Digital Signal Processing

Solved HW for Day 5

Question 1: DTFT and DFS

Consider the Discrete Time Fourier Transform (DTFT) $X(e^{j\omega})$ of a sequence $x(n) \in \ell^2(Z)$, of finite support, L.

- By observing N uniform samples of $X(e^{j\omega})$, one obtains the DFS $X(k)$ of the periodically extended version of the same signal, for all values of N.
- By observing $N \geq L$ uniform samples of $X(e^{j\omega})$, one obtains the DFS of the periodically extended version of the same signal, $X(k)$.
- The DFT of $x[n]$ is one period of an uniformly sampled version of $X(e^{j\omega})$ at an appropriate sampling rate.

Solution of question 1

Q: Show that by observing N uniform samples of $X(e^{j\omega})$, one obtains the DFS $X(k)$ of the periodically extended version of the same signal, for all values of N.

Using the definition of DTFT, $X(\omega) = \sum_{n=-\infty}^{\infty} x(n)e^{-i\omega n}$.
Since $X(\omega)$ is periodic in 2π, the discretization with N samples is obtained with $\omega = 2\pi k/N$:

$$X\left(\frac{2\pi}{N}k\right) = \sum_{n=-\infty}^{\infty} x(n)e^{-i2\pi kn/N} \quad k = 0, 1..N-1$$

$$= \sum_{r=-\infty}^{\infty} \sum_{n=rN}^{(r+1)N-1} x(n)e^{-i2\pi kn/N}$$

$$= \sum_{n=0}^{N-1} \sum_{r=-\infty}^{\infty} x(n+rN)e^{-i2\pi kn/N}$$

Solution of question 1

Q: Show that by observing $N \geq L$ uniform samples of $X(e^{j\omega})$, one obtains the DFS of the periodically extended version of the same signal, $X(k)$.

Let,

$$x_p(n) = \sum_{r=-\infty}^{\infty} x(n + rN)$$

be the periodic extension of the finite length sequence of interest, $x(n)$. Then,

$$X\left(\frac{2\pi}{N}k\right) = \sum_{n=0}^{N-1} x_p(n)e^{-i2\pi kn/N}$$

is the DFS of the periodic sequence. Applying the inverse transform, we get the signal $x_p(n)$.

Solution of question 1

The sequence of interest, $x(n)$ corresponds to one period of the periodic signal, $x_p(n)$ when $N \geq L$

$$x(n) = \begin{cases} x_p(n) & \text{if } 0 \leq n \leq N-1 \\ 0 & \text{otherwise} \end{cases}$$

When $N < L$,

$$x_p(n) = \sum_{r=-\infty}^{\infty} x(n + rN)$$

$x_p(n)$ is no longer the exact periodic repition of $x(n)$. It is now the sum of *N-shifted* versions of the signal $x(n)$ and since $N < L$, each N-tap signal, is corrupted by a shifted version of itself.

Solution of question 1

Q: Show that the DFT of $x[n]$ is one period of an uniformly sampled version of $X(e^{j\omega})$ at an appropriate sampling rate.

Consider the previous result for $N > L$,

$$X\left(\frac{2\pi}{N}k\right) = \sum_{n=0}^{N-1} x_p(n) e^{-i2\pi kn/N}, \tag{1}$$

where we note that $x_p(n) = x(n)$, for $n \in [0, N-1]$. Then, (1) can be rewritten as

$$X\left(\frac{2\pi}{N}k\right) = \sum_{n=0}^{N-1} x(n) e^{-i2\pi kn/N}. \tag{2}$$

The right hand side of the above equation is the DFT sum, while the left hand side is one period of the uniformly sampled version of the DTFT, sampled such that $N > L$.

Question 2: DFT of the Autocorrelation

Consider a sequence $x[n]$ of finite length L.
Let $X(k)$ denote the N point DFT of $x[n]$ and define the circular autocorrelation sequence $r_{xx}[m]$ of $x[n]$ as

$$r_{xx}[m] = \sum_{n=0}^{N-1} x[n]x^*[n - m \bmod N].$$

Express $r_{xx}[m]$ in terms of $X(k)$.

