# Digital Signal Processing

Module 3: from Euclid to Hilbert

# Module Overview:

- Module 3.1: Signal processing as geometry or from Euclid to Hilbert spaces
- Module 3.2: Vectors, vector spaces, inner products, and Hilbert spaces
- Module 3.3: Bases for Hilbert spaces

# Digital Signal Processing

Module 3.1: a tale of two (and more) vectors

# Let's look at $\mathbb{R}^2$

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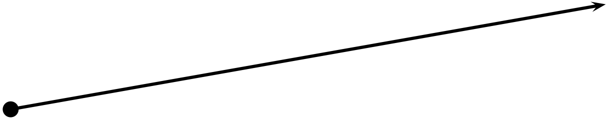

$\mathbf{x} = [x_0\ x_1]^T$

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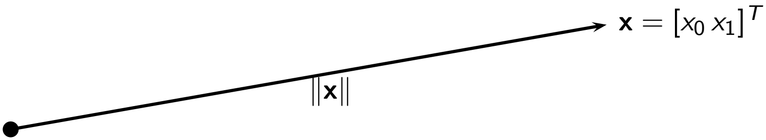

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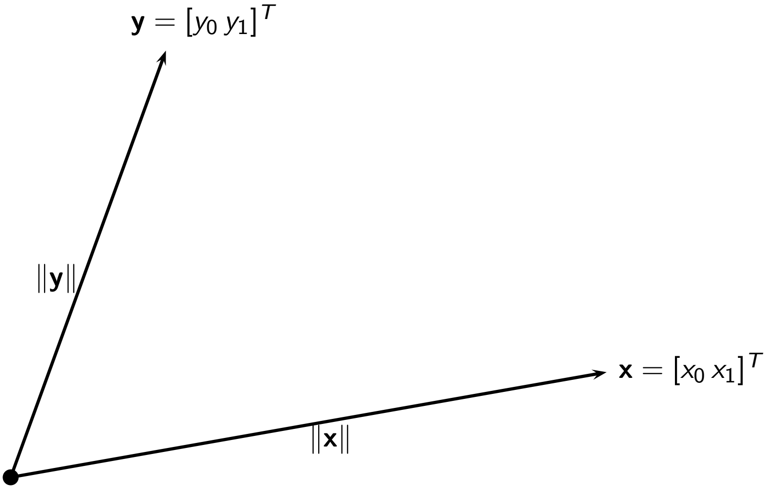

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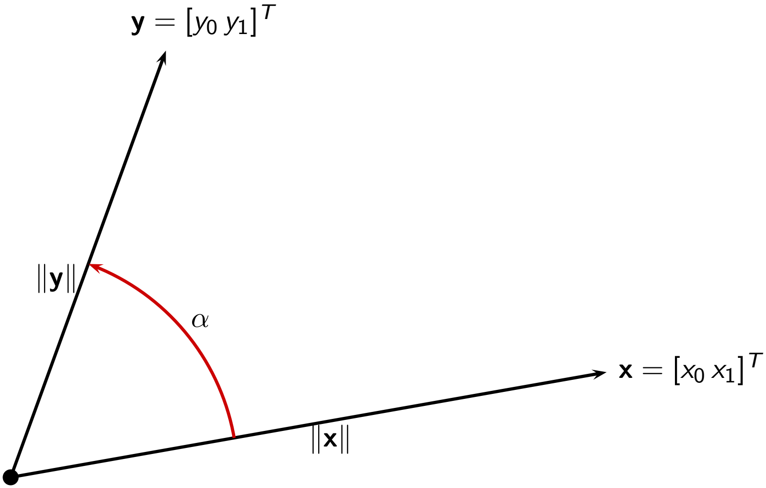

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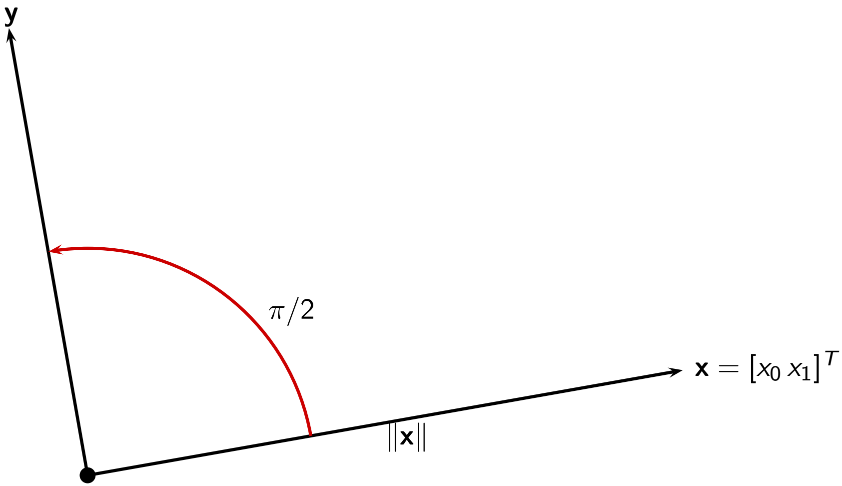
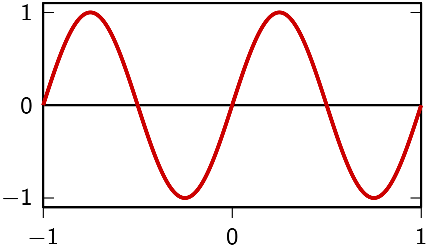
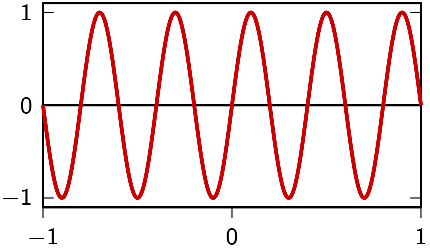


# Vectors can be very general objects!

Example: space of square-integrable functions over $[-1, 1]$: $L_2([-1, 1])$




$\mathbf{x}^{(1)} = \sin(f_1\, t), \quad f_1 = 2\pi$

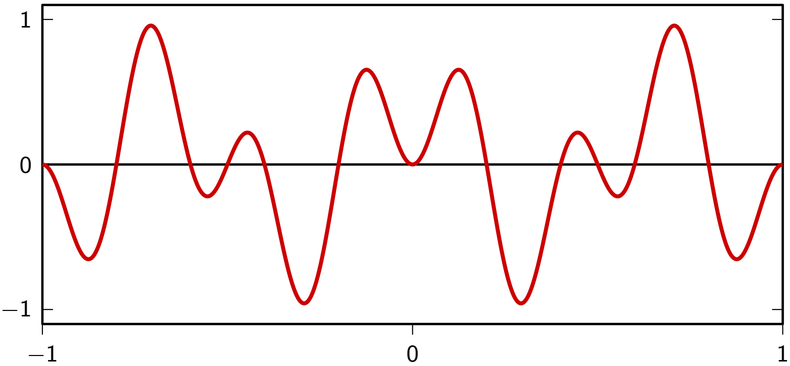



$\mathbf{x}^{(2)} = \sin(f_2\, t), \quad f_2 = 5\pi$

$$\langle \mathbf{x}^{(1)}, \mathbf{x}^{(2)} \rangle = \int_{-1}^{1} \sin(f_1\, t) \sin(f_2\, t) dt$$

# Orthogonality in a functional vector space.

$\mathbf{x}^{(1)} \perp \mathbf{x}^{(2)}$ if $f_1 \neq f_2$ and $f_1, f_2$ integer multiples of a fundamental (harmonically related)

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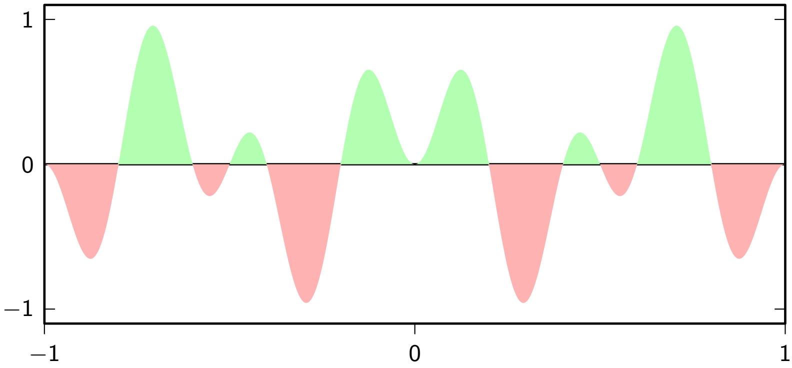


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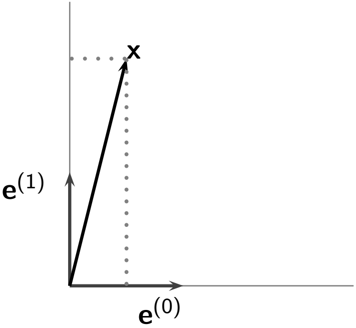
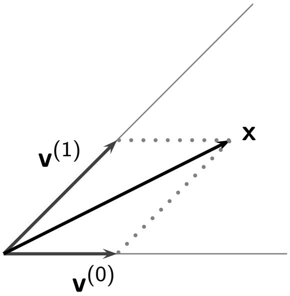
# Vectors spanning a space

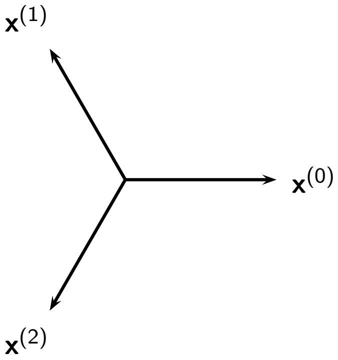



orthogonal basis




biorthogonal basis

# Too many vectors for the space




Linear dependence:

$$\exists\{a_0, a_1, a_2\} \text{ s.t. } a_0\mathbf{x}^{(0)} + a_1\mathbf{x}^{(1)} + a_2\mathbf{x}^{(2)} = 0$$

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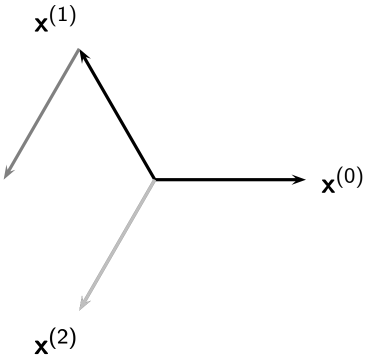


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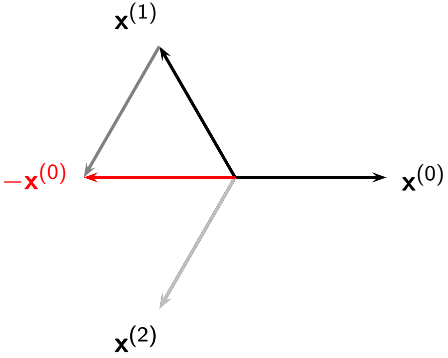


