Introduction to Game Theory 2:

Sequential Games

Eric Pauwels (CWI & VU)

Version: November 11, 2021, v.2

Reading

- **Recommended**
  - Shoham and Leyton-Brown: Chapter 5, sections 5.1-5.3
- **Optional**
  - William Spaniel: Game Theory 101: The Complete Textbook (paperback): *Very accessible and clear, teaching through examples. Accompanying YouTube channel.*
  - N. Nisan, T. Roughgarden, E. Tardos, V. Vazirani: Algorithmic Game Theory. Cambridge UP. *Solid, mathematical. Advanced.*
  - A. Dixit, B. Nalebuff: Thinking Strategically. Norton. *Lots of context and background. Interesting and non-technical.*

Overview

Table of Contents

Sequential games

- **Normal-form games:**
  - Simultaneous moves by players
  - Central solution concept: Nash equilibrium
- **Sequential games:**
  - Players move in succession, observe (at least partially) prior moves by opponents
  - **Perfect versus imperfect information:** what exactly is known about previous moves?
    - players have full knowledge of all the preceding moves (perfect information)
    - players might not know the complete game history till then (imperfect information);
  - **Model for many sequential interactions** in games, politics, economics, etc

Simultaneous vs. Sequential Games

- **Simultaneous games:** players make their moves simultaneously, i.e. without knowing what the other players will do!
  - Rock-paper-scissors
  - Sealed bid auctions
  - Cournot's duopoly model
- **Sequential games:** Sequence of successive moves by players who can see each other's moves (to some extent – see next slide):
  - Chess
  - Card games
  - Open cry auctions
  - Stackelberg's duopoly model
  - Negotiation (Rubinstein's model)

Extensive form representation of sequential game

- Visualisation of temporal relationships (game tree)
- **Extensive form** is finite game representation that **does not assume** that players act **simultaneously**;
- Can be converted in normal form representation (*possibly exponentially larger!*)
- **Game tree:** makes **temporal structure** explicit

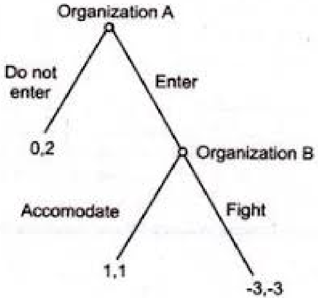


Figure-2: Extensive Form Games

Information in Game Theory

	PERFECT complete history known to all players	**IMPERFECT** unaware of actions taken by others
COMPLETE NO private info agents, actions, payoffs known	E.g. chess	Simultaneous games Information sets
INCOMPLETE private info e.g. private valuation	• Open cry auction • Different types of opponents	Sealed bid auction

Aside: Types of knowledge

- **Mutual knowledge** is Example??
  - known to all players,
  - but players do not know that others know
  - e.g. *the elephant in the room* , solutions to homeworks
- **Common knowledge** is
  - known to all players,
  - and all players know all others know . . .
  - and all players know all others know that all others know . . .
  - and so on . . .
  - e.g. In continental Europe one drives on the RHS of the road

Major Ideas in Non-Cooperative Game Theory

	SIMULTANEOUS (STATIC)	**SEQUENTIAL** (DYNAMIC)
Complete information NO private info	Nash equilibrium	Backwards induction, Subgame-perfect NE: *discard NE based on non-credible threats*
Incomplete information private info	Bayesian Nash eq.	Perfect Nash eq.

Sequential games: perfect vs. imperfect information

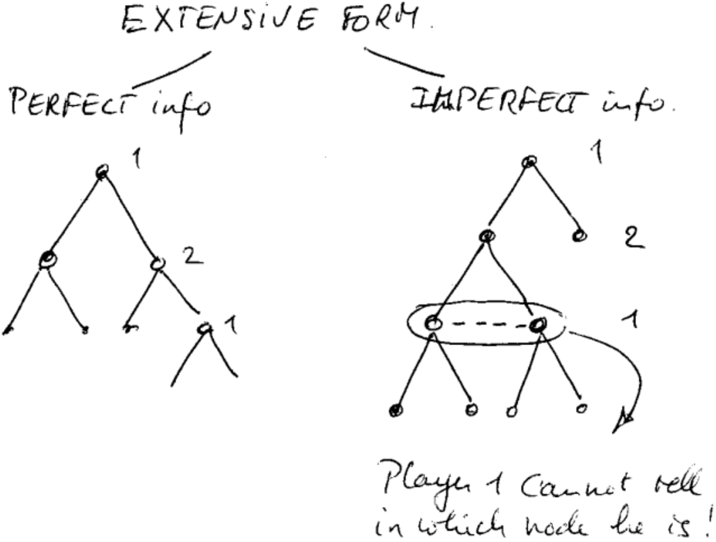


Table of Contents

Sequential Games with Perfect Information

Protoype: two-player, sequential-move game:

