Digital Signal Processing

Solved HW for Day 4

Question 1: Plancherel-Parseval Equality

Let $x[n]$ and $y[n]$ be two complex valued sequences and $X(e^{jw})$ and $Y(e^{jw})$ their corresponding DTFTs.

- Show that
$$\langle x[n], y[n] \rangle = \frac{1}{2\pi} \langle X(e^{jw}), Y(e^{jw}) \rangle,$$
where we use the inner products for $l_2(\mathbb{Z})$ and $L_2([-\pi, \pi])$ respectively.
- What is the physical meaning of the above formula when $x[n] = y[n]$?

Solution of question 1

Q: Show that $\langle x[n], y[n] \rangle = \frac{1}{2\pi} \langle X(e^{jw}), Y(e^{jw}) \rangle$, where we use the inner products for $l_2(\mathbb{Z})$ and $L_2([-\pi, \pi])$ respectively.

The inner product in $l_2(\mathbb{Z})$ is defined as $\langle x[n], y[n] \rangle = \Sigma_n x^*[n] y[n]$, and in $L_2([-\pi, \pi])$ as $\langle X(e^{jw}), Y(e^{jw}) \rangle = \int_{-\pi}^{\pi} X^*(e^{jw}) Y(e^{jw}) dw$.
Thus,

$$\begin{aligned} \frac{1}{2\pi} \int_{-\pi}^{\pi} X^*(e^{jw}) Y(e^{jw}) dw &= \frac{1}{2\pi} \int_{-\pi}^{\pi} \left(\Sigma_n x[n] e^{-jwn} \right)^* \Sigma_m y[m] e^{-jwm} dw \\ &\overset{(1)}{=} \frac{1}{2\pi} \int_{-\pi}^{\pi} \Sigma_n x^*[n] e^{jwn} \Sigma_m y[m] e^{-jwm} dw \end{aligned}$$

where (1) follows from the properties of the complex conjugate...

Solution of question 1

$$
\begin{aligned}
&= \frac{1}{2\pi}\int_{-\pi}^{\pi} \Sigma_n \Sigma_m x^*[n]y[m]e^{jw(n-m)}dw \\
&\overset{(2)}{=} \frac{1}{2\pi}\Sigma_n \Sigma_m x^*[n]y[m]\int_{-\pi}^{\pi} e^{jw(n-m)}dw \\
&\overset{(3)}{=} \Sigma_n x^*[n]y[n],
\end{aligned}
$$

where (2) follows from swapping the integral and the sums and (3) from the fact that

$$
\frac{1}{2\pi}\int_{-\pi}^{\pi} e^{jw(n-m)}dw = \begin{cases} 1 & \text{if } m = n \\ 0 & \text{if } m \neq n \end{cases}.
$$

Solution of question 1

Q: What is the physical meaning of the above formula when $x[n] = y[n]$?

If $x[n] = y[n]$,

- $\langle x[n], x[n] \rangle$ corresponds to the energy of the signal in the time domain;
- $\langle X(e^{jw}), X(e^{jw}) \rangle$ corresponds to the energy of the signal in the frequency domain
- The Plancherel-Parseval equality illustrates an energy conservation property from the time domain to the frequency domain. This property is known as the *Parseval's theorem*.

Question 2: Structure of DFT formulas

The DFT and IDFT formulas are similar, but not identical. Consider a length-N signal $x[n], N = 0, \ldots, N-1$.

Q: What is the length-N signal $y[n]$ obtained as

$$y[n] = \text{DFT}\{\text{DFT}\{x[n]\}\}?$$

In other words, what are the effects of applying twice the DFT transform?

Solution of question 2

Let $f[n] = \text{DFT}\{x[n]\}$. We have:

$$
\begin{aligned}
y[n] &= \sum_{k=0}^{N-1} f[k] e^{-j\frac{2\pi}{N}nk} \\
&= \sum_{k=0}^{N-1} \left\{ \sum_{i=0}^{N-1} x[i] e^{-j\frac{2\pi}{N}ik} \right\} e^{-j\frac{2\pi}{N}nk} \\
&= \sum_{i=0}^{N-1} x[i] \sum_{k=0}^{N-1} e^{-j\frac{2\pi}{N}(i+n)k}.
\end{aligned}
$$

Solution of question 2

Now,

$$\sum_{k=0}^{N-1} e^{-j\frac{2\pi}{N}(i+n)k} = \begin{cases} N & \text{for } (i+n) = 0, N, 2N, 3N, \ldots \\ 0 & \text{otherwise} \end{cases} = N\delta[(i+n) \mod N]$$

so that

$$\begin{aligned} y[n] &= \sum_{i=0}^{N-1} x[i] N\delta[(i+n) \mod N] \\ &= \begin{cases} Nx[0] & \text{for } n = 0 \\ Nx[N-n] & \text{otherwise.} \end{cases} \end{aligned}$$

In other words, if $\mathbf{x} = [1\ 2\ 3\ 4\ 5]^T$ then $\text{DFT}\{\text{DFT}\{\mathbf{x}\}\} = 5[1\ 5\ 4\ 3\ 2]^T = [5\ 25\ 20\ 15\ 10]^T$.

