

# Digital Signal Processing

Module 5: Linear Filters

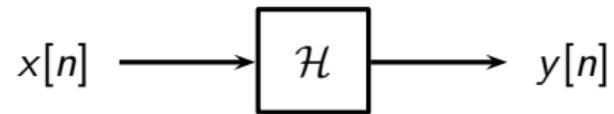
- ▶ **Module 5.1–5.3:** LTI systems and convolution; examples; stability
- ▶ **Module 5.4:** Convolution theorem
- ▶ **Module 5.5–5.6:** Ideal filters and their approximation
- ▶ **Module 5.7–5.8:** Realizable filters; implementation
- ▶ **Module 5.9–5.10:** Filter design

# Digital Signal Processing

Module 5.1: Linear Filters

- ▶ Linearity and time invariance
- ▶ Convolution

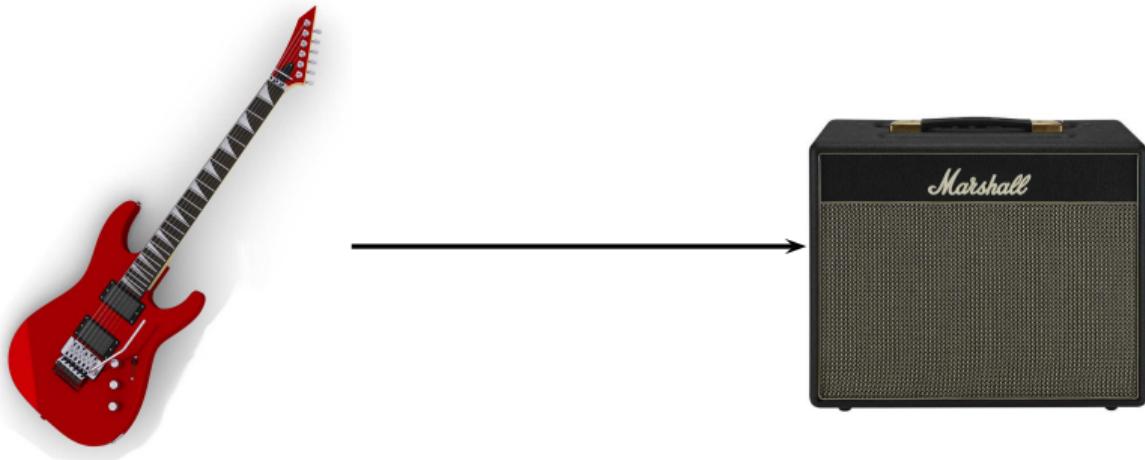
- ▶ Linearity and time invariance
- ▶ Convolution



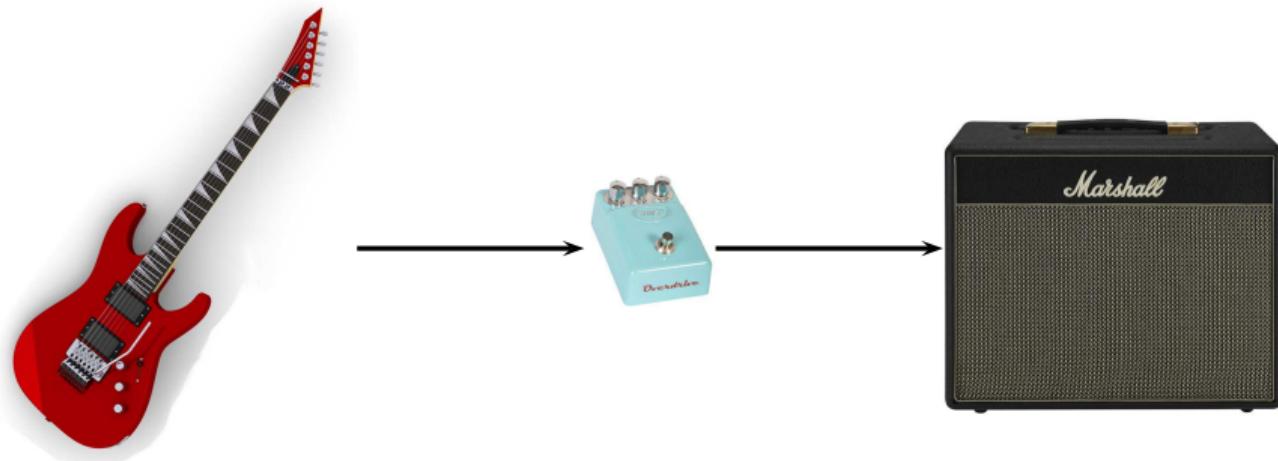
$$y[n] = \mathcal{H}\{x[n]\}$$

$$\mathcal{H}\{\alpha x_1[n] + \beta x_2[n]\} = \alpha \mathcal{H}\{x_1[n]\} + \beta \mathcal{H}\{x_2[n]\}$$

# Linearity

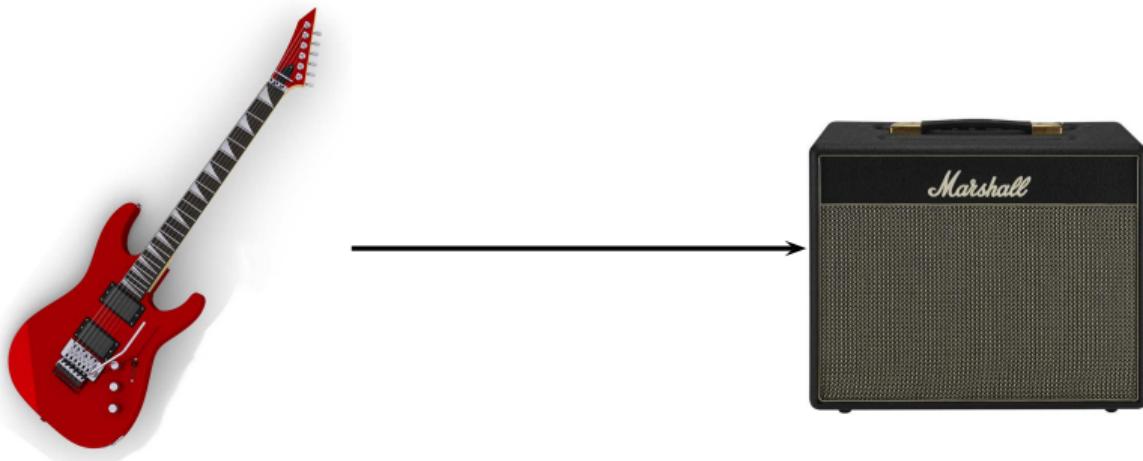


# (Non) Linearity

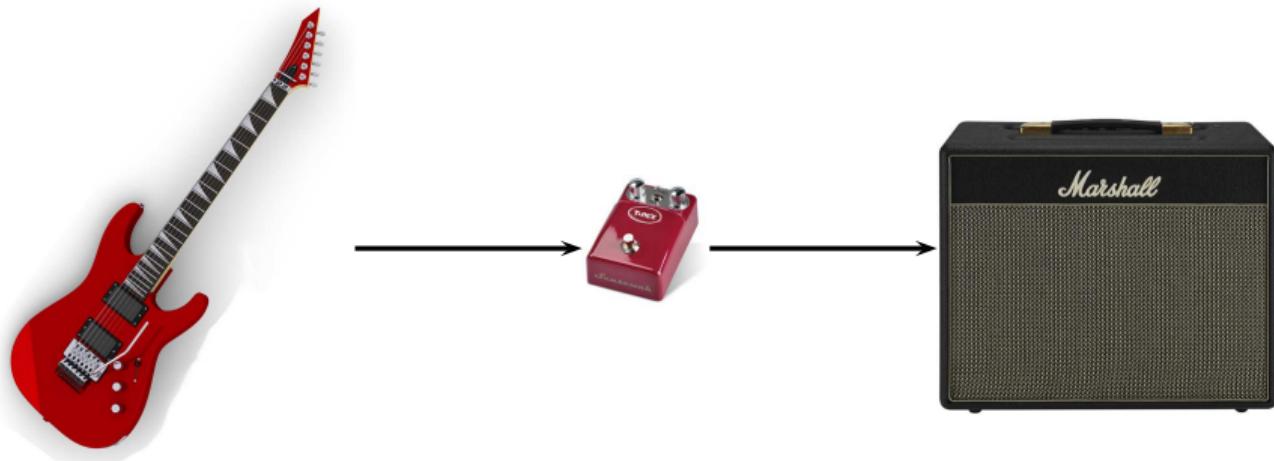


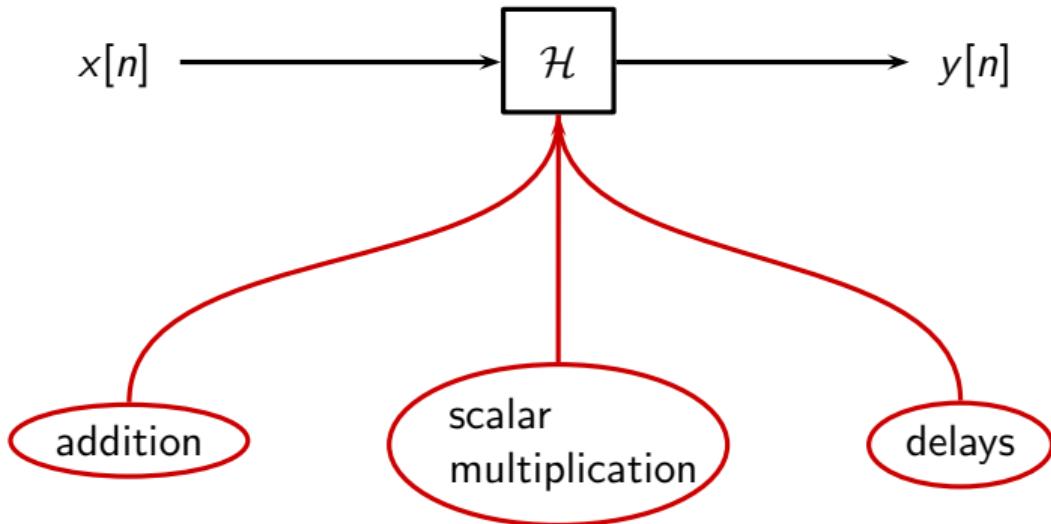
$$y[n] = \mathcal{H}\{x[n]\} \iff \mathcal{H}\{x[n - n_0]\} = y[n - n_0]$$

# Time invariance



# Time (in)variance





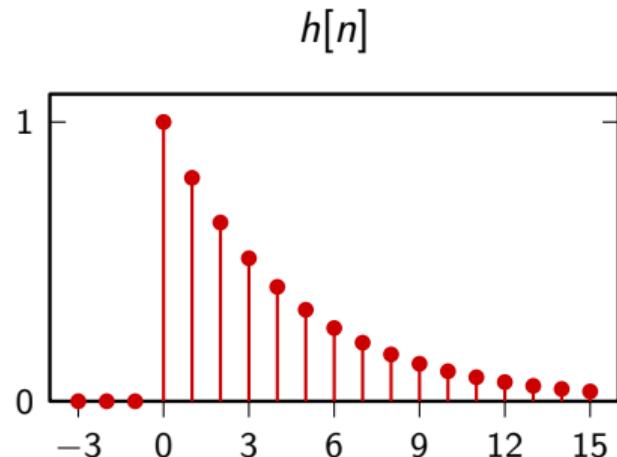
$$y[n] = H(x[n], x[n-1], x[n-2], \dots, y[n-1], y[n-2], \dots)$$

with  $H(\cdot)$  a linear function of its arguments

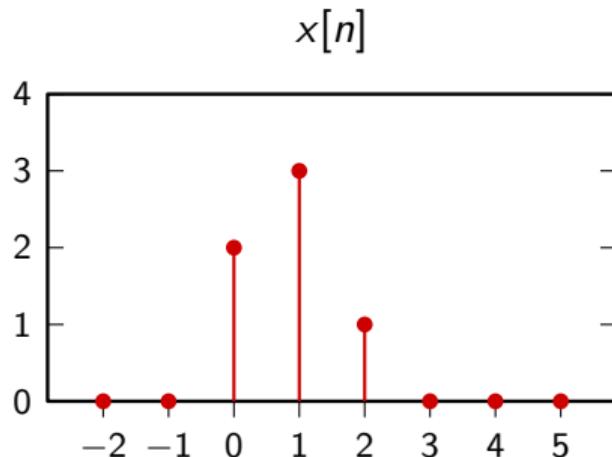
$$h[n] = \mathcal{H}\{\delta[n]\}$$

Fundamental result: impulse response fully characterizes the LTI system!

# Example



$$h[n] = \alpha^n u[n]$$



$$x[n] = \begin{cases} 2 & n = 0 \\ 3 & n = 1 \\ 1 & n = 2 \\ 0 & \text{otherwise} \end{cases}$$

- ▶  $x[n] = 2\delta[n] + 3\delta[n - 1] + \delta[n - 2]$
- ▶ we know the impulse response  $h[n] = \mathcal{H}\{\delta[n]\}$ ;
- ▶ compute  $y[n] = \mathcal{H}\{x[n]\}$  exploiting linearity and time-invariance

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## Example

$$y[n] = \mathcal{H}\{2\delta[n] + 3\delta[n - 1] + \delta[n - 2]\}$$

$$= 2\mathcal{H}\{\delta[n]\} + 3\mathcal{H}\{\delta[n - 1]\} + \mathcal{H}\{\delta[n - 2]\}$$

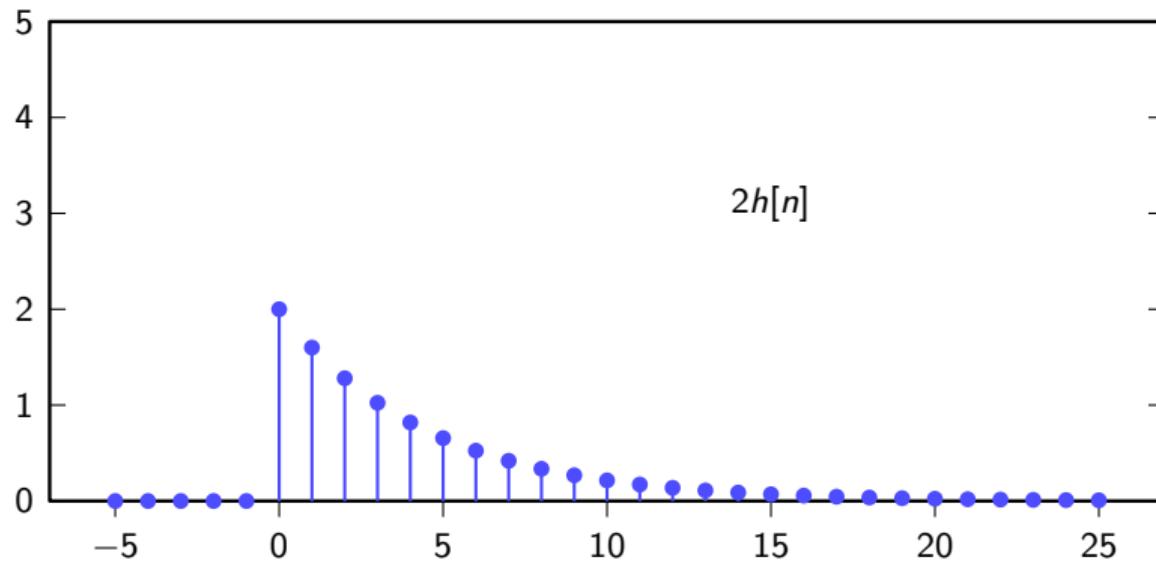
$$= 2h[n] + 3h[n - 1] + h[n - 2]$$

## Example

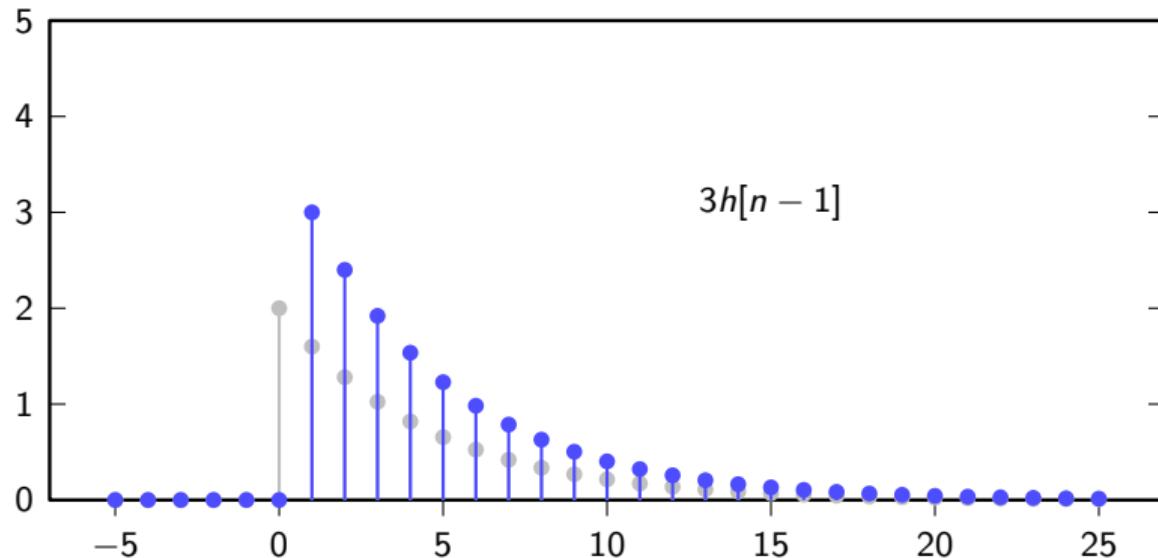
$$\begin{aligned}y[n] &= \mathcal{H}\{2\delta[n] + 3\delta[n - 1] + \delta[n - 2]\} \\&= 2\mathcal{H}\{\delta[n]\} + 3\mathcal{H}\{\delta[n - 1]\} + \mathcal{H}\{\delta[n - 2]\} \\&= 2h[n] + 3h[n - 1] + h[n - 2]\end{aligned}$$

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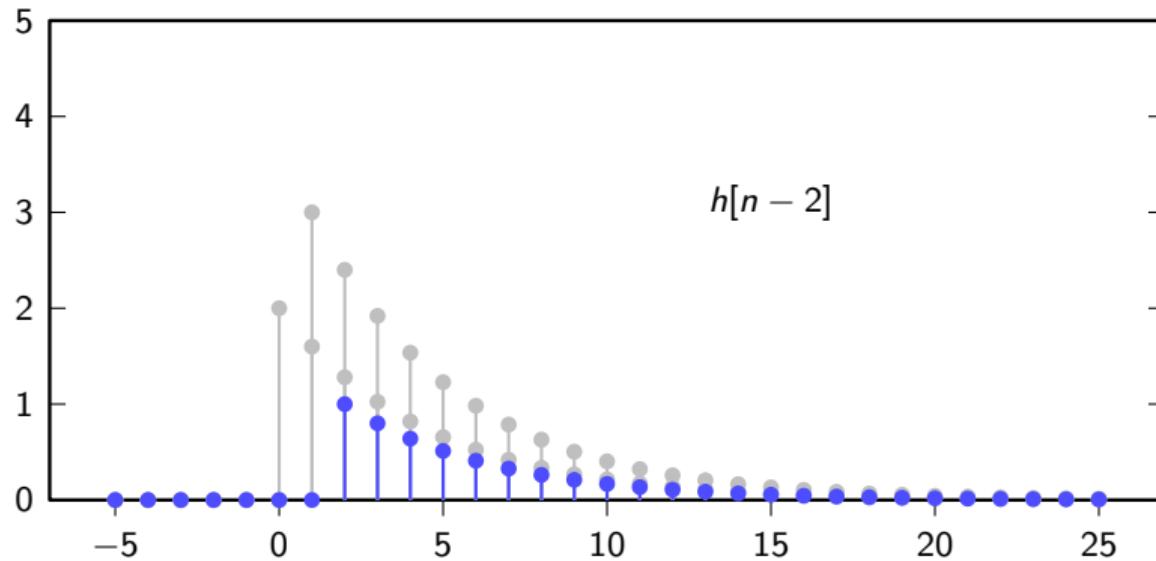
## Example



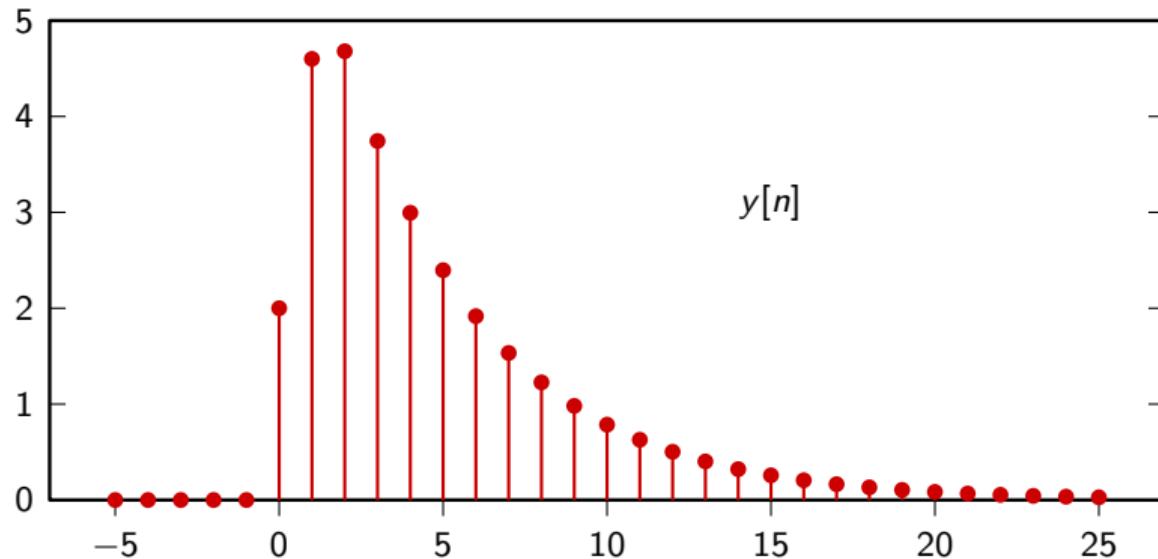
## Example



## Example



## Example



We can always write (remember Module 3.2):

$$x[n] = \sum_{k=-\infty}^{\infty} x[k]\delta[n - k]$$

by linearity and time invariance:

$$\begin{aligned} y[n] &= \sum_{k=-\infty}^{\infty} x[k]h[n - k] \\ &= x[n] * h[n] \end{aligned}$$

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$$\begin{aligned} y[n] &= \sum_{k=-\infty}^{\infty} x[k]h[n - k] \\ &= x[n] * h[n] \end{aligned}$$

$$x[n] * h[n] = \sum_{k=-\infty}^{\infty} x[k]h[n-k]$$

The recipe:

Ingredients:

- ▶ a sequence  $x[m]$
- ▶ a second sequence  $h[m]$

- ▶ time-reverse  $h[m]$
- ▶ at each step  $n$  (from  $-\infty$  to  $\infty$ ):
  - center the time-reversed  $h[m]$  in  $n$   
(i.e. shift by  $-n$ )
  - compute the inner product

$$x[n] * h[n] = \sum_{k=-\infty}^{\infty} x[k]h[n-k]$$

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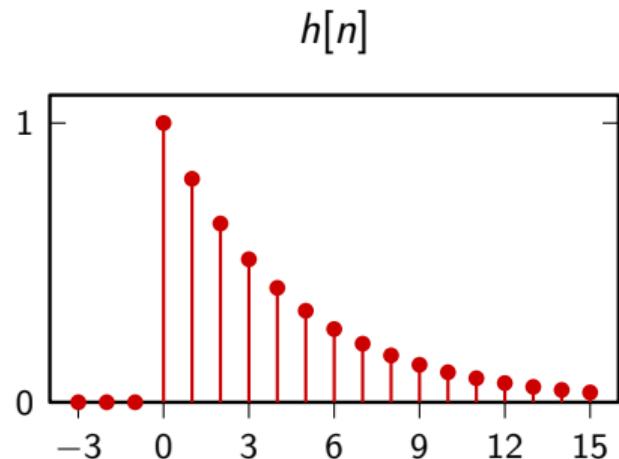
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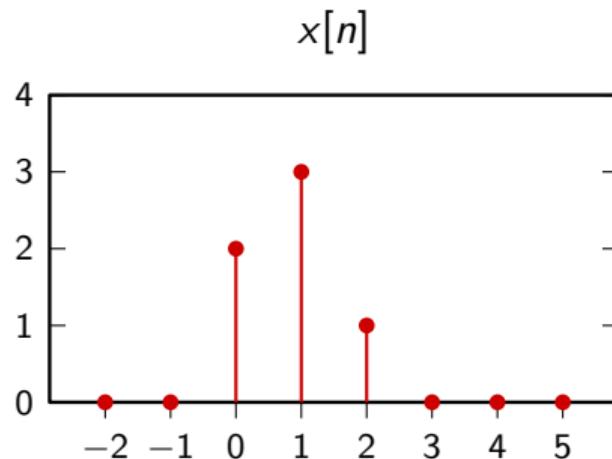
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## Same example, different perspective

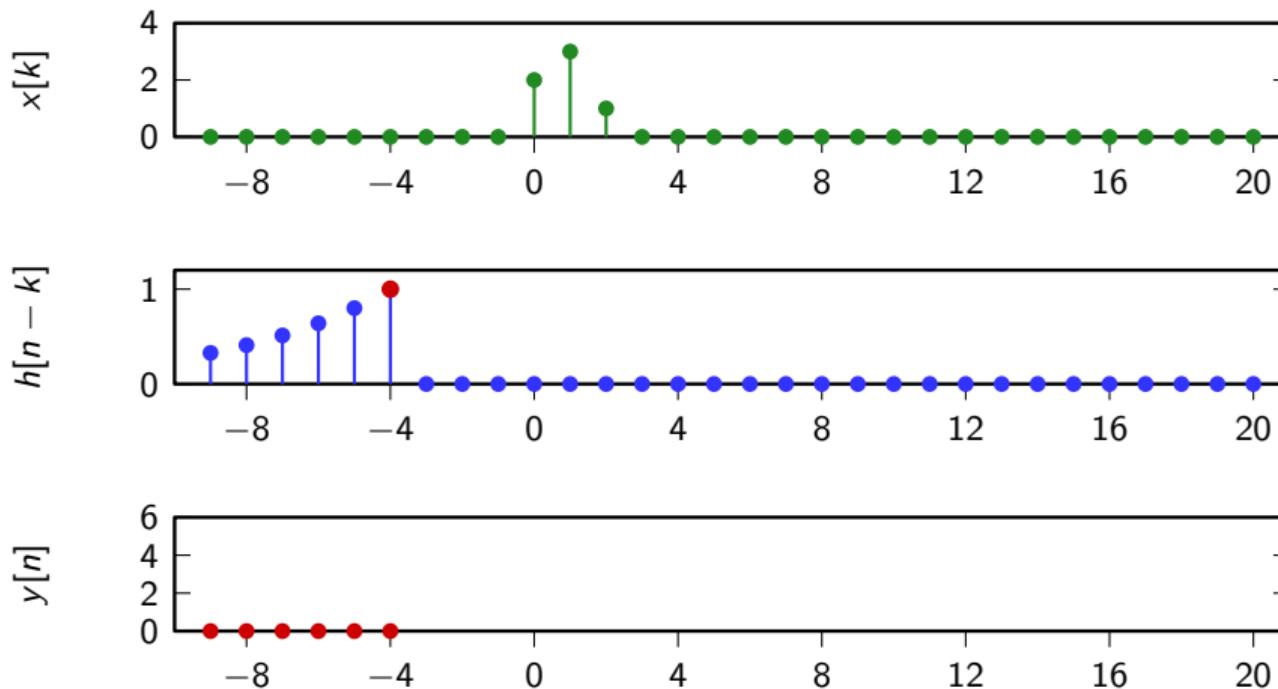


$$h[n] = \alpha^n u[n]$$

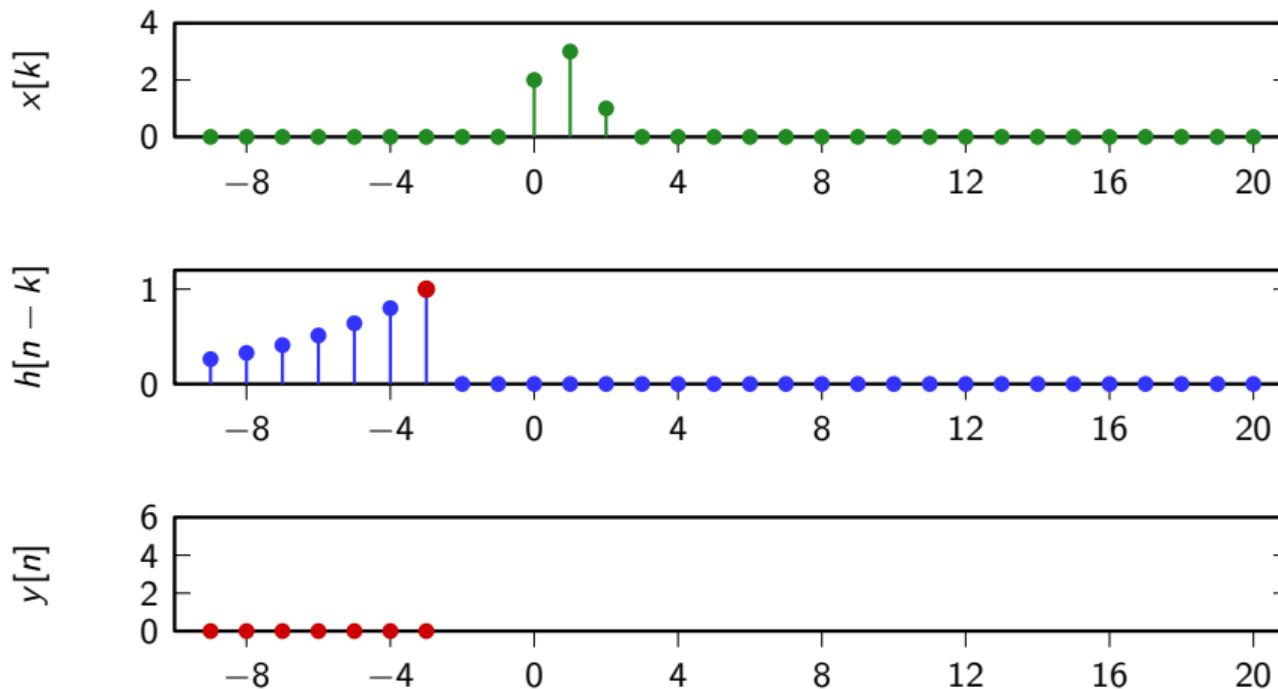


$$x[n] = \begin{cases} 2 & n = 0 \\ 3 & n = 1 \\ 1 & n = 2 \\ 0 & \text{otherwise} \end{cases}$$

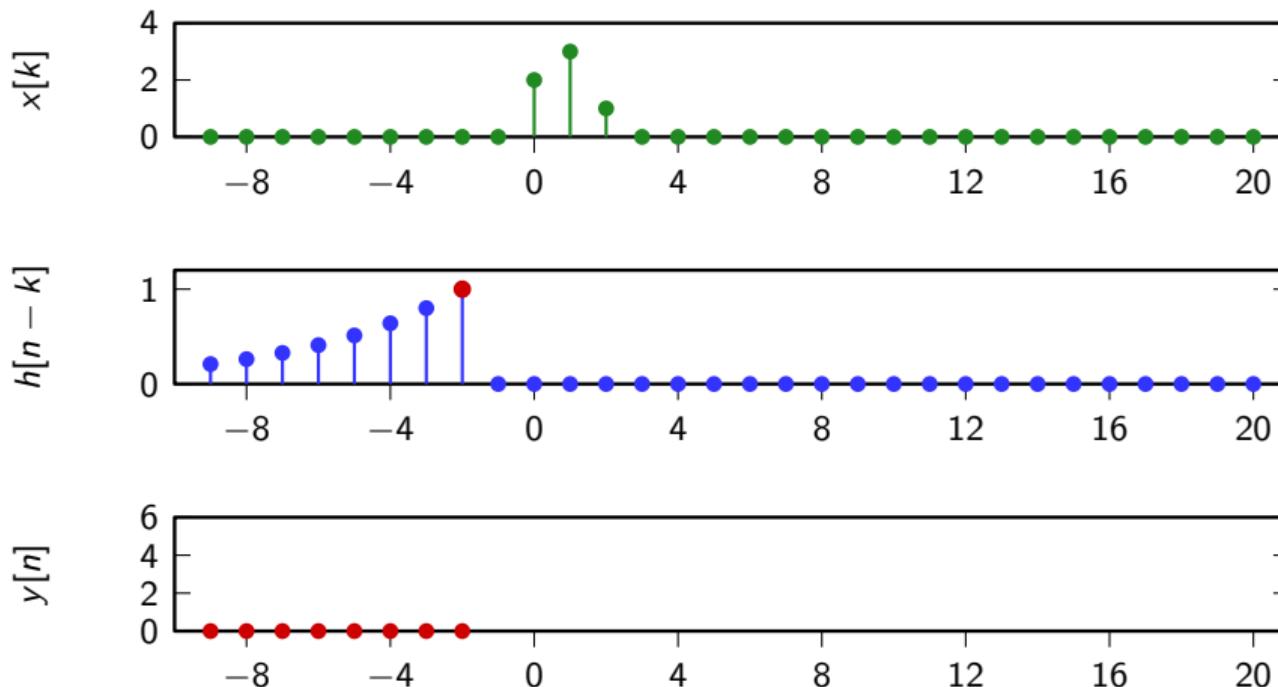
# Convolution example



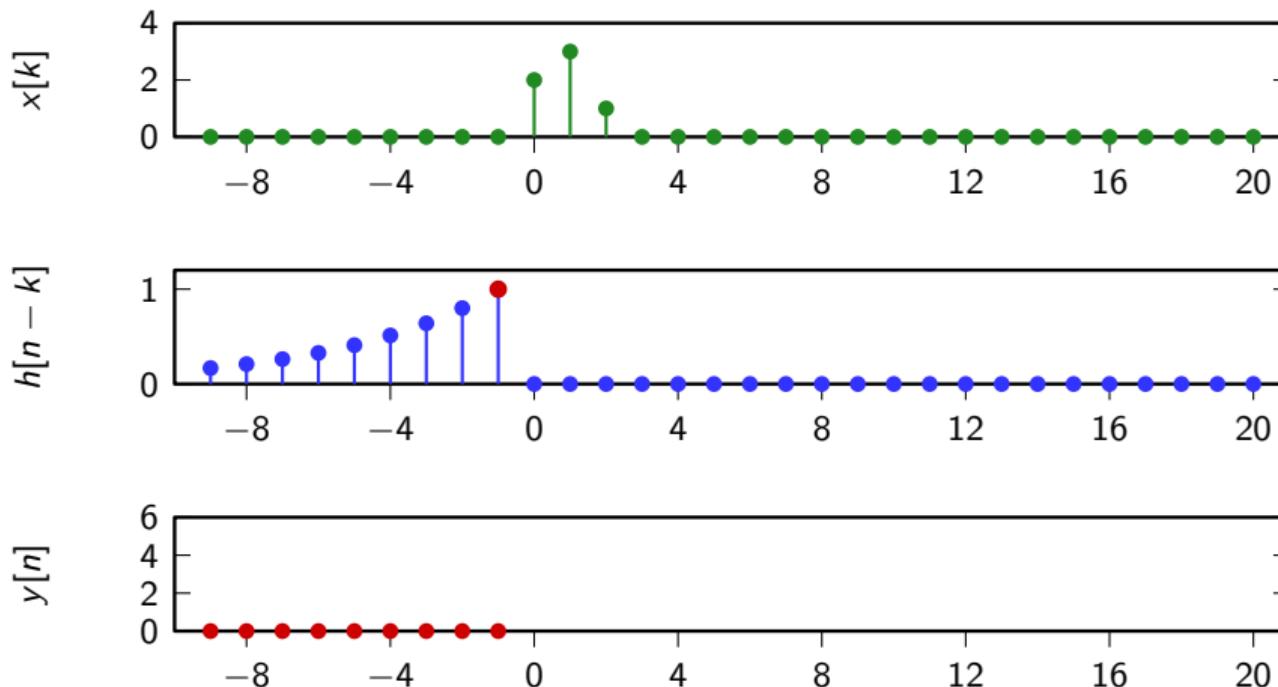
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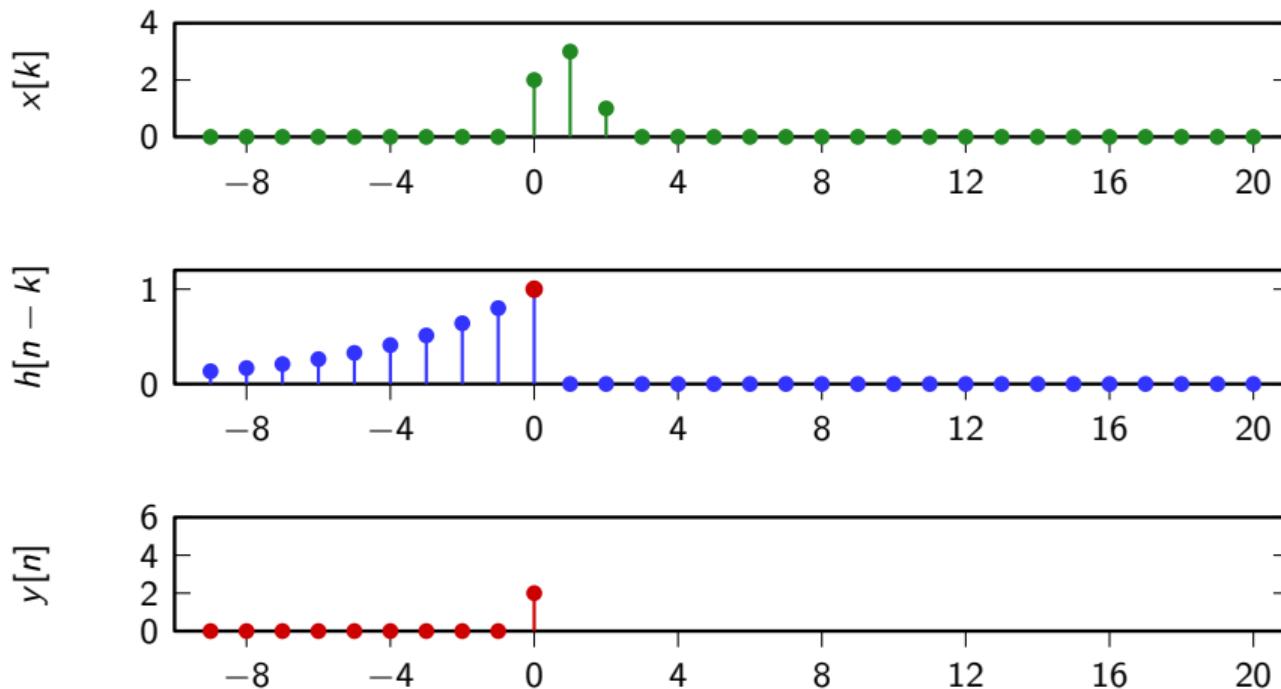
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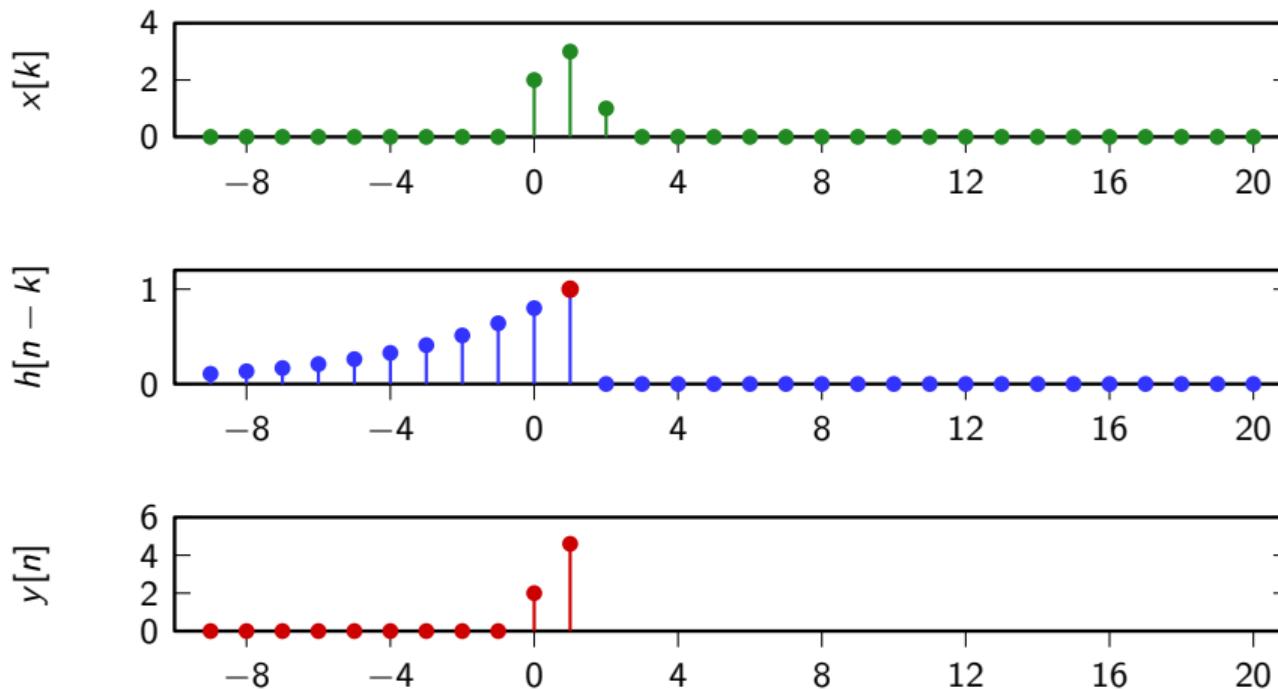
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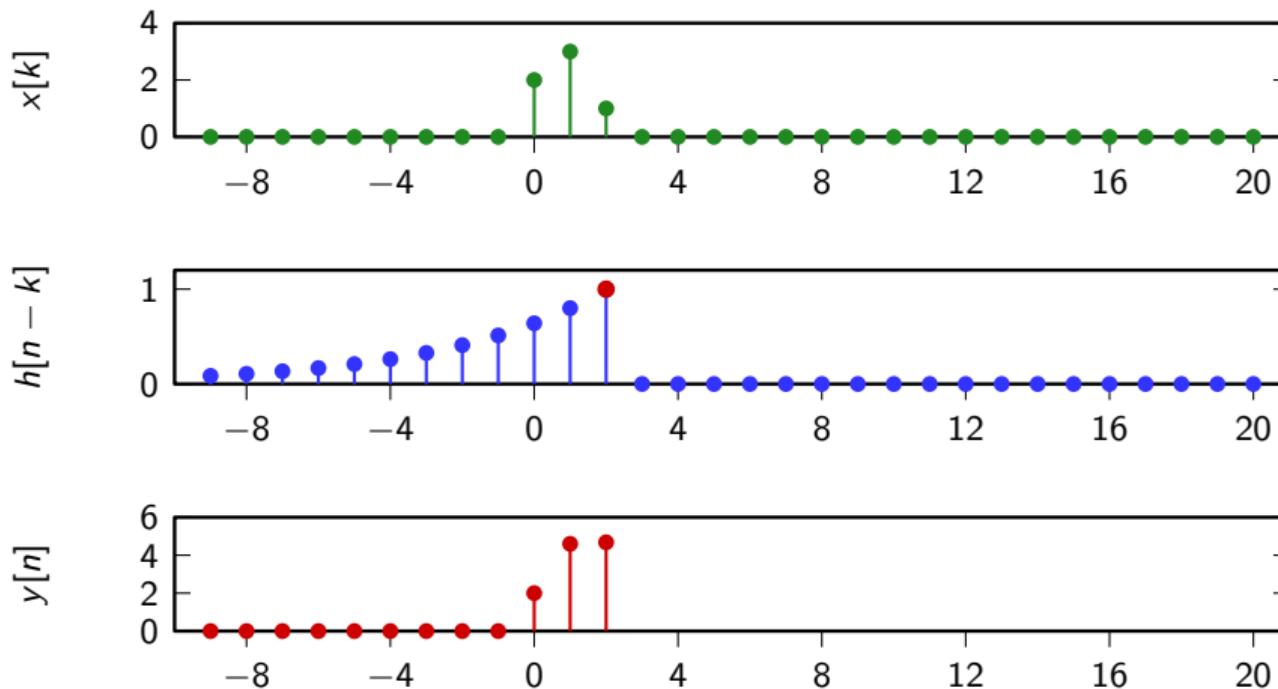
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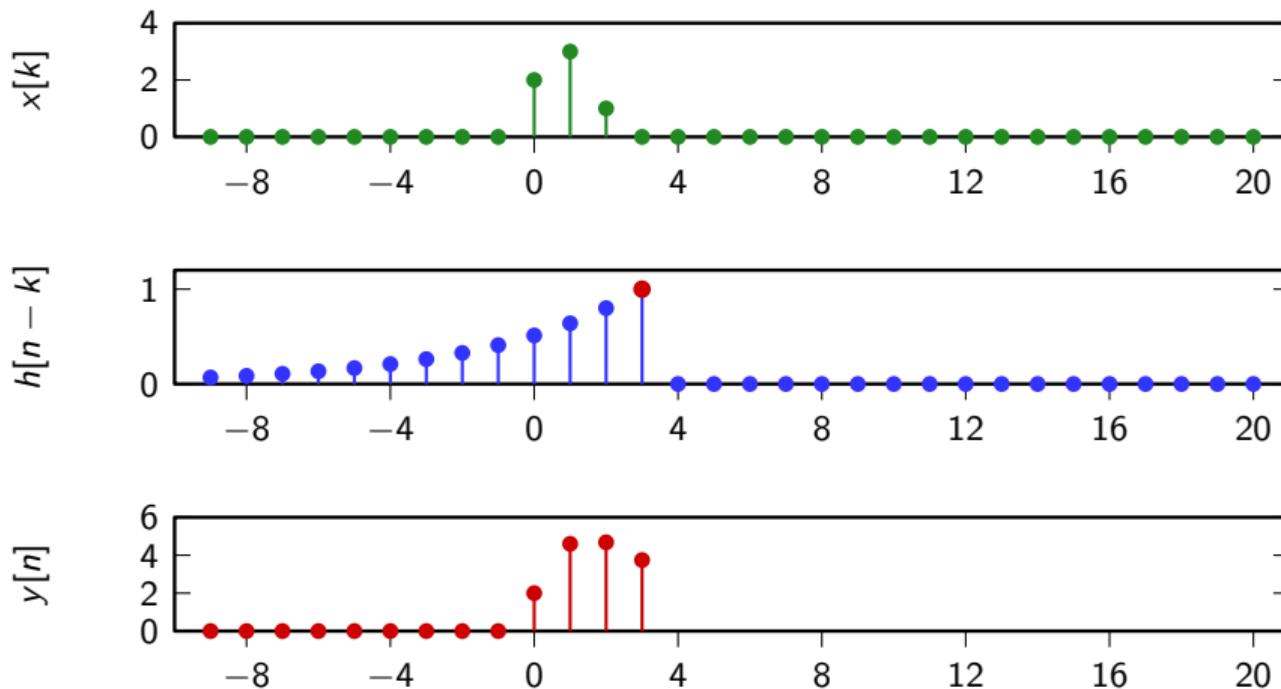
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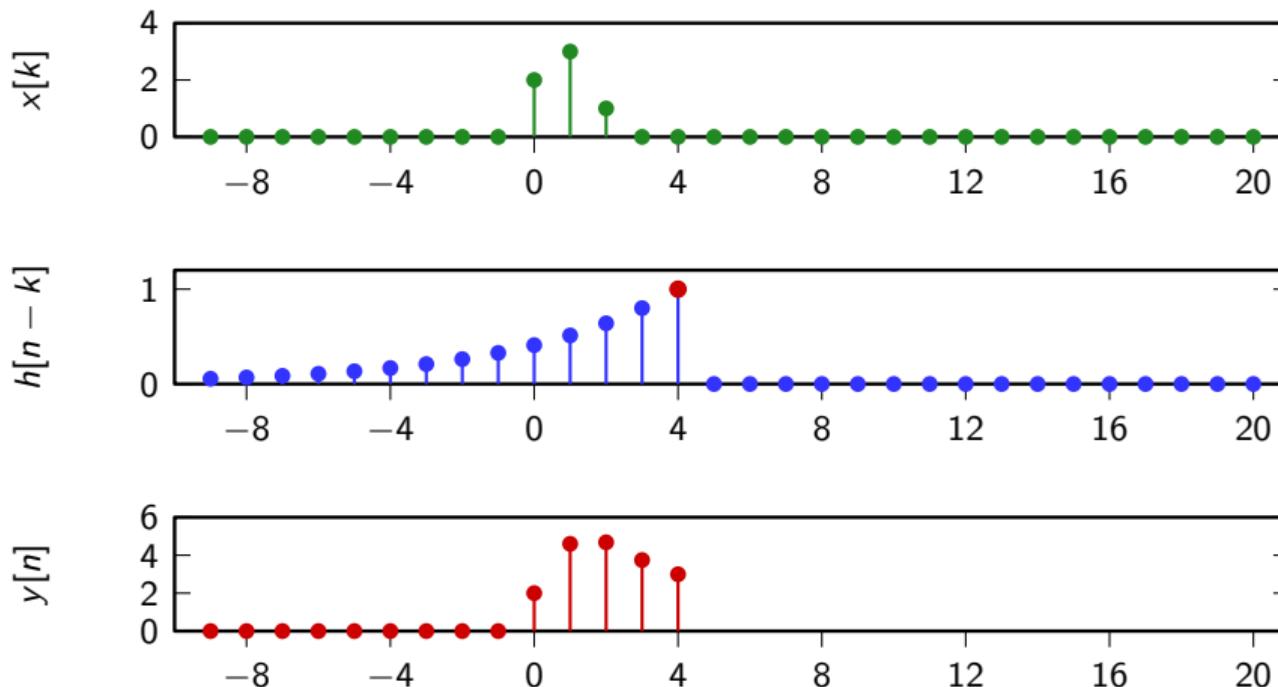
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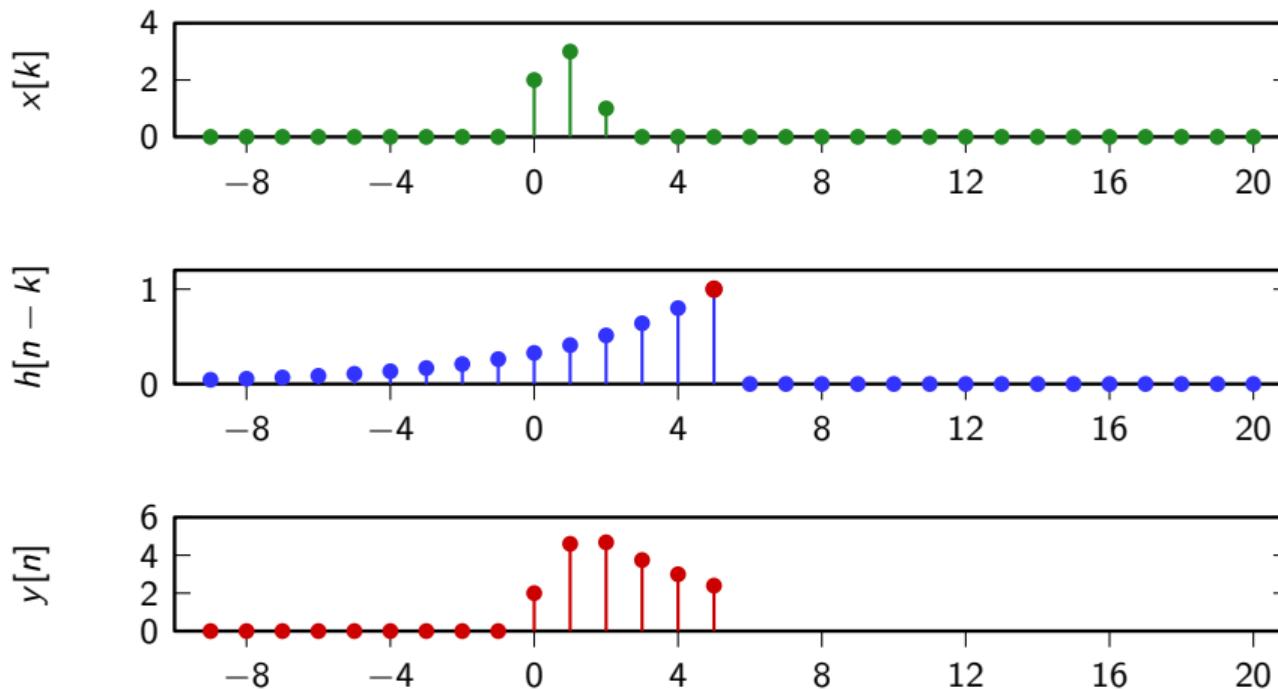
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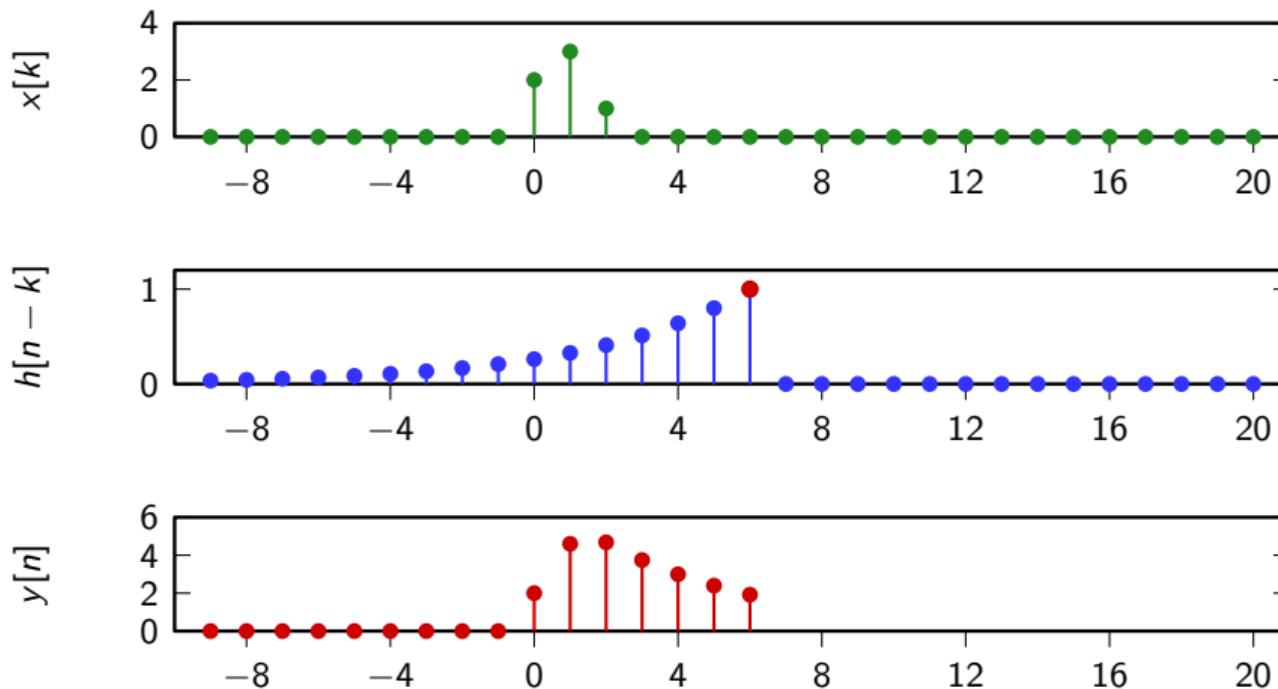
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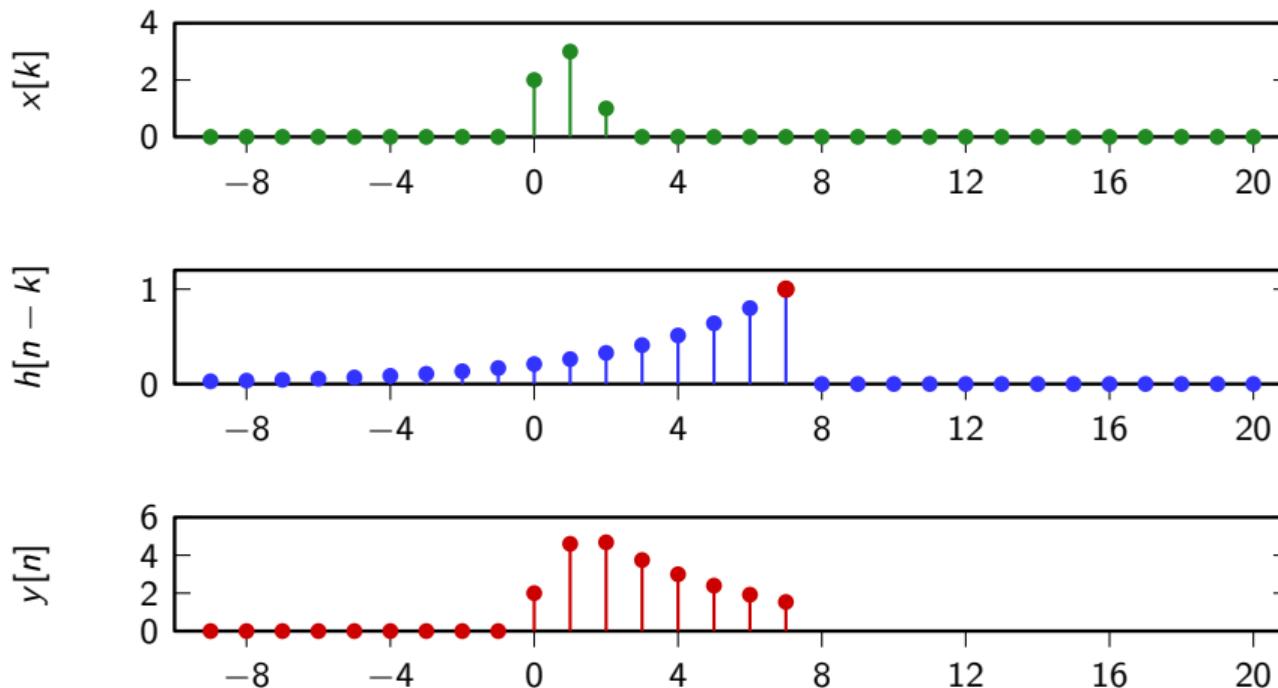
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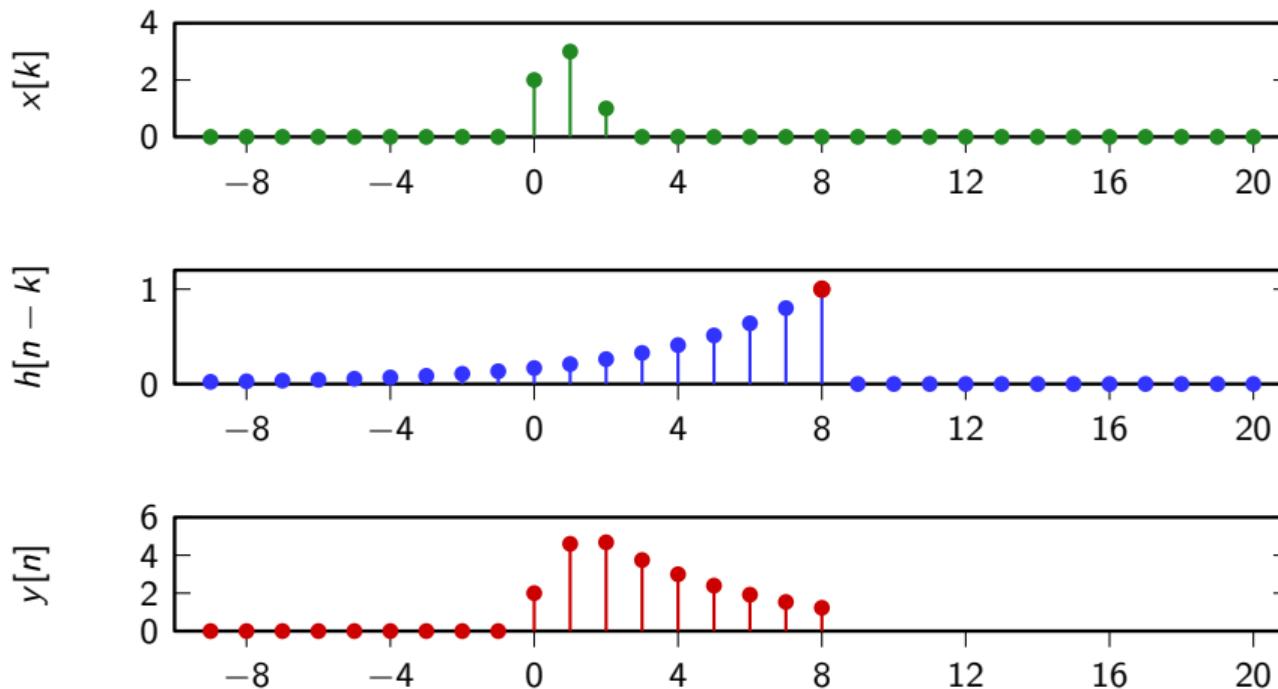
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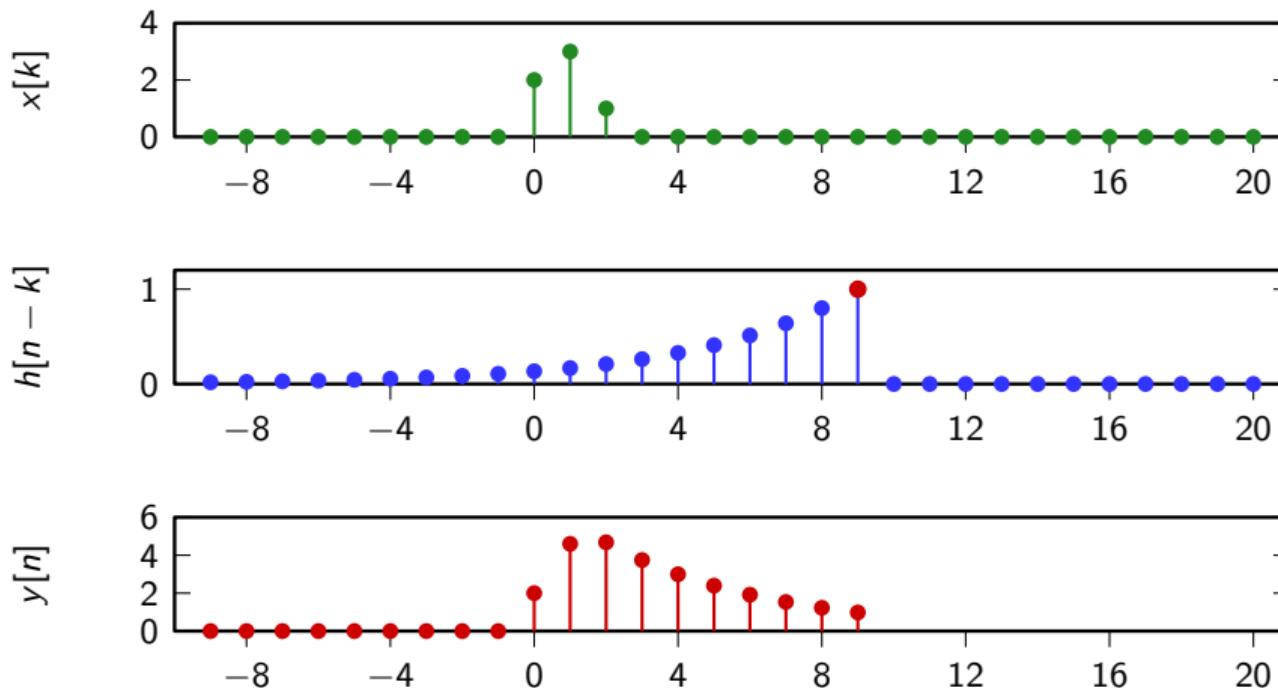
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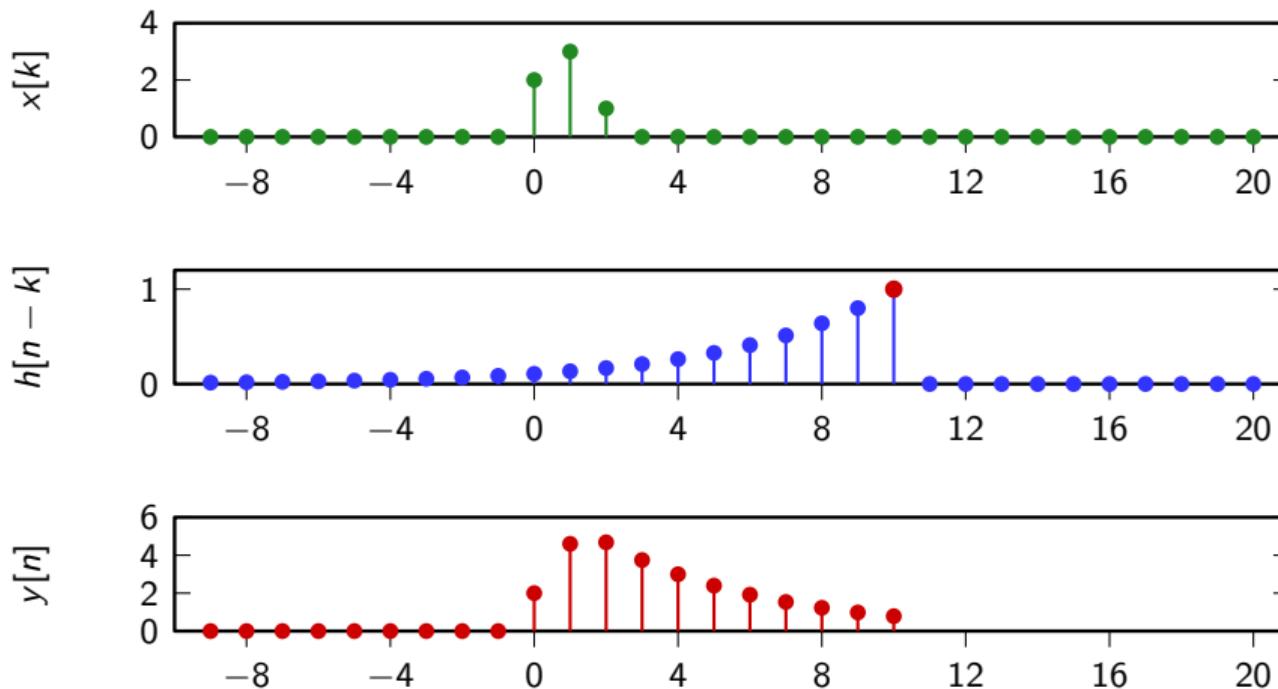
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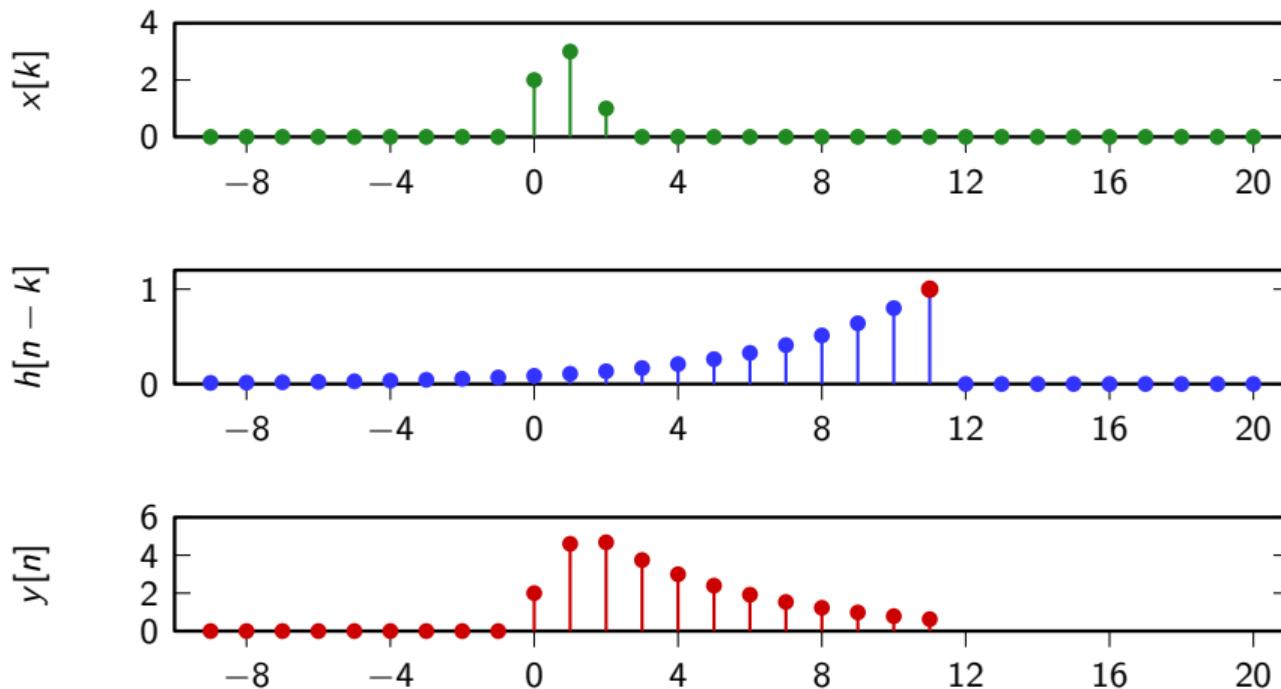
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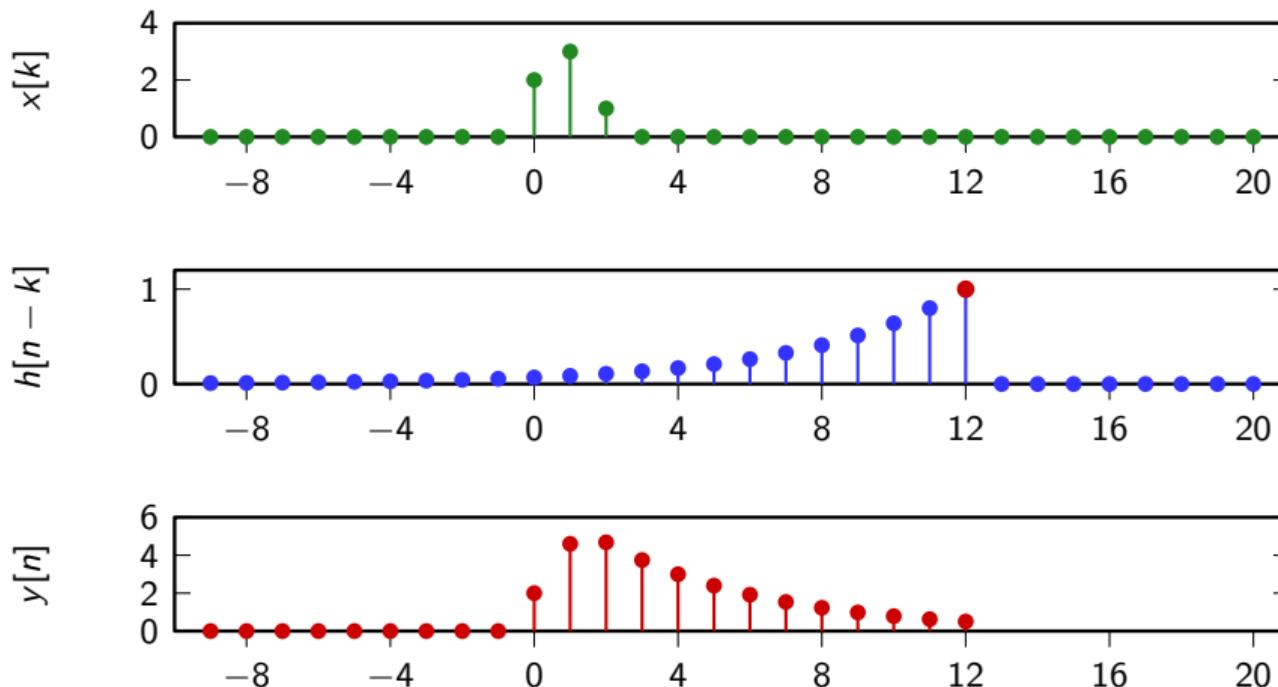
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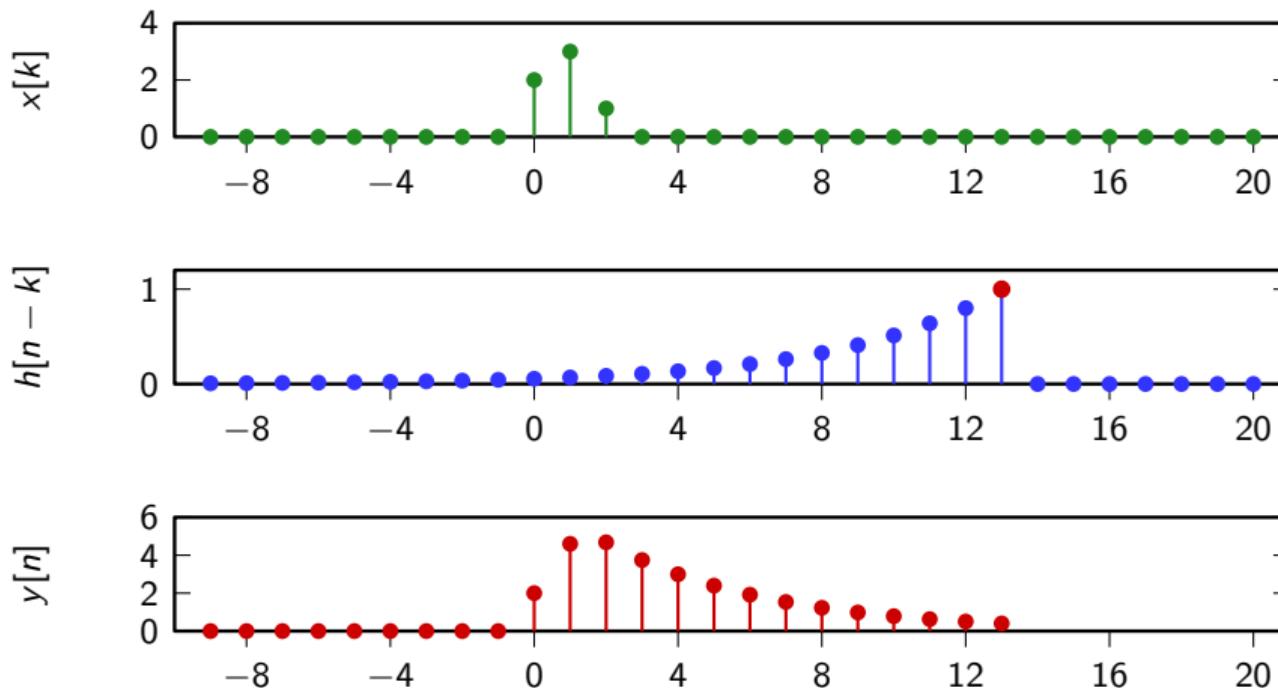
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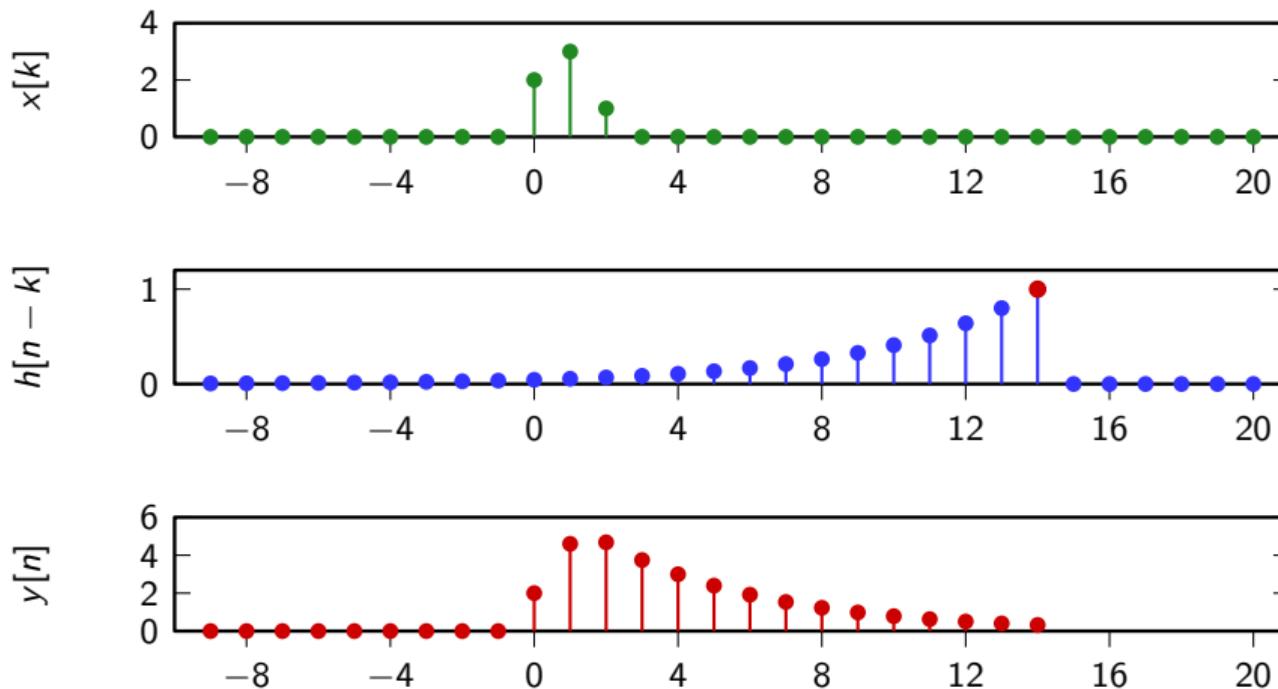
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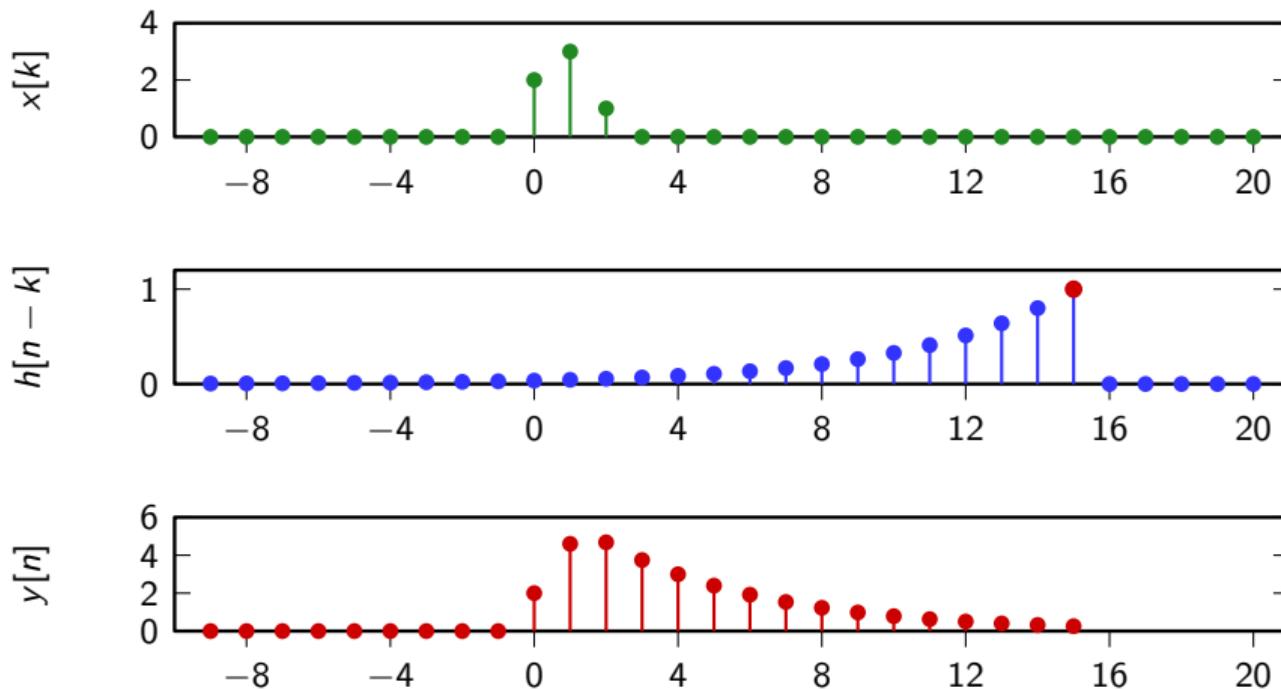
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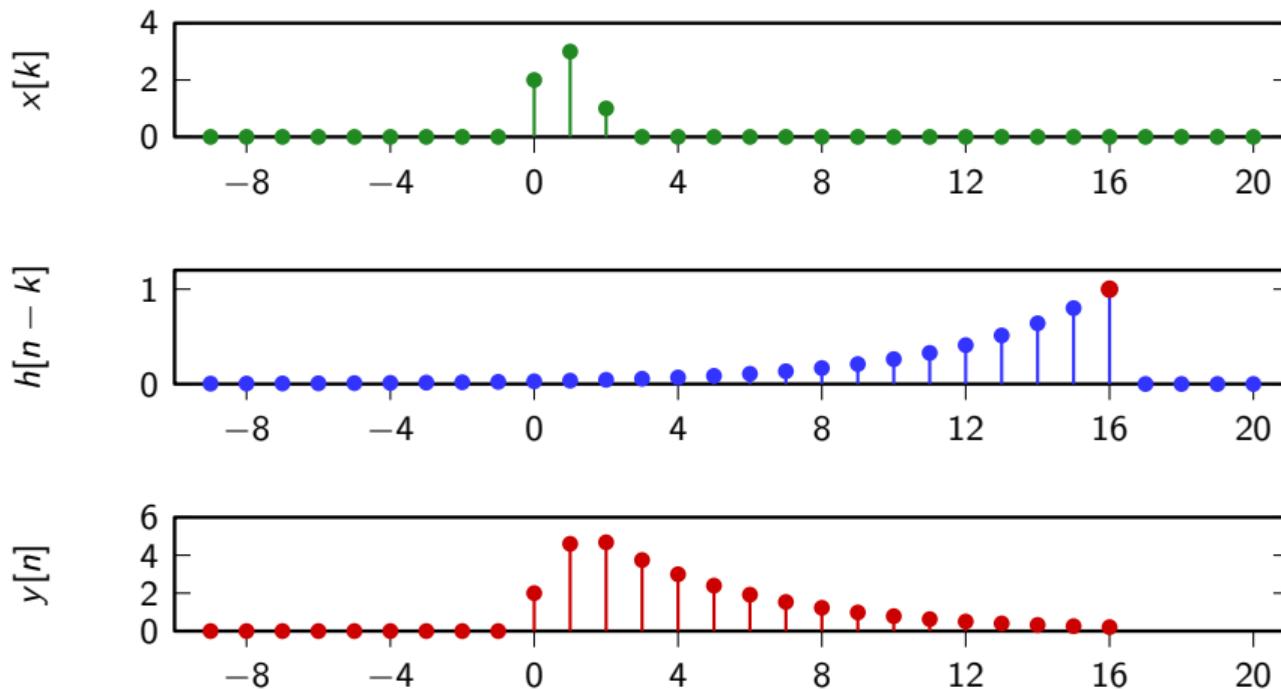
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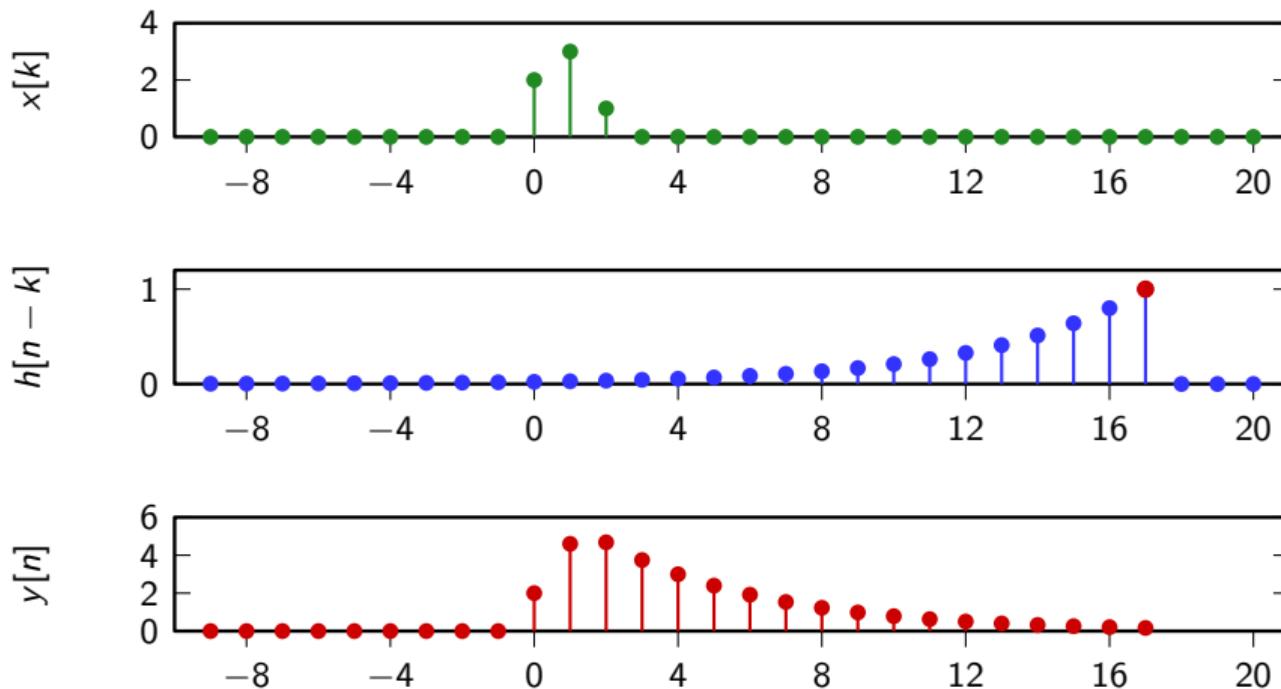
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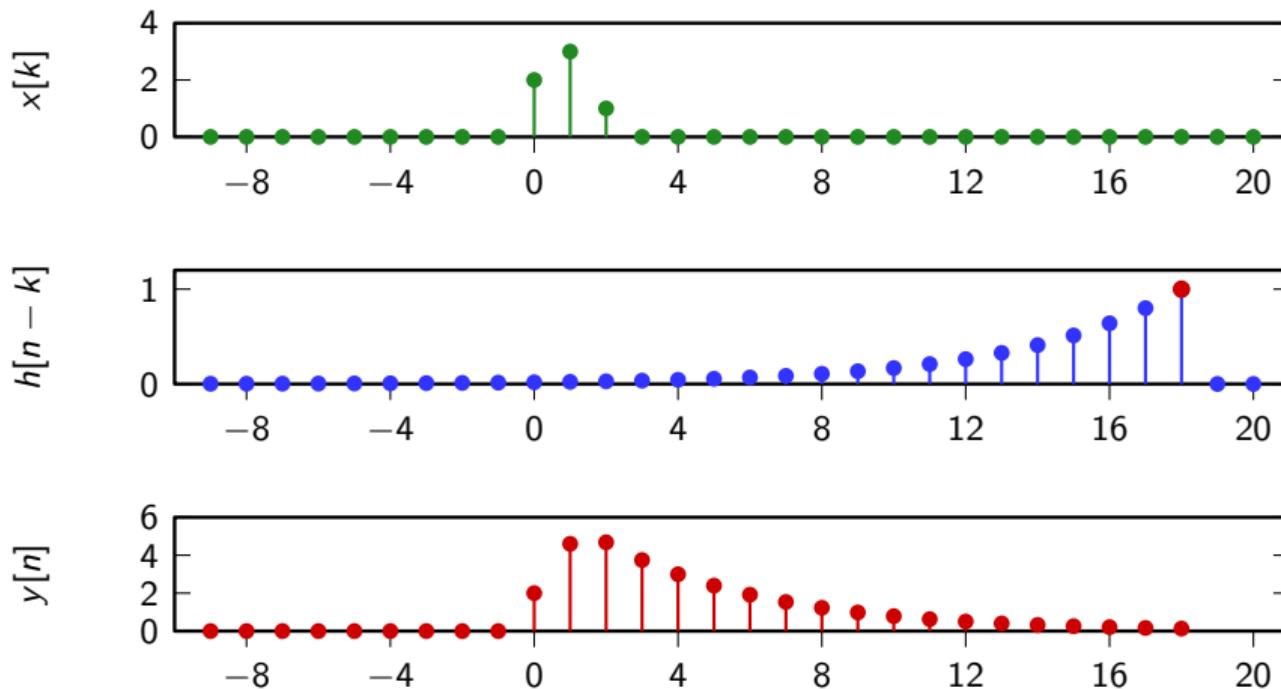
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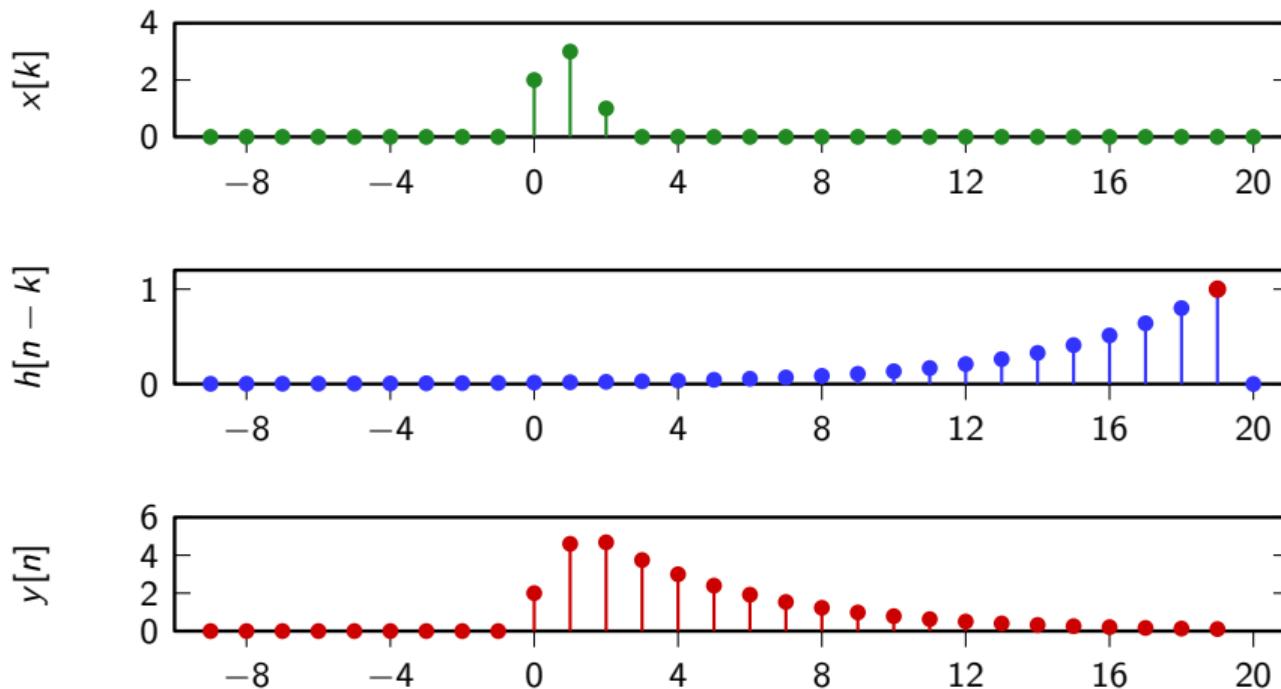
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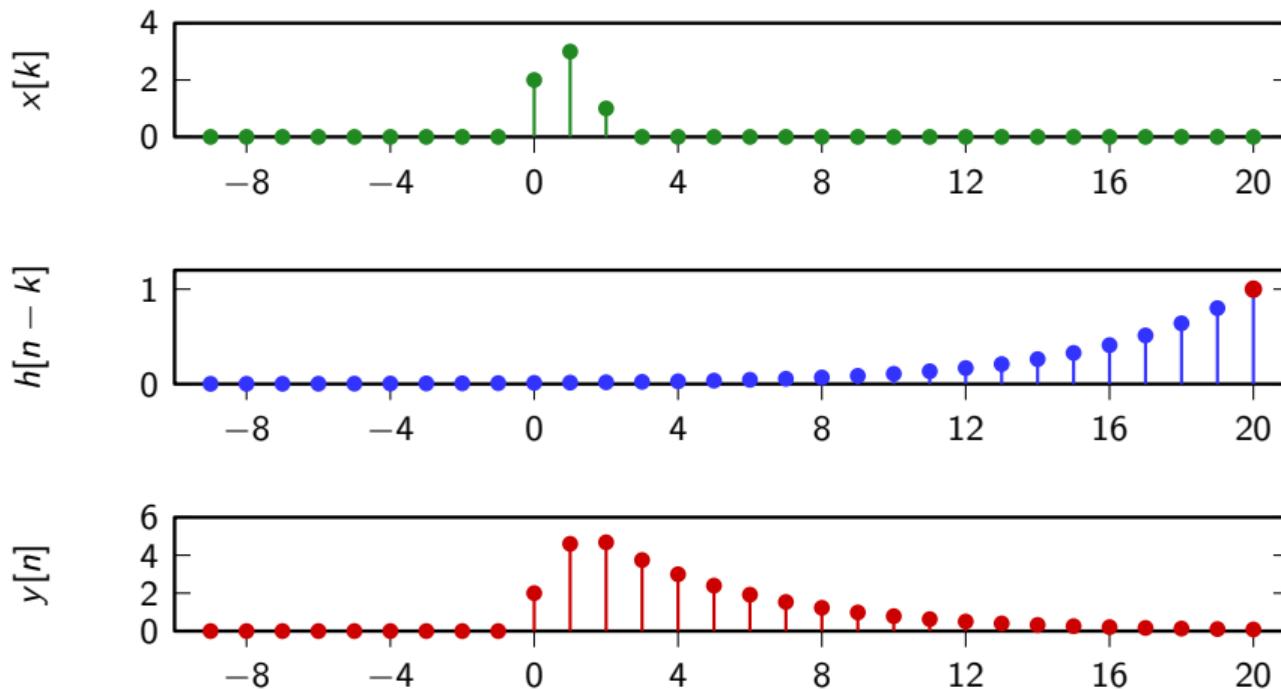
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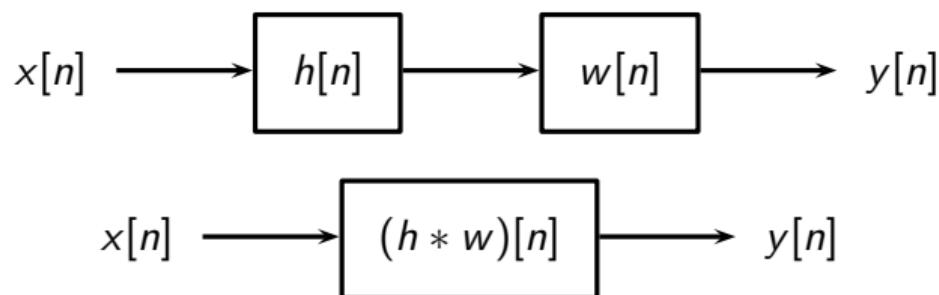
# Convolution example



