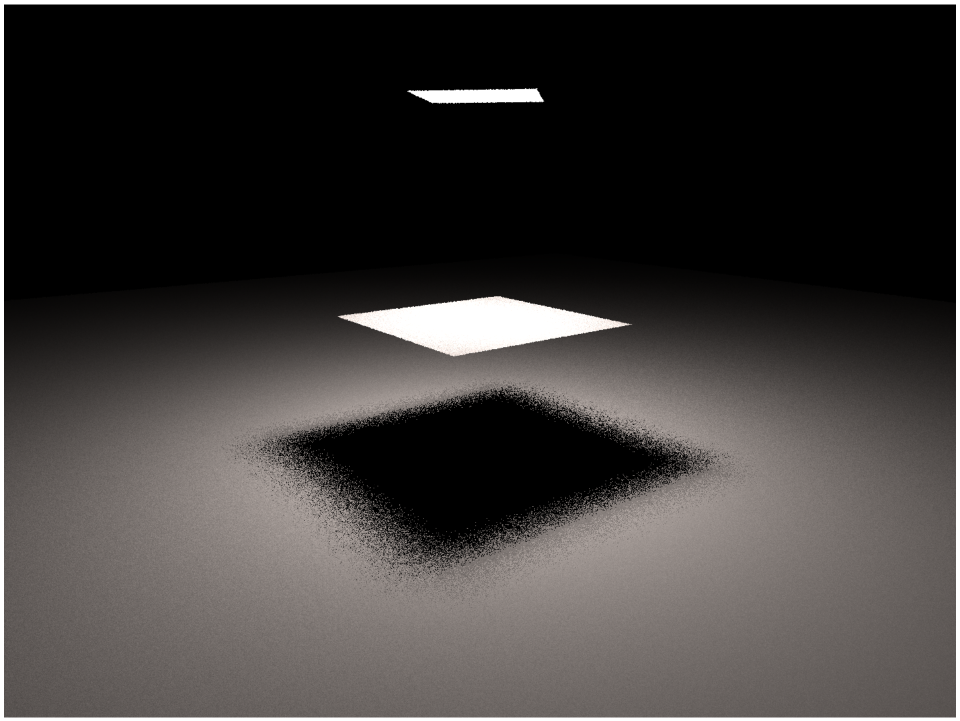
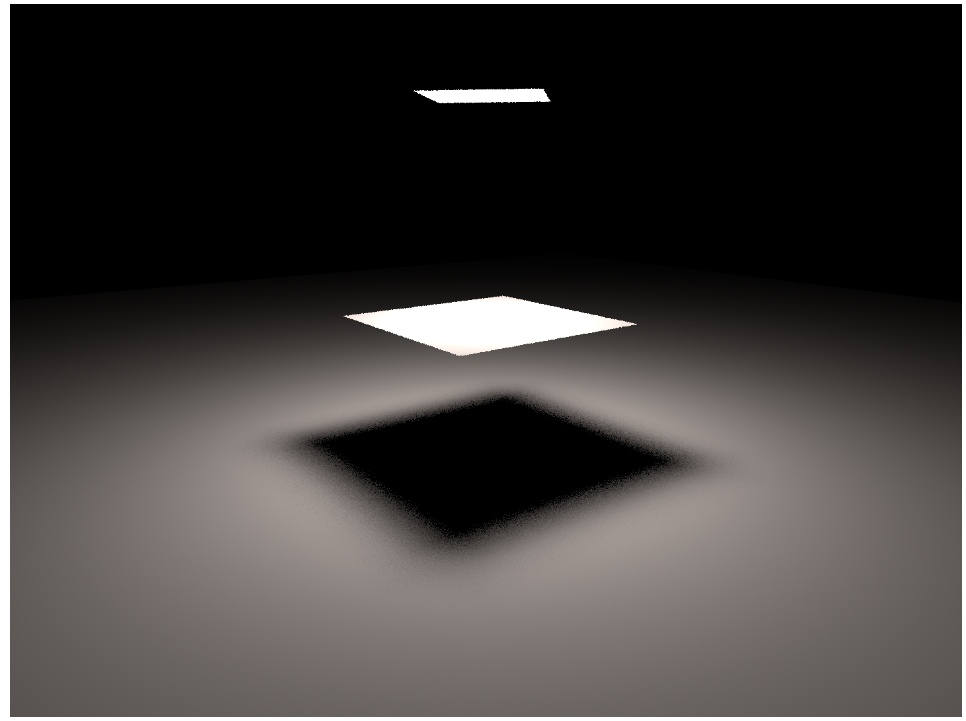
Recap

Quality Improves with More Rays



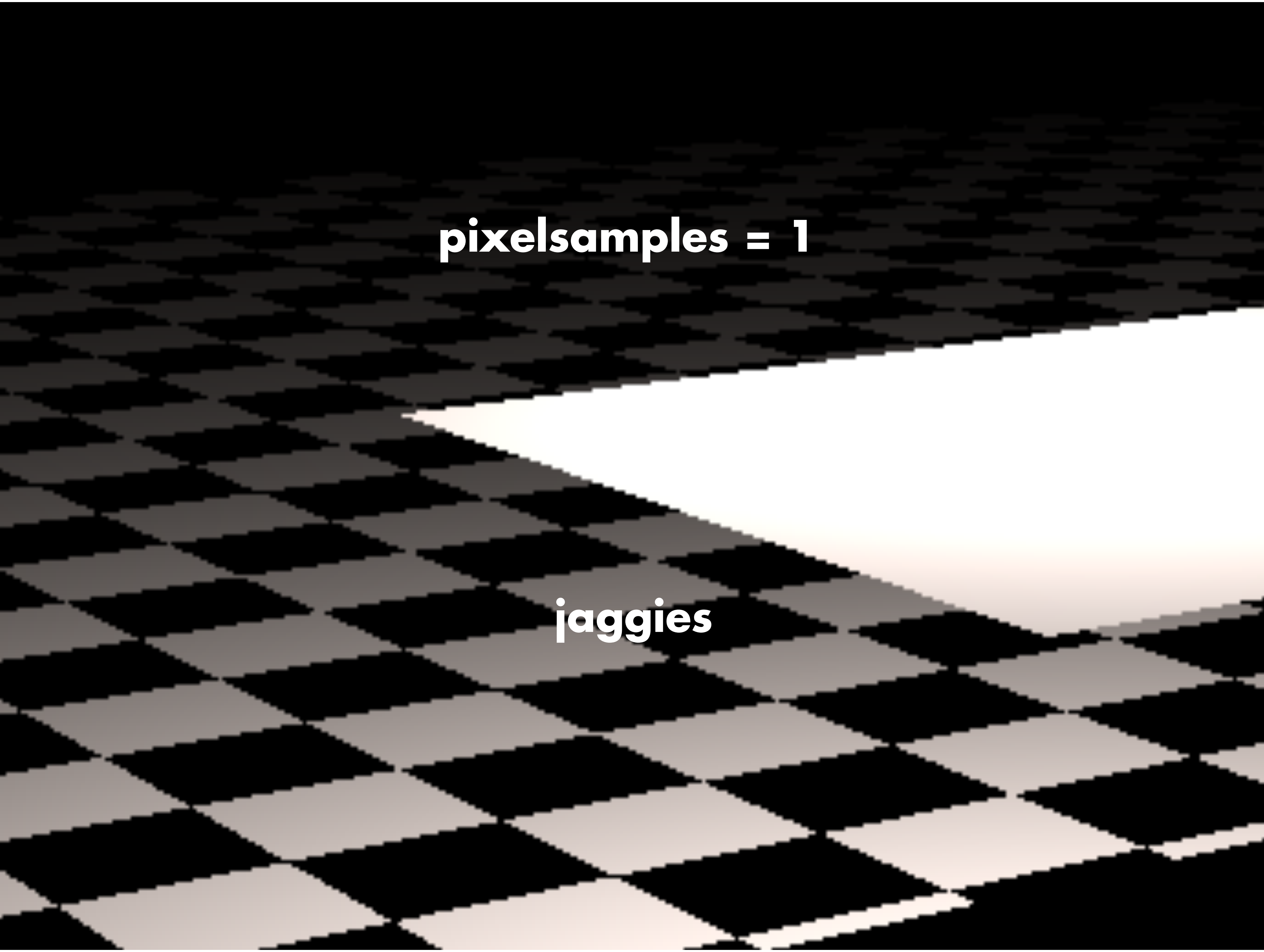
Area

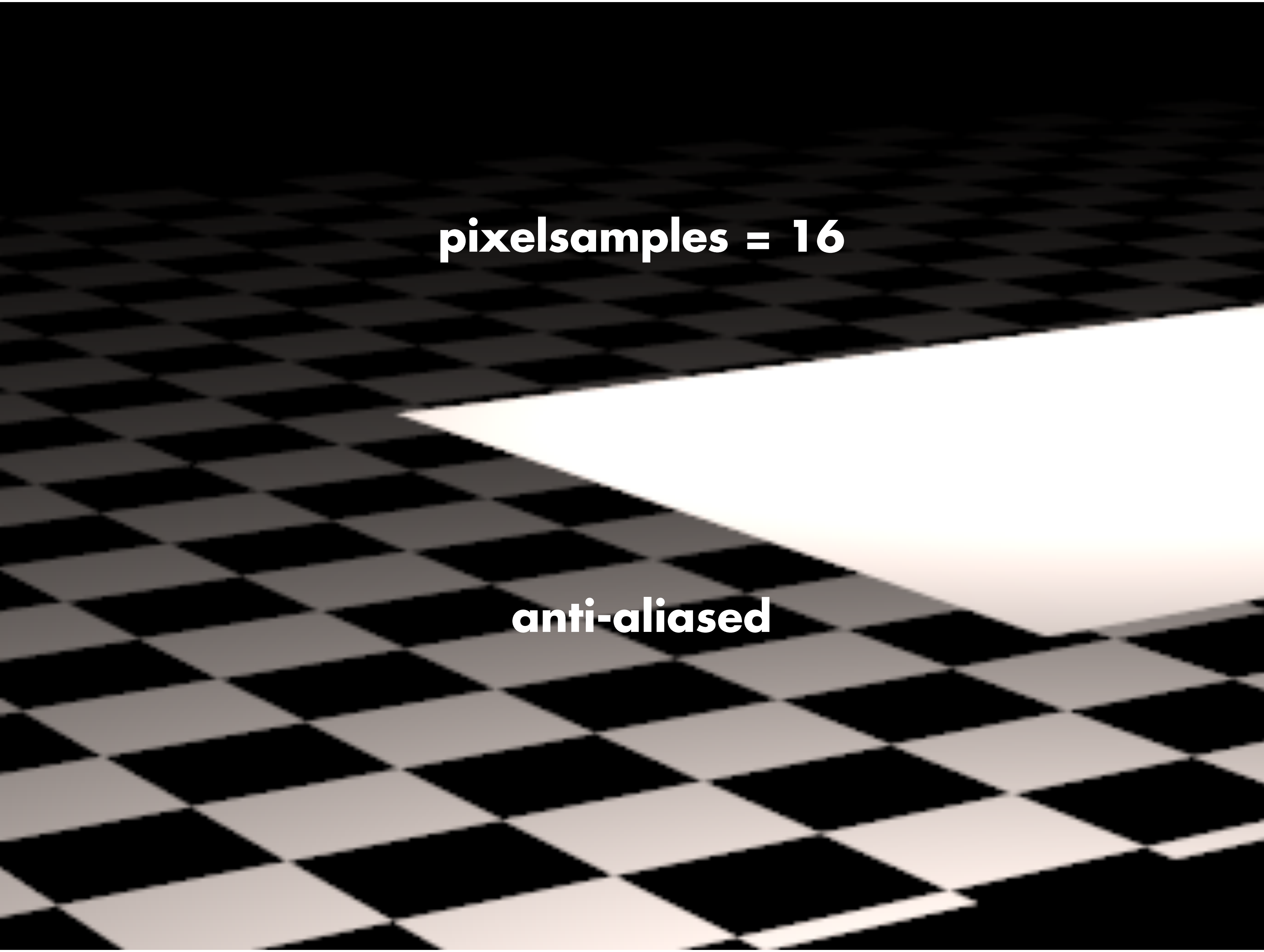
1 shadow ray



Area

16 shadow rays


pixelsamples = 1
jaggies


pixelsamples = 16
anti-aliased

Sampling and Reconstruction

Basic signal processing

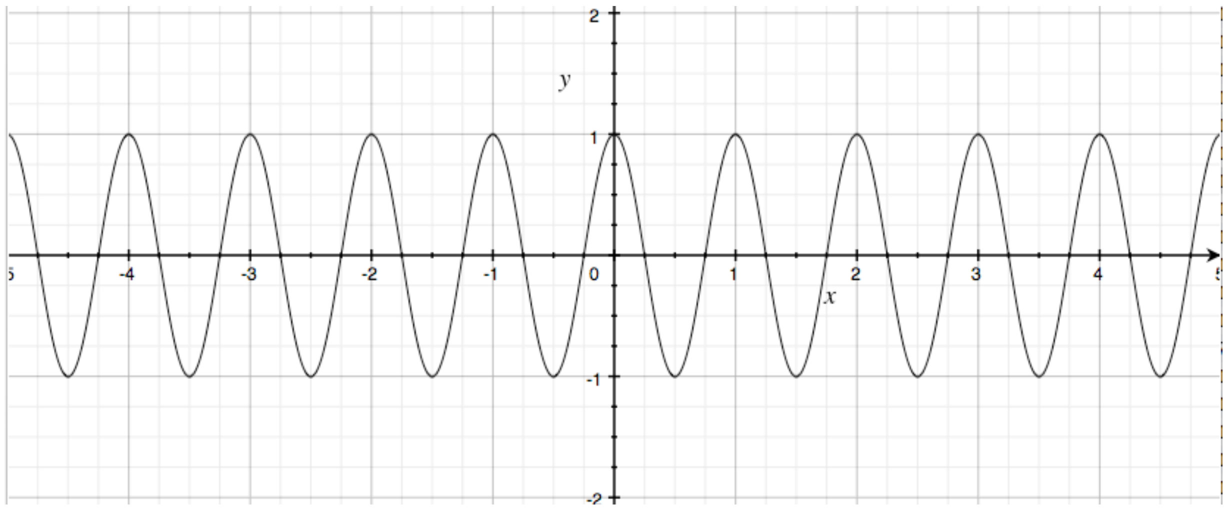
- Fourier transforms
- The convolution theorem
- The sampling theorem

Aliasing and antialiasing

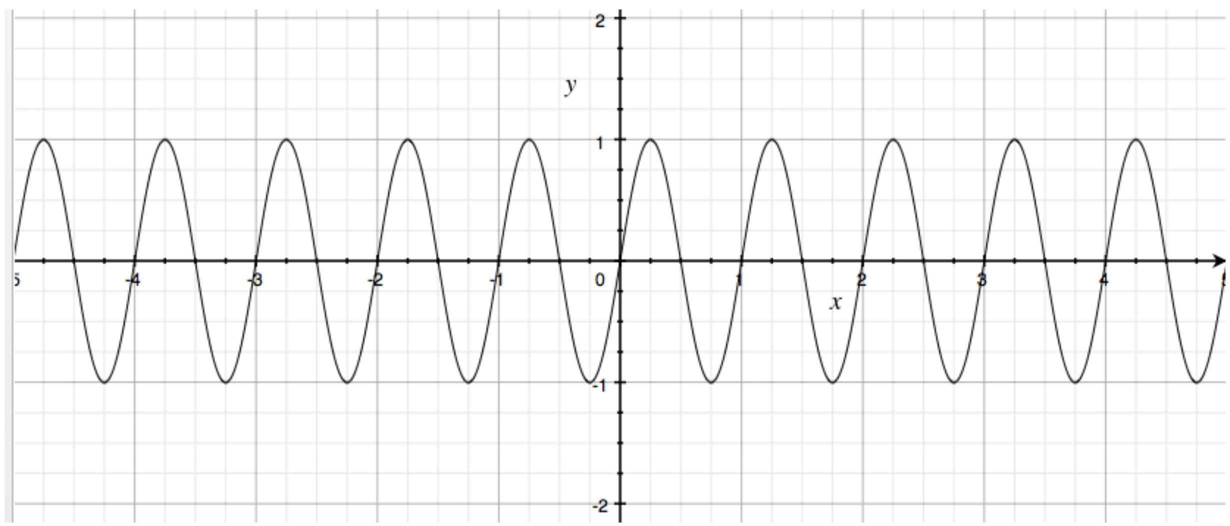
- Uniform supersampling
- Stochastic sampling