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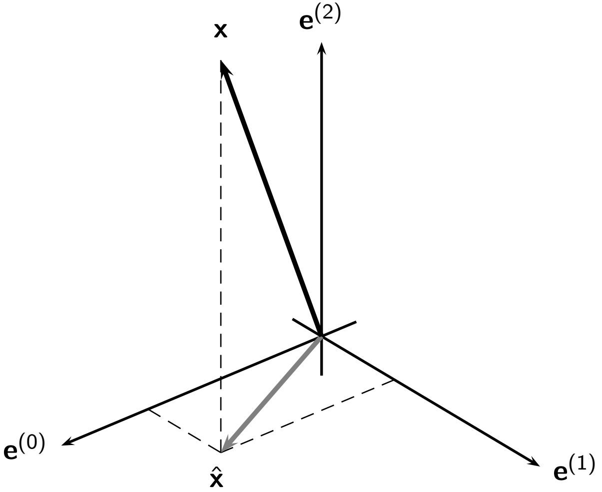
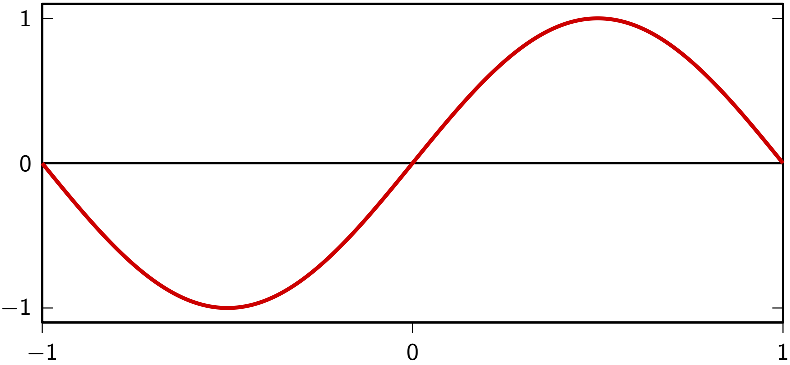
# Not enough vectors for the space

subspace projection: $\hat{\mathbf{x}}$ is the closest approximation to $\mathbf{x}$ in the space spanned by $\{\mathbf{e}^{(0)}, \mathbf{e}^{(1)}\}$



$$\sum_{k=0}^{N} \mathbf{x}^{(2k+1)}, \qquad \mathbf{x}^{(n)} = \sin(\pi\, n\, t)/n, \quad t \in [-1, 1]$$




$N = 0$

# Interesting questions when the space has $\infty$ dimensions

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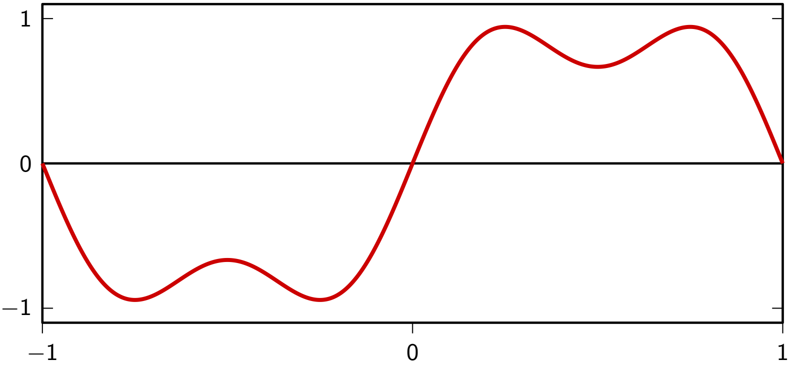


$N = 1$

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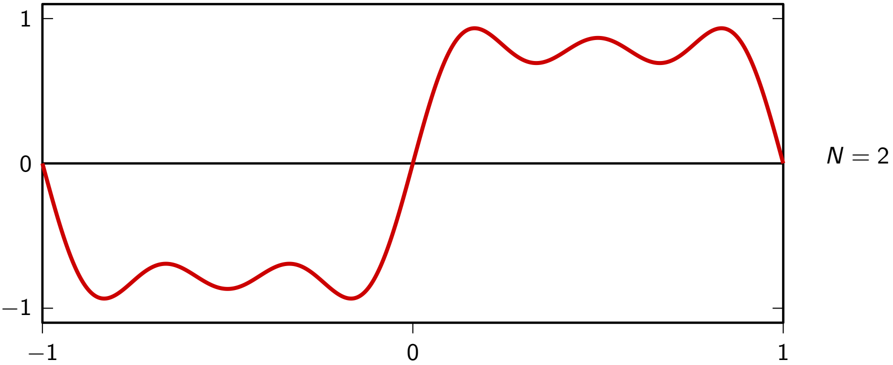

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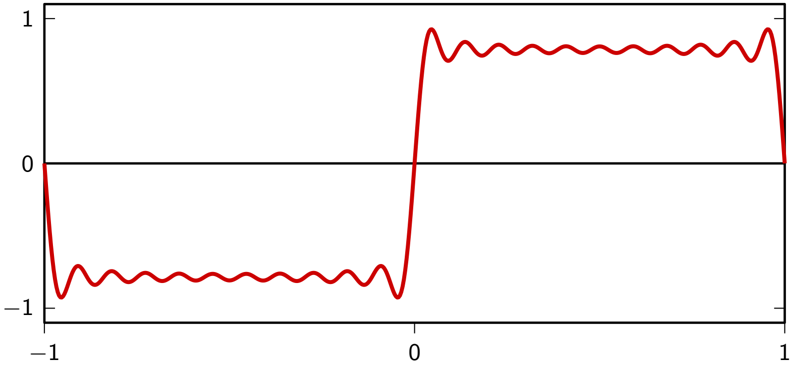
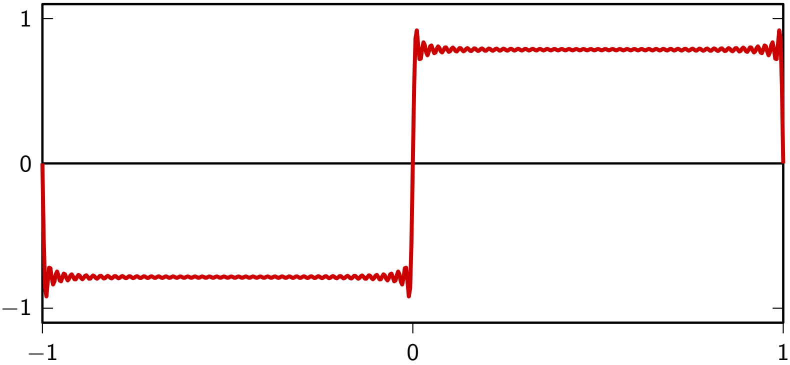



$N = 10$

$$\sum_{k=0}^{N} \mathbf{x}^{(2k+1)}, \qquad \mathbf{x}^{(n)} = \sin(\pi\, n\, t)/n, \quad t \in [-1, 1]$$




$N = 50$

# Interesting questions when the space has $\infty$ dimensions

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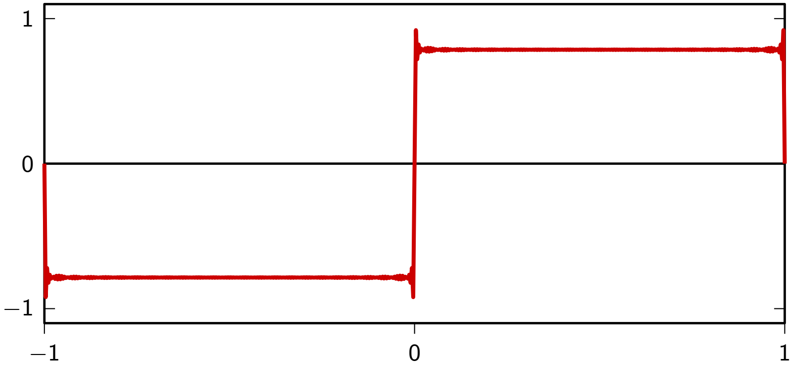


$N = 150$

# END OF MODULE 3.1

# Digital Signal Processing

Module 3.2: Hilbert Space, properties and bases

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- Definition of Hilbert space
- Examples

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# Hilbert Space – the ingredients:

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# 1) Vector space

We need *at least* to:

- resize vectors: scalar multiplication
- combine vectors together: addition

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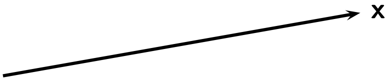
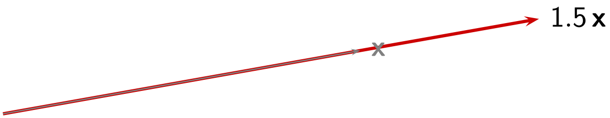
# Scalar multiplication in $\mathbb{R}^2$

$$\mathbf{x} = [x_0 \quad x_1]^T$$



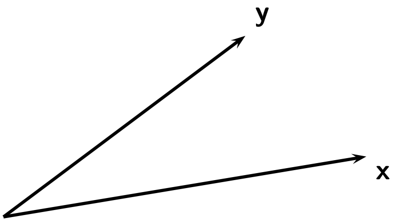
# Scalar multiplication in $\mathbb{R}^2$

$$a\mathbf{x} = [ax_0 \quad ax_1]^T$$



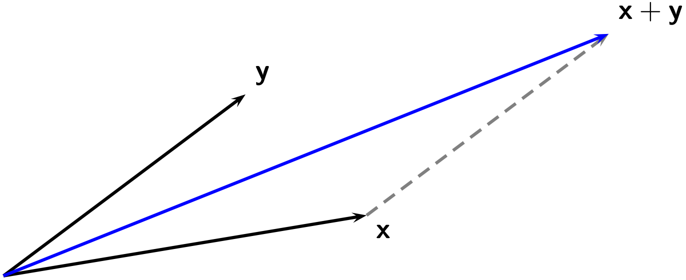
# Addition in $\mathbb{R}^2$

$\mathbf{x}, \mathbf{y}$

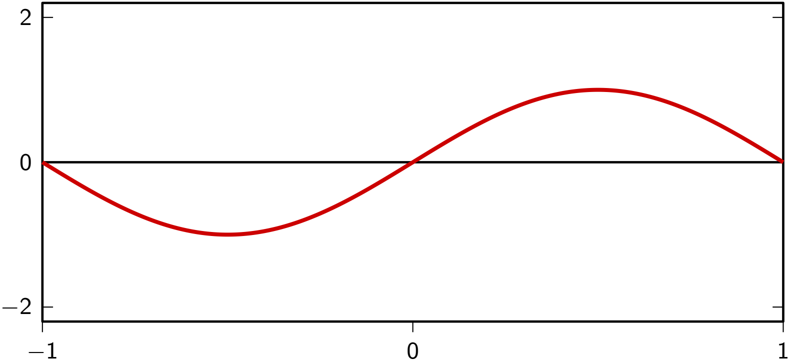


# Addition in $\mathbb{R}^2$

$$\mathbf{x} + \mathbf{y} = [x_0 + y_0 \quad x_1 + y_1]^T$$



# Scalar multiplication in $L_2[-1,1]$




$\mathbf{x} = \sin(\pi t)$

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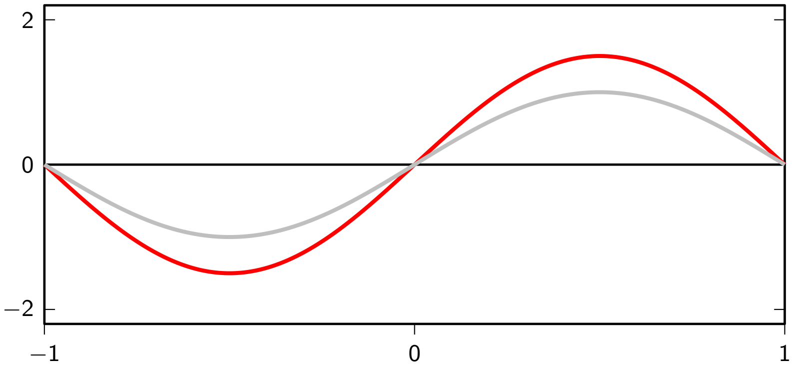


$1.5\,\mathbf{x}$

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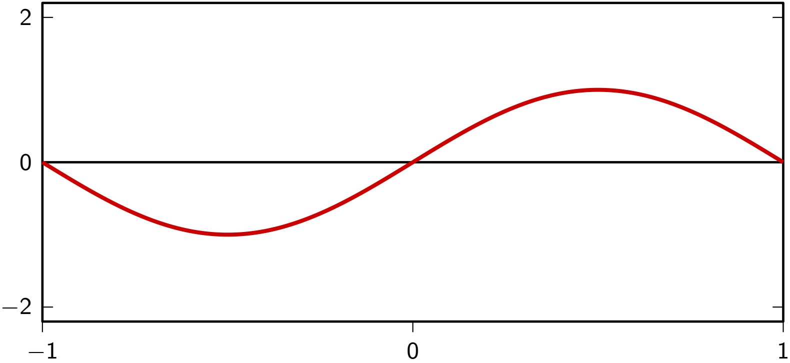


