Chapter 1

The Black-Scholes Model

Let's review some common derivative knowledge first.

1. European option: the payoff is path independent of the underlying asset (in this chapter, we will simply call it the stock).
2. Call options have payoff $(S_T - K)^+ = \max(S_T - K, 0)$ at maturity.
3. Put options have payoff $(K - S_T)^+ = \max(K - S_T, 0)$ at maturity.

Since the put-call parity relates the price of a put option and a call option, we would focus on call options in this chapter.

We assume that the stock price S_t follows a Geometric Brownian Motion (GBM) under the real probability measure $\mathbb{P}$,

$$\mathrm{d}S_t = \mu S_t \mathrm{d}t + \sigma S_t \mathrm{d}B_t^{\mathbb{P}}, \tag{1.1}$$

where $B_t^{\mathbb{P}}$ is the standard Brownian motion (Wiener process) under measure $\mathbb{P}$, μ is the drift rate of S (annualized), and σ is the standard deviation of the stock's return. This equation is understood under the discrete limit

$$S_{t+\Delta t} - S_t = \underbrace{\mu S_t \Delta t}_{\text{drift term}} + \underbrace{\sigma S_t (B_{t+\Delta t}^{\mathbb{P}} - B_t^{\mathbb{P}})}_{\text{diffusion term, } \sim\mathcal{N}(0,\Delta t)} . \tag{1.2}$$

In many cases, it is easier to transform this to the risk neutral measure $\mathbb{Q}$

$$\mathrm{d}S_t = r S_t \mathrm{d}t + \sigma S_t \mathrm{d}B_t^{\mathbb{Q}}, \tag{1.3}$$

where $B_t^{\mathbb{Q}}$ is the standard Brownian motion under measure $\mathbb{Q}$, and r is risk free rate. write something on the measure change, the Girsanov's theorem

The value of a call option $C(t, S)$ at a specific time t is $C(t, S_t)$. Traditionally, there are two approaches to derive the Black-Scholes Model.

1. PDE: Construct a PDE for $C(t, S)$, which can be reduce to a heat equation, then solve the heat equation to get the solution.
2. Probabilistic: the fair price of a call option $C(t, S_t)$ should be the payoff $(S_T - K)^+$ at maturity T discounts to the current time t.

$$C(t, S_t) = e^{-r(T-t)} \mathbb{E}^{\mathbb{Q}}[(S_T - K)^+ | S_t = S]. \tag{1.4}$$

Here we note that the conditional expectation is conditioned on S_t is due to the Markovian of S_t.

1.1 The PDE Approach

1.1.1 The Black-Scholes PDE

Let's start by applying Itô's lemma to a call option $C(t, S_t)$ under the risk-neutral measure $\mathbb{Q}$, we find

$$\begin{aligned}
\mathrm{d}C(t, S_t) &= \frac{\partial C}{\partial t}\mathrm{d}t + \frac{\partial C}{\partial S}\mathrm{d}S_t + \frac{1}{2}\frac{\partial^2 C}{\partial S^2}\mathrm{d}[S]_t, \\
&= \frac{\partial C}{\partial t}\mathrm{d}t + \frac{\partial C}{\partial S}(rS_t\mathrm{d}t + \sigma S_t \mathrm{d}B_t^{\mathbb{Q}}) + \frac{1}{2}\frac{\partial^2 C}{\partial S^2}\sigma^2 S_t^2\mathrm{d}t, \\
&= \left(\frac{\partial C}{\partial t} + rS_t\frac{\partial C}{\partial S} + \frac{1}{2}\sigma^2 S_t^2\frac{\partial^2 C}{\partial S^2}\right)\mathrm{d}t + \sigma S_t\frac{\partial C}{\partial S}\mathrm{d}B_t^{\mathbb{Q}},
\end{aligned} \tag{1.5}$$

where $[S]_t = \int_0^t (\mathrm{d}S_u)^2$ is the quadratic variation of S.