Review: Basic Signal Processing

Sines and Cosines



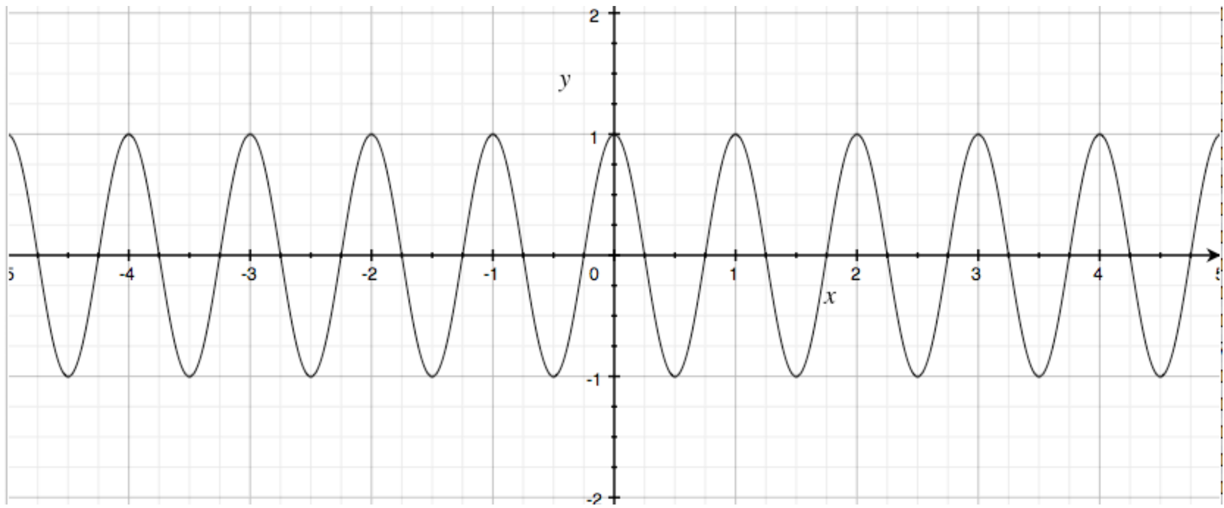
$$\cos 2\pi x$$



$$\sin 2\pi x$$

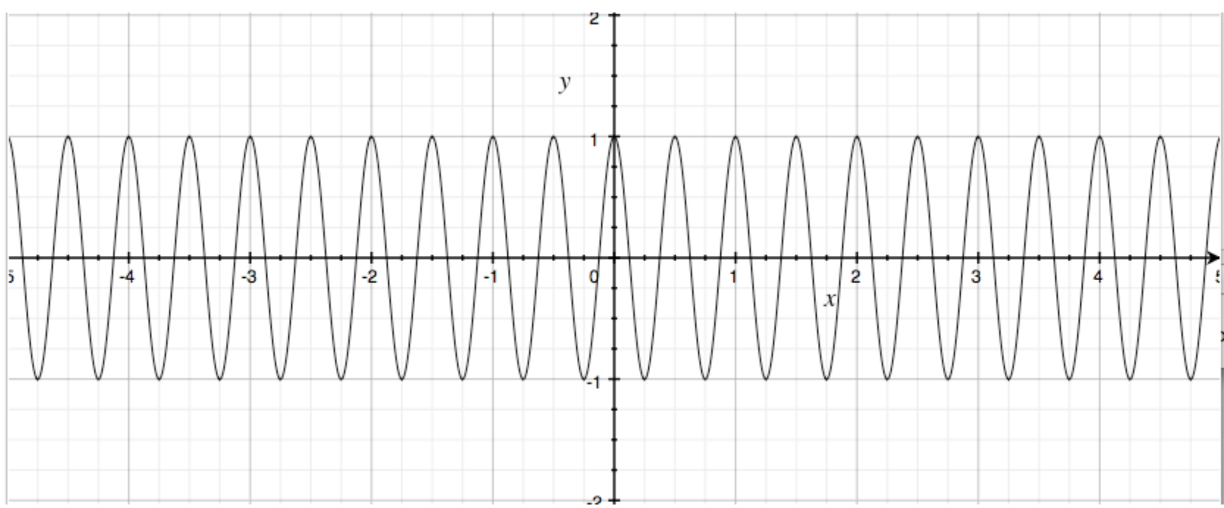
Frequencies $\cos 2\pi f x$

$$f = \frac{1}{T}$$



$$f = 1$$

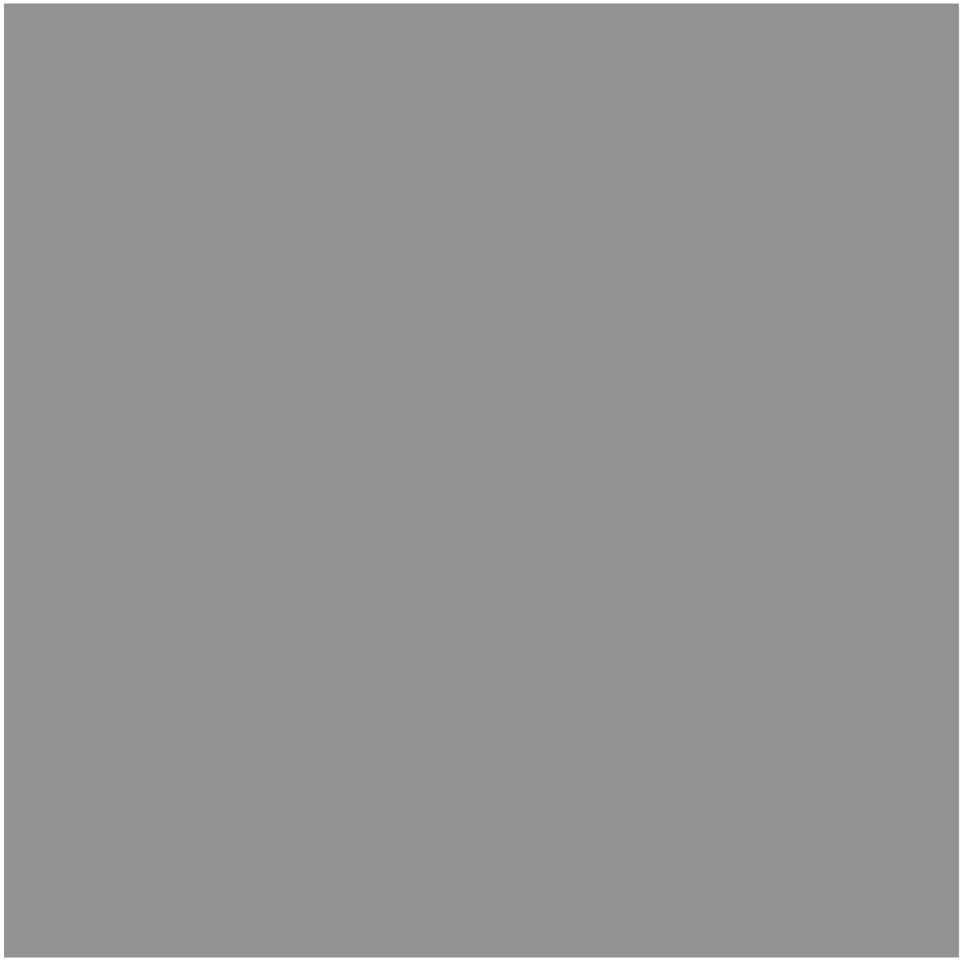
$$\cos 2\pi x$$



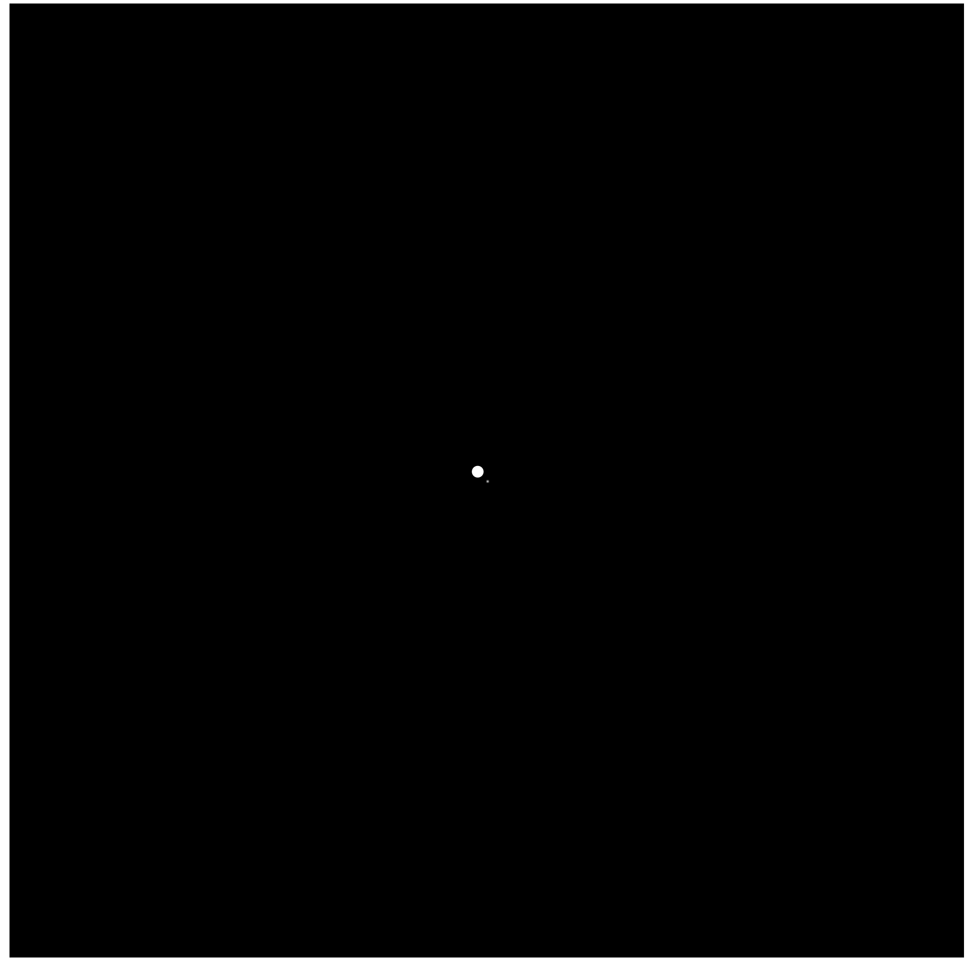
$$f = 2$$

$$\cos 4\pi x$$

Constant

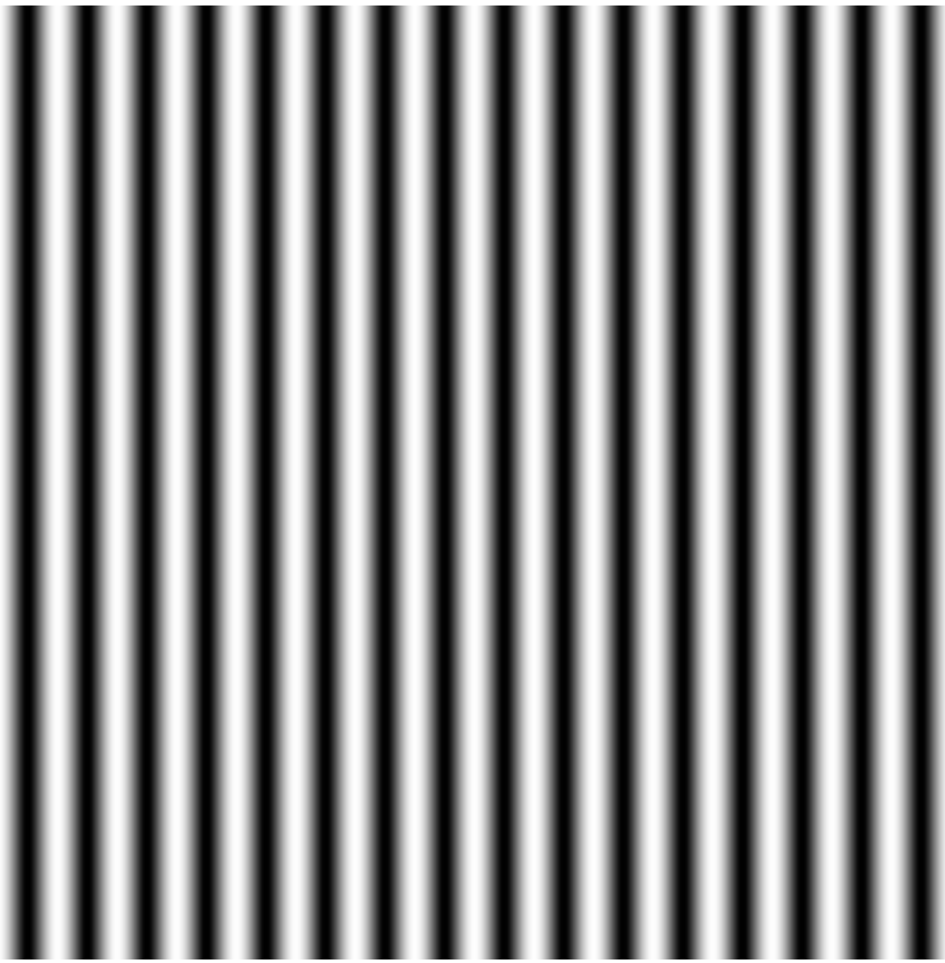


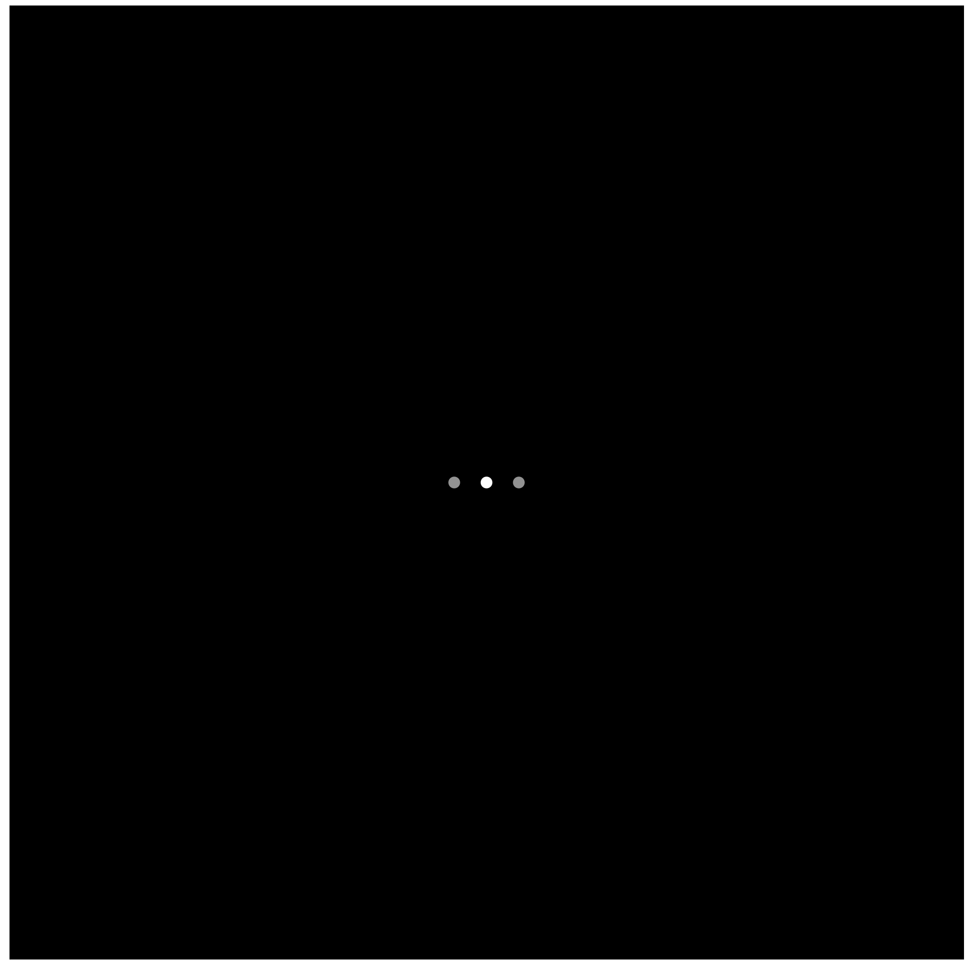
Spatial Domain



Frequency Domain

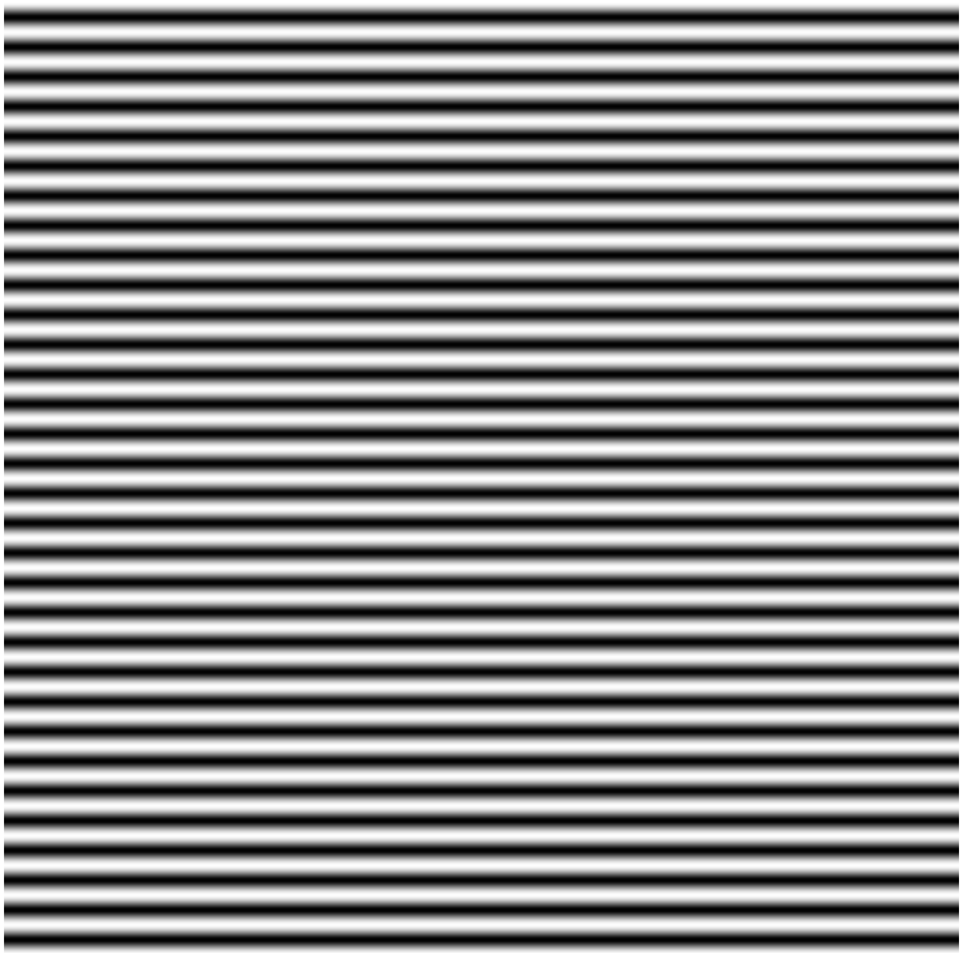
$\sin(2\pi/32)x$ **32 pixels per cycle**





Spatial Domain **Frequency Domain**

$\sin(2\pi/16)y$ **16 pixels per cycle**

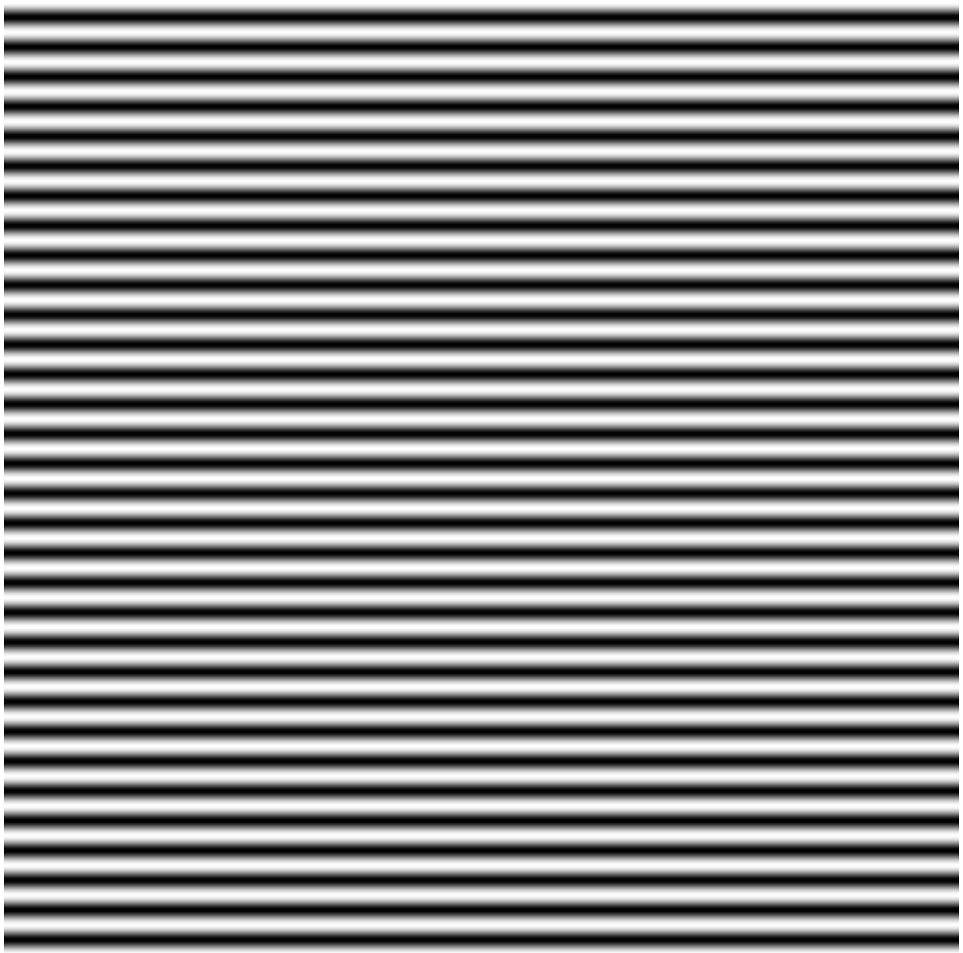


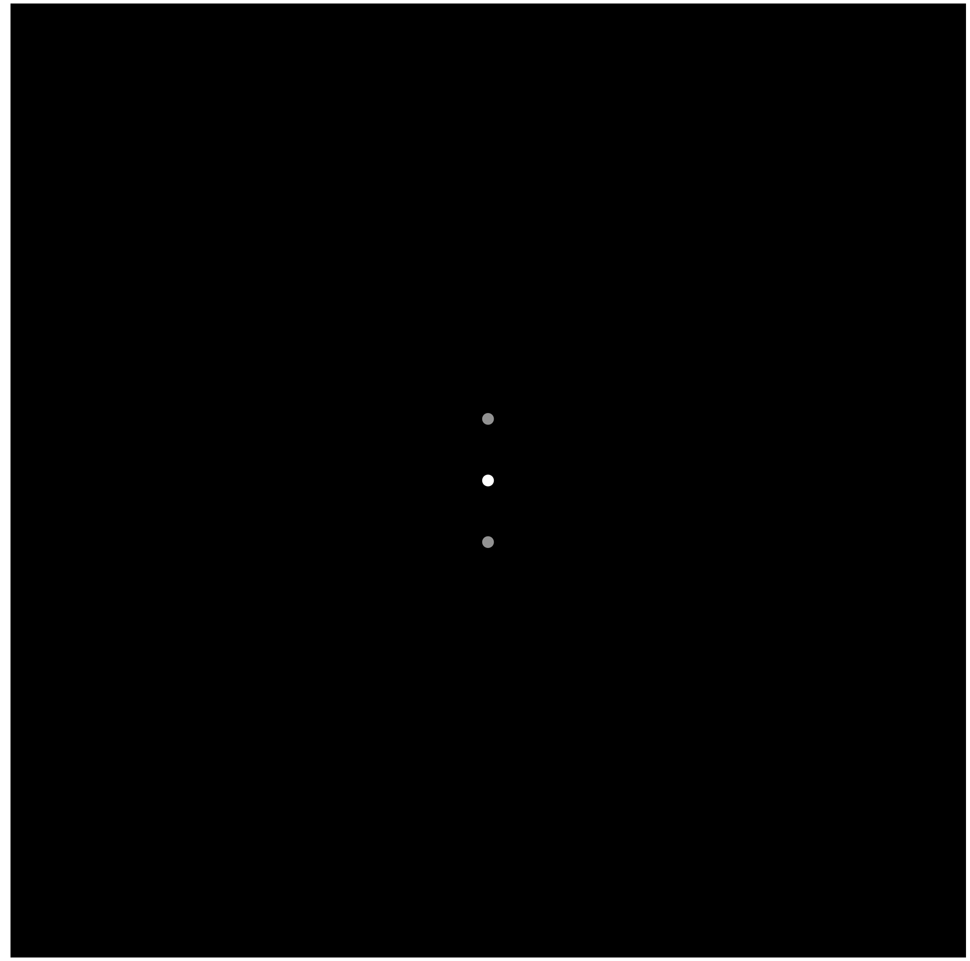
Spatial Domain

?

Frequency Domain

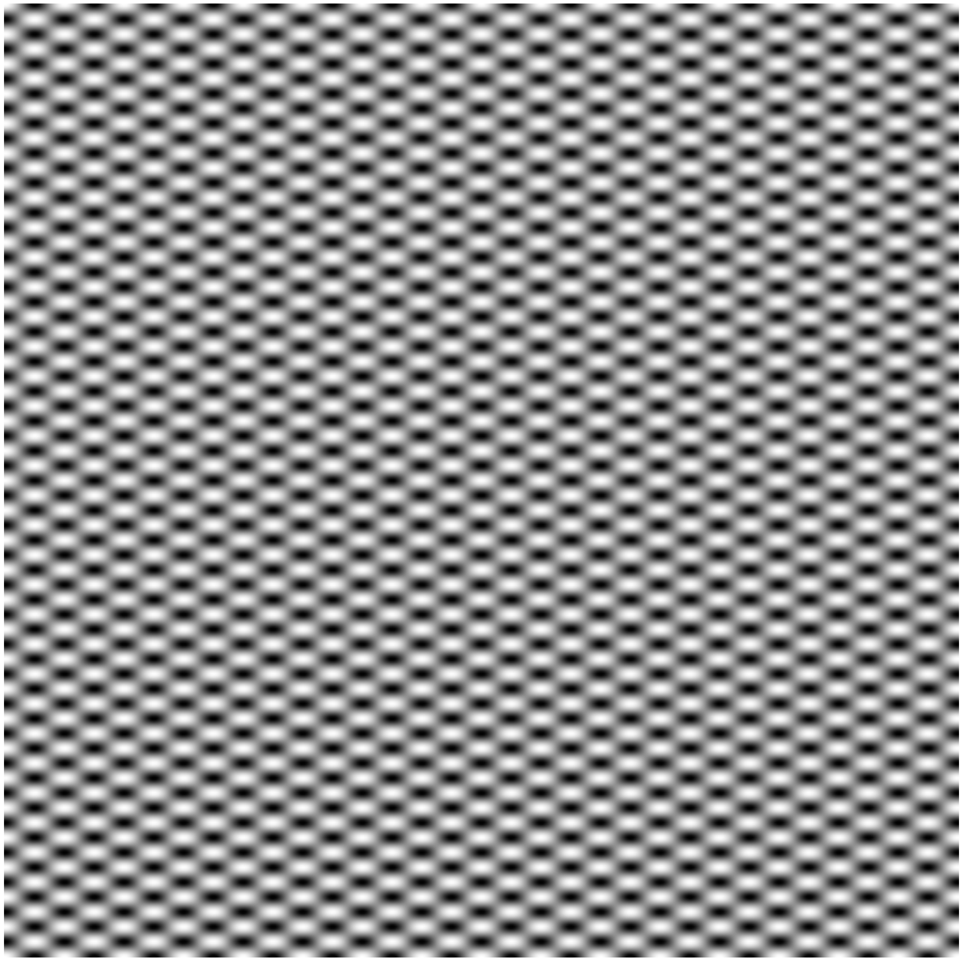
$\sin(2\pi/16)y$ **16 pixels per cycle**

