Xiaoyu's thermostat interpreted as targeting an space or time average

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In Xiaoyu's version of the thermostat the ζ equation reads:

$$\dot{\zeta} = \frac{1}{d} p^T M^{-2}(p) p - tr\left(\frac{d}{dp}\left(\frac{1}{d} M^{-1}(p) p\right)\right) \tag{1}$$

$$= \frac{1}{d} \sum_i^d \left(\left(M^{-1}(p)_{ii} p_i\right)^2 - \frac{d}{dp_i}\left(M^{-1}(p)_{ii} p_i\right)\right) \tag{2}$$

Note that $\frac{dK}{dp_i} = M^{-1}(p)_{ii} p_i$. Hence we have

$$\dot{\zeta} = \frac{1}{d} \sum_i^d \left(\left(\frac{dK}{dp_i}\right)^2 - \frac{d}{dp_i}\frac{dK}{dp_i}\right) \tag{3}$$

Now consider the space average of this quantity:

$$\langle \left(\frac{dK}{dp_i}\right)^2 - \frac{d^2 K}{dp_i^2} \rangle \tag{4}$$

A simple integration by parts gives:

$$\langle \frac{d^2 K}{dp_i^2} \rangle = \int \frac{d^2 K}{dp_i^2} \exp\left(-K(p)\right) dp \tag{5}$$

$$= \underbrace{\int \left[\frac{dK}{dp_i} \exp\left(-K(p)\right)\right]_{p_i=\infty}^{\infty} dp_{-i}}_{=0} - \int \frac{dK}{dp_i}\left(-\frac{dK}{dp_i} \exp\left(-K(p)\right)\right) dp \tag{6}$$

$$= \langle \left(\frac{dK}{dp_i}\right)^2 \rangle \tag{7}$$

Note that because $K(p) = -\log f_p(p)$, the negative log density of p we can interpret $\langle \frac{d^2K}{dp_i^2} \rangle$ as the negative Fisher information of the distribution of the momenta - not entirely sure this helps us but it's nice. In any case this calculation implies that

$$\langle \dot{\zeta} \rangle = 0 \tag{8}$$

I think this makes it clear why this thermostat equation might be expected to work.

Interestingly, the classical thermostat can also be interpreted in this way since $p_i \frac{dK}{dp_i} = \left(\frac{dK}{dp_i}\right)^2$ and $\frac{d^2K}{dp_i^2} = 1$.