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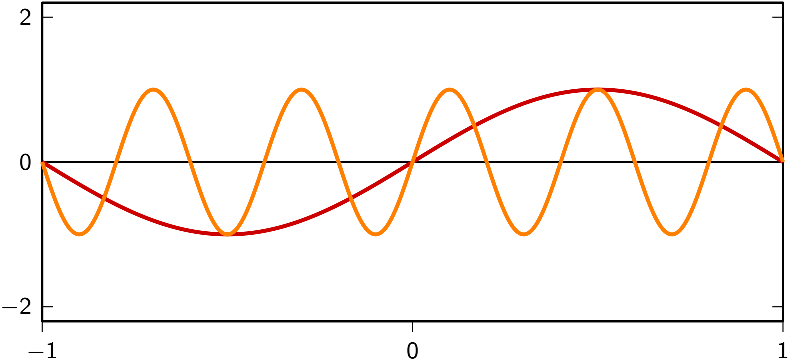


$\mathbf{y} = \sin(5\pi t)$

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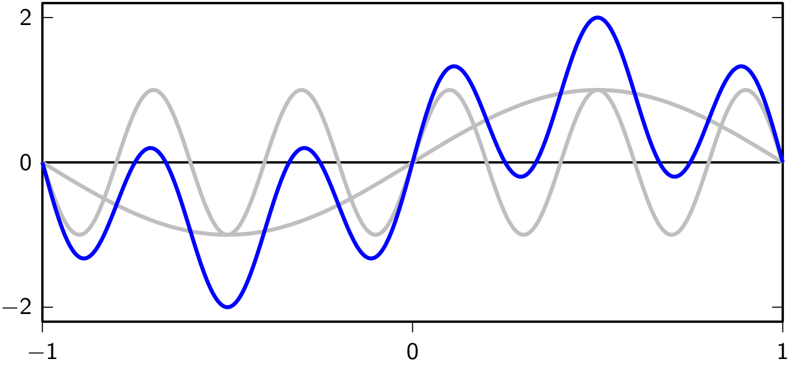


$\mathbf{x} + \mathbf{y}$

# Formal properties of a vector space:

For $\mathbf{x}, \mathbf{y}, \mathbf{z} \in V$ and $\alpha, \beta \in \mathbb{C}$:

- $\mathbf{x} + \mathbf{y} = \mathbf{y} + \mathbf{x}$
- $(\mathbf{x} + \mathbf{y}) + \mathbf{z} = \mathbf{x} + (\mathbf{y} + \mathbf{z})$
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- $\exists 0 \in V \quad | \quad \mathbf{x} + 0 = 0 + \mathbf{x} = \mathbf{x}$
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# Vector subspace:

- intuition: $\mathbb{R}^2 \subset \mathbb{R}^3$
- addition and scaling in subspace remain in subspace

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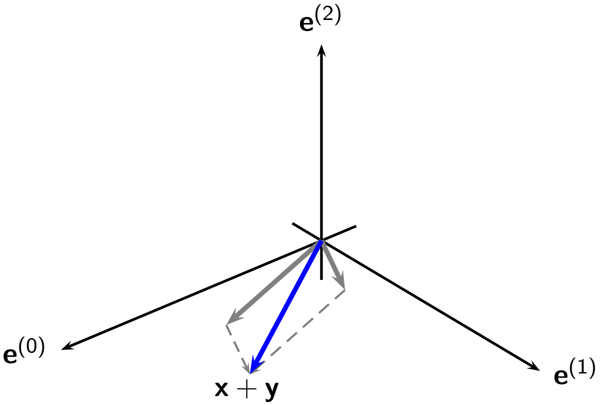
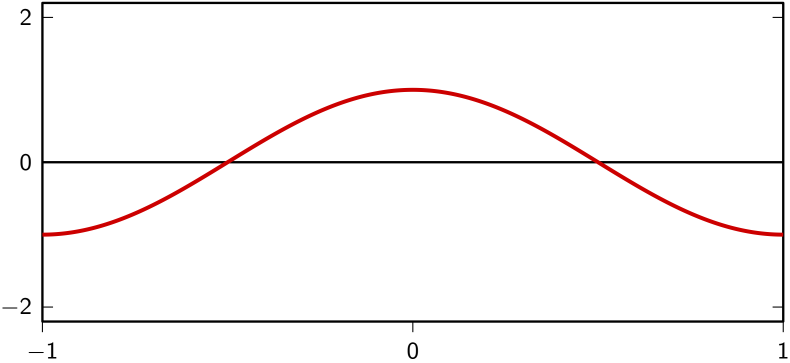
# Addition in subspace:



# Subspace of symmetric functions over $L_2[-1, 1]$




$\mathbf{x} = \cos(\pi t)$

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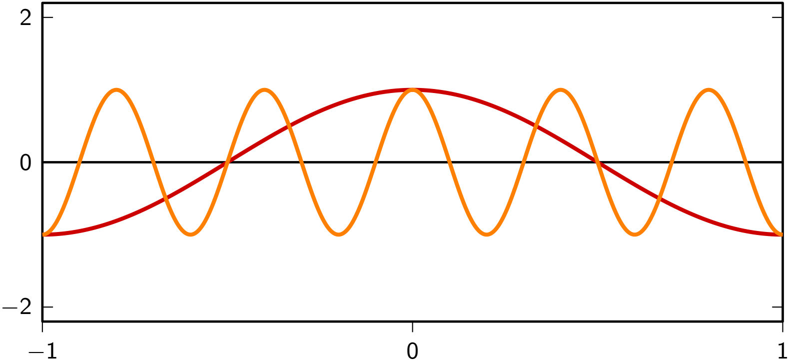


$\mathbf{y} = \cos(5\pi t)$

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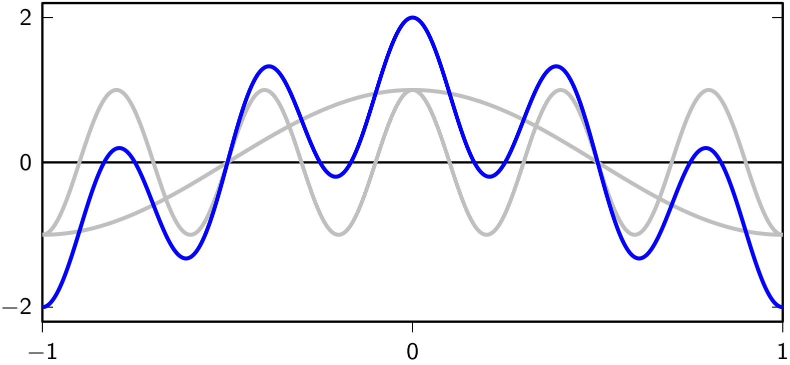


$\mathbf{x} + \mathbf{y}$, symmetric

# 2) Inner product

- measure of similarity between vectors
- when inner product is zero vectors are most different: orthogonal vectors

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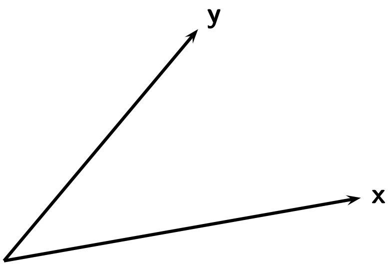
# Inner product in $\mathbb{R}^2$

$$\langle \mathbf{x}, \mathbf{y} \rangle = x_0 y_0 + x_1 y_1$$



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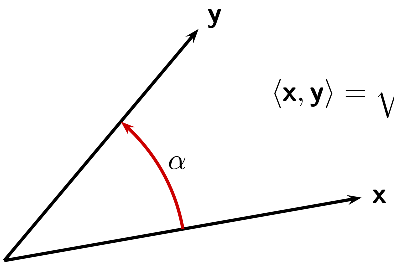


$$\langle \mathbf{x}, \mathbf{y} \rangle = \sqrt{(x_0^2 + x_1^2)(y_0^2 + y_1^2)}\, \cos\alpha$$

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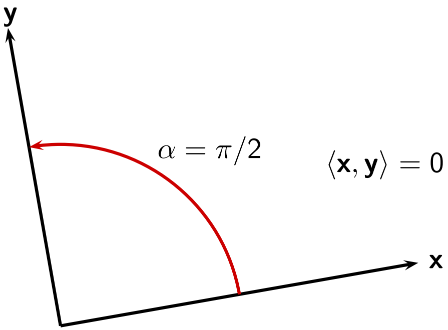
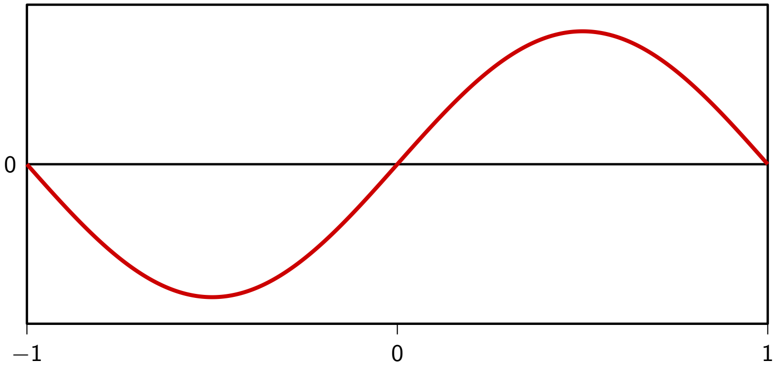



$$\langle \mathbf{x}, \mathbf{y} \rangle = 0$$

# Inner product in $L_2[-1,1]$

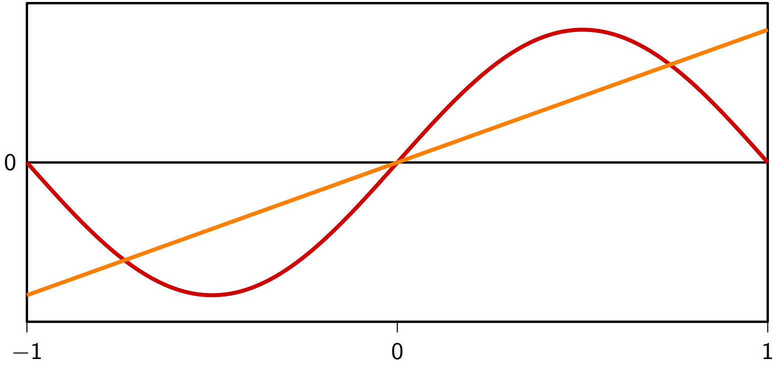
$$\langle \mathbf{x}, \mathbf{y} \rangle = \int_{-1}^{1} x(t)y(t)dt$$




$$\mathbf{x} = \sin(\pi t)$$

# Inner product in $L_2[-1, 1]$

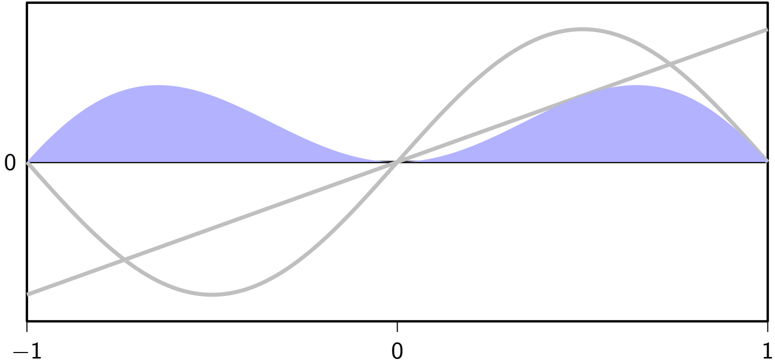
$$\langle \mathbf{x}, \mathbf{y} \rangle = \int_{-1}^{1} x(t)y(t)dt$$