- Player 1 chooses action $a_{11} \in A_1$;
- Player 2 observes a_{11} and then chooses action $a_{21} \in A_2$;
- ...
- Hereafter, both players receive pay-off: $u_1(a_{1n}, a_{2n})$ and $u_2(a_{1n}, a_{2n})$ respectively;

Examples:

- Various board and card games (e.g. chess, go, etc)
- Stackelberg's sequential-move version of Cournot's duopoly;
- Rubinstein's bargaining model

Solving Extensive Form Games using Backward Induction

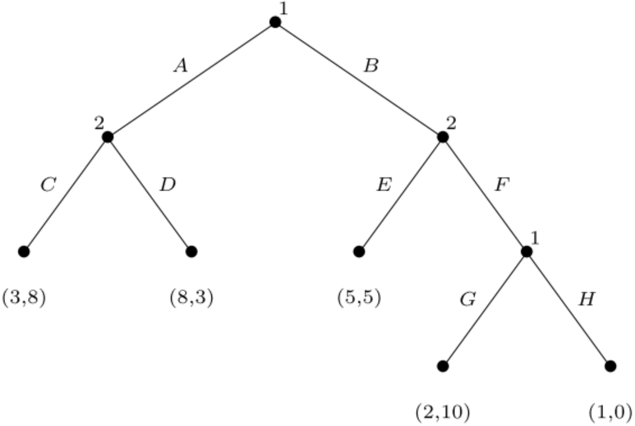
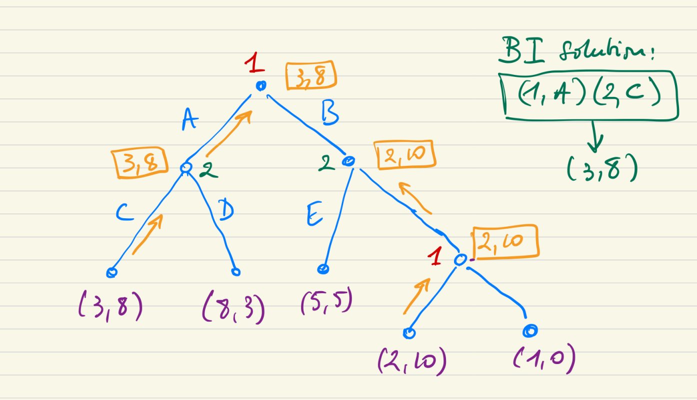


Figure 5.2: A perfect-information game in extensive form.

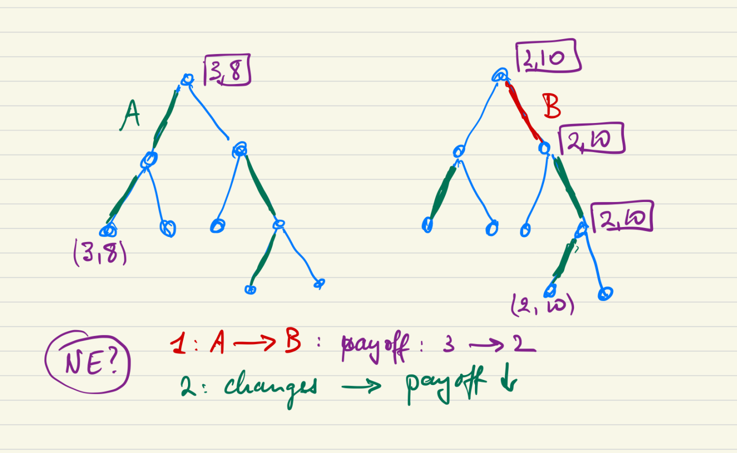
Solving Extensive Form Games using Backward Induction

Backward induction

- Start at leaf-nodes: easy decision as only one player involved;
- Propagate decisions and utilities to root;



