Digital Signal Processing

Solved HW for Day 2

Question 1: Fourier basis

Consider the Fourier basis $\{\mathbf{w}^{(k)}\}_{k=0,\dots,N-1}$, defined as:

$$\mathbf{w}_n^{(k)} = e^{-j\frac{2\pi}{N}nk}.$$

- Prove that it is an orthogonal basis in $\mathbb{C}^N$.
- Normalize the vectors in order to get an orthonormal basis.

Solution of Question 1

Q: Prove that it is an orthogonal basis in $\mathbb{C}^N$.

- Recall one of the most important properties for finite dimensional subspaces: The set of N non-zero orthogonal vectors in an N-dimensional subspace is a basis for the subspace.
- Therefore, it is sufficient to just prove the orthogonality of the vectors $\{\mathbf{w}^{(k)}\}_{k=0,\dots,N-1}$.

Solution of Question 1

Q: Prove that it is an orthogonal basis in $\mathbb{C}^N$.

- Let us compute the inner product, that is:

$$\begin{aligned}
\langle \mathbf{w}^{(k)}, \mathbf{w}^{(h)} \rangle &= \sum_{n=0}^{N-1} \mathbf{w}^{*(k)}[n] \mathbf{w}^{(h)}[n] = \sum_{n=0}^{N-1} e^{j\frac{2\pi}{N}nk} e^{-j\frac{2\pi}{N}nh} \\
&= \sum_{n=0}^{N-1} e^{-j\frac{2\pi}{N}n(h-k)} = \begin{cases} N & \text{if } k = h \\ 0 & \text{otherwise.} \end{cases}
\end{aligned}$$

- Since the inner product of the vectors is equal to zero, we conclude that they are orthogonal.
- However, they do not have a unit norm and therefore are not the orthonormal vectors.

Solution of Question 1

Q: Normalize the vectors in order to get an orthonormal basis.

- In order to obtain the *orthonormal basis* we normalize the vectors with the factor $1/\sqrt{N}$, having:

$$
\begin{aligned}
\langle \mathbf{w}_{norm}^{(k)}, \mathbf{w}_{norm}^{(h)} \rangle &= \sum_{n=0}^{N-1} \frac{1}{\sqrt{N}} e^{j\frac{2\pi}{N}nk} \frac{1}{\sqrt{N}} e^{-j\frac{2\pi}{N}nh} \\
&= \frac{1}{N} \sum_{n=0}^{N-1} e^{-j\frac{2\pi}{N}n(h-k)} = \begin{cases} 1 & \text{if } k = h \\ 0 & \text{otherwise.} \end{cases}
\end{aligned}
$$

Question 2: Vector spaces & signals I

- Show that the set of all ordered n-tuples $[a_1, a_2, \ldots, a_n]$ with the natural definition for the sum:
$$[a_1, a_2, \ldots, a_n] + [b_1, b_2, \ldots, b_n] = [a_1 + b_1, a_2 + b_2, \ldots, a_n + b_n]$$
and the multiplication by a scalar:
$$\alpha[a_1, a_2, \ldots, a_n] = [\alpha a_1, \alpha a_2, \ldots, \alpha a_n]$$
form a vector space. Give its dimension and find a basis.
- Show that the set of signals of the form $y(x) = a\cos(x) + b\sin(x)$ (for arbitrary a, b) with the usual addition and multiplication by a scalar form a vector space. Give its dimension and find a basis.

Solution of Question 2

Q: Show that the set of all ordered n-tuples form a vector space.

It is straight forward to verify that the set of all ordered n-tuples $[a_1, a_2, \ldots, a_n]$ with

- $[a_1, a_2, \ldots, a_n] + [b_1, b_2, \ldots, b_n] = [a_1 + b_1, a_2 + b_2, \ldots, a_n + b_n]$
- $\alpha[a_1, a_2, \ldots, a_n] = [\alpha a_1, \alpha a_2, \ldots, \alpha a_n]$

satisfies all the properties of a vector space, which are

1. Addition is commutative.
2. Addition is associative.
3. Scalar multiplication is distributive.

Solution of Question 2

- There exists a null vector: $[0, 0, \dots, 0]$.
- Additive inverse: $[-a_1, -a_2, \dots, -a_n]$.
- Identity element for scalar multiplication: 1.