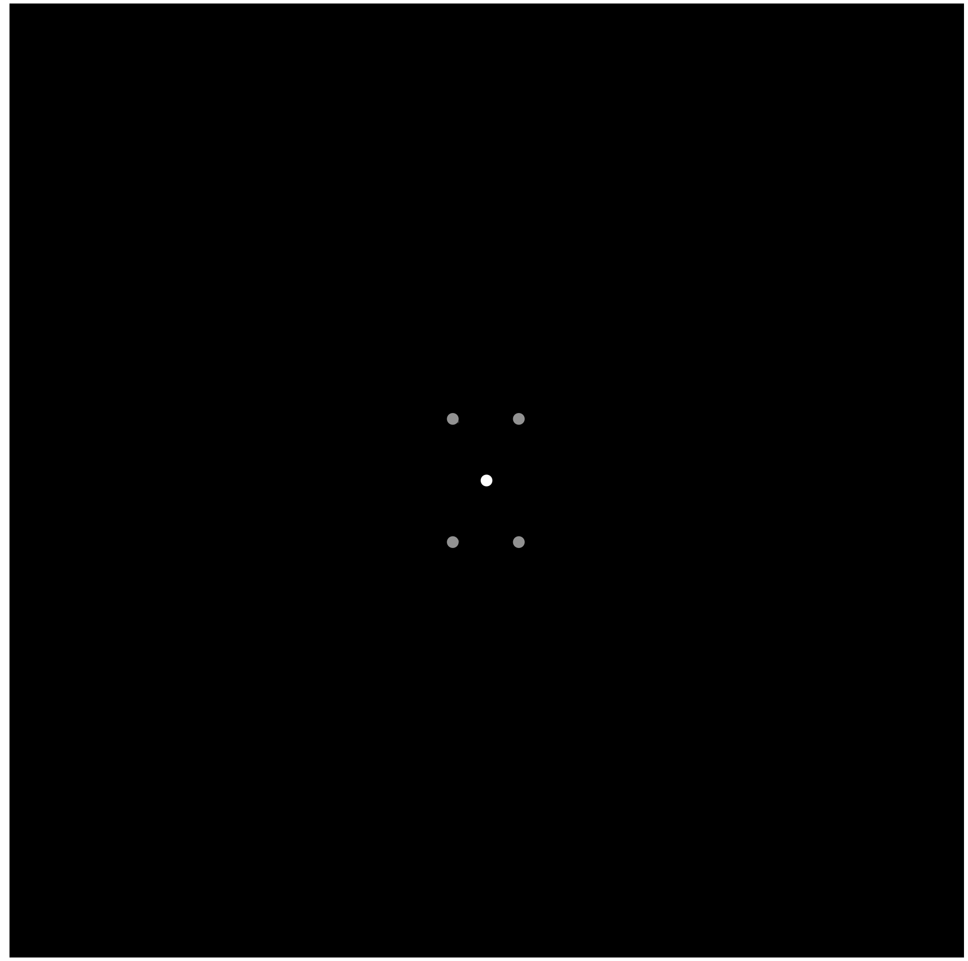




Spatial Domain **Frequency Domain**

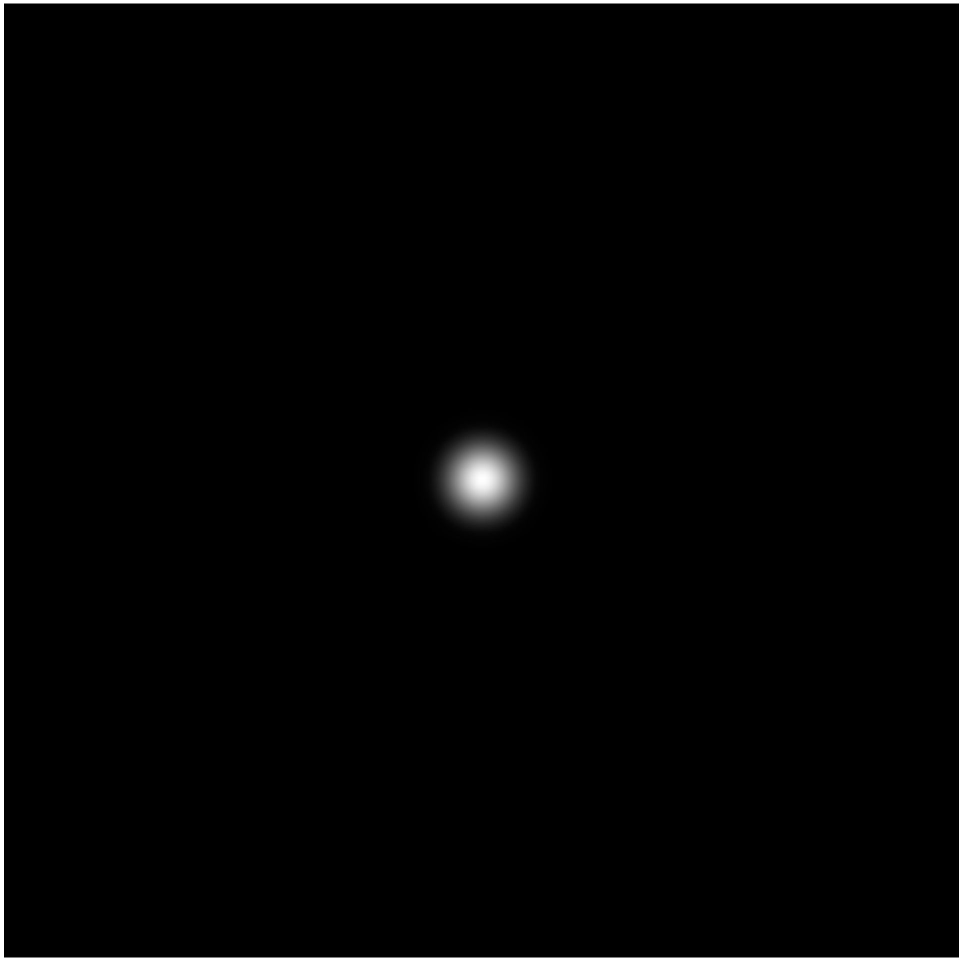
$$\sin(2\pi/32)x \times \sin(2\pi/16)y$$