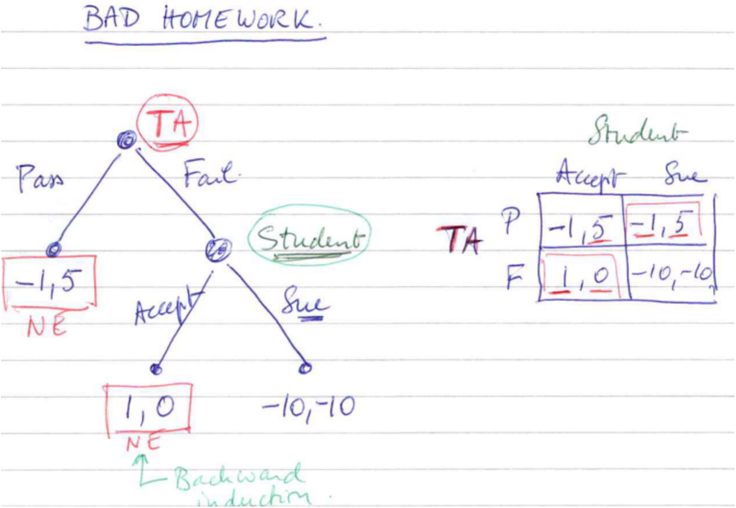
Is Backward Induction solution a Nash Eq.?



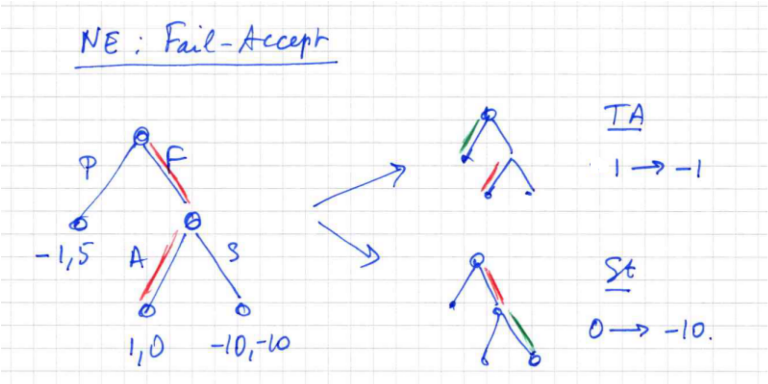
To check for NE: need to know decision at each node!

Bad homework: Backward induction and NE (2)

Student is trying to bully TA into giving passing grade!

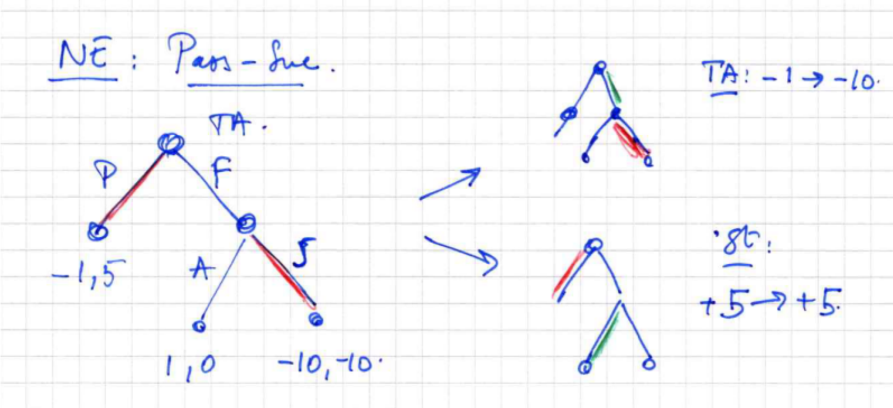


Bad homework: Checking NE 1 in extensive form



- (Fail, Accept): no incentive for unilateral deviation: hence NE!
- This NE corresponds to backward induction (BI) solution

Bad homework: Checking NE 2 in extensive form



(Pass, Sue): no incentive for unilateral deviation: hence NE!

...but due to non-credible threat (sue) by student.

From Extensive for Normal Form (perfect information)

Aim: Transform sequential game in normal form to use standard methods to find NEs.

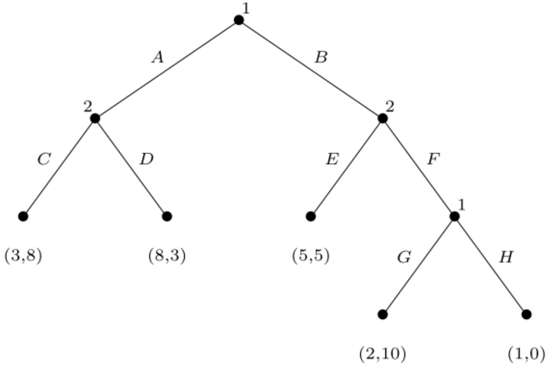
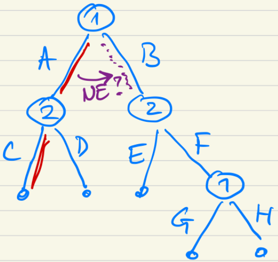


Figure 5.2: A perfect-information game in extensive form.