Consider a self-financing portfolio $\Pi_t = C(t, S_t) - \Delta_t S_t$, which is required to be risk-free, i.e. $\mathrm{d}\Pi_t = r\Pi_t\mathrm{d}t$. From

$$\begin{aligned}
\mathrm{d}\Pi_t &= \mathrm{d}C(t, S_t) - \Delta_t\mathrm{d}S_t, \\
&= \left(\sigma S_t\frac{\partial C}{\partial S} - \Delta_t\sigma S_t\right)\mathrm{d}B_t^{\mathbb{Q}} + \left(\frac{\partial C}{\partial t} + rS_t\frac{\partial C}{\partial S} + \frac{1}{2}\sigma^2 S_t^2\frac{\partial^2 C}{\partial S^2} - \Delta_t r S_t\right)\mathrm{d}t,
\end{aligned} \tag{1.6}$$

eliminating the stochastic term determines Δ_t

$$\sigma S_t\frac{\partial C}{\partial S} - \Delta_t\sigma S_t = 0, \quad \Rightarrow \quad \Delta_t = \frac{\partial C}{\partial S}. \tag{1.7}$$

Δ_t is known as the hedge ratio or Greek Delta. Substituting Δ_t back into the drift term in $\mathrm{d}\Pi_t$ (1.6) gives

$$\frac{\partial C}{\partial t} + \frac{1}{2}\sigma^2 S_t^2\frac{\partial^2 C}{\partial S^2} = r\underbrace{\left(C(t, S_t) - \frac{\partial C}{\partial S}S_t\right)}_{\Pi_t}.$$

Rearranging then yields the classic Black-Scholes PDE

$$\frac{\partial C}{\partial t} + rS\frac{\partial C}{\partial S} + \frac{1}{2}\sigma^2 S^2\frac{\partial^2 C}{\partial S^2} = rC. \tag{1.8}$$

The boundary conditions for the Black-Scholes PDE are

$$C(T, S) = (S - K)^+, \quad C(t, 0) = 0. \tag{1.9}$$

1.1.2 The Black-Scholes PDE and Heat Equations

It is well known that the Black-Scholes equation can be solved by reducing to a standard heat equation. We will verify it in this section. To solve the Black-Scholes PDE, we introduce the log-price x and time variable τ

$$x = \log\frac{S}{K}, \quad \tau = \frac{\sigma^2}{2}(T - t), \tag{1.10}$$

so that

$$S = Ke^x, \quad t = T - \frac{2\tau}{\sigma^2}, \tag{1.11}$$

and define

$$C(t, S) = Kv(x, \tau). \tag{1.12}$$

To express the Black-Scholes PDE in terms of v, x, τ, we first compute the time derivative

$$\frac{\partial v}{\partial \tau} = \frac{1}{K}\frac{\partial C}{\partial \tau} = \frac{1}{K}\frac{\partial C}{\partial t}\frac{\partial t}{\partial \tau} = \frac{-2}{K\sigma^2}\frac{\partial C}{\partial t},$$

that is

$$\frac{\partial C}{\partial t} = -\frac{K\sigma^2}{2}\frac{\partial v}{\partial \tau}. \tag{1.13}$$

Next, compute the first order spatial derivative

$$\frac{\partial v}{\partial x} = \frac{1}{K}\frac{\partial C}{\partial x} = \frac{1}{K}\frac{\partial C}{\partial S}\frac{\partial S}{\partial x} = e^x\frac{\partial C}{\partial S},$$

that is

$$\frac{\partial C}{\partial S} = e^{-x}\frac{\partial v}{\partial x}. \tag{1.14}$$

And compute the second order derivative,

$$\frac{\partial^2 v}{\partial x^2} = \frac{\partial}{\partial x}\left(e^x\frac{\partial C}{\partial S}\right) = e^x\frac{\partial C}{\partial S} + e^x\frac{\partial}{\partial S}\left(\frac{\partial C}{\partial x}\right) = e^x\frac{\partial C}{\partial S} + Ke^{2x}\frac{\partial^2 C}{\partial S^2},$$

that is

$$\frac{\partial^2 C}{\partial S^2} = \frac{1}{K}e^{-2x}\left(\frac{\partial^2 v}{\partial x^2} - \frac{\partial v}{\partial x}\right). \tag{1.15}$$