- ▶ linearity and time invariance (by definition)
- ▶ commutativity:  $(x * h)[n] = (h * x)[n]$
- ▶ associativity for absolutely- and square-summable sequences:  
 $((x * h) * w)[n] = (x * (h * w))[n]$

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END OF MODULE 5.1

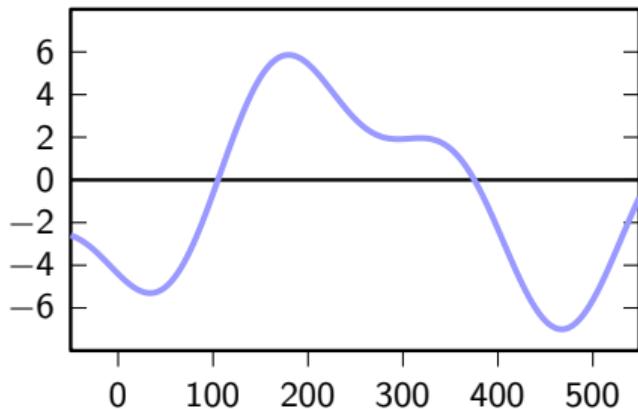
# Digital Signal Processing

Module 5.2: Filtering by example

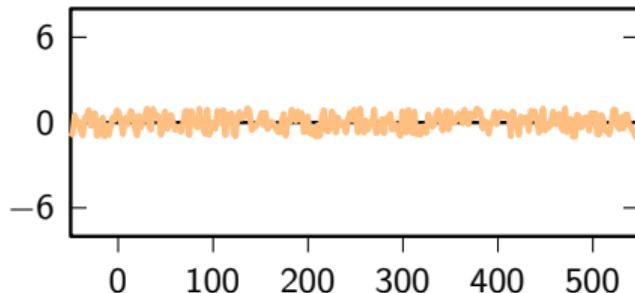
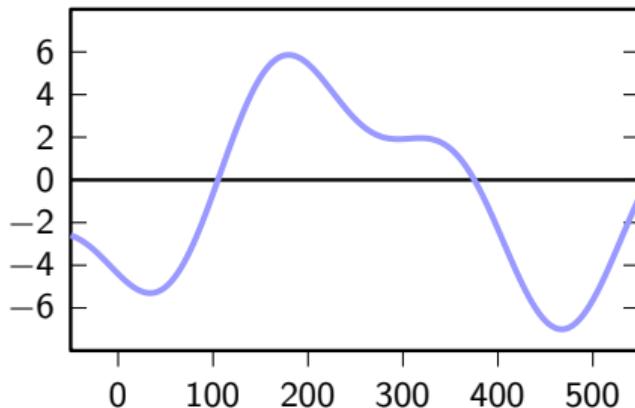
- ▶ Moving average filter
- ▶ Leaky integrator

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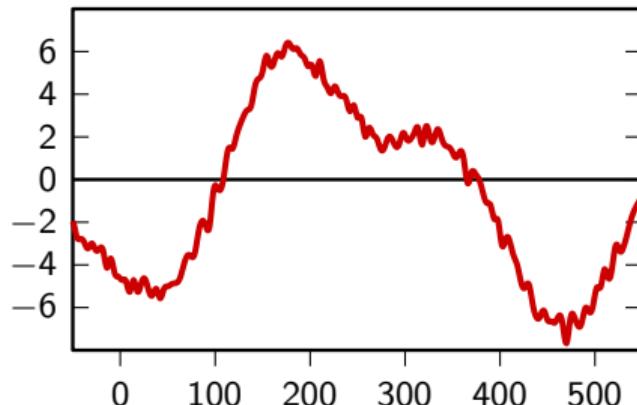
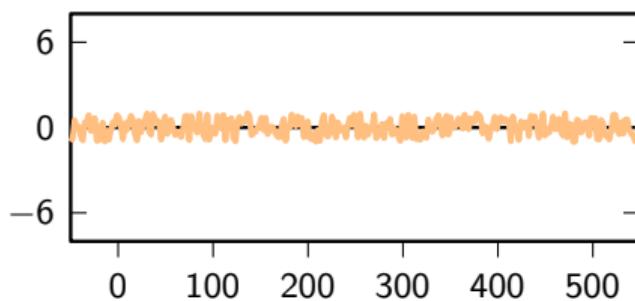
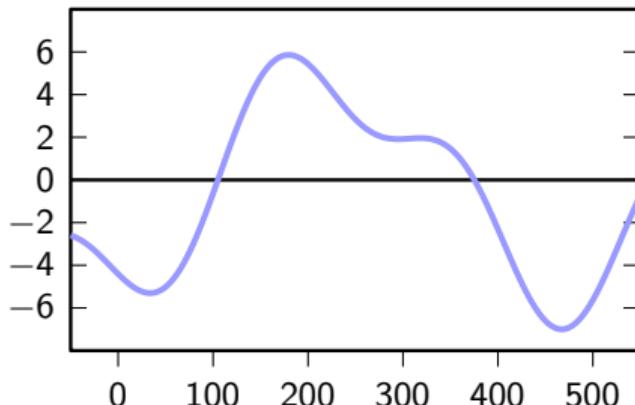
## Typical filtering scenario: denoising



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## Typical filtering scenario: denoising



- ▶ idea: replace each sample by the local average
- ▶ for instance:  $y[n] = (x[n] + x[n - 1])/2$
- ▶ more generally:

$$y[n] = \frac{1}{M} \sum_{k=0}^{M-1} x[n - k]$$

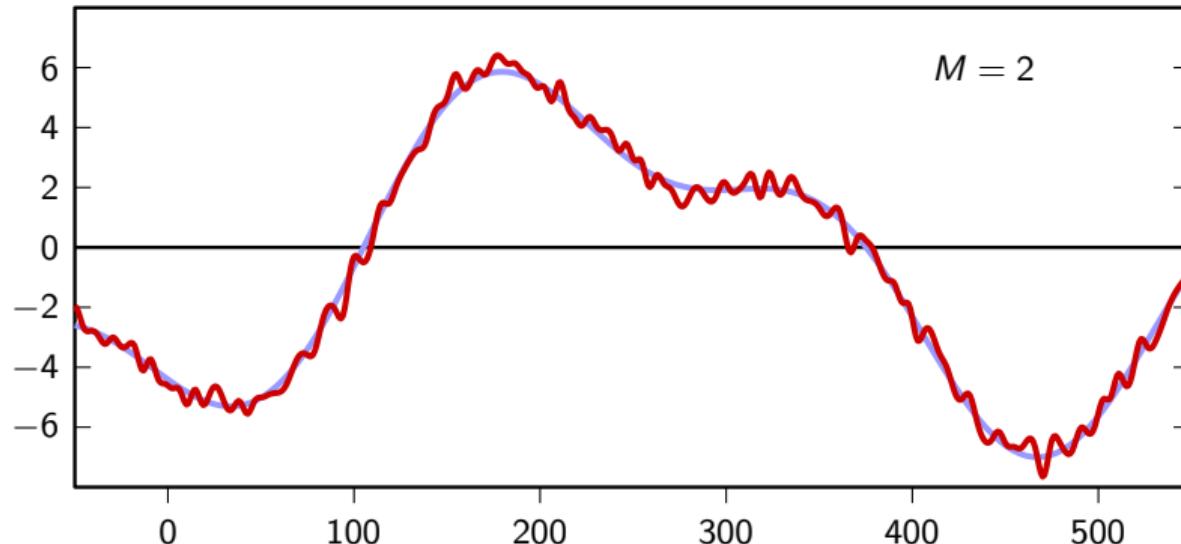
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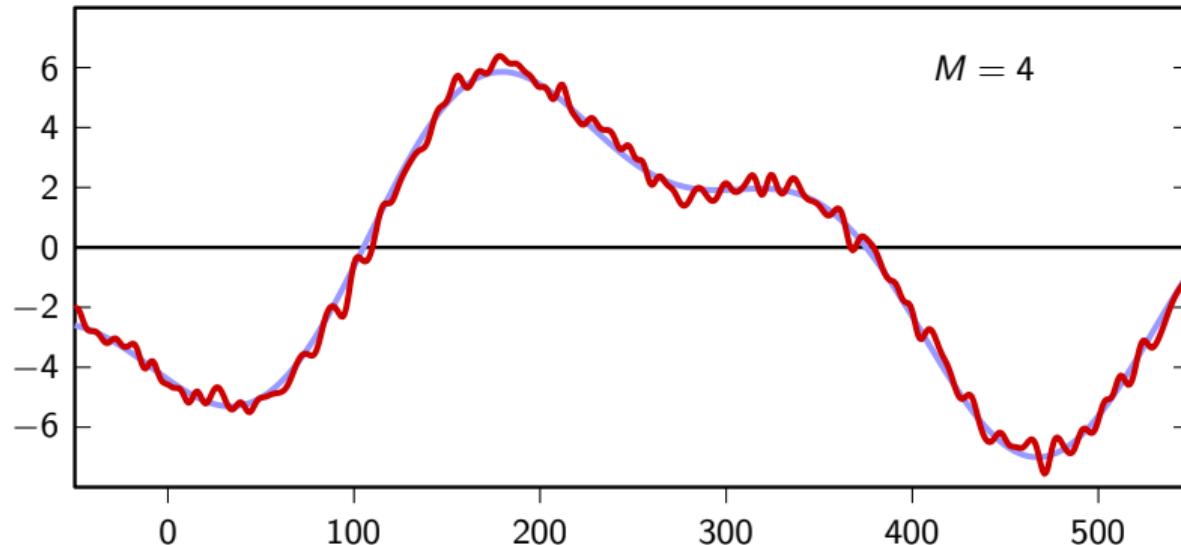
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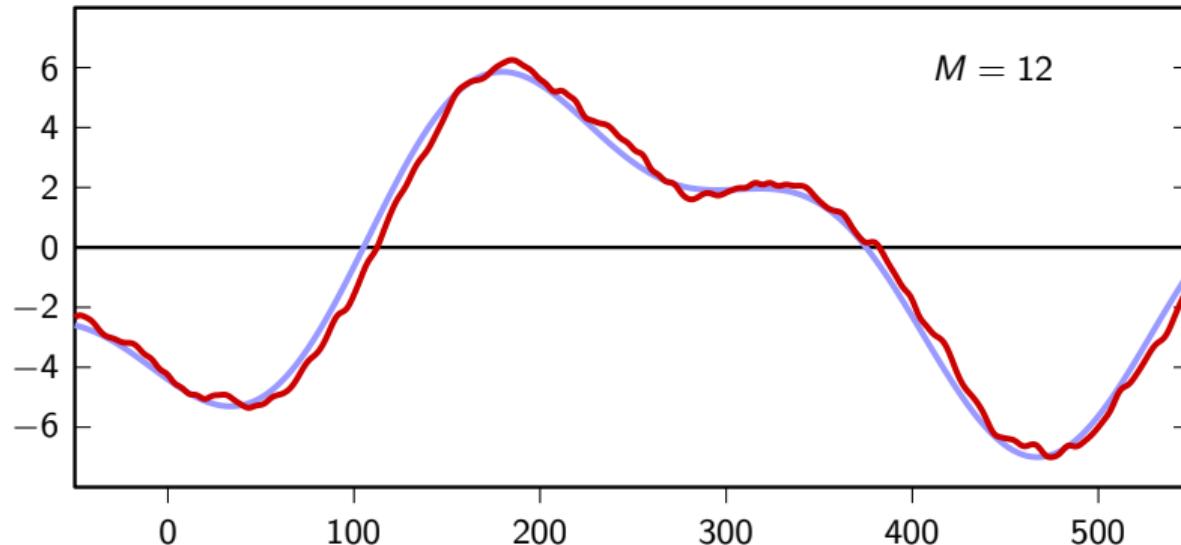
## Denoising by Moving Average



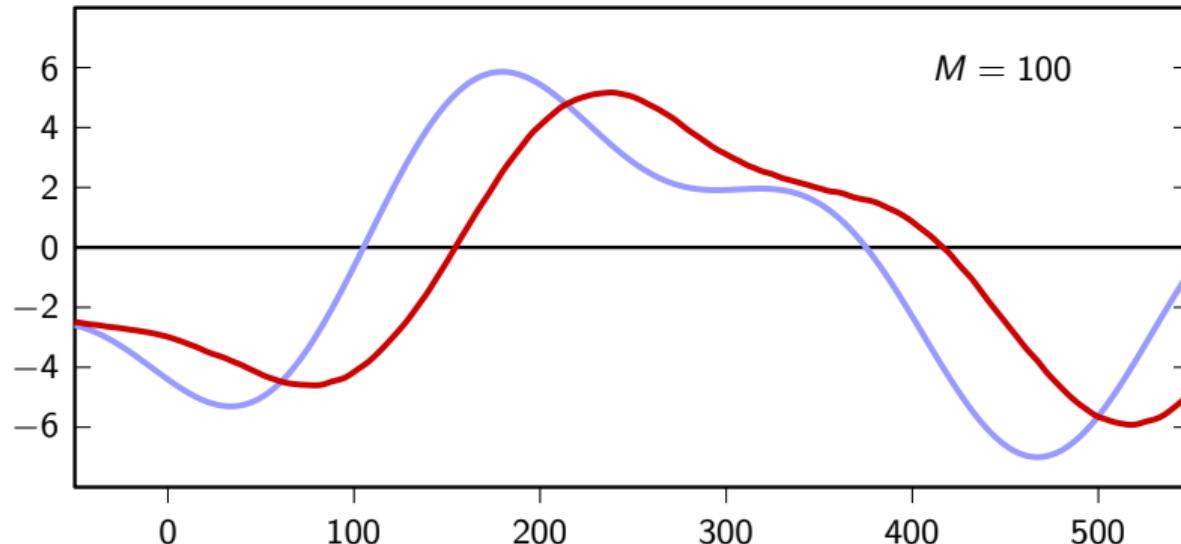
## Denoising by Moving Average



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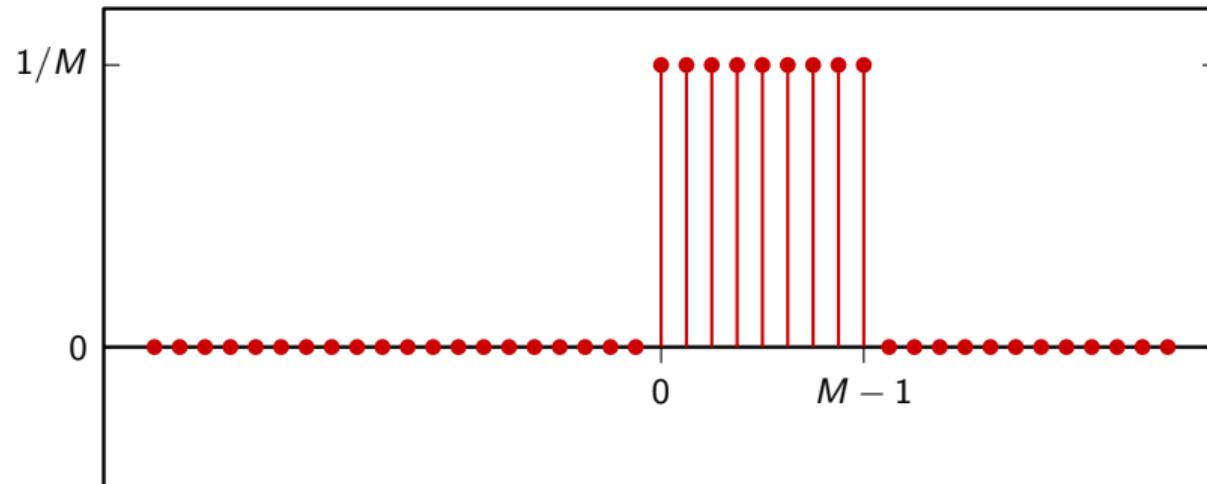
$M = 100$

$$h[n] = \frac{1}{M} \sum_{k=0}^{M-1} \delta[n - k]$$

$$= \begin{cases} 1/M & \text{for } 0 \leq n < M \\ 0 & \text{otherwise} \end{cases}$$

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- ▶ smoothing effect proportional to  $M$
- ▶ number of operations and storage also proportional to  $M$

$$y_M[n] = \frac{1}{M} (x[n] + x[n-1] + x[n-2] + \dots + x[n-M+1])$$

moving average over  $M$  points

$$y_M[n] = \frac{1}{M}x[n] + \frac{1}{M} (x[n-1] + x[n-2] + \dots + x[n-M+1])$$

$$y_M[n] = \frac{1}{M}x[n] + \frac{1}{M}(x[n-1] + x[n-2] + \dots + x[n-M+1])$$

“almost”  $y_{M-1}[n-1]$

i.e., moving average over  $M - 1$  points, delayed by one

Formally:

$$y_M[n] = \frac{1}{M} \sum_{k=0}^{M-1} x[n - k]$$

$$y_M[n - 1] = \frac{1}{M} \sum_{k=0}^{M-1} x[(n - 1) - k]$$

Formally:

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Formally:

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$$y_M[n - 1] = \frac{1}{M} \sum_{k=0}^{M-1} x[n - (k + 1)]$$

Formally:

$$y_M[n] = \frac{1}{M} \sum_{k=0}^{M-1} x[n - k]$$

$$y_M[n - 1] = \frac{1}{M} \sum_{k=0+1}^{M-1+1} x[n - k]$$

Formally:

$$y_M[n] = \frac{1}{M} \sum_{k=0}^{M-1} x[n - k]$$

$$y_M[n - 1] = \frac{1}{M} \sum_{k=1}^M x[n - k]$$

Formally:

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$$y_M[n-1] = \frac{1}{M} \sum_{k=1}^M x[n - k]$$

$$y_{M-1}[n] = \frac{1}{M-1} \sum_{k=0}^{M-2} x[n - k]$$

Formally:

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$$y_{M-1}[n-1] = \frac{1}{M-1} \sum_{k=1}^{M-1} x[n - k]$$

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$$y_{M-1}[n-1] = \frac{1}{M-1} \sum_{k=1}^{M-1} x[n-k]$$

$$\sum_{k=0}^{M-1} x[n-k] = x[n] + \sum_{k=1}^{M-1} x[n-k]$$

$$My_M[n] = x[n] + (M-1)y_{M-1}[n-1]$$

$$y_M[n] = \frac{1}{M} \sum_{k=0}^{M-1} x[n-k] \quad y_{M-1}[n-1] = \frac{1}{M-1} \sum_{k=1}^{M-1} x[n-k]$$

$$\sum_{k=0}^{M-1} x[n-k] = x[n] + \sum_{k=1}^{M-1} x[n-k]$$

$$My_M[n] = x[n] + (M-1)y_{M-1}[n-1]$$

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- ▶ filter is now recursive, since it uses its previous output value

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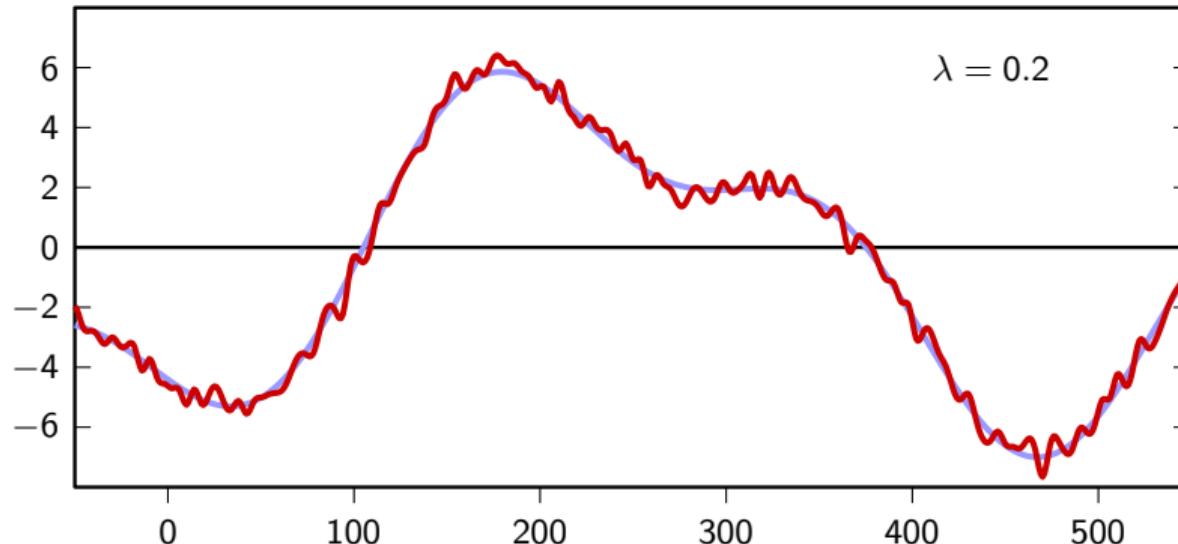
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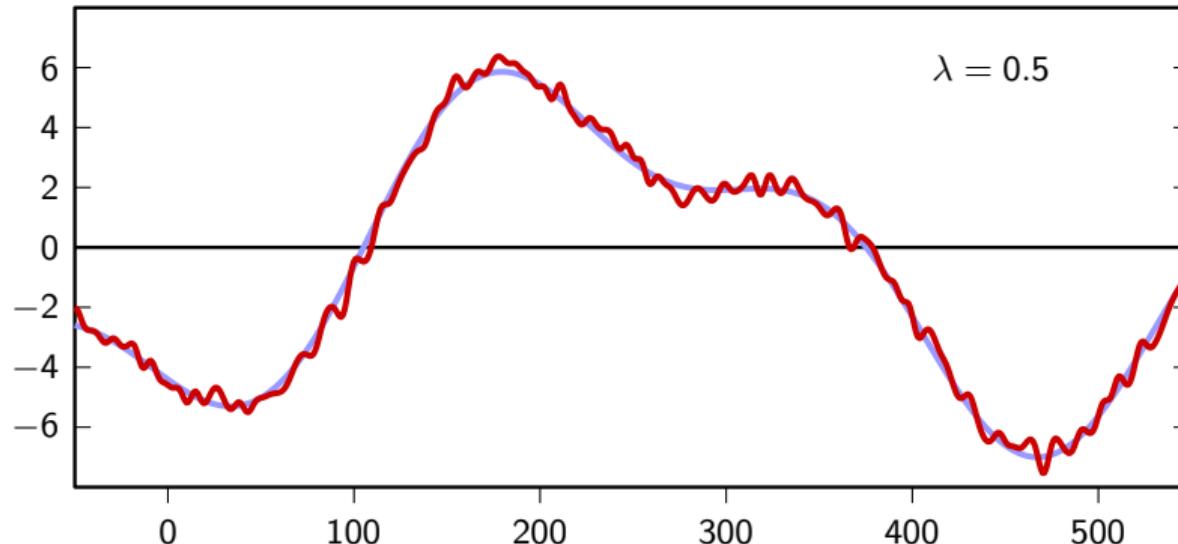
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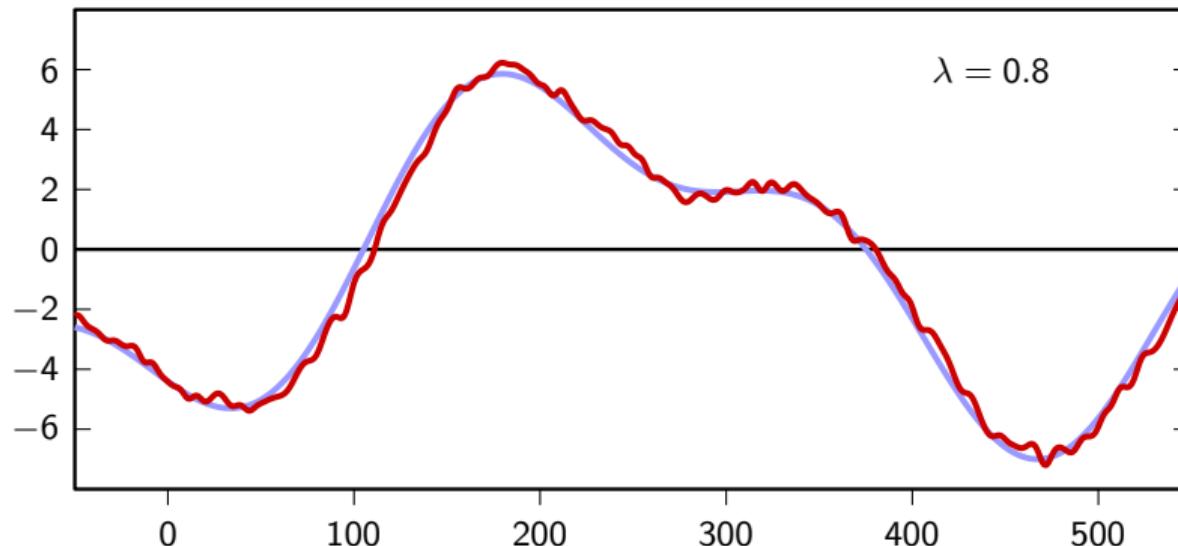
# Denoising recursively with the Leaky Integrator



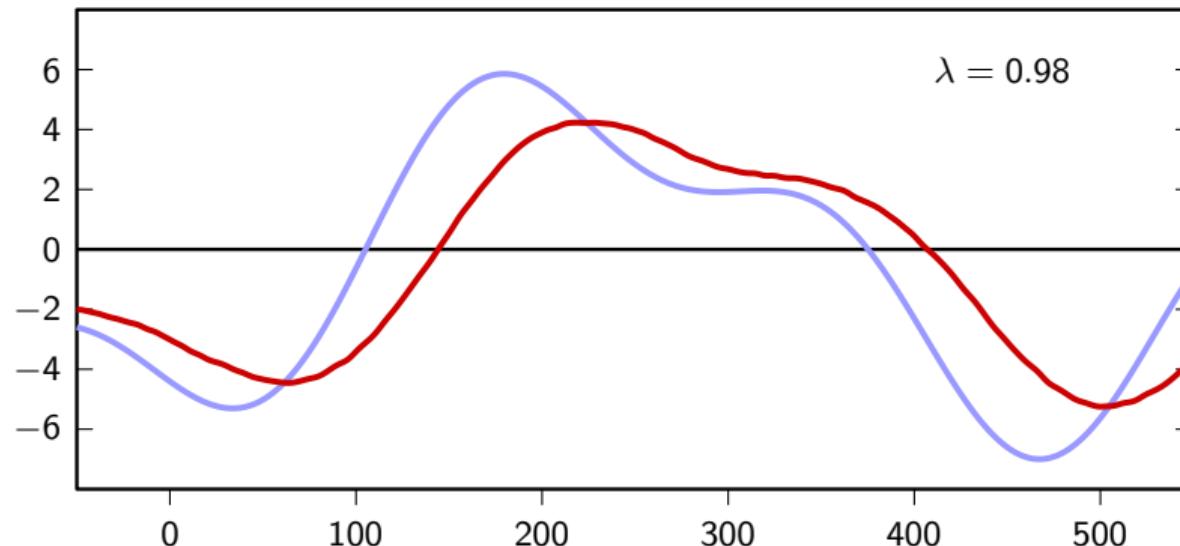
# Denoising recursively with the Leaky Integrator



# Denoising recursively with the Leaky Integrator



# Denoising recursively with the Leaky Integrator



## What about the impulse response?

$$y[n] = \lambda y[n-1] + (1 - \lambda) \delta[n]$$

- ▶  $y[n] = 0$  for all  $n < 0$
- ▶  $y[0] = \lambda y[-1] + (1 - \lambda) \delta[0] = (1 - \lambda)$
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$$y[n] = \lambda y[n - 1] + (1 - \lambda)\delta[n]$$

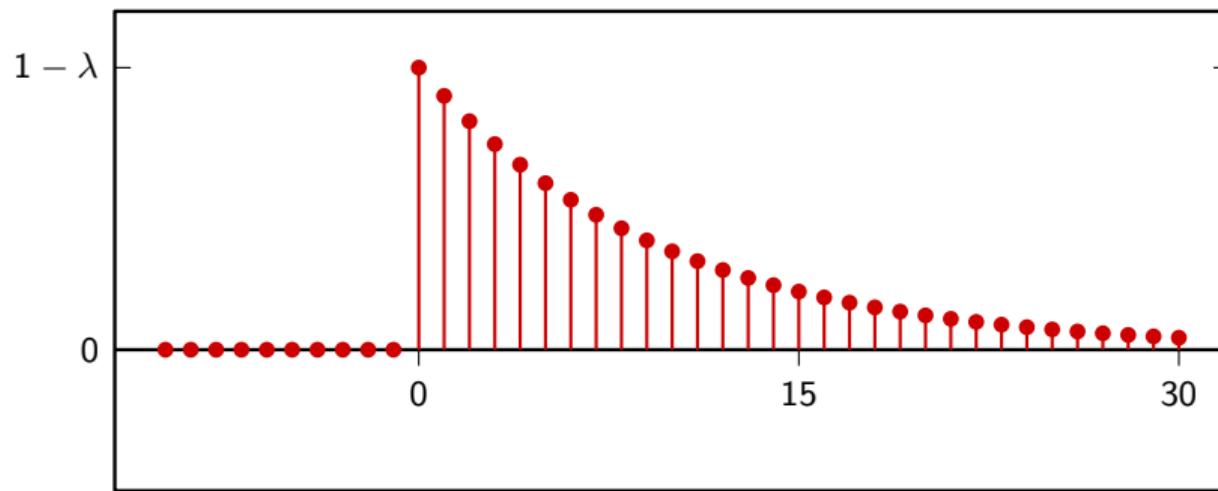
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- ▶ ...

$$h[n] = (1 - \lambda)\lambda^n u[n]$$



Discrete-time integrator is a boundless accumulator:

$$y[n] = \sum_{k=-\infty}^n x[k]$$

We can rewrite the integrator as

$$y[n] = y[n - 1] + x[n]$$

To prevent “explosion” pick  $\lambda < 1$

$$y[n] = \lambda y[n - 1] + (1 - \lambda)x[n]$$

keep only a fraction  $\lambda$  of  
the accumulated value  
so far and forget  
("leak") a fraction  $1 - \lambda$

add only a fraction  $1 - \lambda$   
of the current value to  
the accumulator

END OF MODULE 5.2

# Digital Signal Processing

Module 5.3: Filter Stability

- ▶ Filter classification in the time domain
- ▶ Filter stability

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- ▶ Finite Impulse Response (FIR)
- ▶ Infinite Impulse Response (IIR)
- ▶ causal
- ▶ noncausal

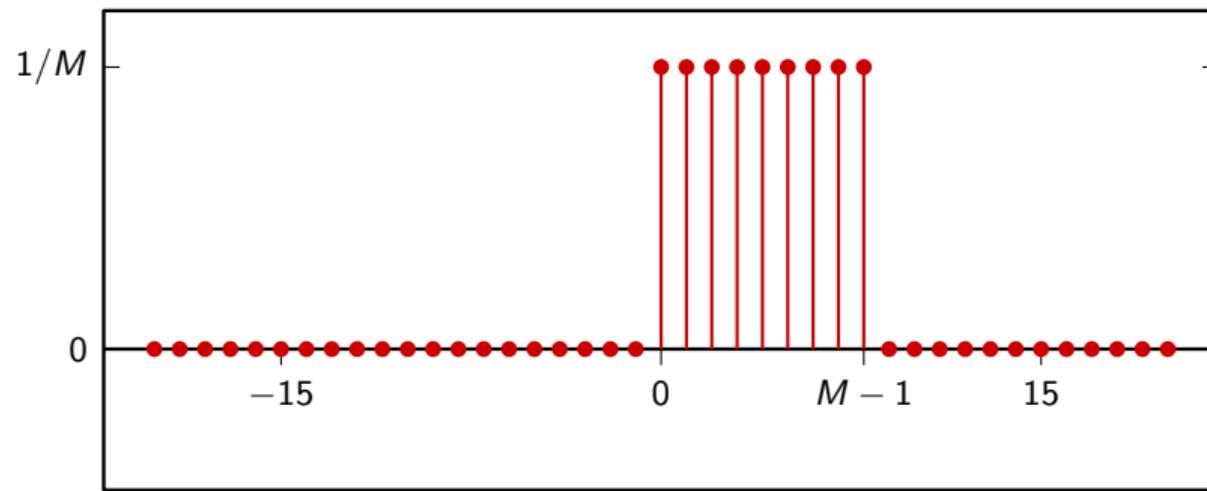
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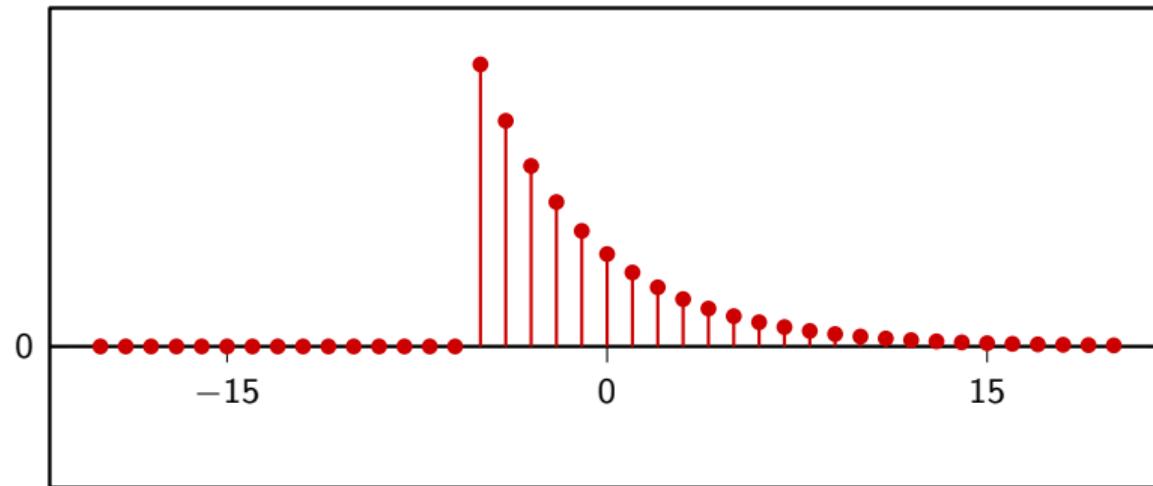
- ▶ impulse response has finite support
- ▶ only a finite number of samples are involved in the computation of each output sample

Moving Average filter



- ▶ impulse response has infinite support
- ▶ a potentially infinite number of samples are involved in the computation of each output sample
- ▶ surprisingly, in many cases the computation can still be performed in a finite amount of steps

Leaky Integrator



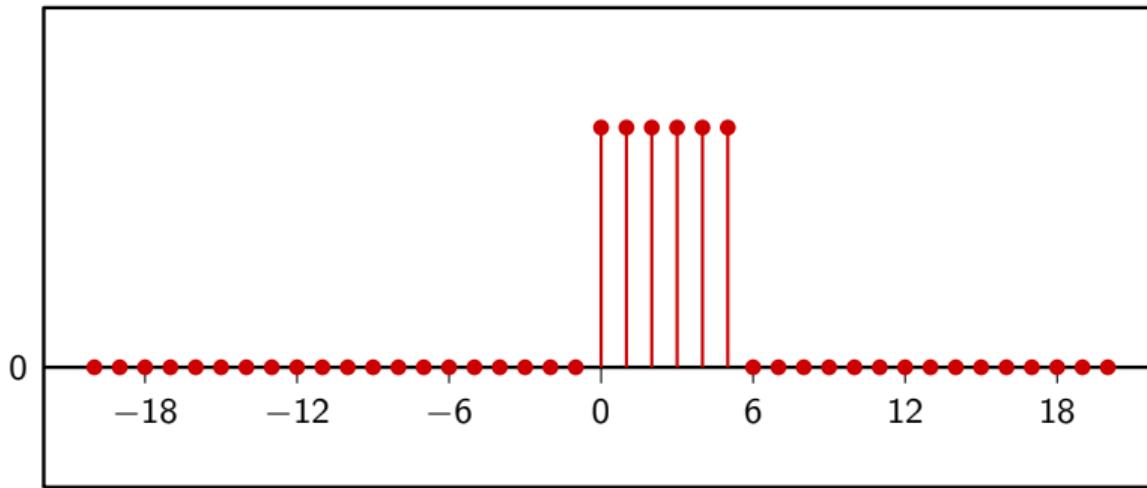
► causal:

- impulse response is zero for  $n < 0$
- only past samples (with respect to the present) are involved in the computation of each output sample
- causal filters can work “on line” since they only need the past

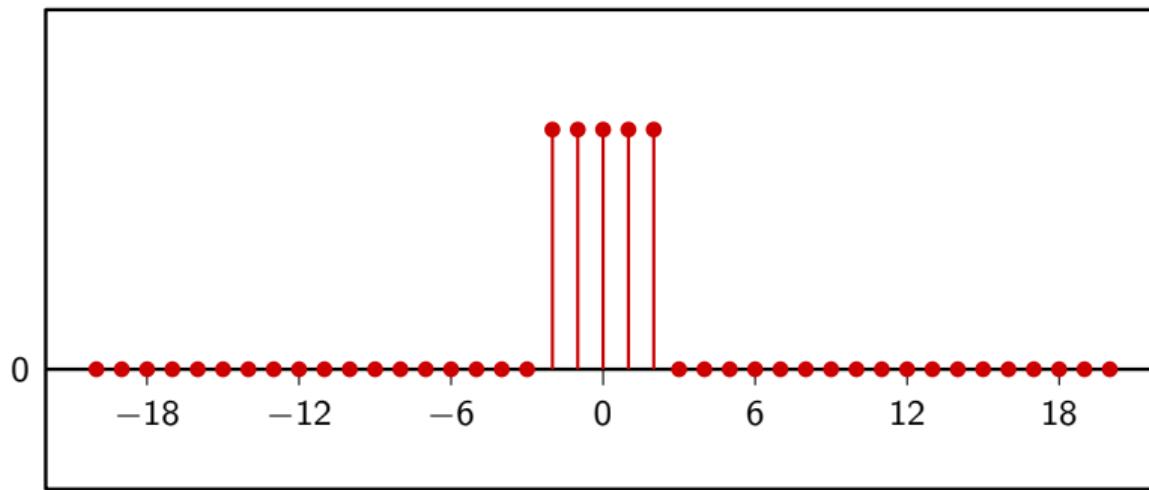
► noncausal:

- impulse response is nonzero for some (or all)  $n < 0$
- can still be implemented in an offline fashion (when all input data is available on storage, e.g. in Image Processing)

## Moving Average filter



## Zero-centered Moving Average filter



- ▶ key concept: avoid “explosions” if the input is nice
- ▶ a nice signal is a bounded signal:  $|x[n]| < M$  for all  $n$
- ▶ Bounded-Input Bounded-Output (BIBO) stability: if the input is nice the output should be nice

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A filter is BIBO stable if and only if its impulse response is absolutely summable

Proof:

Hypotheses:

- ▶  $|x[n]| < M$
- ▶  $\sum_n |h[n]| = L < \infty$

Thesis:

- ▶  $|y[n]|$  bounded

$$\begin{aligned}|y[n]| &= \left| \sum_{k=-\infty}^{\infty} h[k]x[n-k] \right| \\ &\leq \sum_{k=-\infty}^{\infty} |h[k]x[n-k]| \\ &\leq M \sum_{k=-\infty}^{\infty} |h[k]| \\ &\leq ML\end{aligned}$$

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Proof (by contradiction):

Hypotheses:

- ▶  $|x[n]| < M$
- ▶  $|y[n]| < P$

Thesis:

- ▶  $h[n]$  absolutely summable

▶ assume  $\sum_n |h[n]| = \infty$ ▶ build  $x[n] = \begin{cases} +1 & \text{if } h[-n] \geq 0 \\ -1 & \text{if } h[-n] < 0 \end{cases}$ ▶ clearly,  $x[n]$  is bounded

▶ however

$$y[0] = (x * h)[0] = \sum_{k=-\infty}^{\infty} h[k]x[-k] = \sum_{k=-\infty}^{\infty} |h[k]| = \infty$$

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FIR filters are always stable

Let's check the Leaky Integrator:

$$\begin{aligned}\sum_{n=-\infty}^{\infty} |h[n]| &= |1 - \lambda| \sum_{n=0}^{\infty} |\lambda|^n \\ &= \lim_{n \rightarrow \infty} |1 - \lambda| \frac{1 - |\lambda|^{n+1}}{1 - |\lambda|} \\ &< \infty \quad \text{for } |\lambda| < 1\end{aligned}$$

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We will study indirect methods for filter stability later in this Module.