$$\mathbf{y} = t$$

## Inner product in $L_2[-1, 1]$

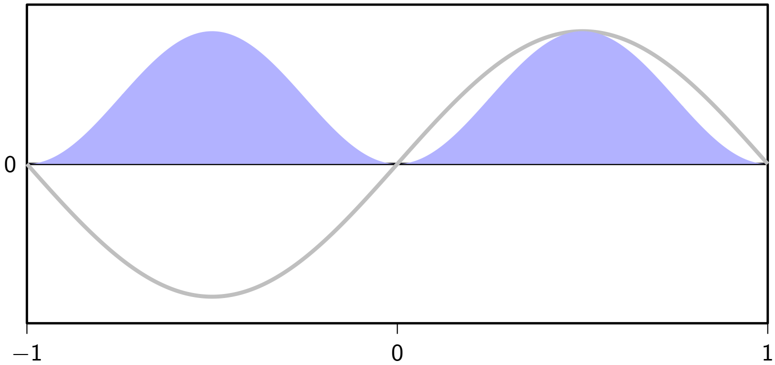
$$\langle \mathbf{x}, \mathbf{y} \rangle = \int_{-1}^{1} t \sin(\pi t) dt$$




$$\langle \mathbf{x}, \mathbf{y} \rangle = 2/\pi \approx 0.6367$$

# Inner product in $L_2[-1,1]$

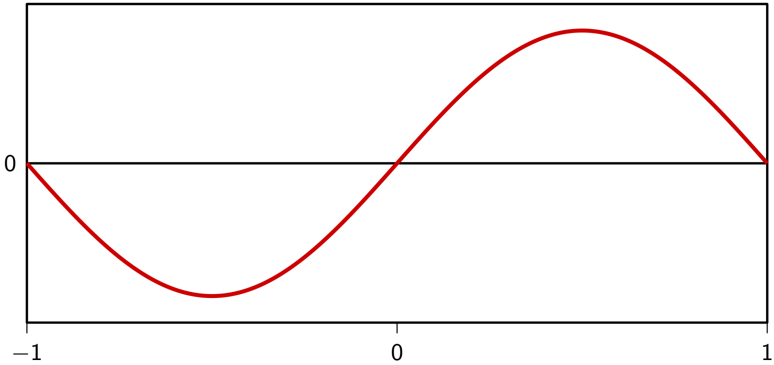
$\langle \mathbf{x}, \mathbf{x} \rangle$




$\mathbf{x} = \sin(\pi t),\ \langle \mathbf{x}, \mathbf{x} \rangle = 1$

# Inner product in $L_2[-1, 1]$

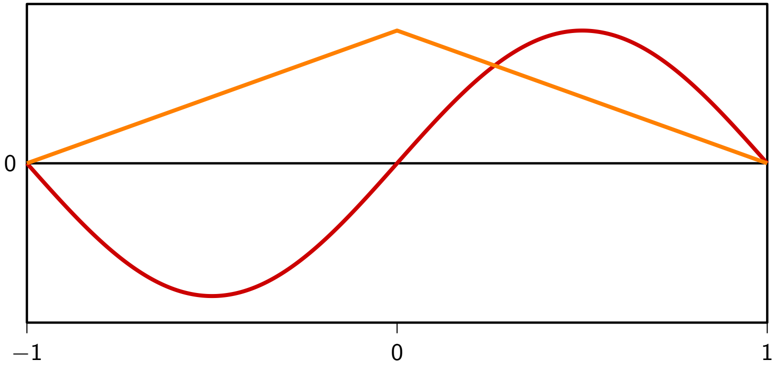
$\mathbf{x}, \mathbf{y}$ from orthogonal subspaces




$\mathbf{x} = \sin(\pi t)$, antisymmetric

# Inner product in $L_2[-1, 1]$

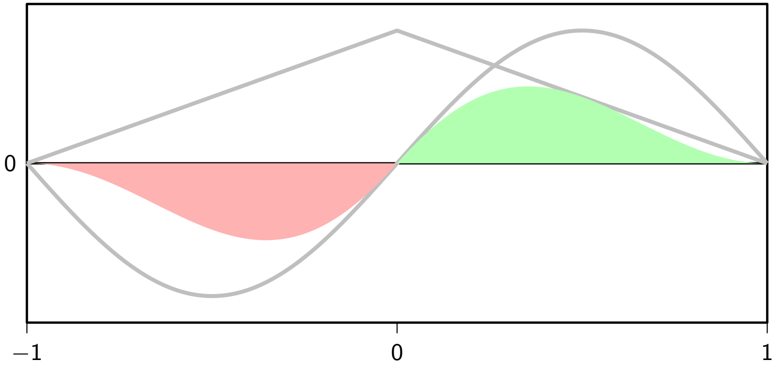
$\mathbf{x}, \mathbf{y}$ from orthogonal subspaces




$\mathbf{y} = 1 - |t|$, symmetric

# Inner product in $L_2[-1, 1]$

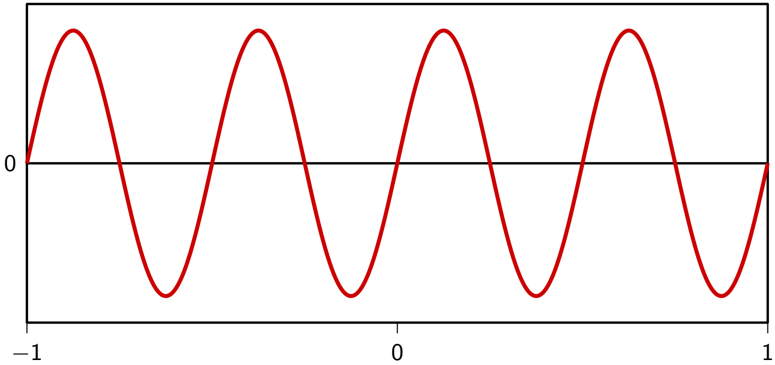
$\mathbf{x}, \mathbf{y}$ from orthogonal subspaces




$$\langle \mathbf{x}, \mathbf{y} \rangle = 0$$

# Inner product in $L_2[-1, 1]$

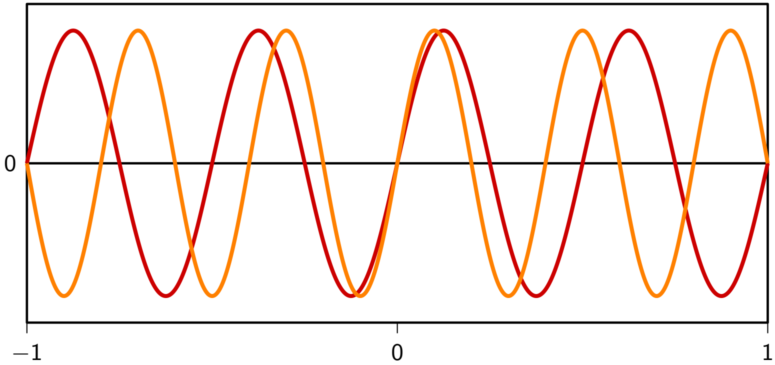
sinusoids with frequencies integer multiples of a fundamental




$\mathbf{x} = \sin(4\pi t)$

# Inner product in $L_2[-1, 1]$

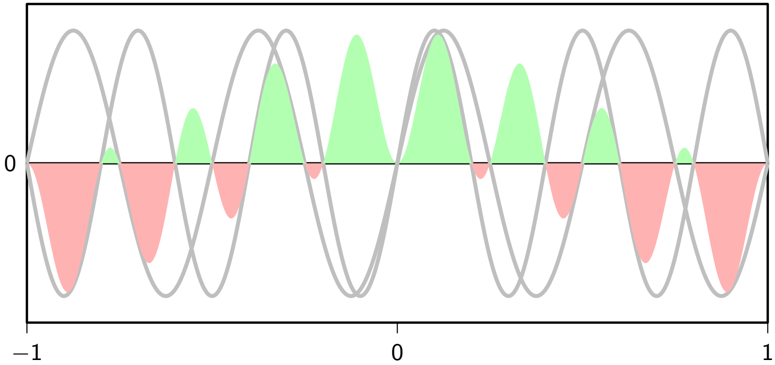
sinusoids with frequencies integer multiples of a fundamental




$\mathbf{x} = \sin(4\pi t)$ , $\quad \mathbf{y} = \sin(5\pi t)$

# Inner product in $L_2[-1,1]$

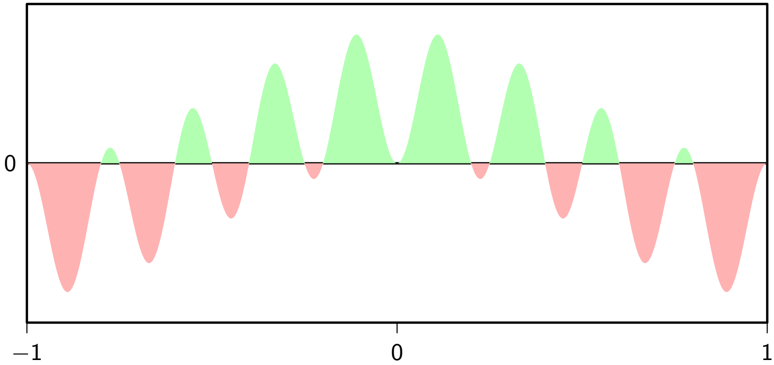
sinusoids with frequencies integer multiples of a fundamental




$\mathbf{x} = \sin(4\pi t)$ , $\quad \mathbf{y} = \sin(5\pi t)$ , $\quad \langle \mathbf{x}, \mathbf{y} \rangle = 0$

# Inner product in $L_2[-1, 1]$

sinusoids with frequencies integer multiples of a fundamental




$$\langle \mathbf{x}, \mathbf{y} \rangle = 0$$

## Formal properties of the inner product

For $\mathbf{x}, \mathbf{y}, \mathbf{z} \in V$ and $\alpha \in \mathbb{C}$:

- $\langle \mathbf{x} + \mathbf{y}, \mathbf{z} \rangle = \langle \mathbf{x}, \mathbf{z} \rangle + \langle \mathbf{y}, \mathbf{z} \rangle$
- $\langle \mathbf{x}, \mathbf{y} \rangle = \langle \mathbf{y}, \mathbf{x} \rangle^*$
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# Inner product for signals

$$\langle \mathbf{x}, \mathbf{y} \rangle = \sum_{n=0}^{N-1} x^*[n]y[n]$$

well defined for all finite-length vectors (i.e. vectors in $\mathbb{C}^N$)

# Inner product for signals

$$\langle \mathbf{x}, \mathbf{y} \rangle = \sum_{n=-\infty}^{\infty} x^*[n]y[n]$$

careful: sum may explode!