Substituting Eqs. (1.13), (1.14) and (1.15) into the Black-Scholes equation (1.8), and use the variable transformation Eqs. (1.11) and (1.12) we get

$$\begin{aligned}
-\frac{K\sigma^2}{2}\frac{\partial v}{\partial \tau} + rK\frac{\partial v}{\partial x} + \frac{1}{2}\sigma^2 K\left(\frac{\partial^2 v}{\partial x^2} - \frac{\partial v}{\partial x}\right) &= rKv, \\
\frac{\partial v}{\partial \tau} - \left(\frac{2r}{\sigma^2} - 1\right)\frac{\partial v}{\partial x} - \frac{\partial^2 v}{\partial x^2} &= -\frac{2r}{\sigma^2}v, \\
\frac{\partial v}{\partial \tau} &= \frac{\partial^2 v}{\partial x^2} + (\lambda - 1)\frac{\partial v}{\partial x} - \lambda v, \quad \underline{\lambda = \frac{2r}{\sigma^2}},
\end{aligned} \tag{1.16}$$

with boundary condition

$$v(x, 0) = \max(e^x - 1, 0). \tag{1.17}$$

The equation (1.16) still contains a first-order spatial derivative and a zeroth-order ("linear") term. To reduce it to the standard heat equation, set $v(x,\tau) = \exp(Ax + B\tau)\theta(x,\tau)$, applying the method of undetermined coefficients to choose A and B so that both the θ and the $\partial_x\theta$ vanish. The derivatives are calculated as

$$\frac{\partial v}{\partial \tau} = B\exp(Ax + B\tau)\theta + \exp(Ax + B\tau)\frac{\partial \theta}{\partial \tau}, \tag{1.18}$$

$$\frac{\partial v}{\partial x} = A\exp(Ax + B\tau)\theta + \exp(Ax + B\tau)\frac{\partial \theta}{\partial x}, \tag{1.19}$$

$$\frac{\partial^2 v}{\partial x^2} = A^2\exp(Ax + B\tau)\theta + 2A\exp(Ax + B\tau)\frac{\partial \theta}{\partial x} + \exp(Ax + B\tau)\frac{\partial^2 \theta}{\partial x^2}. \tag{1.20}$$

Plugging Eqs. (1.18), (1.19) and (1.20) to Eq. (1.16), we get

$$\begin{aligned} B\theta + \frac{\partial\theta}{\partial\tau} &= A^2\theta + 2A\frac{\partial\theta}{\partial x} + \frac{\partial^2\theta}{\partial x^2} + (\lambda-1)\left(A\theta + \frac{\partial\theta}{\partial x}\right) - \lambda\theta, \\ \frac{\partial\theta}{\partial\tau} &= \frac{\partial^2\theta}{\partial x^2} + (2A+\lambda-1)\frac{\partial\theta}{\partial x} + \left[A^2 + (\lambda-1)A - \lambda - B\right]\theta. \end{aligned}$$

To eliminate the $\partial_x\theta$ and θ terms, A and B must satisfy

$$\begin{cases} 2A+\lambda-1=0, \\ A^2+(\lambda-1)A-\lambda-B=0, \end{cases}$$

which yields

$$\begin{cases} A = -\frac{1}{2}(\lambda-1), \\ B = -\frac{1}{4}(\lambda+1)^2. \end{cases} \tag{1.21}$$

Therefore, we have

$$v(x,\tau) = \exp\left[-\frac{1}{2}(\lambda-1)x - \frac{1}{4}(\lambda+1)^2\tau\right]\theta(x,\tau), \tag{1.22}$$

and the Black-Scholes equation reduces to a heat equation

$$\frac{\partial\theta}{\partial\tau} = \frac{\partial^2\theta}{\partial x^2} \tag{1.23}$$

with boundary condition

$$\theta(x,0) = \exp\left[\frac{1}{2}(\lambda-1)x\right]\max(e^x-1,0) = \max\left\{\exp\left[\frac{1}{2}(\lambda+1)x\right] - \exp\left[\frac{1}{2}(\lambda-1)x\right], 0\right\}. \tag{1.24}$$

1.1.3 The Black-Scholes Formula

The solution to the heat equation (1.23) is given by the integral formula write a appendix for heat equation

$$\theta(x,\tau) = \frac{1}{2\sqrt{\pi\tau}}\int_{-\infty}^{\infty}\theta(y,0)\exp\left[-\frac{(x-y)^2}{4\tau}\right]\mathrm{d}y, \tag{1.25}$$

where $\theta(y,0)$ follows from the boundary condition (1.24)

$$\begin{aligned} \theta(y,0) &= \max\left\{\exp\left(\frac{1}{2}\lambda y\right)\left[\exp\left(\frac{1}{2}y\right) - \exp\left(-\frac{1}{2}y\right)\right], 0\right\}, \\ &= \begin{cases} \exp\left(\frac{1}{2}\lambda y\right)\left[\exp\left(\frac{1}{2}y\right) - \exp\left(-\frac{1}{2}y\right)\right], & y > 0, \\ 0, & y < 0. \end{cases} \end{aligned} \tag{1.26}$$