From Extensive for Normal Form (perfect information)



Naive approach:

P_1 [A
B if P2 chooses E
BG, BH if P2 chooses F

P_2 : C → game ends.

NE? : A → B alternative.

Now we need to know what P2 will do : E or F ?

SOLUTION : Systematic approach:

$P_1 \rightarrow \{A, B\} \times \{G, H\}$, $P_2 \rightarrow \{C, D\} \times \{E, F\}$.

$= \{AG, AH, BG, BH\}$ $= \{CE, CF, DE, DF\}$.

From Extensive to Normal Form (perfect information)

- **A pure strategy** for player i in a (perfect information) sequential game is a complete plan of action specifying which action to take at each of its decision nodes ...
- ...irrespective of **whether or not that node can be reached** when playing the strategy!
- Mathematically: it's the **product space** of the possible actions in each decision node:
  - Node 1 has 2 decision nodes: hence $\{A, B\} \times \{G, H\} = \{(A, G), (A, H), (B, G), (B, H)\}$
  - Node 2 has 2 decision nodes: hence $\{C, D\} \times \{E, F\} = \{(C, E), (C, F), (D, E), (D, F)\}$

From extensive to normal form

- **Alternative perspective:**
  1. Extensive form: player "waits" till one of his nodes is reached, then decides what to do;
  2. Normal form: each player makes a complete contingent plan in advance.

局部

- **Informally:**
  - It's a **complete and contingent plan** instructing an assistant playing on your behalf, what to do in **each possible situation;**
  - Suppose that your assistant misunderstood and ended up in another node, then he still needs to know what to do.
  - Allows to explore whether unilateral deviation would be advantageous (Nash criterion)

From Extensive to Normal Form

	(C,E)	(C,F)	(D,E)	(D,F)
(A,G)	3, 8	3, 8	8, 3	8, 3
(A,H)	3, 8	3, 8	8, 3	8, 3
(B,G)	5, 5	2, 10	5, 5	2, 10
(B,H)	5, 5	1, 0	5, 5	1, 0

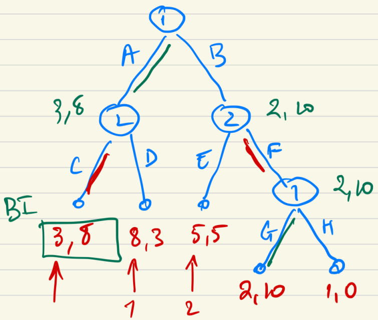
Figure 5.3: The game from Figure 5.2 in normal form.

From Extensive to Normal Form

	(C, E)	(C, F)	(D, E)	(D, F)
(A, G)	3, 8	$\boxed{3, 8}$	8, 3	8, 3
(A, H)	3, 8	$\boxed{3, 8}$	8, 3	8, 3
(B, G)	5, 5	2, 10	5, 5	2, 10
(B, H)	$\boxed{5, 5}$	1, 0	5, 5	1, 0

Figure 5.4: Equilibria of the game from Figure 5.2.

Non-credible threats



NE 1: {AG, CF}

OK

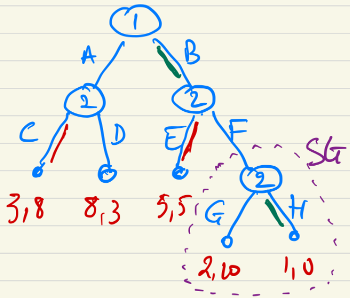
P2: C → D : (3,8) → (8,3)

P1: A → B, then P2 will play F

P1 will respond G

(3,8) → (2,10)

Non-credible threats



NE3: {BH, CE} OK

P1: B → A : (5,5) → (3,8) ↓

G → H : (5,5) → (5,5) =

↳ irrelevant

P2: C → D : irrelevant =

$\boxed{\text{E → F : (5,5) → (1,0)}}$ ↓

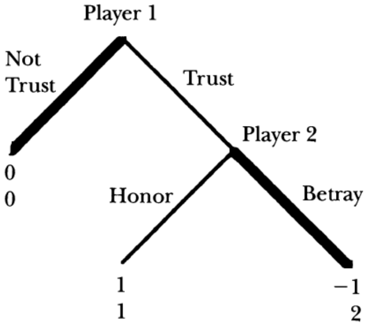
HOWEVER: utility-decrease depends on P1 choosing H in final subgame

→ NON-CREDIBLE THREAT!