END OF MODULE 5.3

# Digital Signal Processing

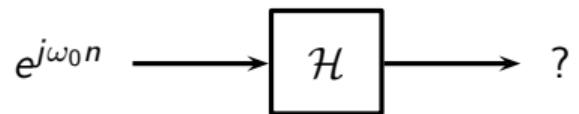
Module 5.4: Frequency Response

- ▶ Eigensequences
- ▶ Convolution theorem
- ▶ Frequency and phase response

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# A remarkable result



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$$\begin{aligned}y[n] &= e^{j\omega_0 n} * h[n] \\&= h[n] * e^{j\omega_0 n} \\&= \sum_{k=-\infty}^{\infty} h[k] e^{j\omega_0(n-k)} \\&= e^{j\omega_0 n} \sum_{k=-\infty}^{\infty} h[k] e^{-j\omega_0 k} \\&= H(e^{j\omega_0}) e^{j\omega_0 n}\end{aligned}$$

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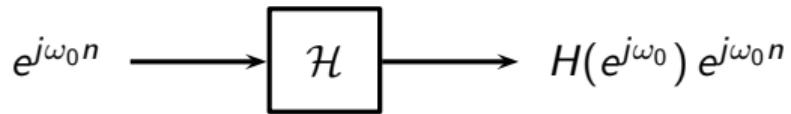
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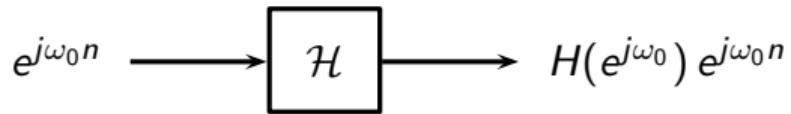
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- ▶ complex exponentials are *eigensequences* of LTI systems, i.e., linear filters cannot change the frequency of sinusoids
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- ▶ DTFT of impulse response determines the frequency characteristic of a filter

If  $H(e^{j\omega_0}) = Ae^{j\theta}$ , then

$$\mathcal{H}\{e^{j\omega_0 n}\} = Ae^{j(\omega_0 n + \theta)}$$

amplitude:

amplification ( $A > 1$ )

or attenuation ( $0 \leq A < 1$ )



phase shift:

delay ( $\theta < 0$ )

or advancement ( $\theta > 0$ )

In general:

$$\text{DTFT } \{x[n] * h[n]\} = ?$$

Intuition: the DTFT reconstruction formula tells us how to build  $x[n]$  from a set of complex exponential “basis” functions. By linearity...

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# The convolution theorem

$$\begin{aligned}\text{DTFT } \{x[n] * h[n]\} &= \sum_{n=-\infty}^{\infty} (x * h)[n] e^{-j\omega n} \\&= \sum_{n=-\infty}^{\infty} \sum_{k=-\infty}^{\infty} x[k] h[n-k] e^{-j\omega n} \\&= \sum_{n=-\infty}^{\infty} \sum_{k=-\infty}^{\infty} x[k] h[n-k] e^{-j\omega(n-k)} e^{-j\omega k} \\&= \sum_{k=-\infty}^{\infty} x[k] e^{-j\omega k} \sum_{n=-\infty}^{\infty} h[n-k] e^{-j\omega(n-k)} \\&= H(e^{j\omega}) X(e^{j\omega})\end{aligned}$$

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# The convolution theorem

$$\begin{aligned}\text{DTFT } \{x[n] * h[n]\} &= \sum_{n=-\infty}^{\infty} (x * h)[n] e^{-j\omega n} \\&= \sum_{n=-\infty}^{\infty} \sum_{k=-\infty}^{\infty} x[k] h[n-k] e^{-j\omega n} \\&= \sum_{n=-\infty}^{\infty} \sum_{k=-\infty}^{\infty} x[k] h[n-k] e^{-j\omega(n-k)} e^{-j\omega k} \\&= \sum_{k=-\infty}^{\infty} x[k] e^{-j\omega k} \sum_{n=-\infty}^{\infty} h[n-k] e^{-j\omega(n-k)} \\&= H(e^{j\omega}) X(e^{j\omega})\end{aligned}$$

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$$H(e^{j\omega}) = \text{DTFT} \{ h[n] \}$$

Two effects:

- ▶ **magnitude:** amplification ( $|H(e^{j\omega})| > 1$ ) or attenuation ( $|H(e^{j\omega})| < 1$ ) of input frequencies
- ▶ **phase:** overall delay and shape changes

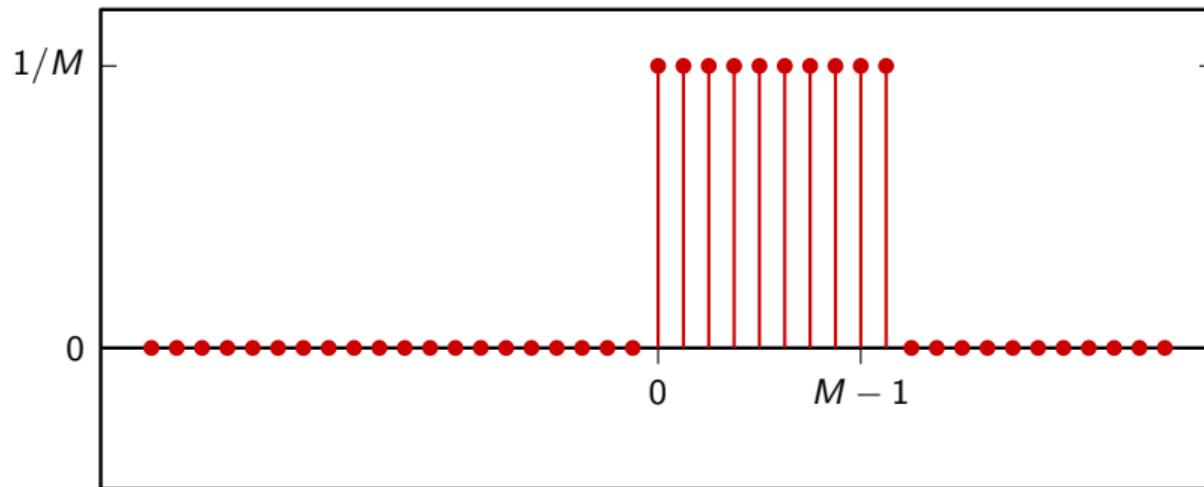
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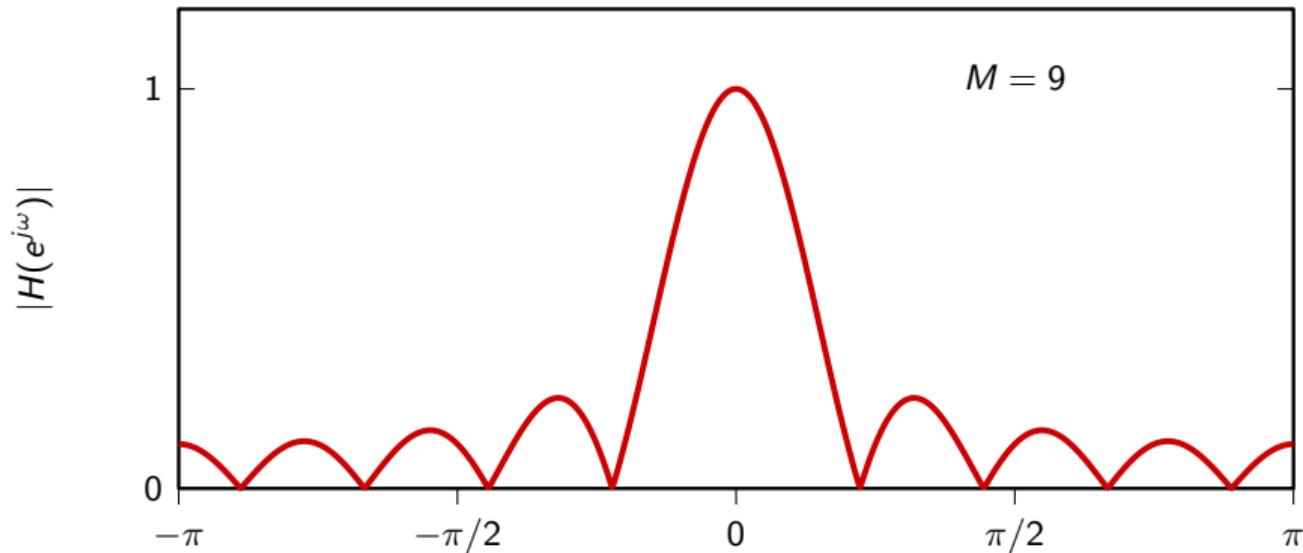
# Moving Average revisited

$$h[n] = (u[n] - u[n - M])/M$$



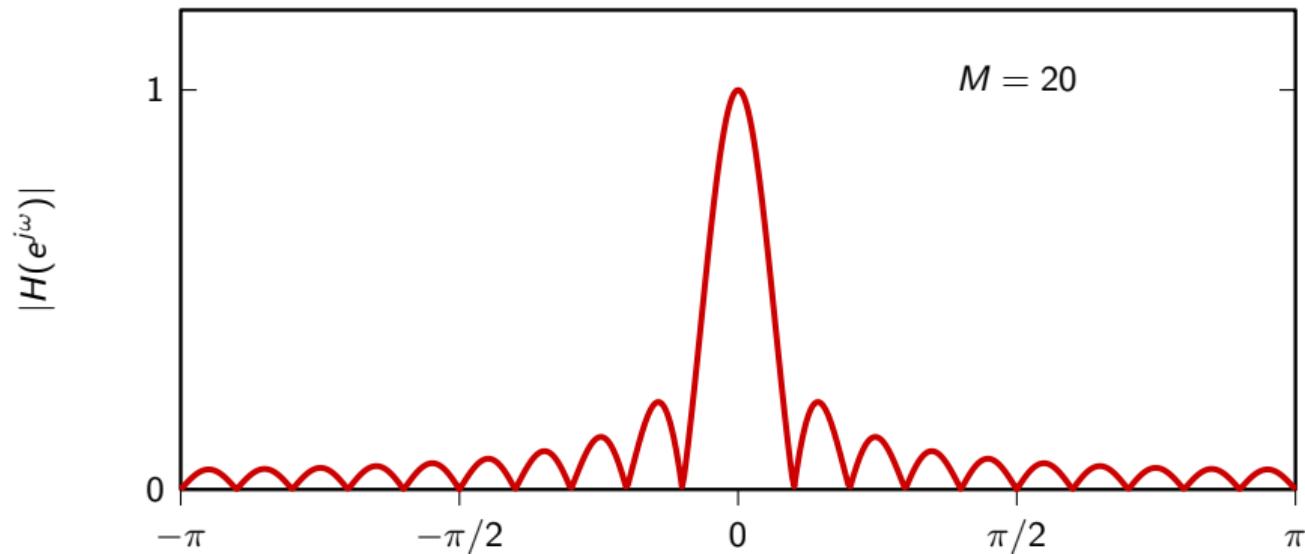
# Moving Average, magnitude response

$$|H(e^{j\omega})| = \frac{1}{M} \left| \frac{\sin(\frac{\omega}{2}M)}{\sin(\frac{\omega}{2})} \right| \quad (\text{see Module 4.7})$$



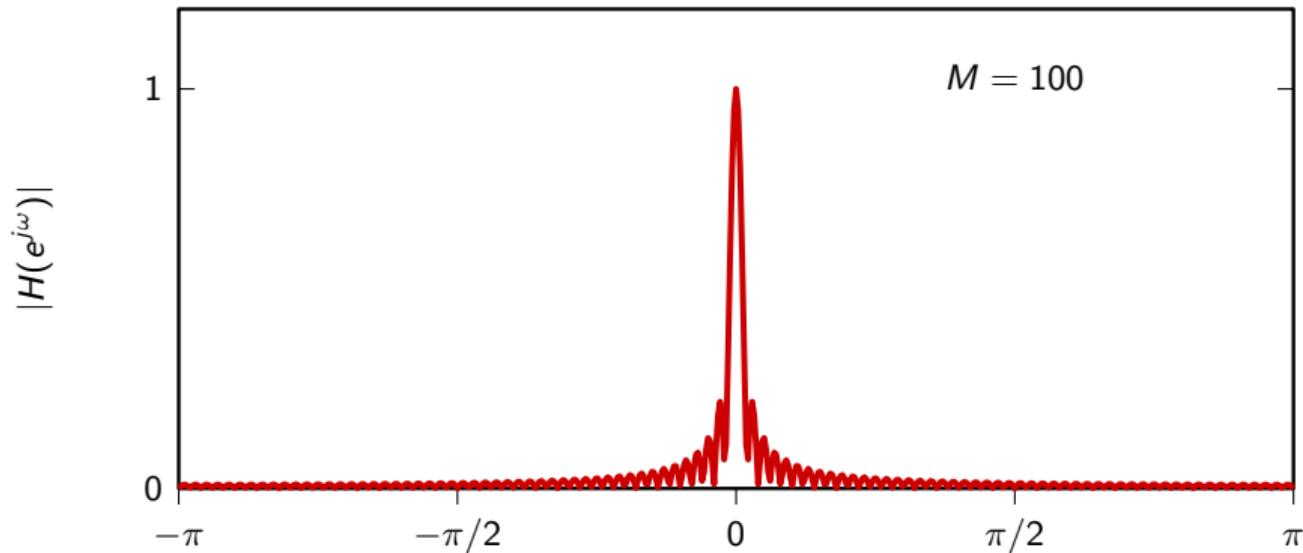
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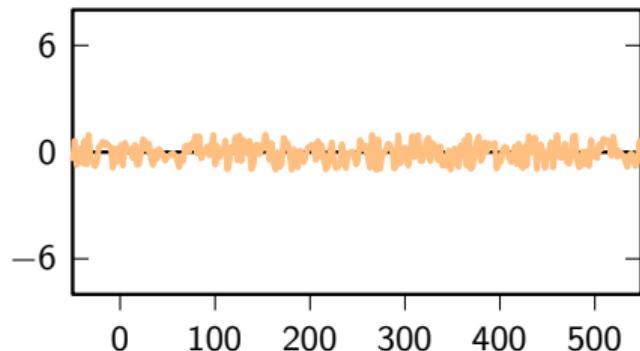
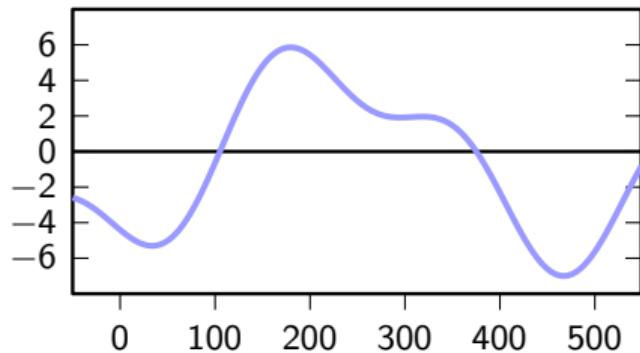


# Moving Average, magnitude response

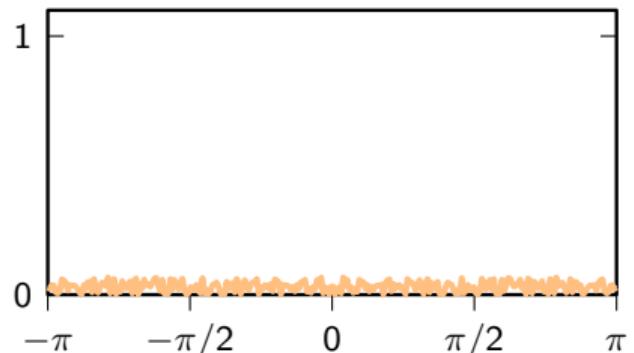
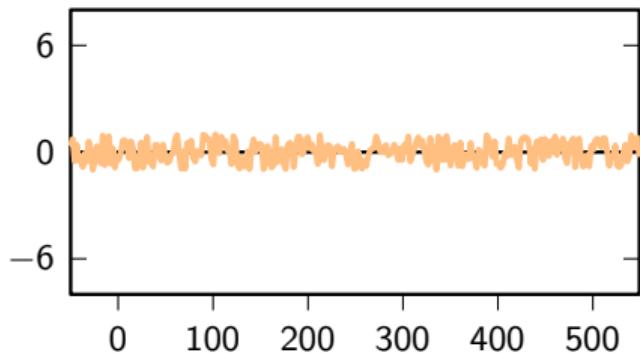
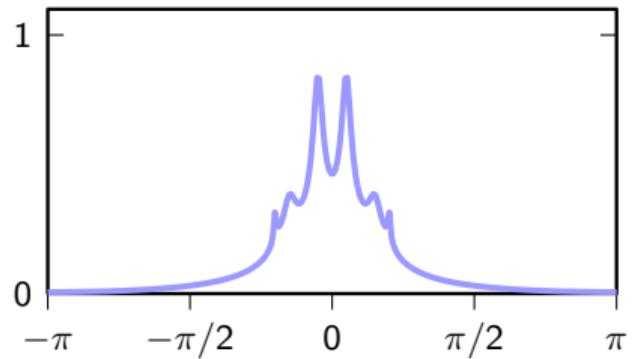
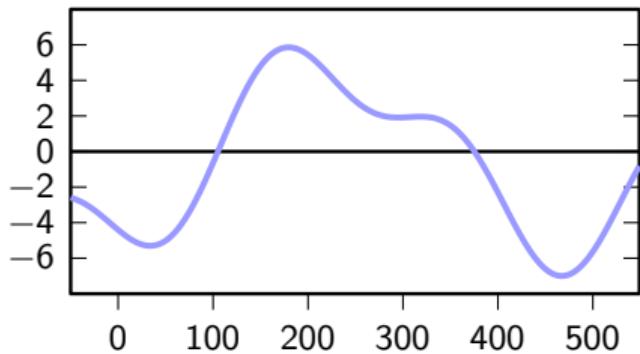
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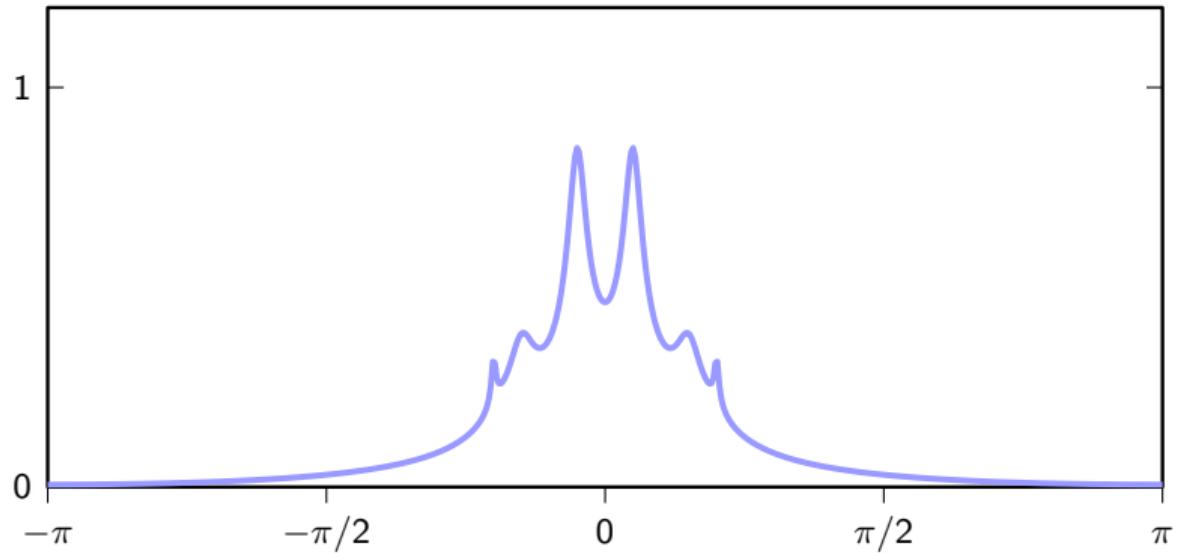


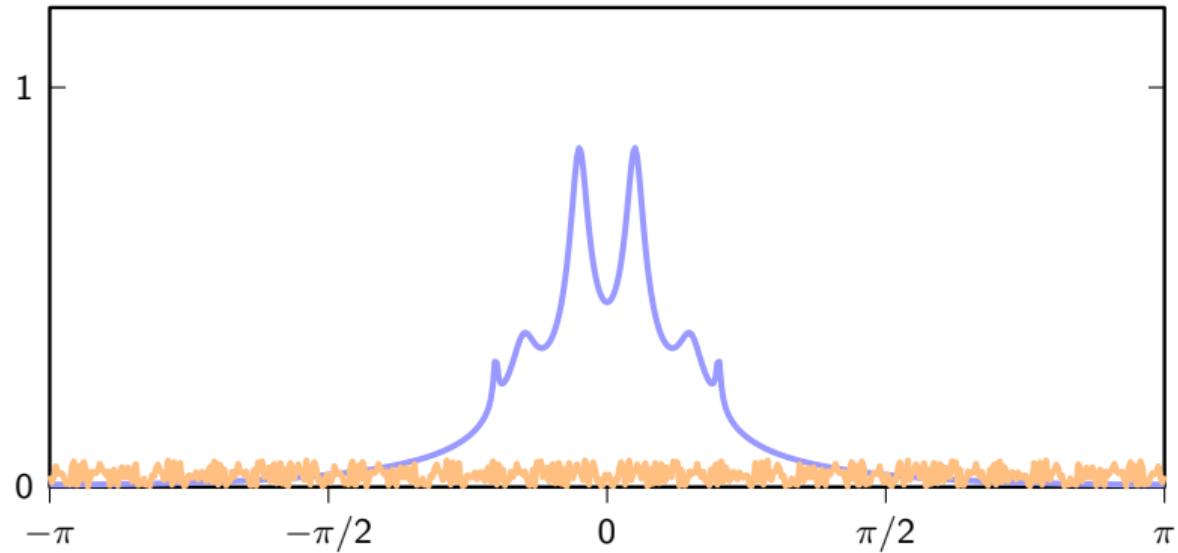
# Denoising revisited

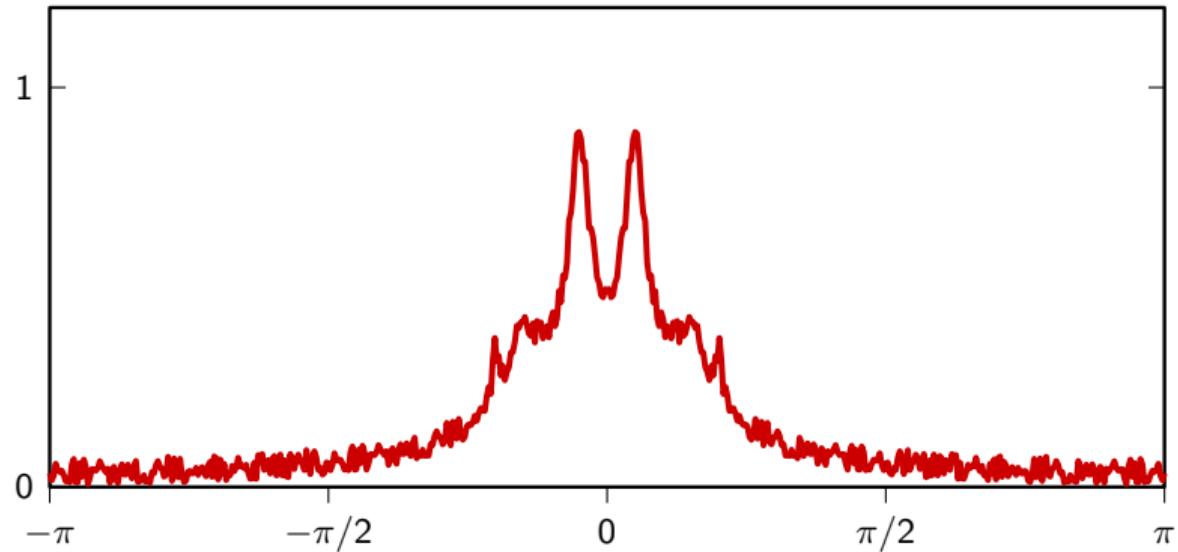


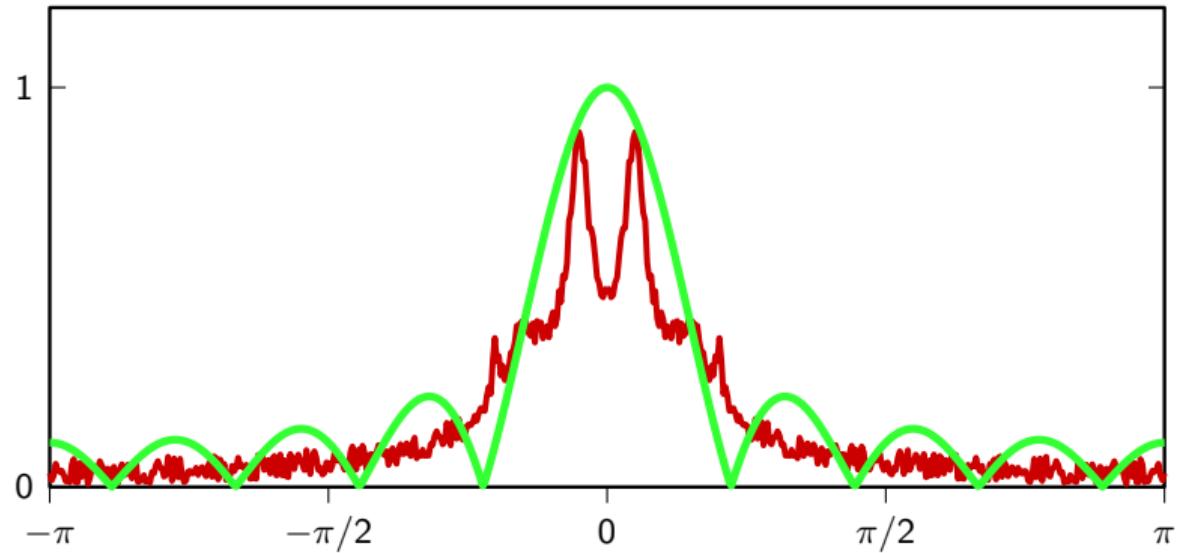
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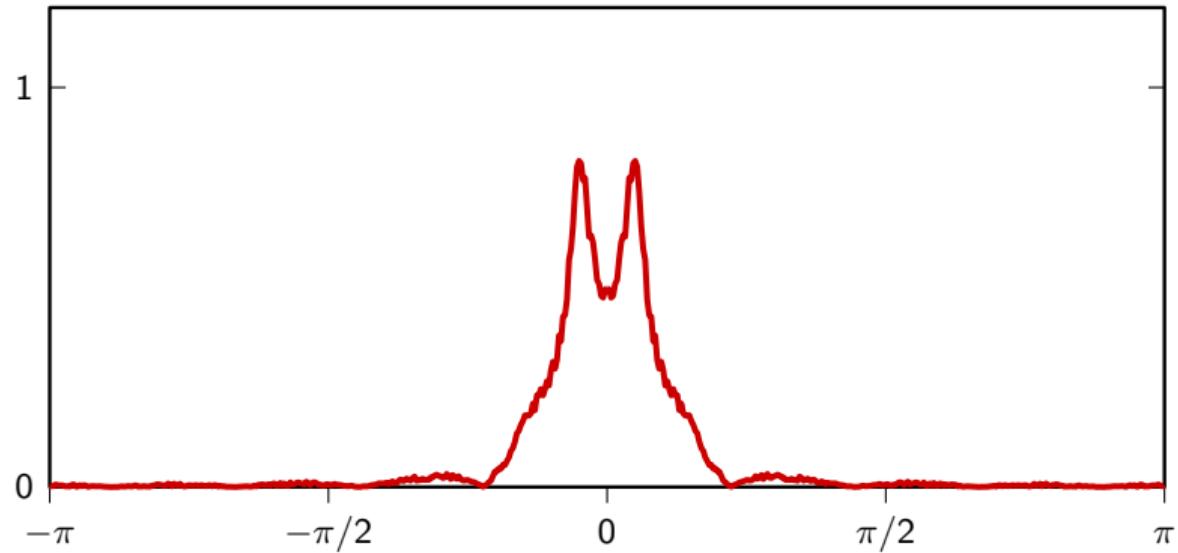


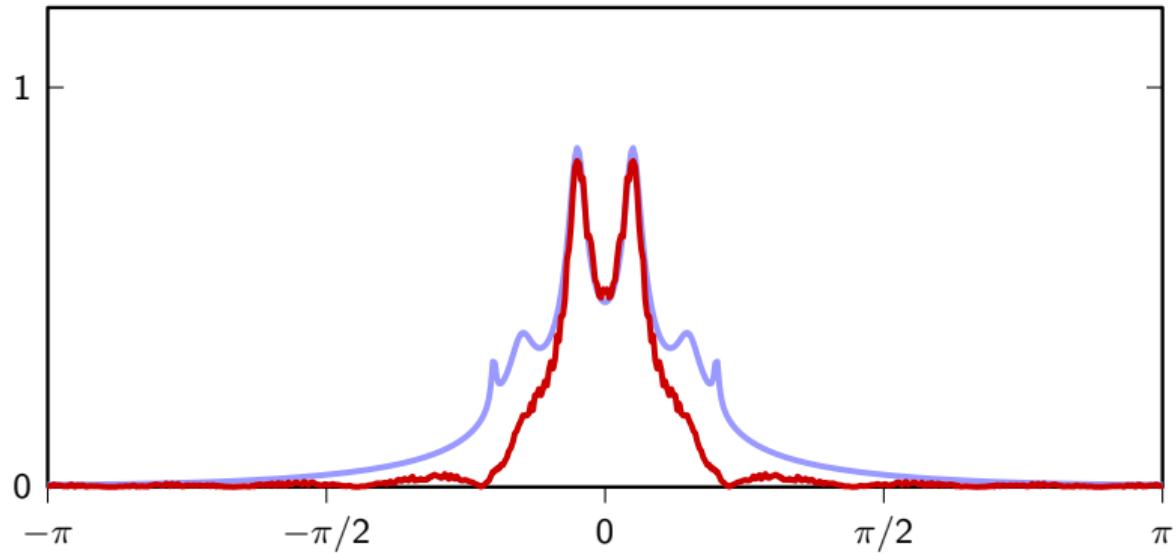




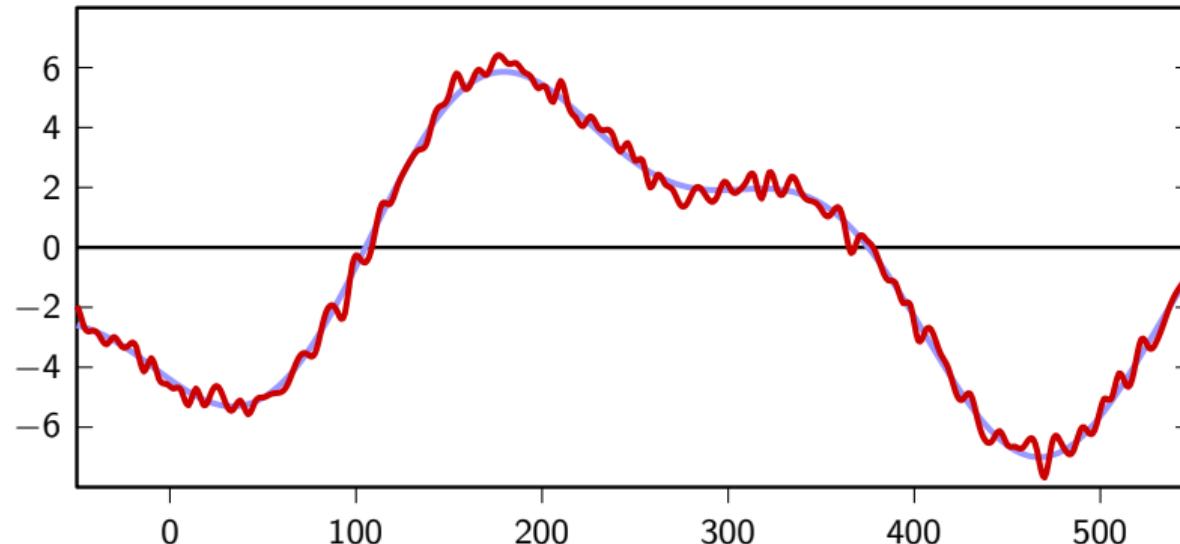




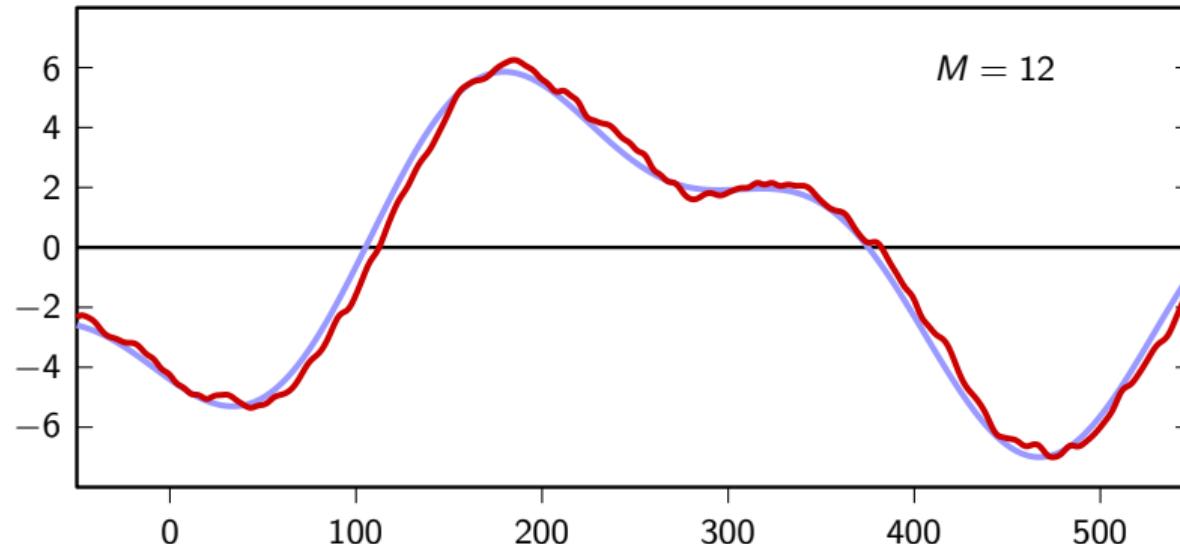




By the way, remember the time-domain analysis...



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# What about the phase?

Assume  $|H(e^{j\omega})| = 1$

- ▶ zero phase:  $\angle H(e^{j\omega}) = 0$
- ▶ linear phase:  $\angle H(e^{j\omega}) = d\omega$
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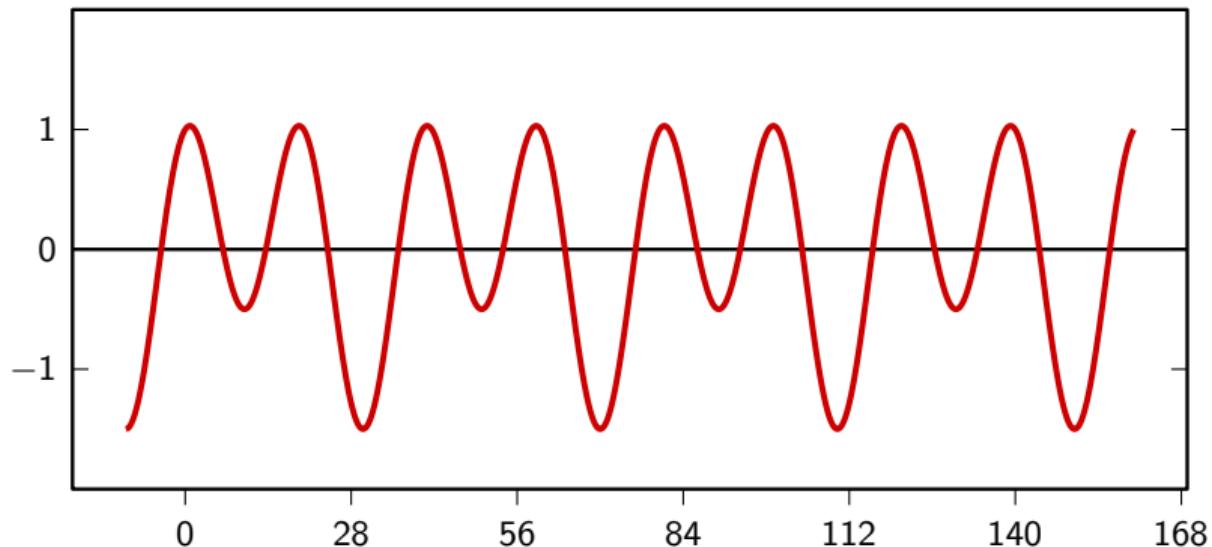
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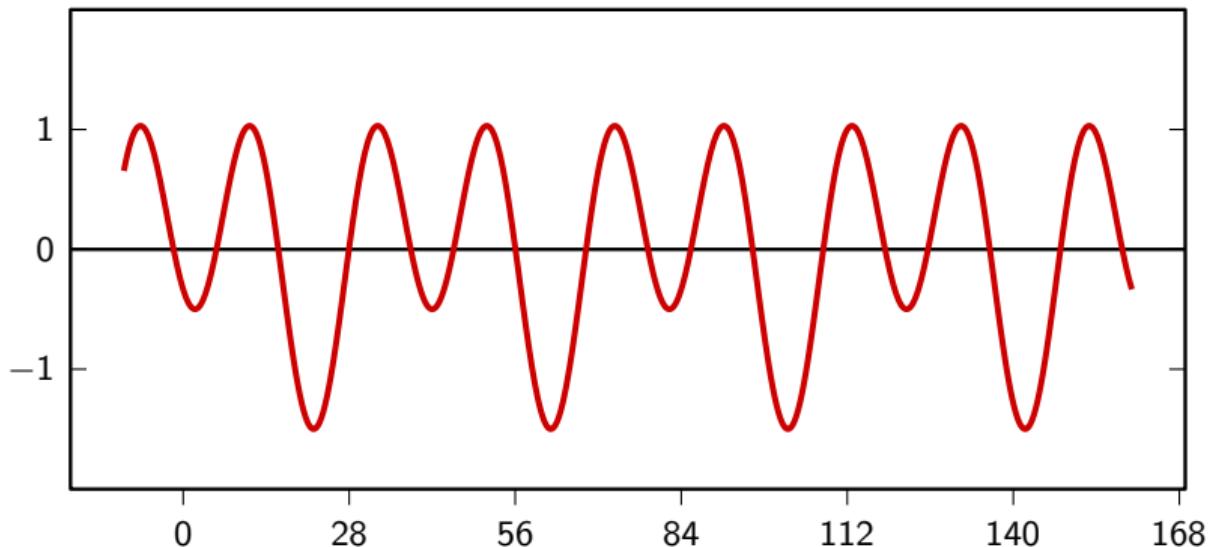
## Phase and signal shape

$$x[n] = \frac{1}{2} \sin(\omega_0 n) + \cos(2\omega_0 n) \quad \omega_0 = \frac{2\pi}{40}$$



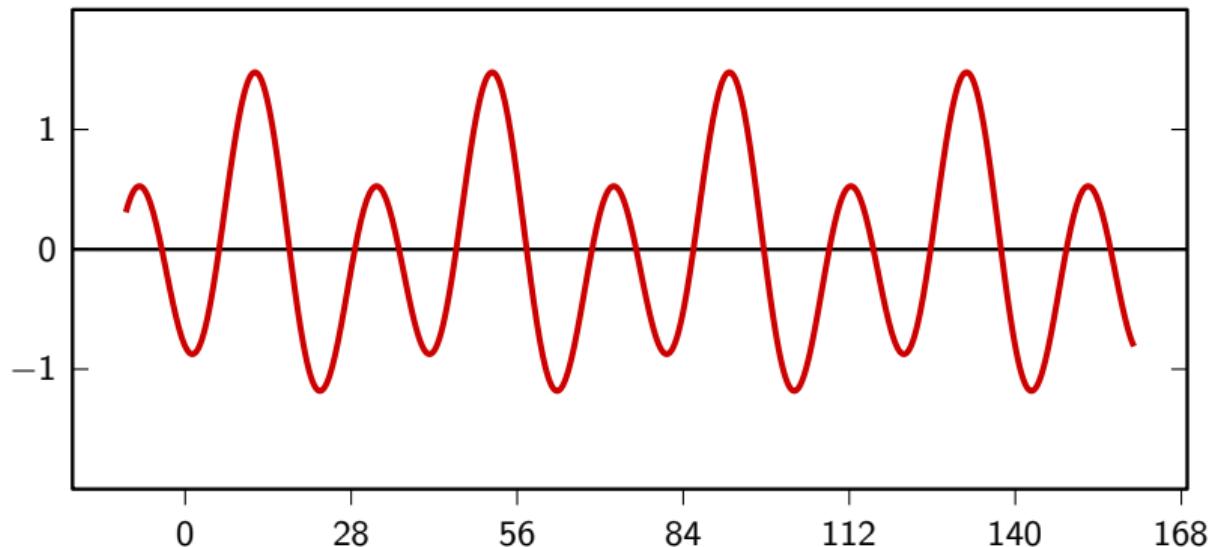
## Phase and signal shape: linear phase

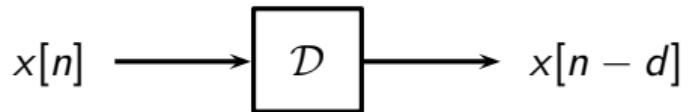
$$x[n] = \frac{1}{2} \sin(\omega_0 n + \theta_0) + \cos(2\omega_0 n + 2\theta_0) \quad \theta_0 = \frac{8\pi}{5}$$



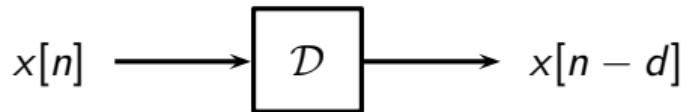
## Phase and signal shape: nonlinear phase

$$x[n] = \frac{1}{2} \sin(\omega_0 n) + \cos(2\omega_0 n + 2\theta_0)$$

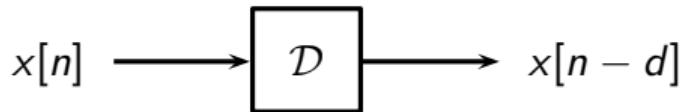




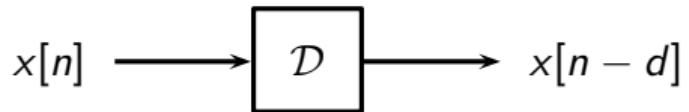
- ▶  $y[n] = x[n - d]$
- ▶  $Y(e^{j\omega}) = e^{-j\omega d} X(e^{j\omega})$
- ▶  $H(e^{j\omega}) = e^{-j\omega d}$
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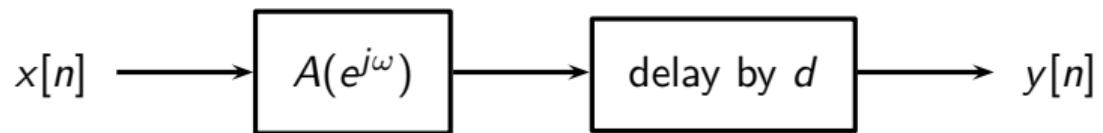


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- ▶  $H(e^{j\omega}) = e^{-j\omega d}$
- ▶ linear phase term

In general, if  $H(e^{j\omega}) = A(e^{j\omega})e^{-j\omega d}$ , with  $A(e^{j\omega}) \in \mathbb{R}$

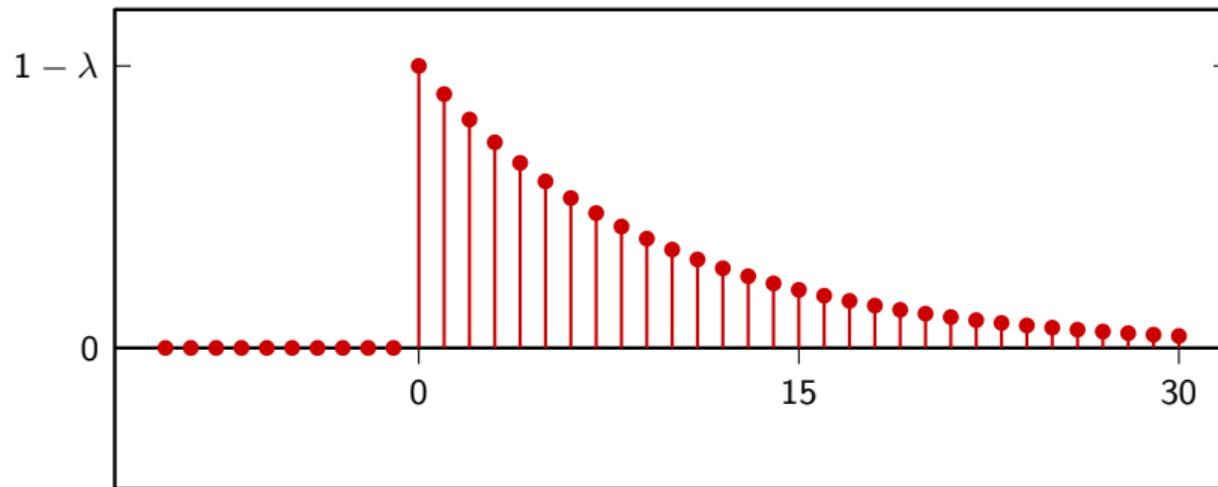


# Moving Average is linear phase

$$H(e^{j\omega}) = \frac{1}{M} \frac{\sin\left(\frac{\omega}{2}M\right)}{\sin\left(\frac{\omega}{2}\right)} e^{-j\frac{M-1}{2}\omega}$$

# Leaky integrator revisited

$$h[n] = (1 - \lambda)\lambda^n u[n]$$



$$H(e^{j\omega}) = \frac{1 - \lambda}{1 - \lambda e^{-j\omega}} \quad (\text{Module 4.4})$$

Finding magnitude and phase require a little algebra...

Recall from complex algebra:

$$\frac{1}{a+jb} = \frac{a-jb}{a^2+b^2}$$

so that if  $x = 1/(a+jb)$ ,

$$|x|^2 = \frac{1}{a^2+b^2}$$

$$\angle x = \tan^{-1} \left[ -\frac{b}{a} \right]$$

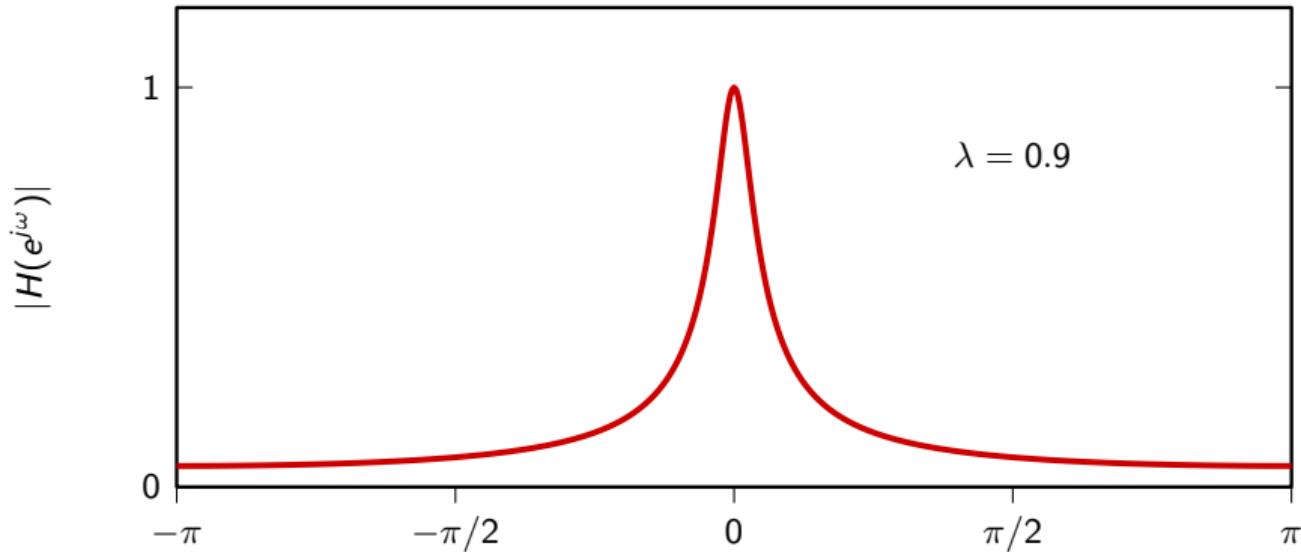
$$H(e^{j\omega}) = \frac{1 - \lambda}{(1 - \lambda \cos \omega) - j\lambda \sin \omega}$$

so that:

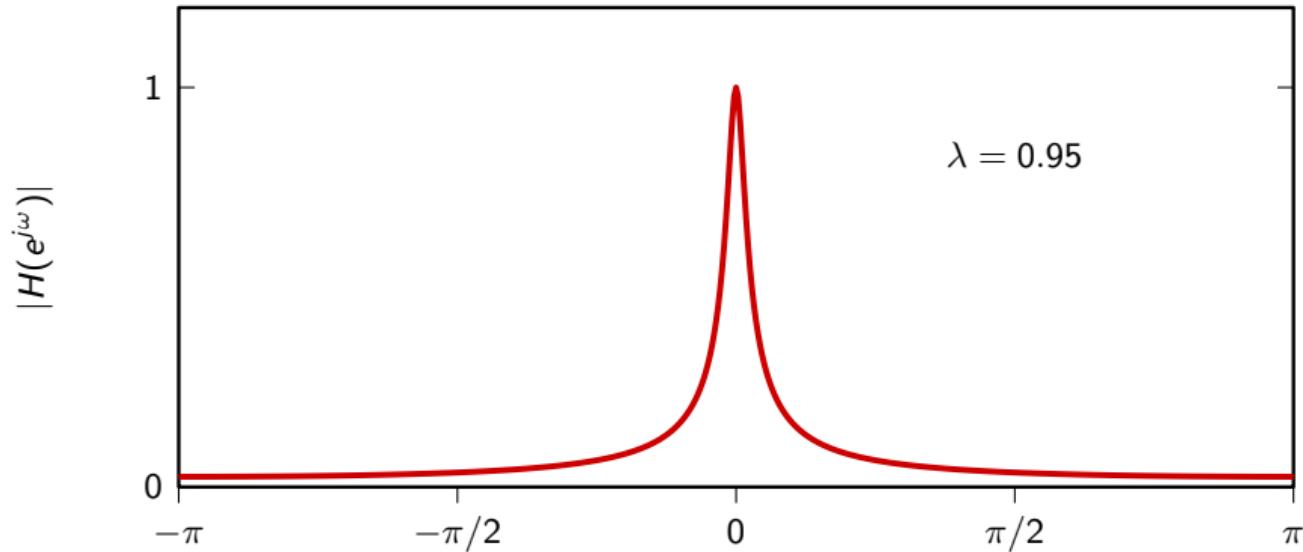
$$|H(e^{j\omega})|^2 = \frac{(1 - \lambda)^2}{1 - 2\lambda \cos \omega + \lambda^2}$$

$$\angle H(e^{j\omega}) = \tan^{-1} \left[ \frac{\lambda \sin \omega}{1 - \lambda \cos \omega} \right]$$

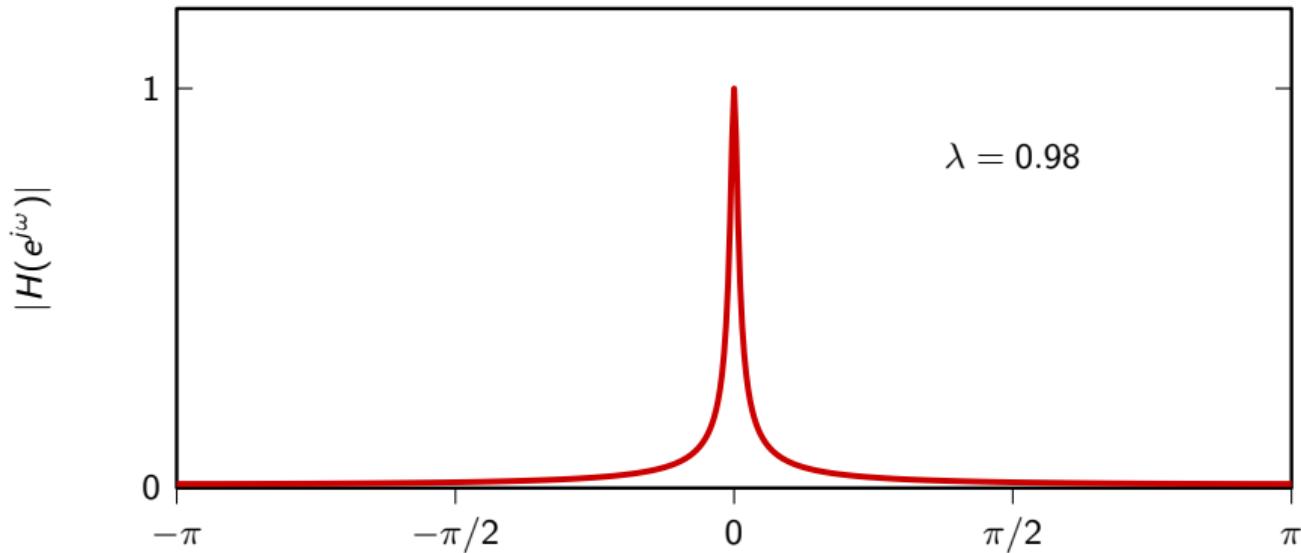
# Leaky integrator, magnitude response



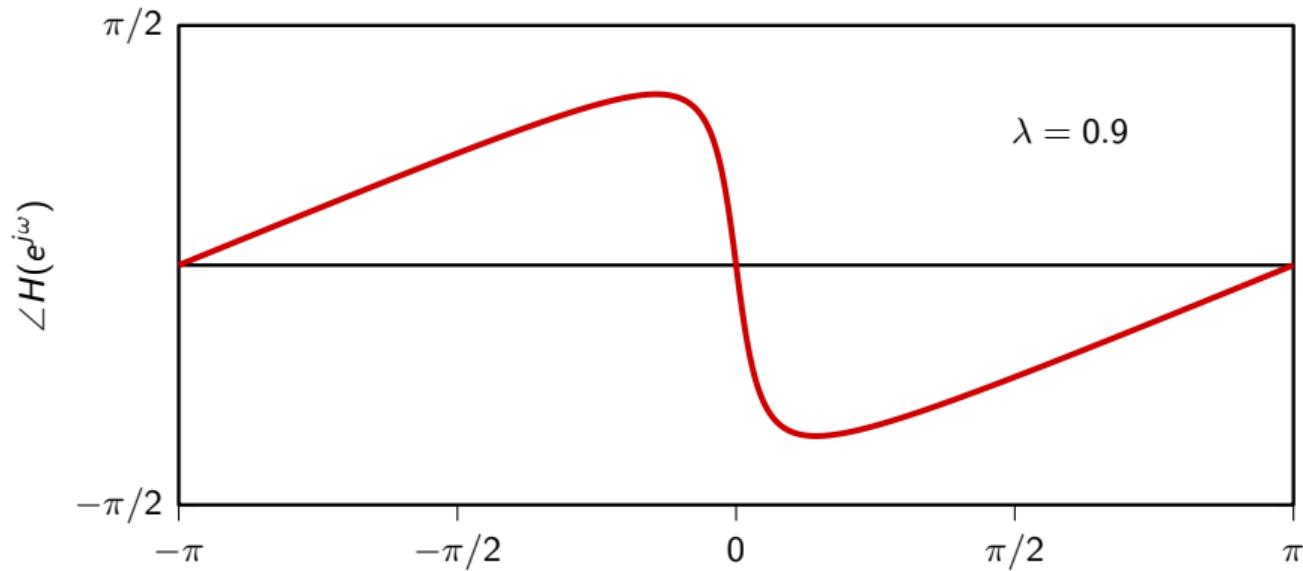
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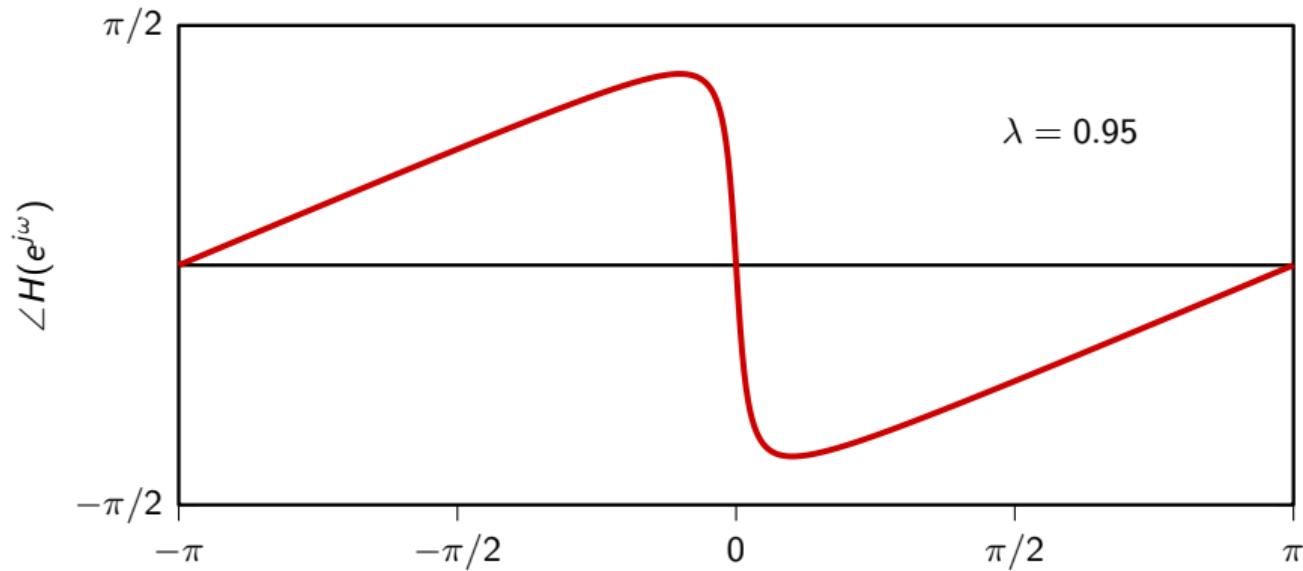
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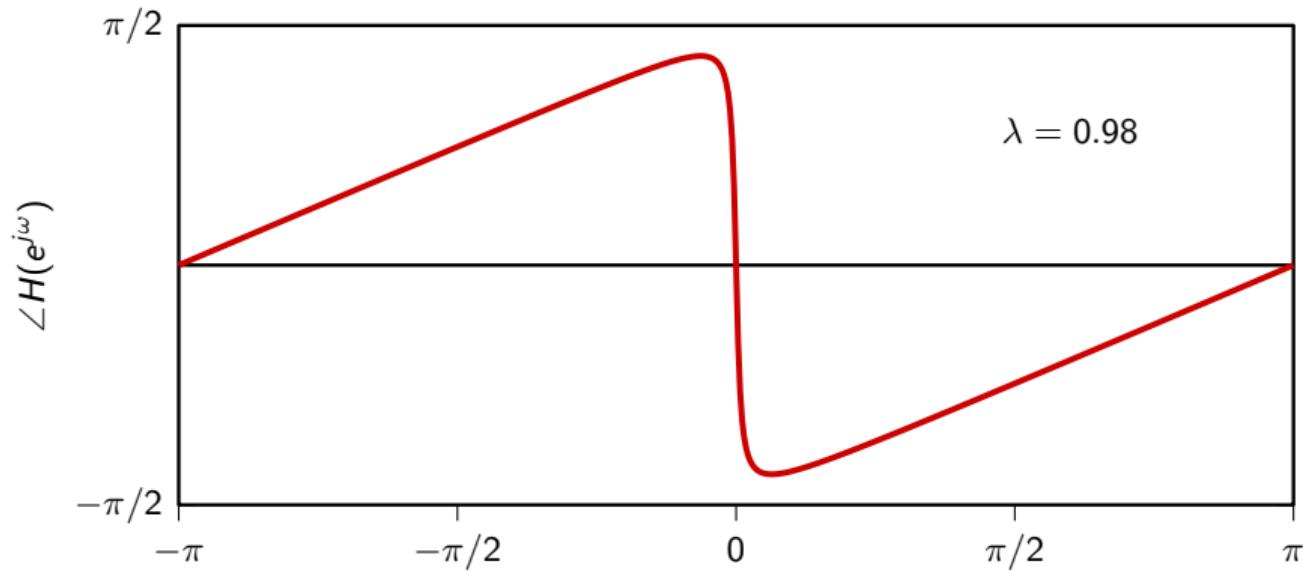
# Leaky integrator, phase response



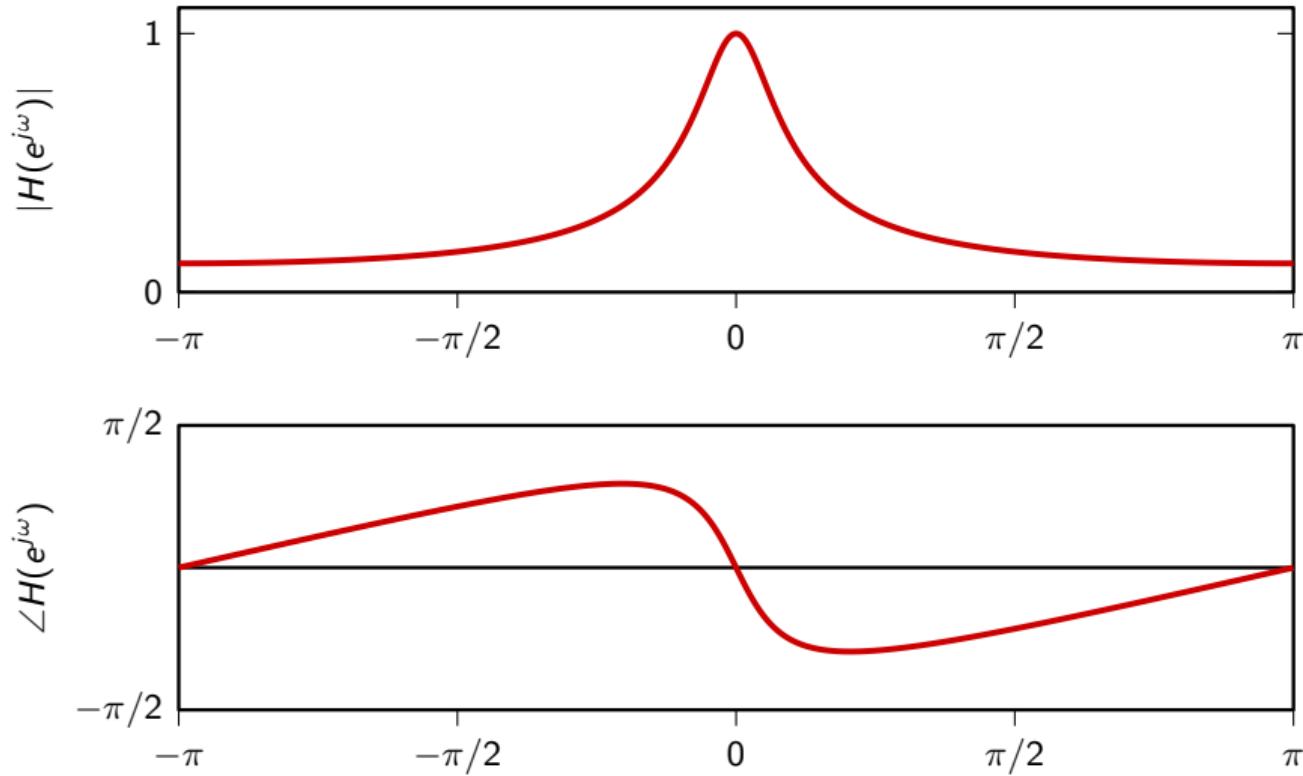
# Leaky integrator, phase response



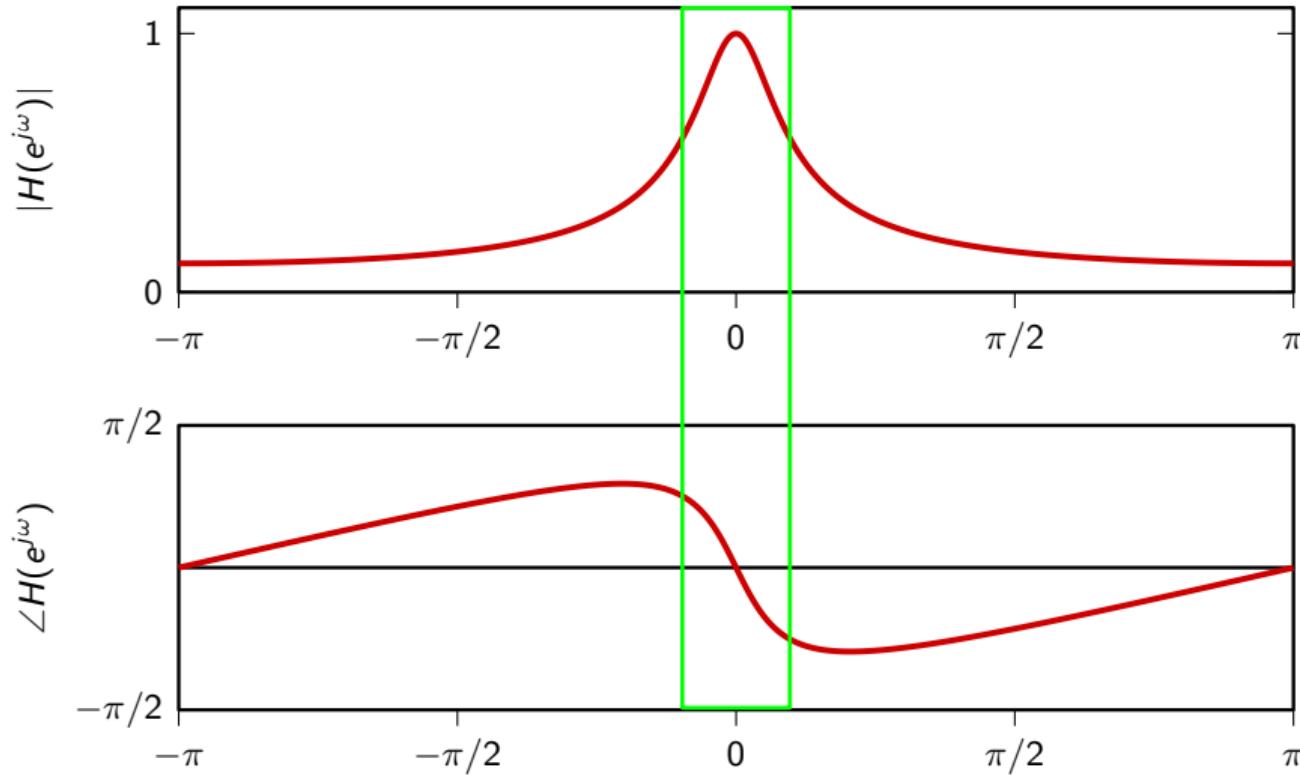
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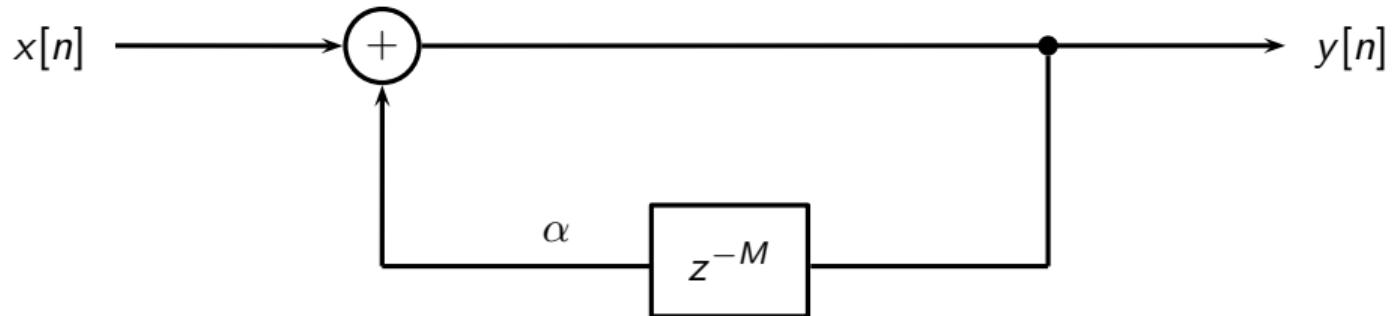
# Phase is sufficiently linear where it matters



