional vs. incremental

Some useful terminology

- Players i and j are **interchangeable** if their contributions to every coalition (subset) S is exactly the same:

$$\forall\, S \subset N \setminus \{i, j\} : \quad v(S \cup i) = v(S \cup j)$$

- A player i is a **dummy player** if the amount he contributes to any coalition is exactly the amount he's able to achieve alone:

$$\forall S \subset N \setminus \{i\} : v(S \cup i) = v(S) + v(i)$$

How to fairly divide the benefits of coalition?

shapley_overview.png

Shapley's Axioms

- **Symmetry:** If i and j are interchangeable then:

$$\psi_i(N, v) = \psi_j(N, v).$$

- **Dummy Player:** will only get what he can achieve on his own:

$$\psi_i(N, v) = v(i).$$

- **Additivity:** Consider two games $G_1 = (N, v_1)$, $G_2 = (N, v_2)$ and combine them in a new game for which $v = v_1 + v_2$. Then:

$$\psi_i(N, v_1 + v_2) = \psi_i(N, v_1) + \psi(N, v_2)$$

Shapley's Axioms

shapley_axioms.png

Shapley value

- Marginal contribution of player i to subset S:

$$\delta_i(S) = v(S \cup i) - v(S)$$

- Shapley value of player i: (denoting $\#N = n, \#S = s$)

$$\varphi_i(N, v) := \frac{1}{n} \sum_{S \subset N \setminus i} \binom{n-1}{s}^{-1} \delta_i(S)$$

- Amplification: see next slides!

Shapley Value: Amplification

- We focus on Shapley value $\varphi_i(N, v)$ for agent i;
- For any existing coalition S not including i, i.e.

$$S \subset N_i := N \setminus i$$

we consider the value increment due to i joining:

$$\delta_i(S) = v(S \cup i) - v(S)$$

- The size $s := \#S$ of the possible coalitions S that i joins, can range between $0 \leq s \leq n - 1$.

Frame Title

- For fixed coalition size s there are

$$N_s := \binom{n-1}{s}$$

coalitions S of that size.

- Hence, the mean contribution $\overline{\Delta}_i$ of i to an existing coalition S of size s is given by:

$$\overline{\Delta}_i(s) := \frac{1}{N_s} \sum_{S:\#S=s} \delta_i(S) = \binom{n-1}{s}^{-1} \sum_{S:\#S=s} \delta_i(S).$$

Frame Title

- Finally, since $0 \leq s \leq n-1$ we compute the average over the n possible choices of s. This average is the Shapley value:

$$
\begin{aligned}
\varphi_i \quad &:= \quad \frac{1}{n}\sum_{s=0}^{n-1}\overline{\Delta}_i(s) = \frac{1}{n}\sum_{s=0}^{n-1}\binom{n-1}{s}^{-1}\sum_{S:\#S=s}\delta_i(S) \\
&= \quad \frac{1}{n}\sum_{s=0}^{n-1}\sum_{S:\#S=s}\binom{n-1}{s}^{-1}\delta_i(S) \\
&= \quad \frac{1}{n}\sum_{S\subset N\setminus i}\binom{n-1}{s}^{-1}\delta_i(S)
\end{aligned}
$$

- Double sum above is actually sum over all subsets $S \subset N \setminus i$.

Shapley's theorem

Shapley (1951)

Given a coalitional game (N, v), the Shapley values $\varphi_i, i = 1, \ldots, n$ specifies the unique distribution of the total value $v(N)$ that is both

- efficient, i.e. $\sum_i \varphi_i = v(N)$
- satisfies Shapley's axioms,
  i.e. *Symmetry, Dummy Player* and *Additivity*.

Shapley value: Alternative computation method

To keep notation simple we focus on three agents ($n = 3$):

- Suppose we are interested Shapley value φ_1 for agent 1:
- Any permutation of 123 (e.g. 231) represents the sequence in which agents join the coalition , e.g.:

$$231 : \quad \emptyset \to 2 \to 23 \to 231$$

$$312 : \quad \emptyset \to 3 \to 31 \to 312$$

- For each permutation, we compute the marginal contribution of agent 1 upon joining, e.g.

$$231 : \delta_1 = v(231) - v(23) \quad \& \quad 312 : \delta_1 = v(31) - v(3)$$

- The Shapley value is obtained by averaging the δ-values over all $n! = 3! = 6$ permutations.

Shapley value: worked example

An AI expert (E) developed a powerful new algorithm. However, in order to implement his ideas, he needs to create a startup and hire a programmer (P) for 2 years. An angel investor (A) provides funding. The value that each coalition of these three stakeholders (E, P, A) can generate satisfies the following rules:

- Without both investor and expert, no value can be generated.
- If he has no assistance from a programmer, the expert's value equals 3, but if he can delegate the programming and focus on R&D, his value rises to 10.
- The value created by the programmer is 5. This is in addition to the rise in value of the expert.

The startup is sold to a large software company for serious money.
How to split this money fairly among the three stakeholder?

Shapley value: Method 1

Shapley value computation: $\#N = n = 3$, $\#S = s$

$$\varphi_i = \frac{1}{n} \sum_{S \subset N \setminus i} \binom{n-1}{s}^{-1} \delta_i(S)$$

Expert (E)

$\underbrace{s = 0}$ $\longrightarrow S = \phi \Rightarrow \delta_E(S) = v(E) - v(\phi) = 0.$

$\hookrightarrow \binom{n-1}{s} = \binom{2}{0} = 1$

$\underbrace{s = 1}$ $\longrightarrow \delta_E(A) = v(AE) - v(A) = 3$

$\delta_E(P) = v(EP) - v(P) = 0$

$\hookrightarrow \binom{n-1}{s} = \binom{2}{1} = 2$

$\underline{s = 2}$ $\longrightarrow \delta_E(AP) = v(APE) - v(AP) = 15$

$\hookrightarrow \binom{n-1}{s} = \binom{2}{2} = 1$

$$\varphi_E = \frac{1}{3}\left[\frac{1}{1} \cdot 0 + \frac{1}{2} 3 + \frac{1}{1} \cdot 15\right] = \frac{11}{2}$$

Shapley value: Method 2

	E	P
A E P	$\underbrace{v(AE)}_{3} - \underbrace{v(A)}_{0} = 3$	$\underbrace{v(AEP)}_{15} - \underbrace{v(AE)}_{3} = 12$
A P E	$\underbrace{v(APE)}_{15} - \underbrace{v(AP)}_{0} = 15$	$v(AP) - v(A) = 0$
E A P	$v(E) - v(\phi) = 0$	$v(AEP) - v(EA) = 12$
E P A	$v(E) - v(\phi) = 0$	$v(EP) - v(E) = 0$
P A E	$\underbrace{v(AEP)}_{15} - \underbrace{v(AP)}_{0} = 15$	$v(P) - v(\phi) = 0$
P E A	$v(EP) - v(P) = 0$	$v(P) - v(\phi) = 0$
	33	24

$$\boxed{\begin{array}{ll} \varphi_E = \frac{33}{6} = \frac{11}{2} & \varphi_P = \frac{24}{6} = 4 \\ \left(\varphi_A = \frac{11}{2}\right) & \text{Notice: } \frac{11}{2} + \frac{11}{2} + 4 = 15 \end{array}}$$

Figure: Notice distribution is efficient: $\varphi_E + \varphi_A + \varphi_P = 15 = v(N)$