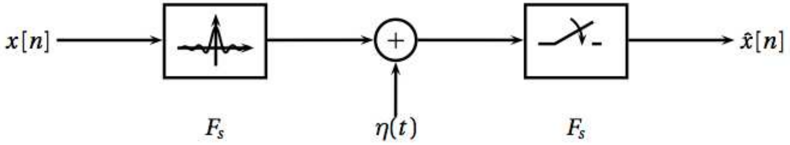
Digital Signal Processing

Solved HW for Day 16

Question 1

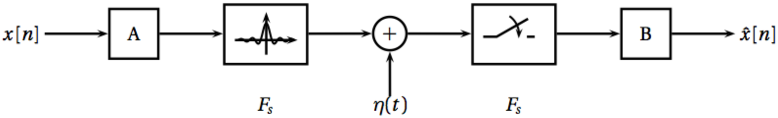
Let $x[n]$ be the discrete-time version of an audio signal, originally bandlimited to $20KHz$ and sampled at $40KHz$; assume that we can model $x[n]$ as an i.i.d. process with variance σ_x^2 . The signal is converted to continuous time, sent over a noisy analog channel and resampled at the receiving end using the following scheme, where both the ideal interpolator and sampler work at a frequency $F_s = 40KHz$:



The channel introduces zero-mean, additive white Gaussian noise.
At the receiving end, after the sampler, assume that the effect of the noise introduced by the channel can be modeled as a zero-mean white Gaussian stochastic signal $\eta[n]$ with power spectral density $P_\eta(e^{j\omega}) = \sigma_0^2$.

Question 1, part b

The SNR obtained with the transmission scheme above is too low for our purposes. Unfortunately the power constraint of the channel prevents us from simply amplifying the audio signal (in other words: the total power $\int_{-\pi}^{\pi} P_x(e^{j\omega})$ cannot be greater than $2\pi\sigma_x^2$). In order to improve the quality of the received signal, we modify the transmission scheme as shown below, by adding pre-processing and post-processing digital blocks at the transmitting and receiving ends, while F_s is still equal to $40000Hz$.



Solution of question 1

Q: What is the signal to noise ratio (SNR) of $\hat{x}[n]$, i.e. the ratio of the power of the good signal and the power of the noise?

- The power of the good signal is simply

$$\int_{-\pi}^{\pi} P_x(e^{j\omega} d\omega) = 2\pi\sigma_x^2$$

- The power of the noise is

$$\int_{-\pi}^{\pi} P_\eta(e^{j\omega} d\omega) = 2\pi\sigma_0^2$$

- so that the SNR is simply σ_x^2/σ_0^2

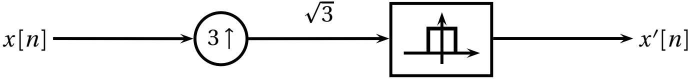
Solution of question 1

- Just as in oversampling, the idea is to send the signal more slowly as to occupy less bandwidth. If the signal has a smaller bandwidth, we can increase its amplitude without exceeding the power constraint, which will allow us to have a better SNR over the band of interest.

Solution of question 1

Q: Design the processing blocks A and B so that the signal to noise ratio of $\hat{x}[n]$ is at least twice that of the simple scheme above. You should use upsamplers, downsamplers and lowpass filters only.

- Consider the following preprocessing chain, where the lowpass filter has a cutoff frequency $\frac{\pi}{3}$:



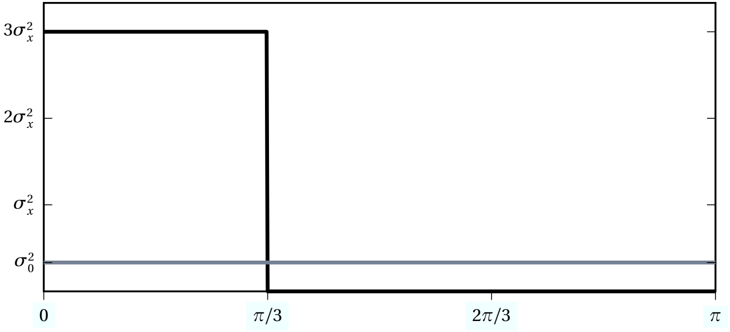
- The power of $x'[n]$ is

$$\int_{-\pi}^{\pi} P_{x'}(e^{j\omega} d\omega) = \int_{-\pi/3}^{\pi/3} 3\sigma_x^2 = 2\pi\sigma_x^2$$

so that the power constraint is fulfilled.

