Propositional Logic

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Table of Content

Basics

- Proposition (declarative sentence): a statement that's either true or false

Example

- It is sunny today
- 2+2=5
- The sun is made of sunflower oil
- $X + Y < 50$
- The sum of two prime numbers is a prime number.

- Not a proposition
  - Are you happy?
  - What a nice sunny day!

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Propositional Variable

- In general, variables are placeholders of values
- Meaning: what the variable symbol means
- Value: is value that the variable takes from its domain

Example

A: The age of Bob (meaning of A)
23: the value of A from the domain D=[0..150]

- Propositional variable represents an arbitrary proposition
- Normally represented by Boolean variables
- The domain of a Propositional variable is True,False

Propositional Logic: Syntax

- Propositional logic deals with *well-formed propositional expressions*
- Constants true and false are propositional expressions
- Propositional variables are propositional expressions
- If A and B are well-formed propositional expressions, then so are
  - $A \vee B$
  - $A \wedge B$
  - $\neg A$
  - $A \rightarrow B$ (If A holds then B holds as well)

Example

$(A \vee B) \wedge (\neg A)$

- Operators precedence order: $\neg$ (tightest), $\wedge$, $\vee$, $\rightarrow$

Propositional Logic: Syntax

- Operators precedence order: $\neg$ (tightest), $\wedge$, $\vee$, $\rightarrow$
- We can check if a formula is well-formed using its parse tree in a recursive way

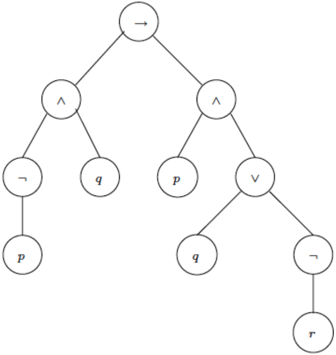

Figure: [1]

Formalizing Sentences in Propositional Logic

- Decompose the set of sentences into a set of *atomic* or indecomposable ones (look for the connectors such as if, then, therefore ect)
- Specify each with a propositional variable
- Use the operators to compose them

Example

- It rains whenever the wind blows from the southeast.
- The peach tree will bloom if it stays warm for a week.

Examples

Example

- Trains run late on exactly those days that I take the train.
- BP To get an A in this course, it is necessary for you to get an A on both parts.
- BP Getting an A on the final and doing all the assignment perfectly is sufficient for getting an A in this course.
- You will get an A in this course if and only if you either do every assignment or you get an A on the final.
- BP The results are not posted unless the exam has been graded.
- BP The exam has not been graded, or it has been graded but the results are not (yet) posted.

Table of Content

Semantics

What does an expression E in propositional logic mean?

- A declarative sentence expresses a fact about something, e.g. a program, a physical system, our thoughts, the real world etc.
- Such factual statements either are true or false.

Definition

A *model* or *valuation* of a propositional expression ϕ is a function from its propositional variables to their corresponding truth values.

Definition

The semantics (meaning) of a propositional expression ϕ is a function from a model to a truth value.

Semantics

- Meaning of a logical operator is a function from its operands values to a truth value.
- We use a truth table to represent this function.
- Semantics of $\wedge$

p	q	$p \wedge q$
$\top$	$\top$	$\top$
$\top$	$\bot$	$\bot$
$\bot$	$\top$	$\bot$
$\bot$	$\bot$	$\bot$

Semantics

- A truth table to describe the semantics of an expressions
- Let ϕ be an expression with k variables and n operators
- 2^k rows (each captures one possible valuations of variables)
- n columns for each operator
- Exponential order: $O(2^k \times n)$

Example

p	q	$\neg p$	$\neg q$	$p \rightarrow \neg q$	$q \vee \neg p$	$(p \rightarrow \neg q) \rightarrow (q \vee \neg p)$
T	T	F	F	F	T	T
T	F	F	T	T	F	F
F	T	T	F	T	T	T
F	F	T	T	T	T	T

Table of Content

Tautologies

- Tautology is an expression that is true under all possible truth assignments.