# Inner product for signals

$$\langle \mathbf{x}, \mathbf{y} \rangle = \sum_{n=-\infty}^{\infty} x^*[n]y[n]$$

We require sequences to be *square-summable*: $\sum |x[n]|^2 < \infty$

Space of square-summable sequences: $\ell_2(\mathbb{Z})$

# Norm

- inner product defines a norm: $\|\mathbf{x}\| = \sqrt{\langle \mathbf{x}, \mathbf{x} \rangle}$
- norm defines a distance: $d(\mathbf{x}, \mathbf{y}) = \|\mathbf{x} - \mathbf{y}\|$

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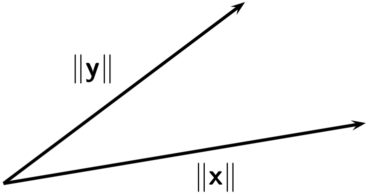
# Norm and distance in $\mathbb{R}^2$

$$\|\mathbf{x}\| = \sqrt{\langle \mathbf{x}, \mathbf{x} \rangle} = \sqrt{x_0^2 + x_1^2}$$

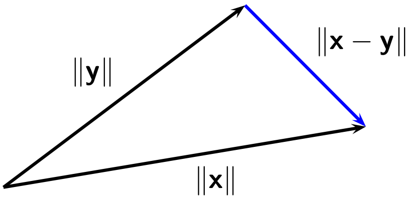
# Norm and distance in $\mathbb{R}^2$

$$\|\mathbf{y}\| = \sqrt{\langle \mathbf{y}, \mathbf{y} \rangle} = \sqrt{y_0^2 + y_1^2}$$



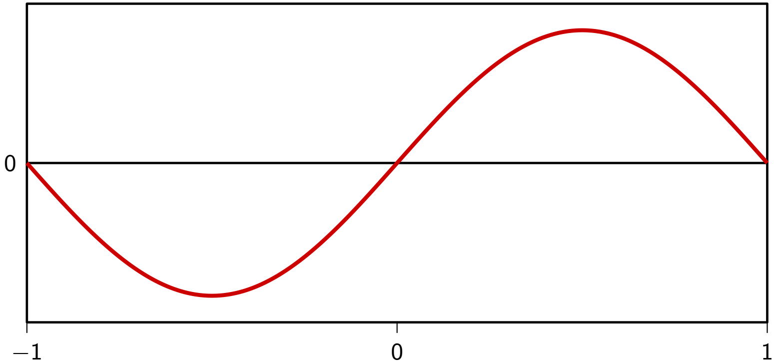
# Norm and distance in $\mathbb{R}^2$

$$\|\mathbf{x} - \mathbf{y}\| = \sqrt{(x_0 - y_0)^2 + (x_1 - y_1)^2}$$



# Norm and distance in $L_2[-1,1]$

$$\|\mathbf{x}-\mathbf{y}\|^2 = \int_{-1}^{1} |x(t)-y(t)|^2 dt \text{ (MSE)}$$




$$\mathbf{x} = \sin(\pi t)$$

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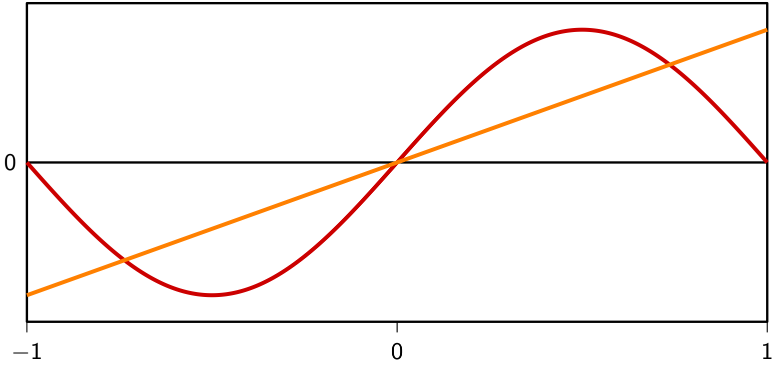


$\mathbf{y} = t$

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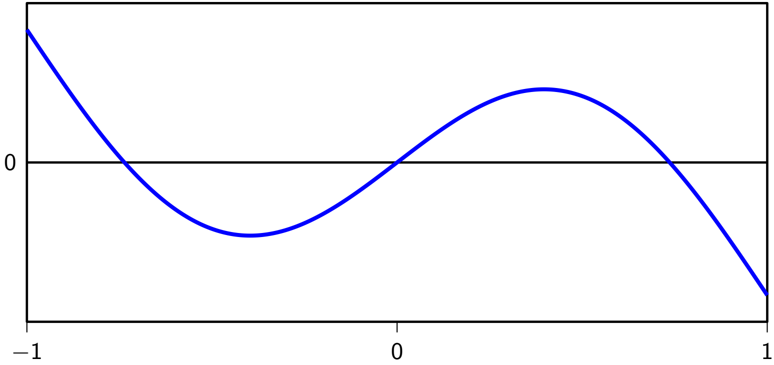


$\mathbf{x}-\mathbf{y}$

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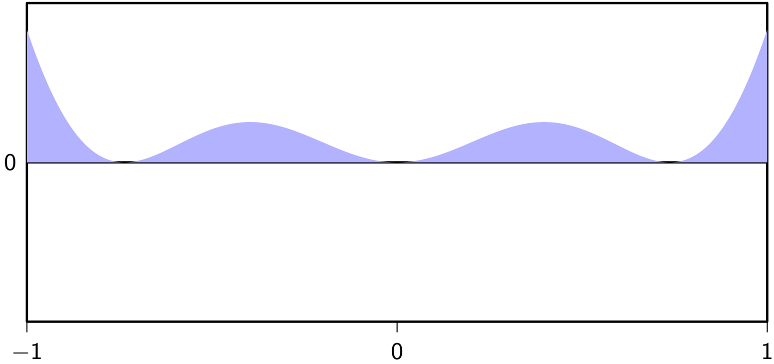
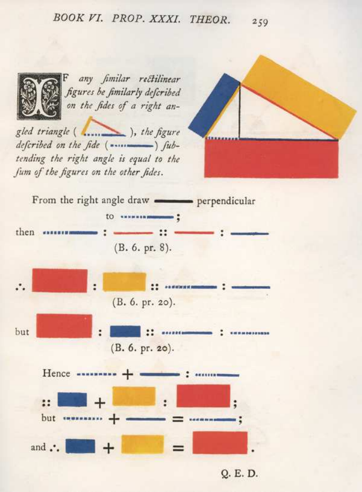



$$\|\mathbf{x}-\mathbf{y}\| = \sqrt{5/3 - 4/\pi} \approx 0.6272$$

# A familiar result

Pythagorean theorem:

$$\|\mathbf{x}+\mathbf{y}\|^2 = \|\mathbf{x}\|^2 + \|\mathbf{y}\|^2 \text{ for } \mathbf{x} \perp \mathbf{y}$$


BOOK VI. PROP. XXXI. THEOR. 259

IF any ſimilar rectilinear figures be ſimilarly deſcribed on the ſides of a right angled triangle ( ), the figure deſcribed on the ſide ( ) ſubtending the right angle is equal to the ſum of the figures on the other ſides.

From the right angle draw perpendicular to ;

then : :: : (B. 6. pr. 8).

∴ : :: : (B. 6. pr. 20).

but : :: : (B. 6. pr. 20).

Hence + : :: + : ;

but + = ;

and ∴ + = .

Q. E. D.

From Euclid's elements by Oliver Byrne (1810 - 1880)

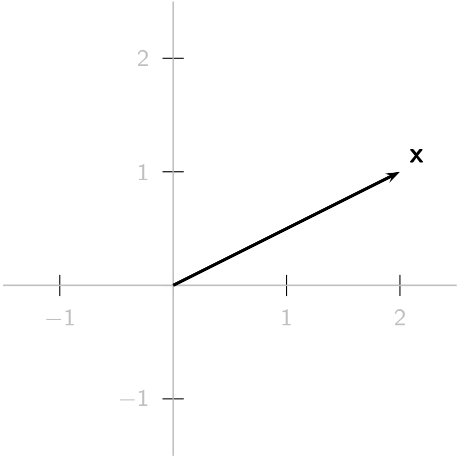
# Bases

$$\mathbf{x} = \begin{bmatrix} 2 \\ 1 \end{bmatrix} \in \mathbb{R}^2$$

$$\mathbf{x} = 2\mathbf{e}^{(0)} + \mathbf{e}^{(1)}$$

$$\mathbf{x} = \mathbf{v}^{(0)} + \mathbf{v}^{(1)}$$

$$\mathbf{x} \neq \alpha_0 \mathbf{g}^{(0)} + \alpha_1 \mathbf{g}^{(1)} \quad \text{for any } \alpha_0, \alpha_1$$



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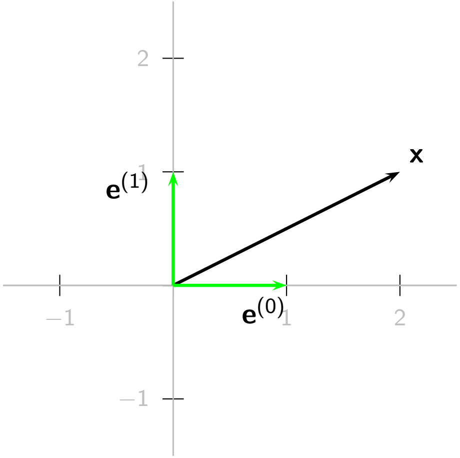

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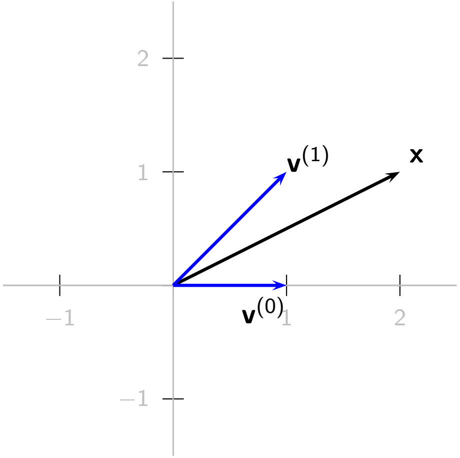

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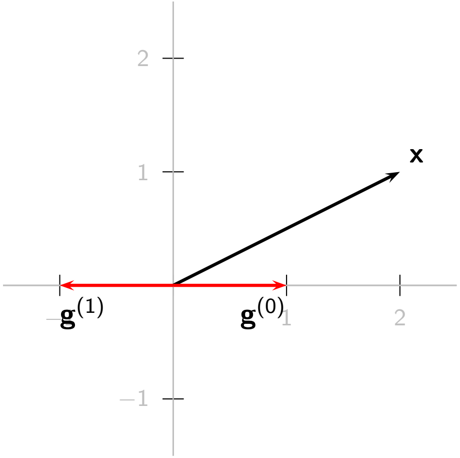

## Bases

- vector space $H$
- set of $K$ vectors from $H$: $W = \{\mathbf{w}^{(k)}\}_{k=0,1,\ldots,K-1}$

$W$ is a basis for $H$ if:

- we can write for *all* $\mathbf{x} \in H$:

$$\mathbf{x} = \sum_{k=0}^{K-1} \alpha_k \mathbf{w}^{(k)}, \quad \alpha_k \in \mathbb{C}$$

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- the coefficients $\alpha_k$ are unique

Unique representation implies linear independence:

$$\sum_{k=0}^{K-1} \alpha_k \mathbf{w}^{(k)} = 0 \qquad \Longleftrightarrow \qquad \alpha_k = 0, \;\; k = 0, 1, \ldots, K-1$$

# Special bases

Orthogonal basis:

$$\langle \mathbf{w}^{(k)}, \mathbf{w}^{(n)} \rangle = 0 \text{ for } k \neq n$$

Orthonormal basis:

$$\langle \mathbf{w}^{(k)}, \mathbf{w}^{(n)} \rangle = \delta[n-k]$$

We can always orthonormalize a basis via the Gram-Schmidt algorithm.

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# Basis expansion

$$\mathbf{x} = \sum_{k=0}^{K-1} \alpha_k \mathbf{w}^{(k)}$$

how do we find the $\alpha$'s ?