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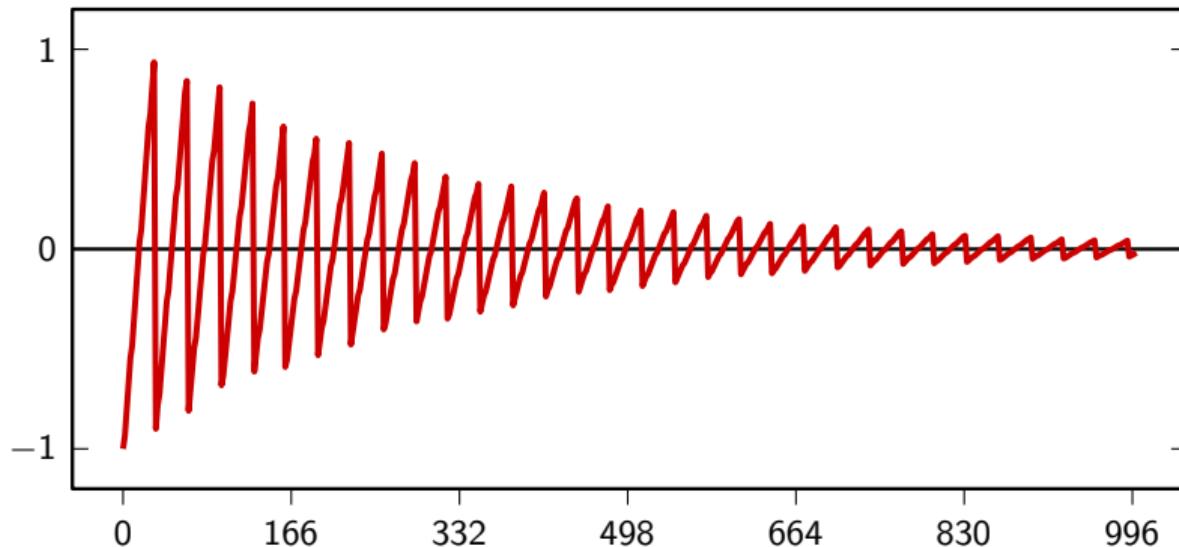


# Karplus-Strong revisited, again!

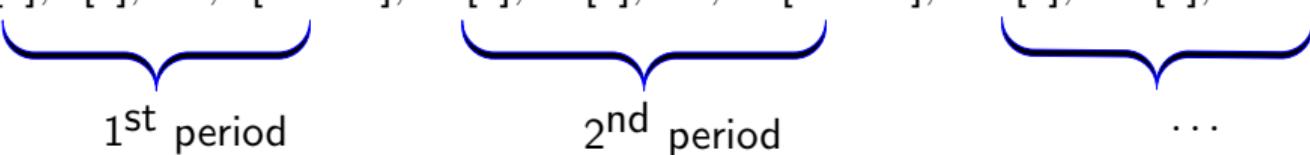


$$y[n] = \alpha y[n - M] + x[n]$$

$$y[n] = \alpha^{\lfloor n/M \rfloor} \bar{x}[n \bmod M] u[n]$$



$$y[n] = \bar{x}[0], \bar{x}[1], \dots, \bar{x}[M-1], \alpha\bar{x}[0], \alpha\bar{x}[1], \dots, \alpha\bar{x}[M-1], \alpha^2\bar{x}[0], \alpha^2\bar{x}[1], \dots$$

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1<sup>st</sup> period                    2<sup>nd</sup> period                    ...

key observation:

$$y[n] = \bar{x}[n] * w[n], \quad w[n] = \begin{cases} \alpha^k & \text{for } n = kM \\ 0 & \text{otherwise} \end{cases}$$

$$Y(e^{j\omega}) = \bar{X}(e^{j\omega})W(e^{j\omega})$$

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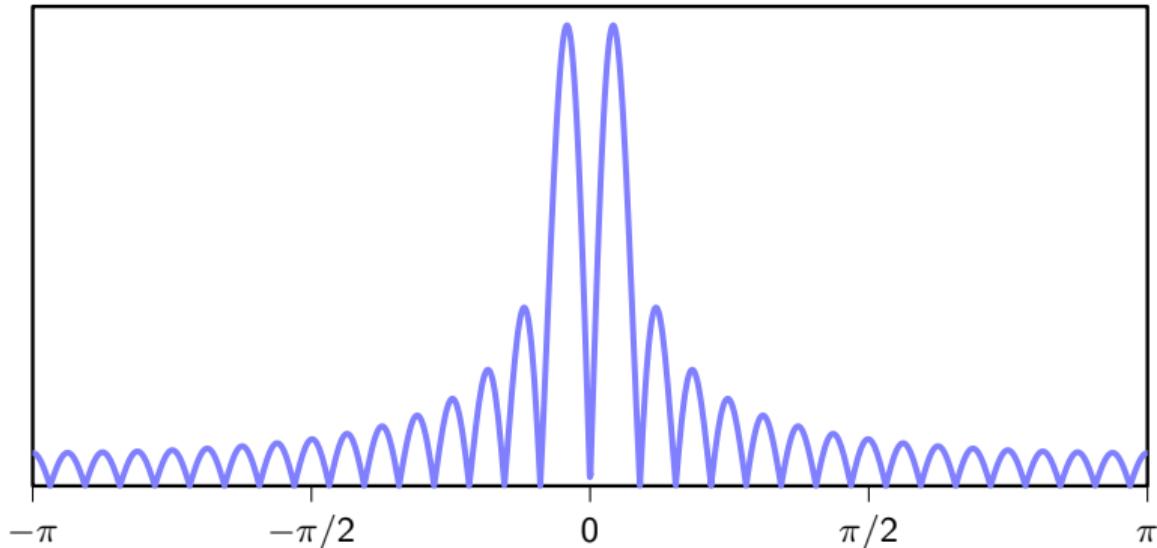
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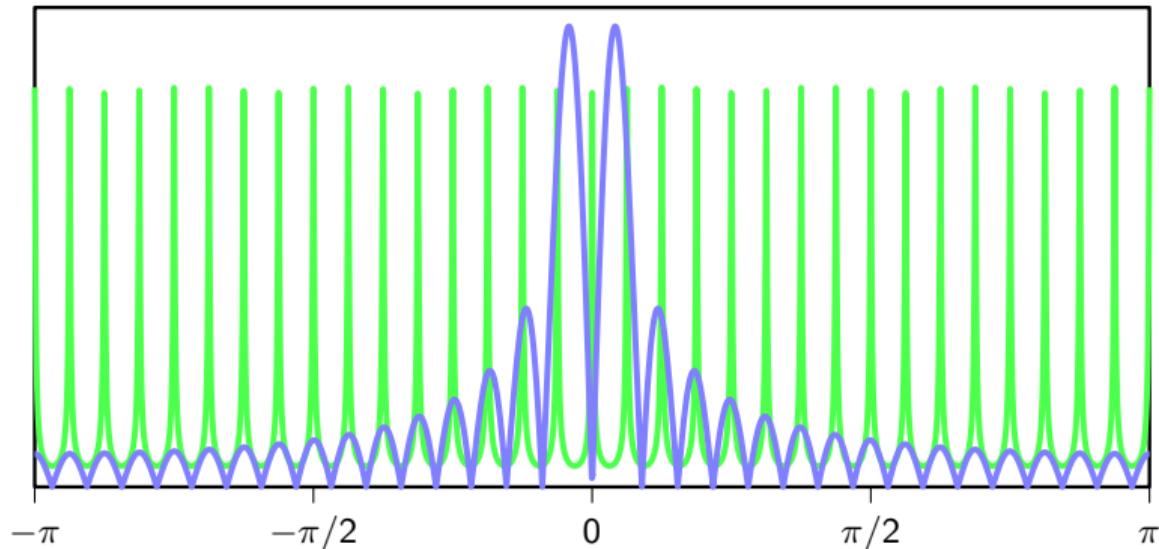
$$\bar{X}(e^{j\omega}) = e^{-j\omega} \left( \frac{M+1}{M-1} \right) \frac{1 - e^{-j(M-1)\omega}}{(1 - e^{-j\omega})^2} - \frac{1 - e^{-j(M+1)\omega}}{(1 - e^{-j\omega})^2}$$

$$W(e^{j\omega}) = \frac{1}{1 - \alpha e^{-j\omega M}}$$

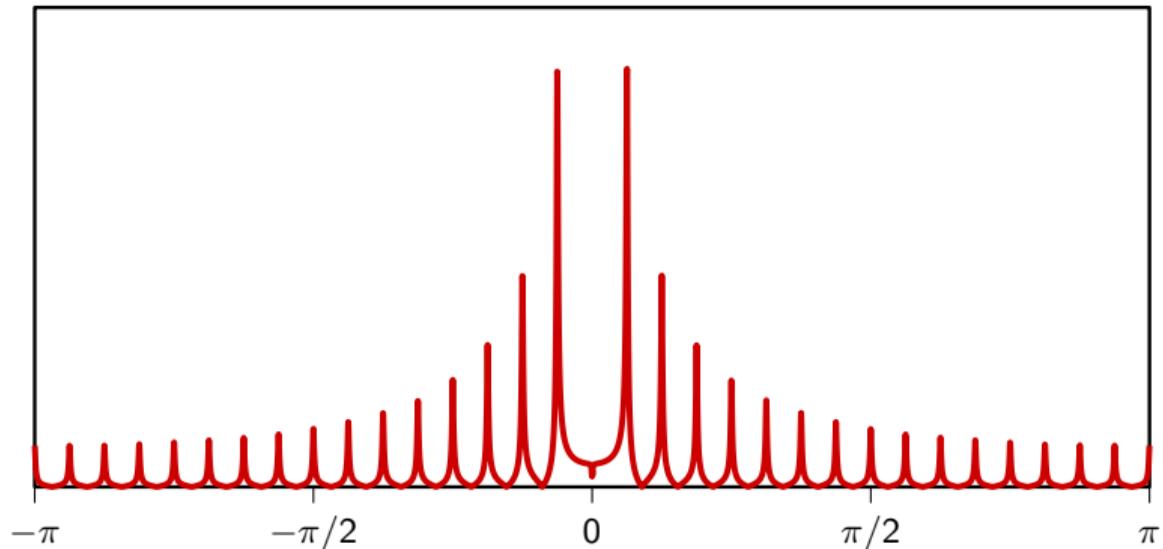
$$|\bar{X}(e^{j\omega})|$$



$$|W(e^{j\omega})|$$



$$|Y(e^{j\omega})|$$



END OF MODULE 5.4

# Digital Signal Processing

Module 5.5: Ideal Filters

- ▶ Filter classification in the frequency domain
  - ▶ Ideal filters
  - ▶ Demodulation revisited

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- ▶ Ideal filters
- ▶ Demodulation revisited

- ▶ Lowpass
- ▶ Highpass
- ▶ Bandpass
- ▶ Allpass

Moving Average and Leaky Integrator are lowpass filters

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- ▶ Highpass
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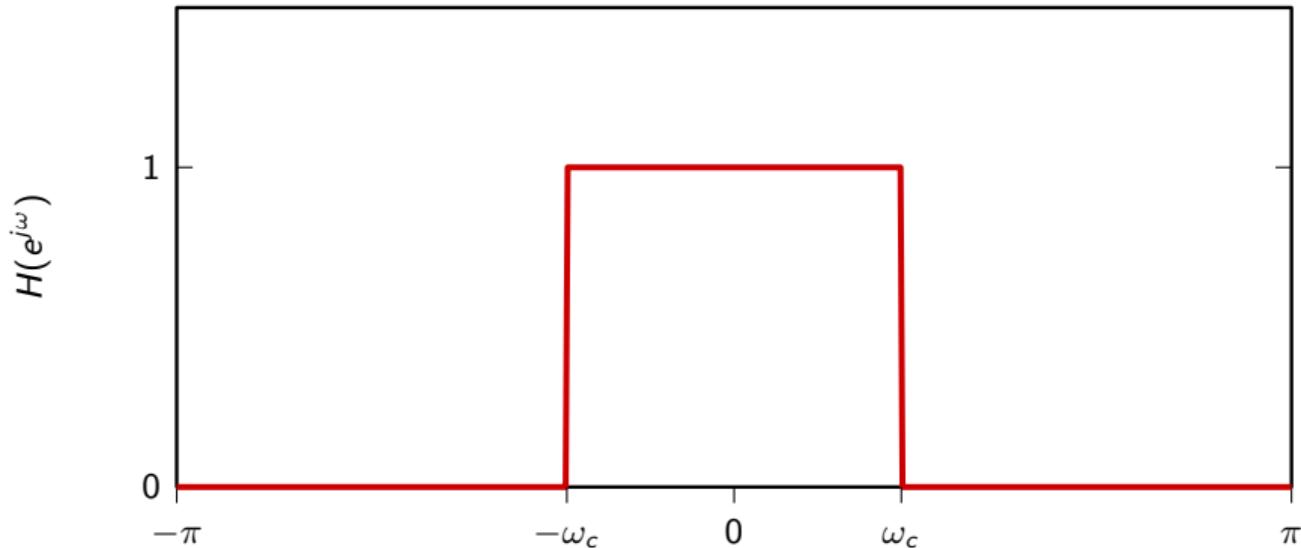
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Moving Average and Leaky Integrator are lowpass filters

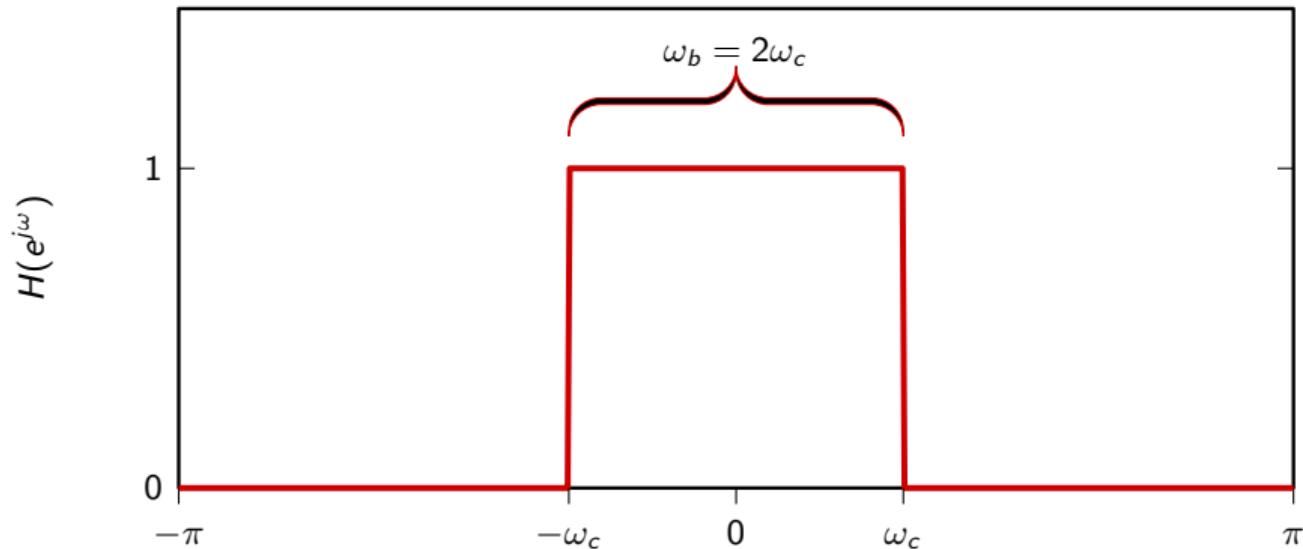
- ▶ Linear phase
- ▶ Nonlinear phase

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- ▶ Nonlinear phase

What is the best lowpass we can think of?



# What is the best lowpass we can think of?



$$H(e^{j\omega}) = \begin{cases} 1 & \text{for } |\omega| \leq \omega_c \\ 0 & \text{otherwise} \end{cases} \quad (2\pi\text{-periodicity implicit})$$

- ▶ perfectly flat passband
- ▶ infinite attenuation in stopband
- ▶ zero-phase (no delay)

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$$\begin{aligned} h[n] &= \text{IDTFT} \{ H(e^{j\omega}) \} \\ &= \frac{1}{2\pi} \int_{-\pi}^{\pi} H(e^{j\omega}) e^{j\omega n} d\omega \\ &= \frac{1}{2\pi} \int_{-\omega_c}^{\omega_c} e^{j\omega n} d\omega \\ &= \frac{1}{\pi n} \frac{e^{j\omega_c n} - e^{-j\omega_c n}}{2j} \\ &= \frac{\sin \omega_c n}{\pi n} \end{aligned}$$

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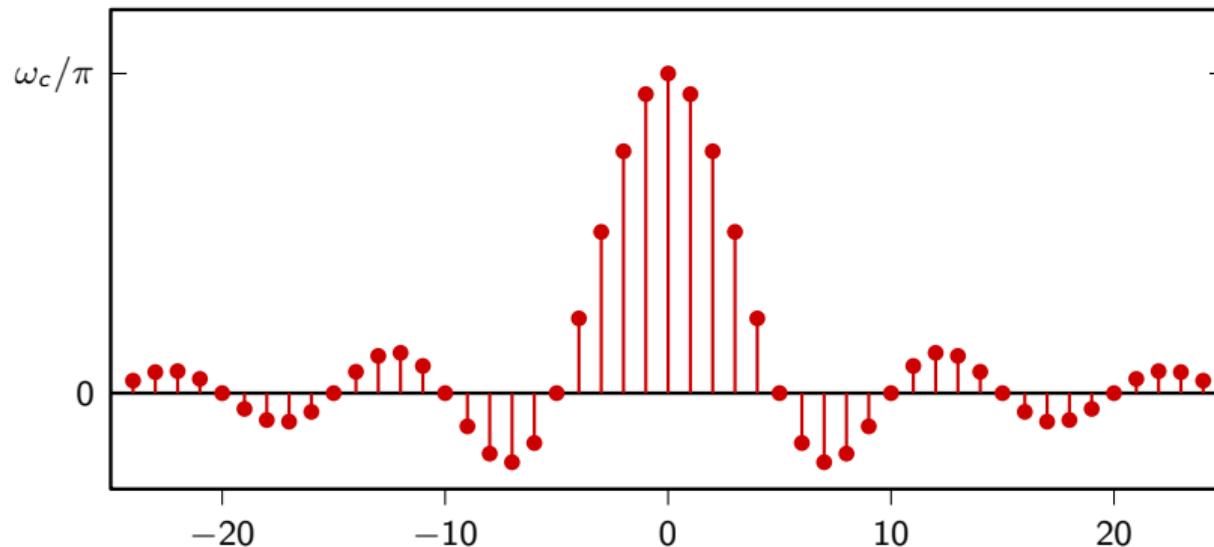
# Ideal lowpass filter: impulse response

$$\begin{aligned} h[n] &= \text{IDTFT} \{ H(e^{j\omega}) \} \\ &= \frac{1}{2\pi} \int_{-\pi}^{\pi} H(e^{j\omega}) e^{j\omega n} d\omega \\ &= \frac{1}{2\pi} \int_{-\omega_c}^{\omega_c} e^{j\omega n} d\omega \\ &= \frac{1}{\pi n} \frac{e^{j\omega_c n} - e^{-j\omega_c n}}{2j} \\ &= \frac{\sin \omega_c n}{\pi n} \end{aligned}$$

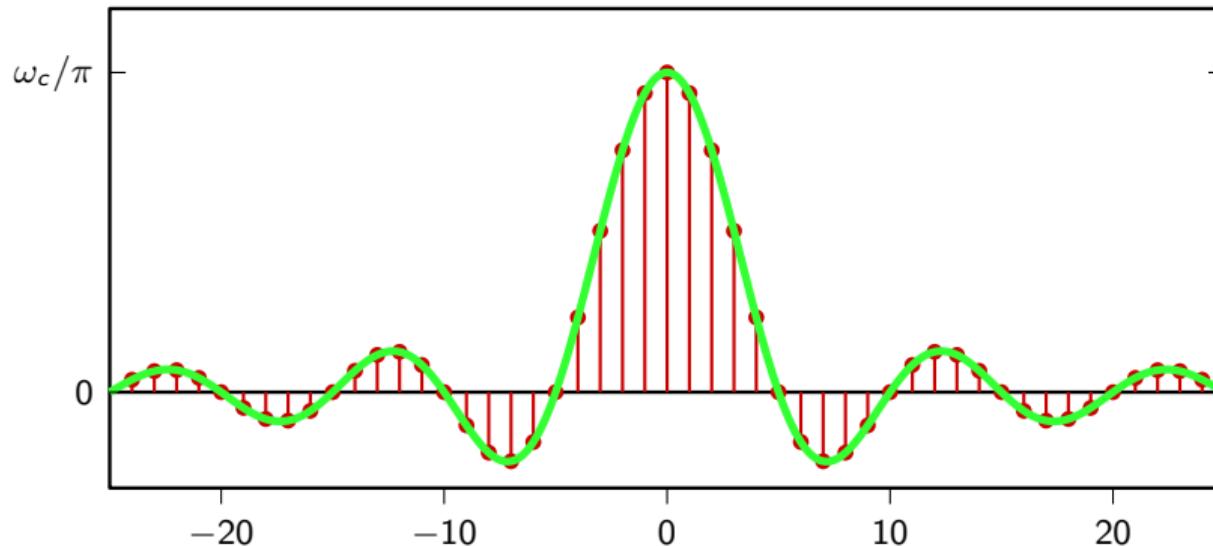
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# Ideal lowpass filter: impulse response



# Ideal lowpass filter: impulse response



- ▶ impulse response is infinite support, two-sided
  - ⇒ cannot compute the output in a finite amount of time
  - ⇒ that's why it's called "ideal"
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The sinc-rect pair:

$$\text{rect}(x) = \begin{cases} 1 & |x| \leq 1/2 \\ 0 & |x| > 1/2 \end{cases}$$

$$\text{sinc}(x) = \begin{cases} \frac{\sin(\pi x)}{\pi x} & x \neq 0 \\ 1 & x = 0 \end{cases}$$

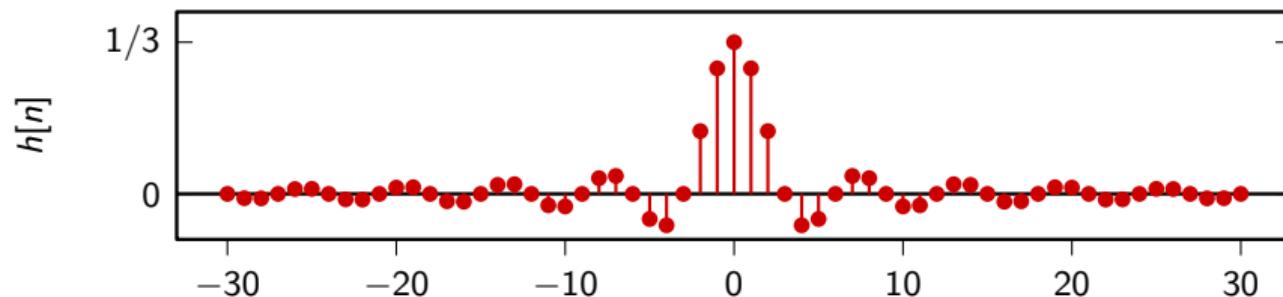
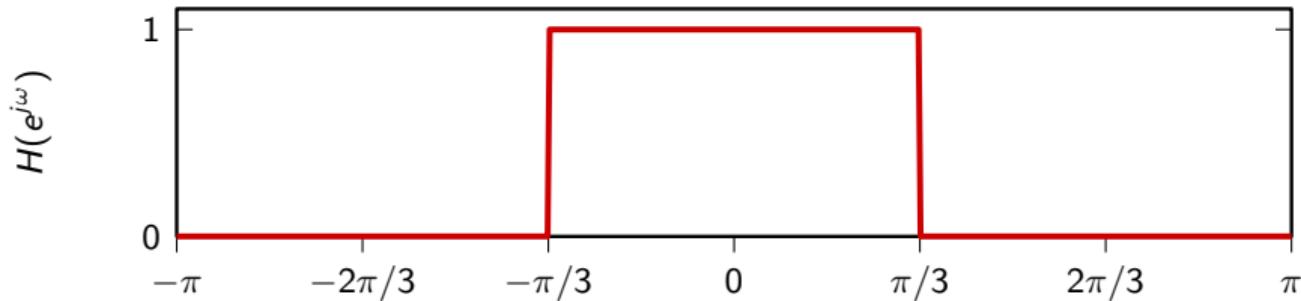
(note that  $\text{sinc}(x) = 0$  when  $x$  is a nonzero integer)

# The ideal lowpass in canonical form

$$\text{rect}\left(\frac{\omega}{2\omega_c}\right) \xleftrightarrow{\text{DTFT}} \frac{\omega_c}{\pi} \text{sinc}\left(\frac{\omega_c}{\pi}n\right)$$

## Example

$$\omega_c = \pi/3: H(e^{j\omega}) = \text{rect}(3\omega/2\pi), h[n] = (1/3)\text{sinc}(n/3)$$



- ▶ the sinc is not absolutely summable
- ▶ the ideal lowpass is not BIBO stable!
- ▶ example for  $\omega_c = \pi/3$ :  $h[n] = (1/3) \operatorname{sinc}(n/3)$
- ▶ take  $x[n] = \operatorname{sign}\{\operatorname{sinc}(-n/3)\}$  and

$$y[0] = (x * h)[0] = \frac{1}{3} \sum_{k=-\infty}^{\infty} |\operatorname{sinc}(k/3)| = \infty$$

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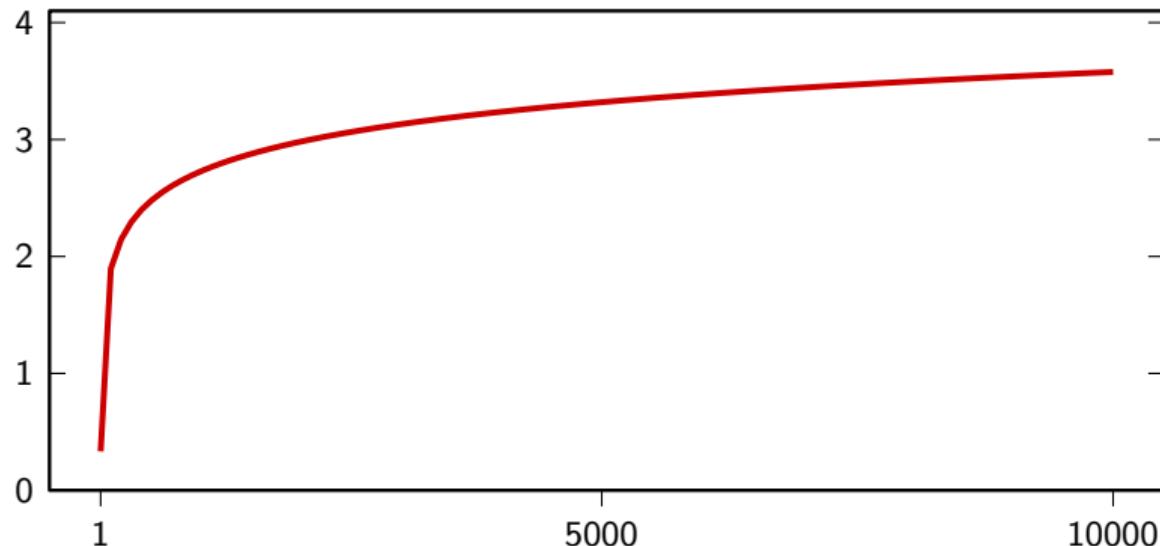
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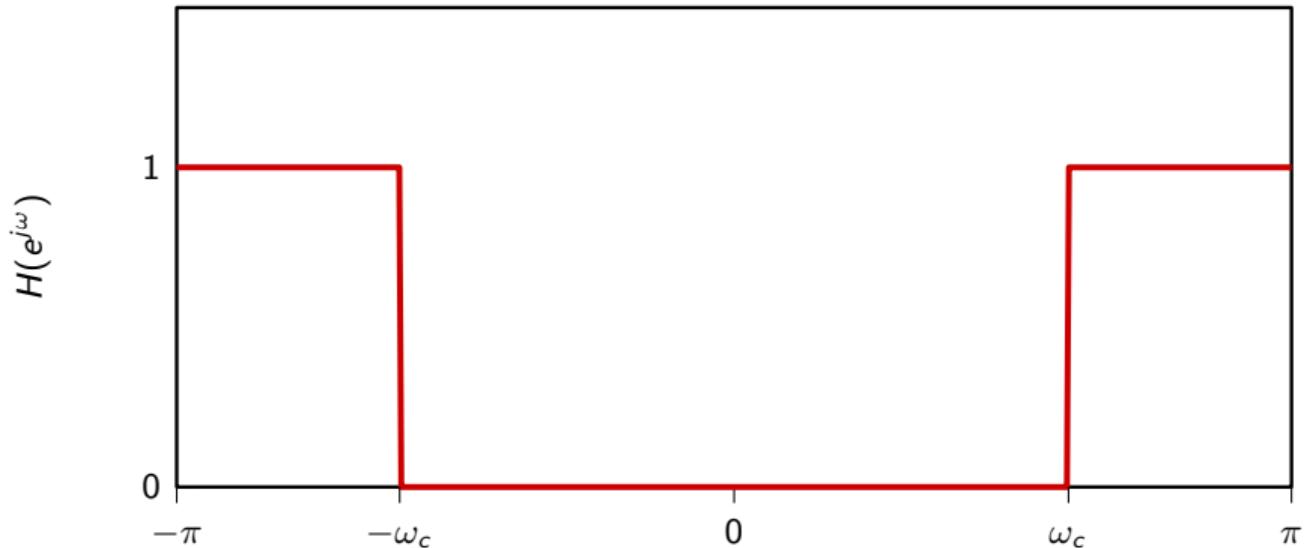
$$y[0] = (x * h)[0] = \frac{1}{3} \sum_{k=-\infty}^{\infty} |\operatorname{sinc}(k/3)| = \infty$$

Divergence is however very slow...

$$s(n) = (1/3) \sum_{k=-n}^n |\operatorname{sinc}(k/3)|$$



# Ideal highpass filter



$$H_{hp}(e^{j\omega}) = \begin{cases} 1 & \text{for } \pi \geq |\omega| \geq \omega_c \\ 0 & \text{otherwise} \end{cases} \quad (2\pi\text{-periodicity implicit})$$

$$H_{hp}(e^{j\omega}) = 1 - H_{lp}(e^{j\omega})$$

$$h_{hp}[n] = \delta[n] - \frac{\omega_c}{\pi} \operatorname{sinc}\left(\frac{\omega_c}{\pi} n\right)$$

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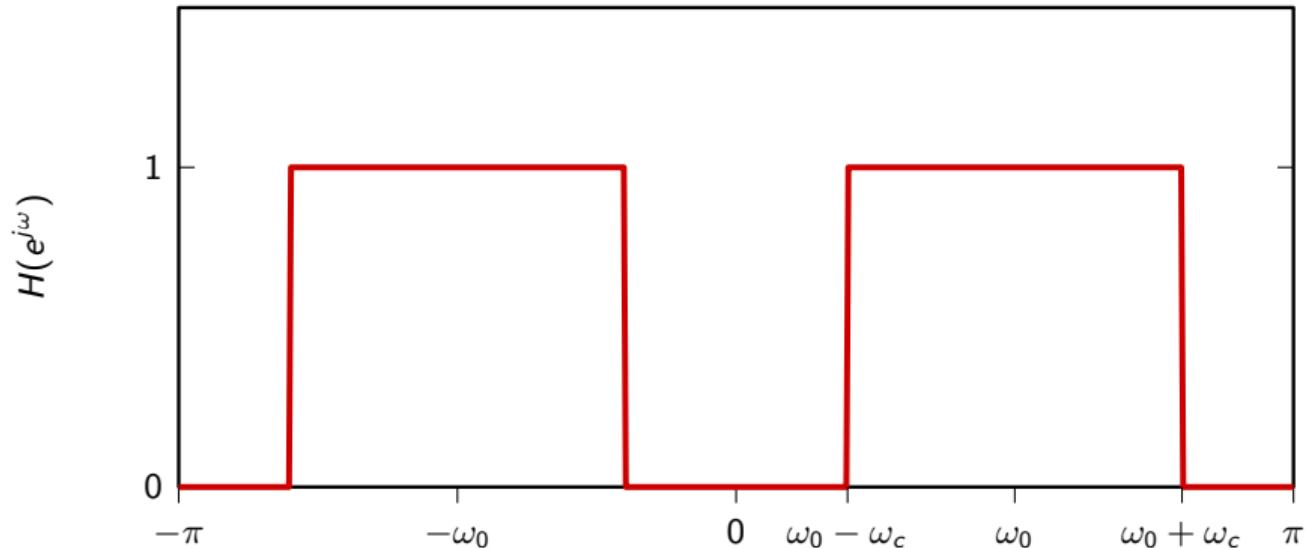
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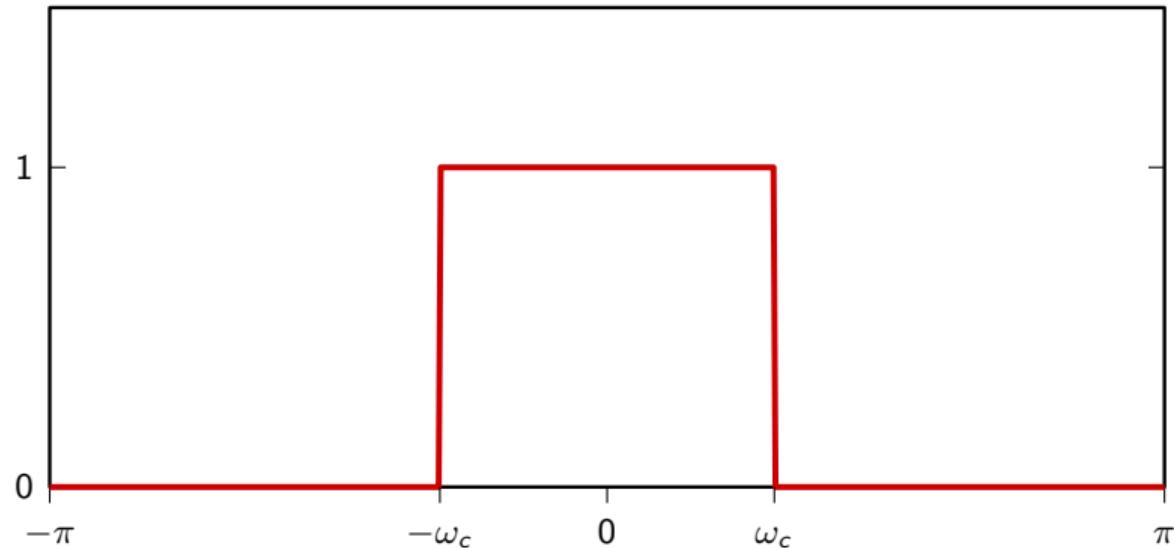
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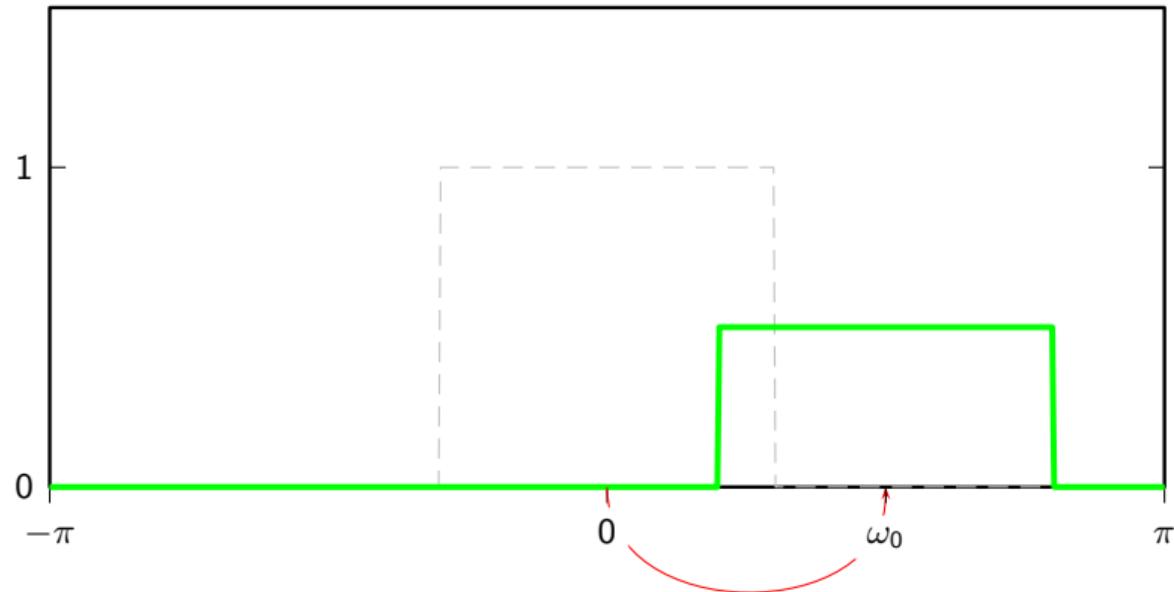
# Ideal bandpass filter



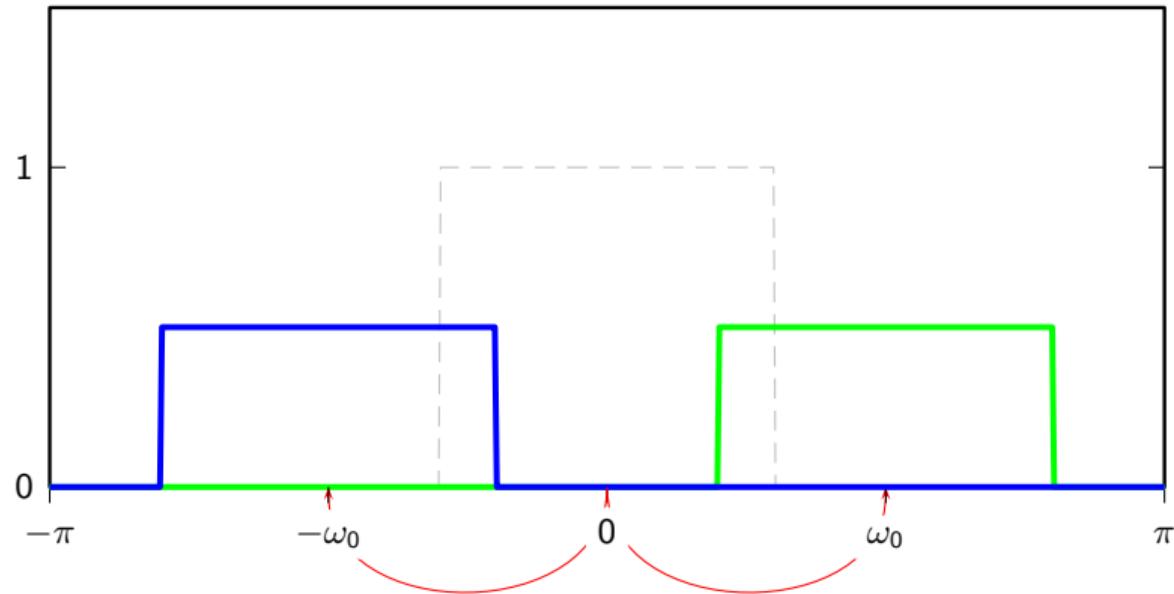
# Ideal bandpass filter



# Ideal bandpass filter



# Ideal bandpass filter



$$H_{bp}(e^{j\omega}) = \begin{cases} 1 & \text{for } |\omega \pm \omega_0| \leq \omega_c \\ 0 & \text{otherwise} \end{cases} \quad (2\pi\text{-periodicity implicit})$$

$$h_{bp}[n] = 2 \cos(\omega_0 n) \frac{\omega_c}{\pi} \operatorname{sinc}\left(\frac{\omega_c}{\pi} n\right)$$

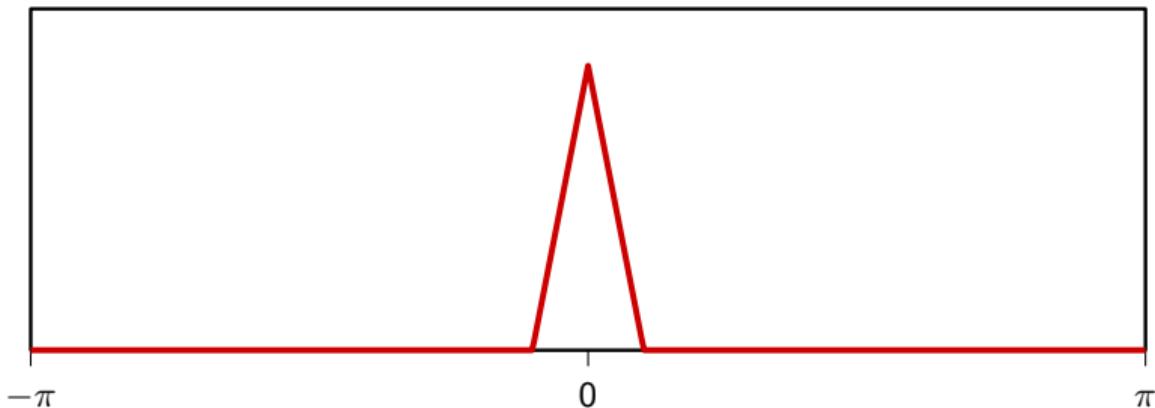
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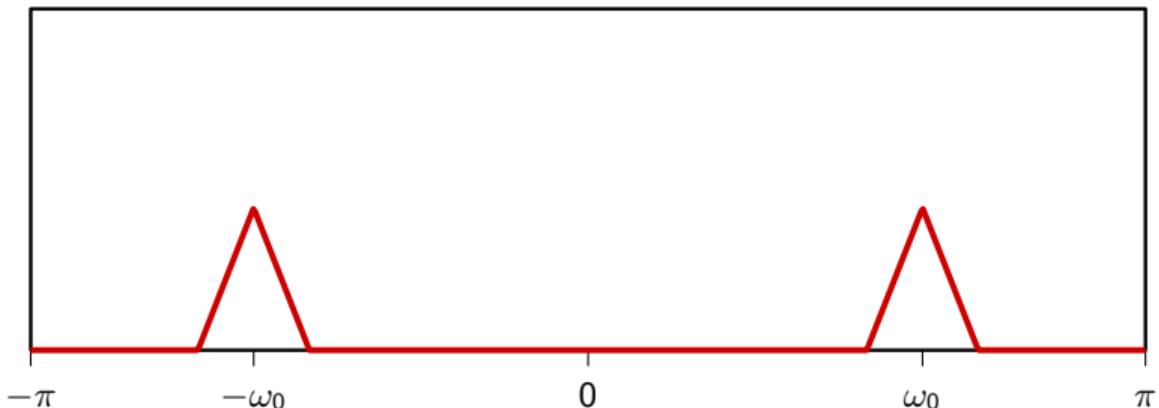
remember Module 4.8:

- ▶ apply sinusoidal modulation to  $x[n]$ :  $y[n] = x[n] \cos \omega_0 n$
- ▶ demodulate by multiplying by the carrier  $x'[n] = y[n] \cos \omega_0 n$
- ▶ demodulated signal contains unwanted high-frequency components

$$X(e^{j\omega})$$

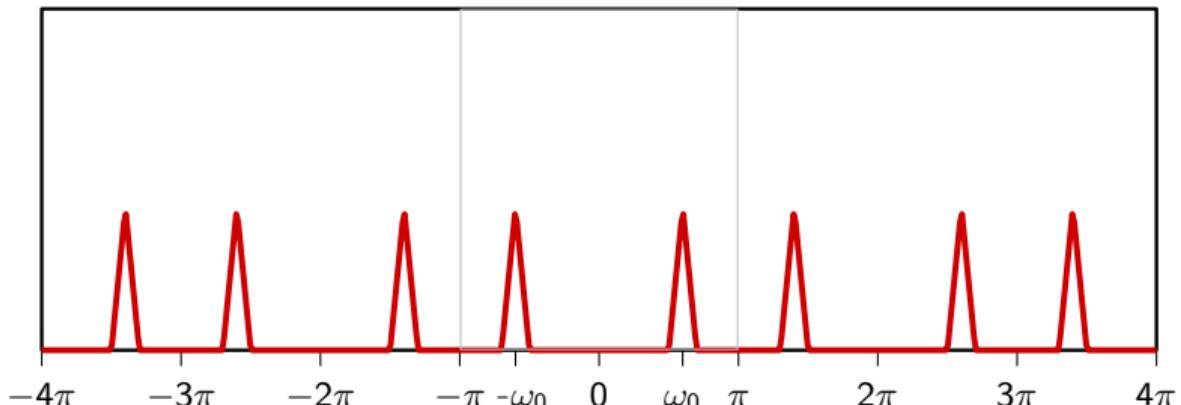


$$Y(e^{j\omega})$$



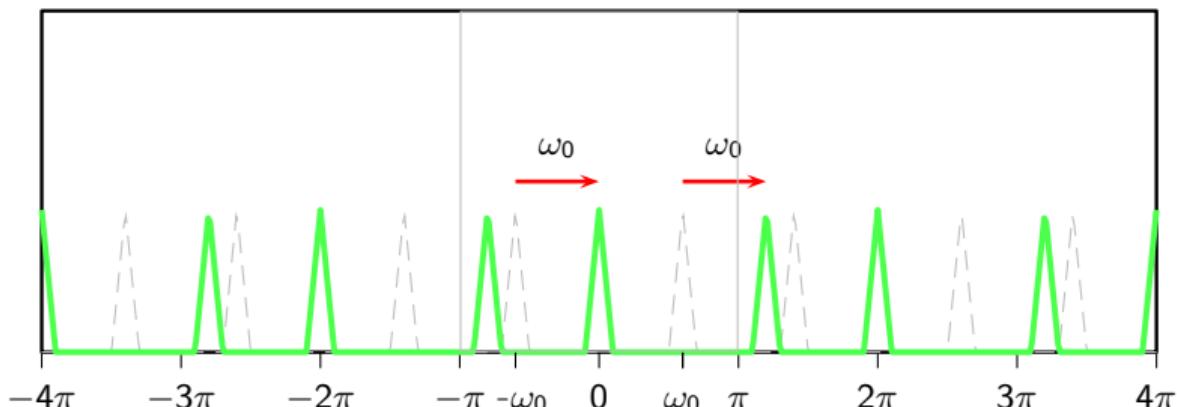
# Demodulation revisited

$$Y(e^{j\omega})$$



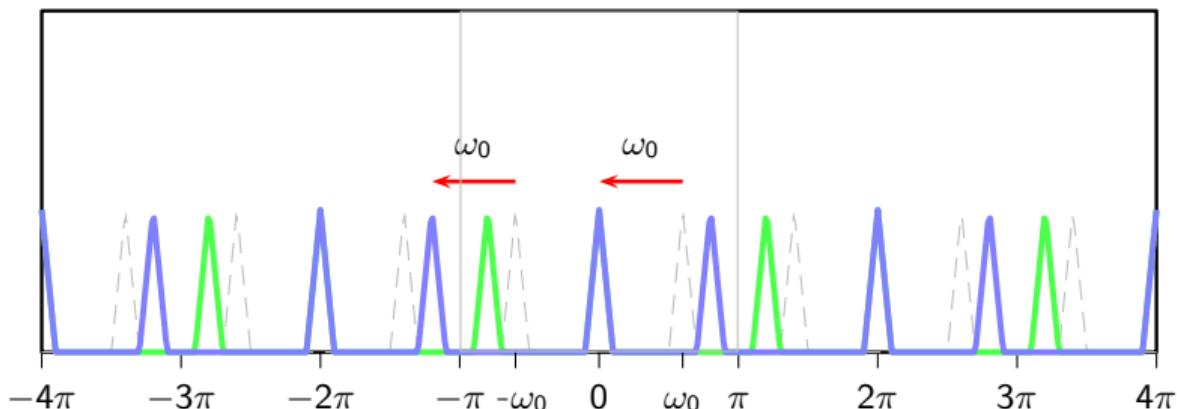
# Demodulation revisited

$$X'(e^{j\omega})$$



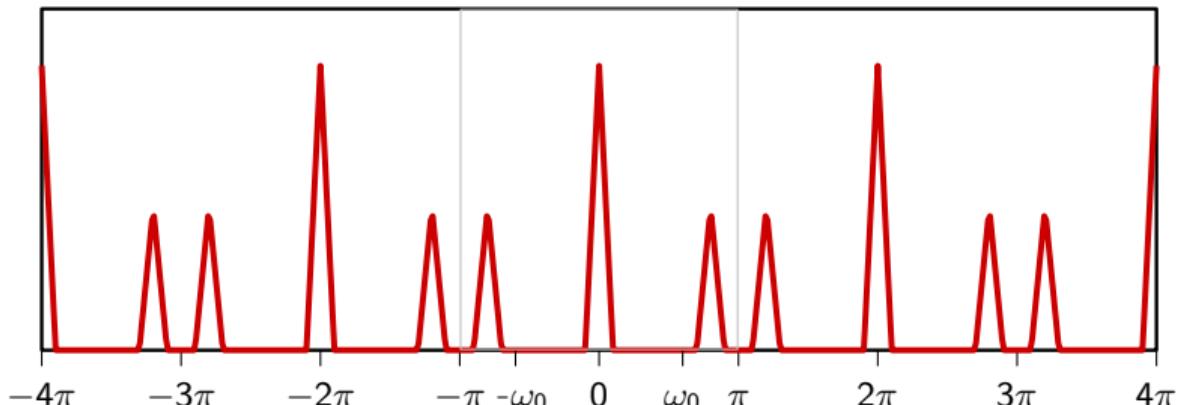
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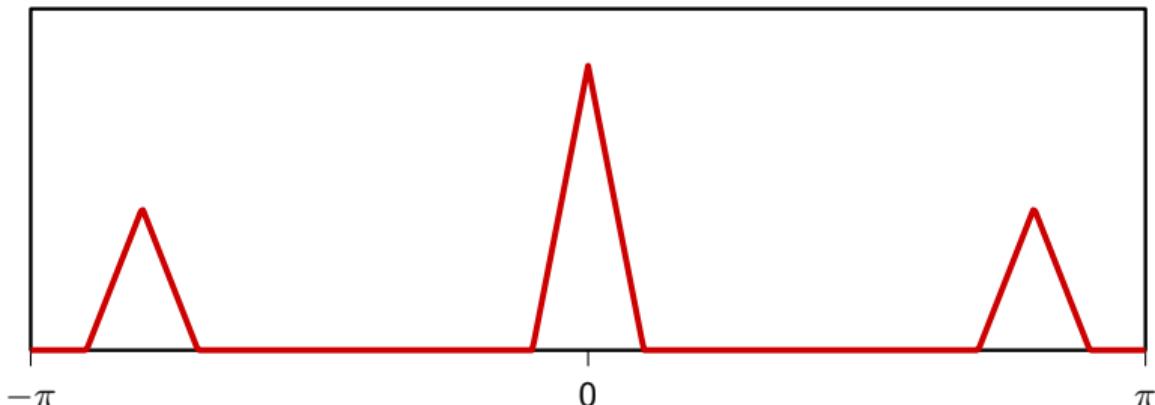
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$$X'(e^{j\omega})$$



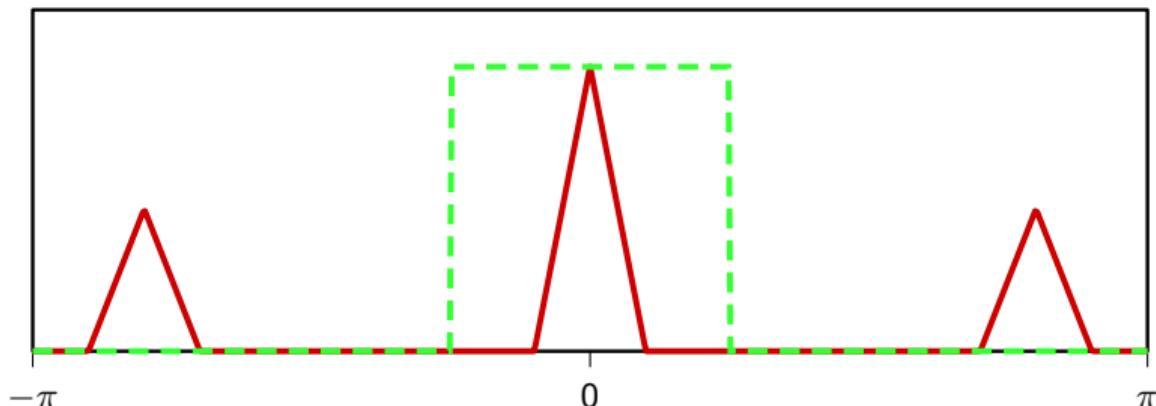
## Solution: lowpass filtering

$$X'(e^{j\omega})$$



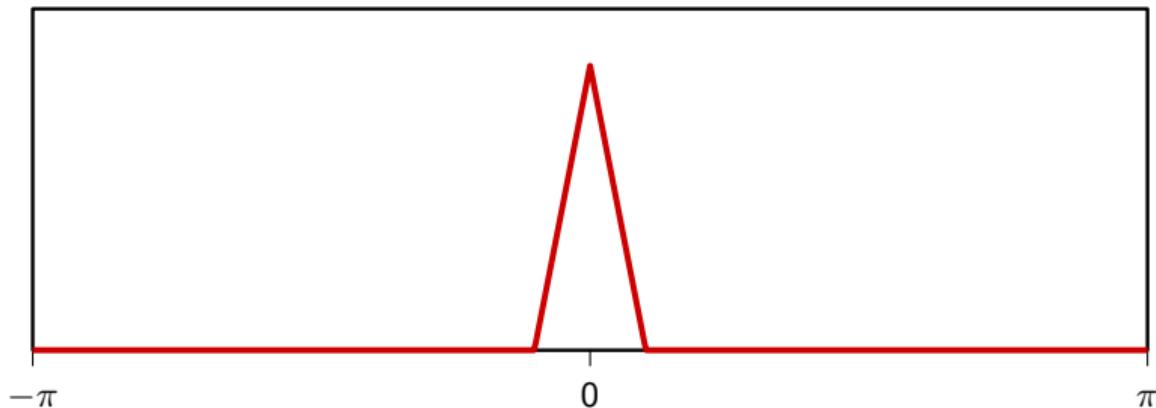
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## Solution: lowpass filtering

$$X(e^{j\omega})$$



END OF MODULE 5.5

# Digital Signal Processing

Module 5.6: Filter Design - Part I: Approximation of Ideal Filters

- ▶ Impulse truncation
- ▶ Window method
- ▶ Frequency sampling

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Idea #1:

- ▶ pick  $\omega_c$
- ▶ compute ideal impulse response  $h[n]$
- ▶ truncate  $h[n]$  to a finite-support  $\hat{h}[n]$
- ▶  $\hat{h}[n]$  defines an FIR filter

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FIR approximation of length  $M = 2N + 1$ :

$$\hat{h}[n] = \begin{cases} \frac{\omega_c}{\pi} \operatorname{sinc}\left(\frac{\omega_c}{\pi} n\right) & |n| \leq N \\ 0 & \text{otherwise} \end{cases}$$

$$\begin{aligned}\text{MSE} &= \frac{1}{2\pi} \int_{-\pi}^{\pi} |H(e^{j\omega}) - \hat{H}(e^{j\omega})|^2 d\omega \\ &= \|H(e^{j\omega}) - \hat{H}(e^{j\omega})\|^2 \\ &= \|h[n] - \hat{h}[n]\|^2 \\ &= \sum_{n=-\infty}^{\infty} |h[n] - \hat{h}[n]|^2\end{aligned}$$

MSE is minimized by symmetric impulse truncation around zero

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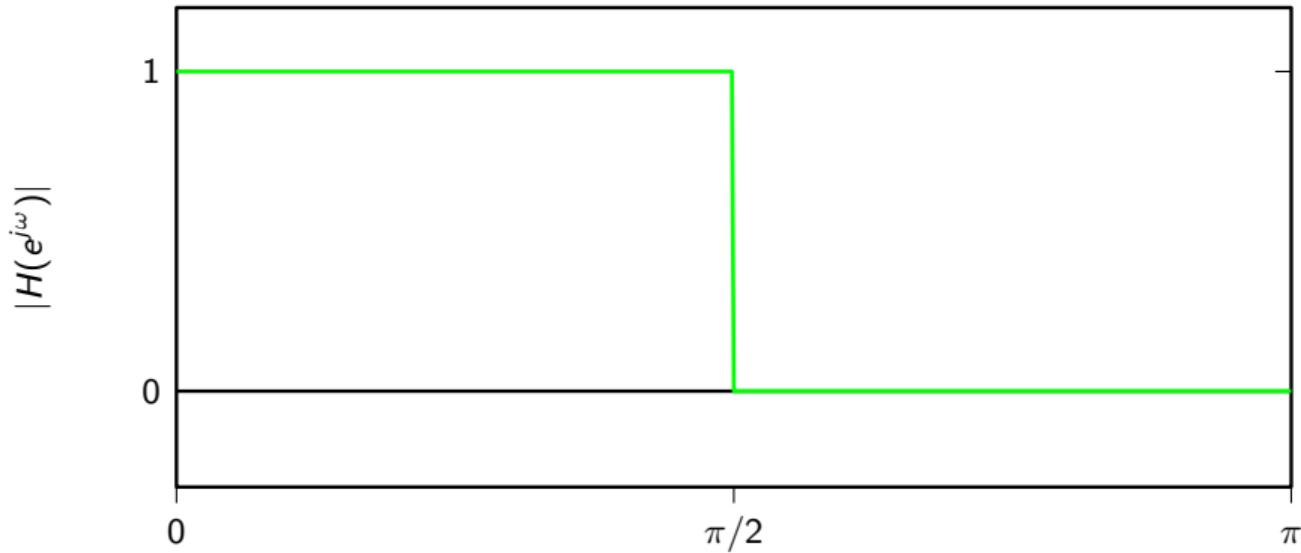
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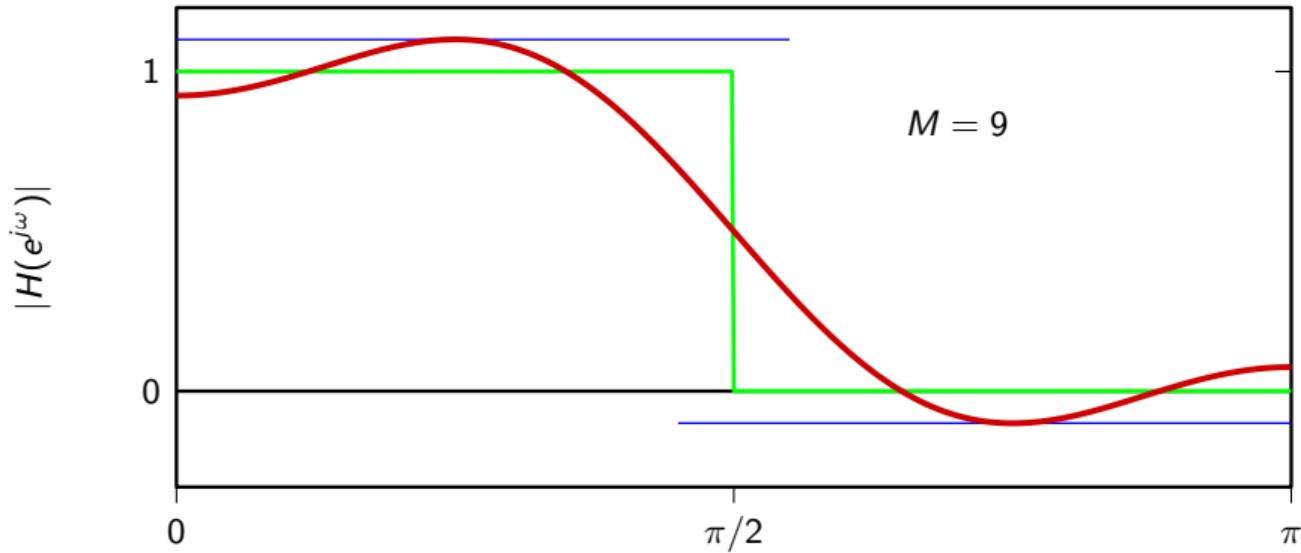
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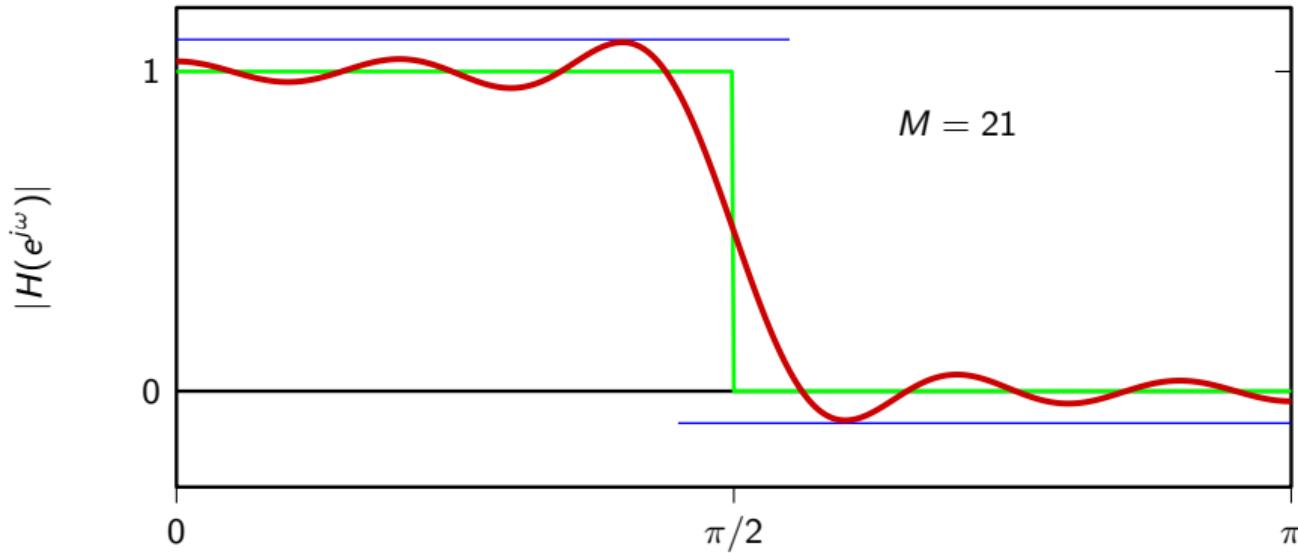
# Why it's not such a good idea



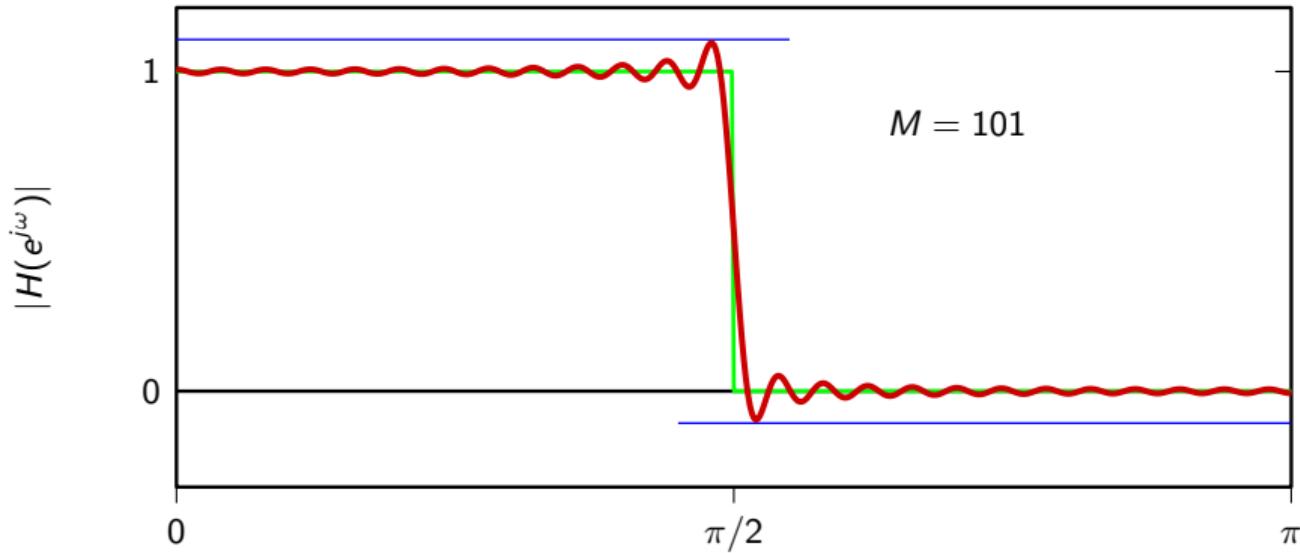
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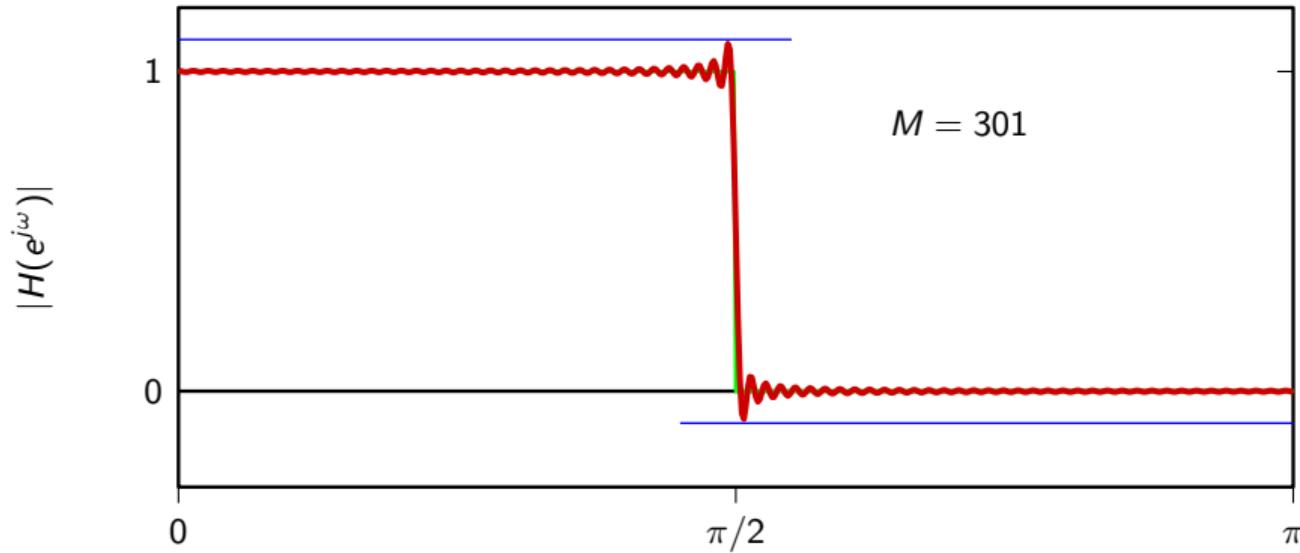
# Why it's not such a good idea