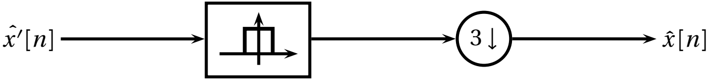
Solution of question 1

The signal and the noise at the receiver after the sampler have the following psd's:



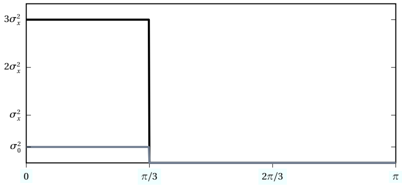
Solution of question 1

At the receiver we can filter out the out-of-band noise with the following scheme, where, once again, the lowpass has cutoff frequency $\pi/3$



Solution of question 1

After the filter the psd is



and after the downsampler

Solution of question 1

and the signal-to-noise ratio becomes

$$\mathrm{SNR}_2 = 2\pi\sigma_x^2/2\pi(\sigma_0^2/3) = 3\mathrm{SNR}_1$$

Solution of question 1

Q: Using your new scheme, how long does it take to transmit a 3-minute song signal?

9 minutes

Question 2: The shape of a constellation

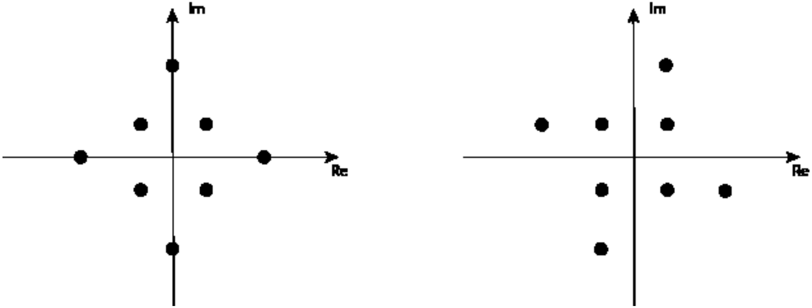
One of the reasons for designing non-regular constellations, or constellations on lattices, different than the upright square grid, is that the energy of the transmitted signal is directly proportional to the parameter σ_α^2, i.e., if we assume that the mapper includes a multiplicative factor G_0, $a[n] = G_0\alpha[n], \alpha[n] \in \mathcal{A}$ and

$$\sigma_\alpha^2 = E \mid a[n] \mid^2 = G_0^2 \sum_{\alpha \in \mathcal{A}} \mid \alpha \mid^2 p_a(\alpha) = G_0^2 \sigma_\alpha^2, \tag{1}$$

where $p_a(\alpha)$ is the probability assigned by the mapper to symbol $\alpha \in \mathcal{A}$. By arranging the same number of alphabet symbols in a different manner, we can sometimes reduce σ_α^2 and therefore use a larger amplification gain while keeping the total output power constant, which in turn lowers the probability of error.

Question 2: The shape of a constellation

Consider this two 8-point constellations

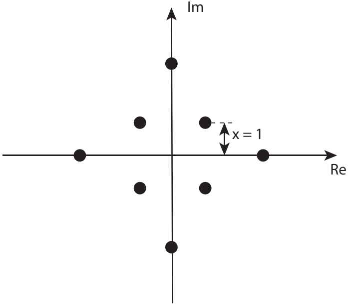


in which the outer points in the irregular constellation (on the left) are at a distance of $1+\sqrt{3}$. In both constellations consider a minimum distance of 1, considering circular decision boundaries centered upon the constellation points. Compute their intrinsic power σ_α^2 for uniform symbol distributions. What do you notice?

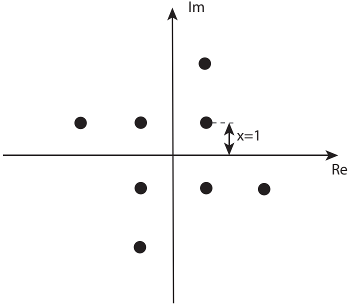
Solution of question 2

- We know that $\sigma_\alpha^2 = \sum_{\alpha \in \mathcal{A}} |\alpha|^2 p_a(\alpha)$
- Then, we have to compute $|\alpha|$ to solve the exercise.

Solution of question 2



(a)



(b)

Constellation (a):

- $|\alpha_1| = \sqrt{x^2 + x^2} = \sqrt{2}x$
- $|\alpha_2| = 1 + \sqrt{3}$

Constellation (b):

- $|\alpha_1| = \sqrt{x^2 + x^2} = \sqrt{2}x$
- $|\alpha_2| = \sqrt{x^2 + (3x)^2} = \sqrt{10}x$

Solution of question 2

Now, we just have to compute σ_α^2:

Constellation(a):

- $\sigma_\alpha^2 = \sum_{\alpha \in \mathcal{A}} |\alpha|^2 p_a(\alpha) = \frac{4(\sqrt{2}x)^2 + 4(1+\sqrt{3})^2}{8} = 4.73$

Constellation(b):

- $\sigma_\alpha^2 = \sum_{\alpha \in \mathcal{A}} |\alpha|^2 p_a(\alpha) = \frac{4(\sqrt{2}x)^2 + 4(\sqrt{10}x)^2}{8} = 6$

In other words, the irregular constellation(a) offers more than a 1 dB gain over the regular one (b). This gain can be translated into a reliability gain by increasing G_0 while the transmitted signal remains within the power constraint.