Substituting the piecewise initial condition from Eq. (1.26) into the integral solution (1.25) yields,

$$\begin{aligned} \theta(x,\tau) &= \frac{1}{2\sqrt{\pi\tau}}\int_0^{\infty}\exp\left(\frac{1}{2}\lambda y\right)\left[\exp\left(\frac{1}{2}y\right) - \exp\left(-\frac{1}{2}y\right)\right]\exp\left[-\frac{(x-y)^2}{4\tau}\right]\mathrm{d}y, \\ &= \underbrace{\frac{1}{2\sqrt{\pi\tau}}\int_0^{\infty}\exp\left[\frac{1}{2}(\lambda+1)y\right]\exp\left[-\frac{(x-y)^2}{4\tau}\right]\mathrm{d}y}_{I_1} - \underbrace{\frac{1}{2\sqrt{\pi\tau}}\int_0^{\infty}\exp\left[\frac{1}{2}(\lambda-1)y\right]\exp\left[-\frac{(x-y)^2}{4\tau}\right]\mathrm{d}y}_{I_2}, \\ &= I_1 - I_2. \end{aligned} \tag{1.27}$$

The solution decomposes into two integrals, I_1 and I_2, which differ only by the sign in $\lambda \pm 1$. Since I_1 and I_2 share the same form, solving one immediately yields the other. To evaluate I_1, perform a Gaussian integral via the following substitution

$$z = \frac{y - x}{\sqrt{2\tau}}, \quad \mathrm{d}z = \frac{1}{\sqrt{2\tau}}\mathrm{d}y. \tag{1.28}$$

Applying this substitution to I_1 gives

$$\begin{aligned}
I_1 &= \frac{1}{\sqrt{2\pi}} \int_{-\frac{x}{\sqrt{2\tau}}}^{\infty} \exp\left[\frac{-z^2}{2}\right] \exp\left[\frac{1}{2}(\lambda+1)(\sqrt{2\tau}z + x)\right] \mathrm{d}z, \\
&= \frac{1}{\sqrt{2\pi}} \int_{-\frac{x}{\sqrt{2\tau}}}^{\infty} \exp\left\{\frac{1}{2}\left[-z^2 + (\lambda+1)\sqrt{2\tau}z - \left((\lambda+1)\sqrt{\frac{\tau}{2}}\right)^2 + (\lambda+1)^2\frac{\tau}{2} + (\lambda+1)x\right]\right\} \mathrm{d}z, \\
&= \exp\left[\frac{1}{4}(\lambda+1)^2\tau + \frac{1}{2}(\lambda+1)x\right] \frac{1}{\sqrt{2\pi}} \int_{-\frac{x}{\sqrt{2\tau}}}^{\infty} \exp\left\{-\frac{1}{2}\left[z - (\lambda+1)\sqrt{\frac{\tau}{2}}\right]^2\right\} \mathrm{d}z, \\
&= \exp\left[\frac{1}{2}(\lambda+1)x + \frac{1}{4}(\lambda+1)^2\tau\right] \frac{1}{\sqrt{2\pi}} \int_{-\frac{x}{\sqrt{2\tau}}-(\lambda+1)\sqrt{\frac{\tau}{2}}}^{\infty} \exp\left\{-\frac{1}{2}u^2\right\} \mathrm{d}u, \quad \underline{u = z - (\lambda+1)\sqrt{\frac{\tau}{2}}}, \\
&= \exp\left[\frac{1}{2}(\lambda+1)x + \frac{1}{4}(\lambda+1)^2\tau\right] N(d_1), \quad \underline{d_1 = \frac{x + (\lambda+1)\tau}{\sqrt{2\tau}}}.
\end{aligned} \tag{1.29}$$

Note that for a normal distribution, we use $\mathcal{N}$ for the probability density function, and N for the cumulative distribution function. Similarly, we obtain

$$I_2 = \exp\left[\frac{1}{2}(\lambda-1)x + \frac{1}{4}(\lambda-1)^2\tau\right] N(d_2), \quad \underline{d_2 = \frac{x + (\lambda-1)\tau}{\sqrt{2\tau}}}. \tag{1.30}$$