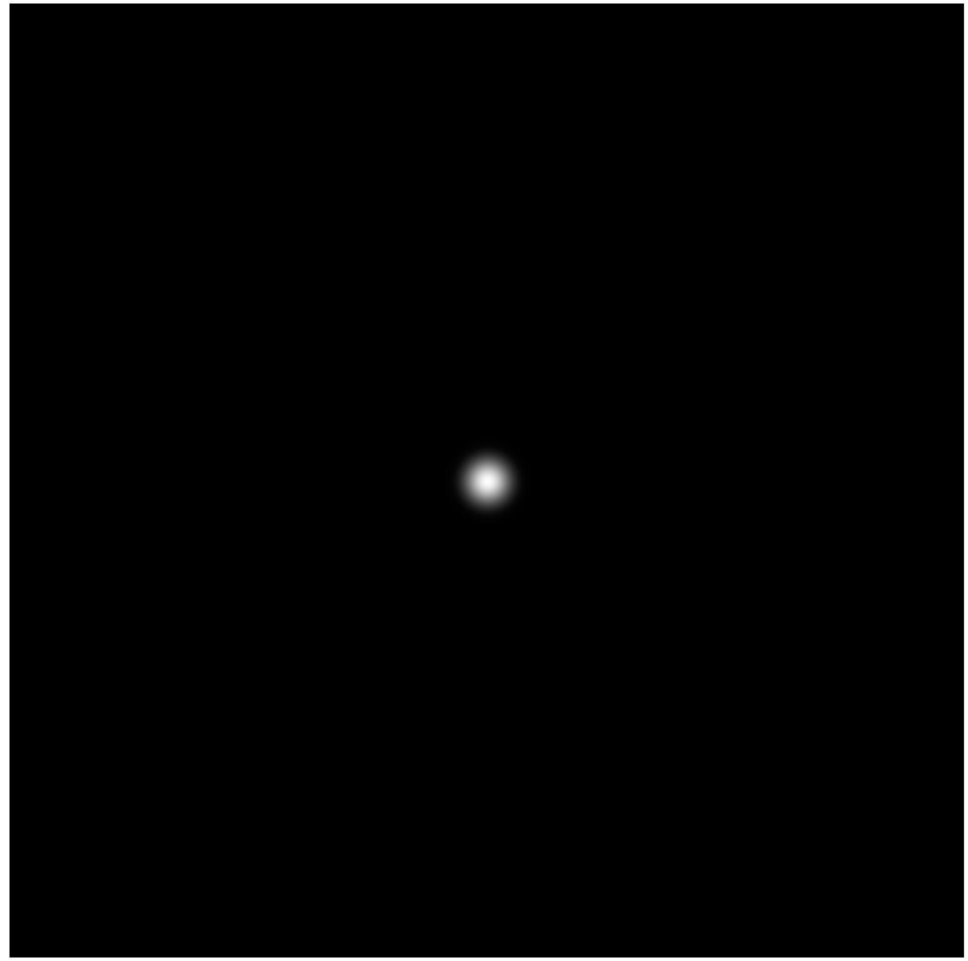


Spatial Domain **Frequency Domain**

$$e^{-r^2/16^2}$$

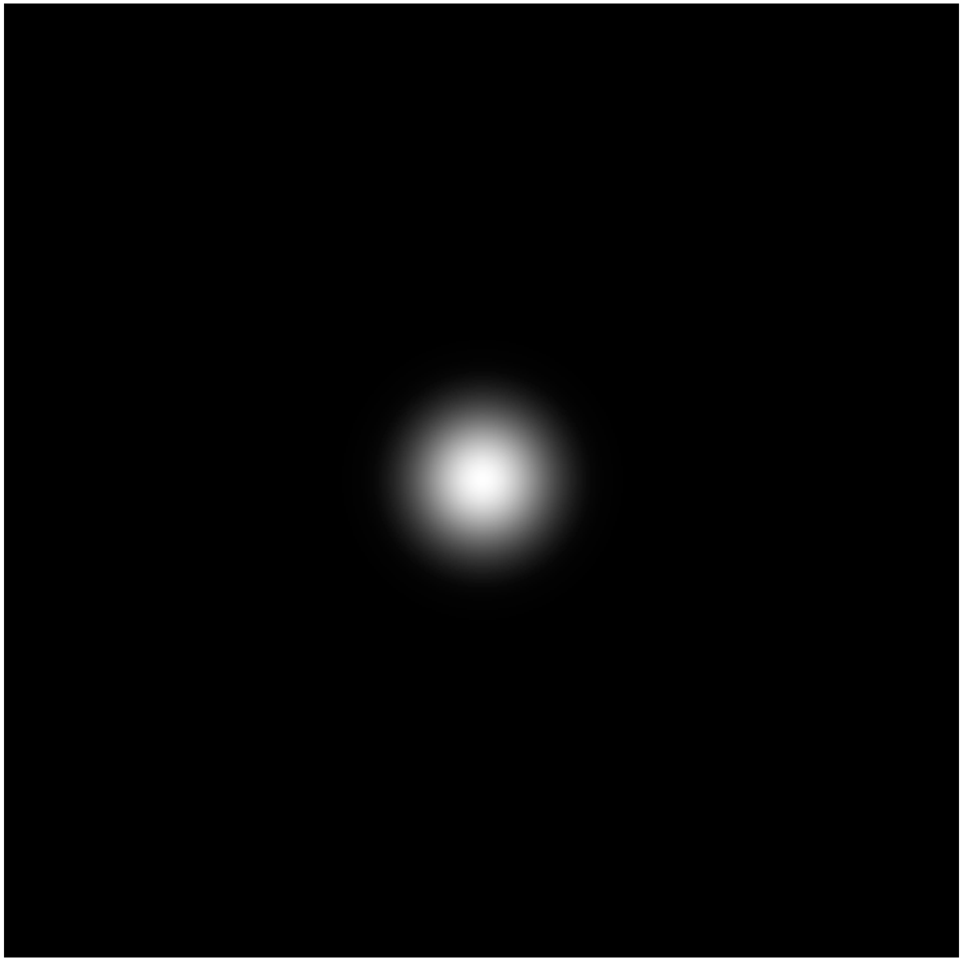


Spatial Domain

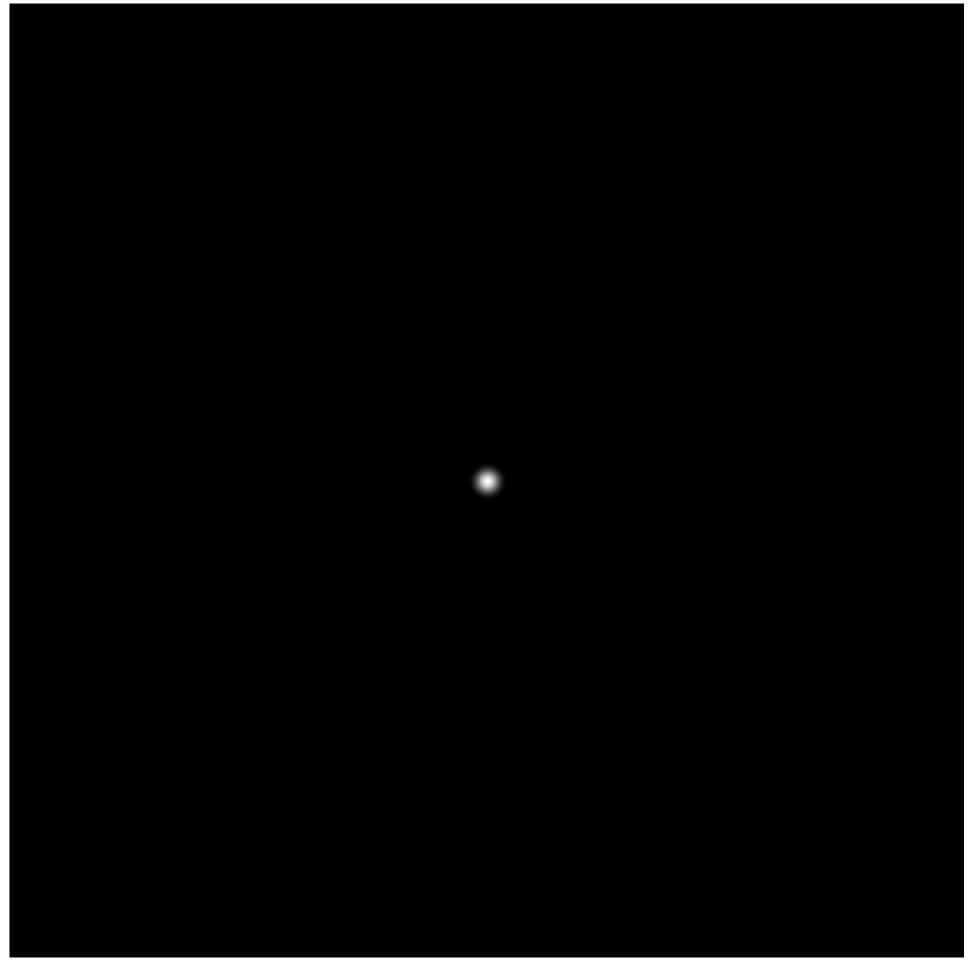


Frequency Domain

$$e^{-r^2/32^2}$$

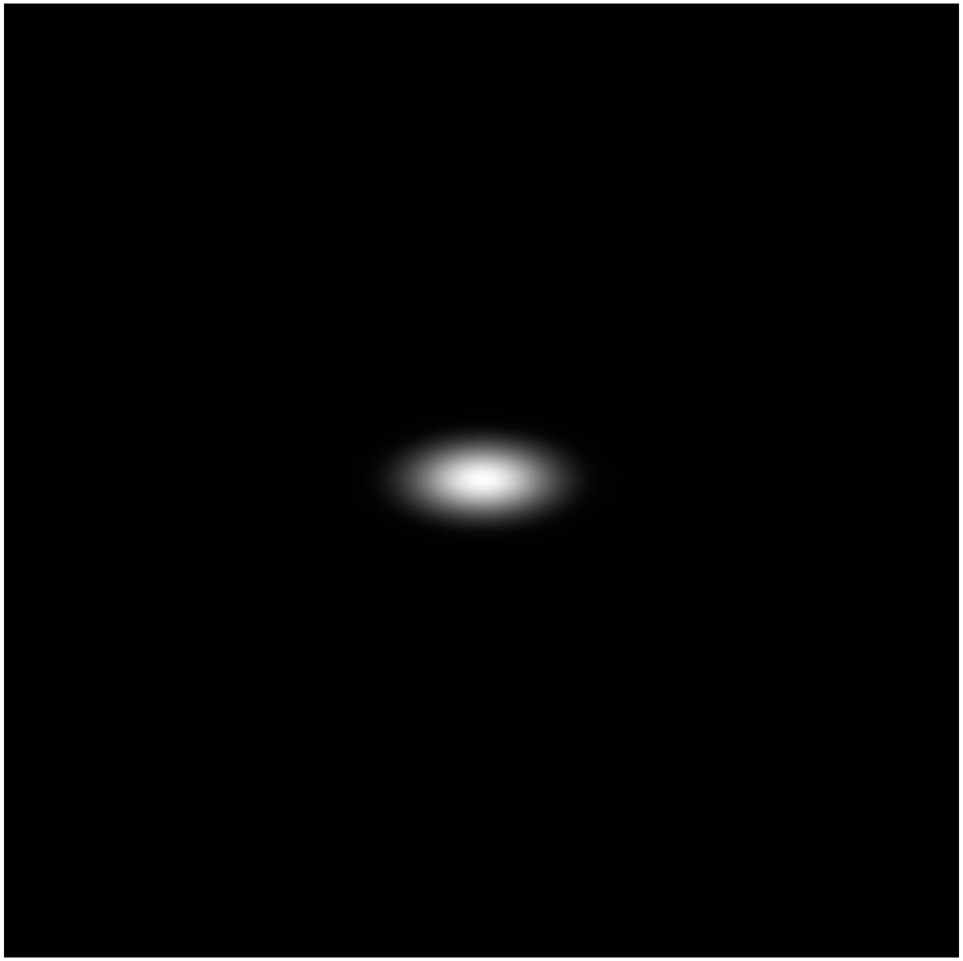


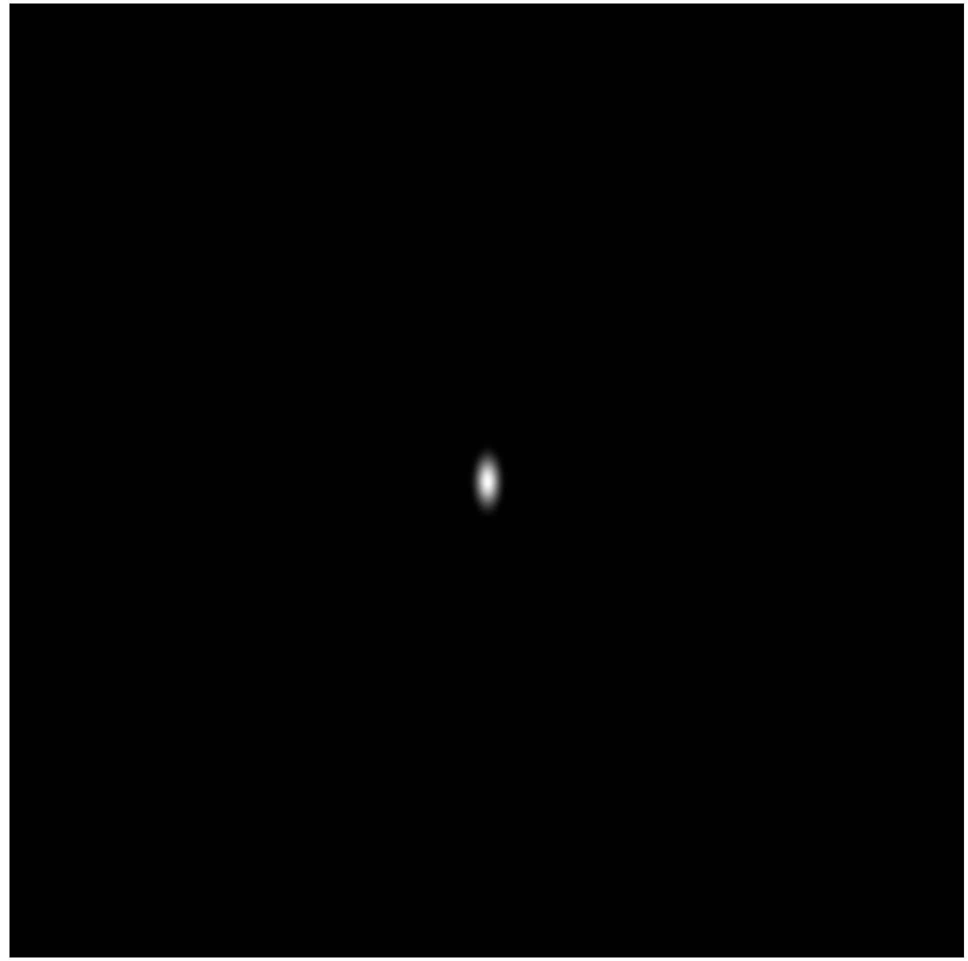
Spatial Domain



Frequency Domain

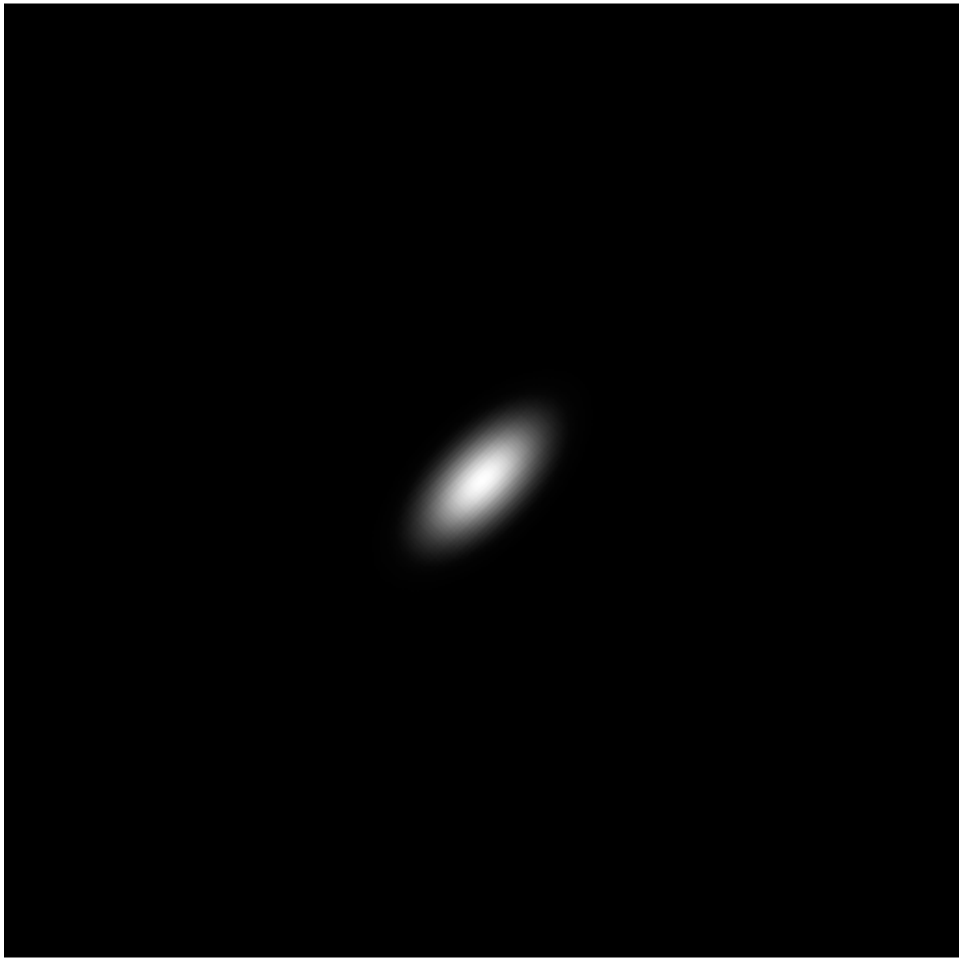
$$e^{-x^2/32^2} \times e^{-y^2/16^2}$$



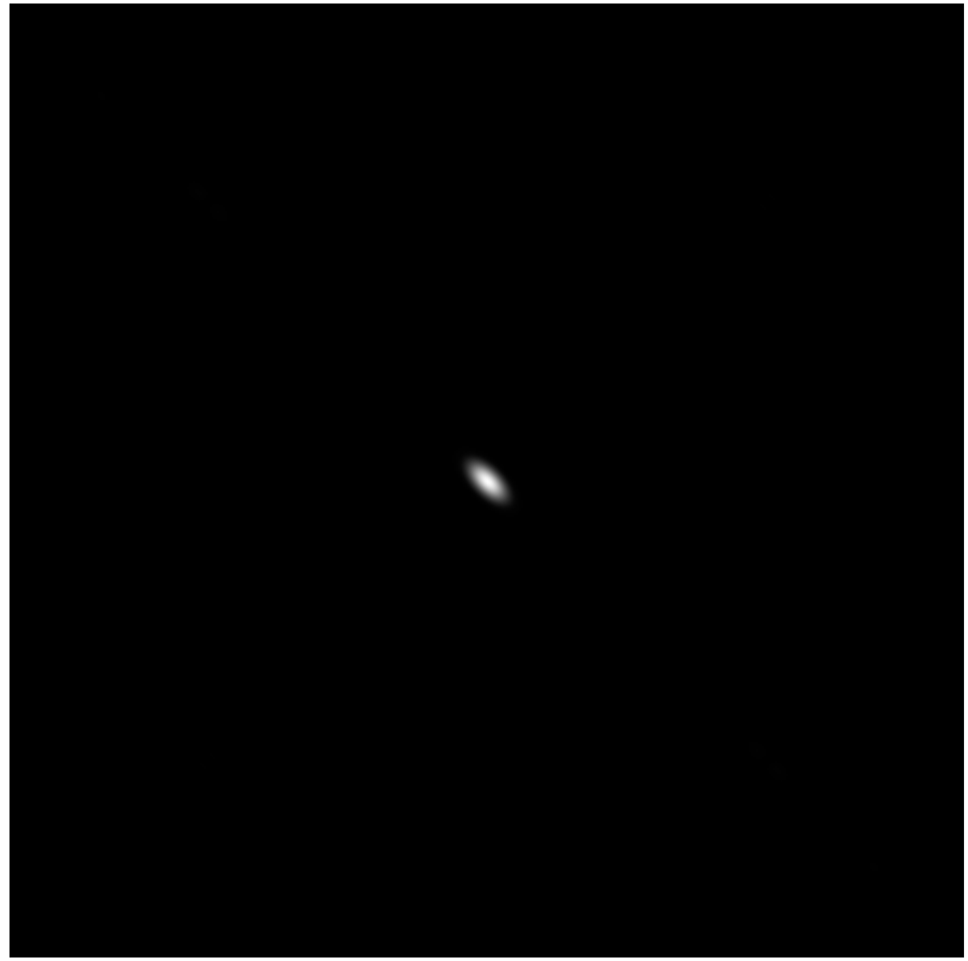


Spatial Domain **Frequency Domain**

Rotate 45 $e^{-x^2/32^2} \times e^{-y^2/16^2}$



Spatial Domain



Frequency Domain

Fourier Transforms

The Fourier transform converts between the spatial and frequency domain

Spatial Domain

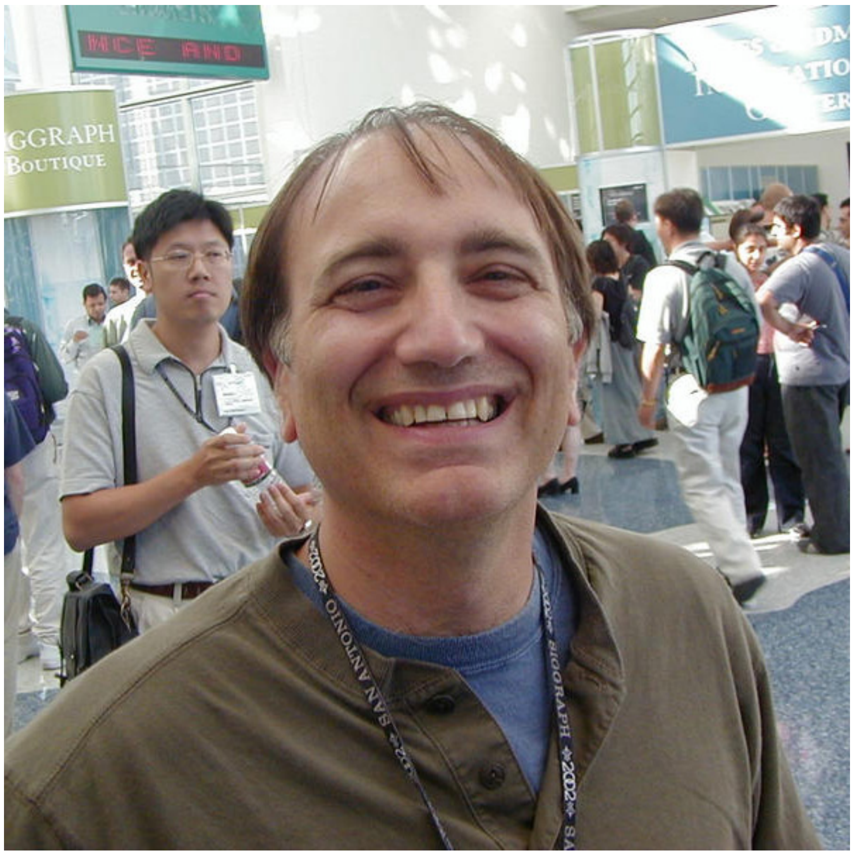
$$\Rightarrow F(\omega) = \int_{-\infty}^{\infty} f(x) e^{-i\omega x}\, dx \Rightarrow$$

Frequency Domain

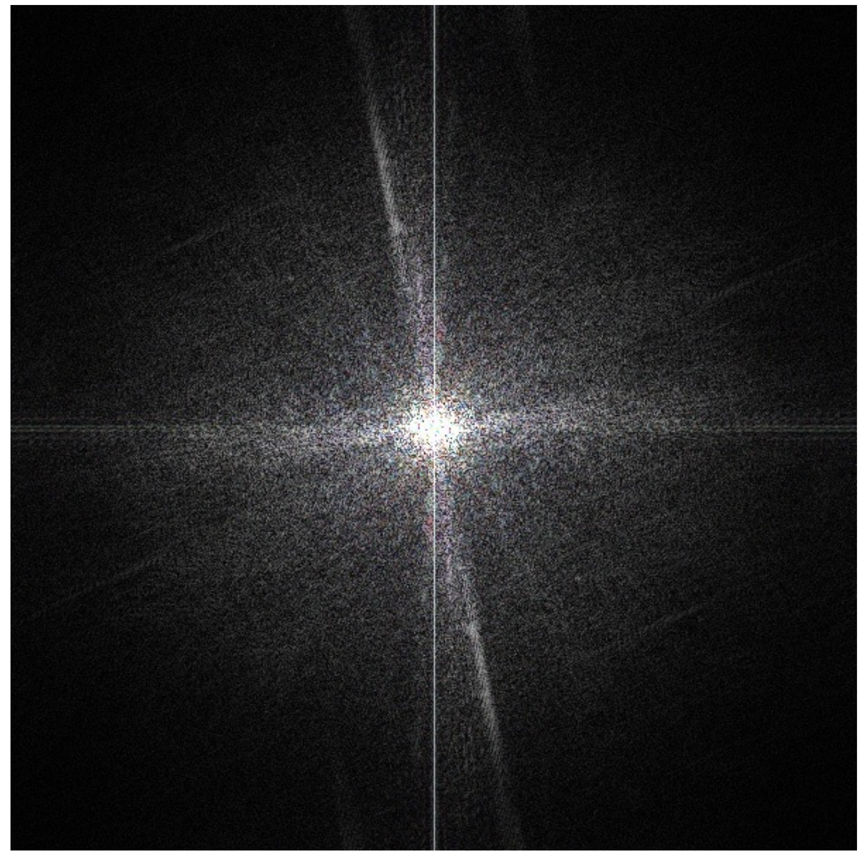
$$\Leftarrow f(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} F(\omega) e^{i\omega x}\, d\omega \Leftarrow$$

Figures generated using `fft2d.py`

Pat's Frequencies



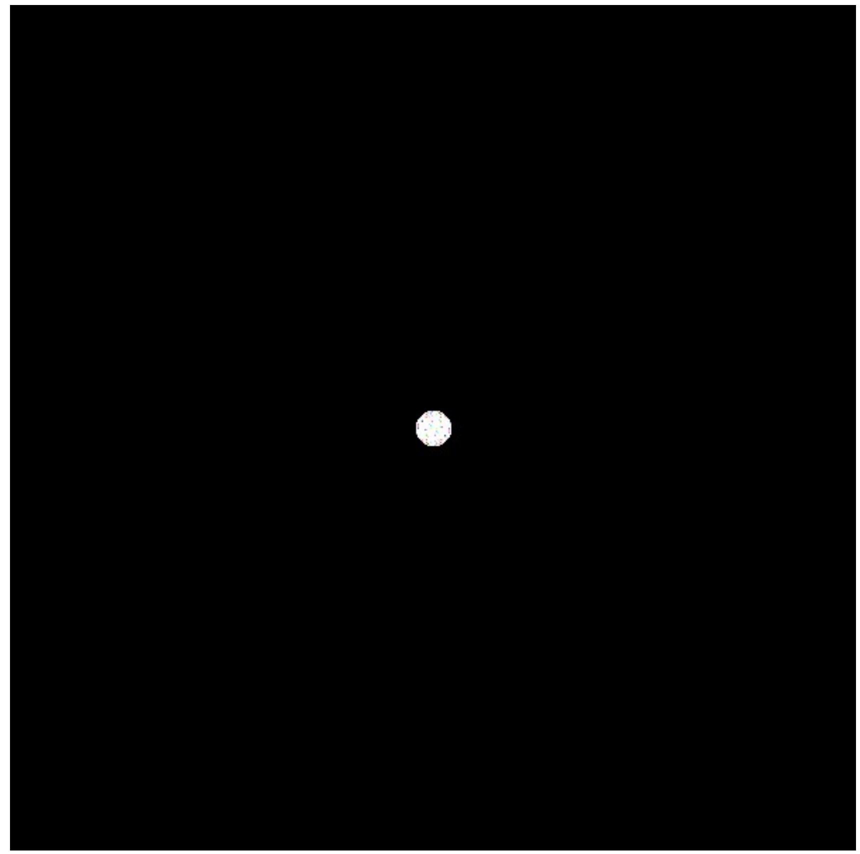
Spatial Domain



Frequency Domain

Filtering: Low Pass Filter

?

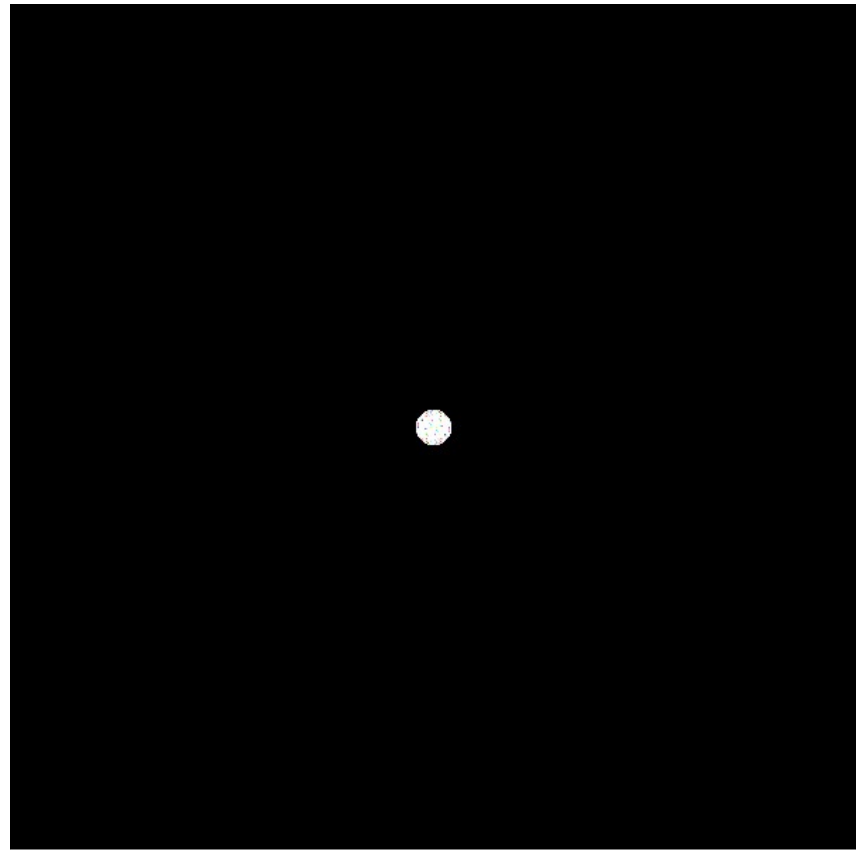


Spatial Domain **Frequency Domain**

Filtering: Low Pass Filter

Keep low frequencies

Spatial Domain

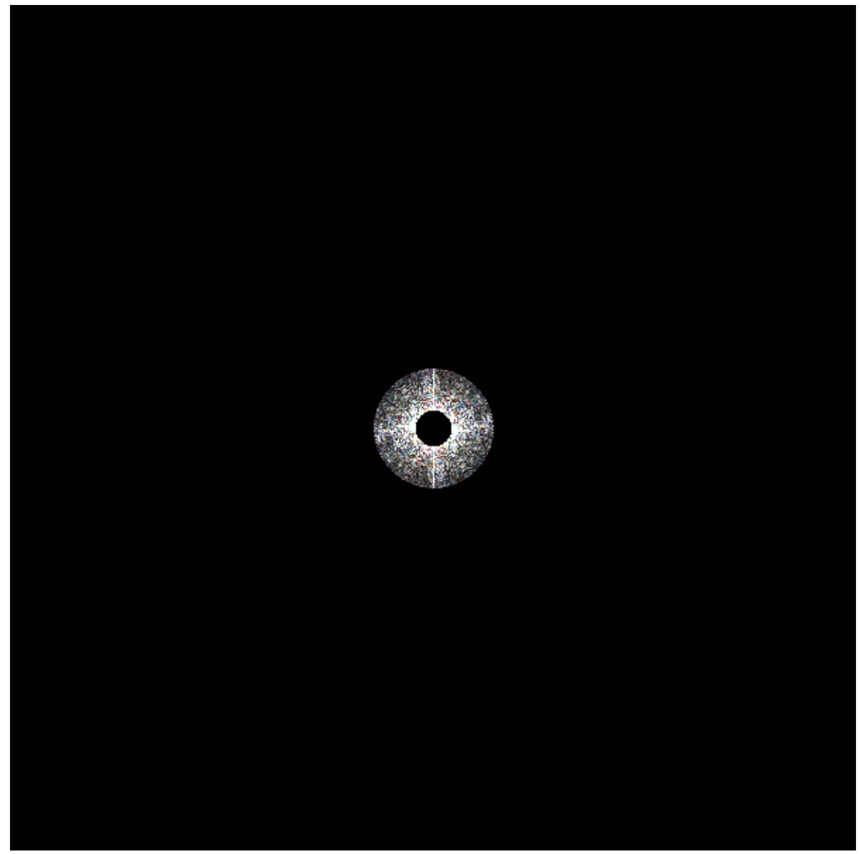


Frequency Domain

Filtering: Band Pass Filter

Keep band of frequencies

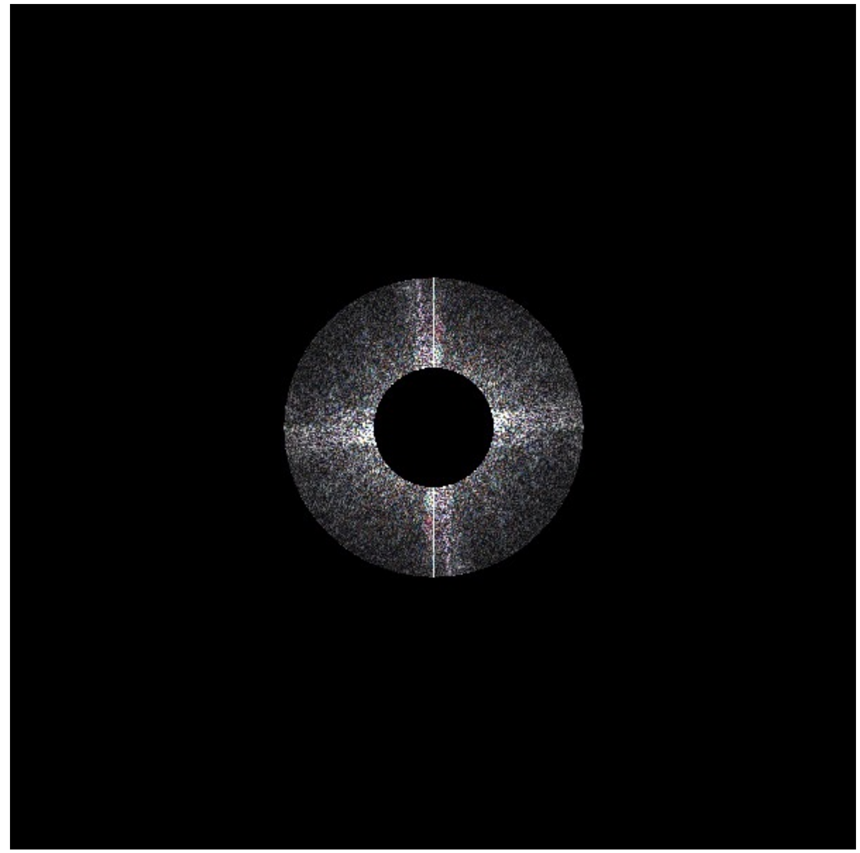
Spatial Domain



Frequency Domain

Filtering: Band Pass Filter

Keep band of frequencies

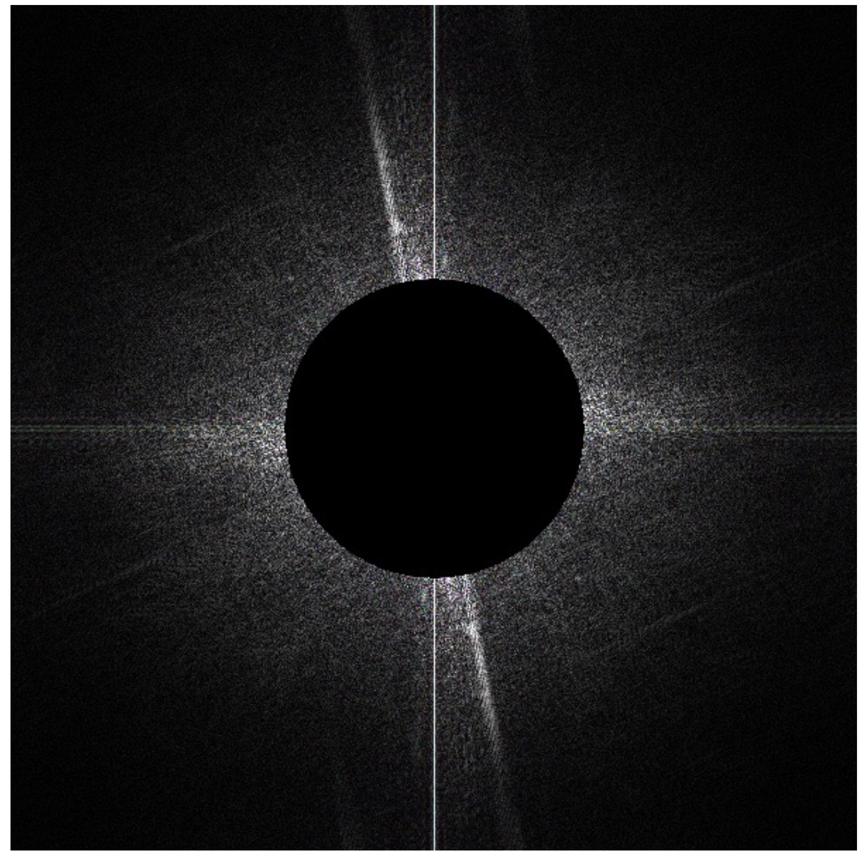


Spatial Domain **Frequency Domain**

Filtering: High Pass Filter

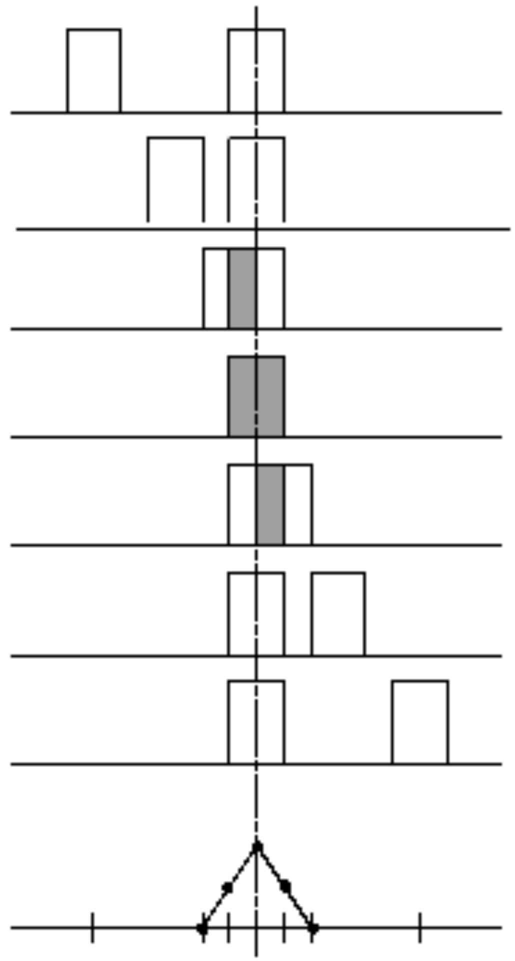
Keep high frequencies

Spatial Domain



Frequency Domain

Filtering by Convolution



$$h(x) = f \otimes g = \int f(x')g(x - x')\,dx'$$

Convolution Theorem

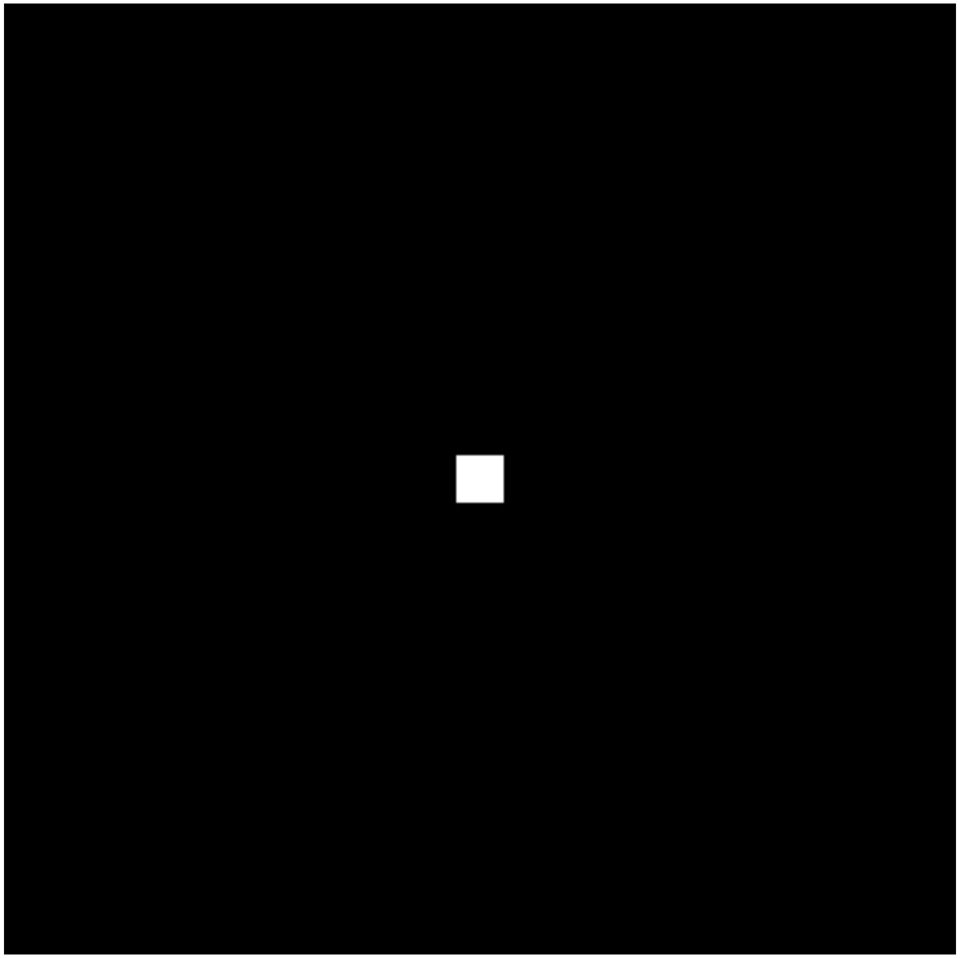
Convolution Theorem: Multiplication in the frequency domain is equivalent to convolution in the space domain.