Example

- $p \vee (\neg p)$
- $p \vee q \equiv q \vee p$
- $p \vee True \equiv True$
- $p \wedge False \equiv False$

- Tautology problem: Checking if a propositional expression is true or not.

Truth Table-based Proof

Definition

If, for all valuation in which all the formulas $\phi_1, \phi_2, ...\phi_n$ evaluate to true, then ψ evaluates to true, we say $\phi_1, \phi_2, ...\phi_n \models \psi$ holds.

- $\models$ is called *the semantic entailment relation*.
- $\phi_1, \phi_2, ...\phi_n \models \psi$
  - equals to $\models (\phi_1 \wedge \phi_2 \wedge ..... \wedge \phi_n) \rightarrow \psi$ or,
  - it means $(\phi_1 \wedge \phi_2 \wedge ..... \wedge \phi_n) \rightarrow \psi$ is a tautology

Example

$\models p \vee q \equiv q \vee p$?

p	q	p $\vee$ q	q $\vee$ p	p $\vee$ q $\equiv$ q $\vee$ p
$\top$	$\top$	$\top$	$\top$	$\top$
$\top$	$\bot$	$\top$	$\top$	$\top$
$\bot$	$\top$	$\top$	$\top$	$\top$
$\bot$	$\bot$	$\bot$	$\bot$	$\top$

Table of Content

Natural Deduction

- Need a set of rules each of which allows us to draw a conclusion from a certain set of premises.
- The proof rules in natural deduction that allow us to infer formulas from other formulas!
- If we apply these rules in succession, a conclusion may be inferred from a set of premises.

Natural Deduction

- Let ϕ_1, ϕ_2, ... ϕ_n be a set of formulas called premises
- Let ψ be another formula which we call conclusion.
- By applying proof rules to the premises, we can get some more formulas.
- If we repeat this process considering the new derived formulas as well, we hope to obtain the conclusion ψ eventually.
- This is denoted by $\phi_1, \phi_2, \ldots, \phi_n \vdash \psi$ and called a sequent.

Deductive Proof

- A sequence of steps
- Each step (line) is either a given hypothesis or is a formula driven from its previous lines using inference rules

$$
\begin{array}{ll}
(1) & \phi_1 \\
(2) & \phi_2 \\
 & \dots \\
(k) & \beta_1 \\
(k+1) & \beta_2 \\
\dots & \\
(m) & \psi
\end{array}
$$

Deductive Proof

- The following notation means that if $\phi_1, \ldots \phi_n$ are in the previous lines, we can add ψ to the proof as well!

$$\frac{\phi_1 \ldots \ \phi_n}{\psi}$$

Basic Inference Rules- Conjunction Rules

- Conjunctive introduction

$$\frac{\phi \qquad \psi}{\phi \wedge \psi} \wedge i$$

- Conjunctive elimination

$$\frac{\phi \wedge \psi}{\phi} \wedge e1$$

$$\frac{\phi \wedge \psi}{\psi} \wedge e2$$

Basic Inference Rules- Conjunction Rules

Example

Prove $(A \wedge B) \wedge C \vdash B \wedge C$

1.	$(A \wedge B) \wedge C$	*Premise*
2.	$(A \wedge B)$	$\wedge e1$ 1
3.	C	$\wedge e2$ 1
4.	B	$\wedge e2$ 2
5.	$B \wedge C$	$\wedge i$ 3, 4

Basic Inference Rules- Double Negation Rules

- Double Negation introduction

$$\frac{\phi}{\neg\neg\phi}\neg\neg i$$

- Double Negation elimination

$$\frac{\neg\neg\phi}{\phi}\neg\neg e$$

Example

Prove $p, \neg\neg(q \wedge r) \vdash \neg\neg p \wedge r$

1	p	premise
2	$\neg\neg(q \wedge r)$	premise
3	$\neg\neg p$	$\neg\neg$i 1
4	$q \wedge r$	$\neg\neg$e 2
5	r	$\wedge$e$_2$ 4
6	$\neg\neg p \wedge r$	$\wedge$i 3, 5