# Why it's not such a good idea



# Why it's not such a good idea



The maximum error around the cutoff frequency  
is around 9% of the height of the jump  
*regardless of  $N$*

$$\hat{h}[n] = h[n] w[n]$$

$$w[n] = \begin{cases} 1 & |n| \leq N \\ 0 & \text{otherwise} \end{cases}$$

$$\hat{H}(e^{j\omega}) = ?$$

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$$(x * y)[n] = \langle x^*[k], y[n - k] \rangle$$

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in  $L_2([-\pi, \pi])$ :

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## Modulation theorem: proof

$$\begin{aligned}\text{IDTFT} \{(X * Y)(e^{j\omega})\} &= \frac{1}{2\pi} \int_{-\pi}^{\pi} (X * Y)(e^{j\omega}) e^{j\omega n} d\omega \\&= \frac{1}{(2\pi)^2} \int_{-\pi}^{\pi} \int_{-\pi}^{\pi} X(e^{j\sigma}) Y(e^{j(\omega-\sigma)}) e^{j\omega n} d\sigma d\omega \\&= \frac{1}{(2\pi)^2} \int_{-\pi}^{\pi} \int_{-\pi}^{\pi} X(e^{j\sigma}) Y(e^{j(\omega-\sigma)}) e^{j\sigma n} e^{j(\omega-\sigma)n} d\sigma d\omega \\&= \frac{1}{2\pi} \int_{-\pi}^{\pi} X(e^{j\sigma}) e^{j\sigma n} d\sigma \frac{1}{2\pi} \int_{-\pi}^{\pi} Y(e^{j(\omega-\sigma)}) e^{j(\omega-\sigma)n} d\omega \\&= x[n] y[n]\end{aligned}$$

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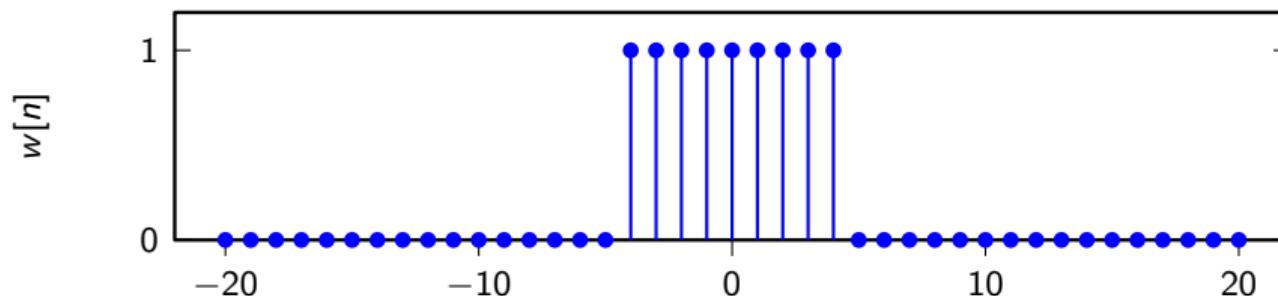
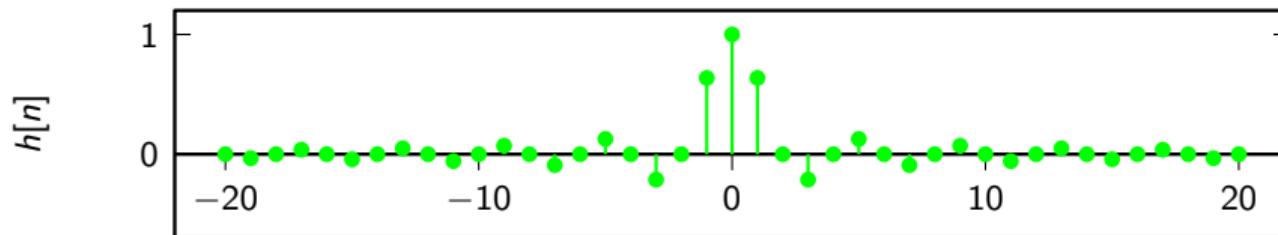
$$\begin{aligned}\text{DTFT } \{x[n] \cos \omega_c n\} &= X(e^{j\omega}) * \frac{1}{2} [\tilde{\delta}(\omega - \omega_c) + \tilde{\delta}(\omega + \omega_c)] \\ &= \frac{1}{4\pi} \int_{-\pi}^{\pi} X(e^{j\sigma}) \tilde{\delta}(\sigma - \omega + \omega_c) d\sigma + \frac{1}{4\pi} \int_{-\pi}^{\pi} X(e^{j\sigma}) \tilde{\delta}(\sigma - \omega - \omega_c) d\sigma \\ &= \frac{1}{2} [X(e^{j(\omega - \omega_c)}) + X(e^{j(\omega + \omega_c)})]\end{aligned}$$

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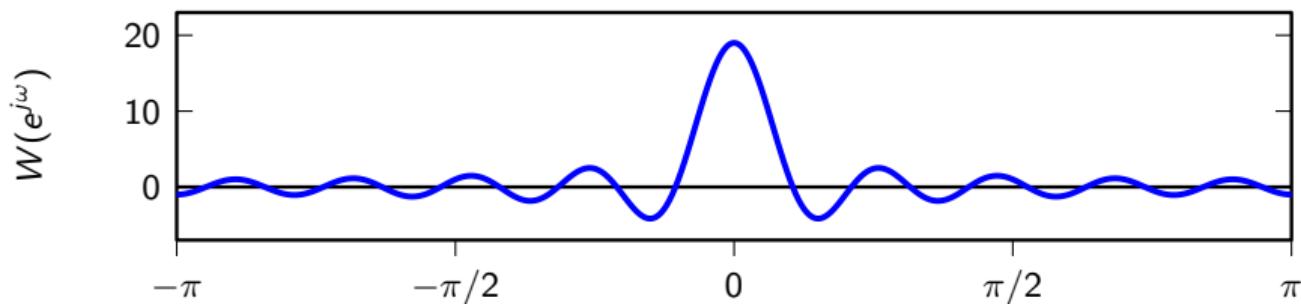
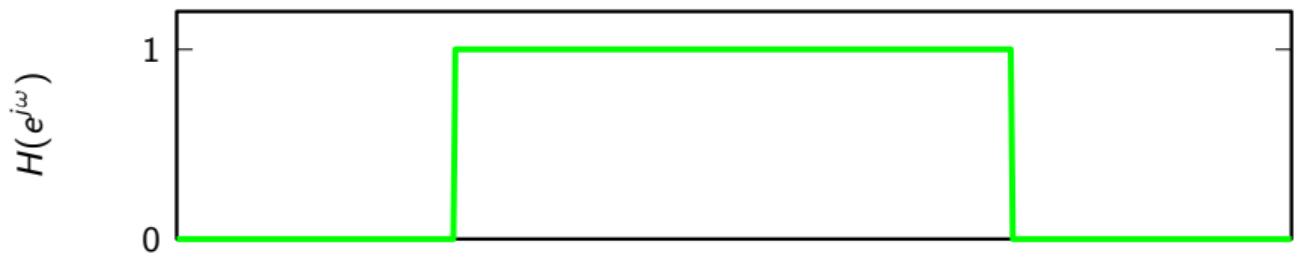
# Understanding the Gibbs phenomenon

$$\hat{h}[n] = h[n] w[n]$$

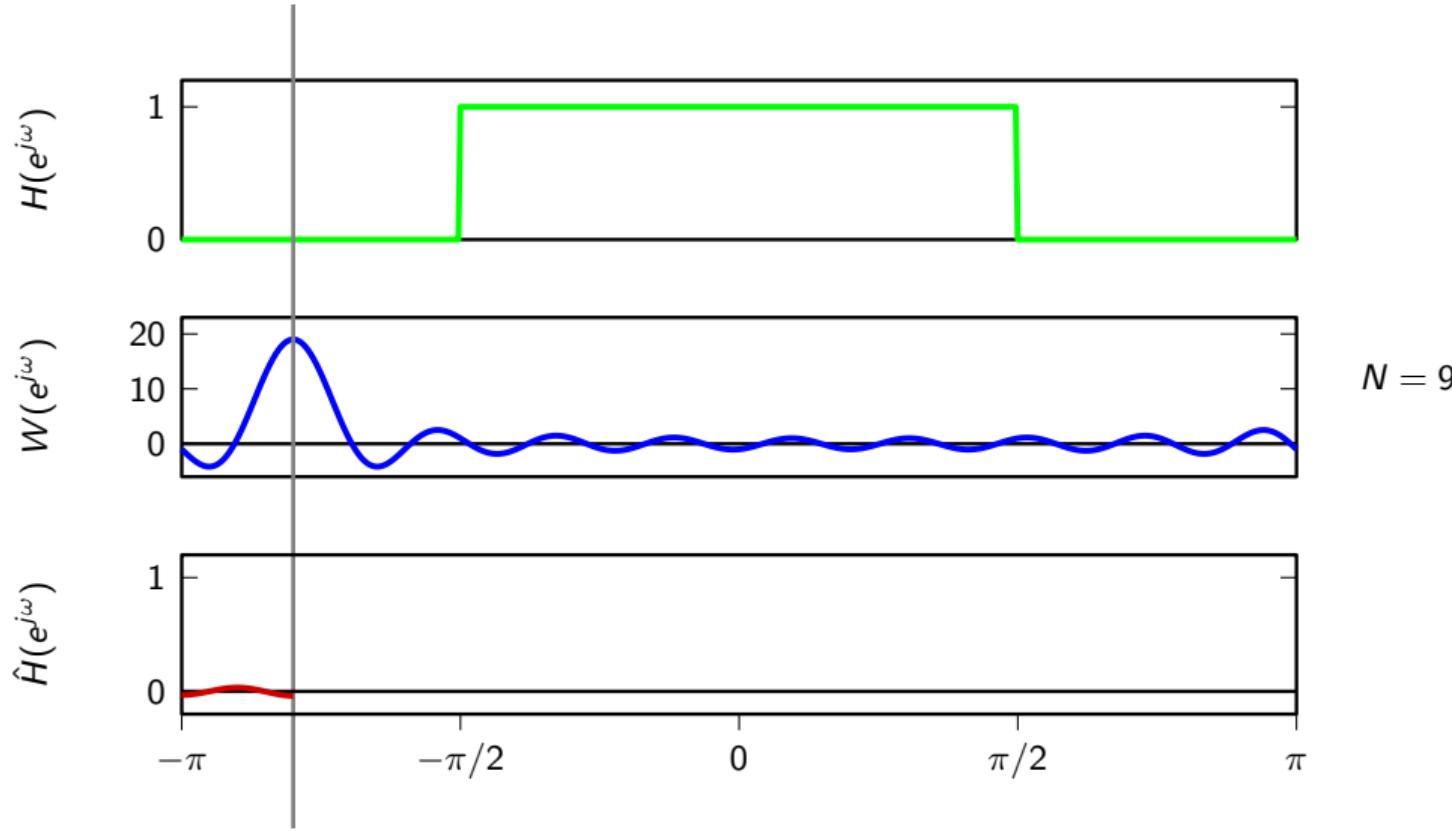


# Understanding the Gibbs phenomenon

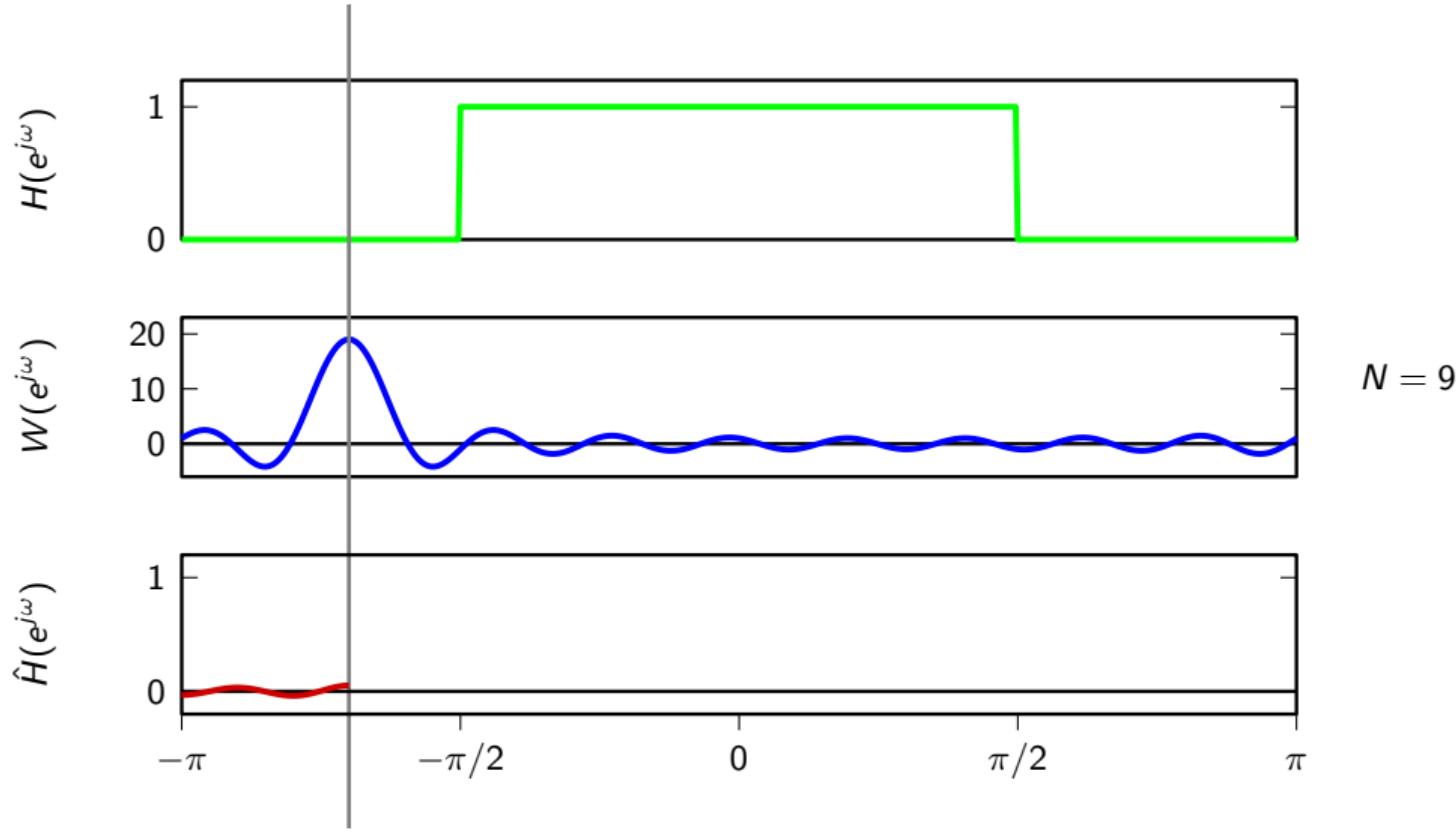
$$\hat{H}(e^{j\omega}) = (H * W)(e^{j\omega}), \quad W(e^{j\omega}) = \sin(\omega(2N+1)/2) / \sin(\omega/2) \quad (\text{Module 4.7})$$



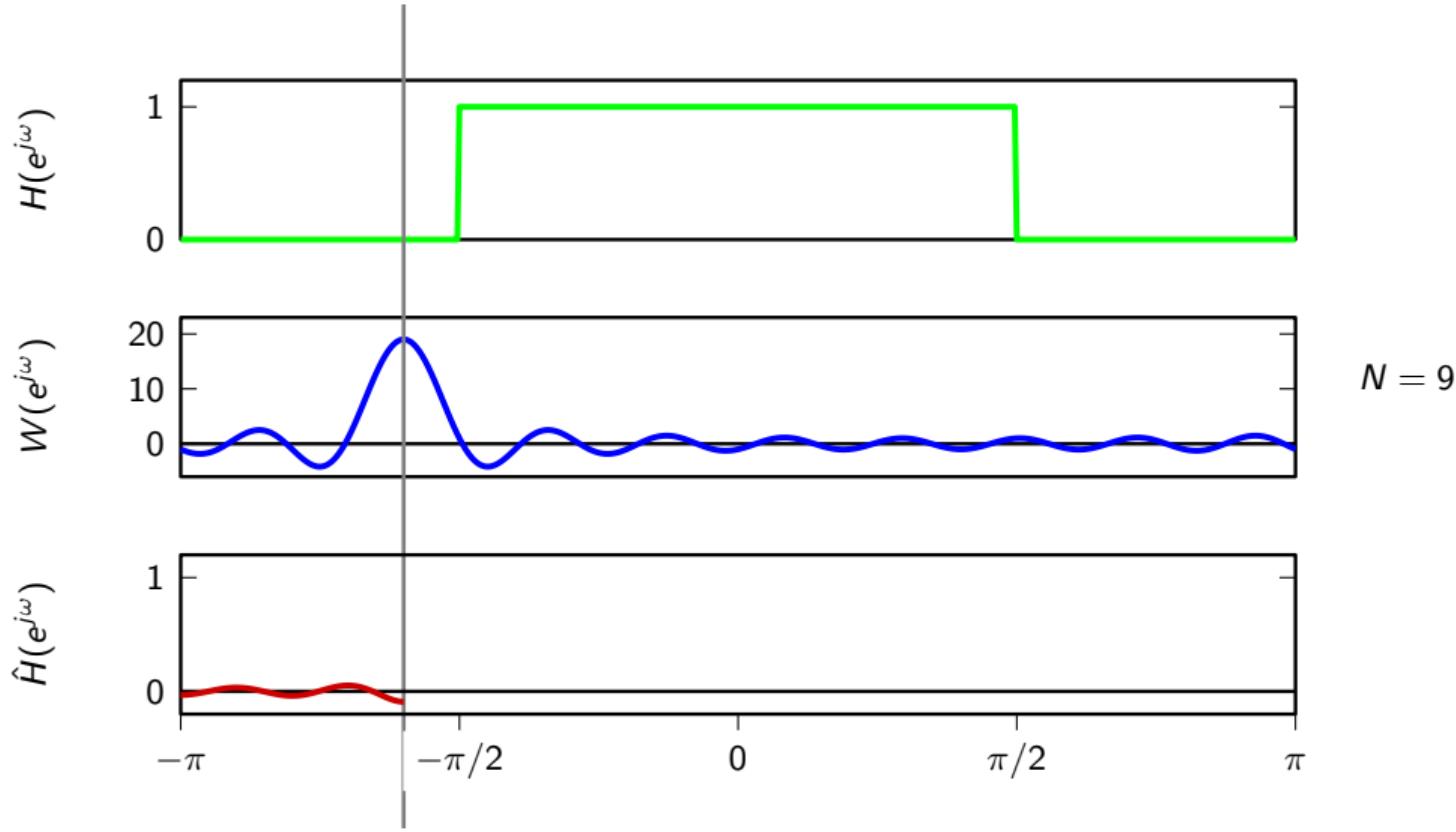
# Implicit frequency-domain convolution



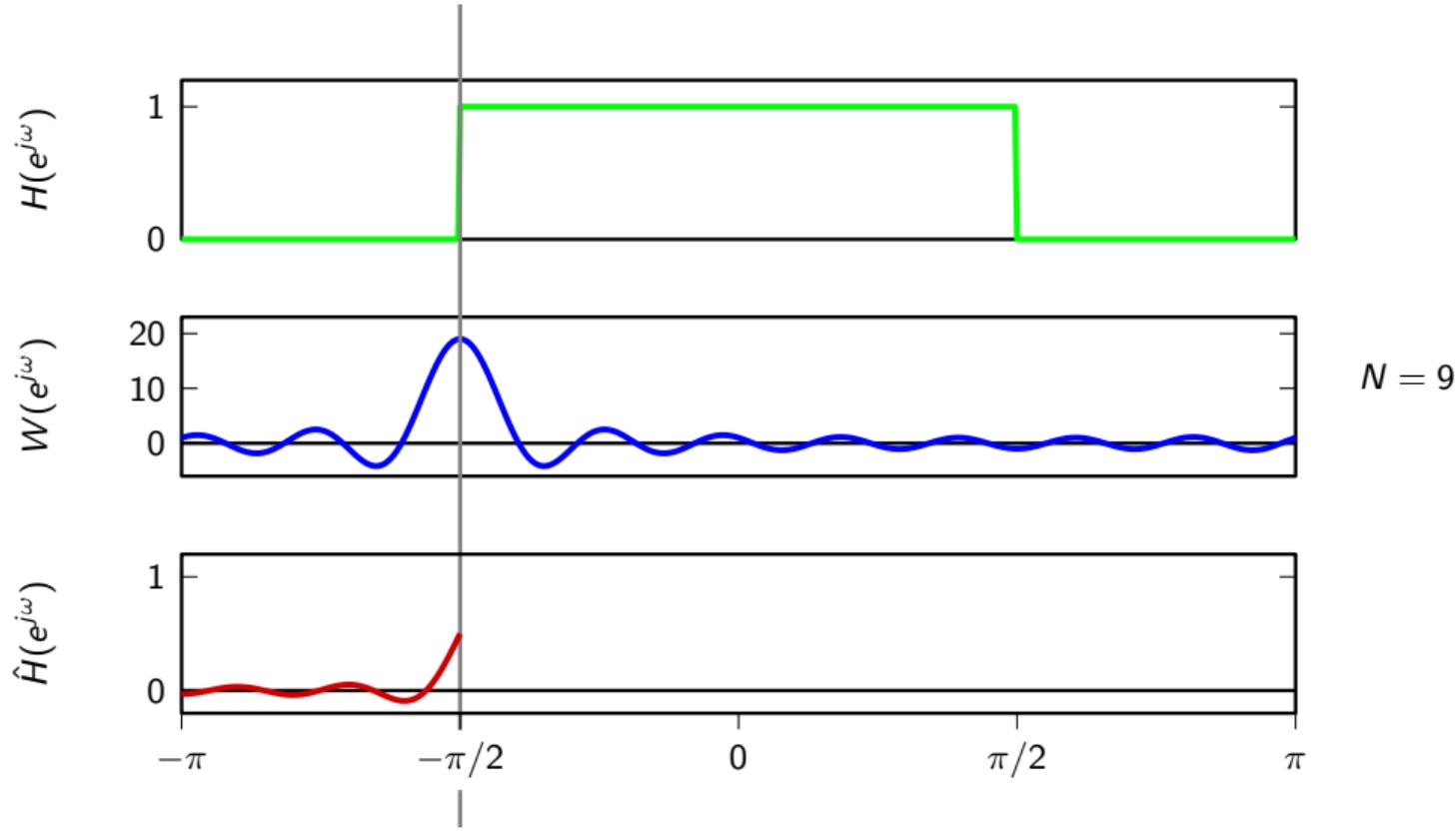
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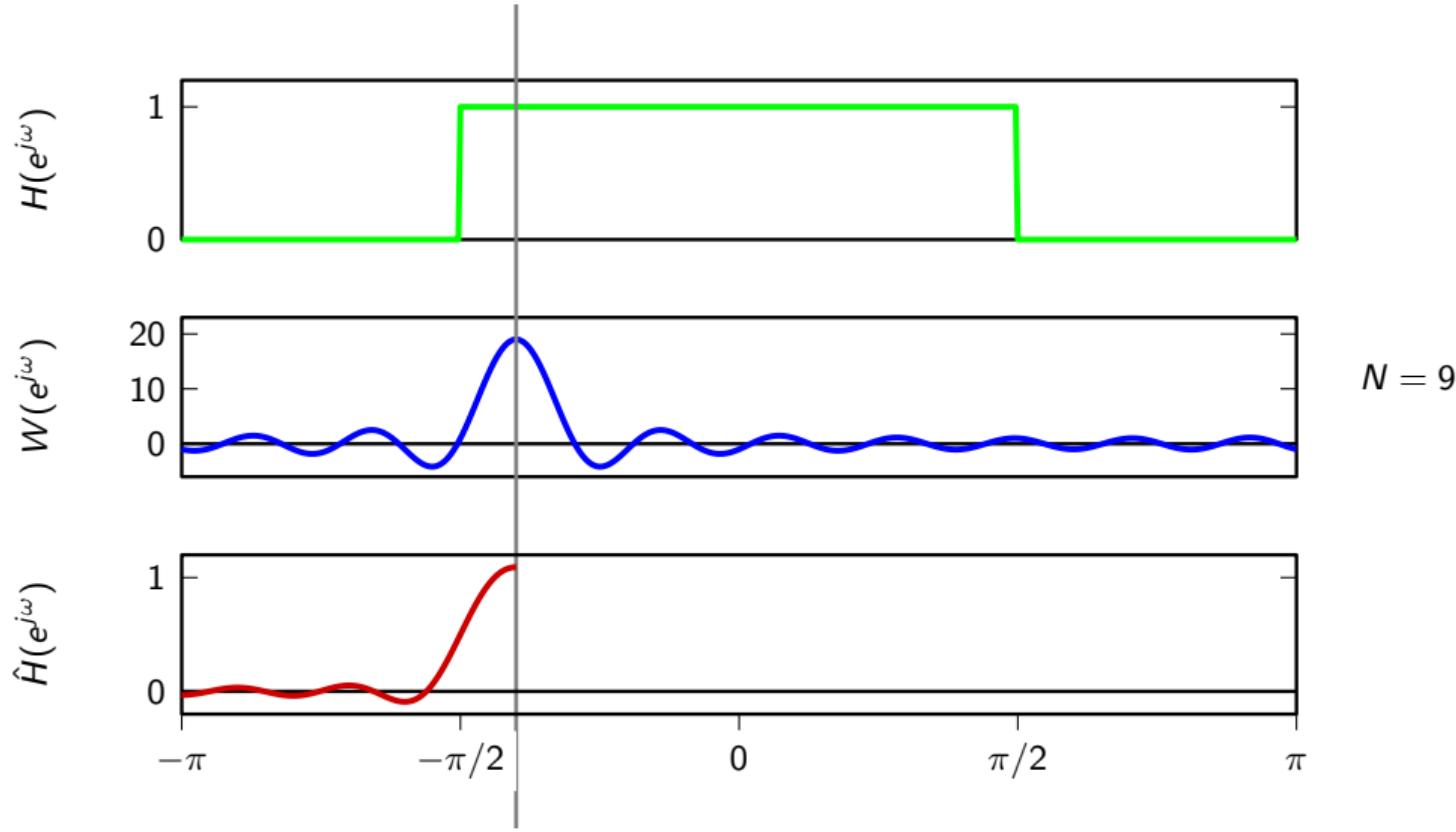
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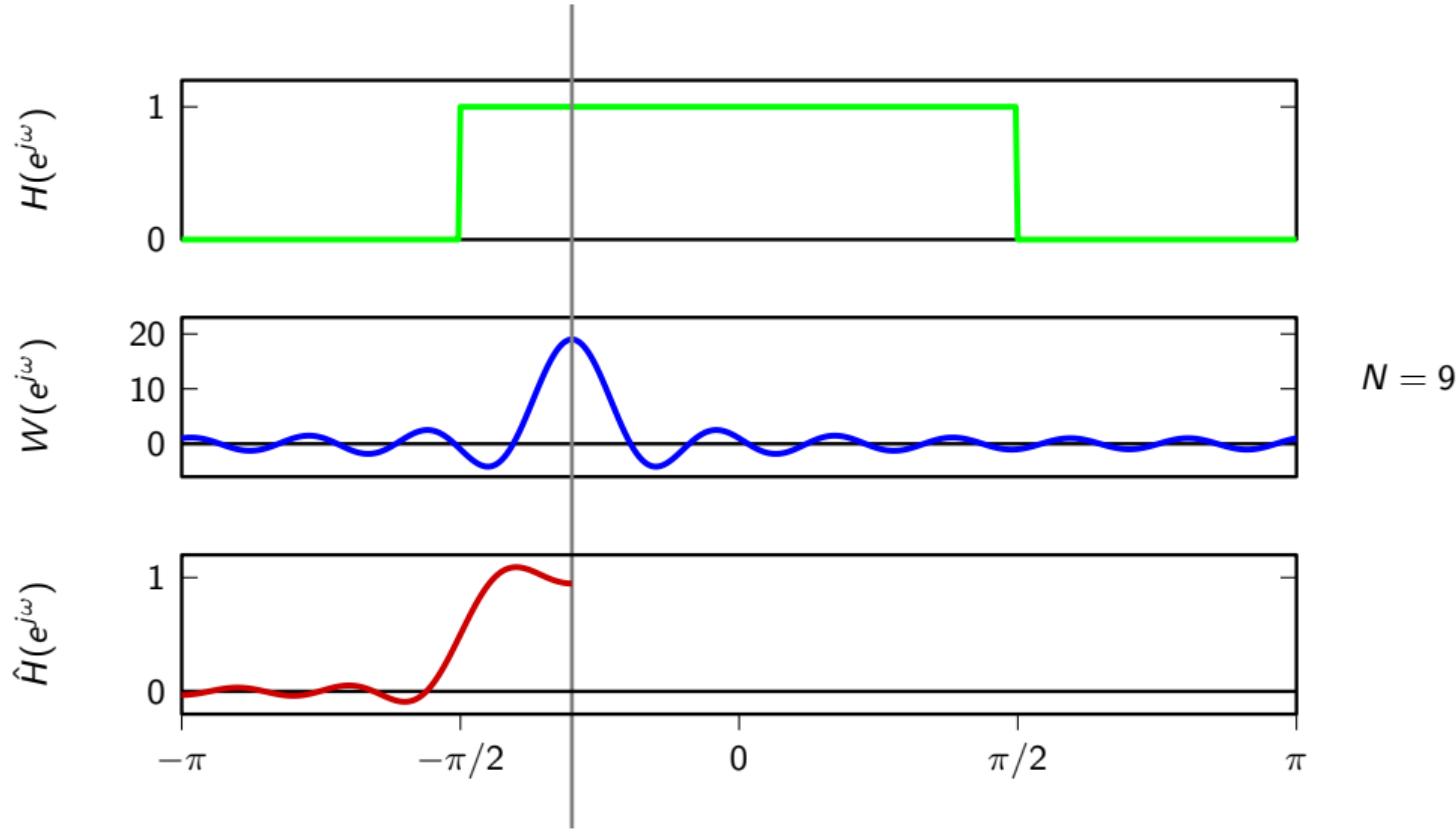
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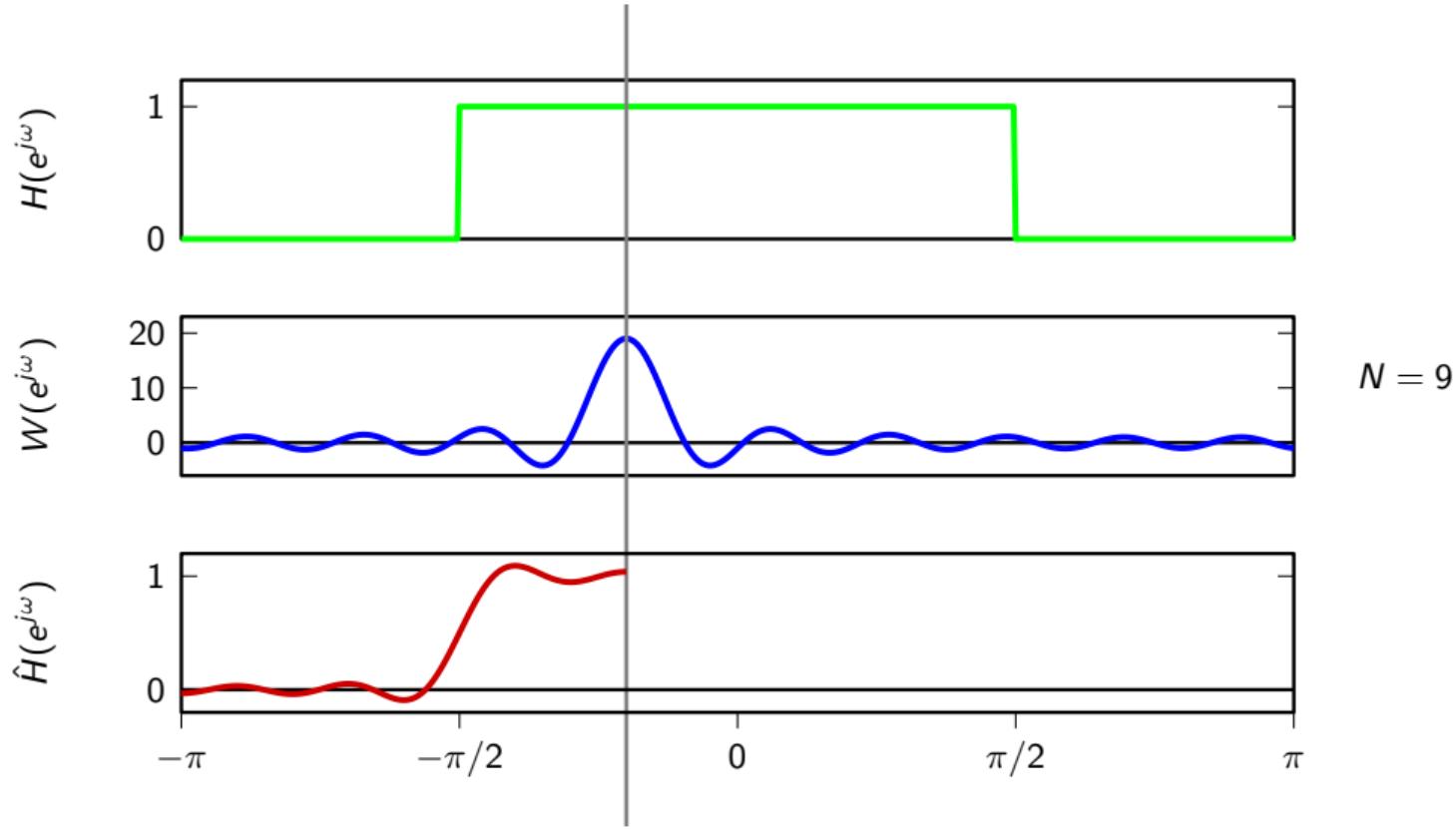
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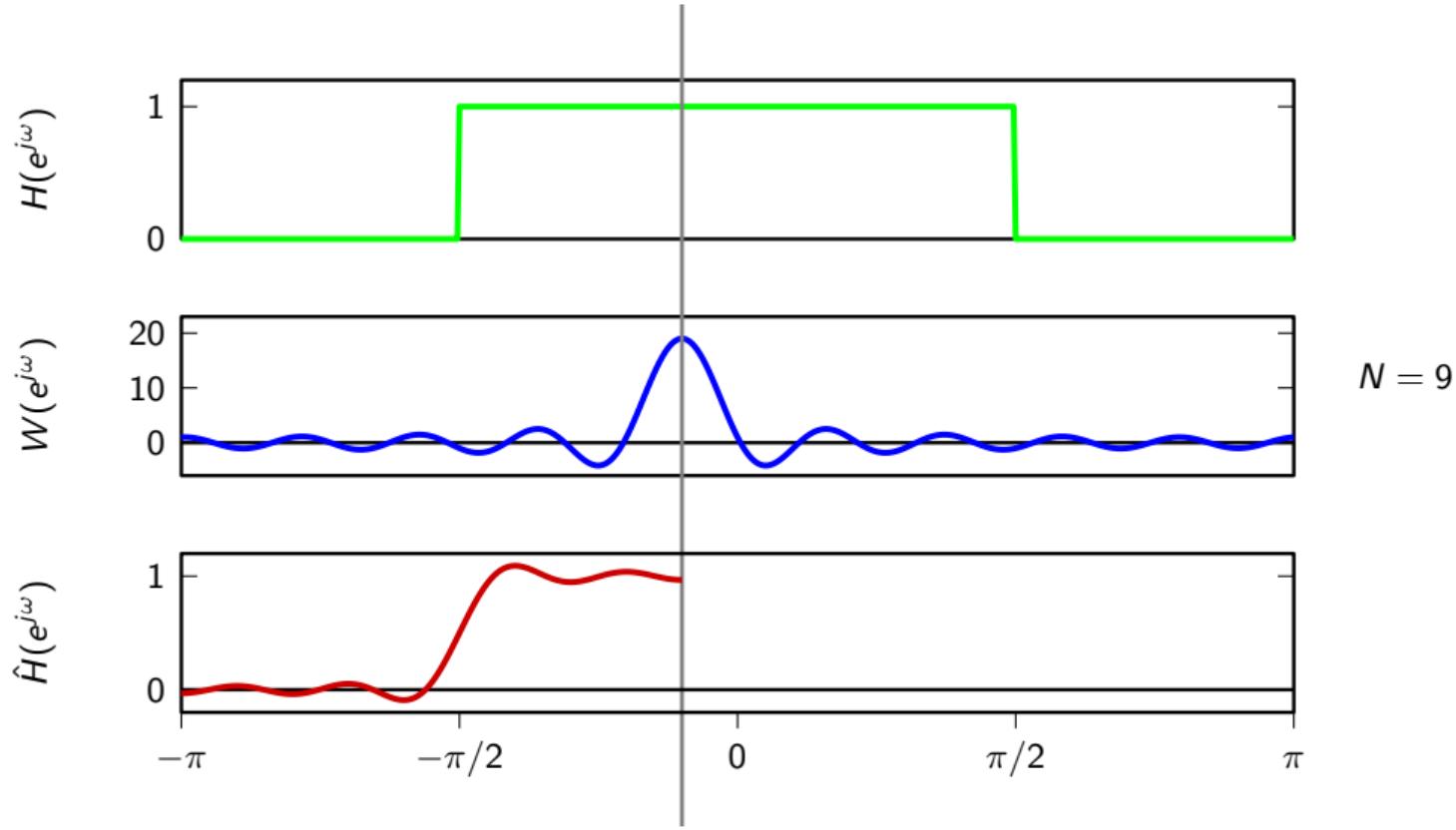
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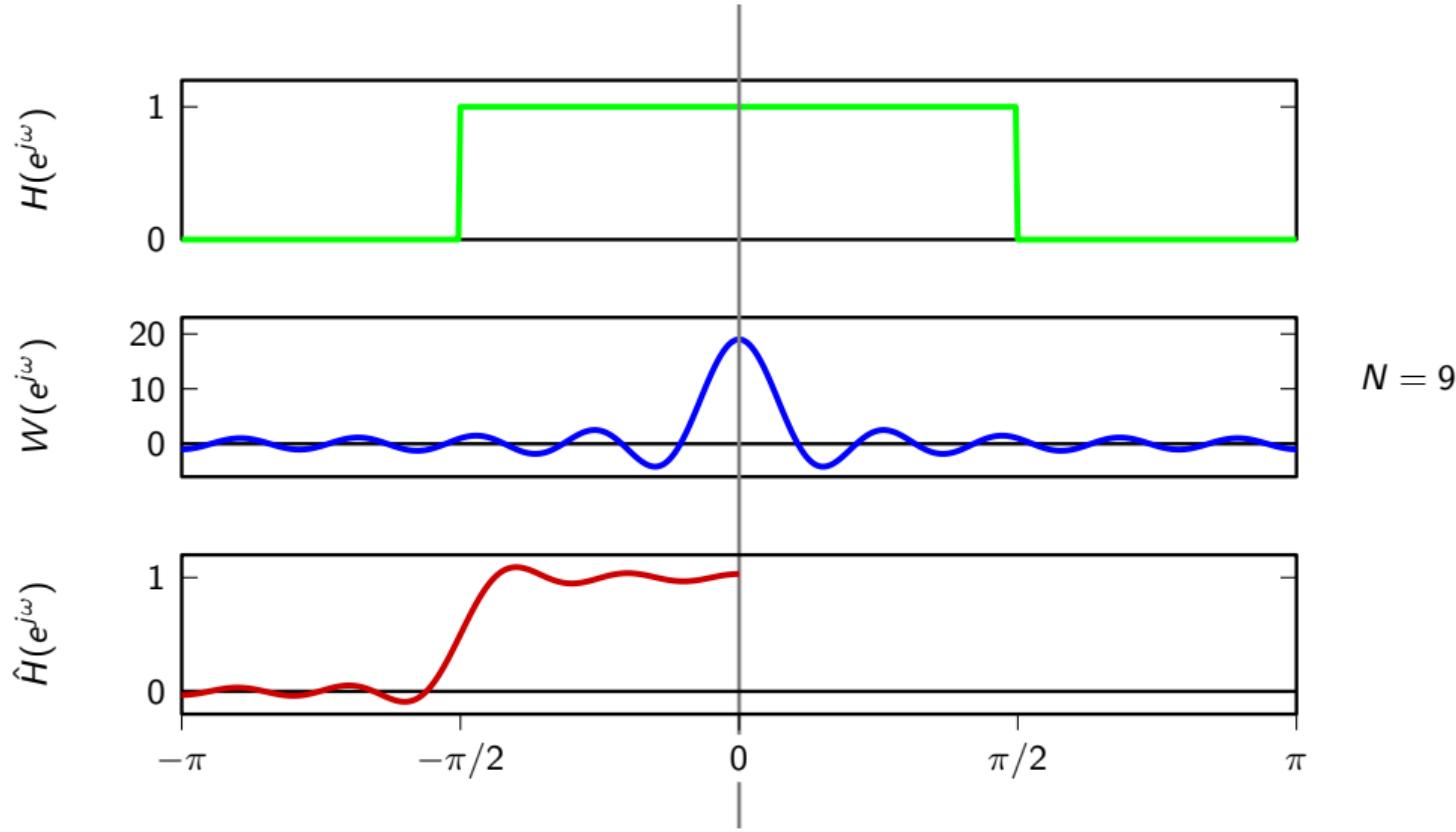
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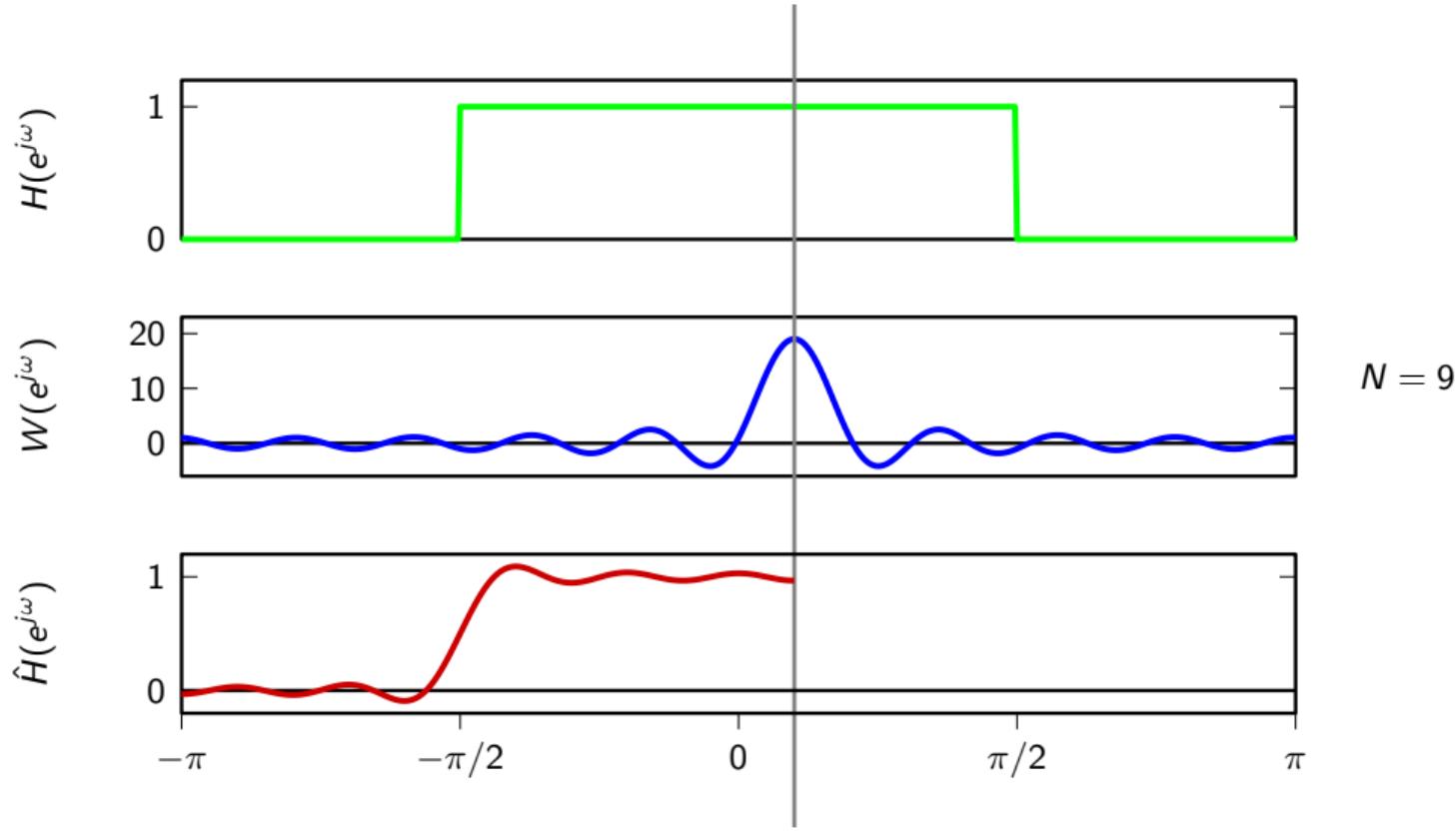
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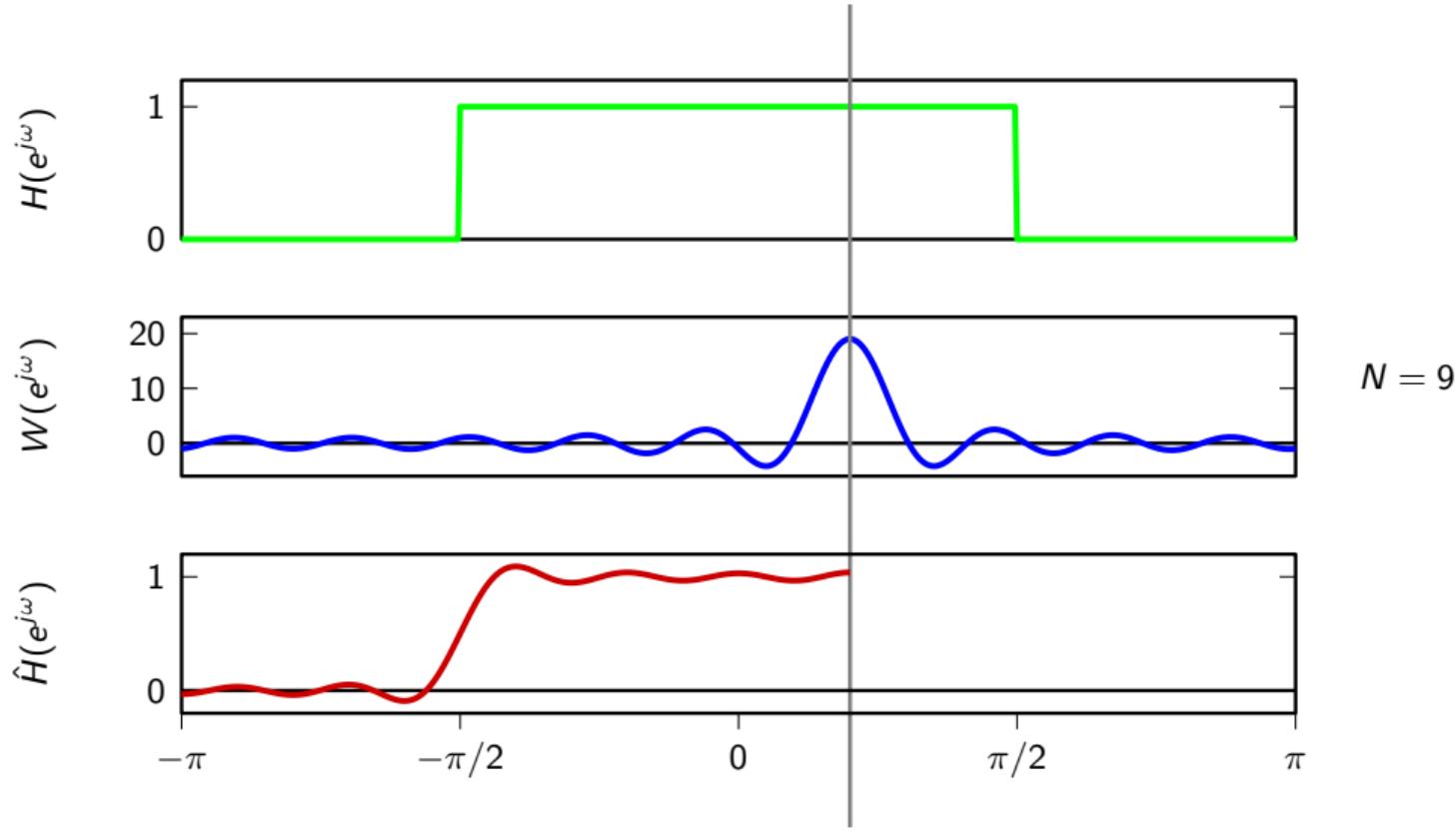
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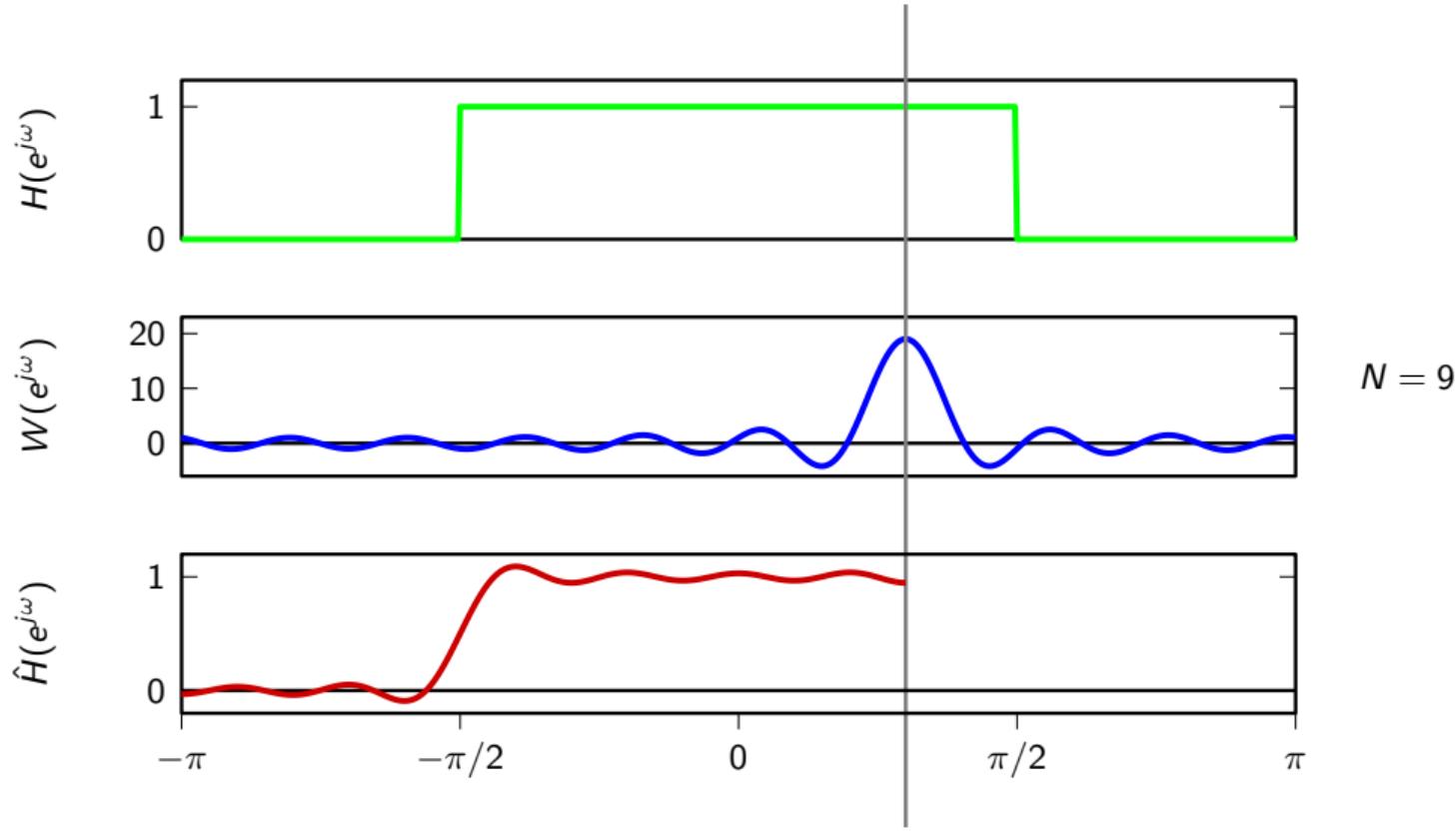
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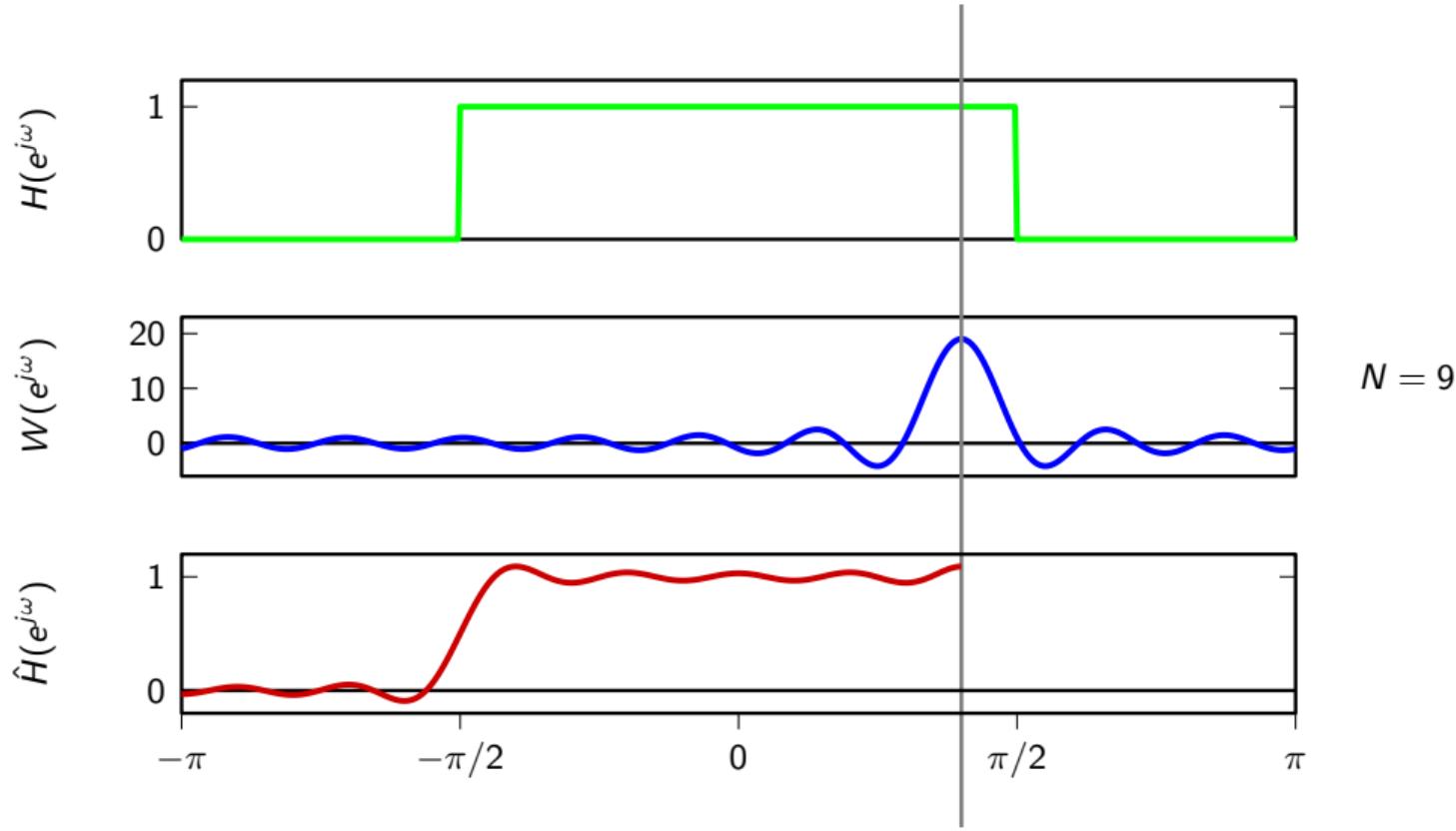
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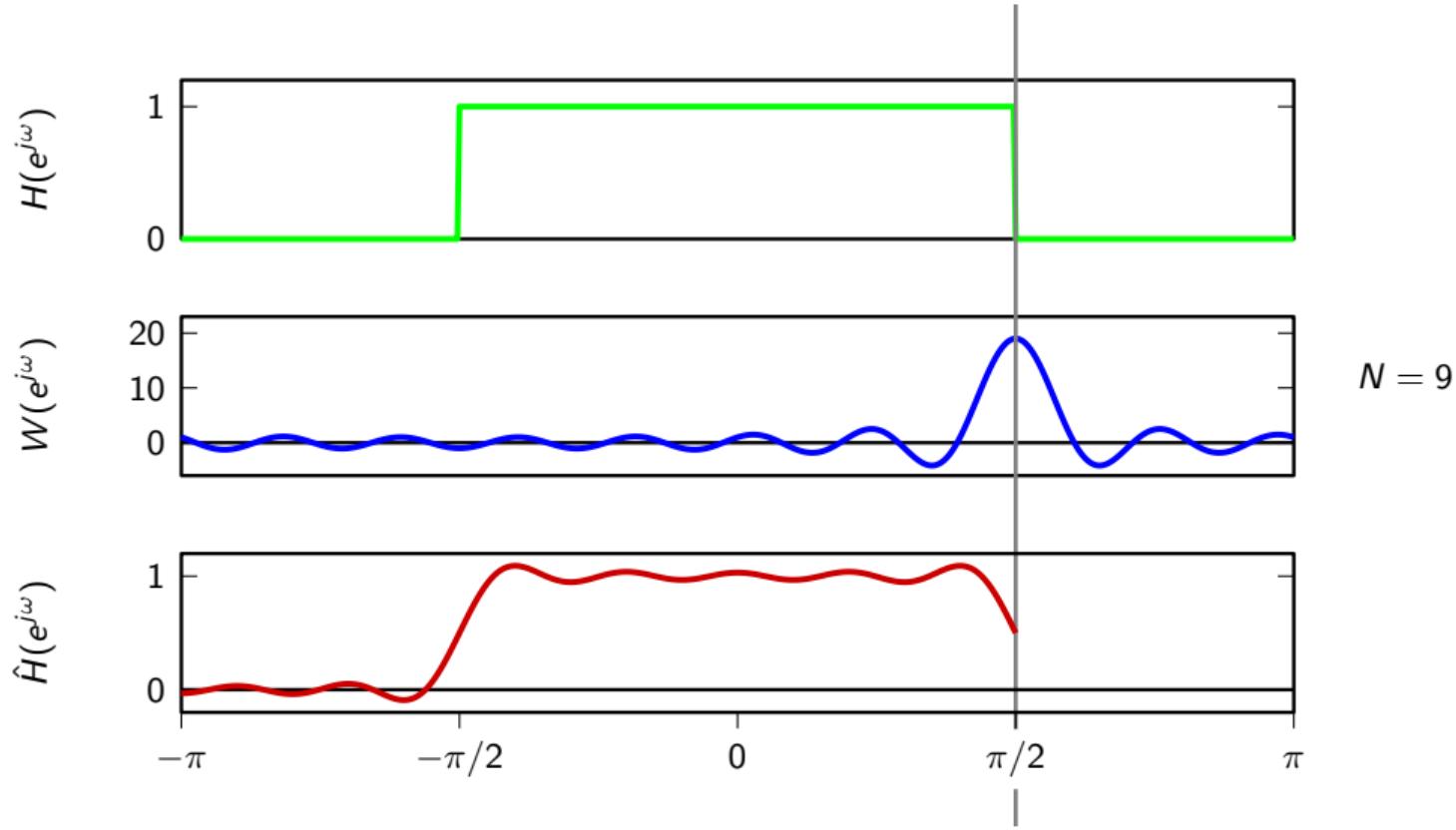
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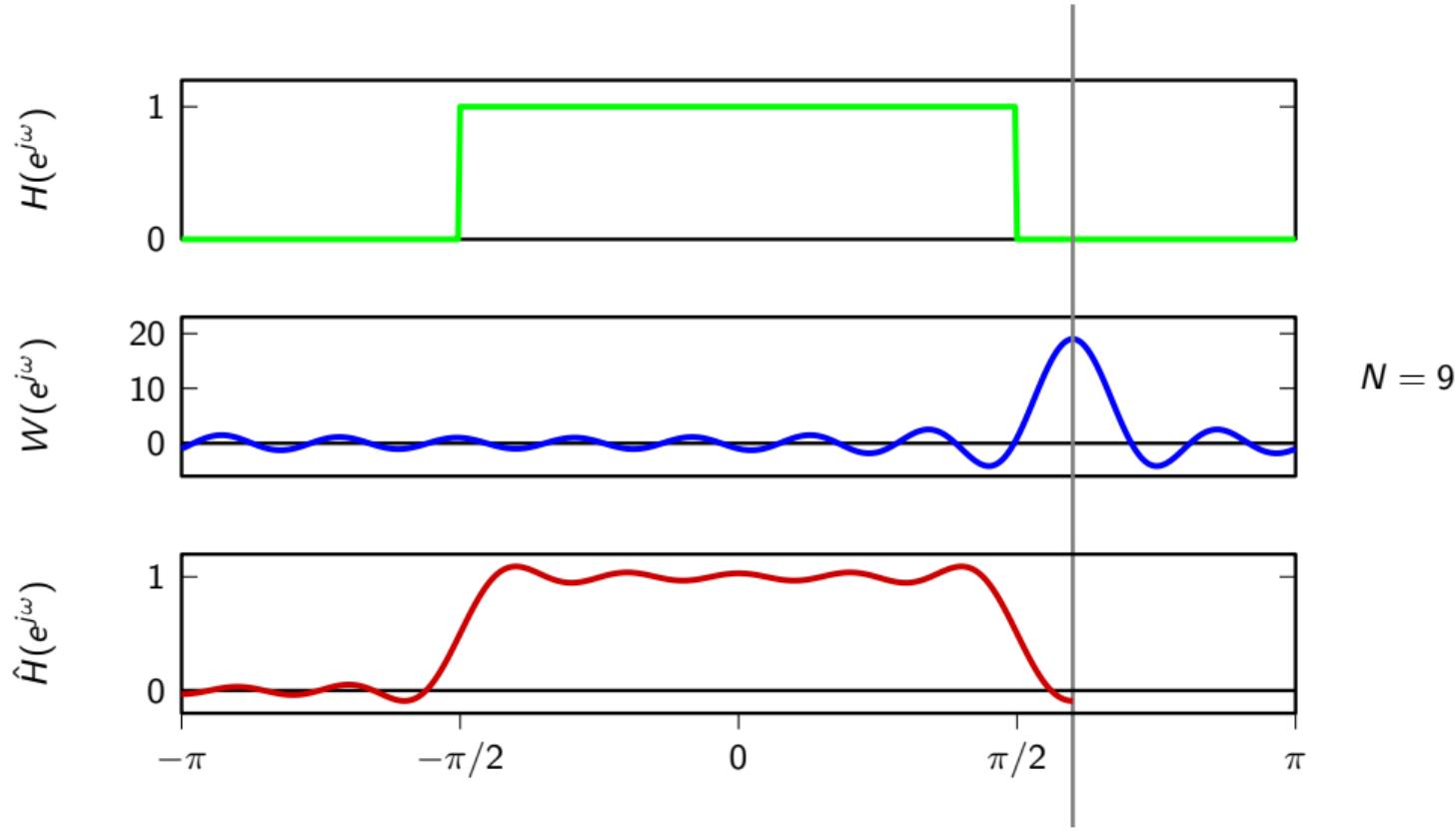
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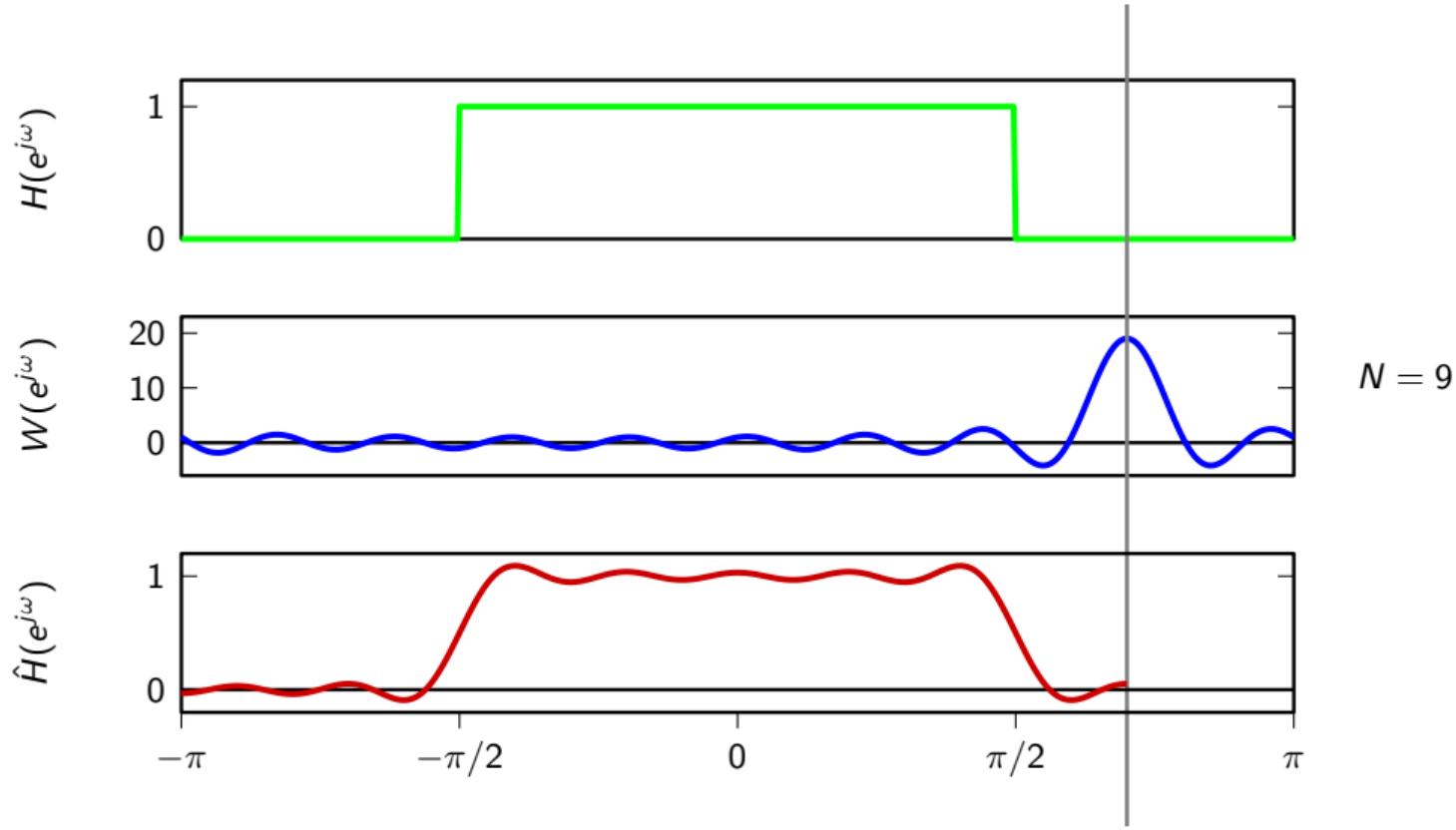
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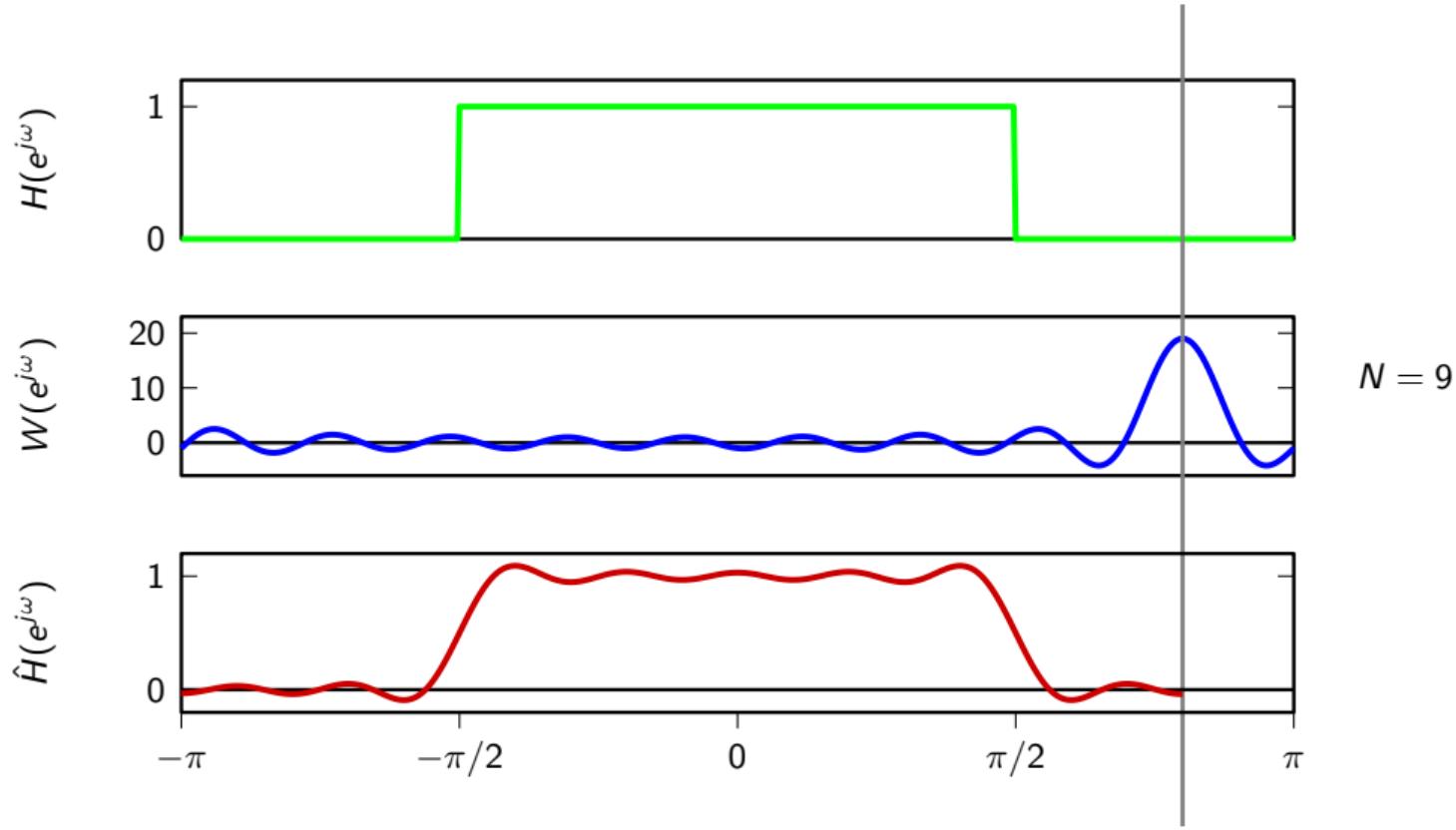
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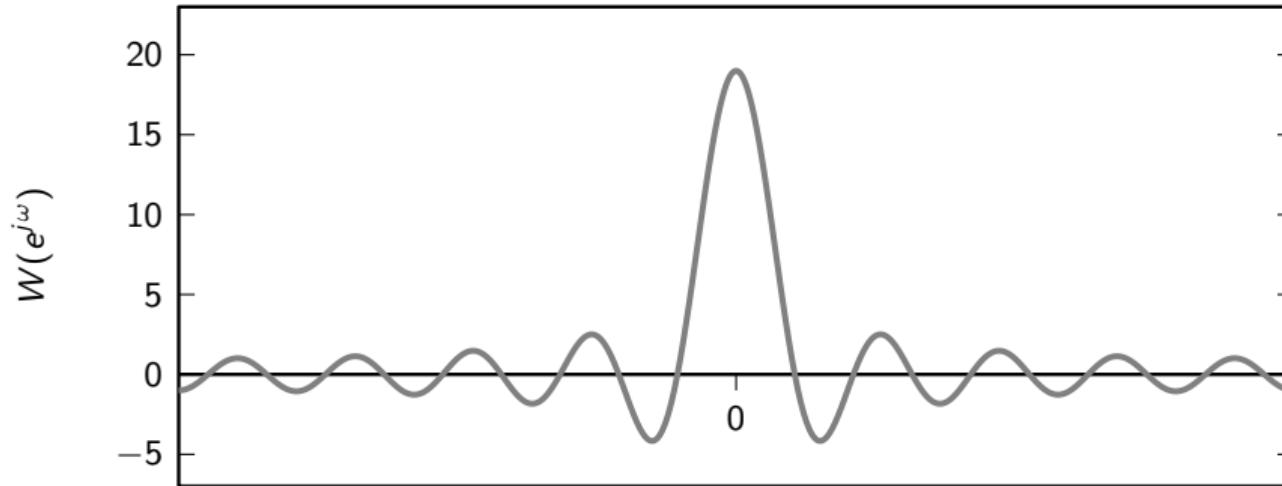
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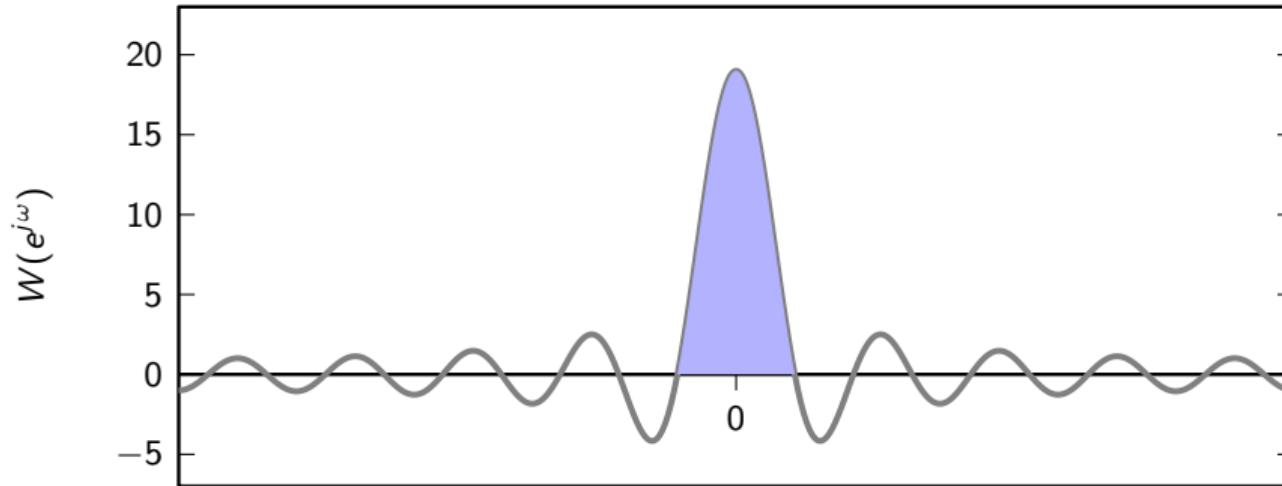
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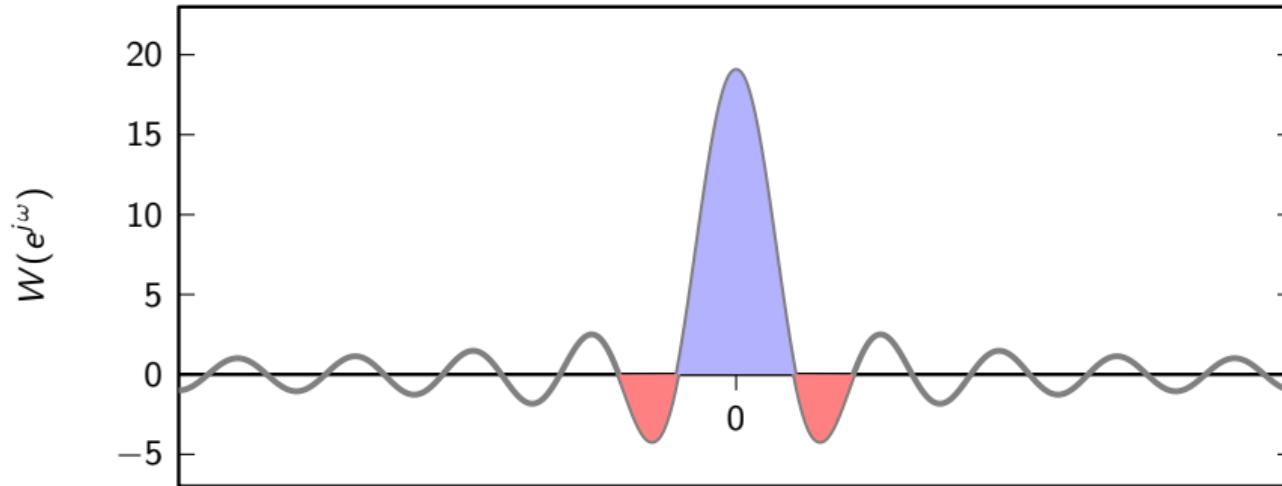
# Mainlobe and sidelobes



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# What if we change the window?