$$f \otimes g \leftrightarrow F \times G$$

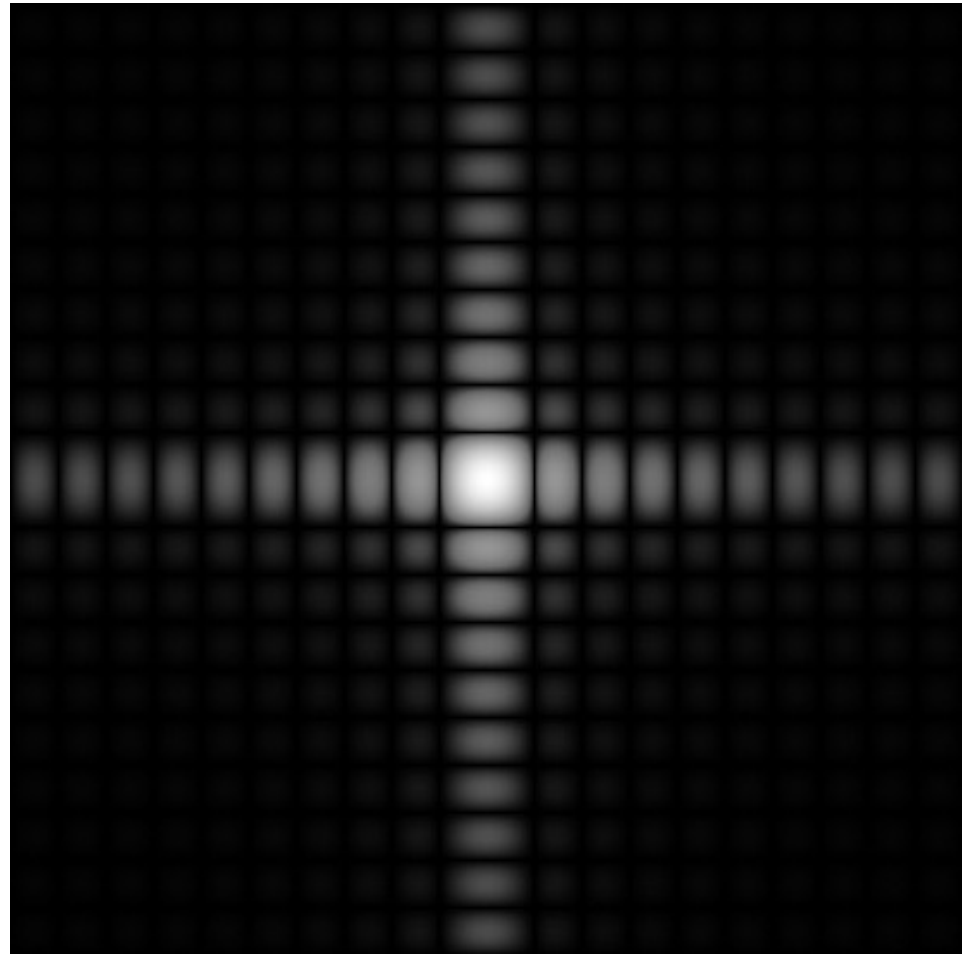
Symmetric Theorem: Multiplication in the space domain is equivalent to convolution in the frequency domain.

$$f \times g \leftrightarrow F \otimes G$$

Spatial and Frequency Domain

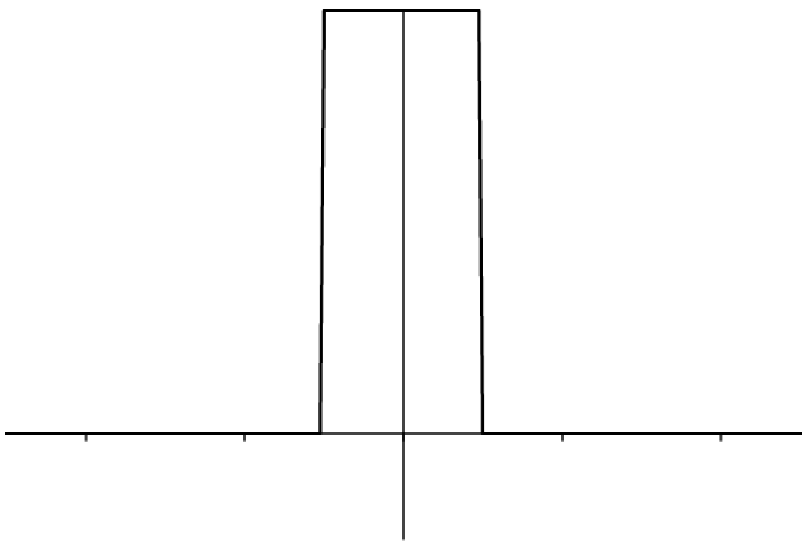


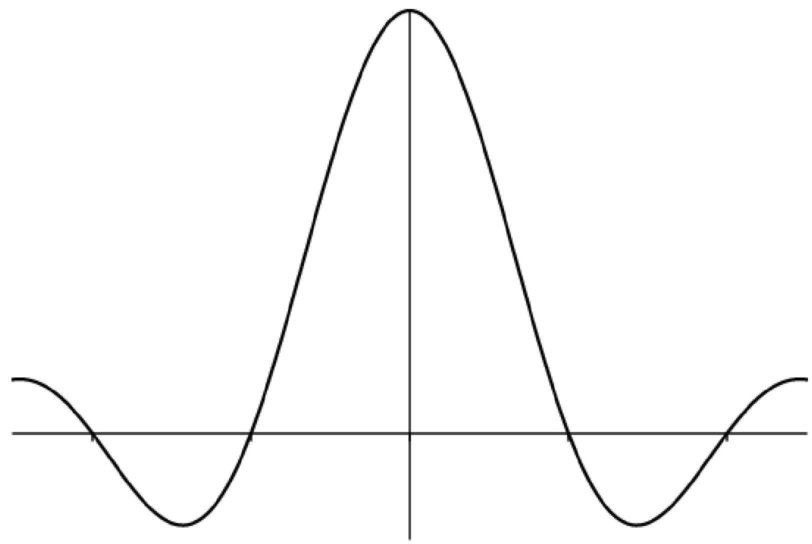
Spatial Domain



Frequency Domain

Math: Box and Sinc Functions



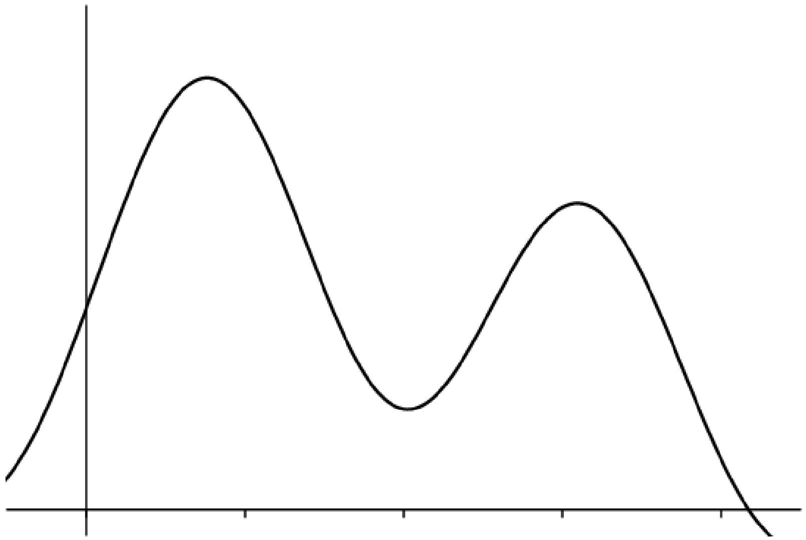


$$\prod_T(x) = \begin{cases} 1 & |x| \le \dfrac{T}{2} \\ 0 & |x| > \dfrac{T}{2} \end{cases}$$

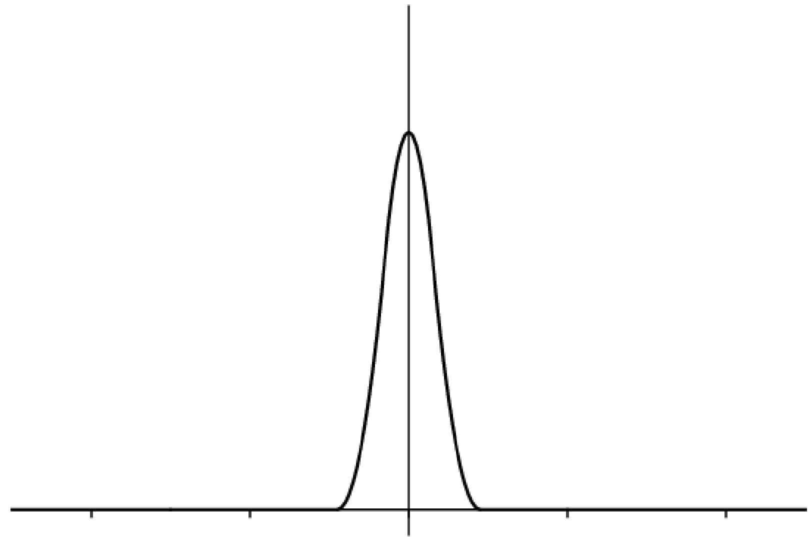
$$\mathrm{sinc}(\mathrm{x}) \equiv \frac{\sin \pi \mathrm{x}}{\pi \mathrm{x}}$$

The Sampling Theorem

Simple Function

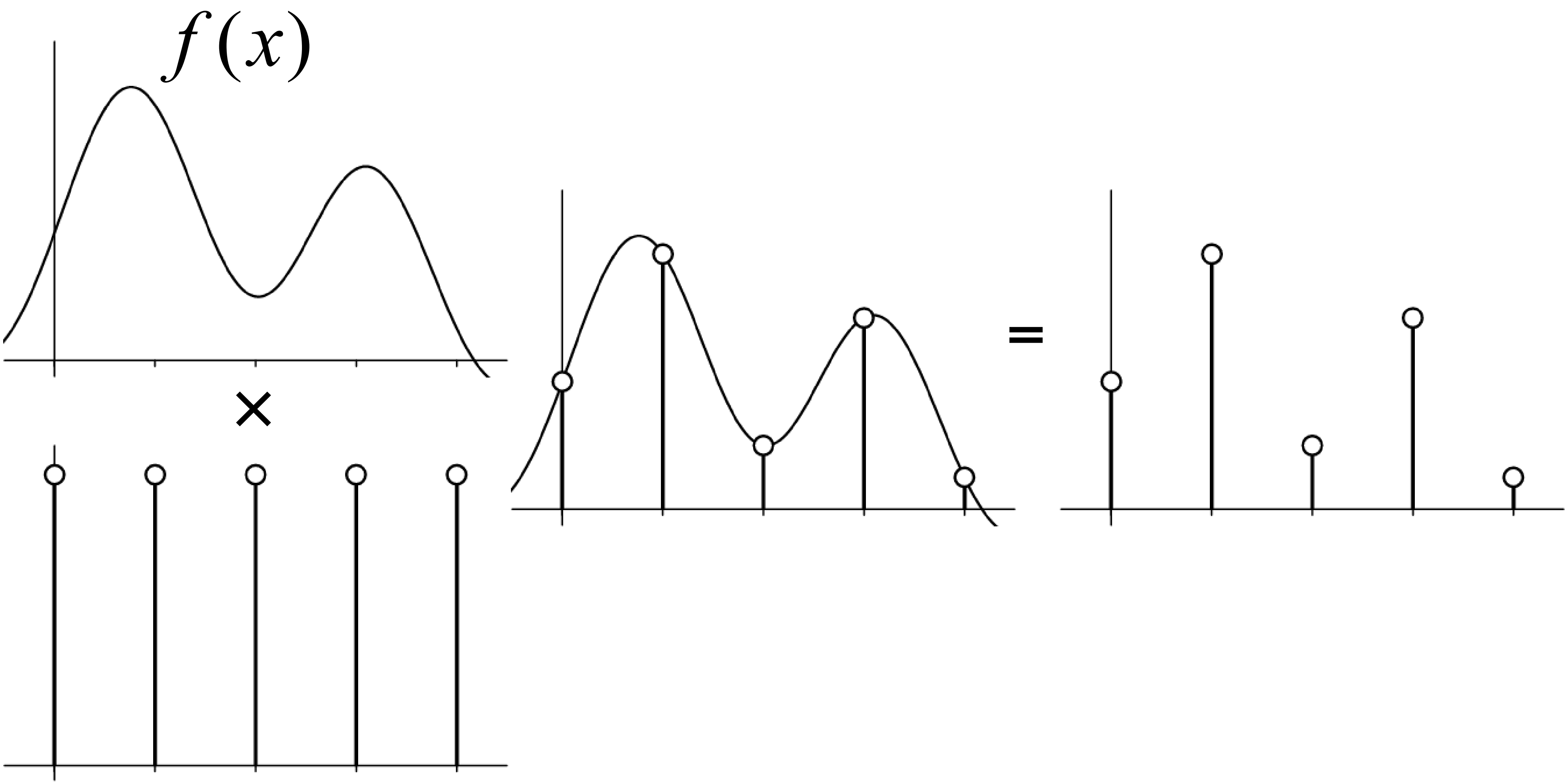


$f(x)$



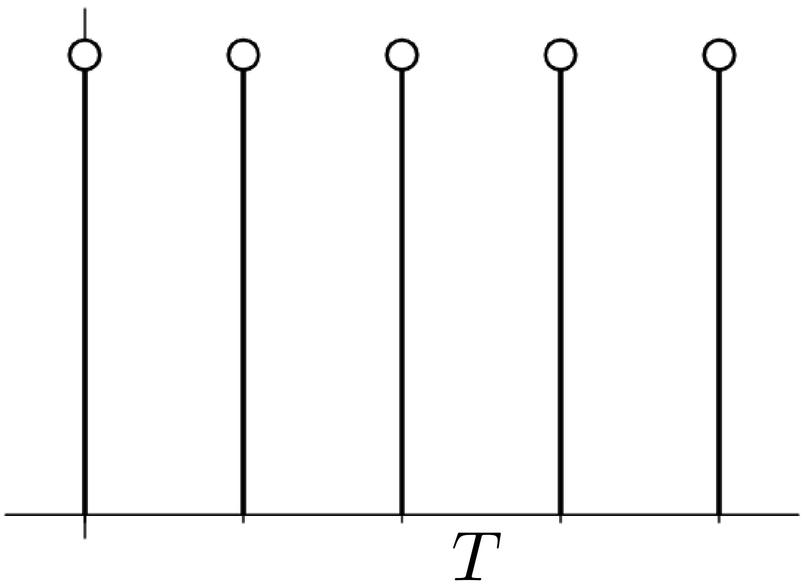
$F(\omega)$

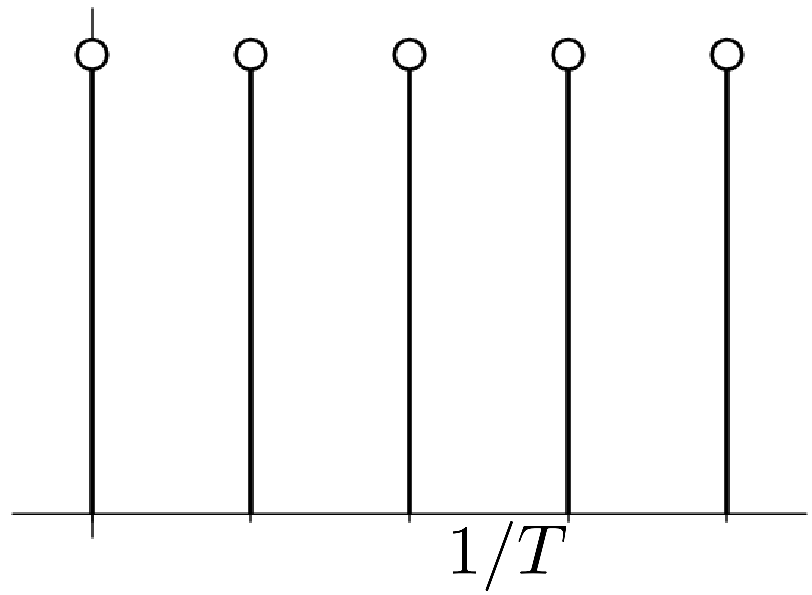
Sampling: Multiply in Spatial Domain



Some Magic

The Fourier transform of a sequence of spikes is a sequence of spikes



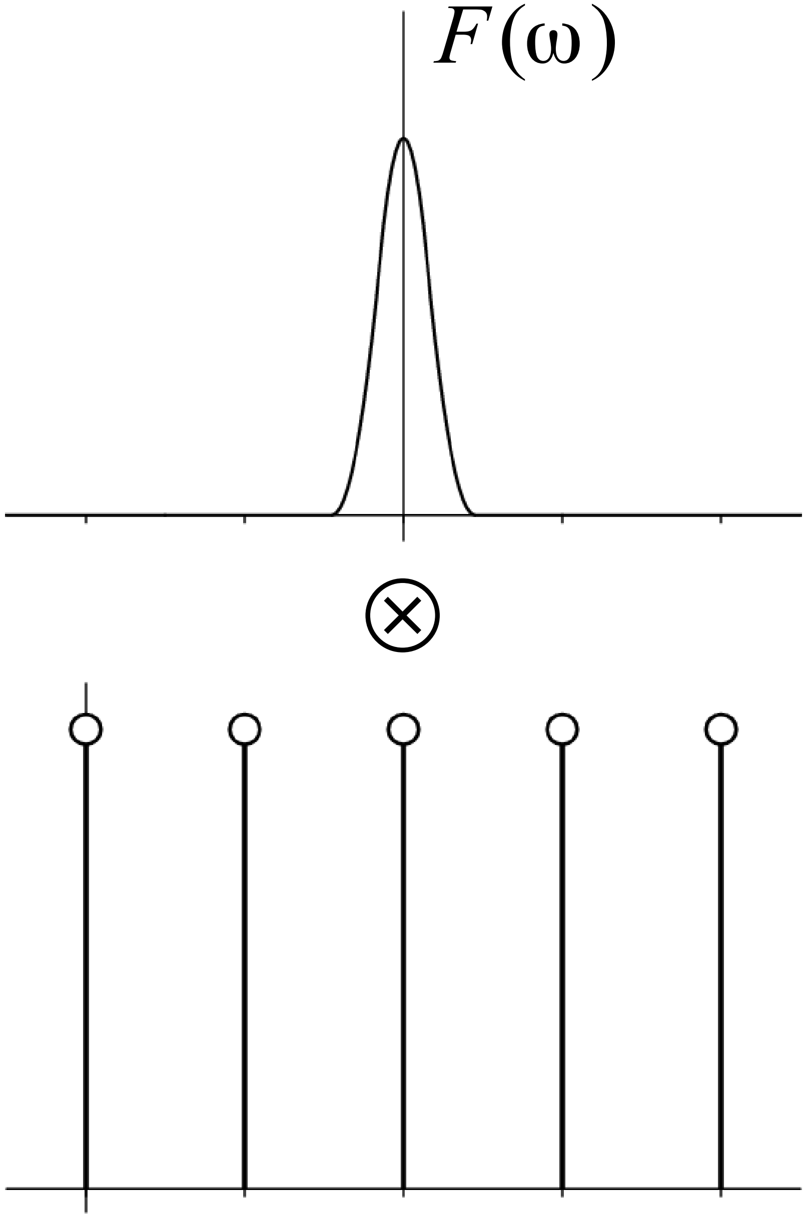


$$\mathrm{III}(\mathrm{t}) = \sum_{\mathrm{n}=-\infty}^{\mathrm{n}=\infty} \delta(\mathrm{t} - \mathrm{nT})$$

$$\mathrm{III}_{1/\mathrm{T}}(\omega) = \sum_{\mathrm{n}=-\infty}^{\mathrm{n}=\infty} \delta(\omega - \mathrm{n}/\mathrm{T})$$