Modus Tollens and Modus Ponens Rules

- Implication elimination (or Modus Ponens Rule)

$$\frac{\phi \quad \phi \to \psi}{\psi} \to e$$

- Modus Tollens Rule

$$\frac{\phi \to \psi \quad \neg\psi}{\neg\phi} MT$$

Example

Prove $p \to (q \to r), p, \neg r \vdash \neg q$

1	$p \to (q \to r)$	premise
2	p	premise
3	$\neg r$	premise
4	$q \to r$	$\to$e 1, 2
5	$\neg q$	MT 4, 3

Basic Inference Rules- Implication Rule and Assumption Introduction

- We can add the new assumptions that are not in the premises, but it should be placed in a box.

$$\begin{array}{l}
\vdots \\
\text{premises} \\
\vdots \\
\boxed{\begin{array}{ll} \phi & \qquad\qquad \text{assumption} \\ \\ \\ \psi & \end{array}} \\
\phi \rightarrow \psi \qquad \rightarrow\text{i}
\end{array}$$

- We use specific rules to use the conclusions derived from an assumption in a box.
- The formulas inside a box cannot always be used outside of that box. Why?

Basic Inference Rules- Implication Rules and Assumption Introduction

- Implication introduction

$$\frac{\boxed{\begin{array}{c}\phi \\ \vdots \\ \psi\end{array}}}{\phi \to \psi} \to\mathrm{i}$$

Example

Prove $p \to q \vdash \neg q \to \neg p$

1	$p \to q$	premise
2	$\neg q$	assumption
3	$\neg p$	MT 1, 2
4	$\neg q \to \neg p$	$\to$i 2–3

Basic Inference Rules- Copying

- This rule allows us to repeat something we already know

Example

Prove $\vdash p \rightarrow (q \rightarrow p)$

1	p	assumption
2	q	assumption
3	p	copy 1
4	$q \rightarrow p$	$\rightarrow$i 2–3
5	$p \rightarrow (q \rightarrow p)$	$\rightarrow$i 1–4

Basic Inference Rules- Negation Rules (and Contradiction)

- Negation introduction

$$\frac{\boxed{\begin{array}{c}\phi \\ \vdots \\ \perp\end{array}}}{\neg\phi}\neg\mathrm{i}$$

- Negation elimination (or contradiction)

$$\frac{\phi \quad \neg\phi}{\perp}\neg e$$

- False elimination (can deduce anything from a false statement)

$$\frac{\perp}{\phi}\perp e$$

Basic Inference Rules- Negation Rules (and Contradiction)

Example

Prove $\neg p \vee q \vdash p \rightarrow q$

1	$\neg p \vee q$			
2	$\neg p$	premise	q	premise
3	p	assumption	p	assumption
4	$\perp$	$\neg$e 3, 2	q	copy 2
5	q	$\perp$e 4	$p \rightarrow q$	$\rightarrow$i 3–4
6	$p \rightarrow q$	$\rightarrow$i 3–5		
7	$p \rightarrow q$			$\vee$e 1, 2–6

Proof By Contradiction

Assume that the conclusion holds and then derive its opposite.

$$\frac{\boxed{\begin{array}{c}\neg\phi \\ \vdots \\ \perp\end{array}}}{\phi}\ \text{PBC.}$$

Proof.

1	$\neg\phi \rightarrow \perp$	given
2	$\neg\phi$	assumption
3	$\perp$	$\rightarrow$e 1, 2
4	$\neg\neg\phi$	$\neg$i 2–3
5	ϕ	$\neg\neg$e 4

Basic Inference Rules- Disjunction Rules

- Disjunctive introduction

$$\frac{\phi}{\phi \vee \psi} \vee i1$$

$$\frac{\psi}{\phi \vee \psi} \vee i2$$

BP Disjunctive elimination. Why?