We want:

- ▶ narrow mainlobe so that transition is sharp
- ▶ small sidelobe so Gibbs error is small
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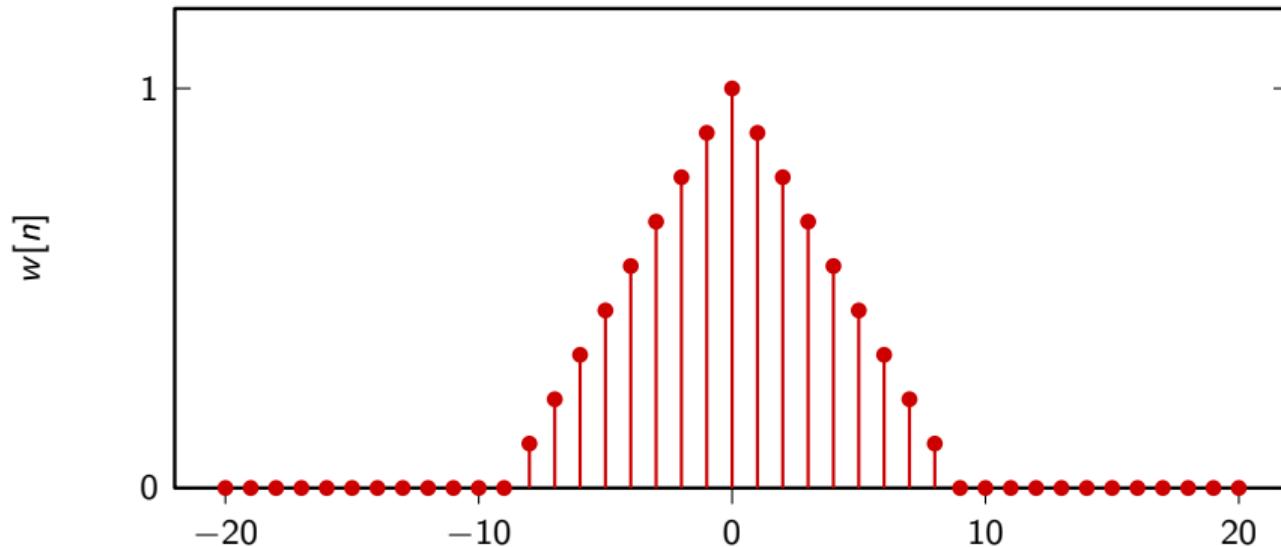
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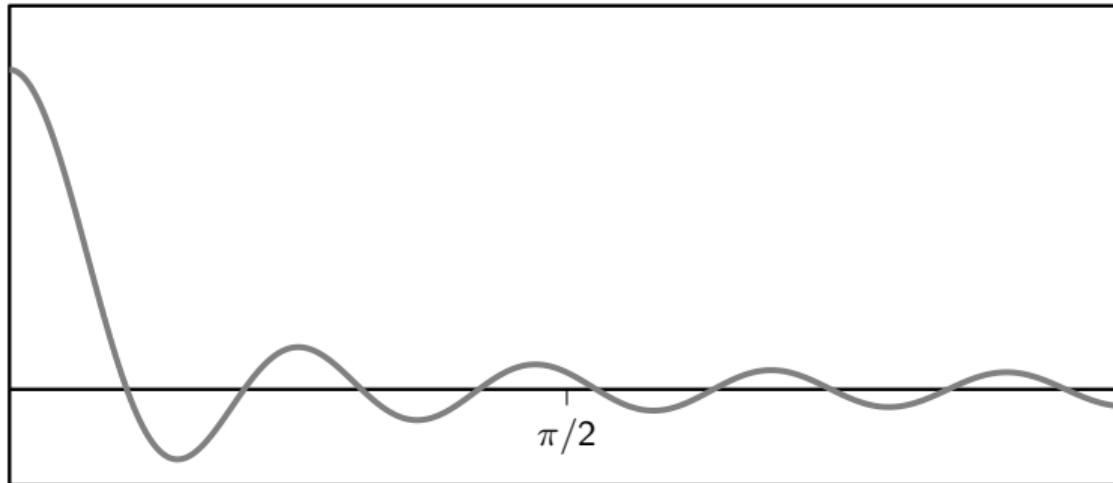
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very conflicting requirements!

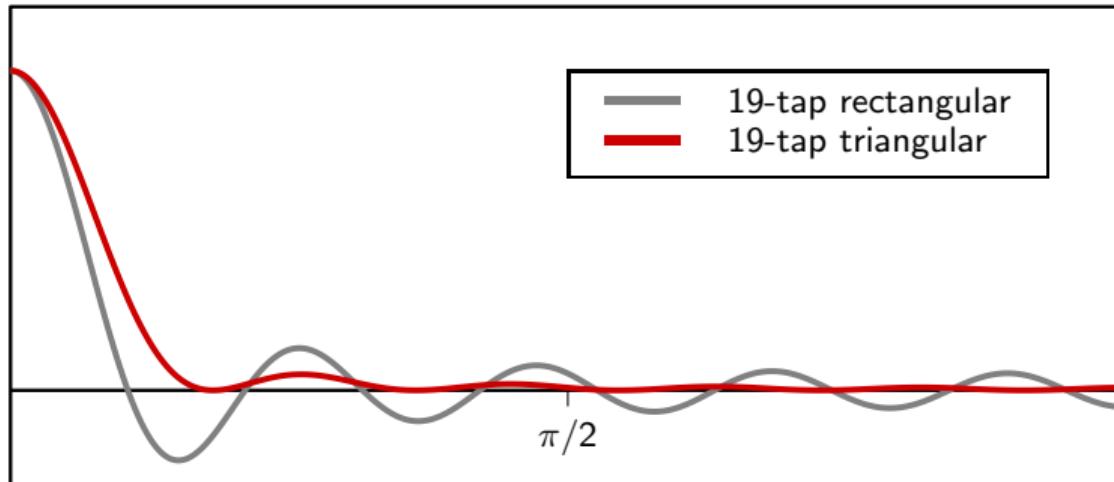
# Triangular window



## Rectangular vs Triangular Window



# Rectangular vs Triangular Window



Idea #2:

- ▶ draw desired frequency response  $H(e^{j\omega})$
- ▶ take  $M$  values at  $\omega_k = (2\pi/M)k$
- ▶ compute IDFT of values
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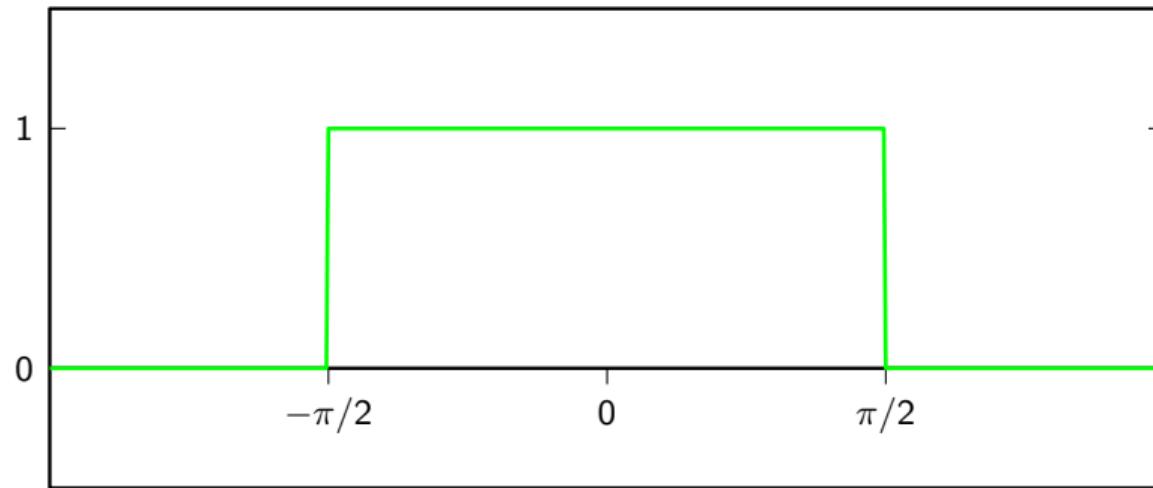
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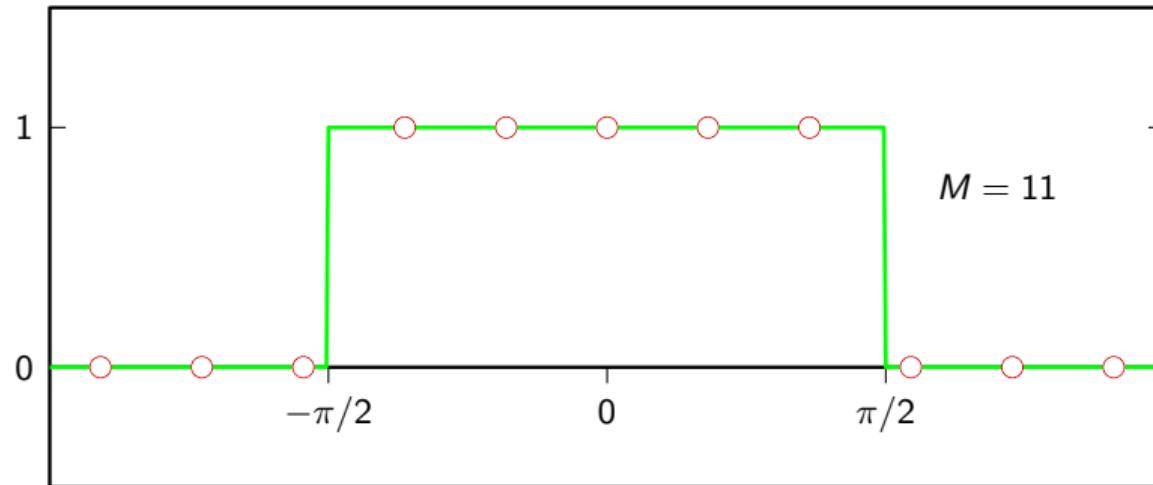
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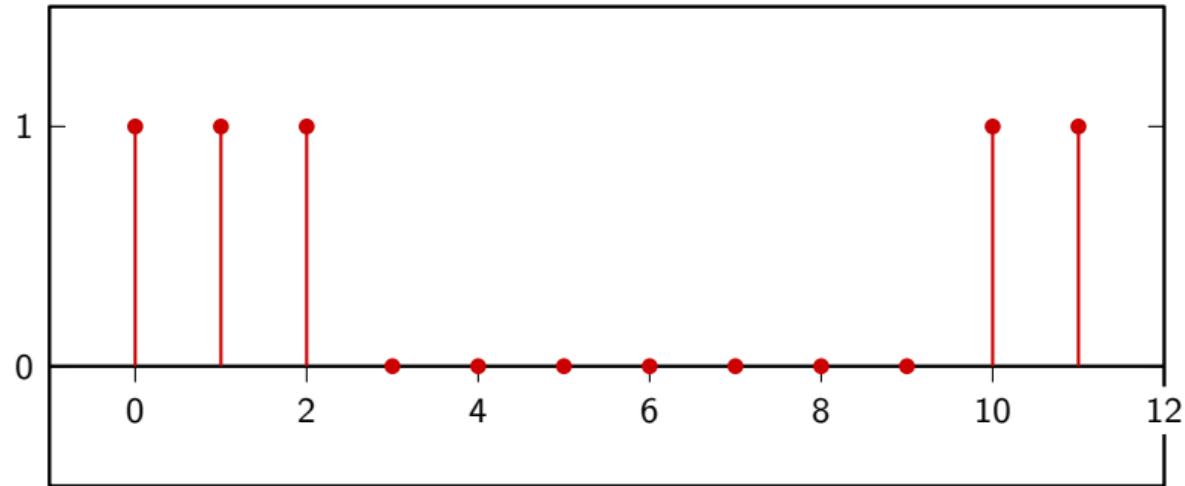


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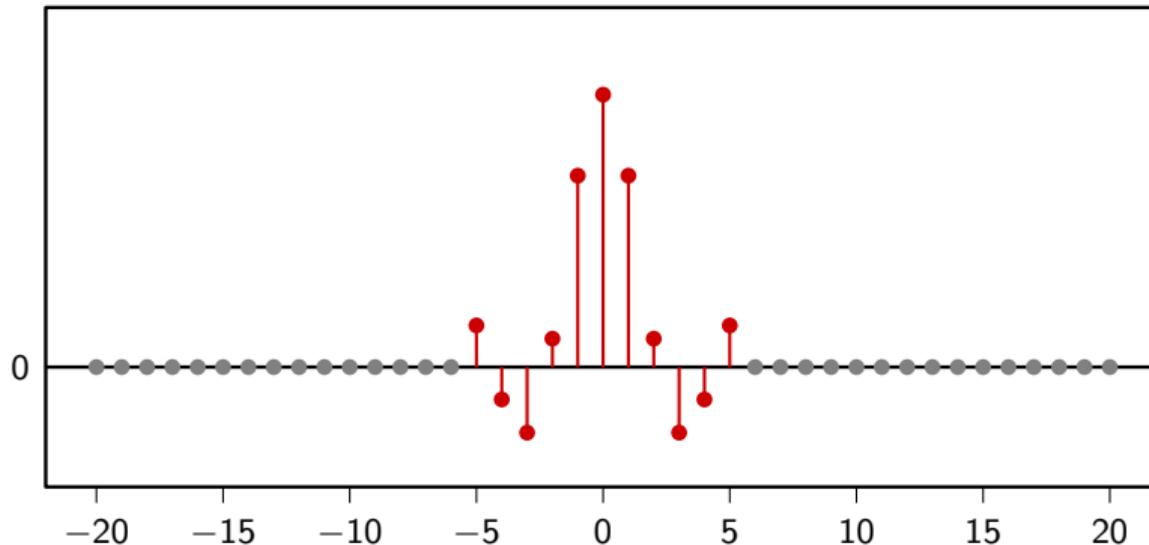


$M = 11$

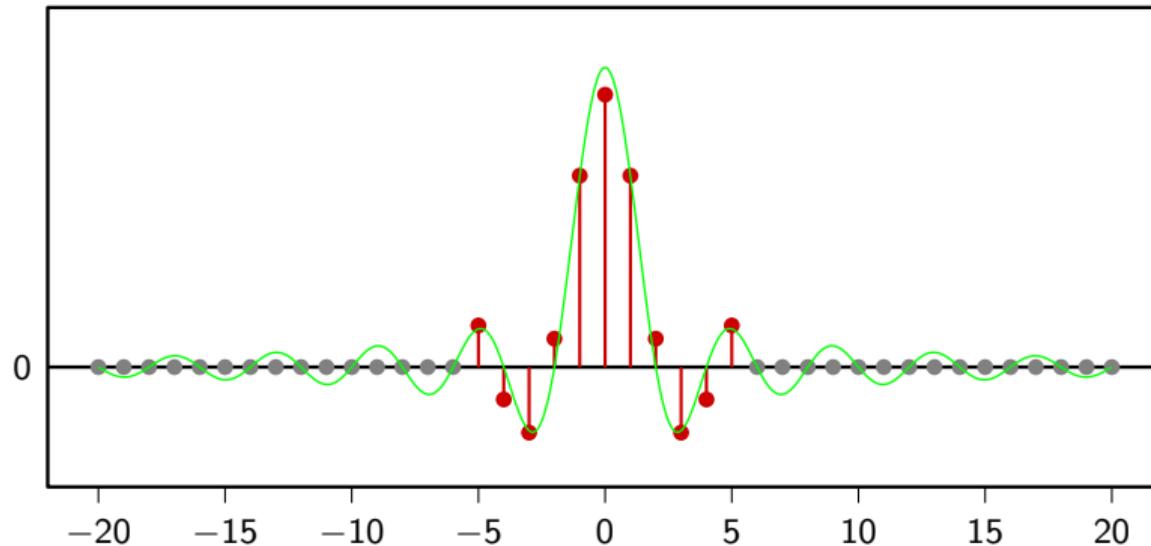
## Frequency sampling: DFT samples



## Frequency sampling: impulse response from IDFT



## Frequency sampling: impulse response from IDFT



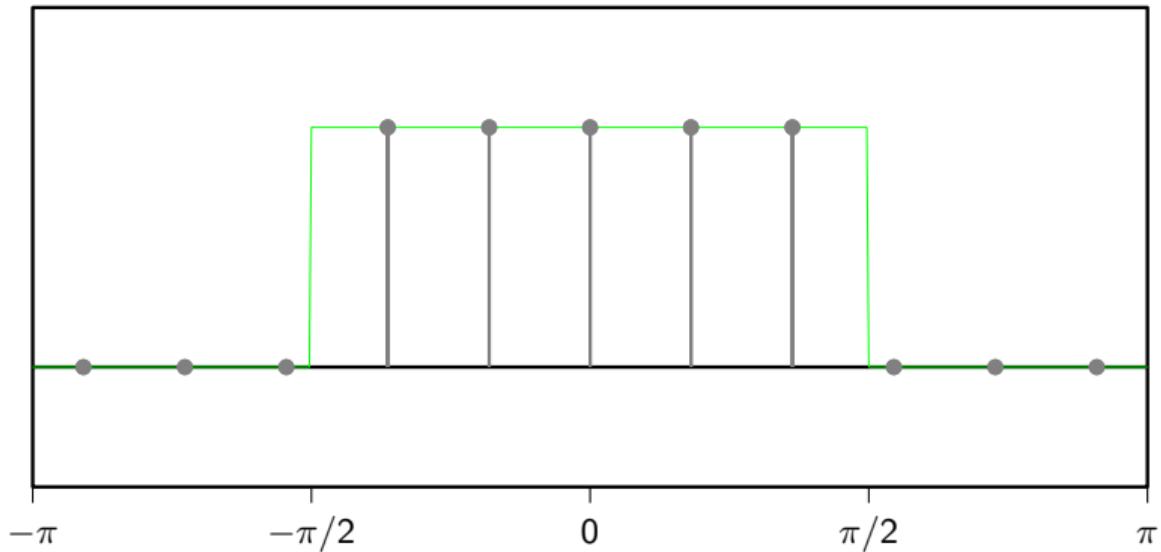
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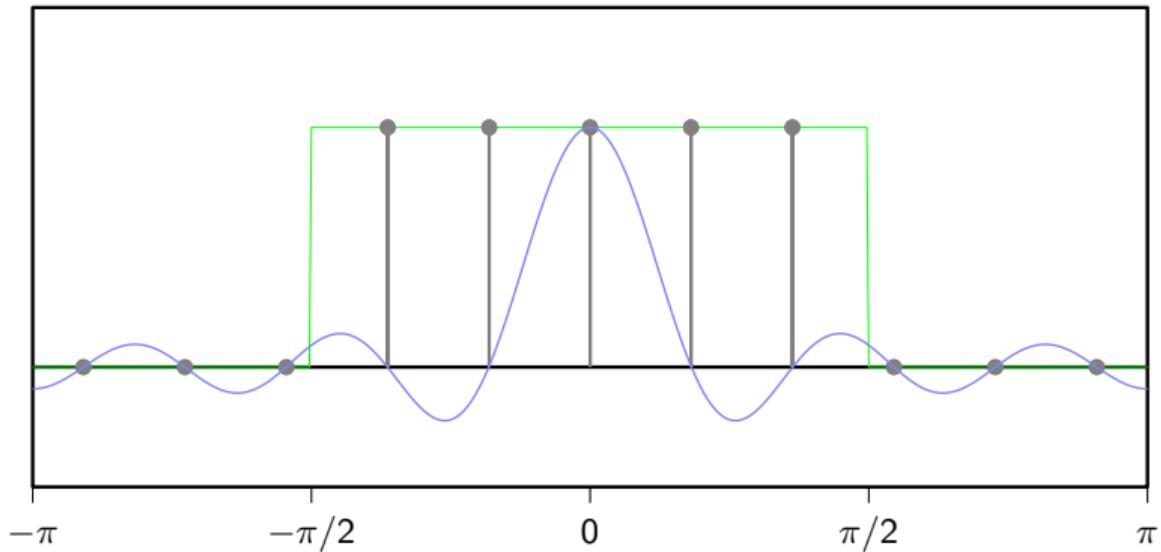
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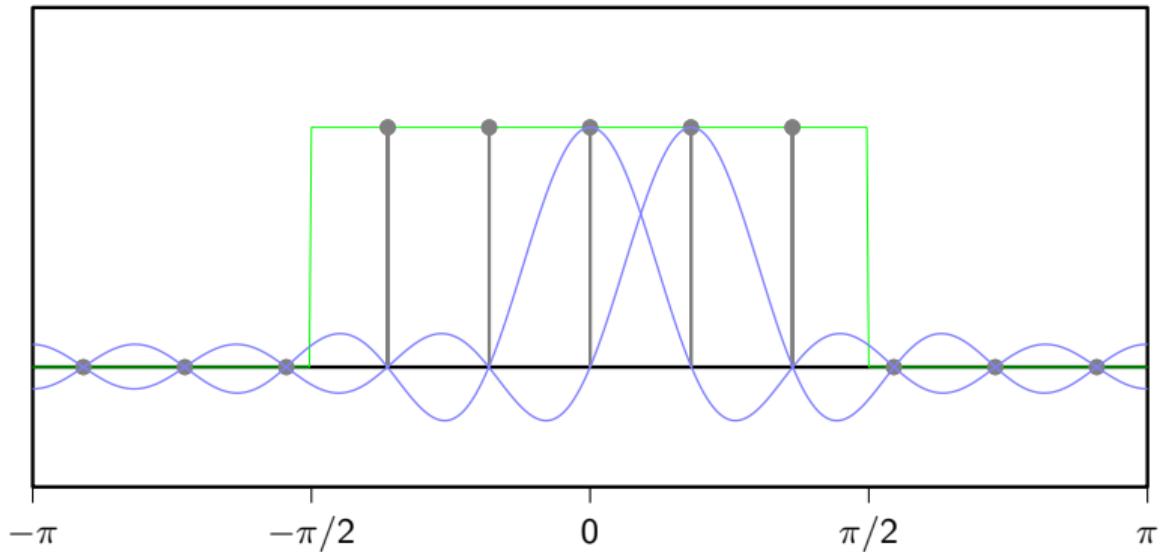
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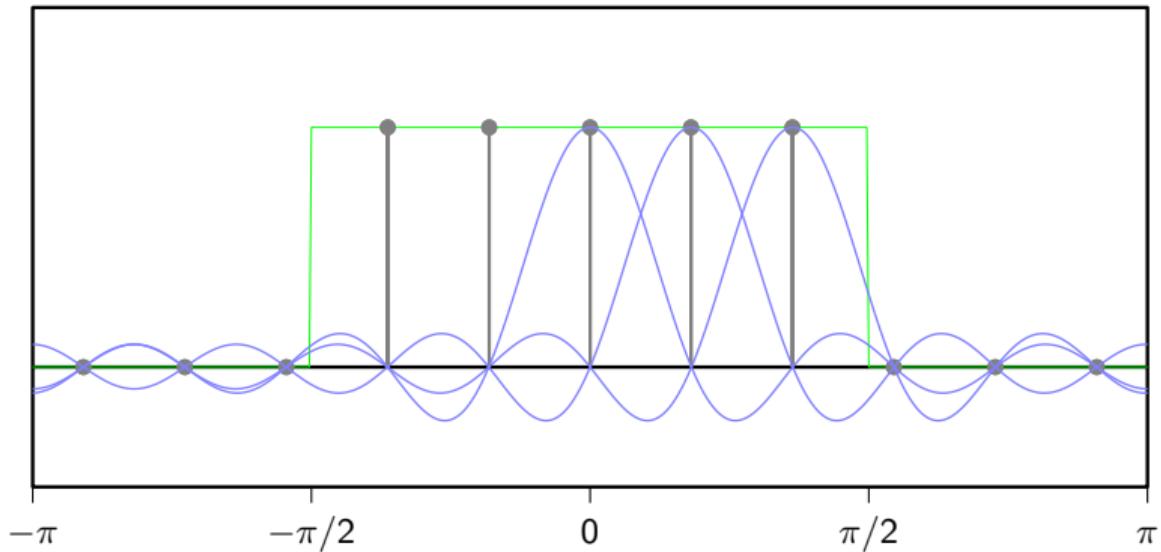
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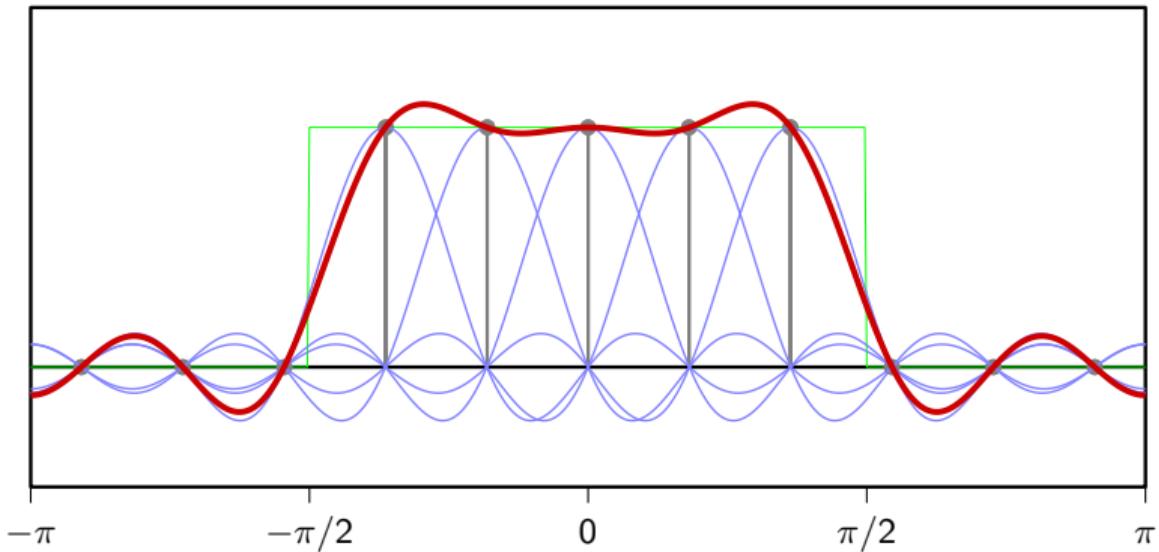
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# Frequency sampling: frequency response



END OF MODULE 5.6

# Digital Signal Processing

Module 5.7: Realizable Filters

- ▶ Constant-Coefficient Difference Equations
- ▶ The  $z$ -transform
- ▶ System transfer function
- ▶ Region of convergence and system stability

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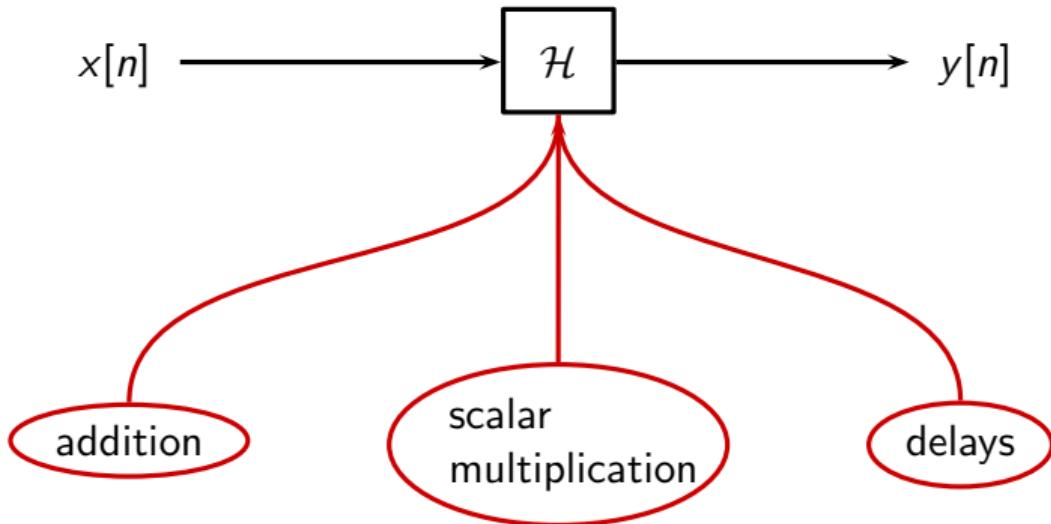
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  - linearity: only sums and multiplications
  - time-invariance: only multiplications by constants
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$$y[n] = (1 - \lambda)x[n] + \lambda y[n - 1]$$

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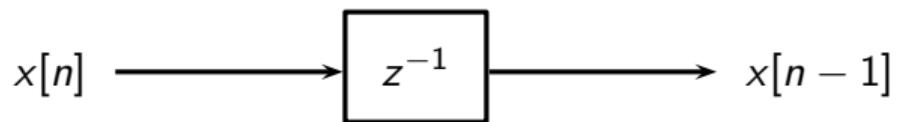
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BTW, remember the delay block?



now the notation should make more sense!

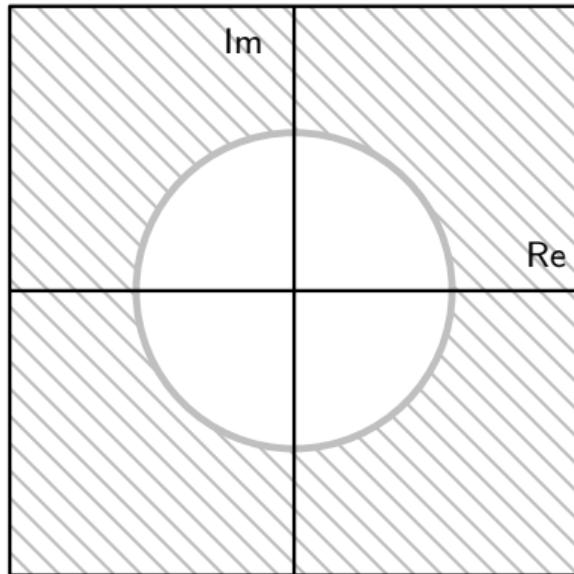
the  $z$ -transform is a power series, so convergence is always absolute

$$|X(z)| < \infty \iff \sum_{n=-\infty}^{\infty} |x[n]z^{-n}| < \infty$$

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- ▶ ROC has circular symmetry (depends only on  $|z|$ )
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Consider the transfer function for an LTI system:

$$H(z) = \frac{b_0 + b_1 z^{-1} + \dots + b_{M-1} z^{-(M-1)}}{a_0 + a_1 z^{-1} + \dots + a_{N-1} z^{-(N-1)}}$$

It can always be factored as:

$$H(z) = C \frac{\prod_{n=1}^{M-1} (1 - z_n z^{-1})}{\prod_{n=1}^{N-1} (1 - p_n z^{-1})}$$

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- ▶  $p_n$ 's: poles of the transfer function
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We know:

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- ▶ ROC cannot include poles

ROC extends outwards from a circle touching the largest-magnitude pole

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- ▶ ROC extends outwards
- ▶ ROC cannot include poles

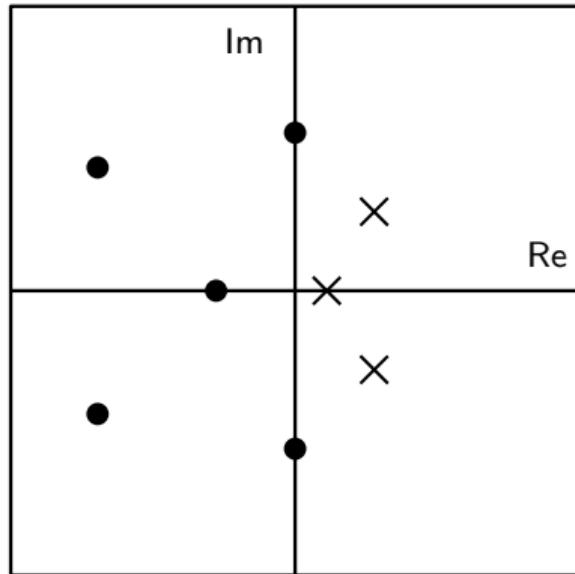
ROC extends outwards from a circle touching the largest-magnitude pole

We know:

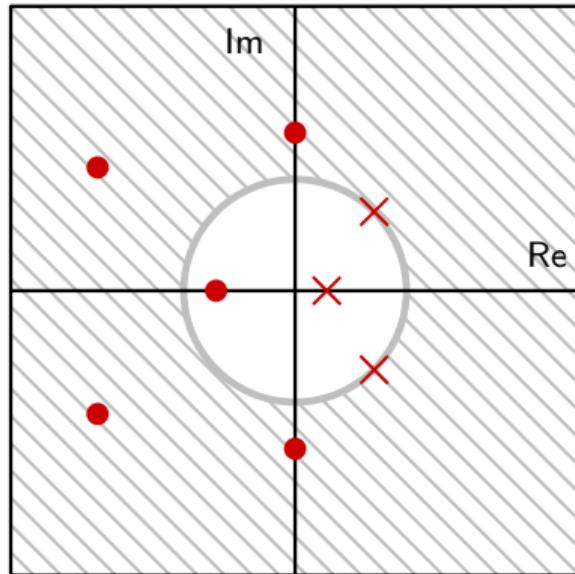
- ▶ ROC extends outwards
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ROC extends outwards from a circle touching the largest-magnitude pole

# ROC for causal systems



# ROC for causal systems



Consider a filter with impulse response  $h[n]$

- ▶ BIBO stability  $\iff \sum_{n=-\infty}^{\infty} |h[n]| < \infty$  (Module 5.3)
- ▶  $H(z)|_{|z|=1} < \infty \iff \sum_{n=-\infty}^{\infty} |h[n]| < \infty$  (absolute convergence of z-transform)

system is stable if and only if ROC includes the unit circle!

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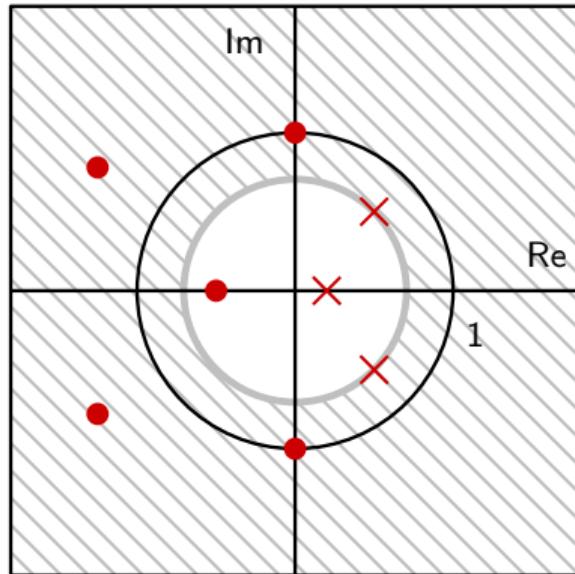
system is stable if and only if ROC includes the unit circle!

Consider a filter with impulse response  $h[n]$

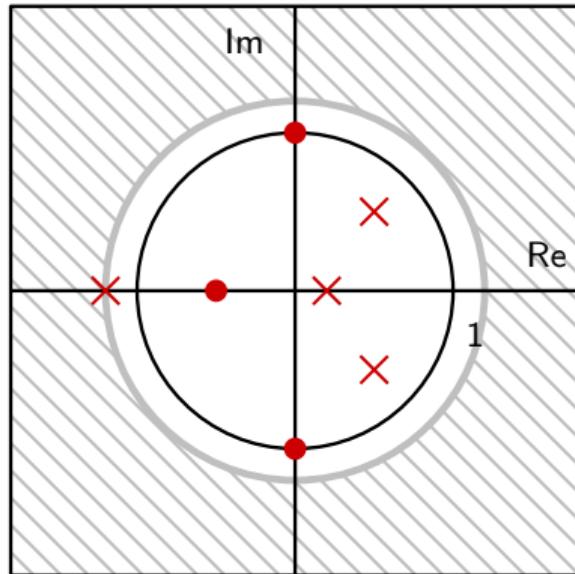
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- ▶  $H(z)|_{|z|=1} < \infty \iff \sum_{n=-\infty}^{\infty} |h[n]| < \infty$  (absolute convergence of z-transform)

system is stable if and only if ROC includes the unit circle!

# Stable system



# Unstable system



The “circus tent” method:

- ▶ magnitude of  $z$  transform is like a rubber sheet over the complex plane
- ▶ zeros glue the sheet to the ground
- ▶ poles are like ... poles, pushing it up
- ▶ frequency response (in magnitude) is sheet profile around the unit circle

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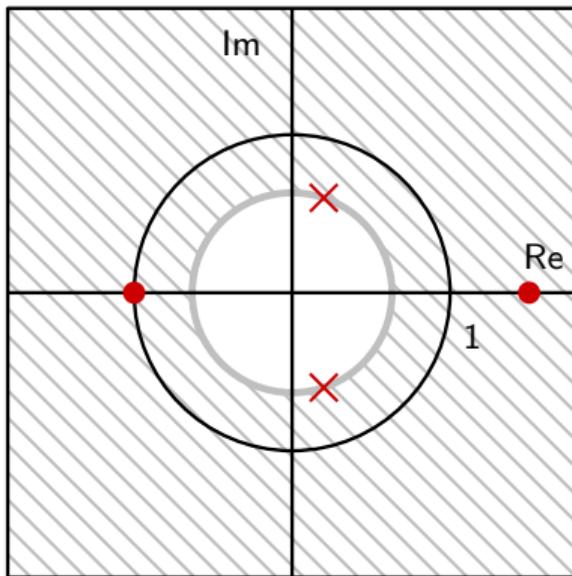
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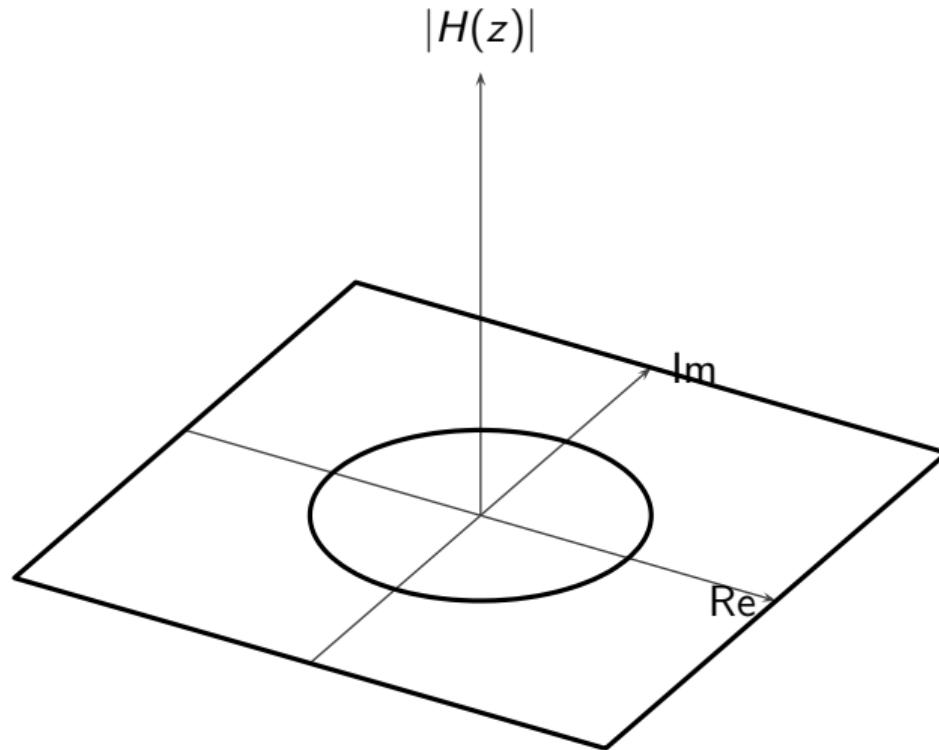
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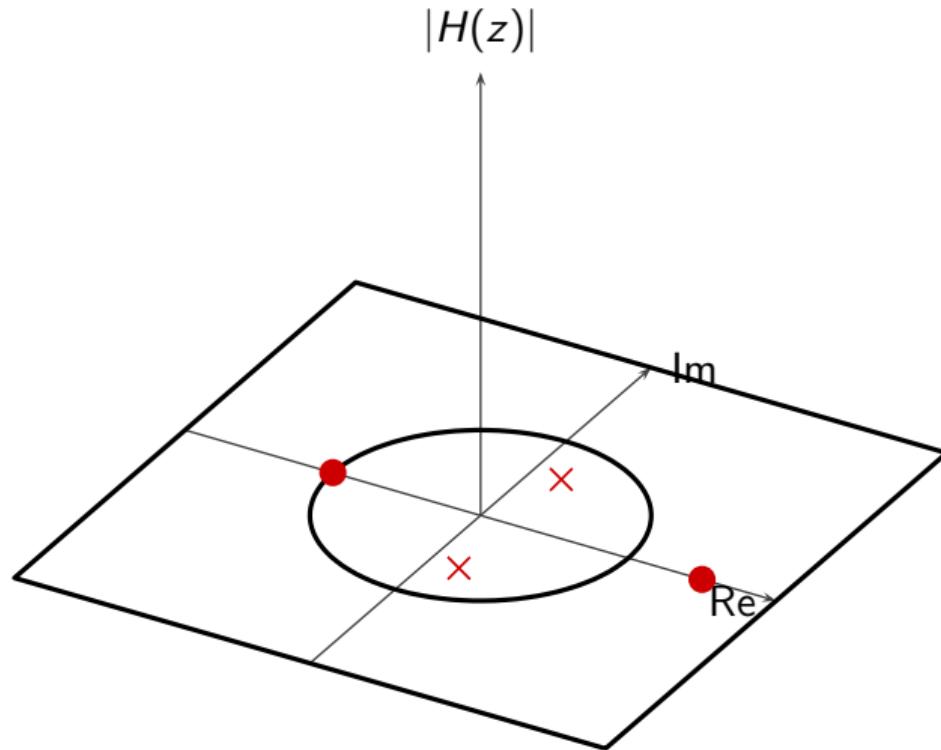
# Estimating the frequency response



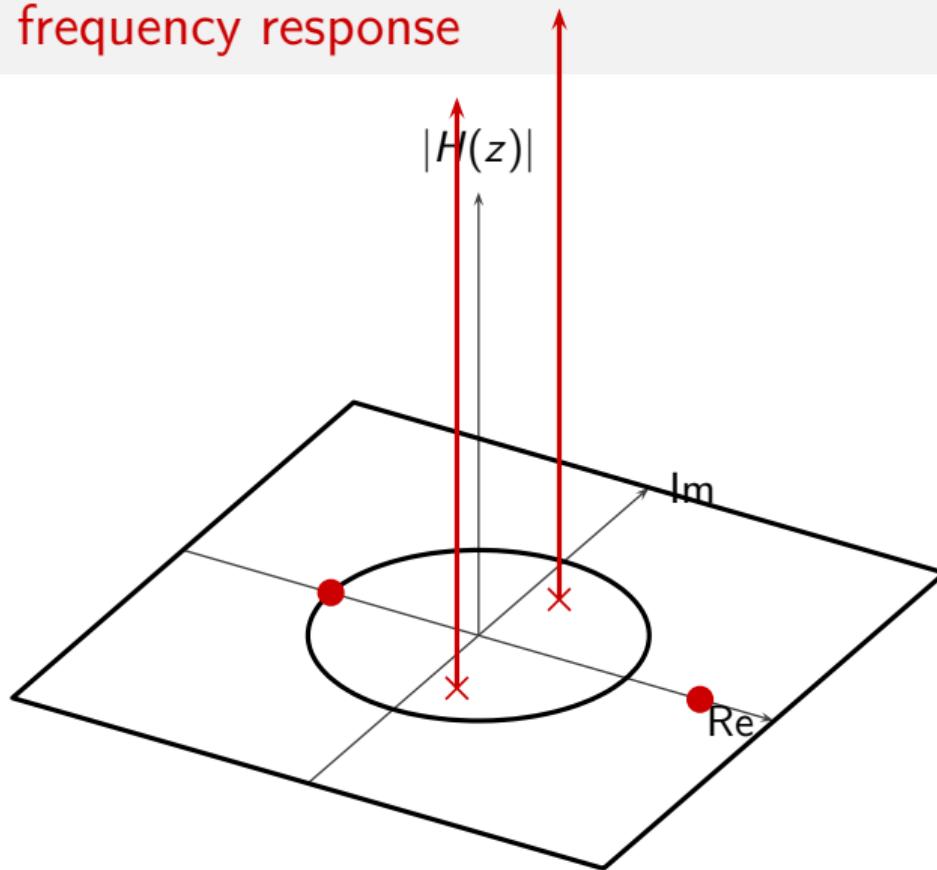
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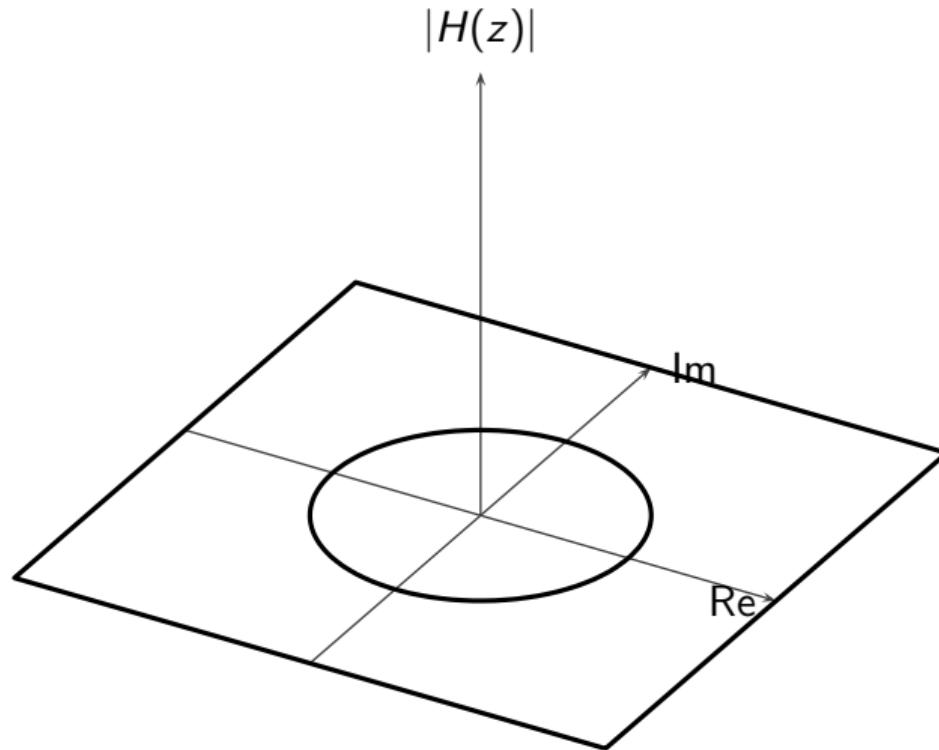
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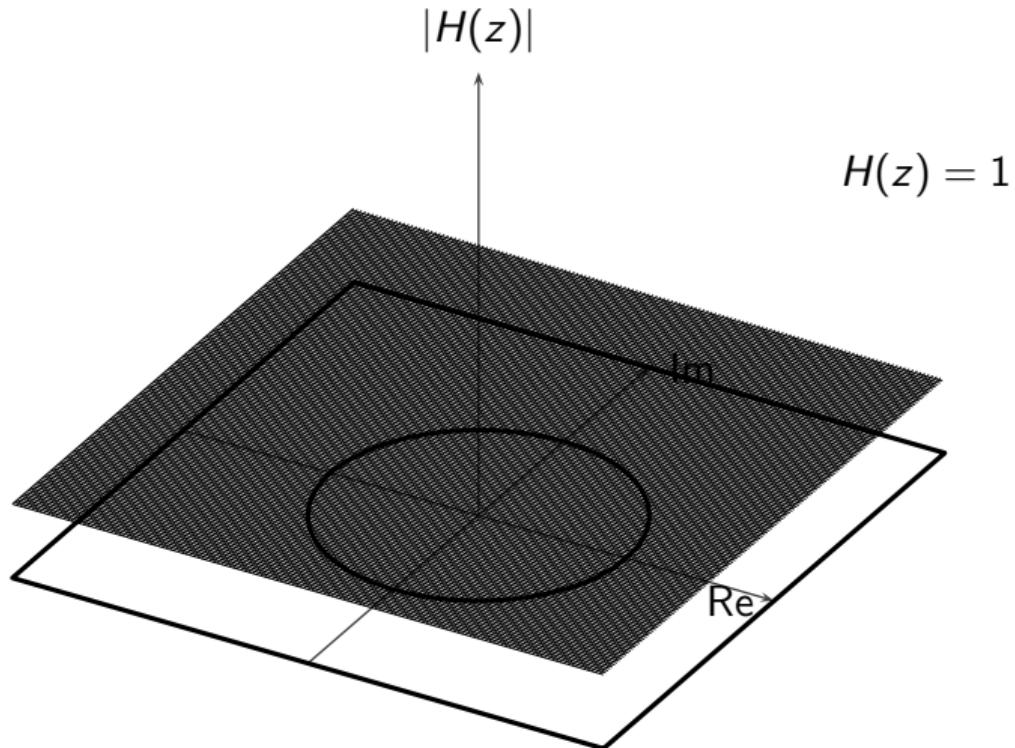
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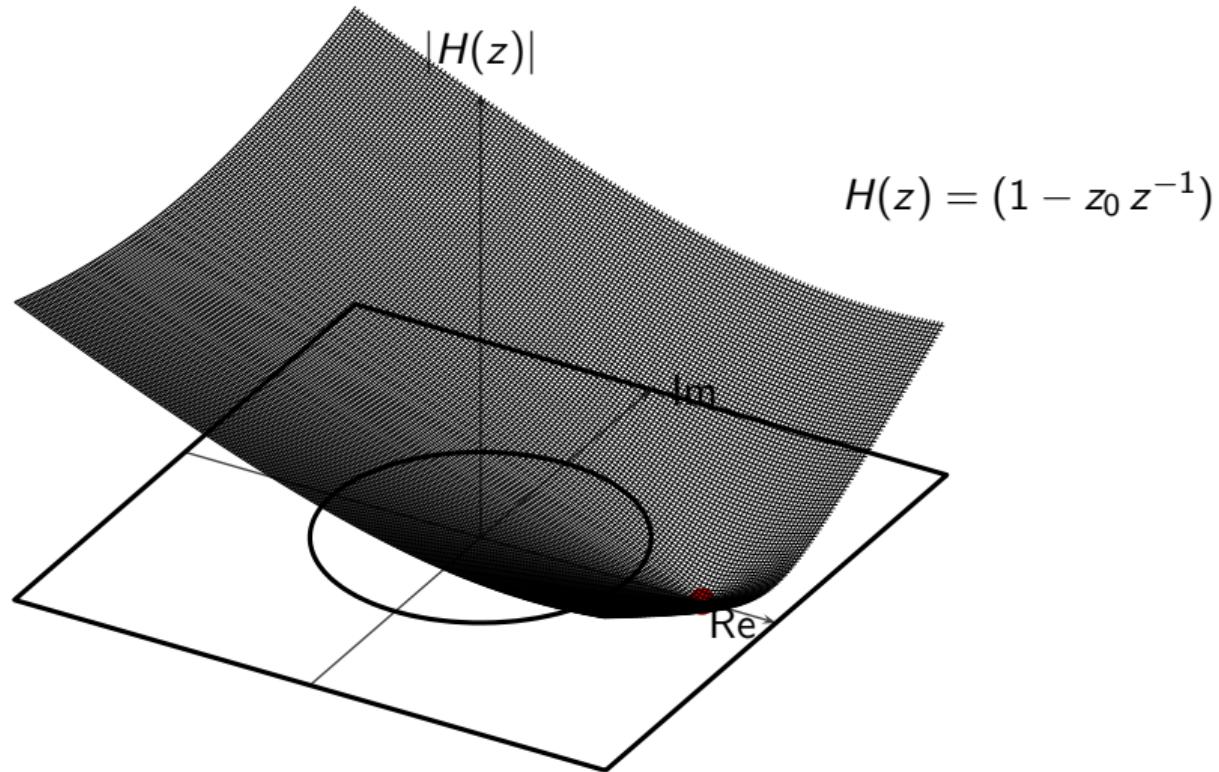
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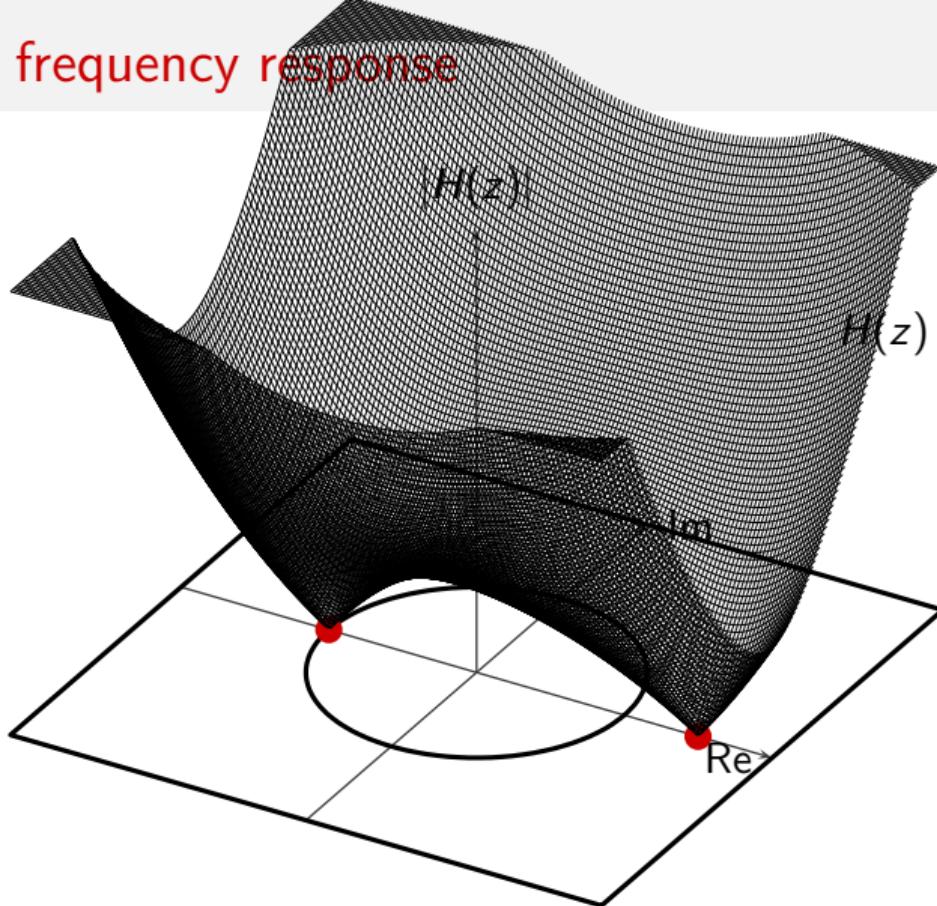
# Estimating the frequency response



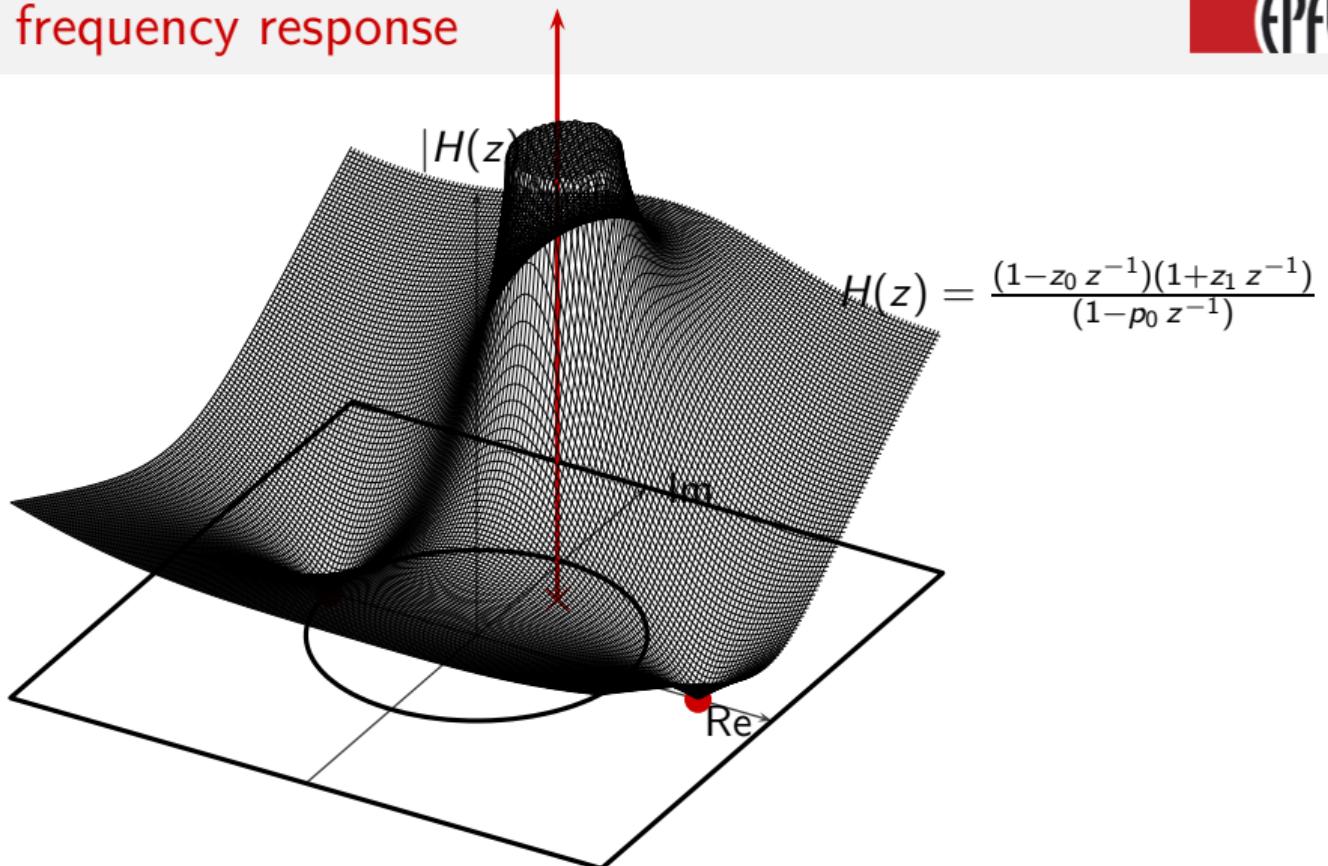
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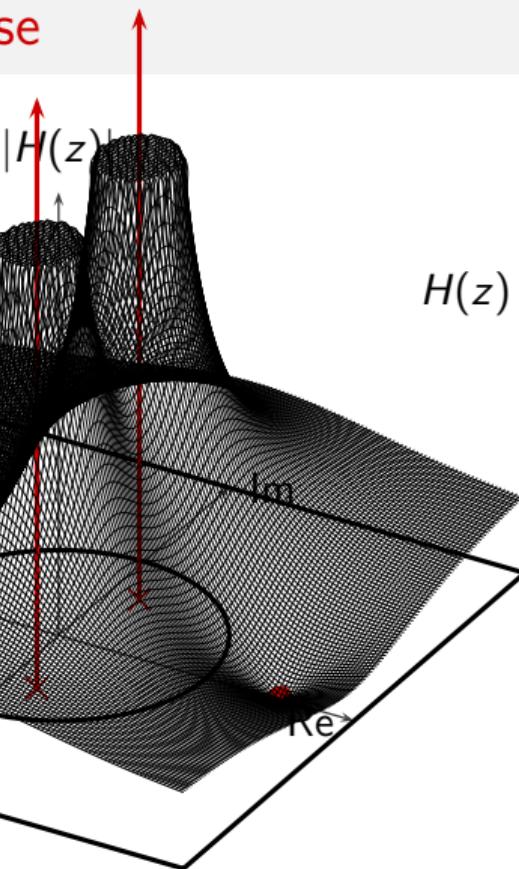
## Estimating the frequency response