Table of Contents

Solution Concept: Backwards Induction vs. Nash

The Trust Game

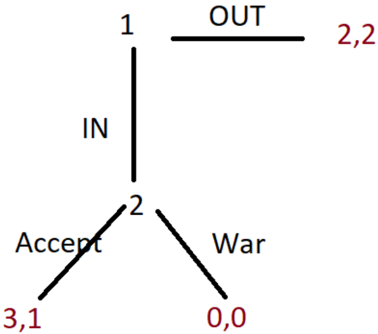


		Player 2	
		Honor	Betray
Player 1	Trust	1, 1	−1, 2
	Not Trust	0, 0	0, 0

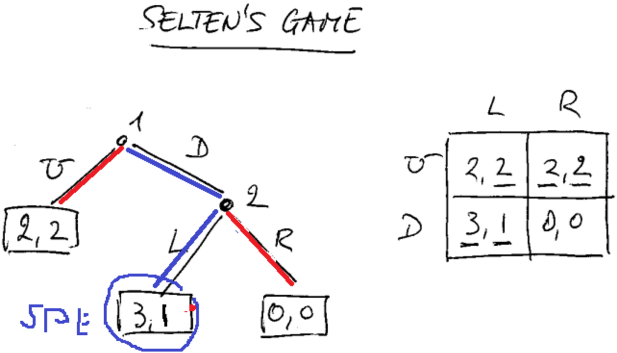
In this case: backward induction (extensive form) agrees with Nash (normal form)

$$NE = (\text{Not Trust}, \text{Betray}) \qquad \text{with utility } (0, 0)$$

Example: Selten's Game (Entry Game in Economics)

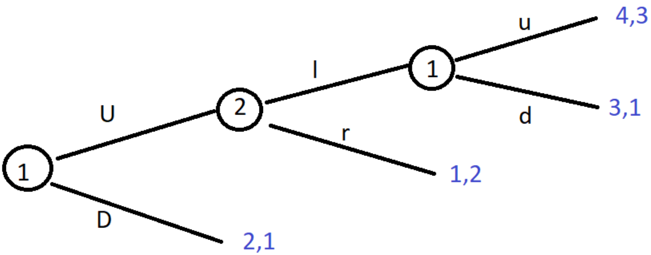


Selten's game: BI (extensive form) vs. NE (normal form)



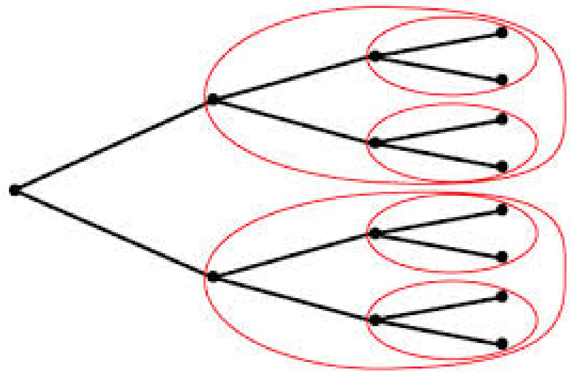
- Ext form: BI yields single optimum (D, L) with util $= (3, 1)$
- Normal form: 2 NE: (D, L) and (U, R) (non-credible threat)

Example: Backward Induction vs. Nash Equilibria



	ℓ	r	
Uu	$\underline{4,3}$	$1,2$	backward induction!
Ud	$3,1$	$1,2$	
Du	$2,1$	$\underline{2,1}$	"r non-credible threat"
Dd	$2,1$	$\underline{2,1}$	"d non-credible threat"

Subgames and Subgame Perfection



Subgame-Perfect Equilibrium (SGPE) :
induces Nash equilibrium in every subgame!

Subgame Perfect Nash Equilibrium

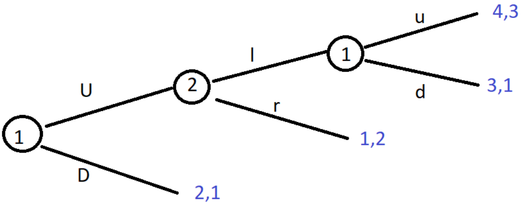
SGPE : refinement of Nash Equilibrium:

Subgame-perfect equilibrium (SPGE, Selten 1965)

A Nash equilibrium s (of game G as a whole) is **subgame-perfect** iff for every subgame G' of G, the restriction of s to G' is also a Nash equilibrium.