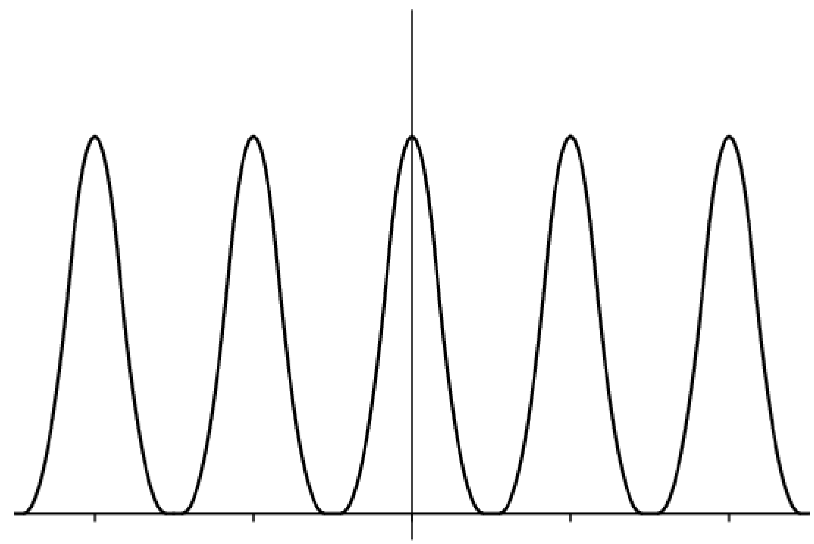
Comb or Shah function

Sampling: Convolve in Freq Domain

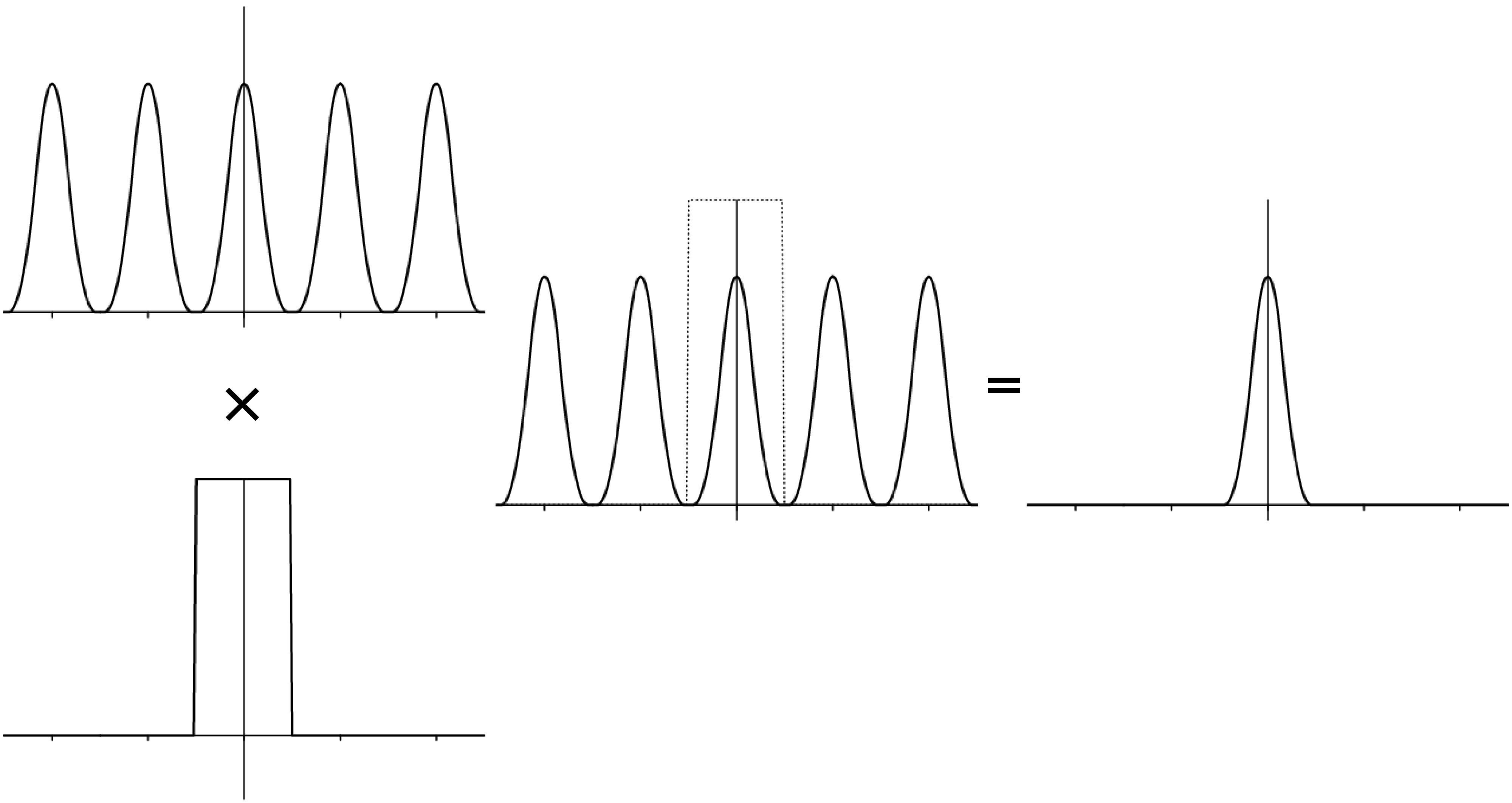


$\otimes$

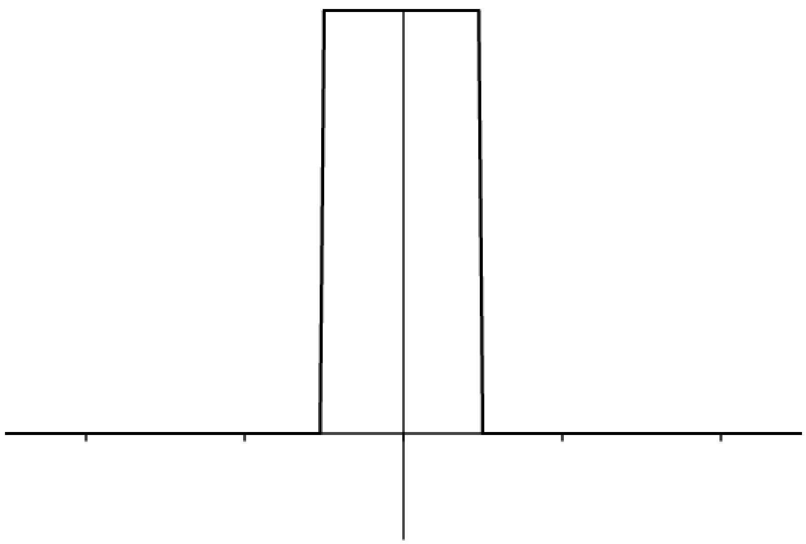
=



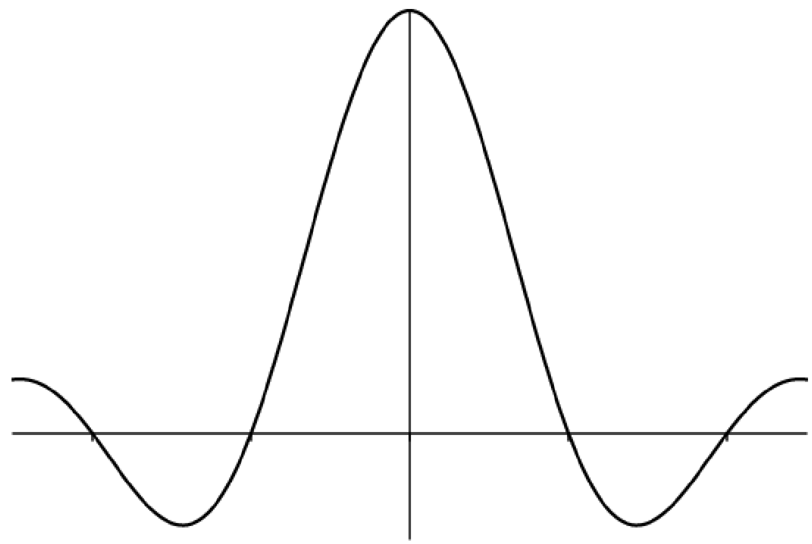
Reconstruction: Frequency Domain



Recall

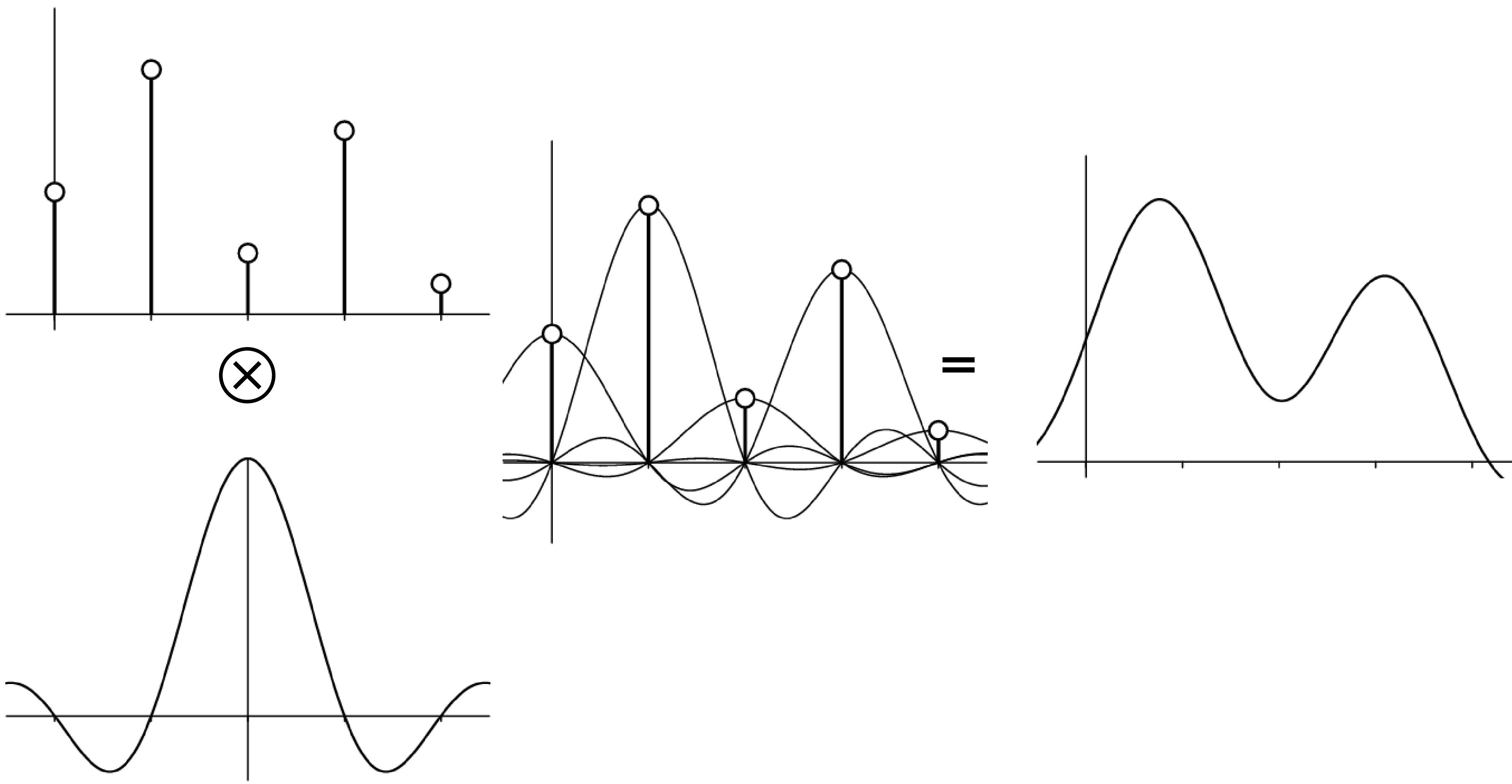


$$\prod_T(x) = \begin{cases} 1 & |x| \leq \frac{T}{2} \\ 0 & |x| > \frac{T}{2} \end{cases}$$

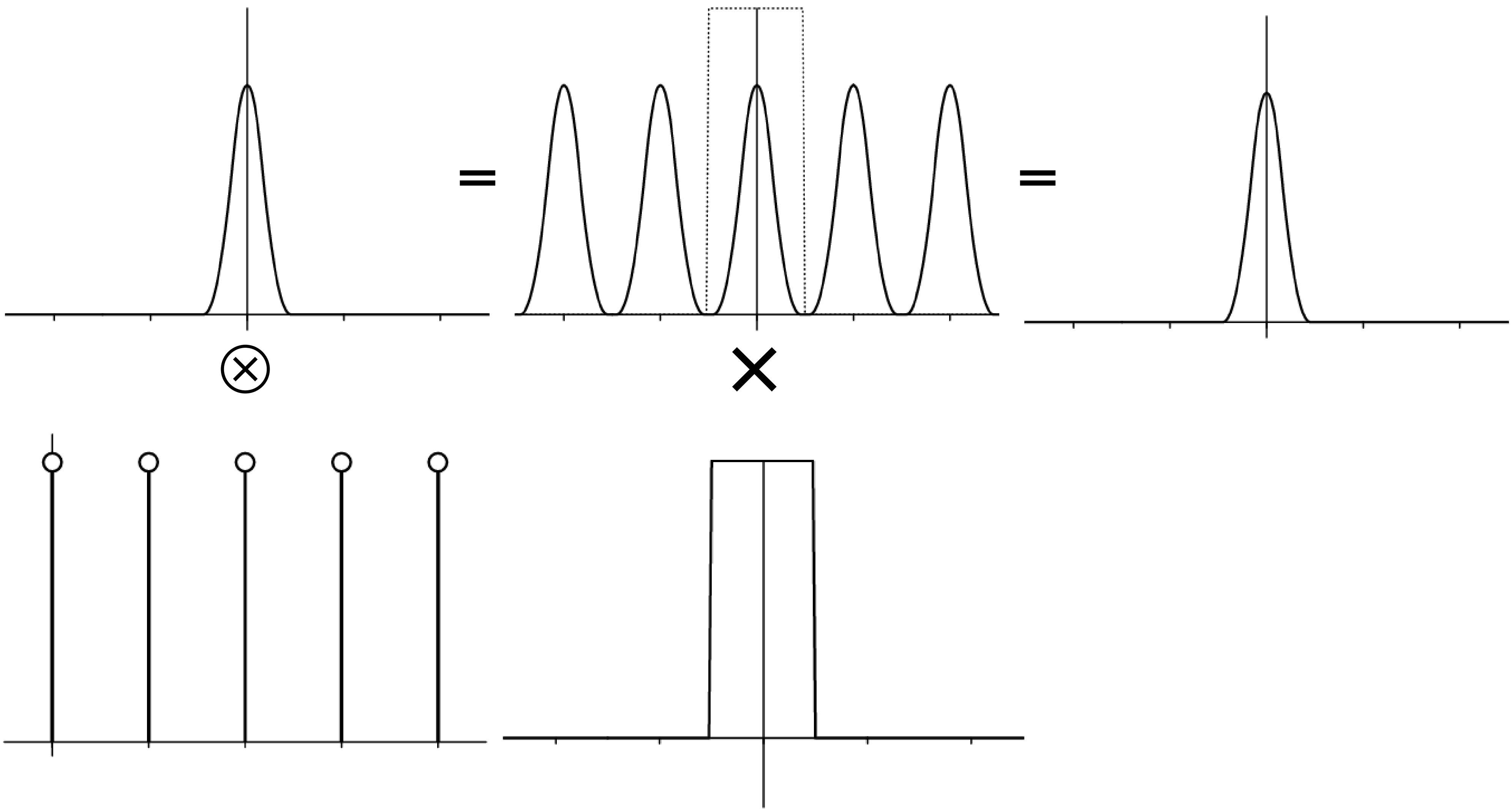


$$\mathrm{sinc}(\mathrm{x}) \equiv \frac{\sin \pi \mathrm{x}}{\pi \mathrm{x}}$$

Reconstruction: Spatial Domain



Recovering a Sampled Signal



Sampling Theorem

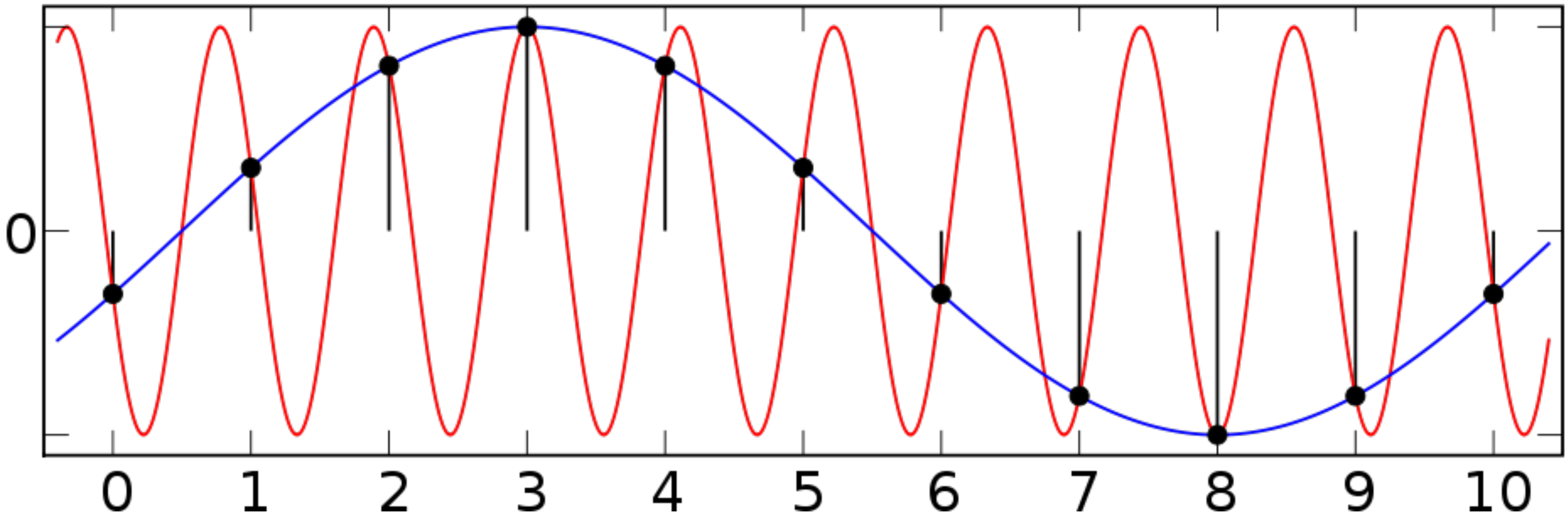
This result is known as the Sampling Theorem, and Claude Shannon is credited with discovering it in 1949

A signal can be reconstructed from its samples without loss of information, if the original signal has no frequencies above 1/2 the sampling frequency

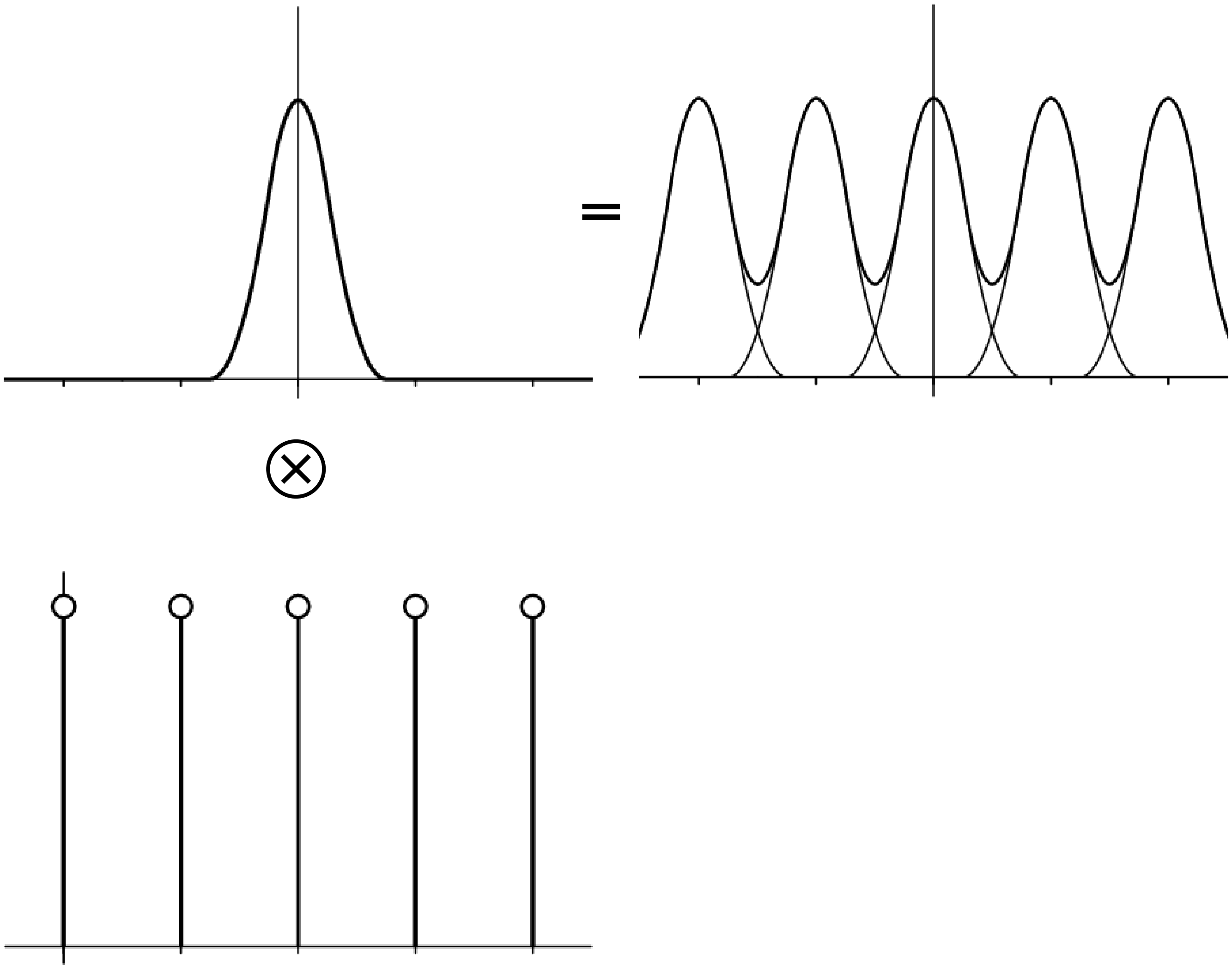
For a given band-limited function, the rate it must be sampled is called the Nyquist Frequency

Aliasing

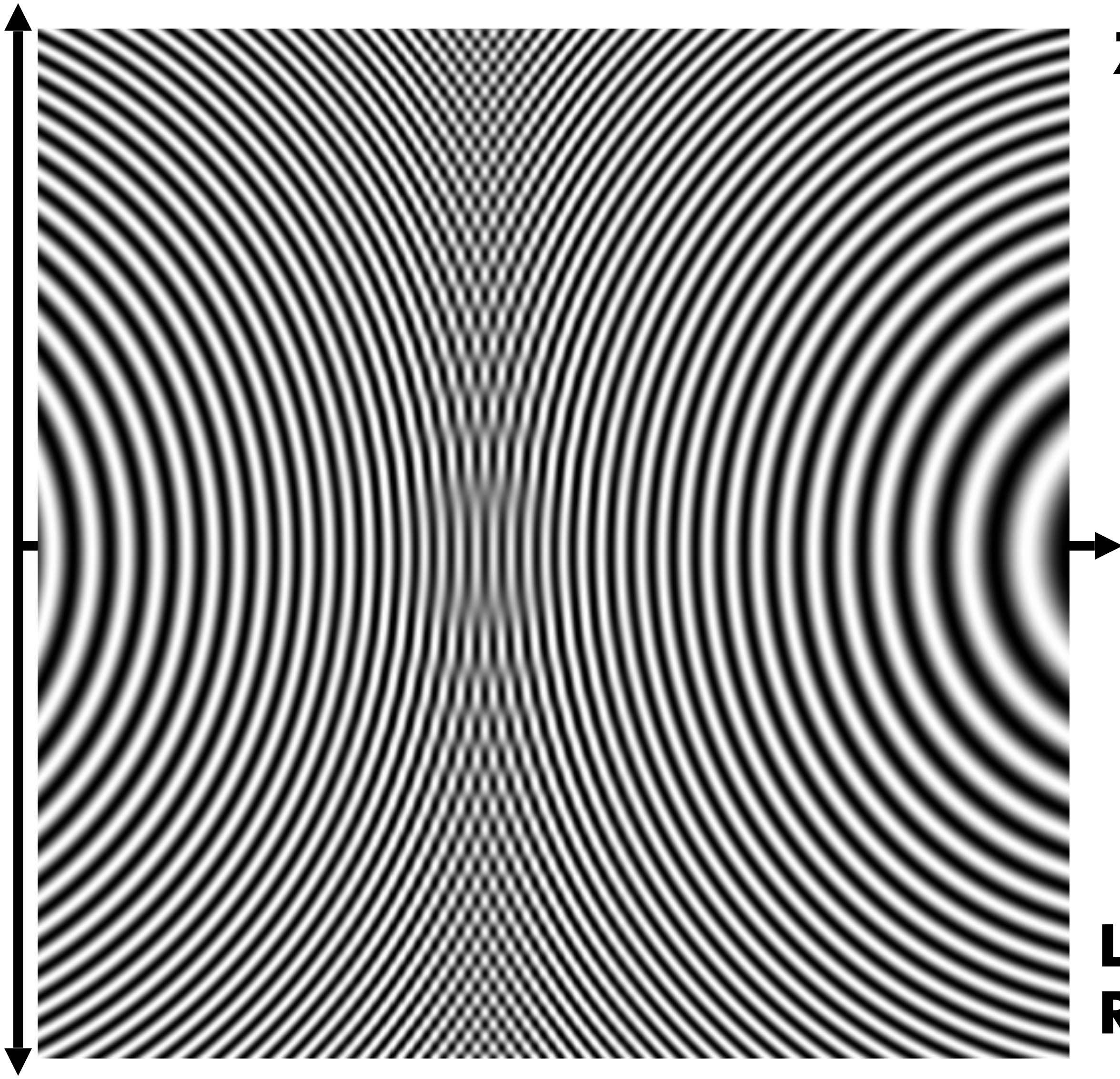
Aliasing



Undersampling: Overlaps/Aliases



Sampling a "Zone Plate"