Question 3: DTFT properties

Q: Derive the time-reverse and time-shift properties of the DTFT.

Define the DTFT transform as

$$x[n] \xleftrightarrow{DTFT} X(e^{j\omega}).$$

1. We have:

$$\sum_{n=-\infty}^{\infty} x[-n]e^{-j\omega n} = \sum_{m=-\infty}^{\infty} x[m]e^{j\omega m} = \sum_{m=-\infty}^{\infty} x[m]e^{-j(-\omega)m} = X(e^{-j\omega})$$

with the change of variable $m = -n$.
Hence, we obtain that the DTFT of the time-reversed sequence $x[-n]$ is:

$$x[-n] \xleftrightarrow{DTFT} X(e^{-j\omega})$$

2. Similarly:

$$\sum_{n=-\infty}^{\infty} x[n-n_0]e^{-j\omega n} = \sum_{m=-\infty}^{\infty} x[m]e^{-j\omega(m+n_0)} = e^{-j\omega n_0} \sum_{m=-\infty}^{\infty} x[m]e^{-j\omega m} = e^{-j\omega n_0} X(e^{j\omega})$$

with the change of variable $m = n - n_0$.
We, therefore, obtain the DTFT of the time-shifted sequence $x[n-n_0]$:

$$x[n-n_0] \xleftrightarrow{DTFT} e^{-j\omega n_0} X(e^{j\omega})$$

Question 4: DFT & Matlab

1. The spectrum of the signal $x[n]$ is obtained using the Matlab commands given below.

```
N=128;fo1=21/128;fo2=21/127;
n=0:N-1;
x1=cos(2*pi*fo1*n);x2=cos(2*pi*fo2*n);
X1=fft(x1);X2=fft(x2);
subplot(223),stem(n-N/2,fftshift(abs(X1)))
subplot(224),stem(n-N/2,fftshift(abs(X2)))
```

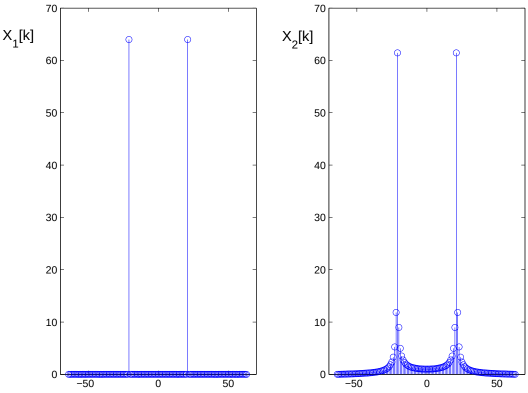
Note that we would expect to see just one sample at the frequency of the signal.

Question 4: DFT & Matlab

- Consider the signal $x(n) = \cos(2\pi f_0 n)$. Use the 'fft' function to compute and draw the DFT of the signal in $N = 128$ points, for: $f_0 = 21/128$ and $f_0 = 21/127$.
  Explain the differences that we can see in these two signal spectra.
- Repeat the process, this time using the 'dftmtrx' function and check that the results are the same. What is the preferred option between the two?

Solution of question 4

Q: Explain the differences that we can see in these two signal spectra.



- On the left figure, our prediction is satisfied and the 21st DFT coefficient represents the exact signal frequency. On the right figure, the frequency of the signal $f_0 = 21/127$ does not coincide with any DFT frequency component.
- The signal energy is spread over all the DFT components. This is called frequency leakage.

Solution of question 4

Q: Repeat the process, this time using the 'dftmtrx' function and check that the results are the same. What is the preferred option between the two?

We can achieve the same results using the *dftmtx* function instead, as illustrated below.

```
W=dftmtx(N);
X3=W*x1;X4=W*x2;
norm(X1-X3)
norm(X2-X4)
subplot(223),stem(n-N/2,fftshift(abs(X3)))
subplot(224),stem(n-N/2,fftshift(abs(X4)))
```

In practice, however, the discrete Fourier transform is computed more efficiently and uses less memory with an FFT algorithm.

Question 5: DTFT, DFT & Matlab

Consider the following infinite non-periodic discrete time signal: $x[n] = \begin{cases} 0 & n < 0, \\ 1 & 0 \leq n < a, \\ 0 & n \geq a. \end{cases}$

- Compute its DTFT $X(e^{jw})$.

We want to visualize the magnitude of $X(e^{jw})$ using Matlab. However, Matlab can not handle continuous sequences as $X(e^{jw})$, thus we need to consider only a finite number of points.