Orthonormal bases are the best:

$$\alpha_k = \langle \mathbf{w}^{(k)}, \mathbf{x} \rangle$$

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# Example: bases for $\mathbb{C}^N$

- a basis will contain $N$ vectors
- canonical (orthonormal) basis:

$$\mathbf{e}^{(k)} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

$k$-th position, $0 \leq k < N$

# Example: bases for sequences in $\ell_2(\mathbb{Z})$

- a basis will contain infinite vectors
- canonical (orthonormal) basis:

$$\mathbf{e}^{(k)} = \begin{bmatrix} \vdots \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \\ \vdots \end{bmatrix}$$

$k$-th position, $k \in \mathbb{Z}$

# Completeness

limiting operations must yield vector space elements

Example of an *incomplete* space: the set of rational numbers

$$x_n = \sum_{k=0}^{n} \frac{1}{k!} \in \mathbb{Q} \qquad \text{but} \qquad \lim_{n\to\infty} x_n = e \notin \mathbb{Q}$$

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# END OF MODULE 3.2

# Digital Signal Processing

## Module 3.3: Hilbert Space and approximation

# Overview:

- Norm conservation, Parseval
- Approximation by projection
- Examples

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# Parseval's Theorem

$$\mathbf{x} = \sum_{k=0}^{K-1} \alpha_k \mathbf{w}^{(k)}$$

For an orthonormal basis:

$$\|\mathbf{x}\|^2 = \sum_{k=0}^{K-1} |\alpha_k|^2$$

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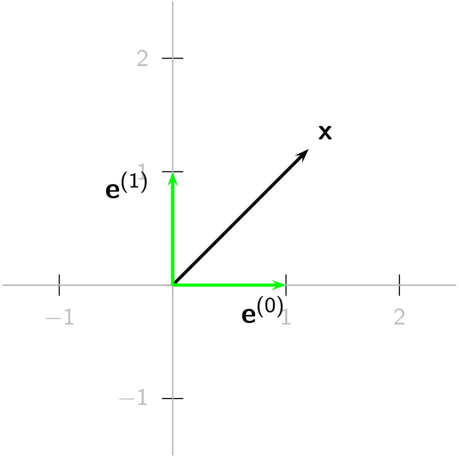
# Example

- canonical basis $E = \{\mathbf{e}^{(0)}, \mathbf{e}^{(1)}\}$
- $\mathbf{x} = \alpha_0 \mathbf{e}^{(0)} + \alpha_1 \mathbf{e}^{(1)}$
- new basis $V = \{\mathbf{v}^{(0)}, \mathbf{v}^{(1)}\}$ with

$$\mathbf{v}^{(0)} = [\cos\theta \;\; \sin\theta]^T$$

$$\mathbf{v}^{(1)} = [-\sin\theta \;\; \cos\theta]^T$$

- $\mathbf{x} = \beta_0 \mathbf{v}^{(0)} + \beta_1 \mathbf{v}^{(1)}$



# Example

- canonical basis $E = \{\mathbf{e}^{(0)}, \mathbf{e}^{(1)}\}$
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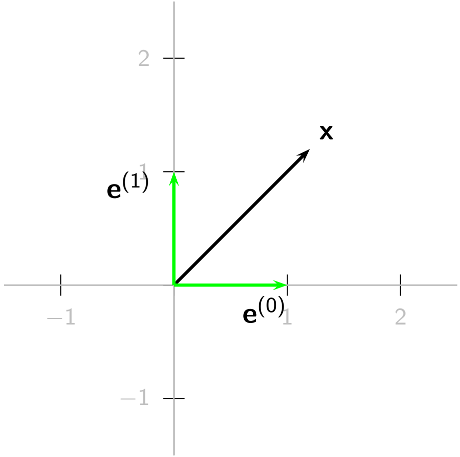

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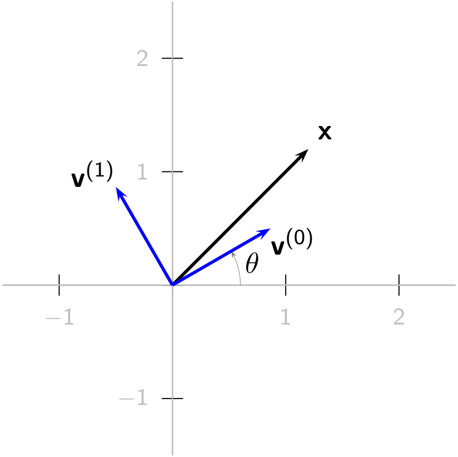

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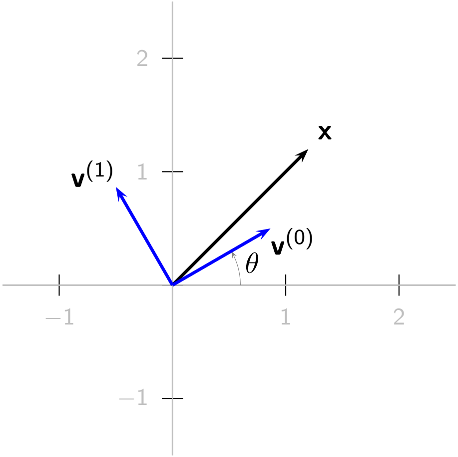

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- new basis is orthonormal:

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$$\beta_1 = \langle \mathbf{v}^{(1)}, \mathbf{x} \rangle$$

- in compact form:

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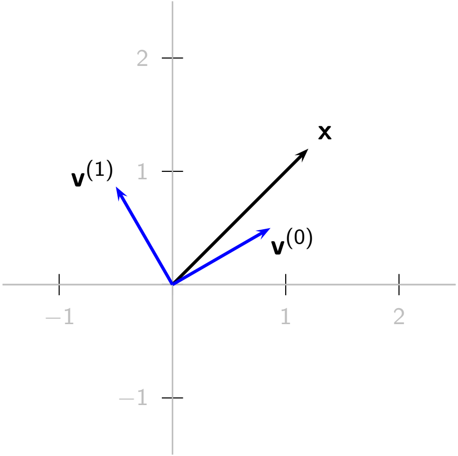
- $\mathbf{R}$: rotation matrix
- key fact: $\mathbf{R}^T\mathbf{R} = \mathbf{I}$



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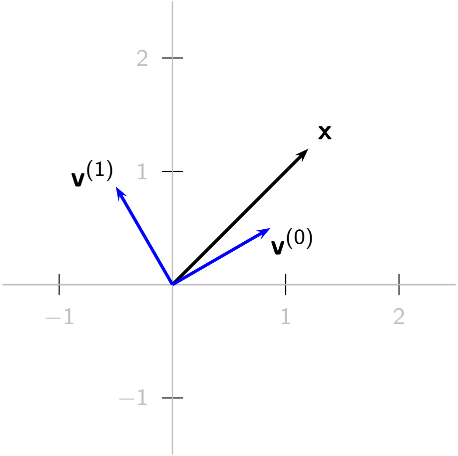

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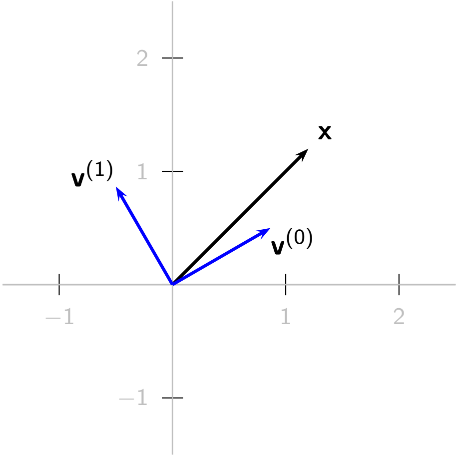

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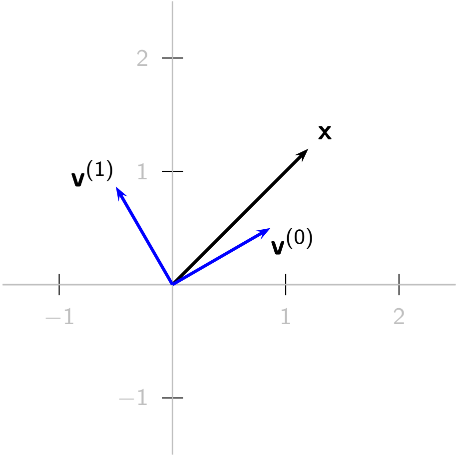

# Example

- square norm in canonical basis: $\|\mathbf{x}\|^2 = \alpha_0^2 + \alpha_1^2$
- square norm in rotated basis: $\|\mathbf{x}\|^2 = \beta_0^2 + \beta_1^2$
- let's verify Parseval:

$$\begin{aligned}
\beta_0^2 + \beta_1^2 &= \boldsymbol{\beta}^T \boldsymbol{\beta} \\
&= (\mathbf{R}\boldsymbol{\alpha})^T (\mathbf{R}\boldsymbol{\alpha}) \\
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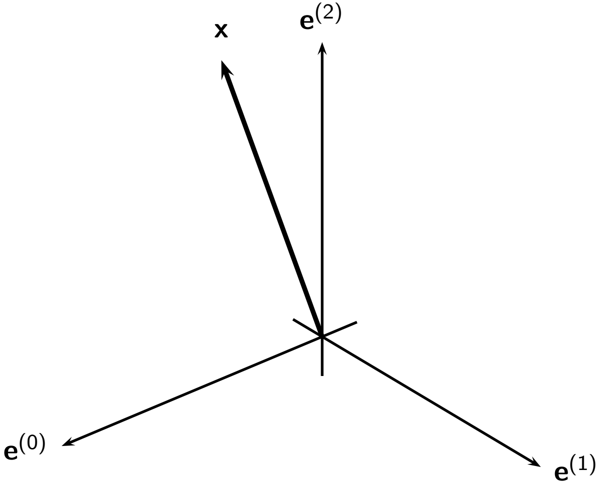
- vector $\mathbf{x} \in V$
- subspace $S \subseteq V$
- approximate $\mathbf{x}$ with $\hat{\mathbf{x}} \in S$



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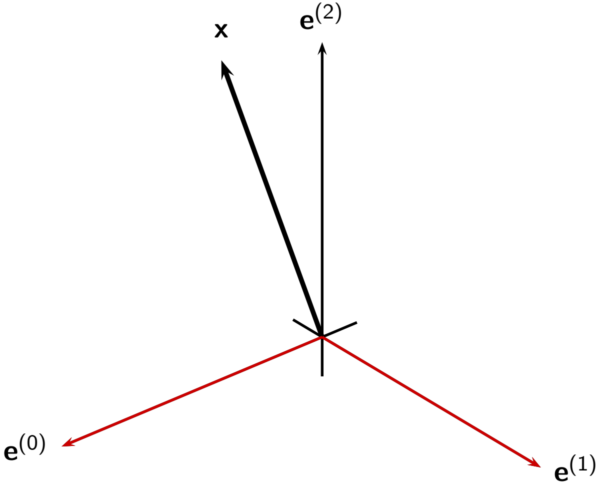

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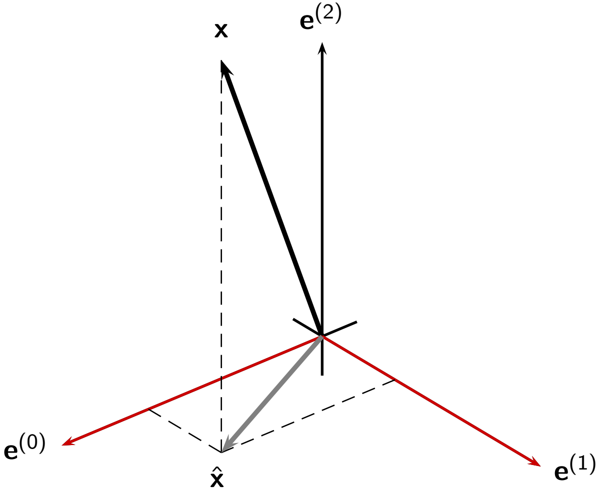

# Least-Squares Approximation

- $\{\mathbf{s}^{(k)}\}_{k=0,1,\ldots,K-1}$ orthonormal basis for $S$
- orthogonal projection:

$$\hat{\mathbf{x}} = \sum_{k=0}^{K-1} \langle \mathbf{s}^{(k)}, \mathbf{x} \rangle \, \mathbf{s}^{(k)}$$

orthogonal projection is the "best" approximation over $S$

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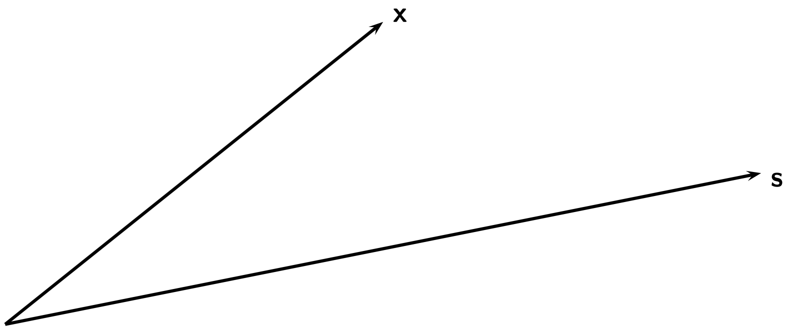