Finally, we express the call option $C(t, S)$ in $\theta(x, \tau)$, using Eqs. (1.12) and (1.22). Then, substitute I_1 (1.29) and I_2 (1.30) into the integral solution of $\theta(x, \tau)$ (1.27) yields

$$\begin{aligned}
C(t, S) &= K \exp\left[-\frac{1}{2}(\lambda-1)x - \frac{1}{4}(\lambda+1)^2\tau\right] \theta(x, \tau), \\
&= K \exp\left[-\frac{1}{2}(\lambda-1)x - \frac{1}{4}(\lambda+1)^2\tau\right] (I_1 - I_2), \\
&= K \exp(x) N(d_1) - K \exp(-\lambda\tau) N(d_2).
\end{aligned}$$

Replacing x, τ, λ by their definitions in terms of S, t, r, σ via Eqs. (1.10) and (1.16), we get the Black-Scholes formula for call options,

$$C(t, S) = SN(d_1) - K \exp[-r(T-t)] N(d_2). \tag{1.31}$$

with parameters d_1 and d_2 given by

$$d_{1,2} = \frac{\log\left(\frac{S}{K}\right) + \left[r \pm \frac{\sigma^2}{2}\right](T-t)}{\sigma\sqrt{T-t}}. \tag{1.32}$$

1.2 Probabilistic Approach

Under the risk neutral measure $\mathbb{Q}$, the stock price follows a Geometric Brownian Motion (previously shown in Eq. (1.3))

$$\mathrm{d}S_t = rS_t\mathrm{d}t + \sigma S_t \mathrm{d}B_t^{\mathbb{Q}},$$

where r is risk free rate. The first step is to solve for S_t. Let $A_t = \log S_t$. Applying Itô's lemma yields

$$\begin{aligned}
\mathrm{d}A_t &= \frac{1}{S_t}\mathrm{d}S_t - \frac{1}{2}\frac{1}{S_t^2}\mathrm{d}[S]_t, \\
&= r\mathrm{d}t + \sigma \mathrm{d}B_t^{\mathbb{Q}} - \frac{1}{2}\sigma^2\mathrm{d}t, \\
&= \left(r - \frac{1}{2}\sigma^2\right)\mathrm{d}t + \sigma\mathrm{d}B_t^{\mathbb{Q}}.
\end{aligned} \tag{1.33}$$

Since the right-hand side is entirely known, we integrate and obtain

$$\begin{aligned}
A_t &= A_0 + \int_0^t \left(r - \frac{1}{2}\sigma^2\right)\mathrm{d}s + \sigma\int_0^t \mathrm{d}B_s^{\mathbb{Q}}, \\
&= A_0 + \left(r - \frac{1}{2}\sigma^2\right)t + \sigma B_t^{\mathbb{Q}}, \\
S_t &= S_0 \exp\left[\left(r - \frac{1}{2}\sigma^2\right)t + \sigma B_t^{\mathbb{Q}}\right].
\end{aligned} \tag{1.34}$$

Thus, looking from the current time t, the solution at the expiration time T is

$$S_T = S_t \exp\left[\left(r - \frac{1}{2}\sigma^2\right)(T-t) + \sigma\left(B_T^{\mathbb{Q}} - B_t^{\mathbb{Q}}\right)\right]. \tag{1.35}$$

To better understand the distribution of S_T, we take the logarithm

$$\log S_T = \underbrace{\log S_t + \left(r - \frac{1}{2}\sigma^2\right)(T-t)}_{\text{deterministic value}} + \sigma\underbrace{\left(B_T^{\mathbb{Q}} - B_t^{\mathbb{Q}}\right)}_{\text{r.v.}\sim\mathcal{N}(0,T-t)}, \tag{1.36}$$

that is, $\log S_T$ is a normal distribution, hence S_t is a log-normal distribution. Define $\tau = T - t$, the time to maturity, and let $Y \sim \mathcal{N}(0,1)$ be the standard normal distribution. We have a cleaner form for S_T

$$S_T = S_t \exp\left[\left(r - \frac{1}{2}\sigma^2\right)\tau + \sigma\sqrt{\tau}Y\right]. \tag{1.37}$$