- Using Matlab, plot 10000 points of one period of$|X(e^{jw})|$ (from 0 to 2π) for $a = 20$.

The DTFT is mostly a theoretical analysis tool, but in many cases, we will compute the DFT. Recall that in Matlab we use the Fast Fourier Transform (FFT), an efficient algorithm to compute the DFT.

- Generate a finite sequence $x_1[n]$ of length $N = 30$ such that $x_1[n] = x[n]$ for $n = 1, \ldots, N$. Compute its DFT and plot its magnitude. Compare it with the plot obtained in (2).

- Repeat now for different values of $N = 50, 100, 1000$. What can you conclude?

Solution of question 5

Q: Compute its DTFT $X(e^{jw})$.

First, we note that $x[n]$ can also be expressed as: $x[n] = u[n] - u[n-a]$.
Using the DTFT shift property:

$$X(e^{jw}) = \frac{1}{1-e^{-jw}} + \frac{1}{2}\tilde{\delta}(w) - \frac{e^{-jaw}}{1-e^{-jw}} - \frac{e^{-jaw}}{2}\tilde{\delta}(w).$$

Note that $e^{-jaw}\tilde{\delta}(w) = \tilde{\delta}(w)$. Therefore,

$$X(e^{jw}) = \frac{1-e^{-jaw}}{1-e^{-jw}}.$$

Solution of question 5

Q: Using Matlab, plot 10000 points of one period of $|X(e^{jw})|$ (from 0 to 2π) for $a = 20$.

We visualize the magnitude of $X(e^{jw})$ using Matlab with the following code:

```
n=1:10000;
w=(n.*2*pi/max(n));
X=((1-exp(-j.*w.*20))./(1-exp(-j.*w)));
plot(w,abs(X));
```

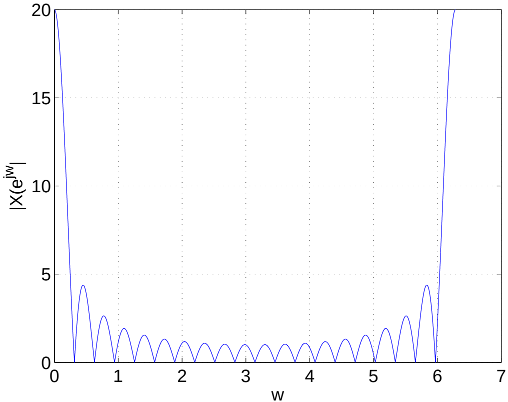


Figure : DTFT of $x[n]$.

Solution of exercise 5

Q: Generate a finite sequence $x_1[n]$ of length $N = 30$ such that $x_1[n] = x[n]$ for $n = 1, \ldots, N$. Compute its DFT and plot its magnitude. Compare it with the plot obtained in (2).

To plot the DFT for $N = 30$, we use the following Matlab code:

```
N=30;
x1=[ones(1,20),zeros(1,N-20)];
X1=fft(x1);
plot(abs(X1));
```

Solution of exercise 5

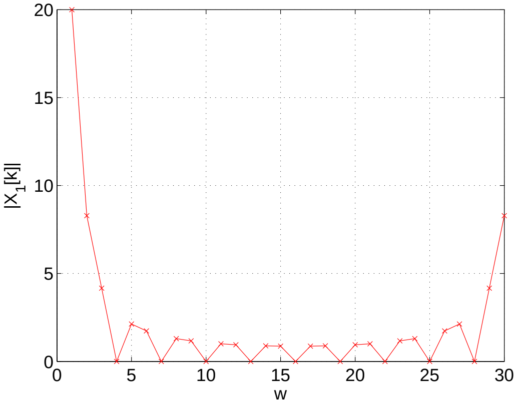
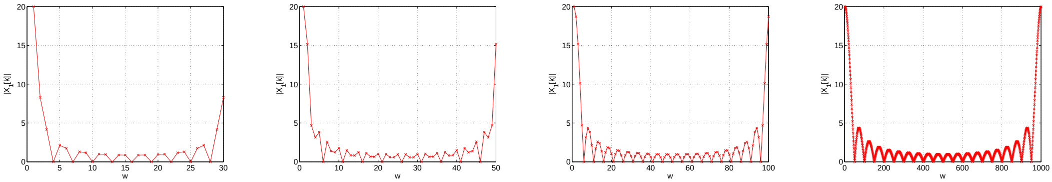


Figure : DFT of $x[n]$ for $N = 30$.

Solution of exercise 5

Q: Repeat now for different values of $N = 50, 100, 1000$. What can you conclude?



(a) N=30 (b) N=50 (c) N=100 (d) N=1000

- As we increase N, the DFT becomes closer and closer to the DTFT of $x[n]$.
- We know that the DFT and the DFS are formally identical, and as N grows, the DFS converges to the DTFT.
- We can use Matlab to approximate the DTFT of any signal.