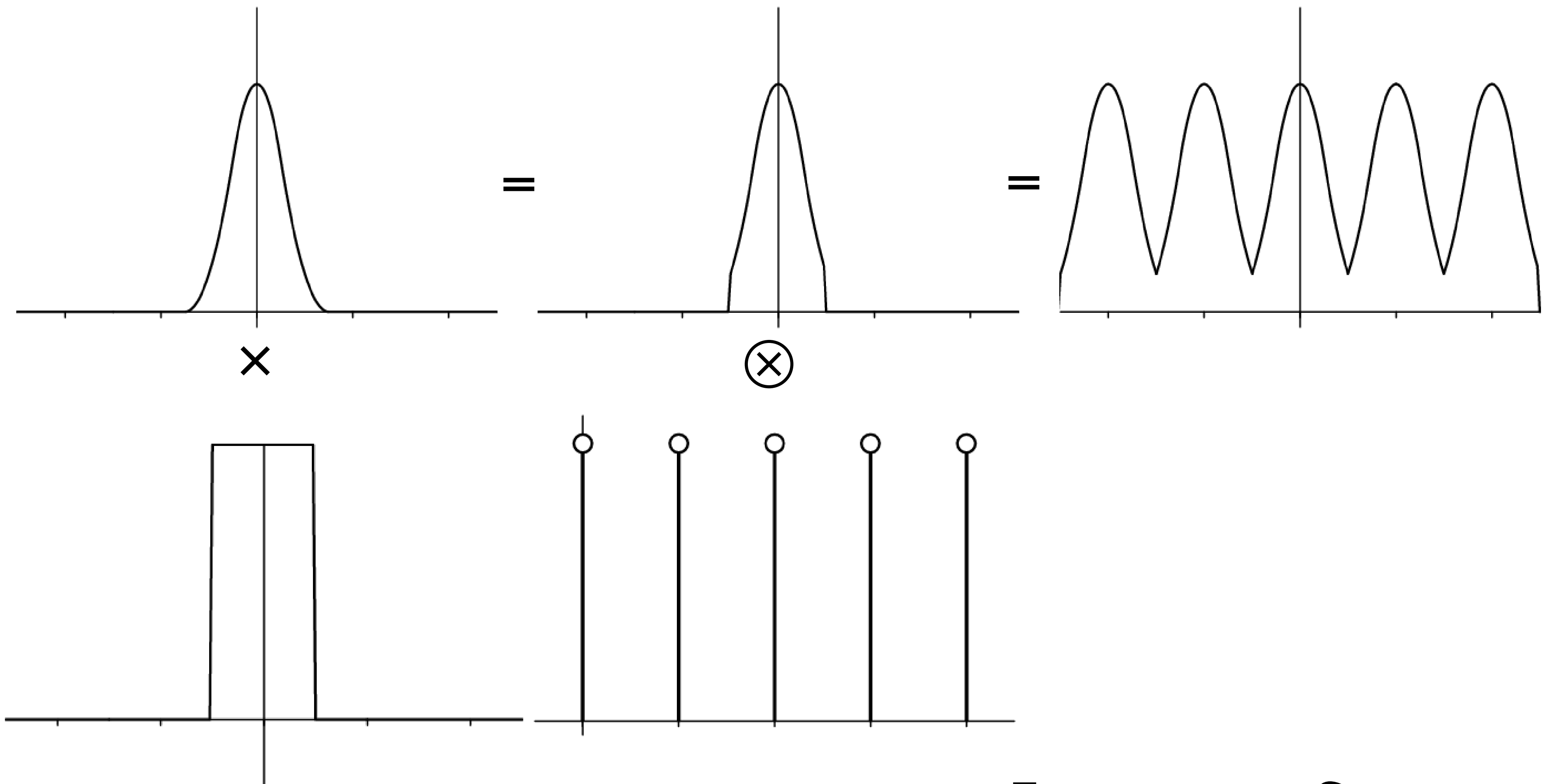
Zone plate: $\sin x^2 + y^2$

Left rings: signal
Right rings: aliasing

Antialiasing

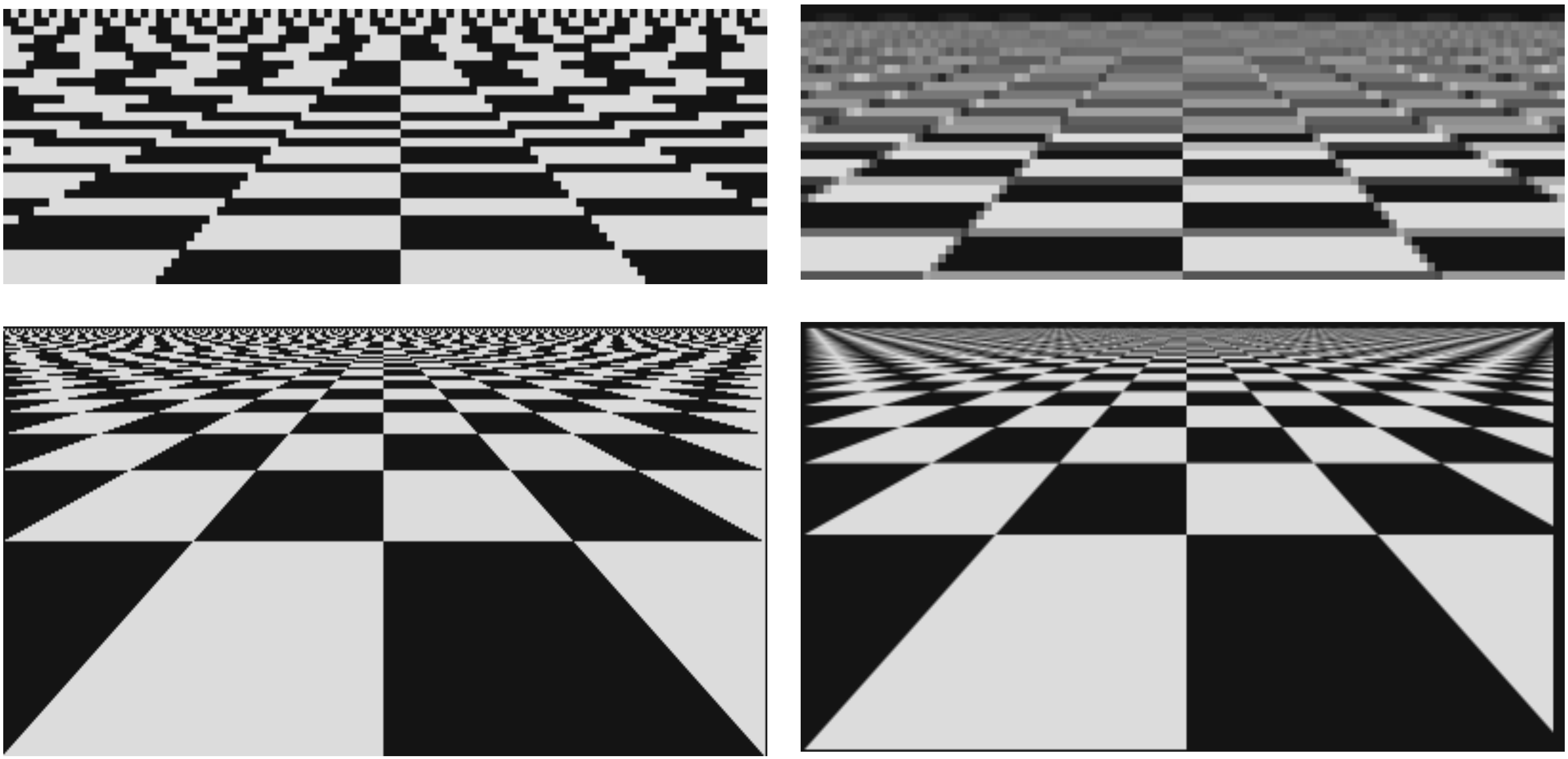
Preventing Aliasing

Antialiasing by Prefiltering



Frequency Space

Point vs. Area Sampling



Point

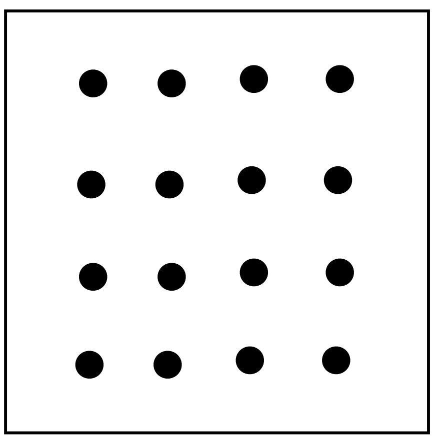
Exact Area

Checkerboard sequence by Tom Duff

Uniform Supersampling

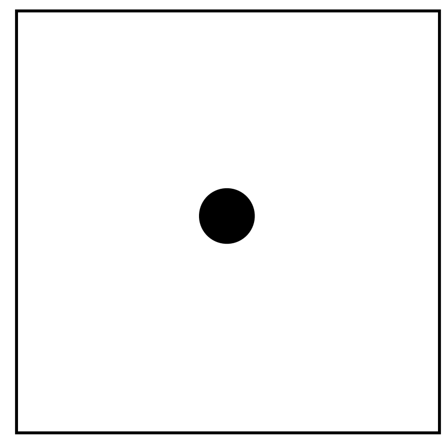
Take *n* samples

Average them together (filter = weighted average)



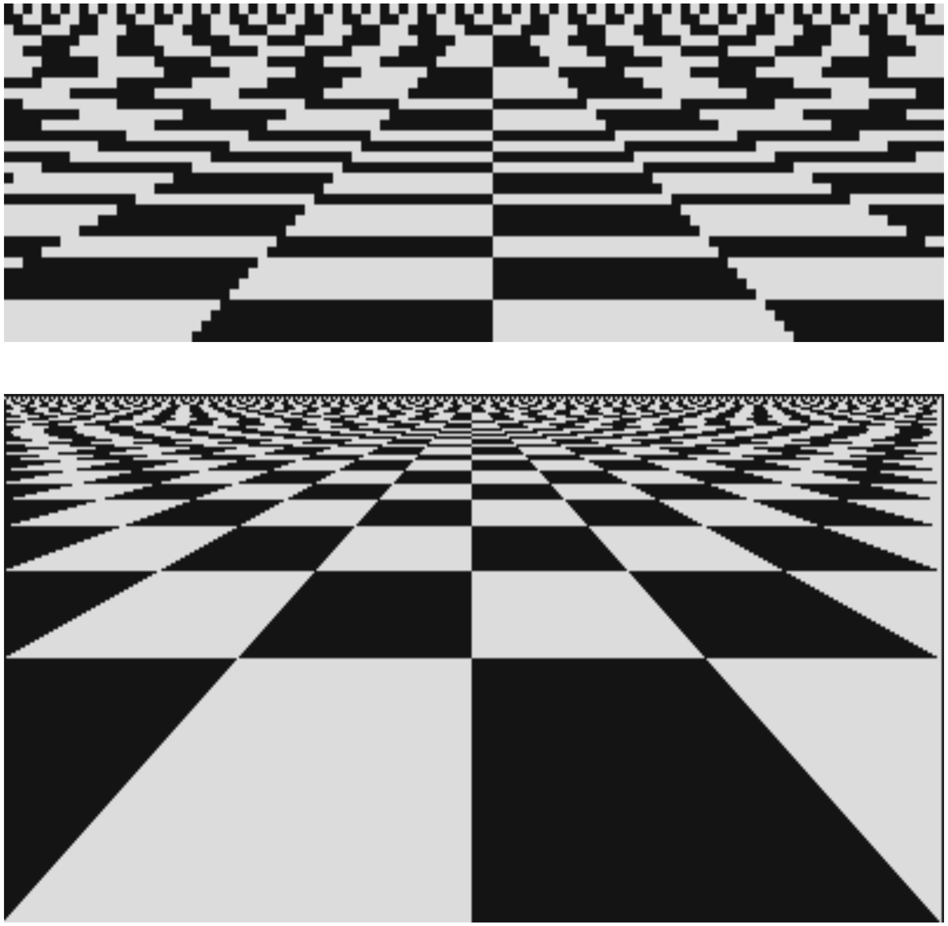
Samples

$$Pixel = \sum_s w_s \cdot Sample_s$$

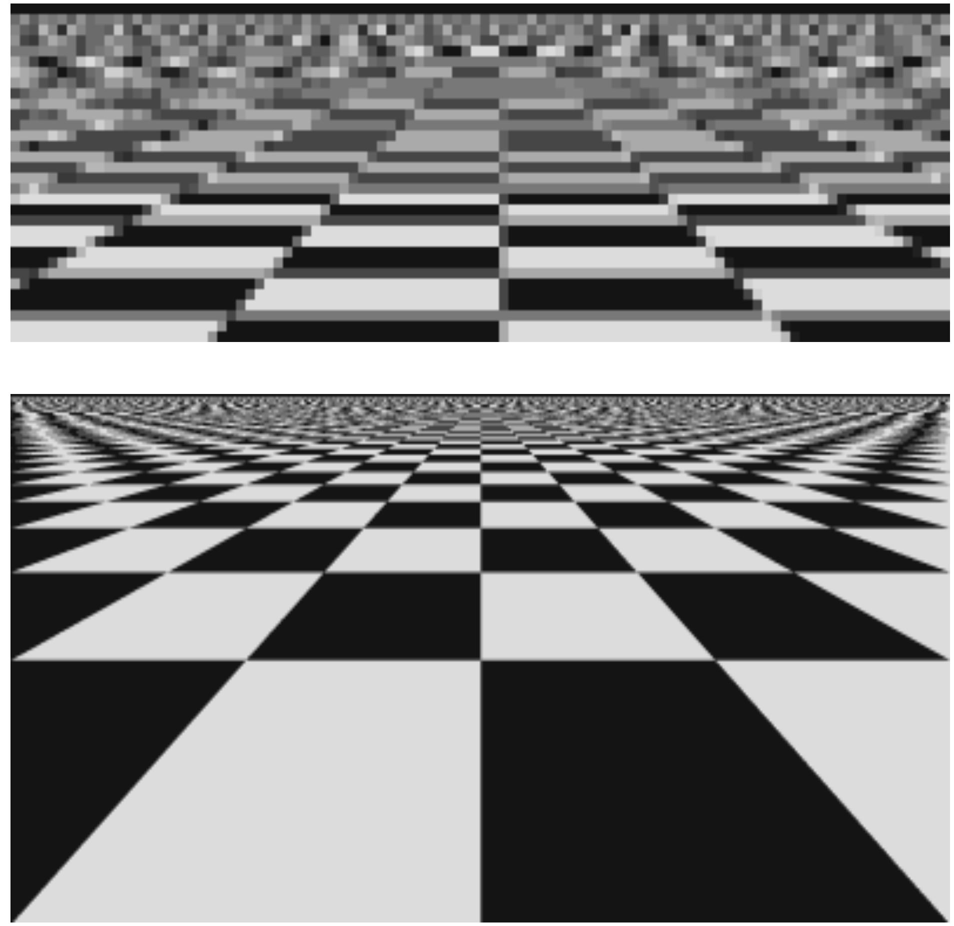


Pixel

Point vs. Supersampled



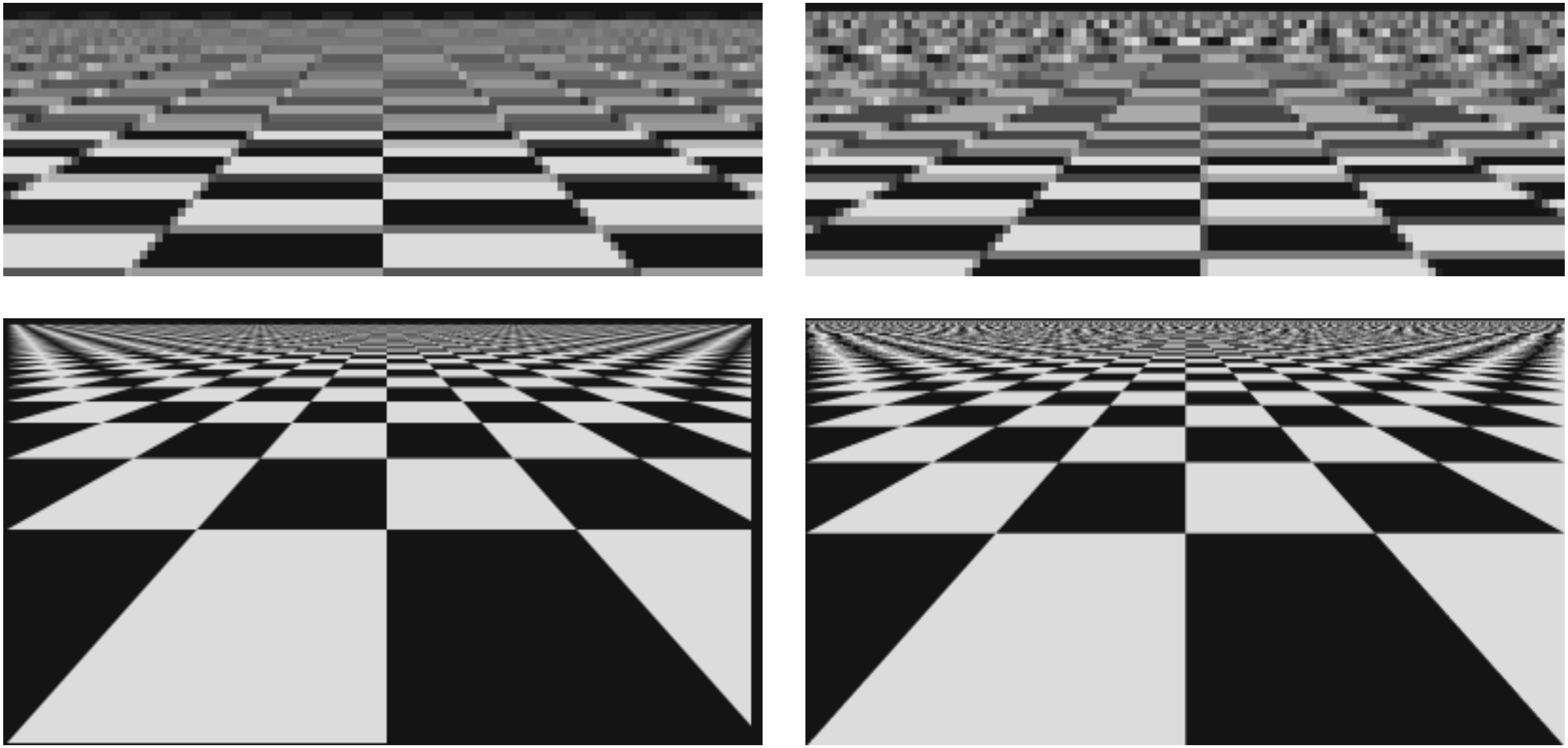
Point



4x4 Uniform

Fewer aliases, but still present

Area vs. Supersampled



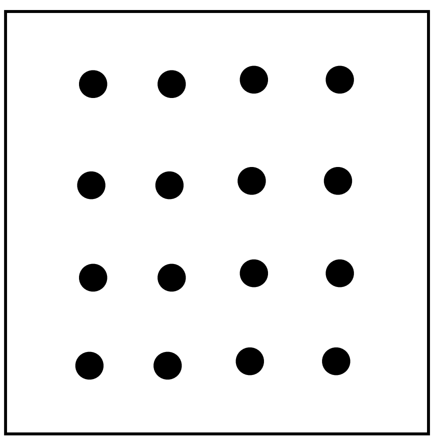
Exact Area | **4x4 Uniform**

Fewer aliases, but still present

Uniform Supersampling

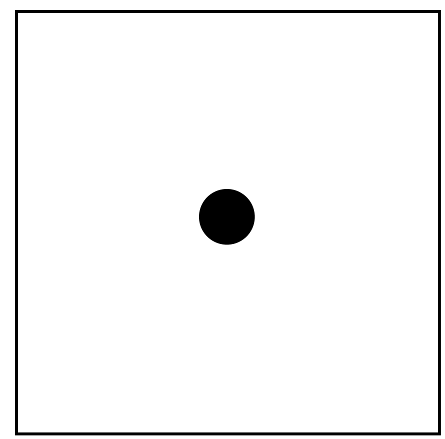
Increasing the number of samples moves each copy of the spectra further apart, thus there is less overlap