$$\frac{\phi \vee \psi \qquad \boxed{\begin{array}{c}\phi \\ \vdots \\ \chi\end{array}} \qquad \boxed{\begin{array}{c}\psi \\ \vdots \\ \chi\end{array}}}{\chi} \vee e$$

Basic Inference Rules- Disjunction Rules

Example

Prove $(A \wedge B) \vee (C \wedge A) \vdash A$

1.	$(A \wedge B) \vee (C \wedge A)$	Premise
2.	$(A \wedge B)$	Assumption
3.	A	$\wedge e1$ 2
4.	$(C \wedge A)$	Assumption
5.	A	$\wedge e2$ 4
6.	A	$\vee e$ 1-5

Compound Inference Rules

- Modus tollens

$$\frac{\phi \to \psi \quad \neg\psi}{\neg\phi}$$

- Syllogism

$$\frac{\phi \to \psi \quad \psi \to \alpha}{\phi \to \alpha}$$

- Disjunctive syllogism

$$\frac{\neg\phi \quad \phi \vee \psi}{\psi}$$

Compound Inference Rules

- Resolution

$$\frac{\phi \vee \psi \quad \neg\phi \vee \alpha}{\psi \vee \alpha}$$

- Contradiction

$$\frac{\neg\phi \rightarrow \bot}{\phi}$$

- Equivalence

$$\frac{\phi \equiv \alpha \quad \phi}{\alpha}$$

$$\frac{\phi \rightarrow \psi \quad \psi \rightarrow \phi}{\phi \equiv \psi}$$

- More on basic rules in the book

Compound Inference Rules- Algebraic Laws

- Associativity

$$\frac{}{(\phi \lor \psi) \lor \gamma \equiv \phi \lor (\psi \lor \gamma)}$$

$$\frac{}{(\phi \land \psi) \land \gamma \equiv \phi \land (\psi \land \gamma)}$$

- Distributivity

$$\frac{}{\phi \lor (\psi \land \gamma) \equiv (\phi \lor \psi) \land (\phi \lor \gamma)}$$

$$\frac{}{\phi \land (\psi \lor \gamma) \equiv (\phi \land \psi) \lor (\phi \land \gamma)}$$

- Commutativity

$$\frac{}{(\psi \land \phi) \equiv (\phi \land \psi)}$$

$$\frac{}{(\psi \lor \phi) \equiv (\phi \lor \psi)}$$

Compound Inference Rules- Algebraic Laws

- DeMorgan's laws:

$$\frac{}{\neg(\phi \vee \psi) \equiv \neg\phi \wedge \neg\psi}$$

$$\frac{}{\neg(\phi \wedge \psi) \equiv \neg\phi \vee \neg\psi}$$

- Implication laws:

$$\frac{}{(\phi \rightarrow \psi) \wedge (\psi \rightarrow \phi) \equiv (\phi \equiv \psi)}$$

$$\frac{}{(\phi \rightarrow \psi) \equiv \neg\phi \vee \psi}$$

How to construct the proof?

- Add all the premises at the top of proof
- Add the conclusion (with a question mark) at the bottom of proof
- Try to fill in the between by applying inference rules
- If several rules can be applied, choose the most promising one
- Try to shape a proof strategy in mind first.
- If the formula to be proven (e.g. the conclusion) is an implication, use the implication introduction rule, i.e. you should do a new sub-proof first.

Example 2

Example

Prove $(E \vee F) \to G, G \to H, \neg H \vdash \neg F$

1.	$(E \vee F) \to G$	*Premise*
2.	$G \to H$	*Premise*
3.	$\neg H$	*Premise*
4.	$(E \vee F) \to H$	*Syllogism* : 1&2
5.	$\neg(E \vee F)$	*Modus tollens* : 3&4
6.	$\neg E \wedge \neg F$	*DeMorgan* : 5
8.	$\neg F$	$\wedge e2$: 6

Table of Content

Summary

- Propositions and proposition logic
- Syntax of propositional logic
- Semantics of propositional logic
- Truth-table -based level (the notation $\models$)
- Inference rules (entailment $\vdash$)
- Next lecture on predicate logic