- SGPE rules out Nash equilibria that rely on non-credible threats;
- Put differently: SGPE is the study of credible threats.

Example 2: Nash equilibria for subgames



	ℓ	r	
Uu	$\underline{4,3}$	$1,2$	backward induction!
Ud	$3,1$	$1,2$	
Du	$2,1$	$\underline{2,1}$	"r non-credible threat"
Dd	$2,1$	$\underline{2,1}$	"d non-credible threat"

$SG2$:

	ℓ	r
u	$\underline{4}, \underline{3}$	$\underline{1}, 2$
d	$3,1$	$\underline{1}, \underline{2}$

Notes

- d non-credible threat implies r non-credible threat;
- Subgames (e.g. SG2) can have additional NE (compared to full game)

Example 2, continued

Game has two non-trivial subgames:

- SG1 rooted at 2nd decision node of 1,
- SG2 rooted at decision node of 2

Normal form yields 3 NE's. Do they induce NE in all subgames?

- $NE1 = (Dd, r)$ with utility $(2, 1)$:
  induces action d in SG1 (not NE!)
- $NE2 = (Du, r)$ with utility $(2, 1)$:
  induces action (u, r) in SG2 (not NE!)
- $NE3 = (Uu, \ell)$ with utility $(4, 3)$:
  induces actions u in SG1, u, ℓ in SG2 (OK!)

Finding SGPE in perfect information games

Two approaches:

- **Matrix form**
  1. Convert game-tree into matrix;
  2. Find all Nash equilibria;
  3. Eliminate the ones that depend on non-credible threats, i.e. do not induce a NE for each subgame;
- **Backward induction (See next slide)**
  - works if NO simultaneous moves or infinite horizons!

Backwards induction

Algorithm to find subgame perfect equilibrium

- Consider each subgame of the game (in increasing order of inclusion)
- Find the NE for the subgame;
- Replace the subgame by a new terminal node that has the equilibrium pay-offs;

Zermelo's Thm (1913)

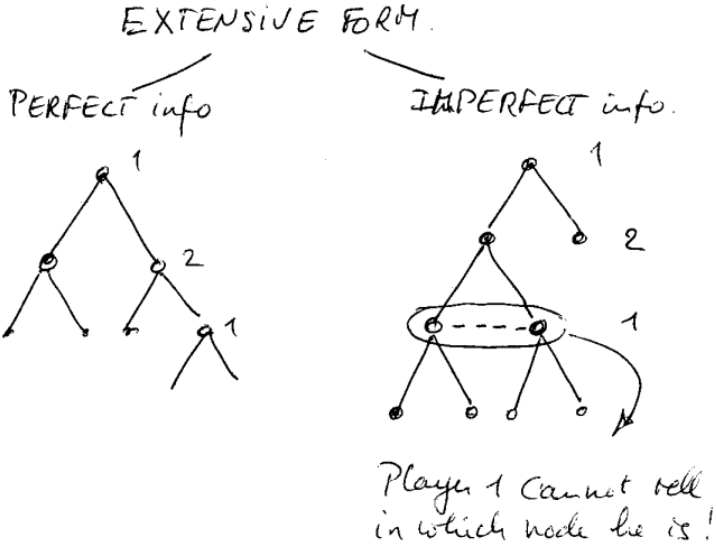
- With perfect information (one player in each iteration), a deterministic move is optimal. Hence there is a SGPE where each player uses a pure strategy.
- **For games with imperfect information**, a SGPE may require mixed strategies.

Table of Contents

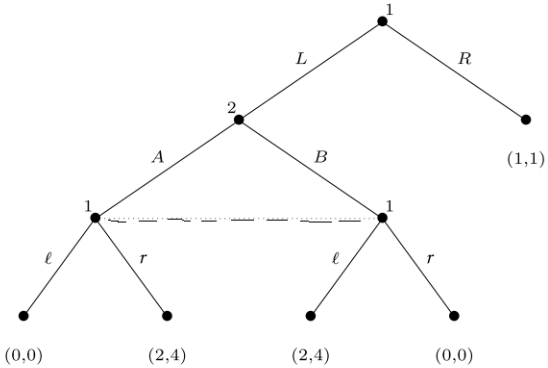
Imperfect information and information sets

- **Imperfect information: Intuition** Players need to act
  - with partial or no knowledge of actions taken by others,
  - with partial recall, i.e. limited memory of own past actions.
- An **imperfect- information game** is an extensive-form game in which each player's decision nodes are partitioned into **information sets;**
- Intuitively, if two decision nodes are in the same information set then the agent cannot distinguish between them.

