Boundary Value Problem.

Consider the boundary value problem

$$y'' + y = 0, \quad y(0) = 1, \quad y(\pi) = a.$$

a arbitrary

$y = C_1 \cos x + C_2 \sin x.$

$y(0) = 1 \rightarrow C_1 = 1$

$y(\pi) = a \rightarrow C_1 = -a.$

if a is not equal to 1, there is no solution.

if $a = -1$,

$y = \cos x + C_2 \sin x$, C_2 arbitrary.

non-homogeneous boundary value problem may have no solution, and it may have infinitely many solutions.

$y'' + \lambda y = 0$, $y(0) = 0$, $y(\pi) = 0$.

let $\lambda = \mu^2$

$y = C_1 \cos \mu x + C_2 \sin \mu x$

$C_1 = 0$. $C_2 \sin \mu\pi = 0$.

$\mu\pi = n\pi \rightarrow \underline{y = C_2 \sin nx}$

$\therefore$ ~~$y = C_2$~~ $\lambda = n^2 \rightarrow$ eigenvalue.

$y'' - \lambda y = 0$ $y(0) = 0$, $y(\pi) = 0$.

$y = C_1 \cosh \sqrt{\lambda} y + C_2 \sinh \sqrt{\lambda} y$

$C_1 = C_2 = 0$: Unique solution.

$$\begin{cases} y'' = f(x, y, y') \\ y(a) = \alpha, \quad y(b) = \beta \end{cases} \qquad (1)$$

Thm 1) Suppose f in (1) is continuous on the set

$$D = \{(x, y, y') \mid a \le x \le b, -\infty < y < \infty, -\infty < y' < \infty\}$$

and that $\frac{\partial f}{\partial y}$ and $\frac{\partial f}{\partial y'}$ are also continuous on D. If

1. $\frac{\partial f}{\partial y}(x, y, y') > 0$ for all $(x, y, y') \in D$ and
2. $\exists M$, with $\left|\frac{\partial f}{\partial y'}(x, y, y')\right| \le M$.

then (1) has a unique solution.

Corollary 1. If the linear two-point boundary-value problem

$$\begin{cases} y'' = p(x) y' + q(x) y + r(x) \\ y(a) = \alpha, \quad y(b) = \beta. \end{cases}$$

satisfies

1. $p(x)$, $q(x)$ and $r(x)$ are continuous on $[a, b]$, and
2. $q(x) > 0$ on $[a, b]$

then the problem has a unique solution.

* Finite Difference Method for Linear Problems

$$\begin{cases} y'' = p(x)y' + q(x)y + r(x) \\ y(a) = \alpha, \quad y(b) = \beta \end{cases}$$

$$a = x_0 < x_1 < \cdots < x_n = b$$

$$x_i = a + i \times h$$

$$i = 0, 1, 2, \ldots, n$$

with $h = \dfrac{b-a}{n}$

For $i = 1, 2, \ldots, n-1$

$$y''(x_i) = p(x_i)y'(x_i) + q(x_i)y(x_i) + r(x_i)$$

$$y'(x_i) = \frac{y(x_{i+1}) - y(x_{i-1})}{2h} - \frac{h^2}{6}y^{(3)}(\eta_i)$$

$$\eta_i \in (x_{i-1}, x_{i+1})$$

$$y''(x_i) \simeq \frac{y(x_{i+1}) - 2y(x_i) + y(x_{i-1})}{h^2} - \frac{h^2}{12}y^{4}(\xi_i)$$

$$\xi_i \in (x_{i-1}, x_{i+1})$$

$$y(x_{i+1}) = y(x_i) + hy' + \frac{1}{2}h^2y'' + \cdots$$

$$+ \; y(x_{i-1}) = y(x_i) - hy' + \frac{1}{2}h^2y'' + \cdots$$

$$y(x_{i+1}) + y(x_{i-1}) = 2y(x_i) + h^2y'' + \frac{1}{12}h^4y^{(4)}(\xi_i)$$

u_i : the approximate value of $y_i = y(x_i)$

$$y'(x_i) \approx \frac{u_{i+1} - u_{i-1}}{2h}$$

$$y''(x_i) \approx \frac{u_{i+1} - 2u_i + u_{i-1}}{h^2}$$

$p_i = p(x_i)$, $q_i = q(x_i)$, $r_i = r(x_i)$

$$\frac{u_{i+1} - 2u_i + u_{i-1}}{h^2} = p_i \frac{u_{i+1} - u_{i-1}}{2h} + q_i u_i + r_i$$

$i = 1, 2, \dots, n-1.$

$u_0 = \alpha$, $u_n = \beta$

$$\left(1 + \frac{h}{2} p_i\right) u_{i-1} - \left(2 + h^2 q_i\right) u_i + \left(1 - \frac{h}{2} p_i\right) u_{i+1}$$

$$= h^2 r_i$$

for $i = 1, 2, \dots \quad n-1$ $\longrightarrow$ $(n-1)$ equations.