This reduces, but does not eliminate, aliasing



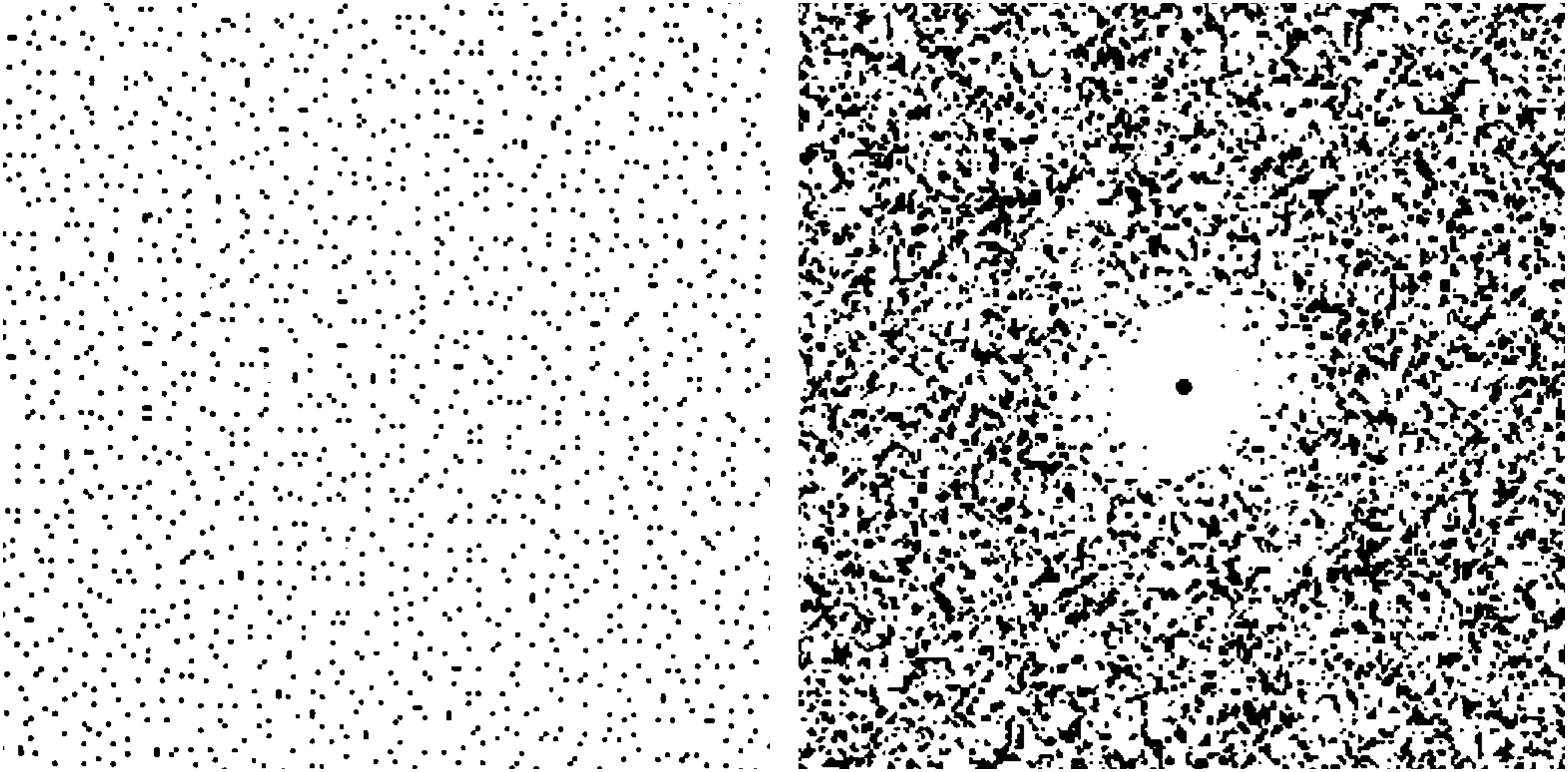
Samples

$$Pixel = \sum_s w_s \cdot Sample_s$$

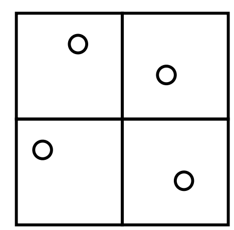


Pixel

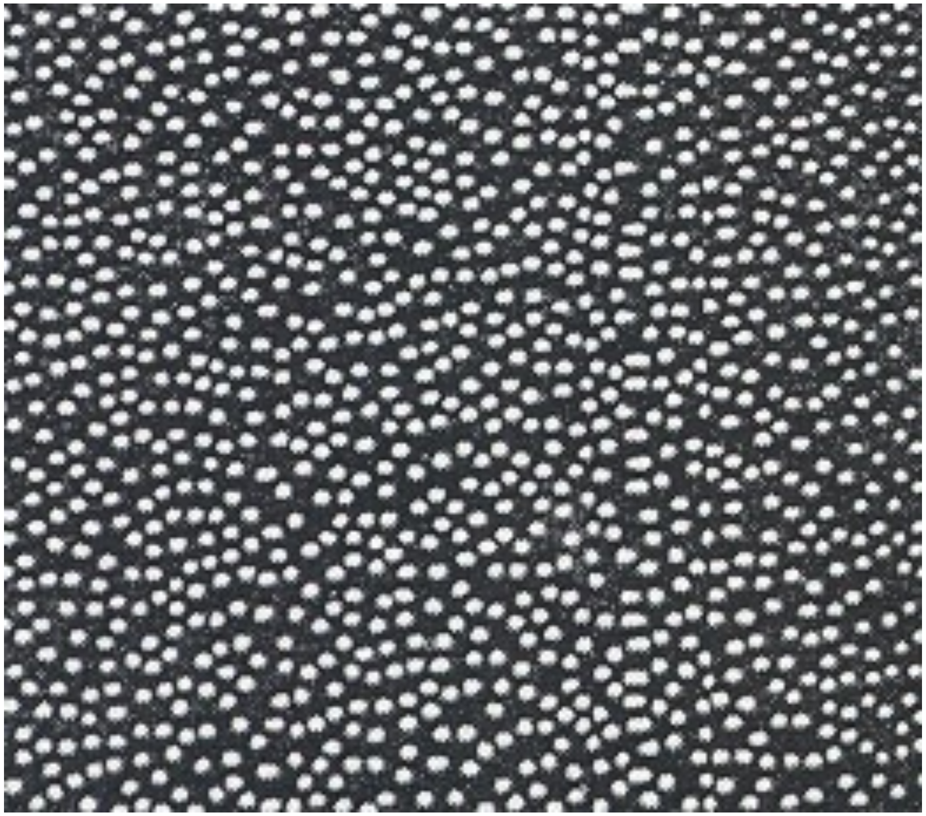
Jittered Sampling



Add uniform random jitter to each sample



Distribution of Extrafoveal Cones

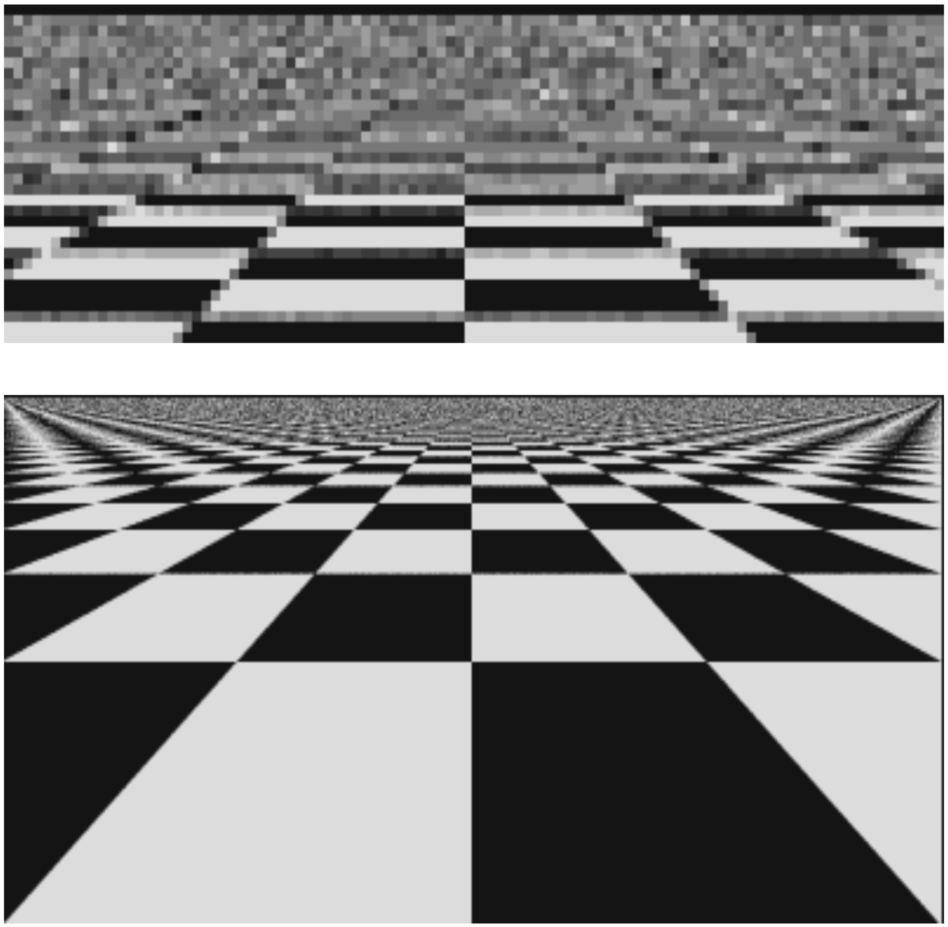


Monkey eye cone distribution

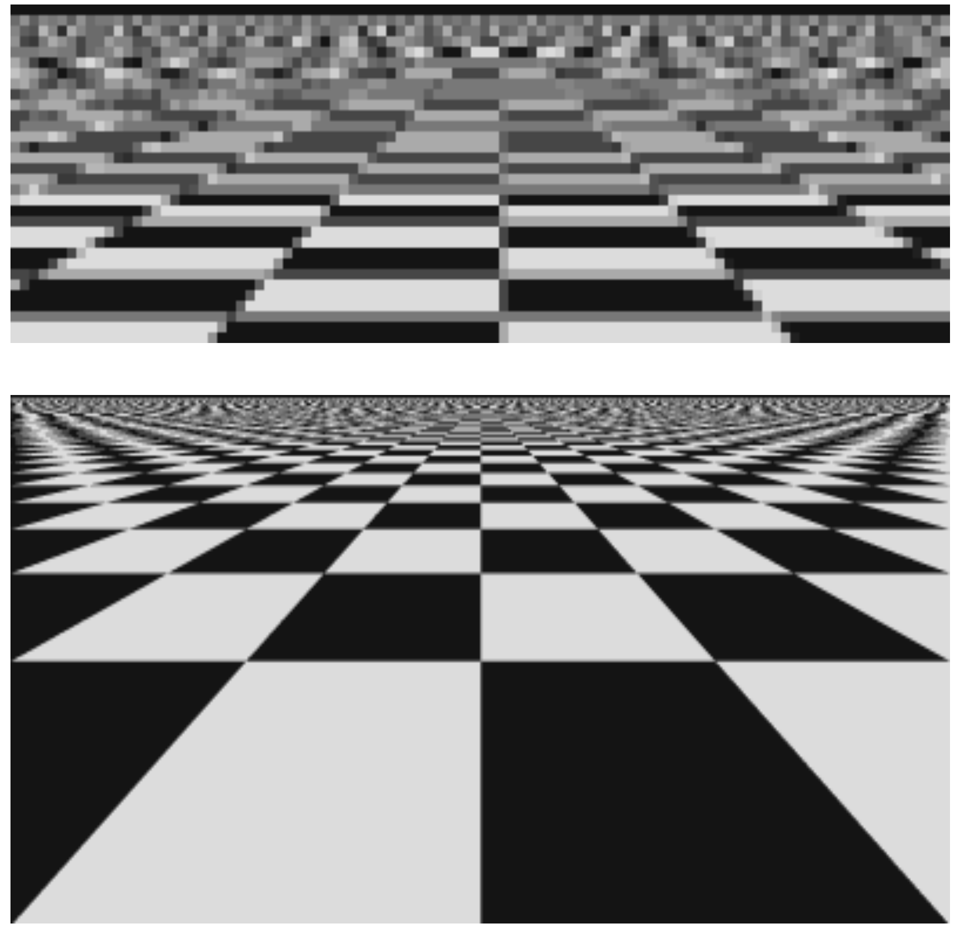
Fourier transform

Yellot

Jittered vs. Uniform Supersampling



4x4 Jittered Sampling



4x4 Uniform

Theory: Analysis of Jitter

Non-uniform sampling

$$s(x) = \sum_{n=-\infty}^{n=\infty} \delta(x - x_n)$$

$$x_n = nT + j_n$$

Jittered sampling

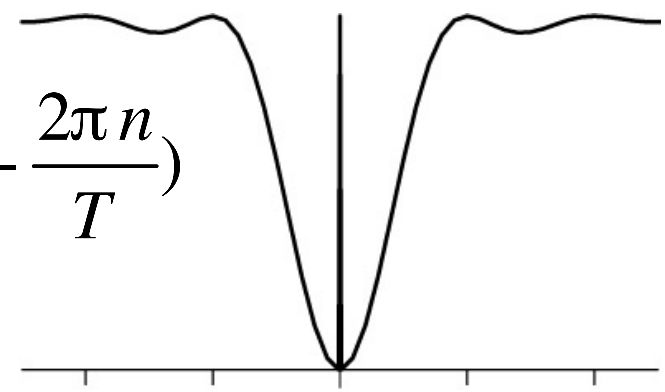
$$j_n \sim j(x)$$

$$j(x) = \begin{cases} 1 & |x| \le 1/2 \\ 0 & |x| > 1/2 \end{cases}$$

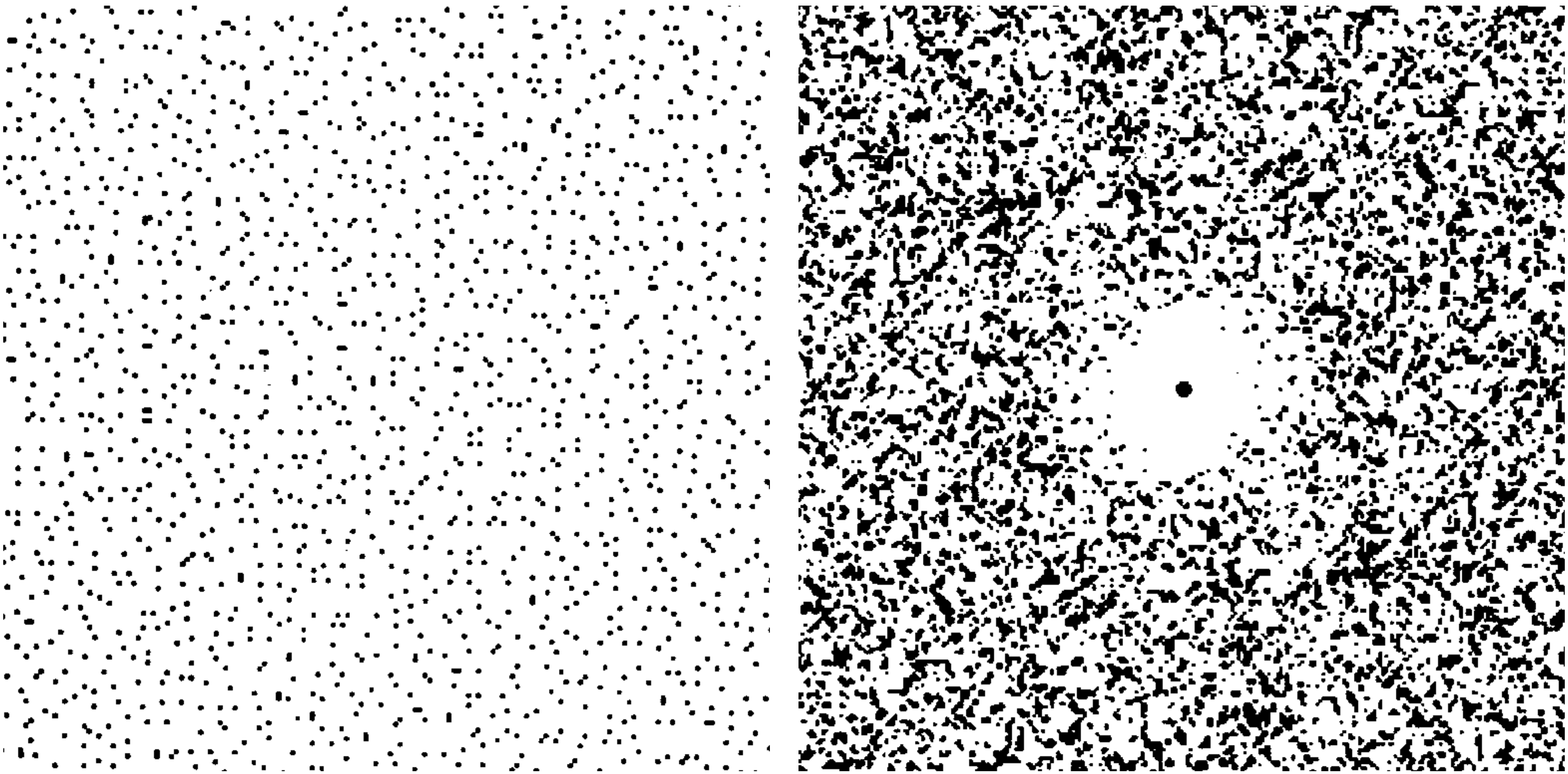
$$J(\omega) = \operatorname{sinc} \omega$$

$$S(\omega) = \frac{1}{T}\left[1 - |J(\omega)|^2\right] + \frac{2\pi}{T^2}|J(\omega)|^2 \sum_{n=-\infty}^{n=-\infty} \delta\left(\omega - \frac{2\pi n}{T}\right)$$

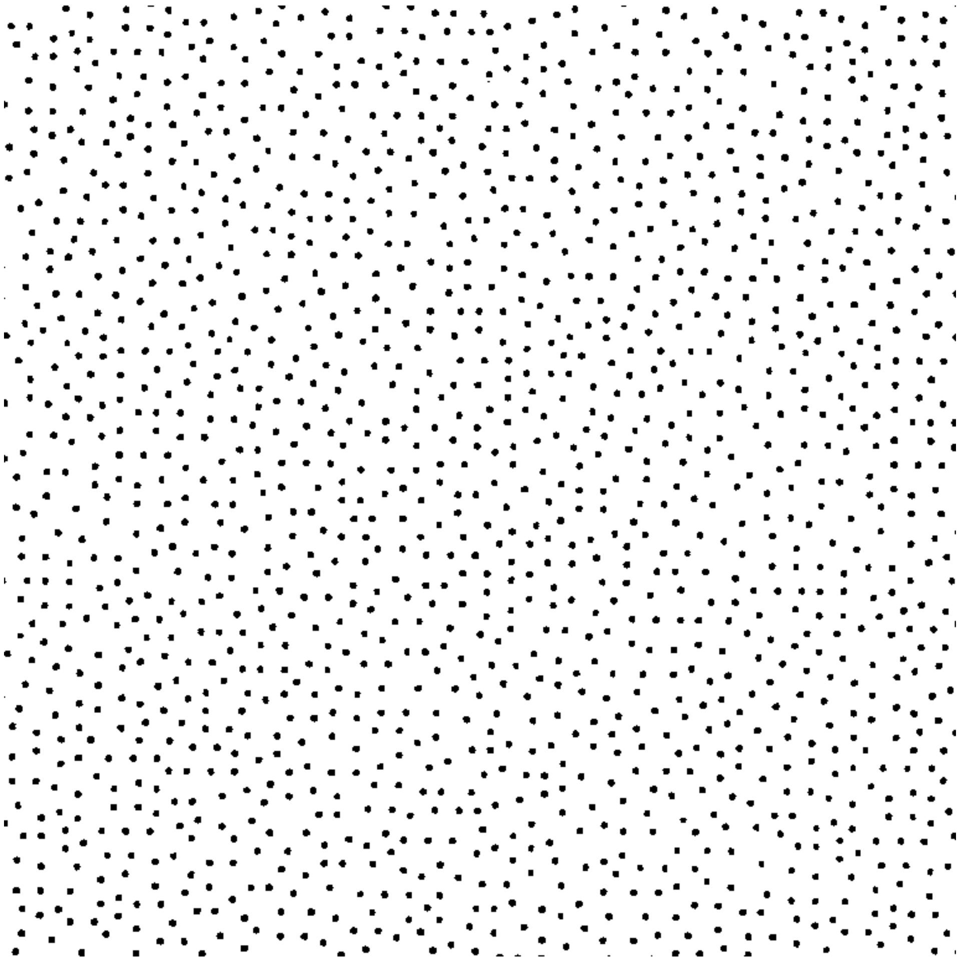
$$= \frac{1}{T}\left[1 - \operatorname{sinc}^2 \omega\right] + \delta(\omega)$$

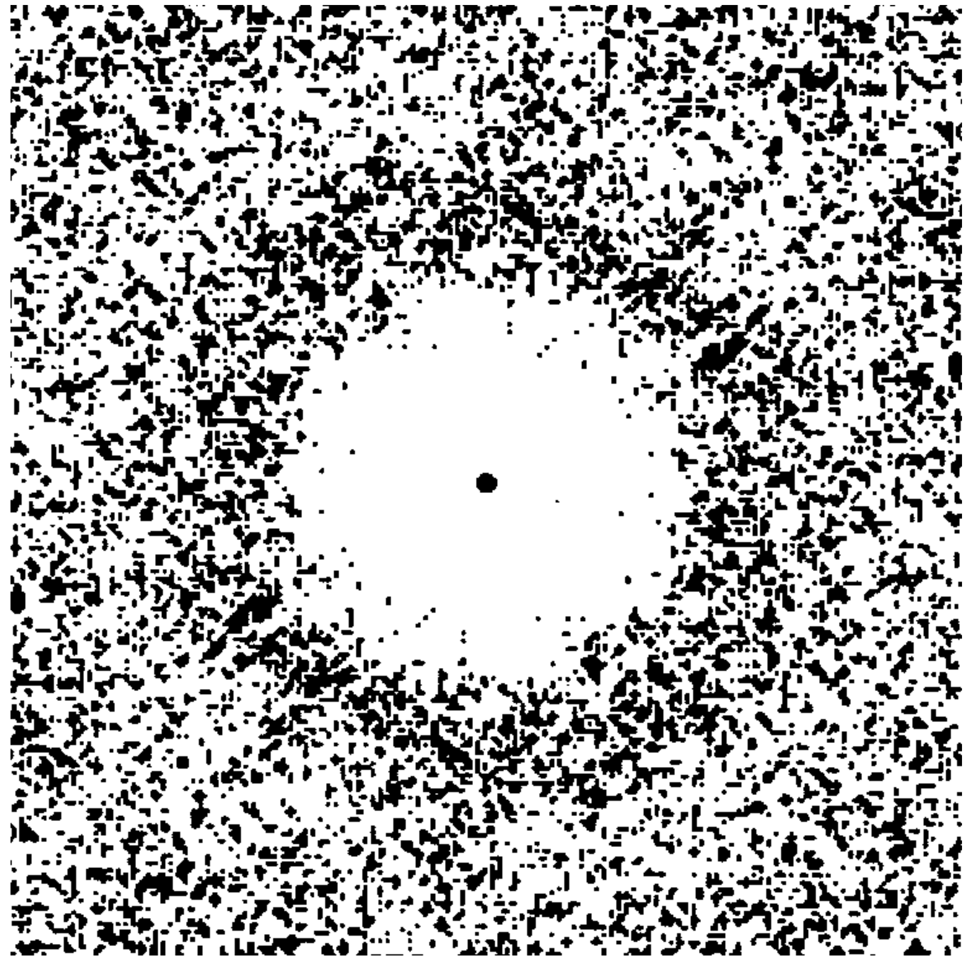


Jittered Sampling



Poisson Disk Sampling





Integration Error when Sampling

Integral

$$I(f) = \int f(x)\,dx = F(0)$$

Sampled integral

$$I_s(f) = \int f(x)s(x)\,dx = \frac{1}{N}\sum f(x_i)$$

$$f(x)s(x) = F(\omega) \oplus S(\omega)$$

$$I_s(f) = \int f(x)s(x)\,dx = F(\omega) \oplus S(\omega)|_{\omega=0}$$

Error

$$\Delta = F(0) - F(\omega) \oplus S(\omega)|_{\omega=0}$$

Non-uniform Sampling

Uniform sampling

- The spectrum of uniformly spaced samples is also a set of uniformly spaced spikes
- Multiplying the signal by the sampling pattern corresponds to placing a copy of the spectrum at each spike (in freq. space)
- Aliases are coherent, and very noticable

Non-uniform sampling

- Samples at non-uniform locations have a different spectrum; a single spike plus noise
- Sampling a signal in this way converts aliases into broadband noise
- Noise is incoherent, and much less objectionable
- May cause error in the integral