Sequential games: perfect vs. imperfect information



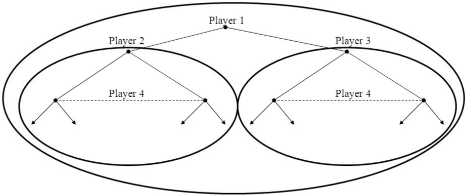
Subgames: Condition on information set



- at any information set, a player must have the same strategies regardless of how the player arrived there;
- Subgames cannot break up information sets!

Subgame of a Sequential Game with Imperfect Information

Example Subgames



How many subgames are there in this game?

1 + 1 + 1 = 3

Subgame definition:

- SG's initial node has singleton informationset;
- All successors are in SG;
- Any information-set is either completely in or out;

Pure strategies and induced normal form

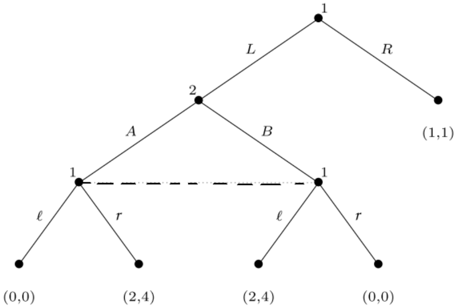


Figure 5.10: An imperfect-information game.

	A	B
$L\ell$	$0, 0$	$2, 4$
Lr	$2, 4$	$0, 0$
$R\ell$	$1, 1$	$1, 1$
Rr	$1, 1$	$1, 1$

Figure 5.14: The induced normal form of the game from Figure 5.10.

Pure actions are cartesian products over actions in information sets.

Subgame Perfect Nash Equilibrium for Imperfect Information Games

SGPE (aka SPE/SPNE) : refinement of Nash Equilibrium:

Subgame-perfect equilibrium (SPGE, Selten 1965)

A Nash equilibrium s (of game G as a whole) is **subgame-perfect** iff for every subgame G' of G, the restriction of s to G' is also a Nash equilibrium.

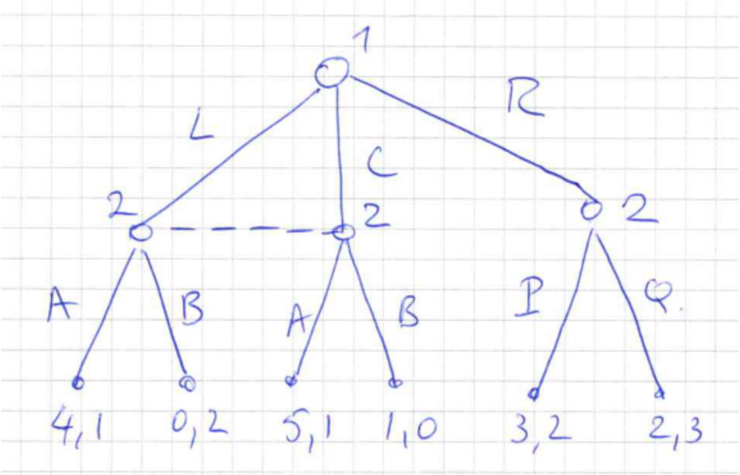
- SGPE rules out Nash equilibria that rely on non-credible threats;
- Put differently: SGPE is the study of credible threats.

Generalized Backwards Induction for Imperfect Information Games

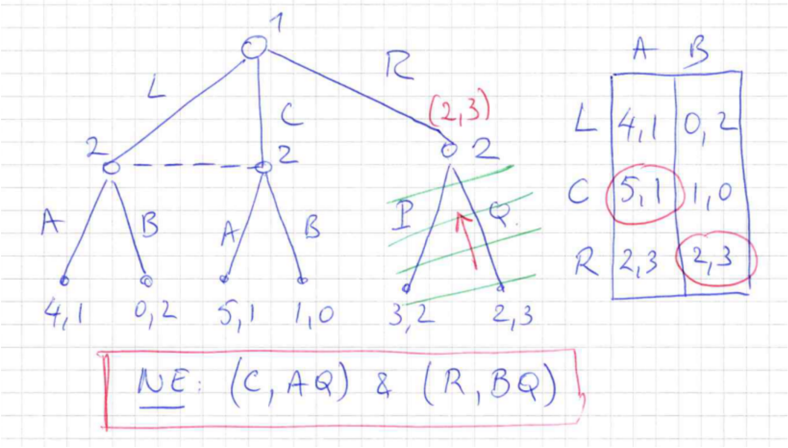
Systematically proceed as follows (if possible)