$\boxed{i=1}$

$$\left(1 + \frac{h}{2} p_1\right) u_0 - \left(2 + h^2 q_1\right) u_1 + \left(1 - \frac{h}{2} p_1\right) u_2 = h^2 r_1$$

$$\Leftrightarrow -\left(2 + h^2 q_1\right) u_1 + \left(1 - \frac{h}{2} p_1\right) u_2 = h^2 r_1 - \left(1 + \frac{h}{2} p_1\right) \alpha.$$

$\boxed{1 < i < n-1}$

$$\left(1 + \frac{h}{2} p_i\right) u_{i-1} - \left(2 + h^2 q_i\right) u_i + \left(1 - \frac{h}{2} p_i\right) u_{i+1} = h^2 r_i$$

$\boxed{i = n-1}$

$$\left(1 + \frac{h}{2} p_{n-1}\right) u_{n-2} - \left(2 + h^2 q_{n-1}\right) u_{n-1} + \left(1 - \frac{h}{2} p_{n-1}\right) \beta = h^2 r_{n-1}$$

$$\left(1 + \frac{h}{2} p_{n-1}\right) u_{n-2} - \left(2 + h^2 q_{n-1}\right) u_{n-1} = h^2 r_{n-1} - \left(1 - \frac{h}{2} p_{n-1}\right) \beta$$

$$Au = f$$

$$A = \begin{bmatrix} -2-h^2 q_1 & 1-\frac{h}{2}p_1 & & & & \\ 1+\frac{h}{2}p_2 & -2-h^2 q_2 & 1-\frac{h}{2}p_2 & & & \\ & & \ddots & & & \\ & & & \ddots & & \\ & & & 1+\frac{h}{2}p_{n-2} & -2-h^2 q_{n-2} & 1-\frac{h}{2}p_{n-2} \\ & & & & 1+\frac{h}{2}p_{n-1} & -2-h^2 q_{n-1} \end{bmatrix}$$

$$u = \begin{pmatrix} u_1 \\ u_2 \\ \vdots \\ u_{n-2} \\ u_{n-1} \end{pmatrix} \quad \text{and} \quad f = \begin{pmatrix} h^2 r_1 - (1+\frac{h}{2}p_1)\alpha \\ h^2 r_2 \\ \vdots \\ h^2 r_{n-2} \\ h^2 r_{n-1} - (1-\frac{h}{2}p_{n-1})\beta \end{pmatrix}$$

Tridiagonal Matrix.

try Gaussian Elimination

$$\begin{pmatrix} b_1 & c_1 & & & & \\ a_2 & b_2 & c_2 & & & \\ & a_3 & b_3 & c_3 & & \\ & & \ddots & \ddots & \ddots & \\ & & & & & c_{n-1} \\ & & & & a_n & b_n \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} d_1 \\ d_2 \\ \vdots \\ \vdots \\ d_n \end{pmatrix}$$

$$\begin{cases} w_i = \dfrac{a_i}{b_{i-1}} \\ b_i := b_i - w_i c_{i-1} \\ d_i := d_i - w_i d_{i-1} \end{cases} \qquad i = 2, 3, \cdots, n$$

$$\begin{cases} x_n = \dfrac{d_n}{b_n} \\ x_i = \dfrac{d_i - c_i x_{i+1}}{b_i}, \quad i = n-1, n-2, \cdots, 1 \end{cases}$$

- Shooting Methods

$y'' = f(x, y, y')$

$y(a) = \alpha, \quad y(b) = \beta$.

$\downarrow$

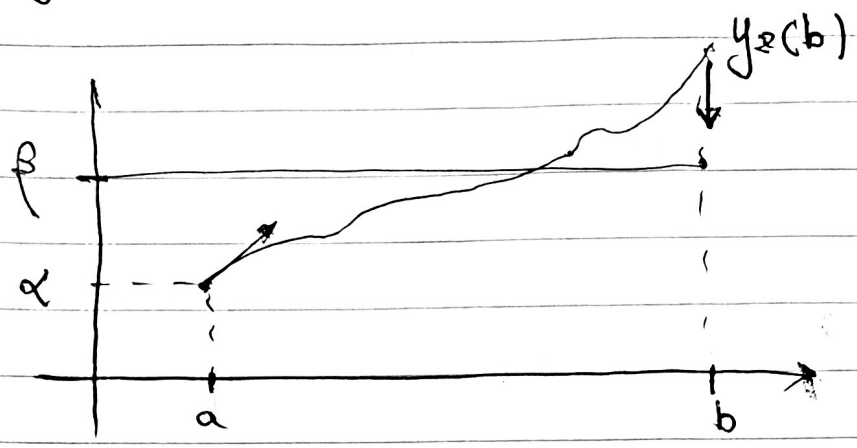
guess $y'(\alpha)$ and solve until $x = b$.
ask $y(b)$ is close to β

Corresponding IVP

$$\begin{cases} y'' = f(x, y, y') \\ y(a) = \alpha, \quad y'(a) = z \end{cases}$$

can be solved by, for example, Runge-Kutta method. We denote this approximate solution y_z and hope $y_z(b) = \beta$.
If not, we use another guess for $y'(a)$ try to solve an altered IVP again.

Object: select z, so that $y_z(b) = \beta$.



Let $\phi(z) = y_z(b) - \beta$.

objective $\rightarrow \phi(z) = 0$

$\boxed{y_z(b) - \beta = 0} \rightarrow$ $\boxed{\text{nonlinear algebraic equation.}}$

Scant method.

$$x_n = x_{n-1} - f(x_{n-1}) \frac{x_{n-1} - x_{n-2}}{f(x_{n-1}) - f(x_{n-2})}$$

· Guess z_1, z_2 and obtain $\phi(z_1)$ and $\phi(z_2)$

$$z_3 = z_2 - \phi(z_2) \frac{z_2 - z_1}{\phi(z_2) - \phi(z_1)}$$

$\vdots$ in general

$$z_{k+1} = z_k - \phi(z_k) \frac{z_k - z_{k-1}}{\phi(z_k) - \phi(z_{k-1})}$$