Recall that the price for call options is the conditional expectation of $(S-K)^+$ at maturity discounted to the current time t, i.e., Eq. (1.4)

$$C(t, S_t) = e^{-r\tau}\mathbb{E}^{\mathbb{Q}}[(S_T - K)^+ | S_t = S].$$

We perform the following calculation

$$\begin{aligned}
C(t,S_t) &= \exp(-r\tau)\mathbb{E}^{\mathbb{Q}}\left[\left(S_t \exp\left\{\left(r - \frac{1}{2}\sigma^2\right)\tau + \sigma\sqrt{\tau}Y\right\} - K\right)^+ \middle| S_t = S\right], \\
&= \exp(-r\tau)\mathbb{E}^{\mathbb{Q}}\left[\left(S \exp\left\{\left(r - \frac{1}{2}\sigma^2\right)\tau + \sigma\sqrt{\tau}Y\right\} - K\right)^+\right], \\
&= \exp(-r\tau)\int_{-\infty}^{+\infty}\left(S \exp\left\{\left(r - \frac{1}{2}\sigma^2\right)\tau + \sigma\sqrt{\tau}y\right\} - K\right)^+ \frac{1}{\sqrt{2\pi}}\exp\left(-\frac{y^2}{2}\right)\mathrm{d}y.
\end{aligned}$$

To get rid of the positive-part operator $()^+$, we calculate the critical point of y

$$
\begin{aligned}
S\exp\left\{\left(r-\frac{1}{2}\sigma^2\right)\tau+\sigma\sqrt{\tau}y\right\}-K=0,\\
\left(r-\frac{1}{2}\sigma^2\right)\tau+\sigma\sqrt{\tau}y=\log\frac{K}{S},\\
y=\underbrace{\frac{\log\frac{K}{S}-\left(r-\frac{1}{2}\sigma^2\right)\tau}{\sigma\sqrt{\tau}}}_{=l}.
\end{aligned}
\tag{1.38}
$$

With this information, we proceed with our calculation

$$
\begin{aligned}
C(t,S_t)&=\exp(-r\tau)\int_l^{+\infty}\left(S\exp\left\{\left(r-\frac{1}{2}\sigma^2\right)\tau+\sigma\sqrt{\tau}y\right\}-K\right)\frac{1}{\sqrt{2\pi}}\exp\left(-\frac{y^2}{2}\right)\mathrm{d}y,\\
&=S\int_l^{+\infty}\exp\left(-\frac{1}{2}\sigma^2\tau+\sigma\sqrt{\tau}y\right)\frac{1}{\sqrt{2\pi}}\exp\left(-\frac{y^2}{2}\right)\mathrm{d}y-\exp(-r\tau)KN(-l),\\
&=S\int_l^{+\infty}\frac{1}{\sqrt{2\pi}}\exp\left[-\frac{1}{2}\left(y^2-2\sigma\sqrt{\tau}y+\left(\sigma\sqrt{\tau}\right)^2\right)\right]\mathrm{d}y-K\exp(-r\tau)N(-l),\\
&=S\int_{l-\sigma\sqrt{\tau}}^{+\infty}\frac{1}{\sqrt{2\pi}}\exp\left(-\frac{1}{2}u^2\right)\mathrm{d}u-K\exp(-r\tau)N(-l),\quad \underline{u=y-\sigma\sqrt{\tau},}\\
&=SN(\sigma\sqrt{\tau}-l)-K\exp(-r\tau)N(-l),\\
&=SN(d_1)-K\exp(-r\tau)N(d_2),
\end{aligned}
\tag{1.39}
$$

where in the last step, we simply verify that

$$
\begin{aligned}
\sigma\sqrt{\tau}-l&=\frac{\sigma^2\tau+\log\frac{S}{K}+\left(r-\frac{1}{2}\sigma^2\right)\tau}{\sigma\sqrt{\tau}}=\frac{\log\frac{S}{K}+\left(r+\frac{1}{2}\sigma^2\right)\tau}{\sigma\sqrt{\tau}}=d_1,\\
-l&=\frac{\log\frac{S}{K}+\left(r-\frac{1}{2}\sigma^2\right)\tau}{\sigma\sqrt{\tau}}=d_2.
\end{aligned}
$$

This result Eq. (1.39) calculated from expectation matches the result Eq. (1.31) from solving the Black-Scholes PDE.