- Consider in turn each subgame of the game in increasing order of inclusion)
  Start at end of game, (no strategic interaction left)
  work backwards to beginning!
- Apply BI as far as you can;
- Replace the subgame by a new terminal node(s) that has the equilibrium pay-offs
  (we might need to consider different possibilities);
- If BI is not possible, use normal form solution techniques to find NE(s) for the remaining game (including mixed ones);

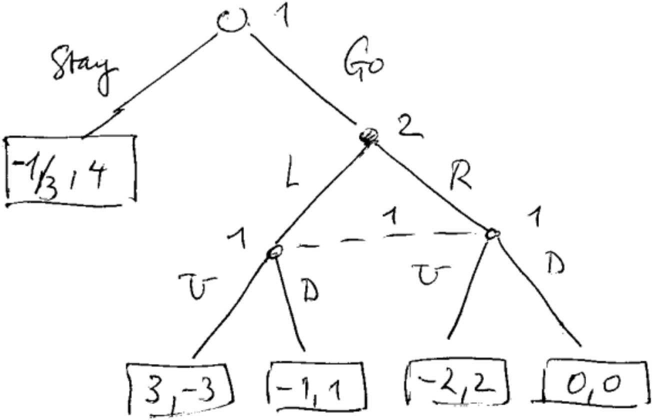
Generalized BI: Example 1



Generalized BI: Example 1, continued



Generalized BI, Example 2



Generalized BI, Example 2 (cont'd)

		q L	$1-q$ R
p	U	$3, -3$	$-2, 2$
$1-p$	D	$-1, 1$	$0, 0$

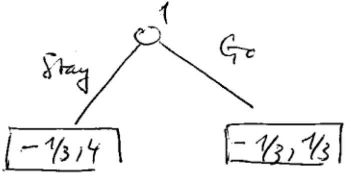
$p^* = 1/6$ $\qquad$ $q^* = 1/3$.

Generalized BI, Example 2 (cont'd)

$$EU_1 = \frac{1}{6} \cdot \frac{1}{3} \cdot 3 + \frac{1}{6} \cdot \frac{2}{3}(-2) + \frac{5}{6} \cdot \frac{1}{3}(-1) + \frac{5}{6} \cdot \frac{2}{3}\, 0$$

$$= \frac{1}{6} + \left(-\frac{2}{9}\right) + \left(\frac{-5}{18}\right) + 0$$

$$= \frac{3 - 4 - 5}{18} = -\frac{6}{18} = -1/3 .$$



Backward Induction vs. Subgame Perfection

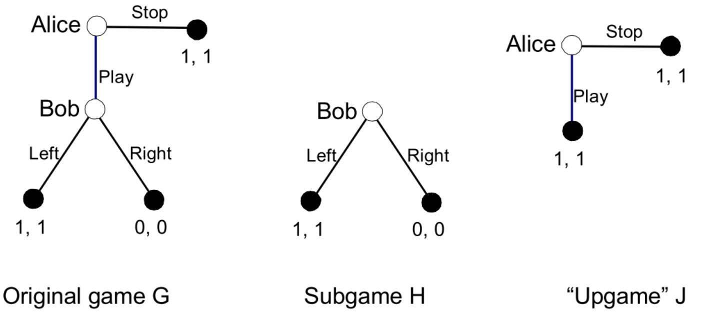


Figure 1. Backward induction versus subgame perfection.

The backward induction "upgame" J is NOT a subgame!

Ref: M. Kaminski: Generalized Backward induction, *Games* 2019, 10, 34

Backward Induction vs. Subgame Perfection

Finite sequential games with perfect information:

- All SPE can be found by backward pruning, ie.
  - systematic and incremental substitution of terminal subgames with Nash-eq pay-offs
- All BI solutions (backward pruning) are SPE

Backward Induction also works in some more general cases
e.g. some infinite games (e.g. Rubinstein)

More complex games require more restrictions on the Nash eq. solution to eliminate unreasonable solutions.
E.g. sequential rationality, perfect equilibrium