Q: Give its dimension and find a basis.

The dimension of this vector space is n and a basis is:

$$[1, 0, \dots, 0], [0, 1, \dots, 0], ..., [0, 0, \dots, 1].$$

Solution of Question 2

Q: Show that the set of signals of the form $y(x) = a\cos(x) + b\sin(x)$ (for arbitrary a, b) with the usual addition and multiplication by a scalar form a vector space.

The set of signals of the form $y(x) = a\cos(x) + b\sin(x)$ (for arbitrary a, b) with the usual addition and multiplication by a scalar form a vector space:

1. Addition is commutative.
2. Addition is associative.
3. Scalar multiplication is distributive.

Solution of Question 2

- There exists a null vector: $a, b = 0$.
- Additive inverse: $-y(x) = -a\cos(x) - b\sin(x)$.
- Identity element for scalar multiplication: 1.

Q: Give its dimension and find a basis.

The dimension of this vector space is 2 and a possible basis is:

$$y_1(x) = \cos(x), y_2(x) = \sin(x).$$

Question 3: Vector spaces & signals II

- Are the four diagonals of a cube orthogonal?
- Express the discrete-time impulse $\delta[n]$ in terms of the discrete-time unit step $u[n]$ and conversely.
- Show that any function $f(t)$ can be written as the sum of an odd and an even function, i.e. $f(t) = f_o(t) + f_e(t)$ where $f_o(-t) = -f_o(t)$ and $f_e(-t) = f_e(t)$.

Solutions of Question 3

Q: Are the four diagonals of a cube orthogonal?

The eight vertices of the cube can be represented by the following four vectors:

$$v_1 = [0,0,0],\ v_2 = [1,0,0],\ v_3 = [0,1,0],\ v_4 = [1,1,0],$$
$$v_5 = [0,0,1],\ v_6 = [1,0,1],\ v_7 = [0,1,1],\ v_8 = [1,1,1].$$

and the four associated diagonals:

$$d_1 = \{v_1, v_8\} = v_8 - v_1 = [1,1,1].$$
$$d_2 = \{v_2, v_7\} = v_7 - v_2 = [-1,1,1].$$
$$d_3 = \{v_3, v_6\} = v_6 - v_3 = [1,-1,1].$$
$$d_4 = \{v_4, v_5\} = v_5 - v_4 = [-1,-1,1].$$

Solutions of Question 3

- Two vectors are orthogonal if their inner product is zero. In this case,

$$< \mathbf{d}^i, \mathbf{d}^j > \neq 0 \text{ for all } i, j.$$

- Therefore, the four diagonals of a cube are not orthogonal.
- Remark also that it is not possible to have 4 orthogonal vectors in a space of dimension 3. So this also explains that the 4 diagonals cannot be orthogonal.

Solutions of Question 3

Q: Express the discrete-time impulse $\delta[n]$ in terms of the discrete-time unit step $u[n]$ and conversely.

- $\delta[n] = u[n] - u[n-1]$
- $u[n] = \Sigma_{k=0}^{\infty} \delta[n-k]$.

Solutions of Question 3

Q: Show that any function $f(t)$ can be written as the sum of an odd and an even function, i.e. $f(t) = f_o(t) + f_e(t)$ where $f_o(-t) = -f_o(t)$ and $f_e(-t) = f_e(t)$.

By solving the equations,

$$\begin{cases} f(t) = f_o(t) + f_e(t) \\ f(-t) = f_o(-t) + f_e(-t) = -f_o(t) + f_e(t) \end{cases}$$

We have

- $f_o(t) = \frac{f(t)-f(-t)}{2}$
- $f_e(t) = \frac{f(t)+f(-t)}{2}$.