Solution of question 2

$$
\begin{aligned}
IDFT[X(k)X^*(k)] &= \frac{1}{N}\sum_{k=0}^{N-1} X(k)X^*(k)e^{j\frac{2\pi}{N}km} \\
&= \frac{1}{N}\sum_{k=0}^{N-1}[\sum_{n=0}^{N-1} x(n)e^{-j\frac{2\pi}{N}kn}][\sum_{l=0}^{N-1} x^*(l)e^{j\frac{2\pi}{N}lk}]e^{j\frac{2\pi}{N}km} \\
&= \frac{1}{N}\sum_{n=0}^{N-1} x(n)\sum_{l=0}^{N-1} x^*(l)\sum_{k=0}^{N-1} e^{j\frac{2\pi}{N}k(l+m-n)} \\
&= \sum_{n=0}^{N-1} x(n)x^*(n-m \bmod N) \\
&= r_{xx}(m)
\end{aligned}
$$

Question 3: Sinusoidal Modulation

Given an ideal low pass filter

$$H_{lp}(e^{j\omega}) = \begin{cases} 1 & |\omega| \leq \omega_b/2, \\ 0 & \text{elsewhere.} \end{cases} \qquad ,$$

relate the impulse response of a bandpass filter with center frequency ω_0 and passband ω_b:

$$H_{bp}(e^{j\omega}) = \begin{cases} 1 & \omega_0 - \omega_b/2 \leq \omega \leq \omega_0 + \omega_b/2, \\ 1 & -\omega_0 - \omega_b/2 \leq \omega \leq -\omega_0 + \omega_b/2, \\ 0 & \text{elsewhere.} \end{cases}$$

with the lowpass filter.

Solution of question 3

Consider a lowpass filter $h_{lp}[n]$ with bandwidth ω_b. If we consider the sequence

$$h[n] = 2\cos(\omega_0 n) h_{lp}[n]$$

the Modulation theorem tells us that its Fourier transform is

$$H(e^{j\omega}) = H_{lp}(e^{j(\omega-\omega_0)}) + H_{lp}(e^{j(\omega+\omega_0)}) = H_{bp}(e^{j\omega})$$

Therefore the impulse response of the bandpass filter is

$$h_{bp}[n] = 2\cos(\omega_0 n) h_{lp}[n] = 2\cos(\omega_0 n)\,\frac{\omega_b}{2\pi}\,\mathrm{sinc}\left(\frac{\omega_b}{2\pi} n\right)$$

Question 4: DFT of a subsampled signal

Consider an N periodic sequence $x[n]$ and its N point DFS $X(k)$. Define $Y(k) = X(2k)$ to be the $N/2$ point sequence that corresponds to the odd terms of $X(k)$. Relate the $N/2$ point inverse DFS of $Y(k)$, $y[n]$ with $x[n]$.

Solution of question 4

Given $Y(k) = X(2k)$, $y[n]$, the $\frac{N}{2}$ point inverse of $Y(k)$ can be expressed as

$$
\begin{aligned}
y[n] &= \frac{1}{N/2}\sum_{l=0}^{\frac{N}{2}-1} Y(l)e^{j\frac{2\pi}{N/2}nl} \\
&= \frac{2}{N}\sum_{l=0}^{\frac{N}{2}-1} X(2l)e^{j\frac{2\pi}{N}2nl}
\end{aligned}
$$

Now we note that

$$
1 + (-1)^n = \begin{cases} 2 & \text{if } n \text{ is even} \\ 0 & \text{if } n \text{ is even} \end{cases}
$$

Solution of question 4

$y[n]$ can now be rewritten in terms of $X(k)$ as

$$
\begin{aligned}
y[n] &= \frac{1}{N}\sum_{k=0}^{N-1}(1+(-1)^n)X(k)e^{j\frac{2\pi}{N}nk} \\
&= \frac{1}{N}\sum_{k=0}^{N-1}X(k)e^{j\frac{2\pi}{N}nk} + \frac{1}{N}\sum_{k=0}^{N-1}X(k)e^{j\frac{2\pi}{N}nk}e^{j\pi k} \\
&= \frac{1}{N}\sum_{k=0}^{N-1}X(k)e^{j\frac{2\pi}{N}nk} + \frac{1}{N}\sum_{k=0}^{N-1}X(k)e^{j\frac{2\pi}{N}k(n+\frac{N}{2})} \\
&= x[n] + x[n+\frac{N}{2}] \qquad n = 0...(\frac{N}{2}-1)
\end{aligned}
$$