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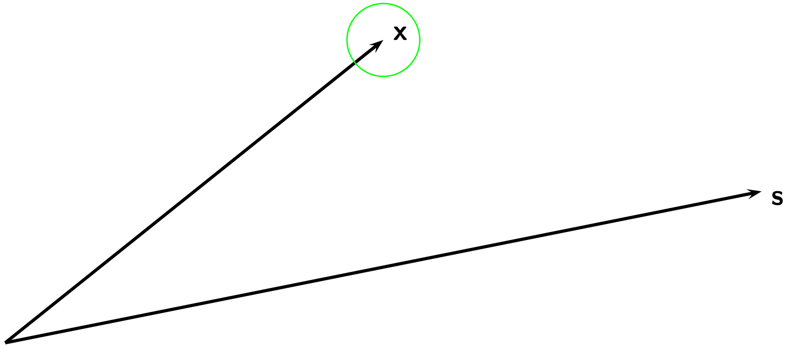

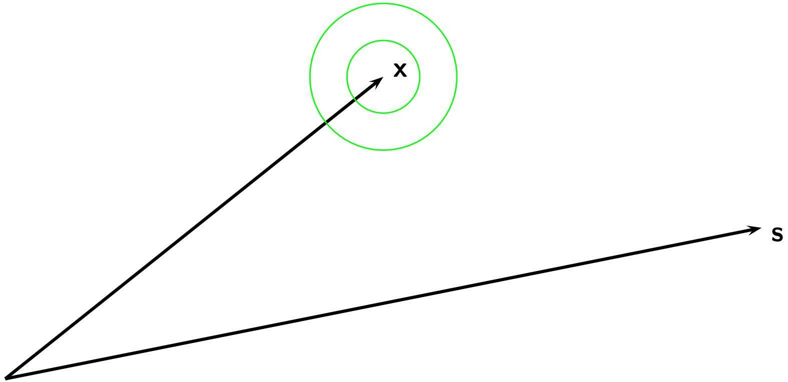
x
s

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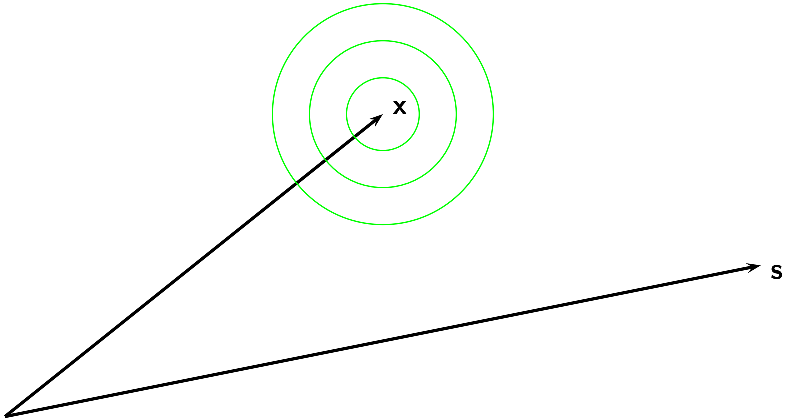

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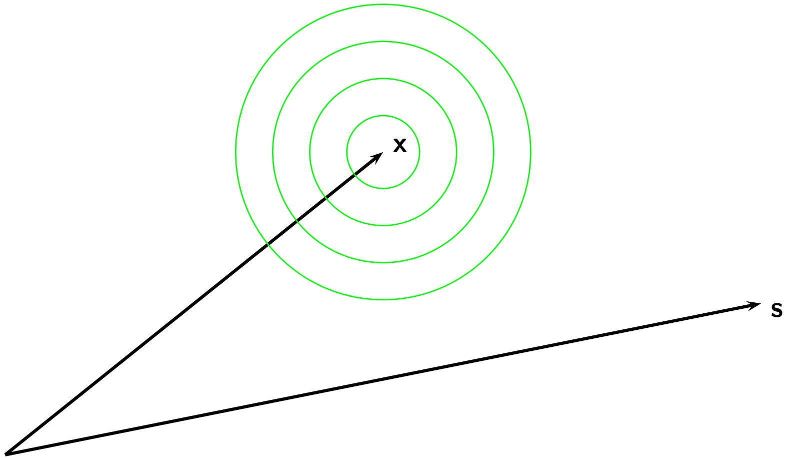

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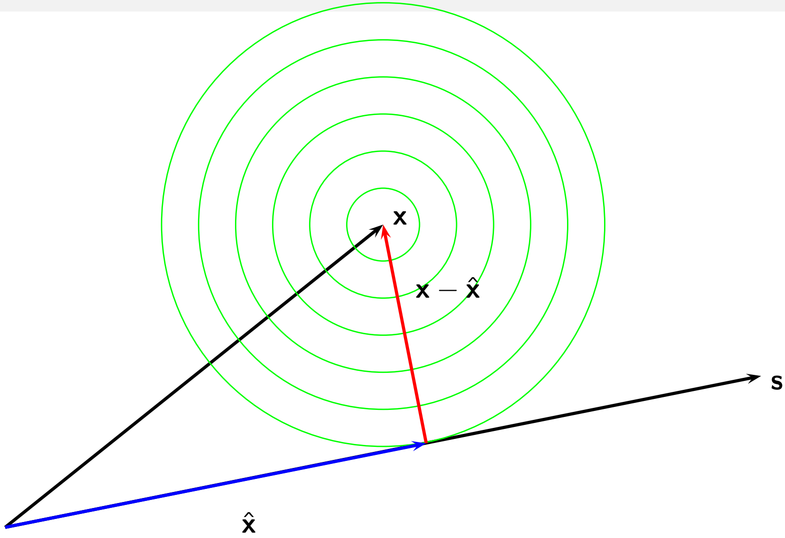

# Example: polynomial approximation

- Hilbert space $P_N[-1, 1] \subset L_2[-1, 1]$
- a self-evident, naive basis: $\mathbf{s}^{(k)} = t^k, \quad k = 0, 1, \ldots, N-1$
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goal: approximate $\mathbf{x} = \sin t$ over $P_3[-1, 1]$

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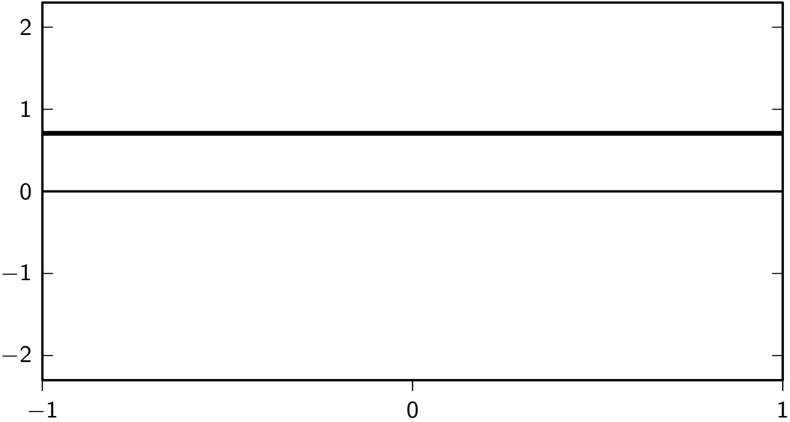
The Gram-Schmidt algorithm leads to an orthonormal basis for $P_N([-1,1])$
(see appendix if interested in details)

$$\mathbf{u}^{(0)} = \sqrt{1/2}$$

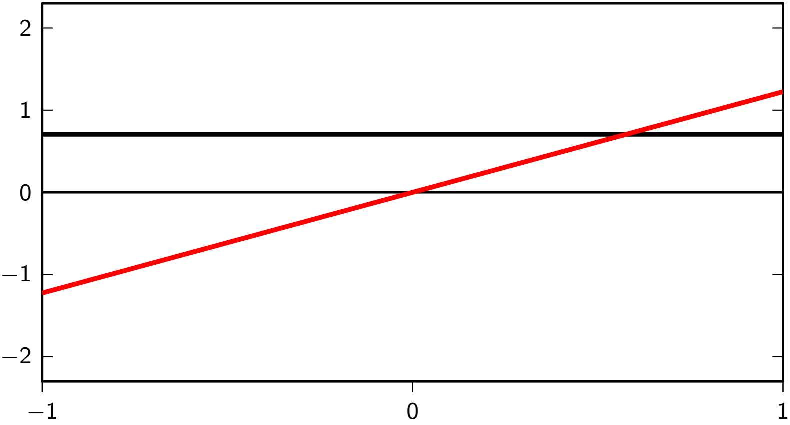
$$\mathbf{u}^{(1)} = \sqrt{3/2}\,t$$

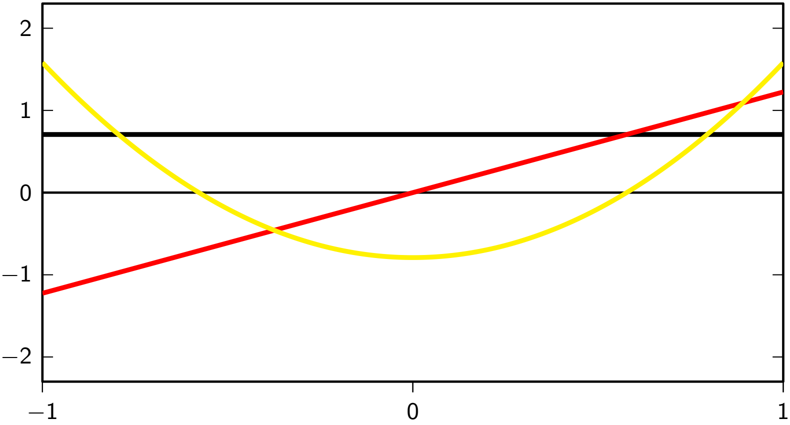
$$\mathbf{u}^{(2)} = \sqrt{5/8}(3t^2 - 1)$$

$$\mathbf{u}^{(3)} = \ldots$$


2
1
0
−1
−2
−1
0
1

# Legendre Polynomials





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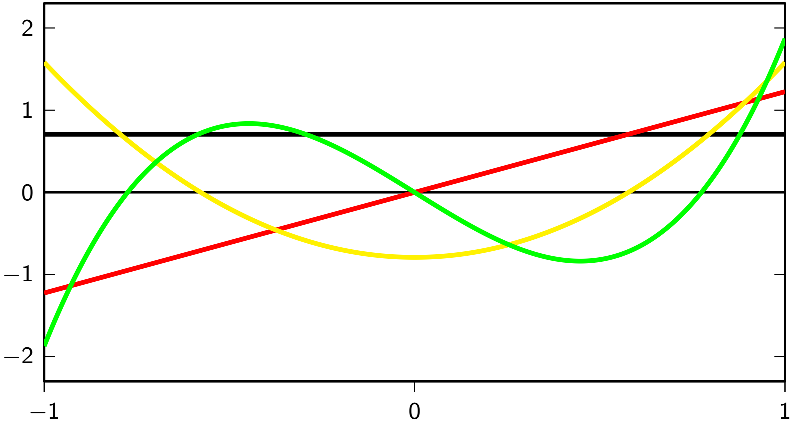