## Estimating the frequency response

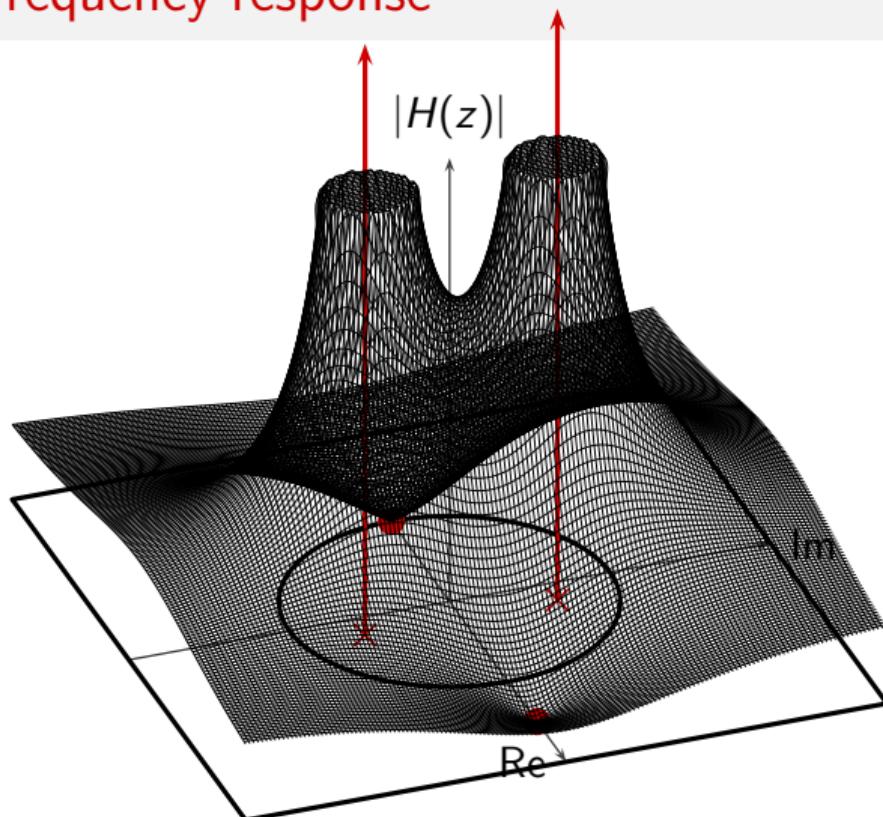


# Estimating the frequency response

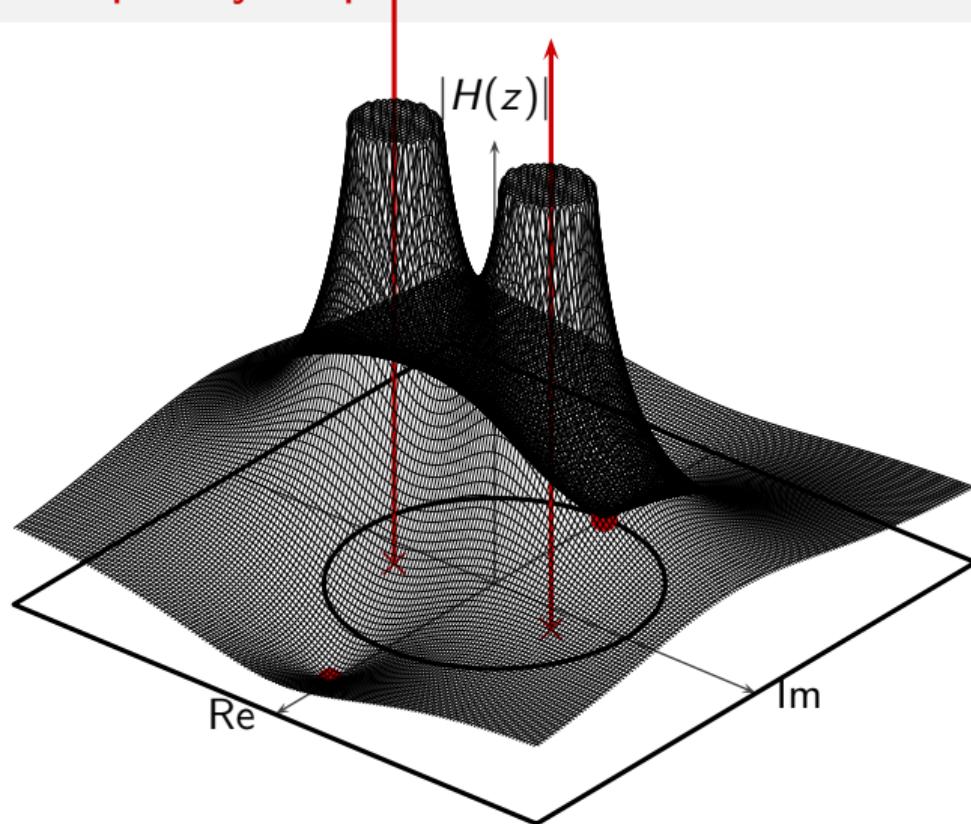


$$H(z) = \frac{(1-z_0 z^{-1})(1+z_1 z^{-1})}{(1-p_0 z^{-1})(1-p_0^* z^{-1})}$$

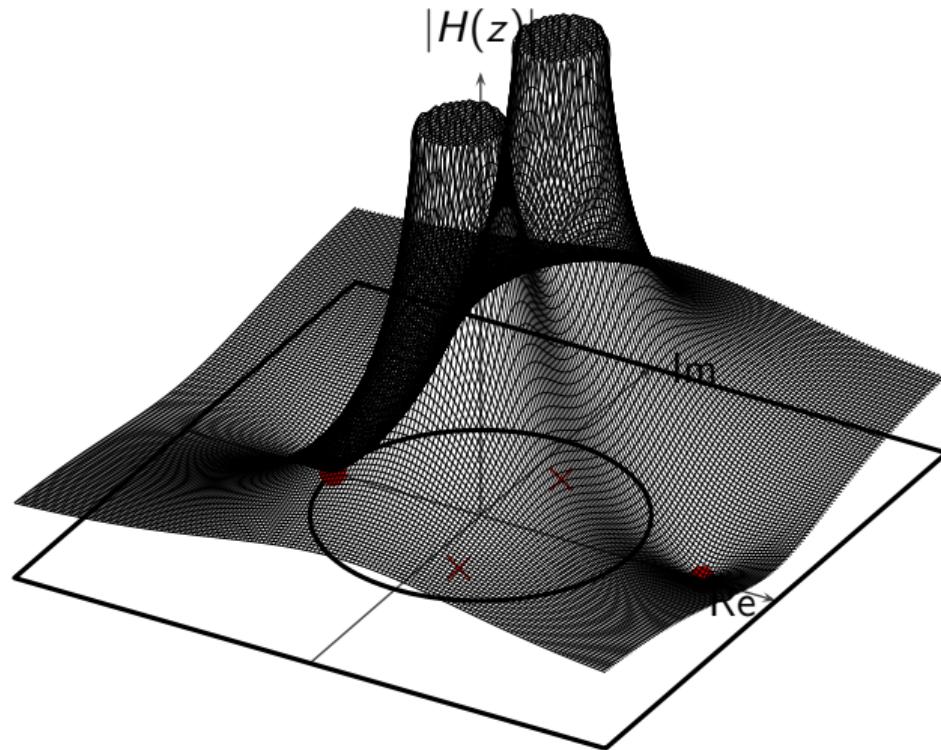
# Estimating the frequency response



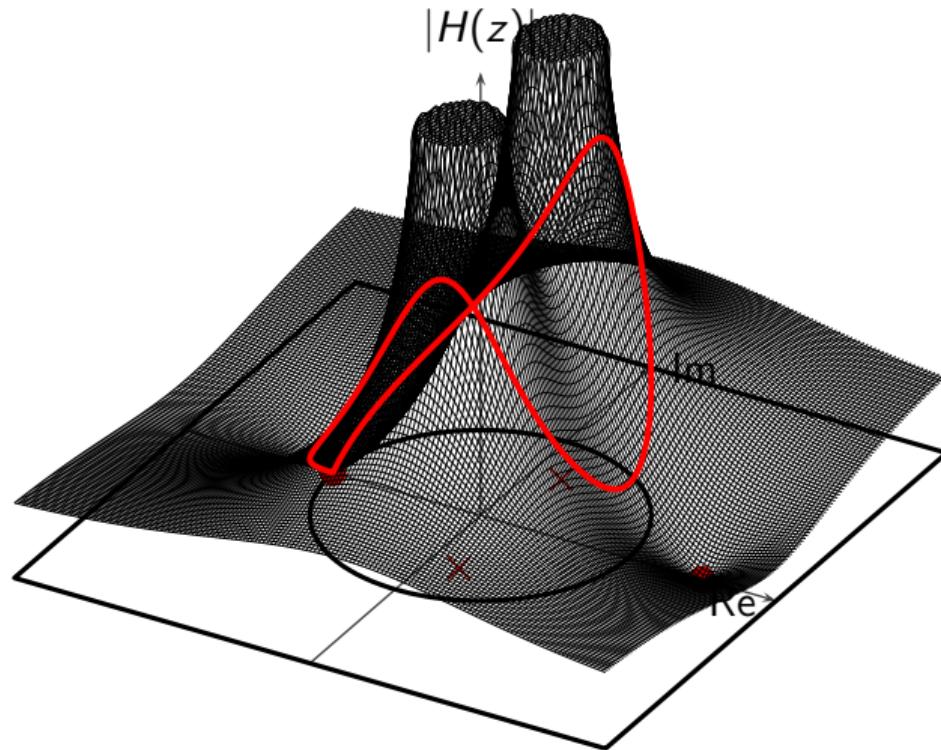
## Estimating the frequency response



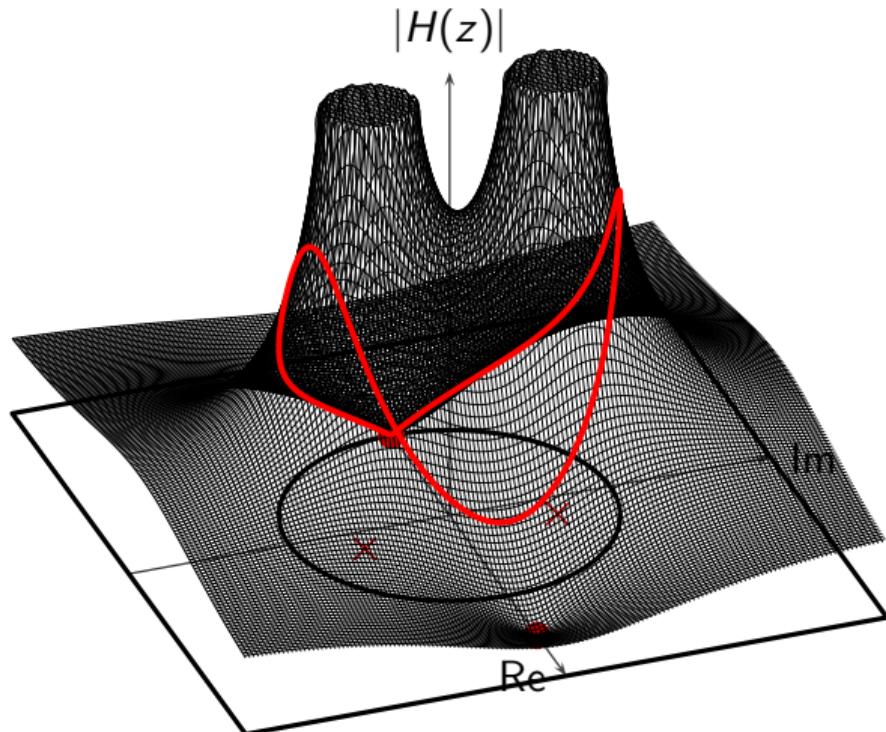
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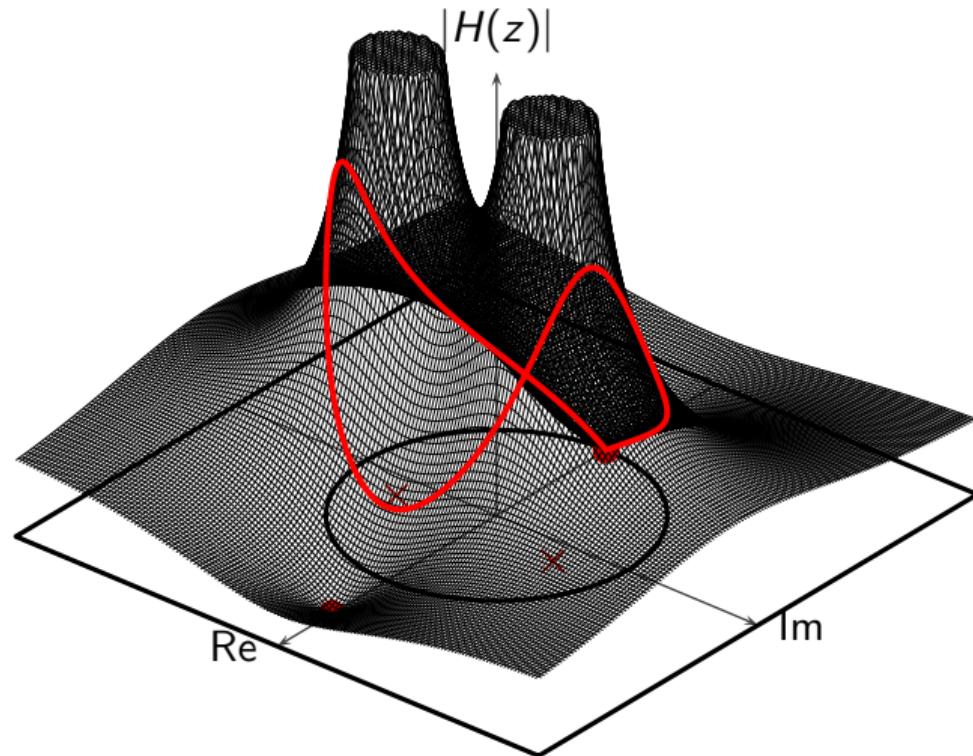
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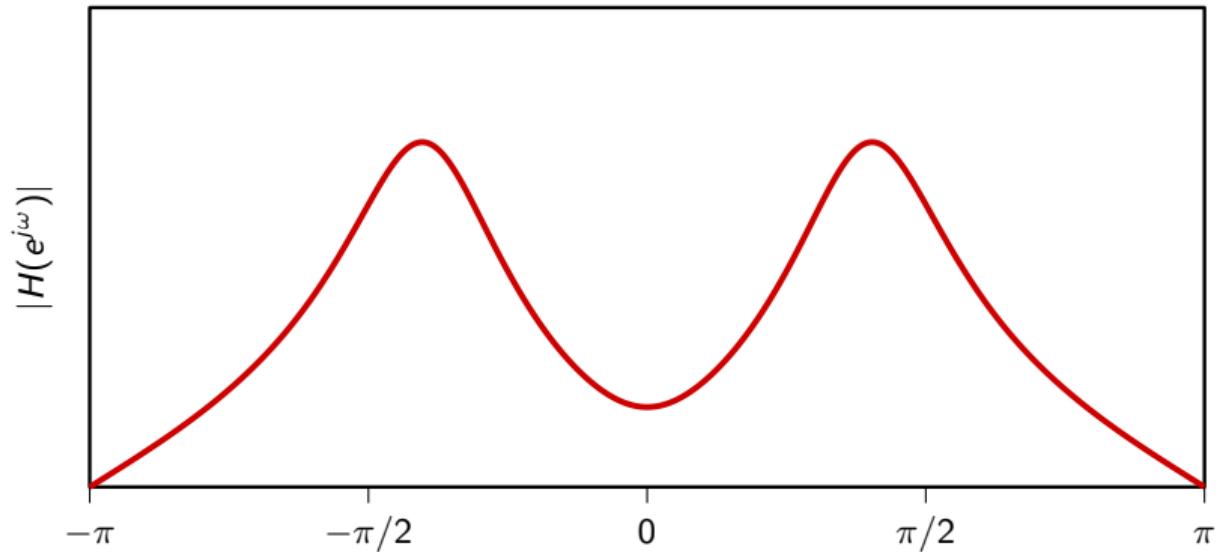
# Estimating the frequency response



# Estimating the frequency response



# Estimating the frequency response



END OF MODULE 5.7

# Digital Signal Processing

Module 5.8: Implementation of Digital Filters

- ▶ Algorithms for CCDE's
- ▶ Block diagram
- ▶ Real-time processing

```
double Leaky(double x) {
    static const double lambda = 0.95;
    static double y = 0;

    y = lambda * y + (1 - lambda) * x;
    return y;
}
```

```
int main() {
    int n;
    for (n = 0; n < 20; n++)
        printf("%.4f ", Leaky(n==0 ? 1.0 : 0.0));
}
```

```
0.0500 0.0475 0.0451 0.0429 0.0407 0.0387 0.0368 0.0349 0.0332 0.0315
0.0299 0.0284 0.0270 0.0257 0.0244 0.0232 0.0220 0.0209 0.0199 0.0189
```

- ▶ we need a “memory cell” to store previous output
- ▶ we need to initialize the storage before first use
- ▶ we need 2 multiplications and one addition per output sample

```
class Leaky:  
    def __init__(self, lmb):  
        self.lmb=lmb  
        self.y=0  
  
    def compute(self, x):  
        res = []  
        for v in x:  
            self.y = self.lmb * self.y + (1 - self.lmb) * v  
            res.append(self.y)  
        return res
```

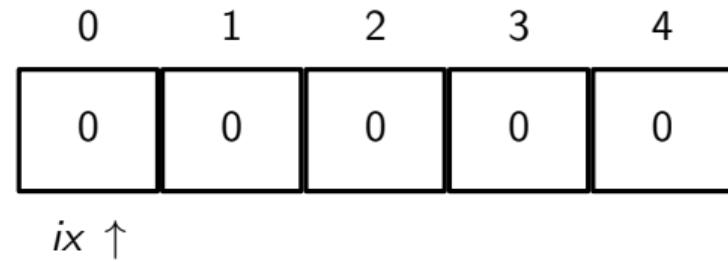
```
>>> from leaky import Leaky  
>>> L=Leaky(0.95)  
>>> print L.compute([0, 0, 0, 0, 1, 0, 0, 0, 0, 0, 0])  
[0.0, 0.0, 0.0, 0.0, 0.0500000000000000, 0.0475000000000000,  
 0.0451250000000000, 0.0428687500000000, 0.0407253125000000,  
 0.038689046875000, 0.0367545945312500]  
>>>
```

$$y[n] = \frac{1}{M} \sum_{k=0}^{M-1} x[n - k]$$

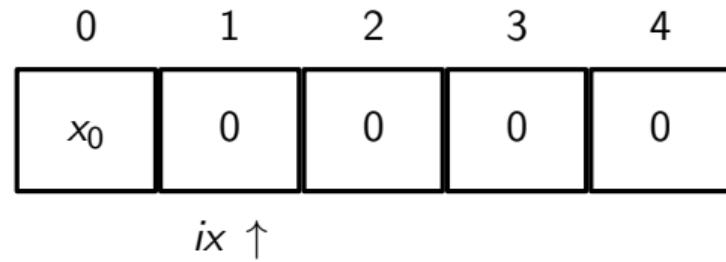
```
double MA(double x) {  
    static const int M = 10;  
    static double z[M];  
    static int ix = -1;  
  
    int n;  
    double avg = 0;
```

```
        if (ix == -1) {  
            for (n = 0; n < M; n++)  
                z[n] = 0;  
            ix = 0;  
        }  
  
        z[ix] = x;  
        ix = (ix + 1) % M;  
  
        for (n = 0; n < M; n++)  
            avg += z[n];  
        return avg / M;  
    }
```

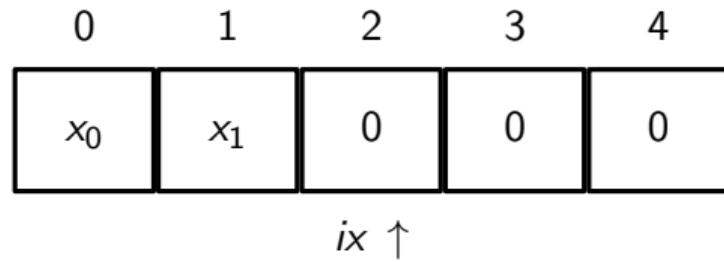
# The circular buffer



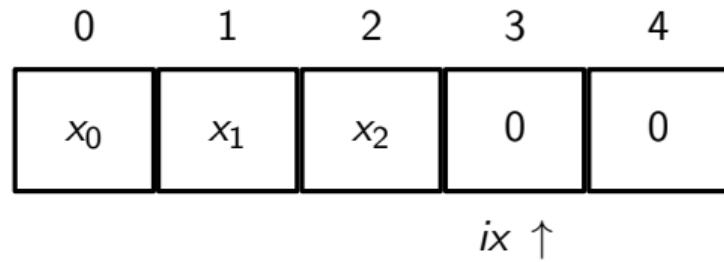
# The circular buffer



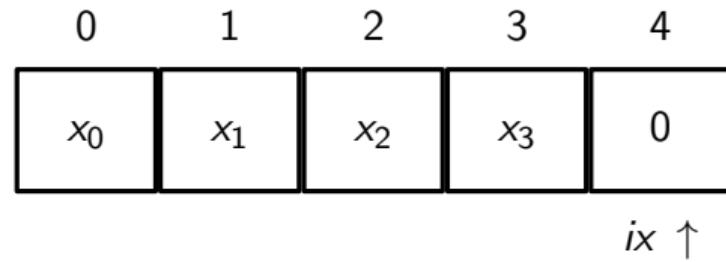
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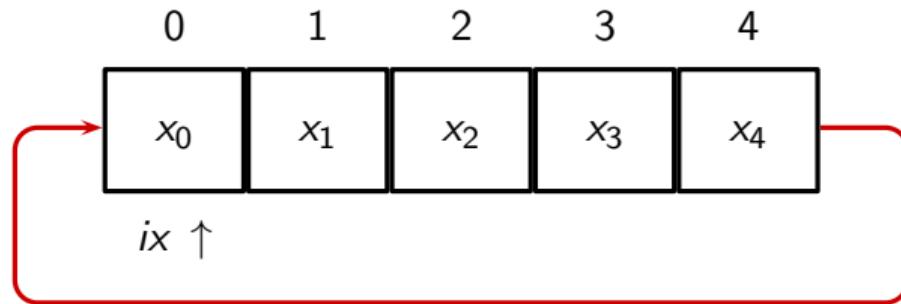
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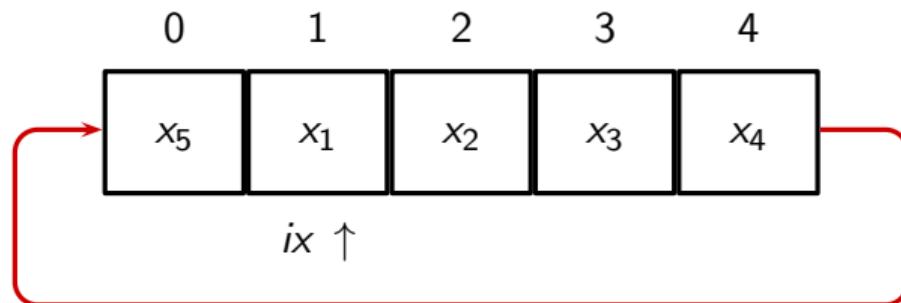
# The circular buffer



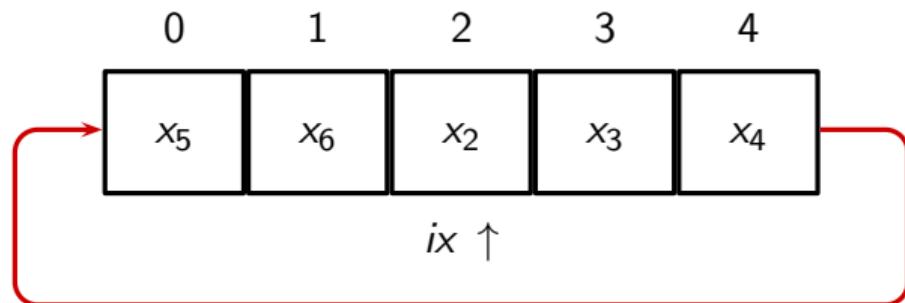
# The circular buffer



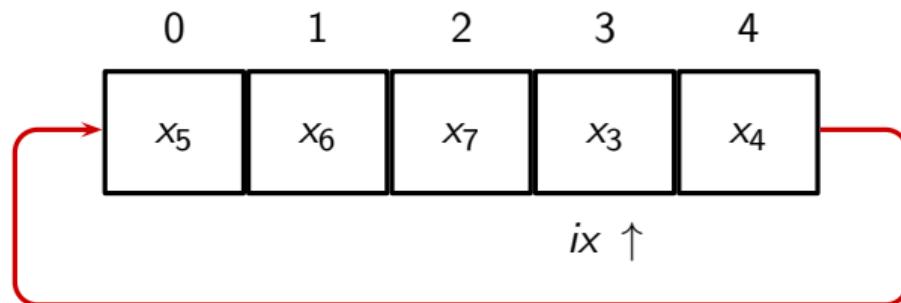
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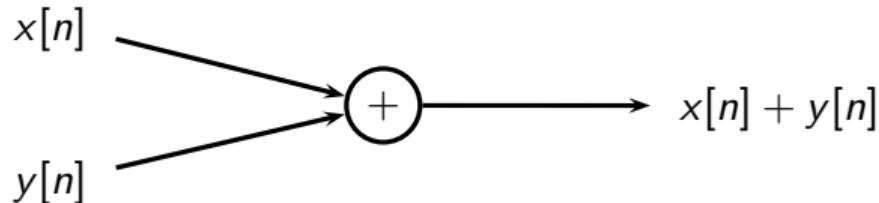


# The circular buffer

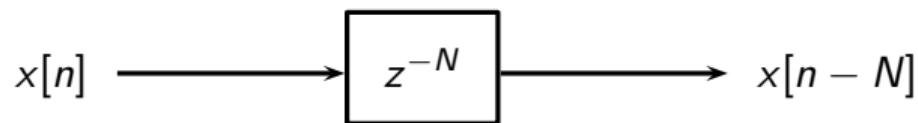


- ▶ we now need  $M$  memory cells to store previous input values
- ▶ we need to initialize the storage before first use
- ▶ we need 1 division and  $M$  additions per output sample

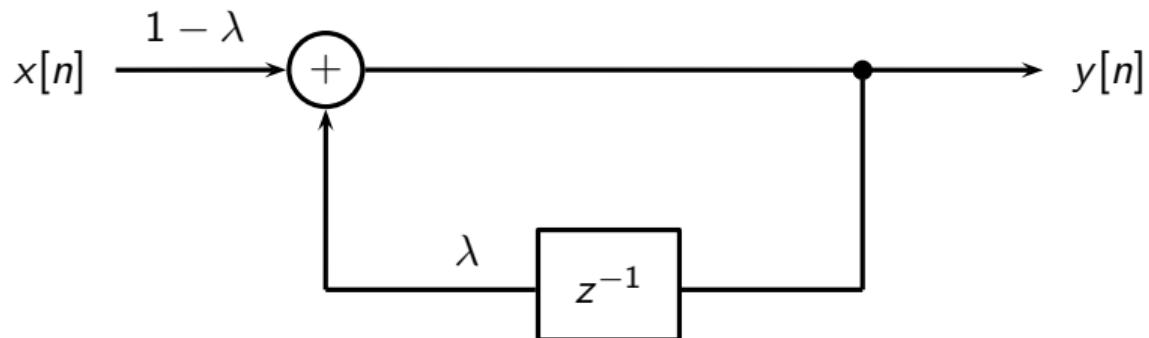
We can abstract from the implementation



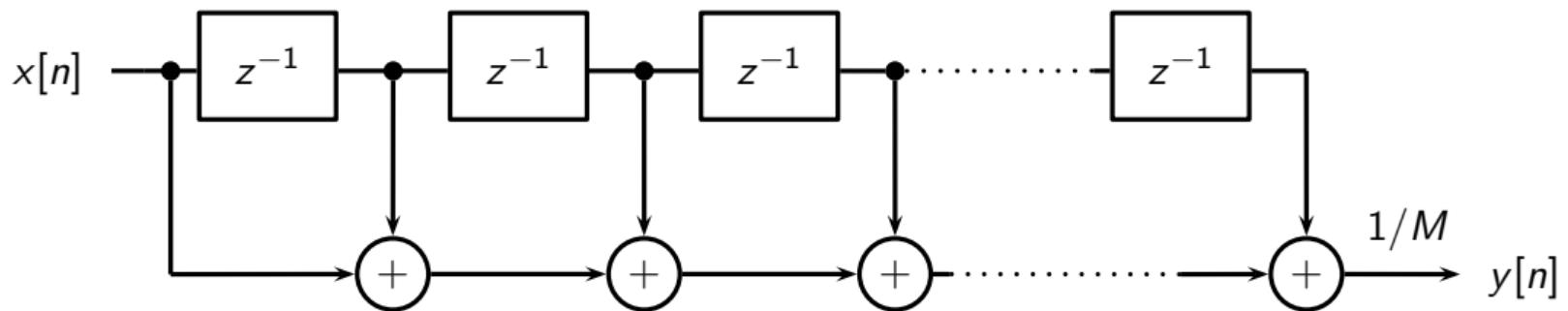
$$x[n] \xrightarrow{\alpha} \alpha x[n]$$



$$y[n] = \lambda y[n - 1] + (1 - \lambda)x[n]$$



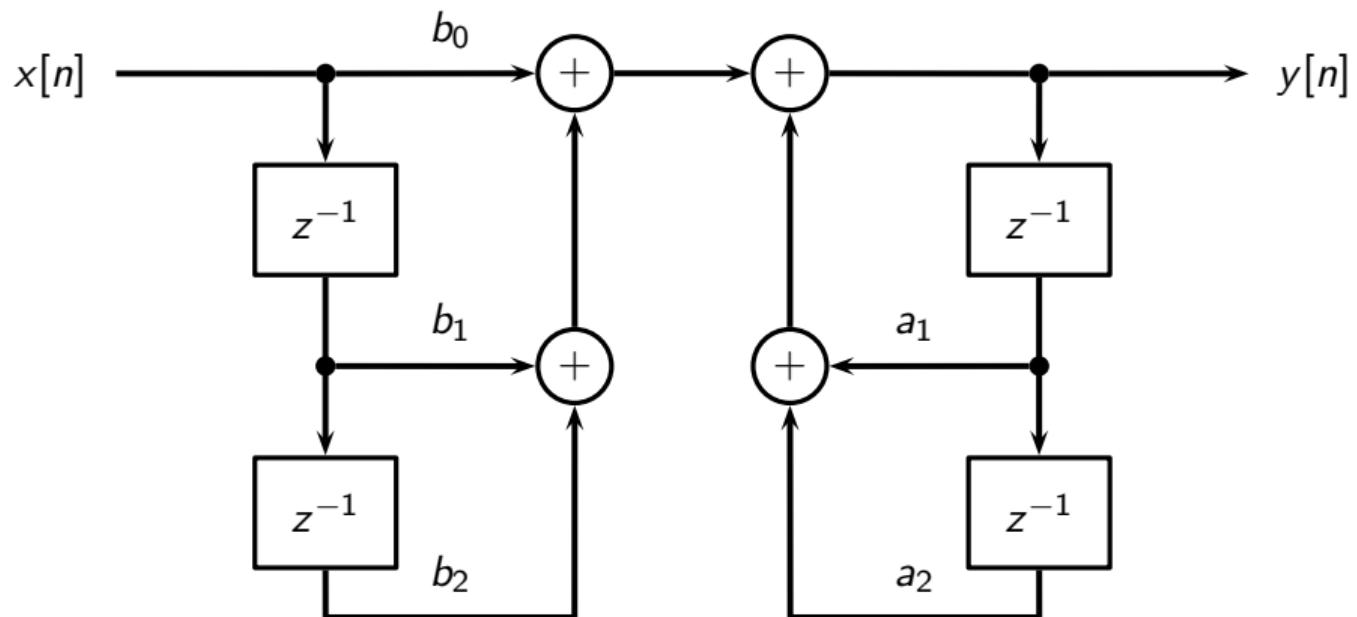
$$y[n] = \frac{1}{M} \sum_{k=0}^{M-1} x[n-k]$$



$$H(z) = \frac{b_0 + b_1 z^{-1} + b_2 z^{-2}}{a_0 - a_1 z^{-1} - a_2 z^{-2}} = \frac{B(z)}{A(z)}$$

$$H(z) = \frac{b_0 + b_1 z^{-1} + b_2 z^{-2}}{1 - a_1 z^{-1} - a_2 z^{-2}} = \frac{B(z)}{A(z)}$$

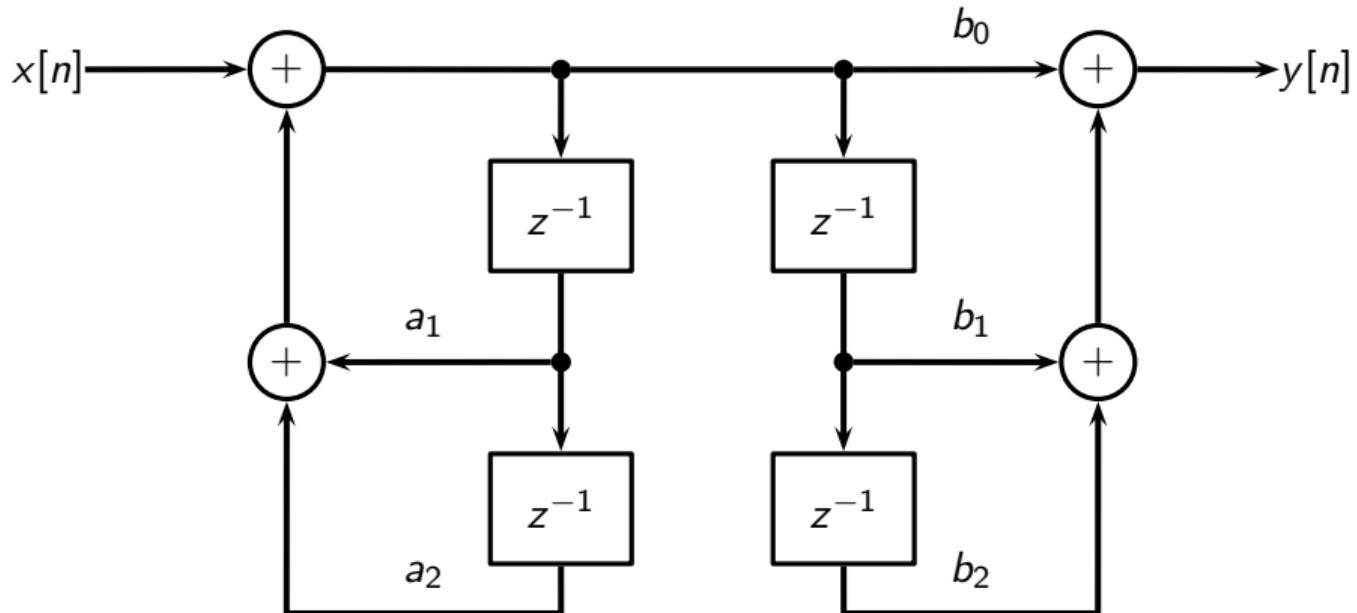
## Second-order section, direct form I



$$B(z)$$

$$1/A(z)$$

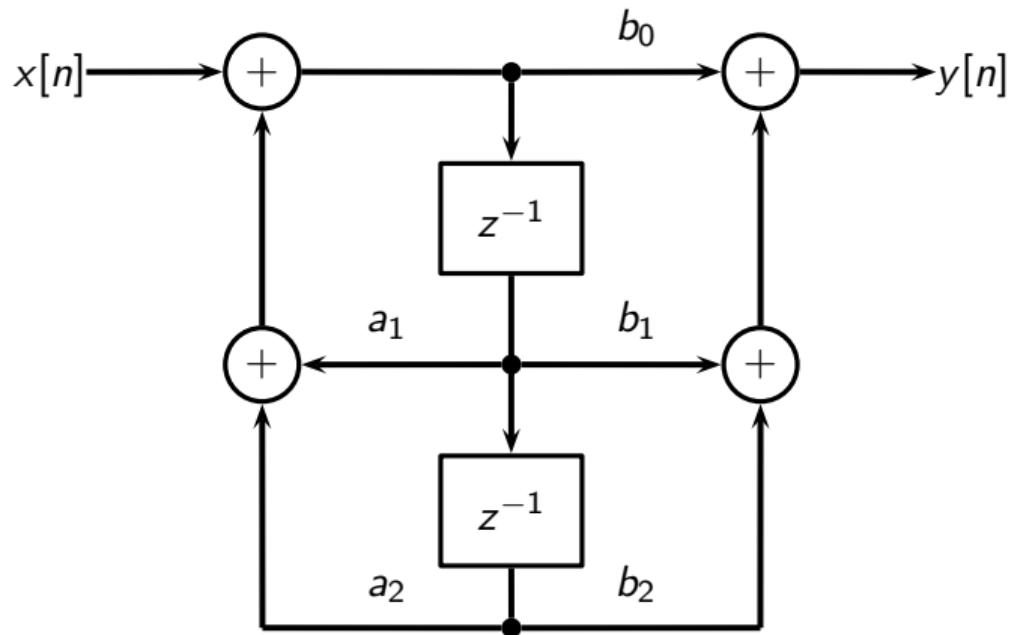
# Second-order section, direct form I, inverted order



$$1/A(z)$$

$$B(z)$$

## Second-order section, direct form II



If input samples arrive every  $T$  seconds,

- ▶ each output sample must be computed in at most  $T$  seconds
- ▶ number of operations becomes important (IIR vs FIR)
- ▶ some common tricks:
  - circular buffers are size  $2^K$  (mod operation faster)
  - exploit the parallelism some processor operations (fetch and compute)

Processing unit have finite precision arithmetic:

- ▶ overflow or underflow are a real problem
- ▶ more complex structures are more resilient (but less efficient)
- ▶ some common tricks:
  - use double precision in the accumulator
  - split the filter in low-order sections
  - use floating point

END OF MODULE 5.8

# Digital Signal Processing

Module 5.9: Filter Design - Part II: Intuitive Filters

- ▶ General problem
- ▶ “Intuitive” IIR design

- ▶ many signal processing problems can be solved using simple filters
- ▶ we have seen simple lowpass filters already (Moving Average, Leaky Integrator)
- ▶ simple (low order) transfer functions allow for intuitive design and tuning

- ▶ let only low frequencies pass
- ▶ used to remove high frequency components (e.g. noise)
- ▶ useful in audio, communication, control systems
- ▶ we know a simple answer: leaky integrator

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## Leaky Integrator

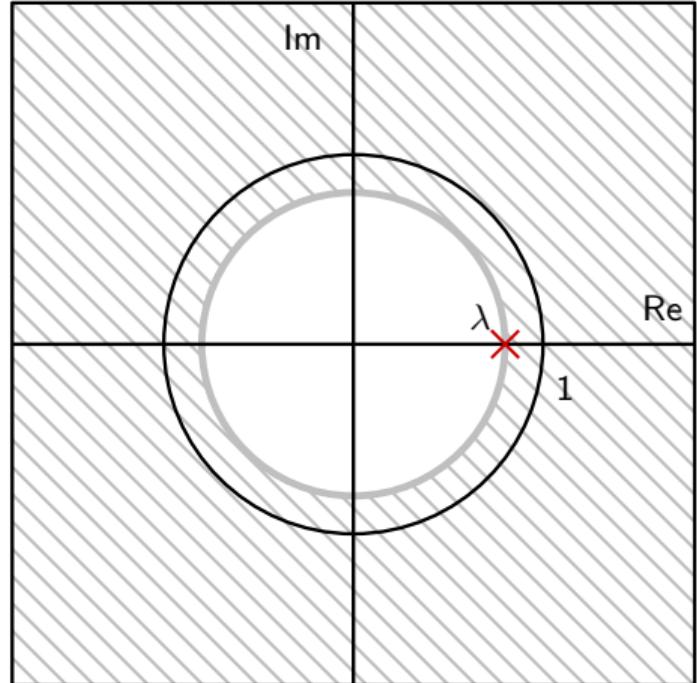
$$H(z) = \frac{(1 - \lambda)}{1 - \lambda z^{-1}}$$

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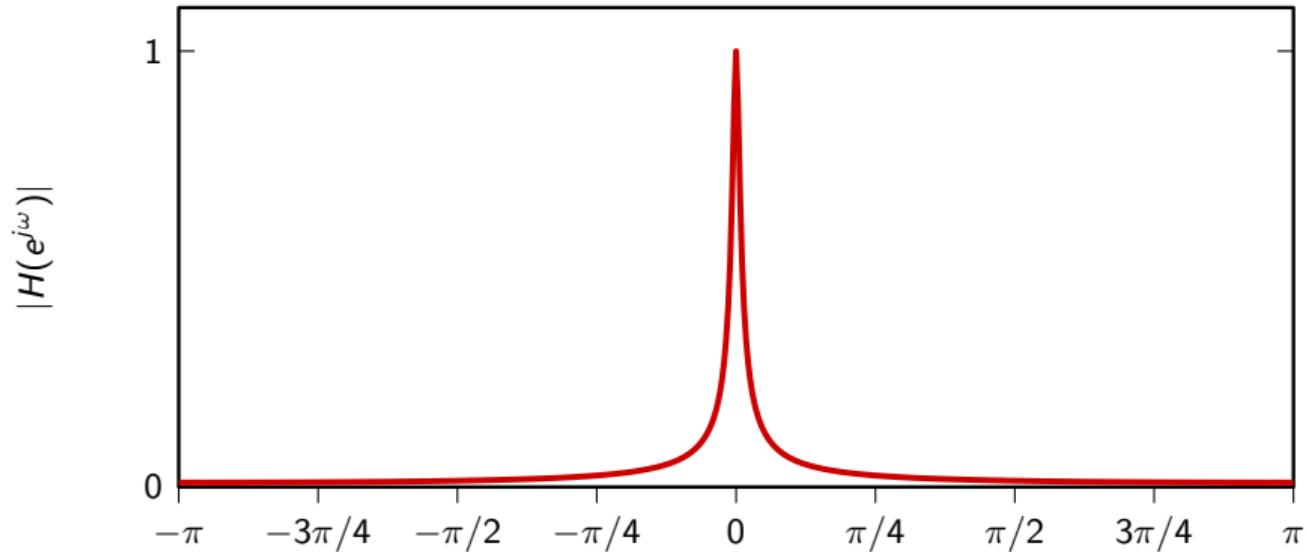
$$y[n] = (1 - \lambda)x[n] + \lambda y[n - 1]$$

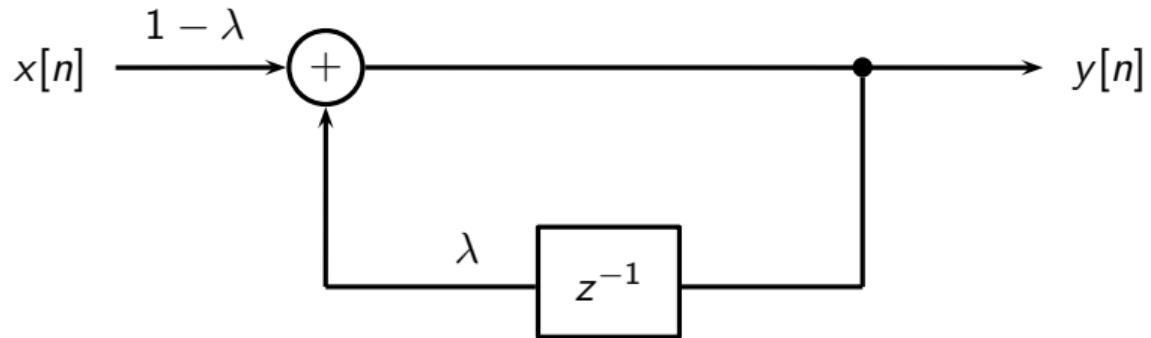
$$H(z) = \frac{(1 - \lambda)}{1 - \lambda z^{-1}}$$

$$y[n] = (1 - \lambda)x[n] + \lambda y[n - 1]$$



# Leaky Integrator, $\lambda = 0.98$



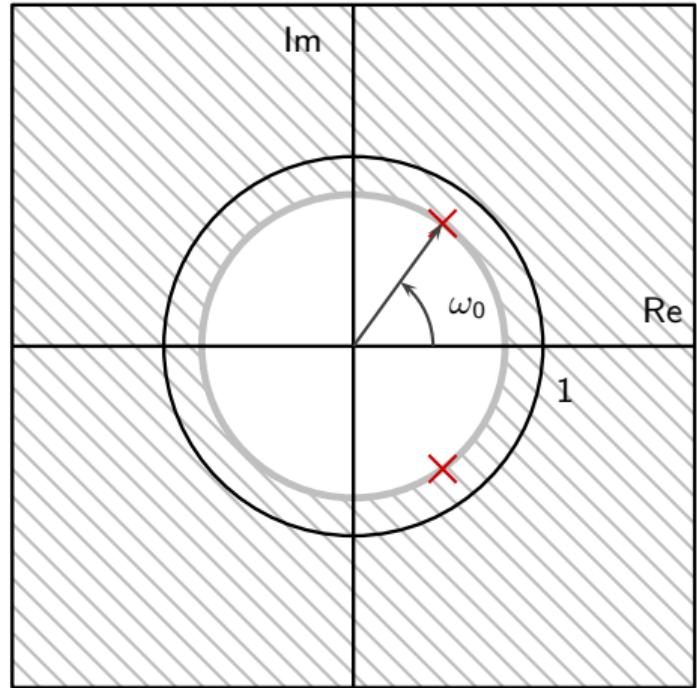


- ▶ a resonator is a narrow bandpass filter
- ▶ used to detect the presence of a sinusoid of a given frequency
- ▶ useful in communication systems and telephony (DTMF)
- ▶ idea: shift the passband of the Leaky Integrator!

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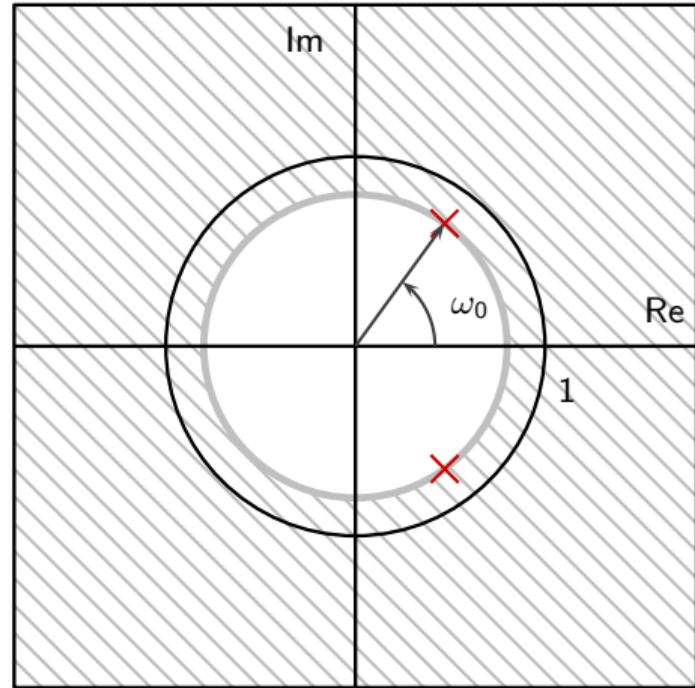
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$$H(z) = \frac{G_0}{(1 - pz^{-1})(1 - p^*z^{-1})}$$

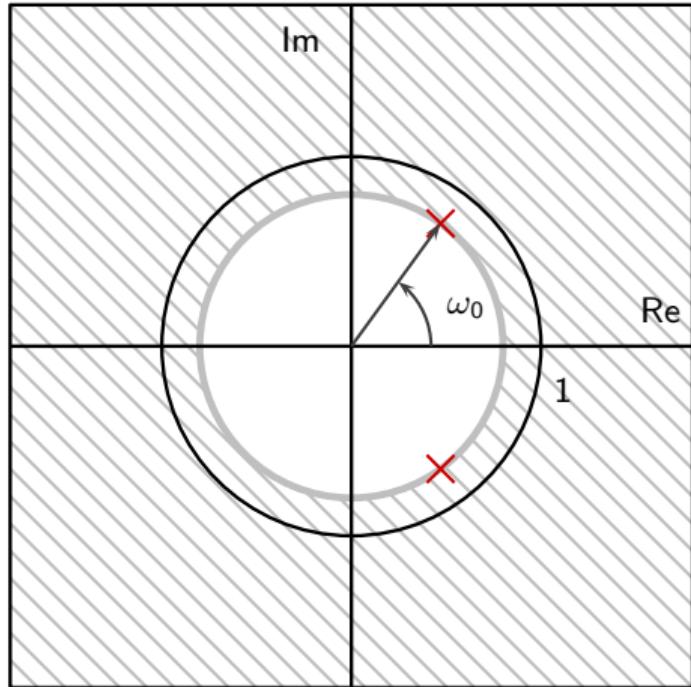
$$p = \lambda e^{j\omega_0}$$



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$$p = \lambda e^{j\omega_0}$$

$$y[n] = G_0x[n] - a_1y[n-1] - a_2y[n-2]$$



$$\begin{aligned} H(z) &= \frac{G_0}{(1 - pz^{-1})(1 - p^*z^{-1})}, \quad p = \lambda e^{j\omega_0} \\ &= \frac{G_0}{1 - 2\Re\{p\} z^{-1} + |p|^2 z^{-2}} \\ &= \frac{G_0}{1 - 2\lambda \cos \omega_0 z^{-1} + |\lambda|^2 z^{-2}} \end{aligned}$$

$$a_1 = 2\lambda \cos \omega_0$$

$$a_2 = -|\lambda|^2$$

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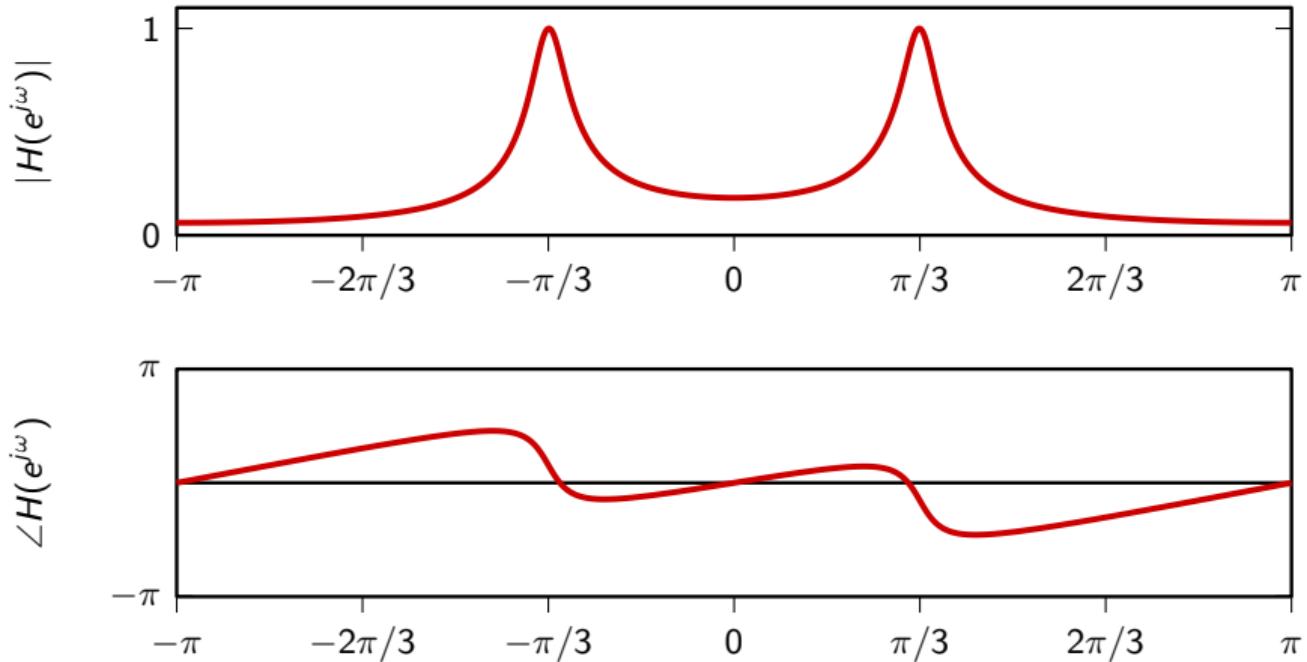
$$a_2 = -|\lambda|^2$$

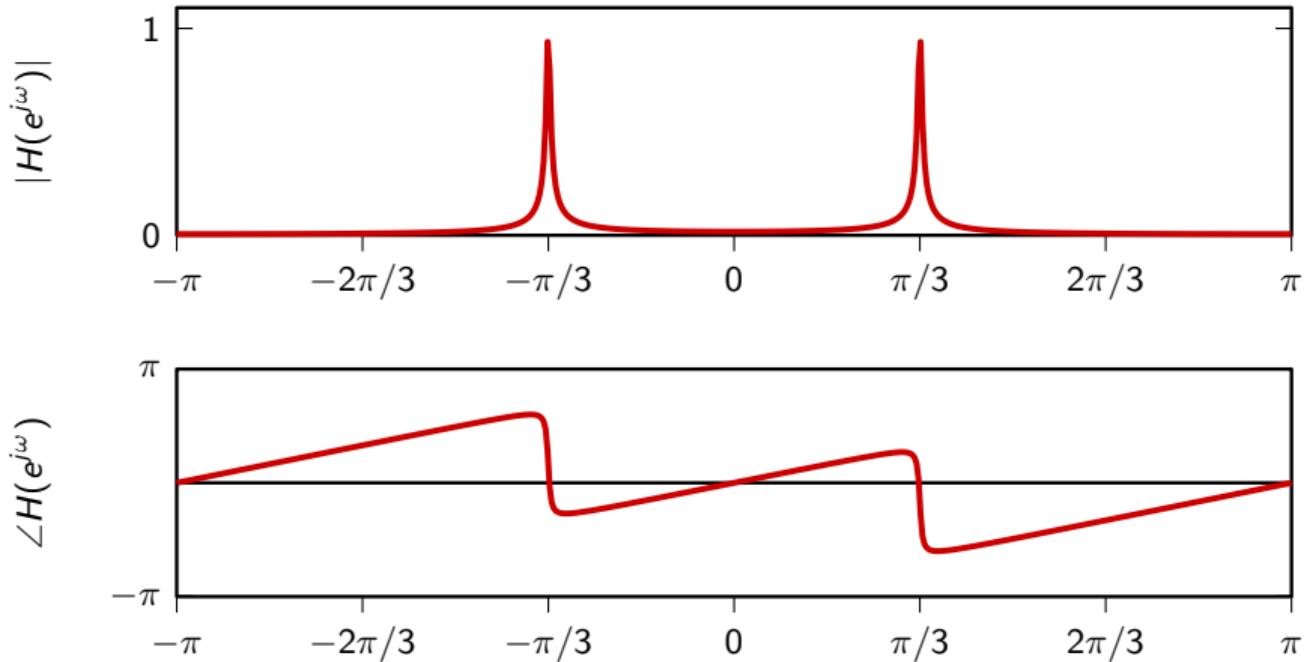
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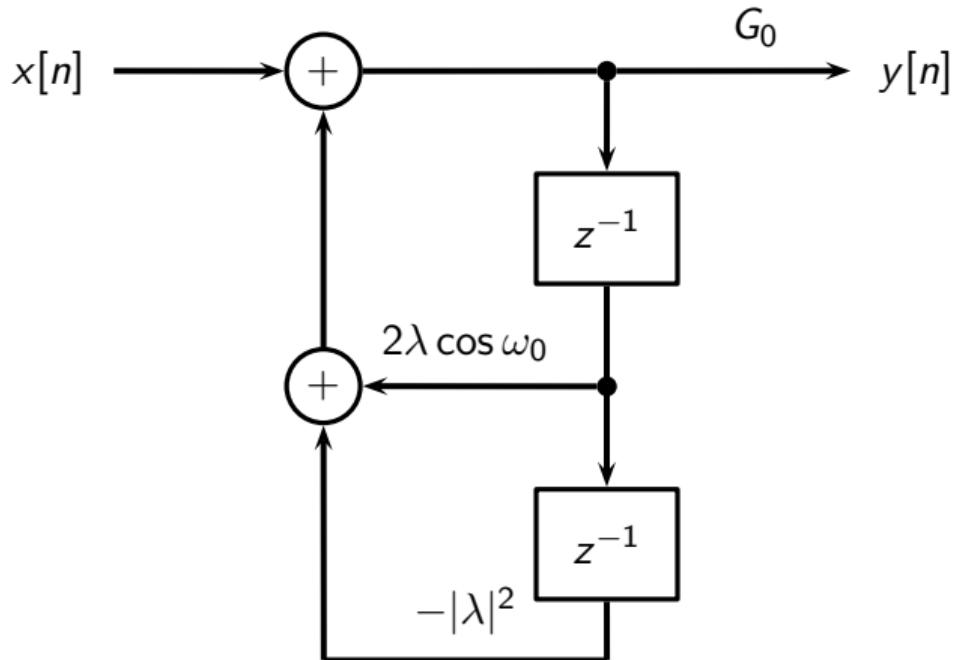
$$a_1 = 2\lambda \cos \omega_0$$

$$a_2 = -|\lambda|^2$$

# Resonator, $\lambda = 0.9, \omega_0 = \pi/3$





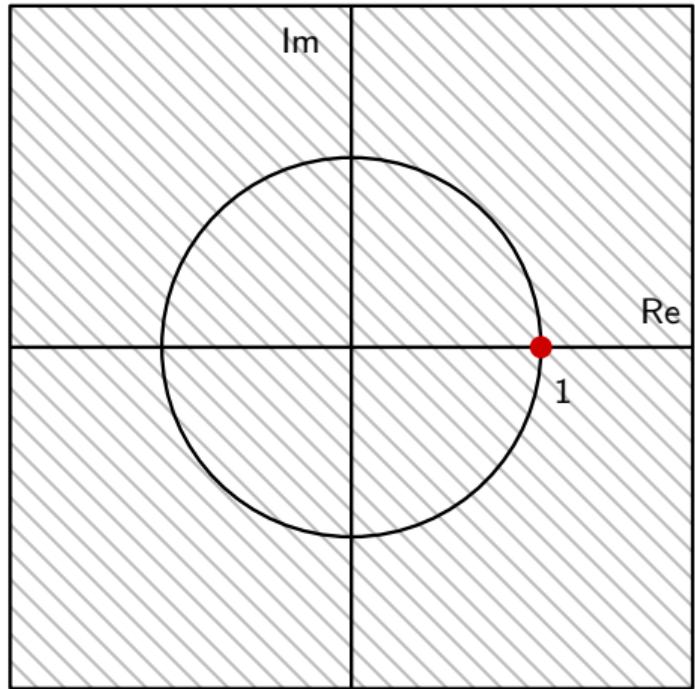


- ▶ a DC-balanced signal has zero sum:  $\lim_{N \rightarrow \infty} \sum_{n=-N}^N x[n] = 0$   
i.e. there is no Direct Current component
- ▶ its DTFT value at zero is zero
- ▶ we want to remove the DC bias from a non zero-centered signal
- ▶ we want to kill the frequency component at  $\omega = 0$

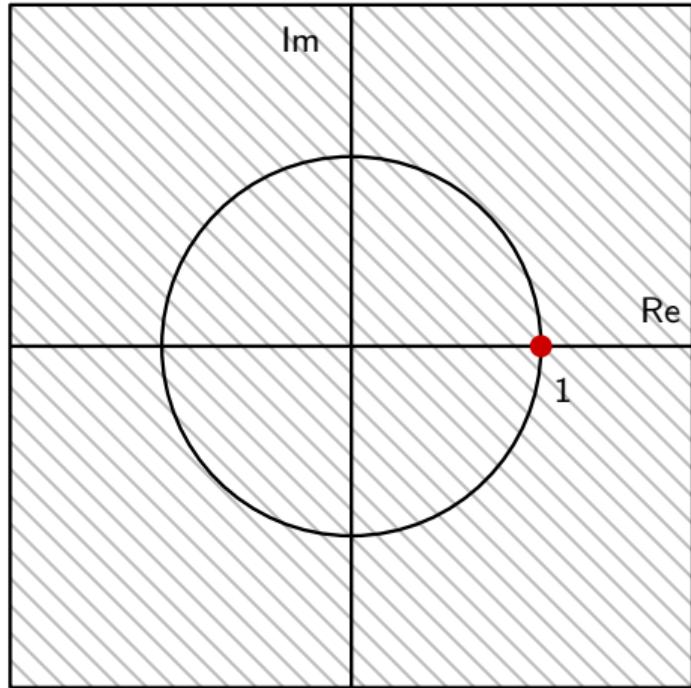
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- ▶ we want to remove the DC bias from a non zero-centered signal
- ▶ we want to kill the frequency component at  $\omega = 0$

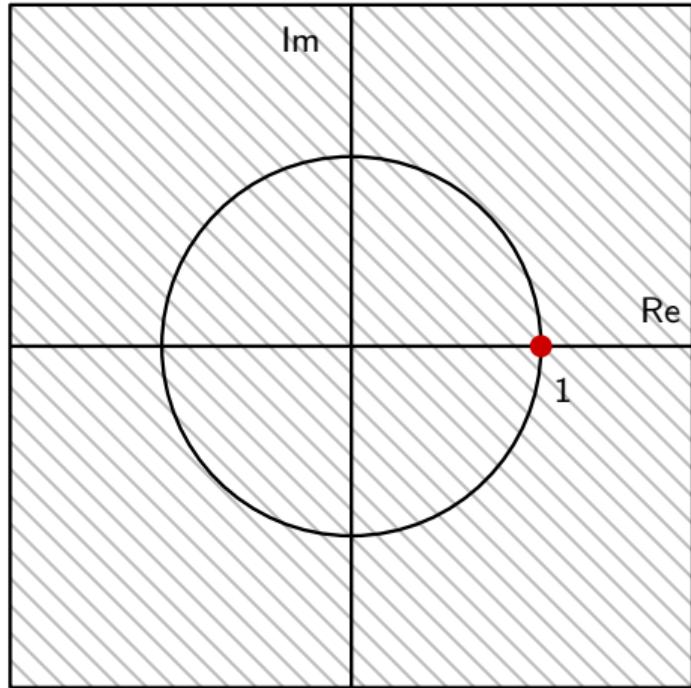


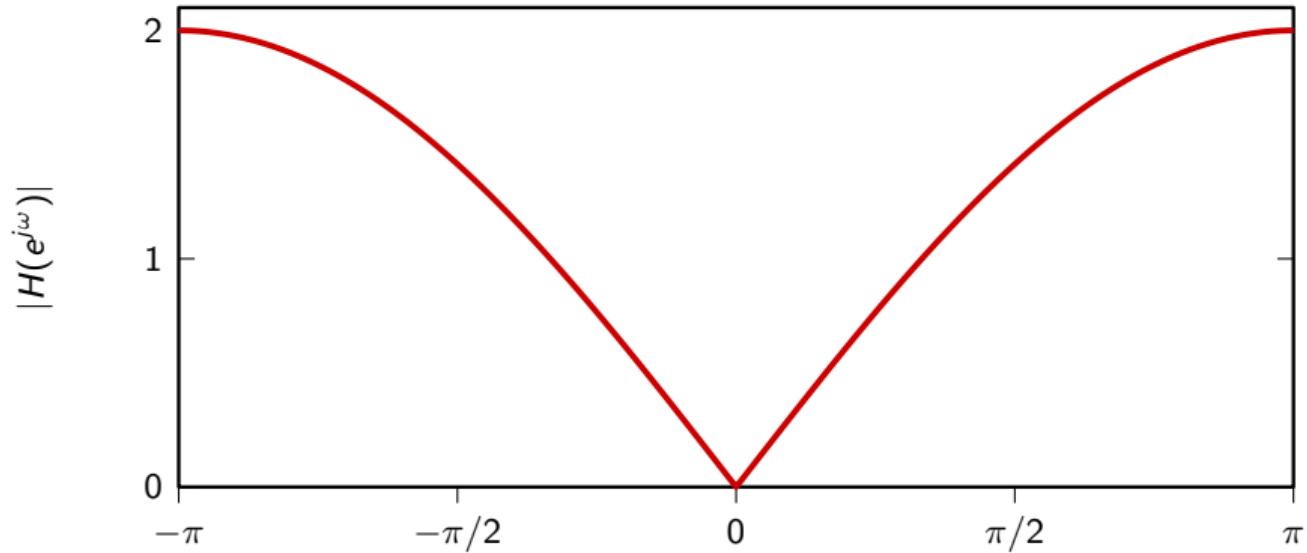
$$H(z) = 1 - z^{-1}$$

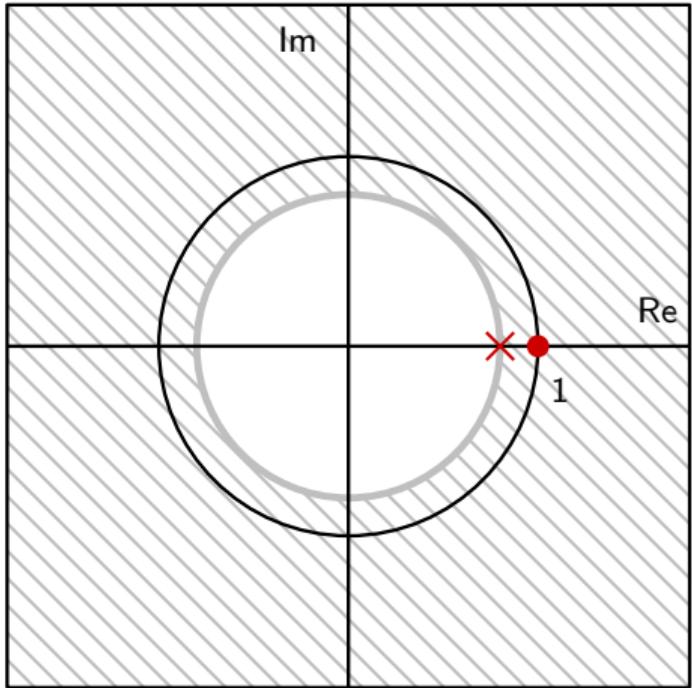


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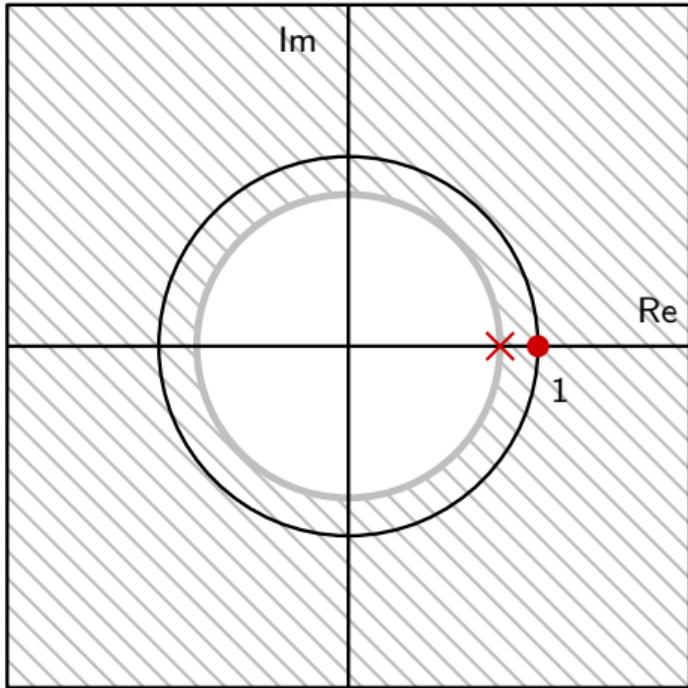
$$y[n] = x[n] - x[n - 1]$$