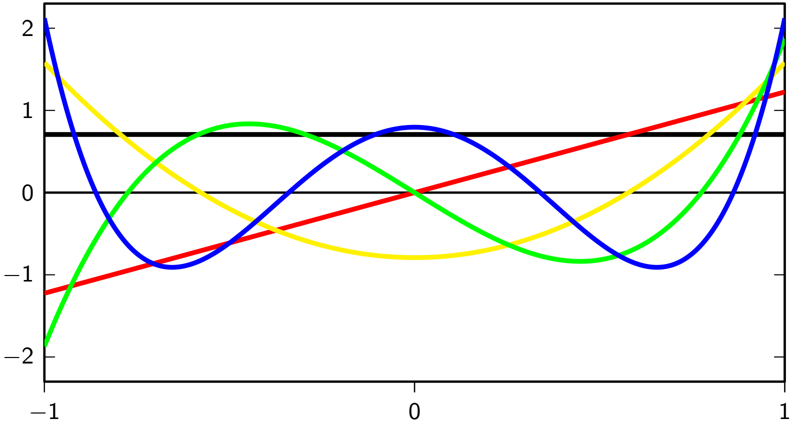
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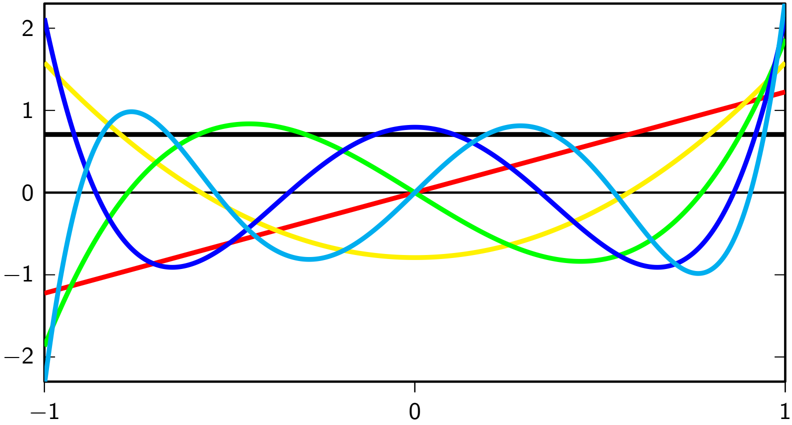

# Orthogonal projection over $P_3[-1,1]$

$$\alpha_k = \langle \mathbf{u}^{(k)}, \mathbf{x} \rangle = \int_{-1}^{1} u_k(t) \sin t \, dt$$

- $\alpha_0 = \langle \sqrt{1/2}, \sin t \rangle = 0$
- $\alpha_1 = \langle \sqrt{3/2}\, t, \sin t \rangle \approx 0.7377$
- $\alpha_2 = \langle \sqrt{5/8}(3t^2 - 1), \sin t \rangle = 0$

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Using the orthogonal projection over $P_3[-1,1]$:

$$\sin t \to \alpha_1 \mathbf{u}^{(1)} \approx 0.9035\, t$$

Using Taylor's series:

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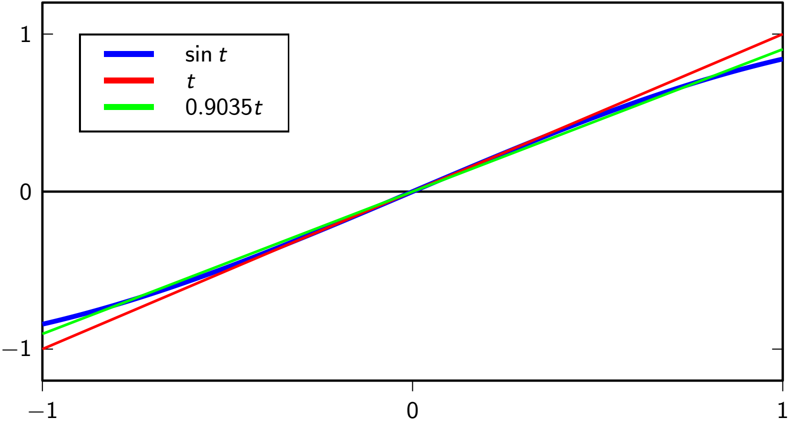
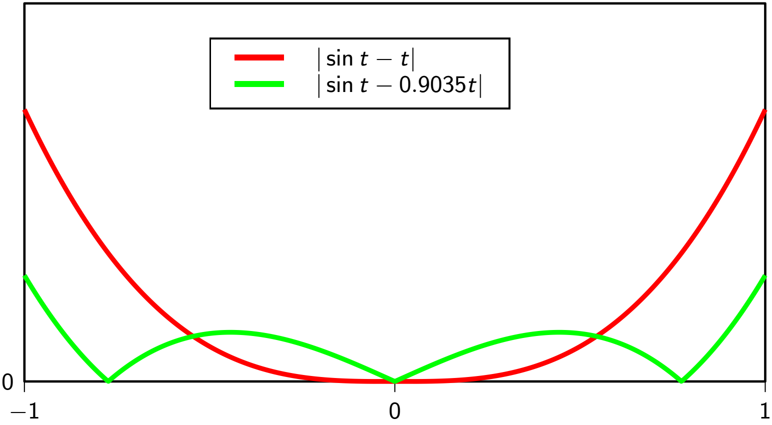
# Sine approximation



# Approximation error



# Error norm

Orthogonal projection over $P_3[-1,1]$:

$$\|\sin t - \alpha_1\, \mathbf{u}^{(1)}\| \approx 0.0337$$

Taylor series:

$$\|\sin t - t\| \approx 0.0857$$

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- finite-length and periodic signals live in $\mathbb{C}^N$
- infinite-length signals live in $\ell_2(\mathbb{Z})$
- different bases are different observation tools for signals
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# END OF MODULE 3.3

# Appendix: orthonormalization of the naive polynomial basis

# Building an orthonormal basis

Gram-Schmidt orthonormalization procedure:

$$\{\mathbf{s}^{(k)}\} \longrightarrow \{\mathbf{u}^{(k)}\}$$

original set orthonormal set

Algorithmic procedure: at each step $k$

1. $\mathbf{p}^{(k)} = \mathbf{s}^{(k)} - \sum_{n=0}^{k-1} \langle \mathbf{u}^{(n)}, \mathbf{s}^{(k)} \rangle \, \mathbf{u}^{(n)}$

2. $\mathbf{u}^{(k)} = \mathbf{p}^{(k)} / \|\mathbf{p}^{(k)}\|$

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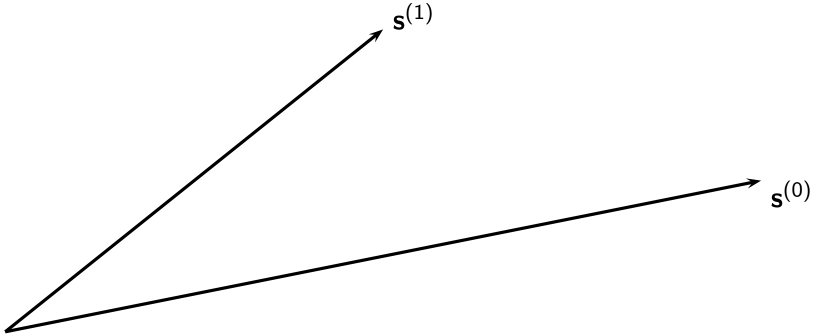

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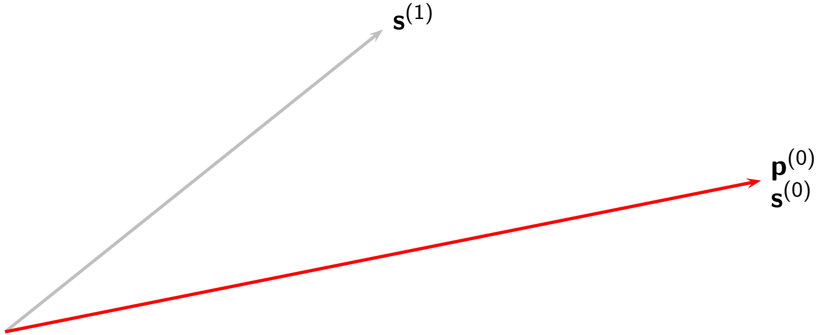

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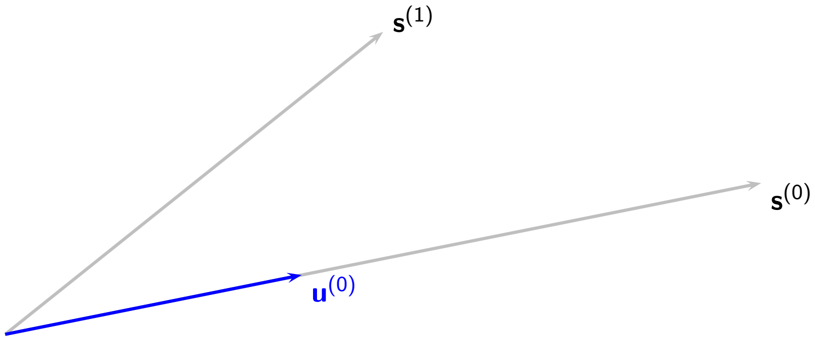

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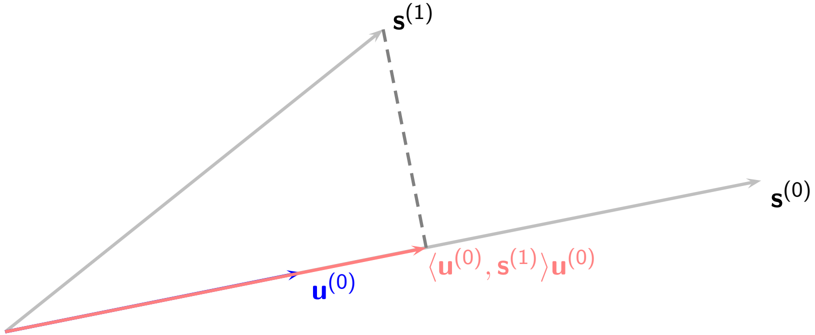

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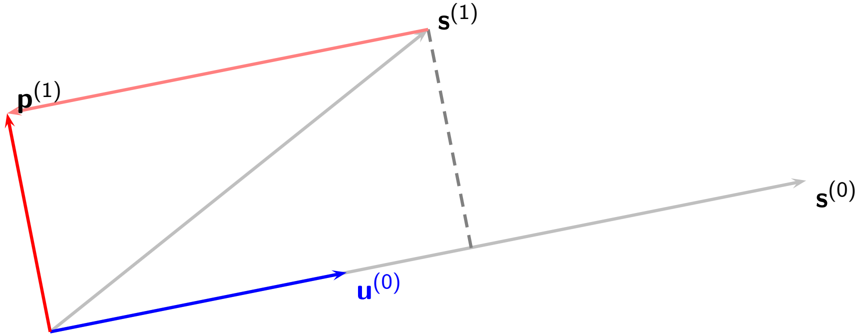

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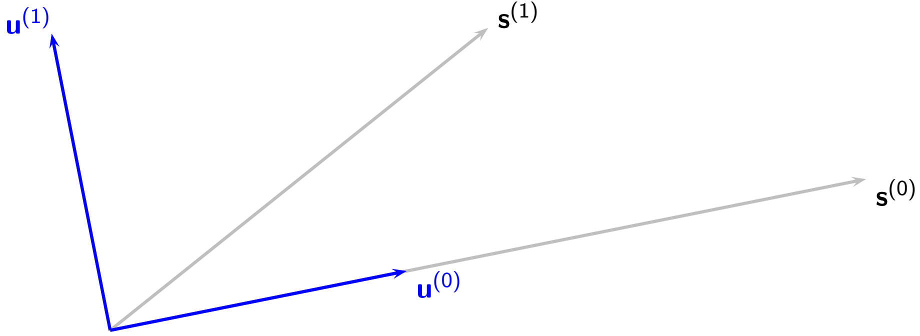

# Building an orthonormal basis

Gram-Schmidt orthonormalization of the naive basis: $\{\mathbf{s}^{(k)}\} \rightarrow \{\mathbf{u}^{(k)}\}$

- $\mathbf{s}^{(0)} = 1$
  - $\mathbf{p}^{(0)} = \mathbf{s}^{(0)} = 1$
  - $\|\mathbf{p}^{(0)}\|^2 = 2$
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# Legendre polynomials

$$\mathbf{u}^{(0)} = \sqrt{1/2}$$

$$\mathbf{u}^{(1)} = \sqrt{3/2}\, t$$

$$\mathbf{u}^{(2)} = \sqrt{5/8}(3t^2 - 1)$$

$$\mathbf{u}^{(3)} = \ldots$$

# END OF MODULE 3