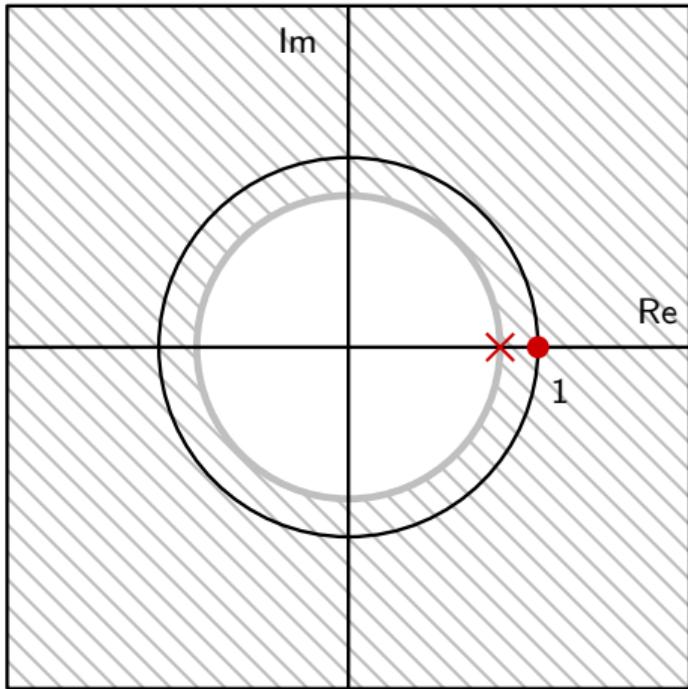


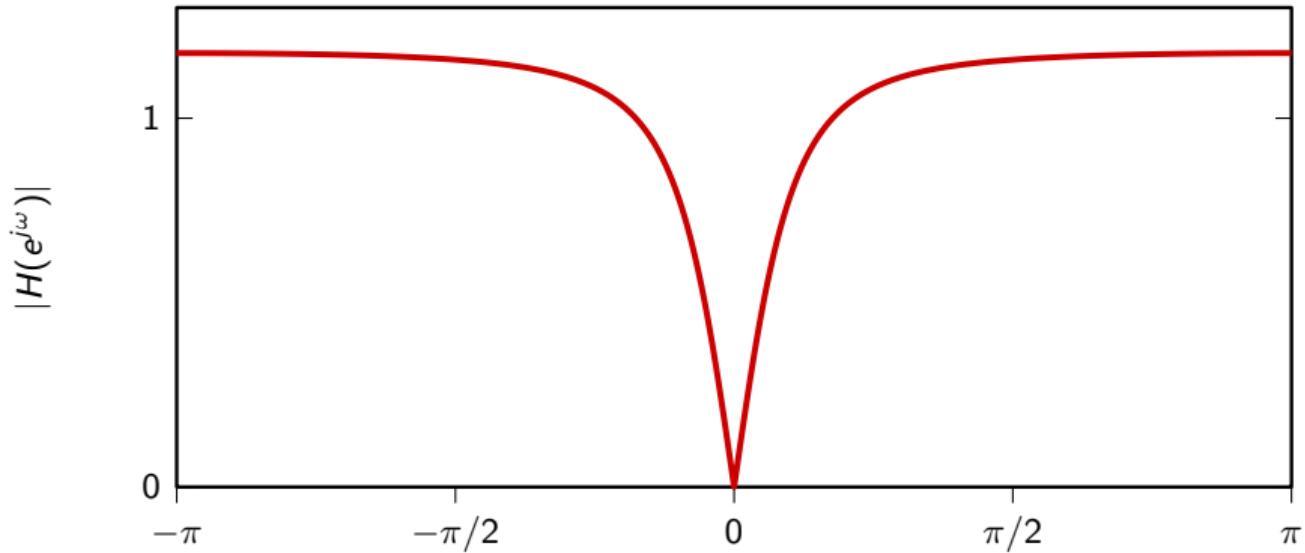
$$H(z) = \frac{1 - z^{-1}}{1 - \lambda z^{-1}}$$

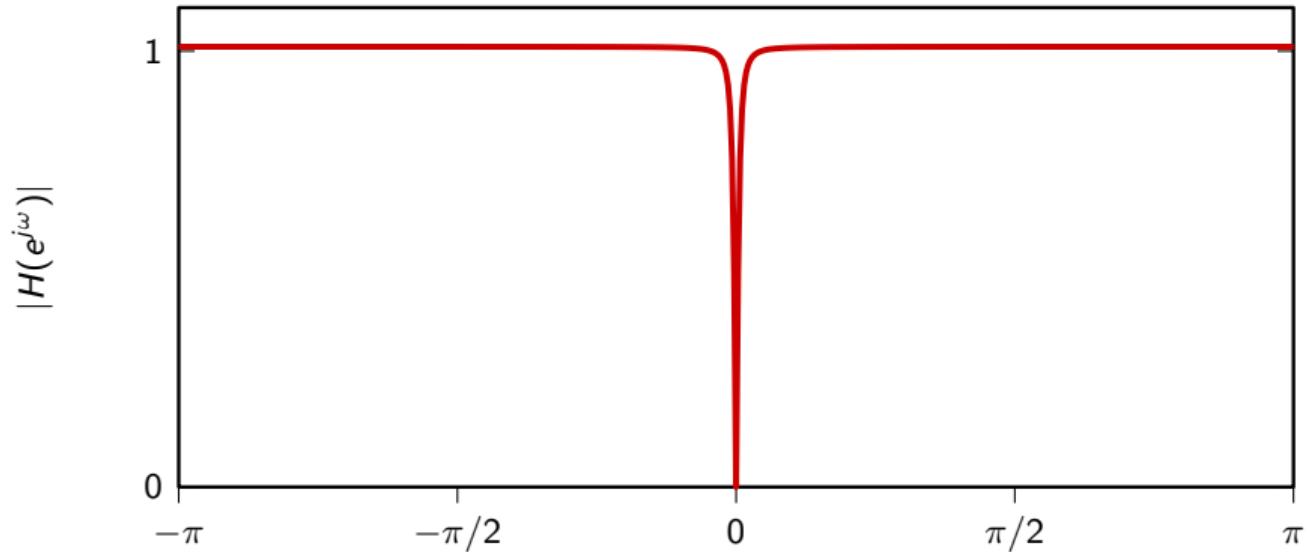


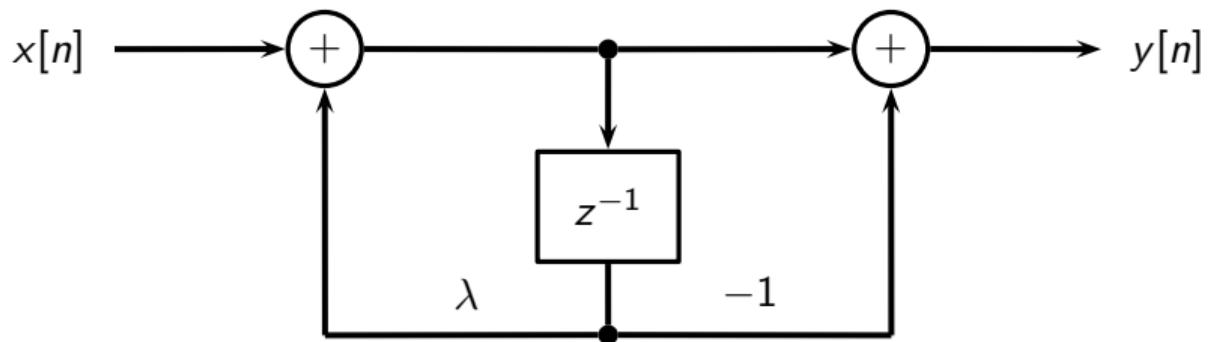
$$H(z) = \frac{1 - z^{-1}}{1 - \lambda z^{-1}}$$

$$y[n] = \lambda y[n-1] + x[n] - x[n-1]$$





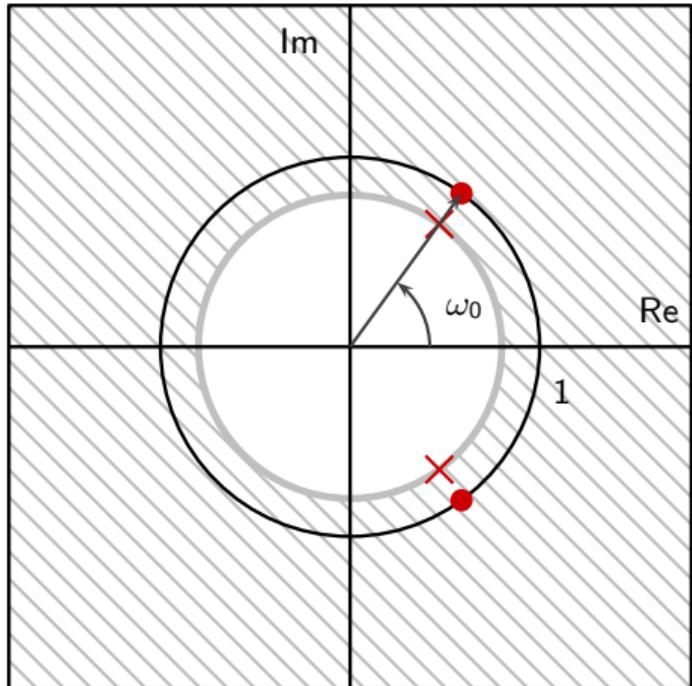




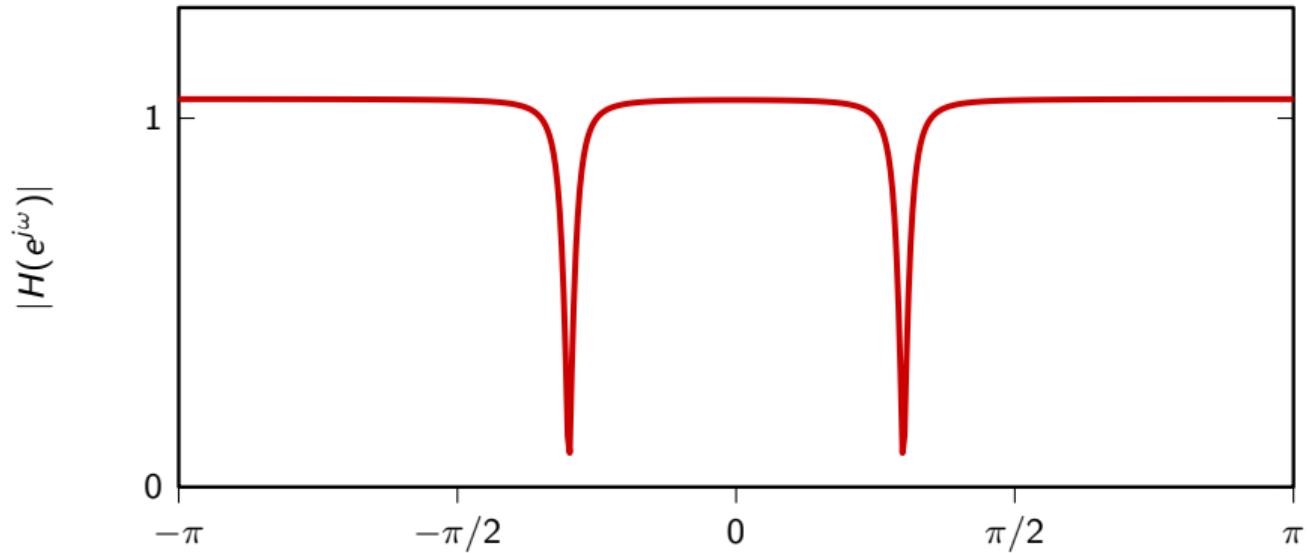
- ▶ similar to DC removal but we want to remove a specific nonzero frequency
- ▶ very useful for musicians: amplifiers for electric guitars pick up the hum from the electric mains (50Hz in Europe and 60Hz in North America)
- ▶ we need to tune the hum removal according to country

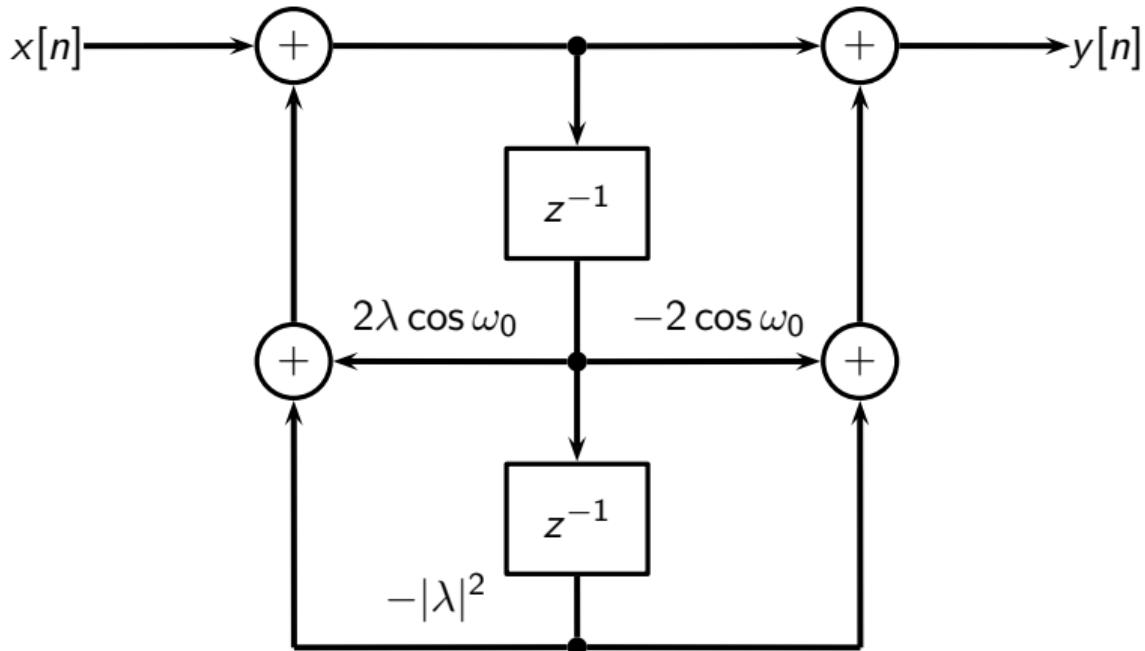
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$$H(z) = \frac{(1 - e^{j\omega_0} z^{-1})(1 - e^{-j\omega_0} z^{-1})}{(1 - \lambda e^{j\omega_0} z^{-1})(1 - \lambda e^{-j\omega_0} z^{-1})}$$





END OF MODULE 5.9

# Digital Signal Processing

Module 5.10: Filter Design - Part III: Design from Specs

- ▶ Filter specifications
- ▶ IIR design
- ▶ FIR design

- ▶ Filter specifications
- ▶ IIR design
- ▶ FIR design

- ▶ Filter specifications
- ▶ IIR design
- ▶ FIR design

You are given a set of requirements:

- ▶ frequency response: passband(s) and stopband(s)
- ▶ phase: overall delay, linearity
- ▶ some limit on computational resources and/or numerical precision

You must determine  $N$ ,  $M$ ,  $a_k$ 's and  $b_k$ 's in

$$H(z) = \frac{b_0 + b_1 z^{-1} + \dots + b_{M-1} z^{-(M-1)}}{a_0 + a_1 z^{-1} + \dots + a_{N-1} z^{-(N-1)}}$$

in order to best fulfill the requirements

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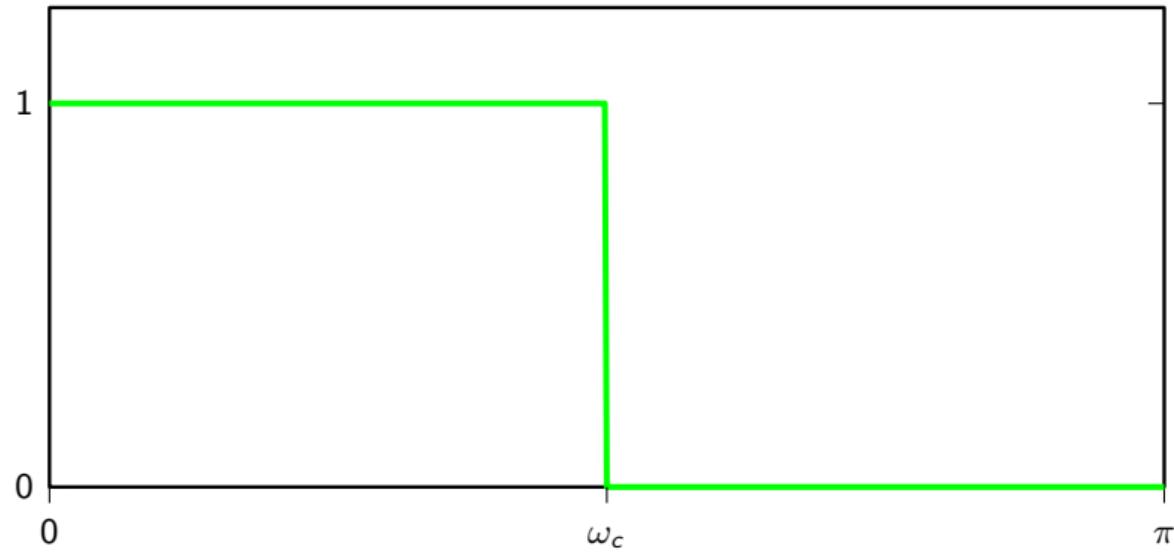
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in order to best fulfill the requirements

## Example: lowpass specs



- ▶ passband/stopband transitions cannot be infinitely sharp  
⇒ use *transition bands*
- ▶ magnitude response cannot be constant over an interval  
⇒ specify magnitude *tolerances* over bands
- ▶ in general:
  - smaller transition bands ⇒ higher filter order
  - smaller error tolerances ⇒ higher filter order
  - higher filter order ⇒ more expensive, larger delay

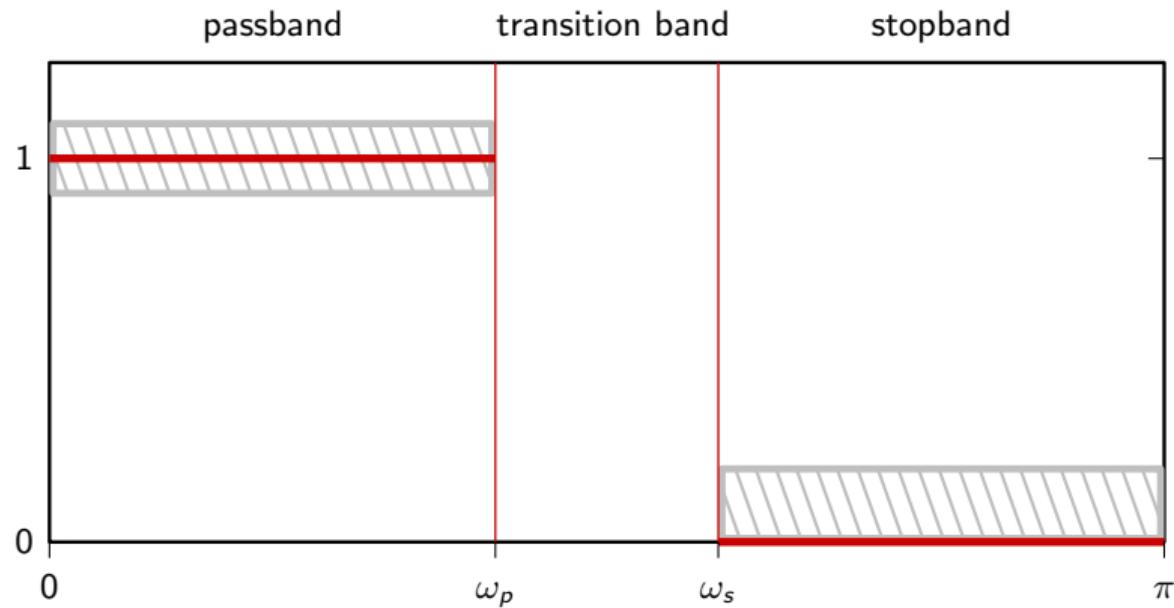
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## Example: realistic lowpass specs



$H(z) = B(z)/A(z)$ , with  $A$  and  $B$  polynomials

$$H(z) = B(z)/A(z), \quad \text{with } A \text{ and } B \text{ polynomials}$$

- $H(e^{j\omega}) = c$  over an interval  $\Rightarrow B(z) - cA(z) = 0$  over an interval  
 $\Rightarrow B(z) - cA(z)$  has an infinite number of roots  
 $\Rightarrow B(z) - cA(z) = 0$  for all values of  $z$   
 $\Rightarrow H(e^{j\omega}) = c$  over the entire  $[-\pi, \pi]$  interval.

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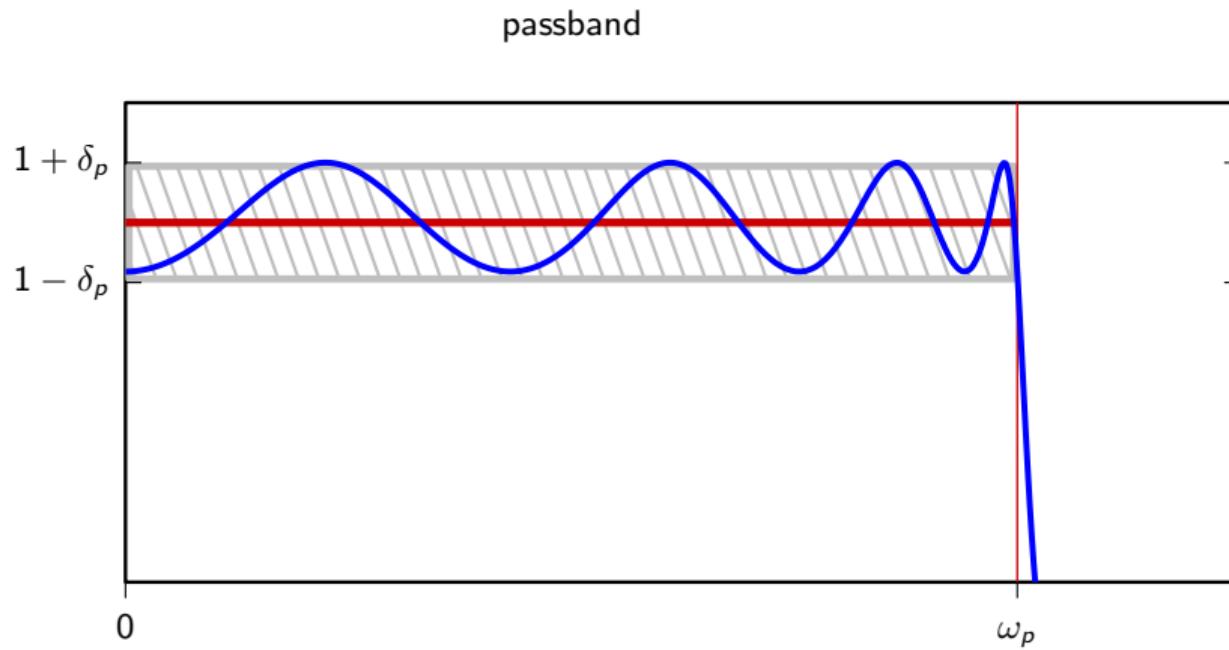
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 \end{aligned}$$

## Important case: equiripple error



- ▶ IIR or FIR?
- ▶ how to determine the coefficients?
- ▶ how to evaluate the performance?

Pros:

- ▶ computationally efficient
- ▶ strong attenuation easy
- ▶ good for audio

Cons:

- ▶ stability issues
- ▶ difficult to design for arbitrary response
- ▶ nonlinear phase

Pros:

- ▶ always stable
- ▶ optimal design techniques exist
- ▶ can be designed with linear phase

Cons:

- ▶ computationally much more expensive
- ▶ may “sound” harsh

- ▶ finding  $N$ ,  $M$ ,  $a_k$ 's and  $b_k$ 's from specs is a hard nonlinear problem
- ▶ established methods:
  - IIR: conversion of analog design
  - FIR: optimal minimax filter design

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Filter design was an established art long before digital processing appeared

- ▶ lots of nice analog filters exist
- ▶ methods exist to “translate” the analog design into a rational transfer function
- ▶ most numerical packages (Matlab, etc.) provide ready-made routines
- ▶ design involves specifying some parameters and testing that the specs are fulfilled

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Magnitude response:

- ▶ maximally flat
- ▶ monotonic over  $[0, \pi]$

Design parameters:

- ▶ order  $N$
- ▶ cutoff frequency

Test values:

- ▶ width of transition band
- ▶ passband error

Magnitude response:

- ▶ maximally flat
- ▶ monotonic over  $[0, \pi]$

Design parameters:

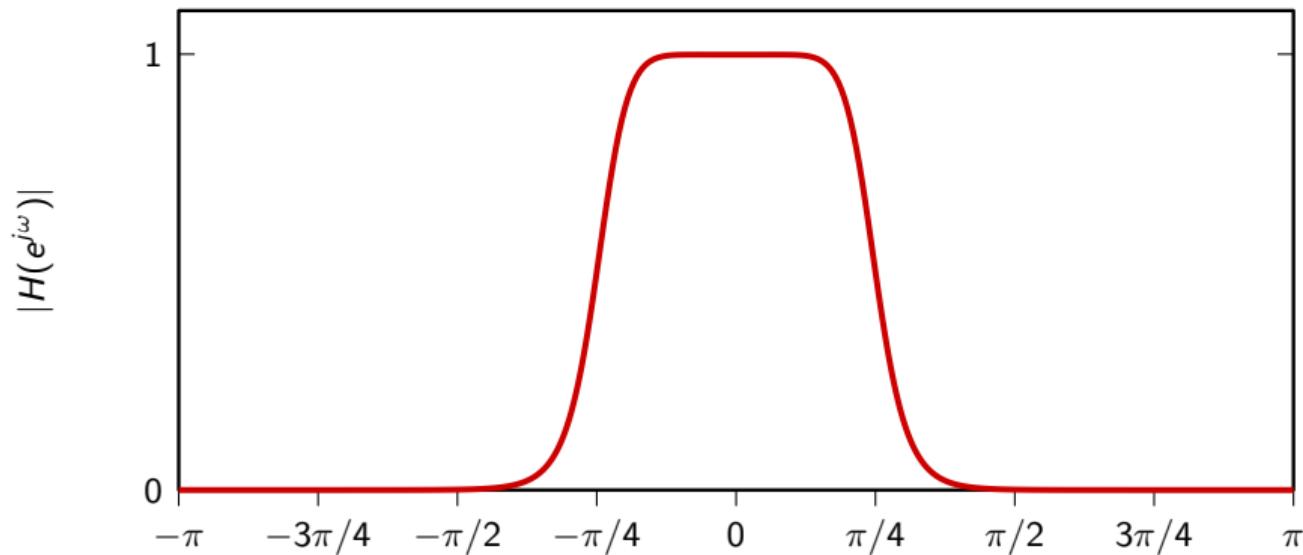
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Test values:

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# Butterworth lowpass example

$$N = 4, \omega_c = \pi/4$$



Magnitude response:

- ▶ equiripple in passband
- ▶ monotonic in stopband

Design parameters:

- ▶ order  $N$
- ▶ passband max error
- ▶ cutoff frequency

Test values:

- ▶ width of transition band
- ▶ stopband error

Magnitude response:

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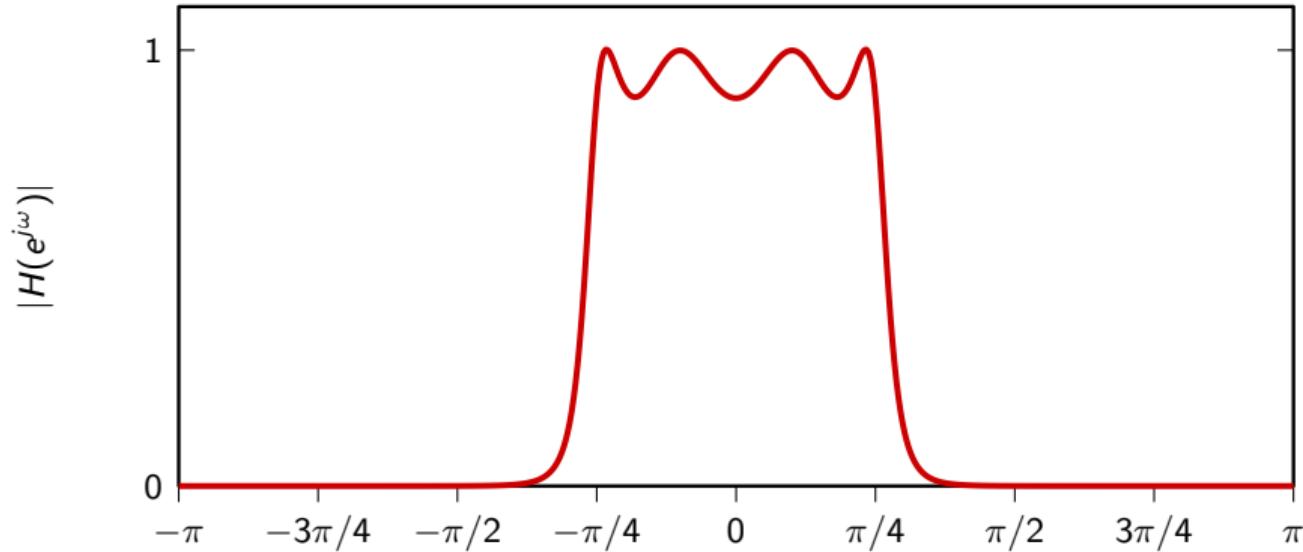
- ▶ order  $N$
- ▶ passband max error
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Test values:

- ▶ width of transition band
- ▶ stopband error

# Chebyshev lowpass example

$$N = 4, \omega_c = \pi/4, e_{\max} = 12\%$$



Magnitude response:

- ▶ equiripple in passband and stopband

Design parameters:

- ▶ order  $N$
- ▶ cutoff frequency
- ▶ passband max error
- ▶ stopband min attenuation

Test value:

- ▶ width of transition band

Magnitude response:

- ▶ equiripple in passband and stopband

Design parameters:

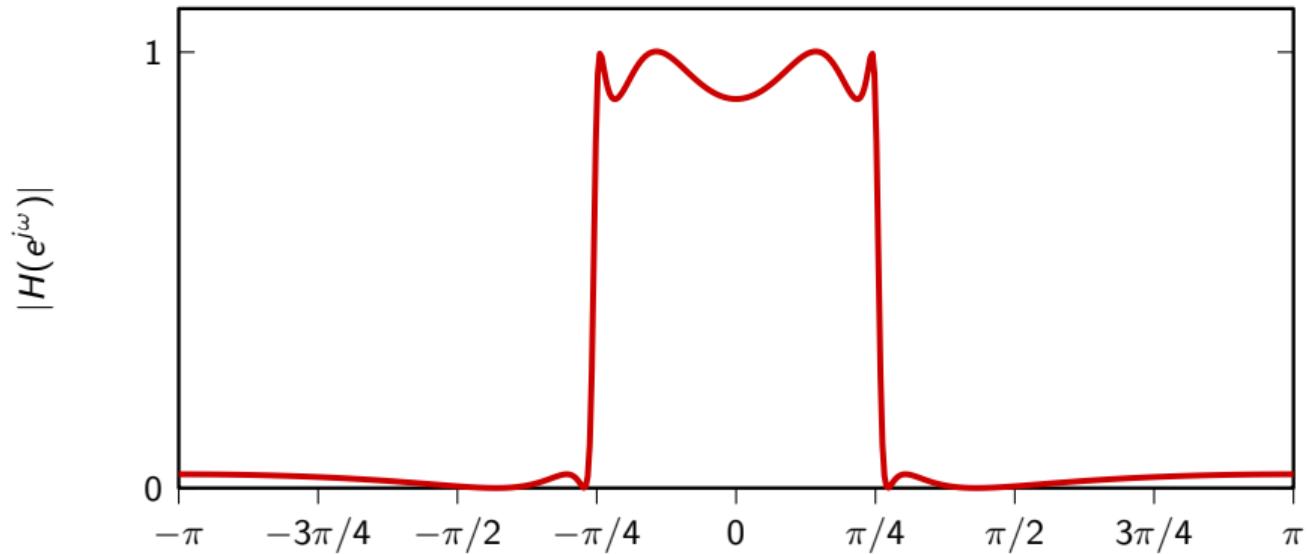
- ▶ order  $N$
- ▶ cutoff frequency
- ▶ passband max error
- ▶ stopband min attenuation

Test value:

- ▶ width of transition band

# Elliptic lowpass example

$$N = 4, \omega_c = \pi/4, e_{\max} = 12\%, \text{att}_{\min} = 0.03$$



FIR filters are a digital signal processing “exclusivity”.

In the 70s Parks and McClellan developed an algorithm to design optimal FIR filters:

- ▶ linear phase
- ▶ equiripple error in passband and stopband

algorithm proceeds by **minimizing the maximum error** in passband and stopband

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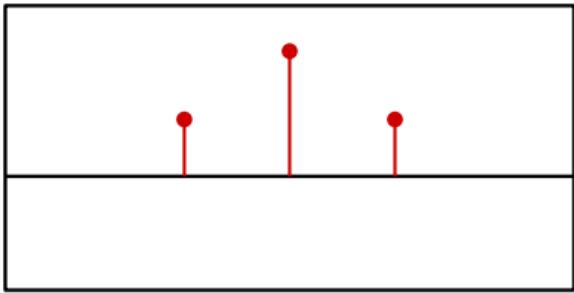
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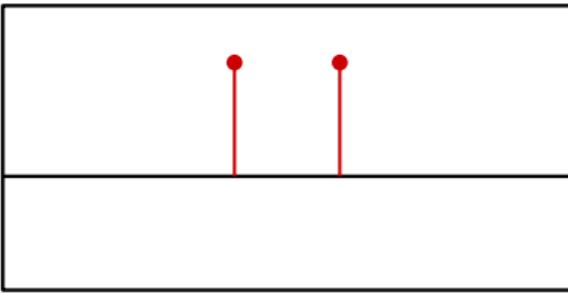
algorithm proceeds by **minimizing** the **maximum** error in passband and stopband

Linear phase derives from a symmetric or antisymmetric impulse response

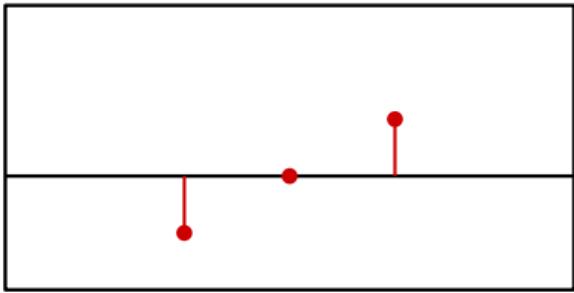
Type I



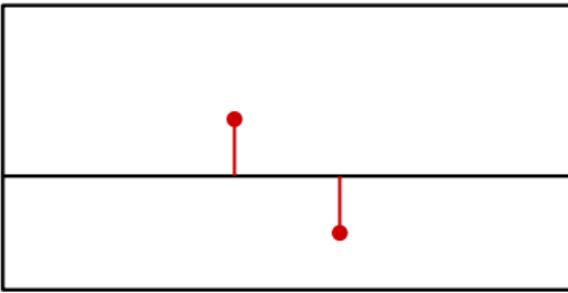
Type II



Type III



Type IV



$$h[C + n] = h[C - n]$$

$$h'[n] = h[n + C]$$

$$h'[n] = h'[-n]$$

$$H(z) = z^{-C} H'(z)$$

$$h[C + n] = h[C - n]$$

$$h'[n] = h[n + C]$$

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$$H(z) = z^{-C} H'(z)$$

$$H'(z) = \sum_{n=-M}^M h'[n]z^{-n}$$

$$= h'[0] + \sum_{n=1}^M h'[n](z^n + z^{-n})$$

$$H'(e^{j\omega}) = h'[0] + \sum_{n=1}^M h'[n](e^{j\omega n} + e^{-j\omega n})$$

$$= h'[0] + 2 \sum_{n=1}^M h'[n] \cos \omega n \quad \in \mathbb{R}$$

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$$H(e^{j\omega}) = \left[ h[C] + 2 \sum_{n=1}^M h[n+C] \cos n\omega \right] e^{-j\omega C}$$

Magnitude response:

- ▶ equiripple in passband and stopband

Design parameters:

- ▶ order  $N$  (number of taps)
- ▶ passband edge  $\omega_p$
- ▶ stopband edge  $\omega_s$
- ▶ ratio of passband to stopband error  $\delta_p/\delta_s$

Test value:

- ▶ passband max error
- ▶ stopband max error

Magnitude response:

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Design parameters:

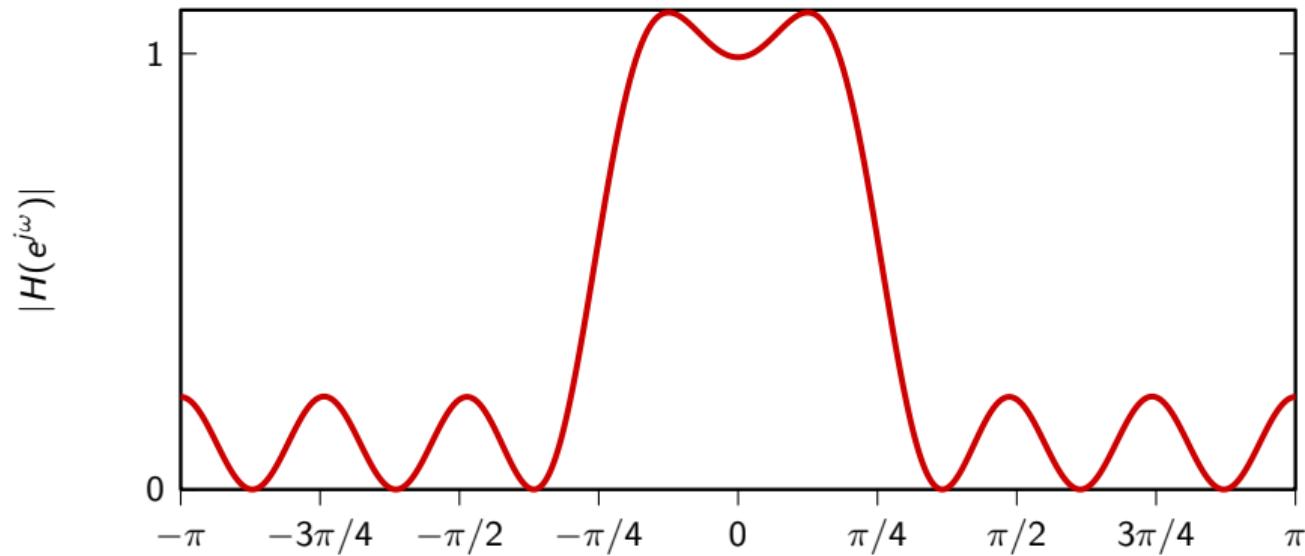
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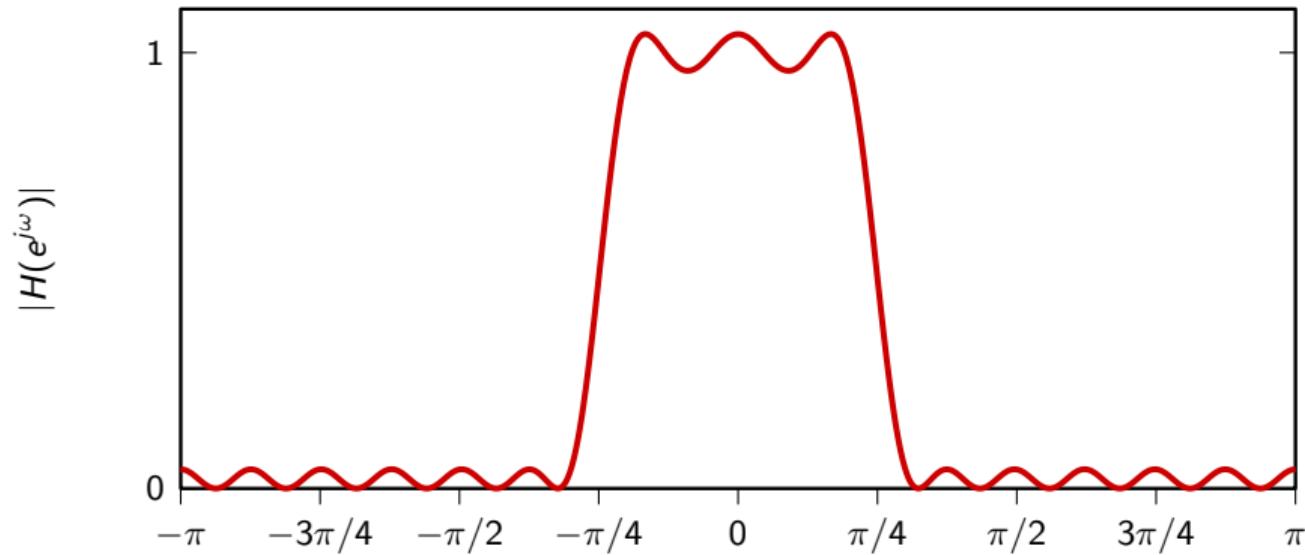
## Minimax lowpass example

$$N = 9, \omega_s = 0.2\pi, \omega_p = 0.3\pi, \delta_p/\delta_s = 10$$



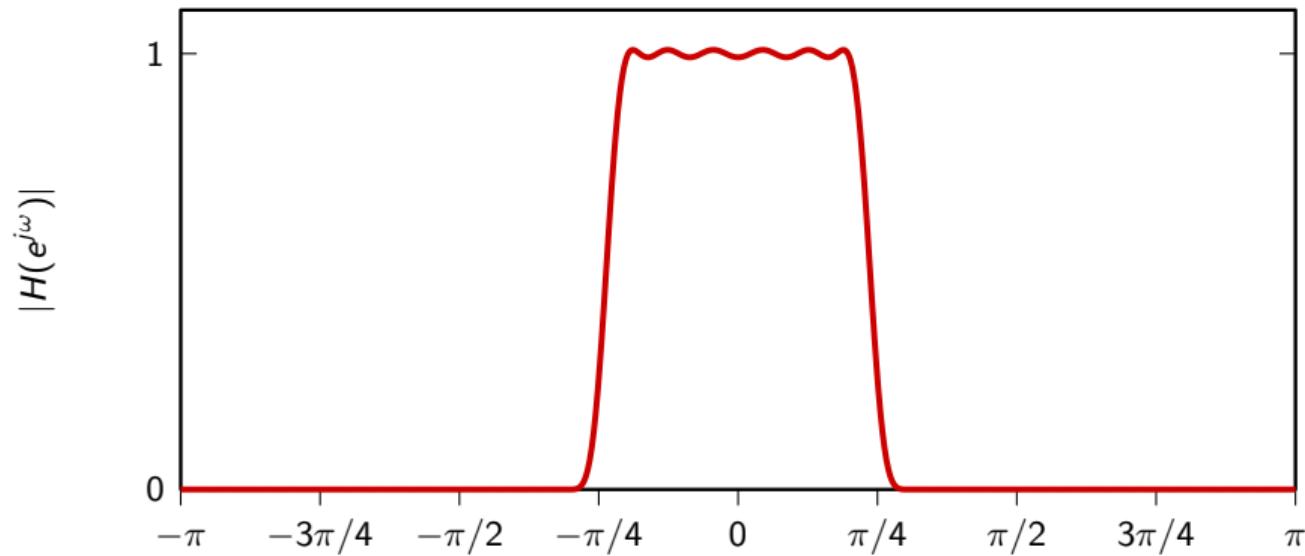
## Minimax lowpass example

$$N = 19, \omega_s = 0.2\pi, \omega_p = 0.3\pi, \delta_p/\delta_s = 10$$



## Minimax lowpass example

$$N = 51, \omega_s = 0.2\pi, \omega_p = 0.3\pi, \delta_p/\delta_s = 1$$

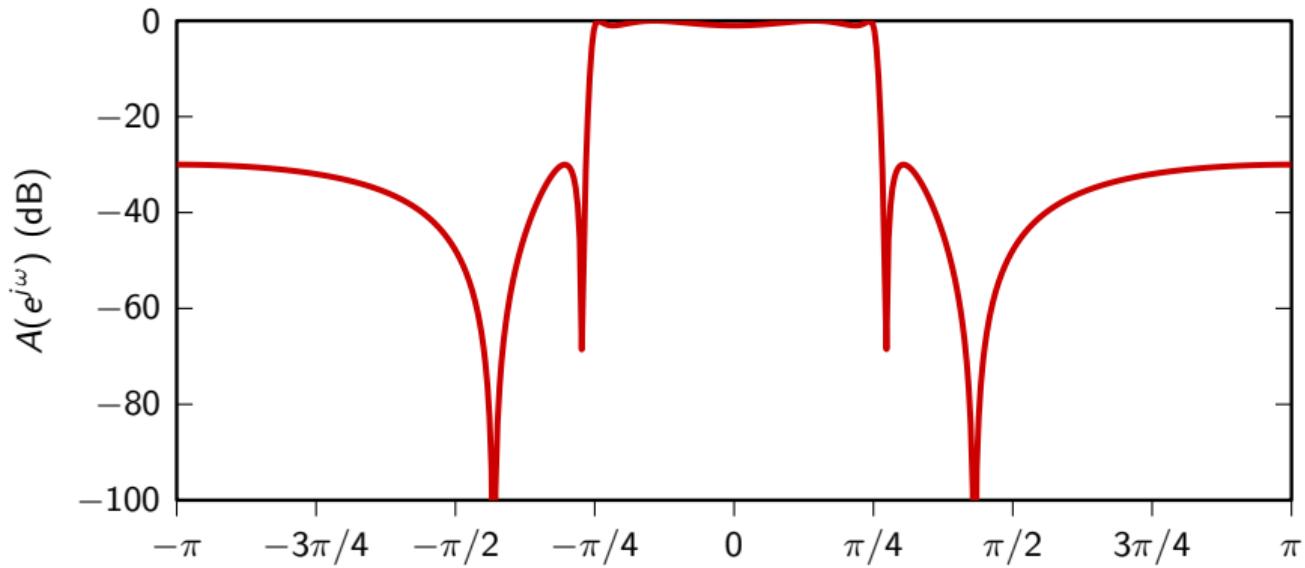


- ▶ filter max passband magnitude  $G$
- ▶ filter attenuation expressed in decibels as:

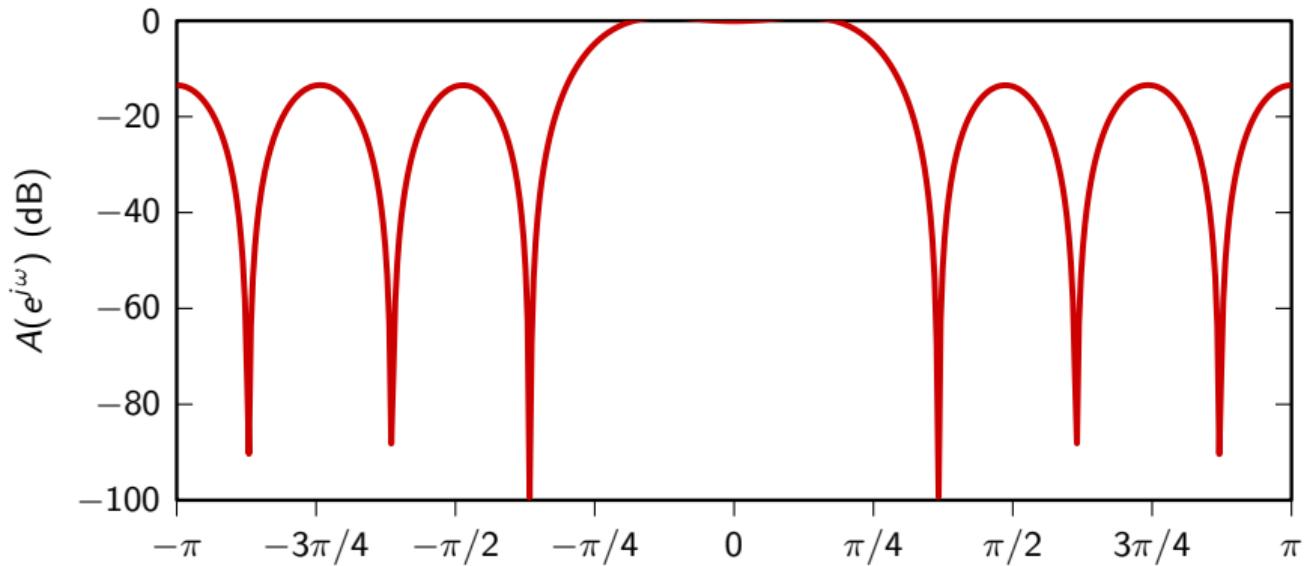
$$A_{\text{dB}} = 20 \log_{10}(|H(e^{j\omega})|/G)$$

- ▶ useful to compare attenuations between filters

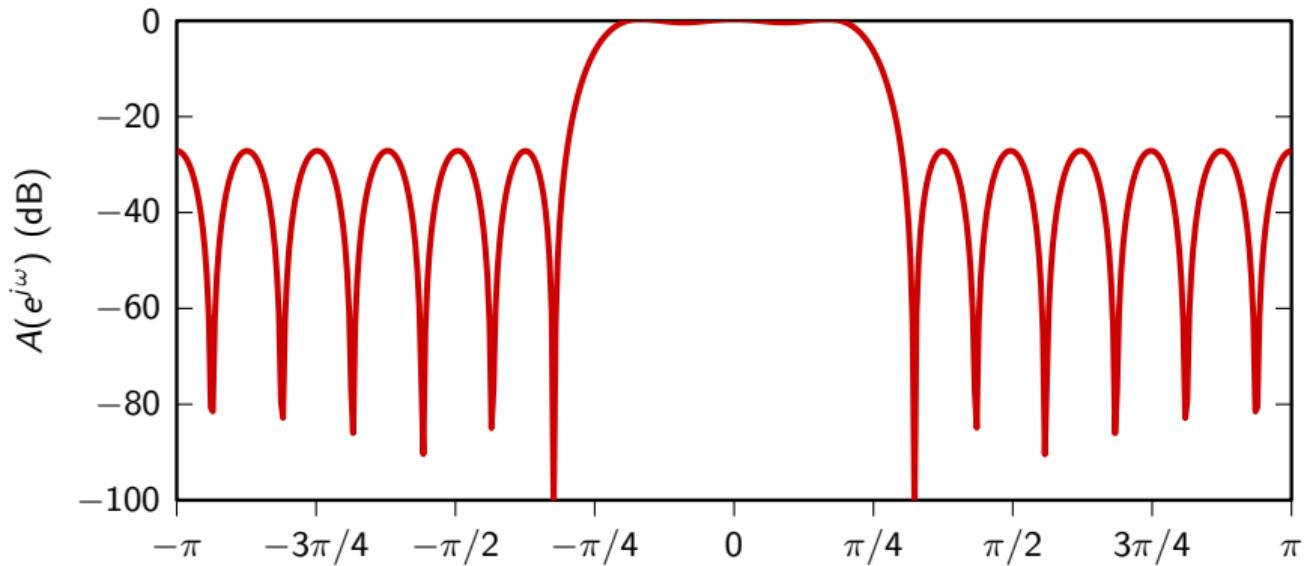
# 4th-order elliptic lowpass, $\omega_c = \pi/4$ , log scale



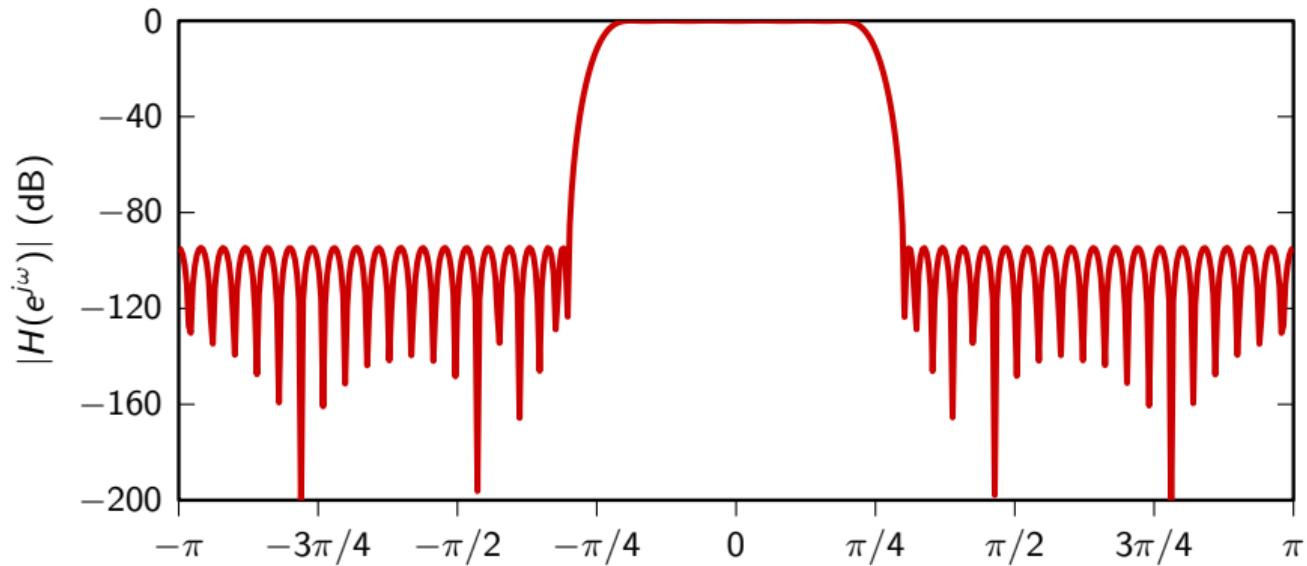
# 9-tap minimax lowpass, $\omega_c = \pi/4$ , log scale



# 19-tap minimax lowpass, $\omega_c = \pi/4$ , log scale



# 51-tap minimax lowpass, $\omega_c = \pi/4$ , log scale



The IIR and FIR methods we just described can be used to design more general filter types than lowpass, with only minor modifications

- ▶ IIR bandpass and highpass can be obtain by modulating the lowpass response
- ▶ optimal FIR bandpass and highpass can be designed by the Parks-McClellan algorithm
- ▶ optimal FIR can also be designed with piecewise linear magnitude response
- ▶ the literature on filter design is vast: this is just the tip of the iceberg!

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**END OF MODULE 5.10**

# Digital Signal Processing

Module 5.11: Real-Time Processing

# Summary:

- ▶ I/O and DMA
  - ▶ multiple buffering
  - ▶ implementation framework
  - ▶ some guitar effects

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- ▶ some guitar effects

Everything works in sync with a *system clock* of period  $T_s$ :

- ▶ “record” a value  $x_i[n]$
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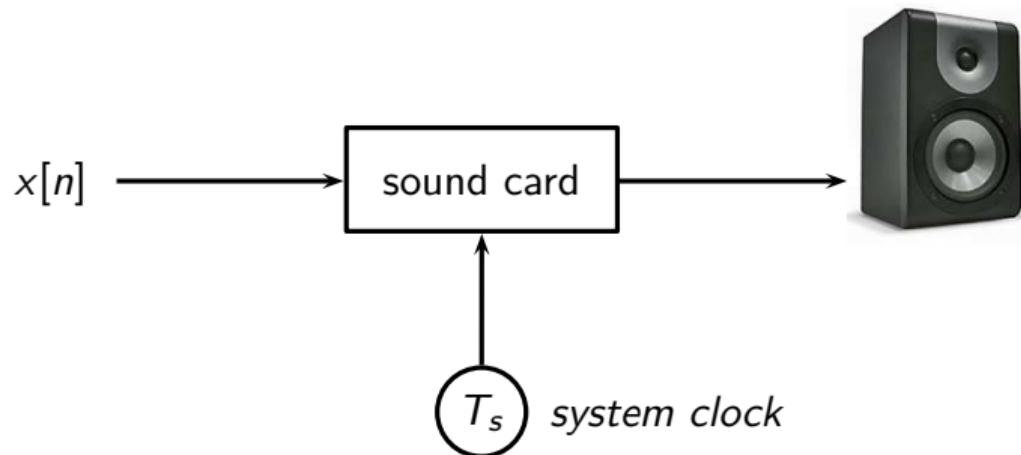
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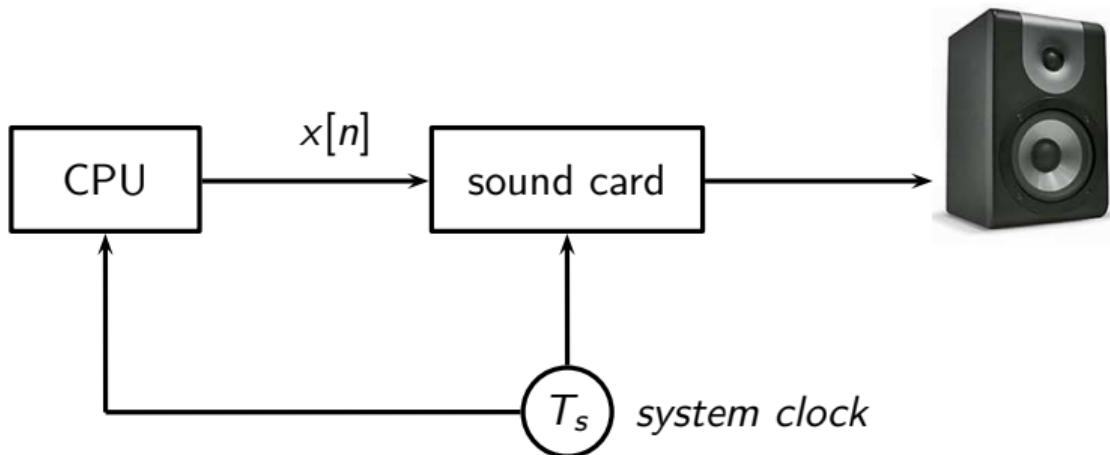
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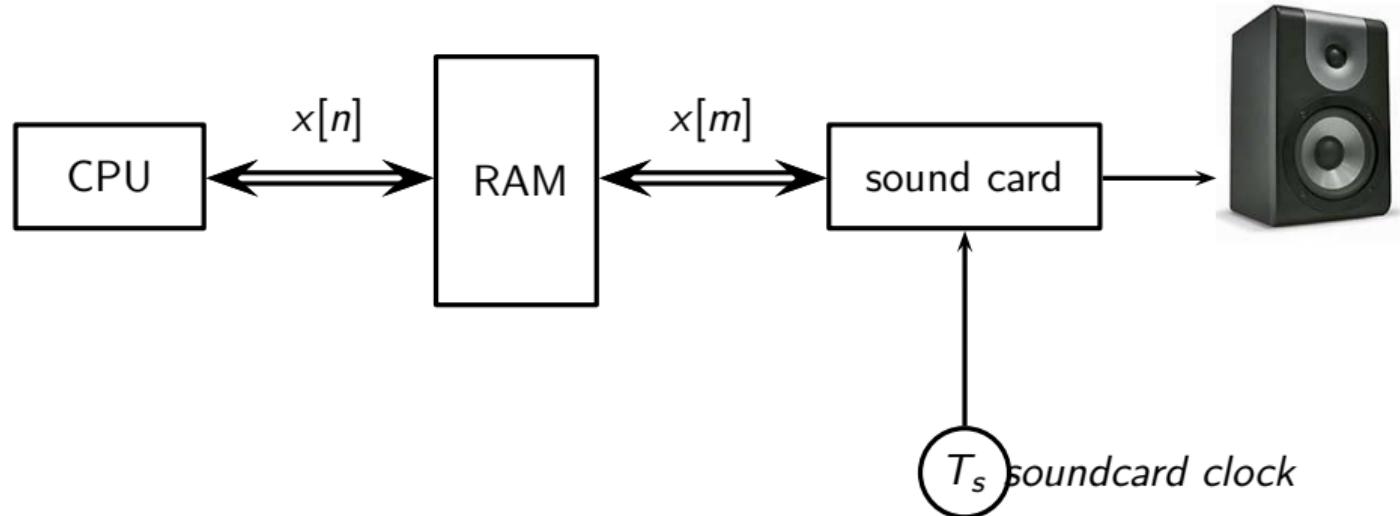
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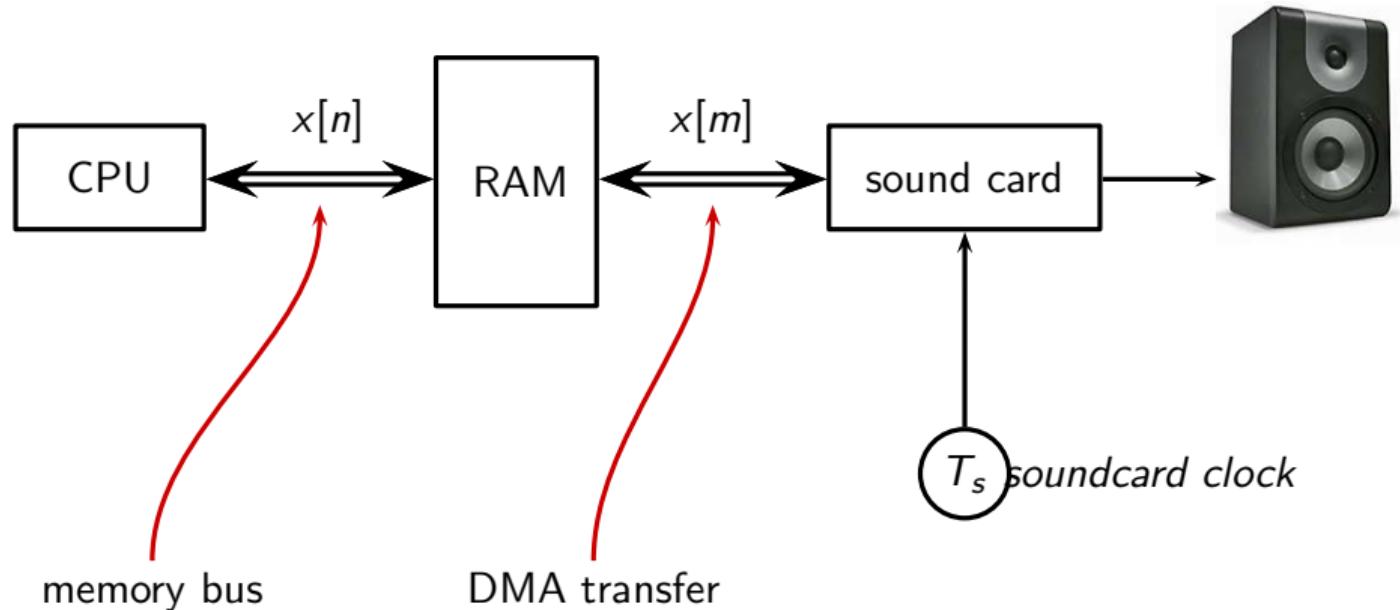
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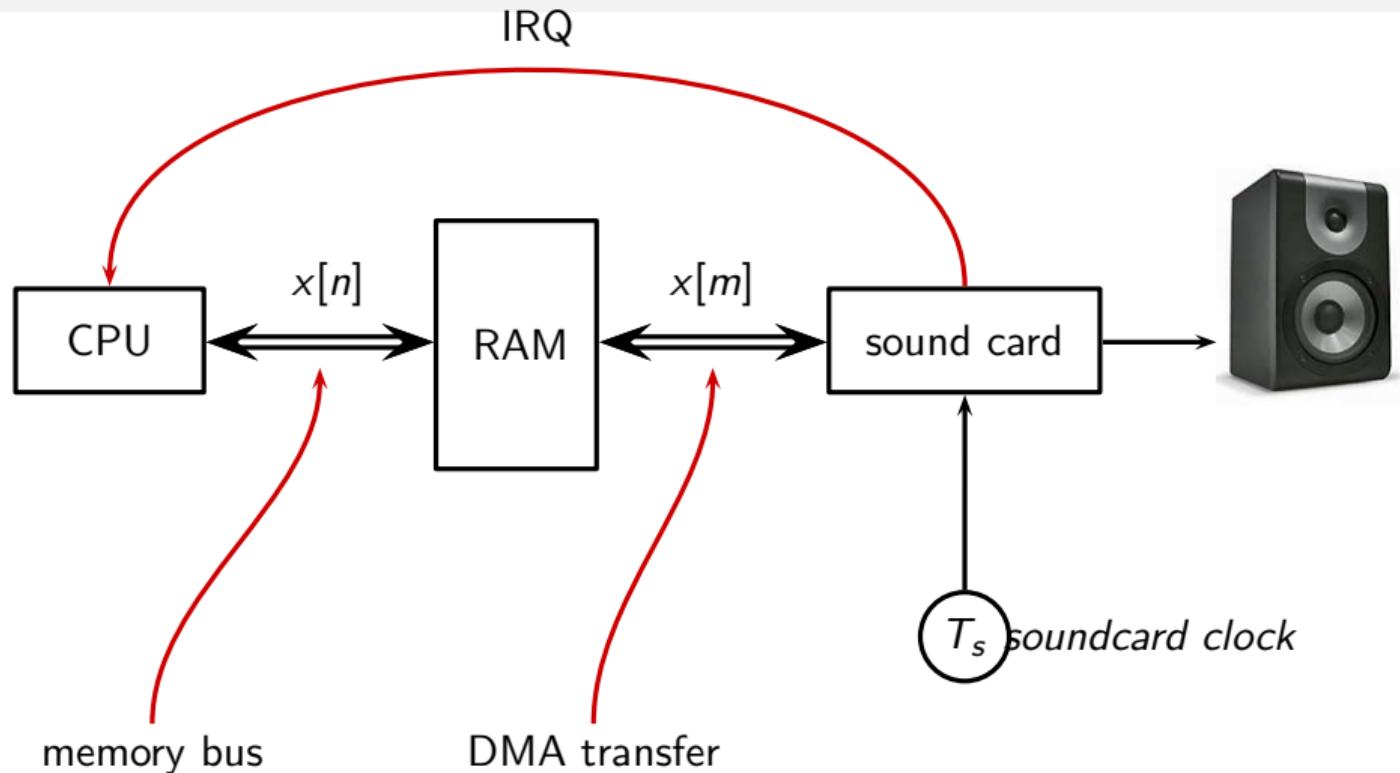
# Playing a sound











- ▶ interrupt for each sample would be too much overhead
- ▶ soundcard consumes sample in buffers
- ▶ soundcard notifies when buffer used up
- ▶ CPU can fill a buffer in less time than soundcard can empty it

buffering introduces delay!

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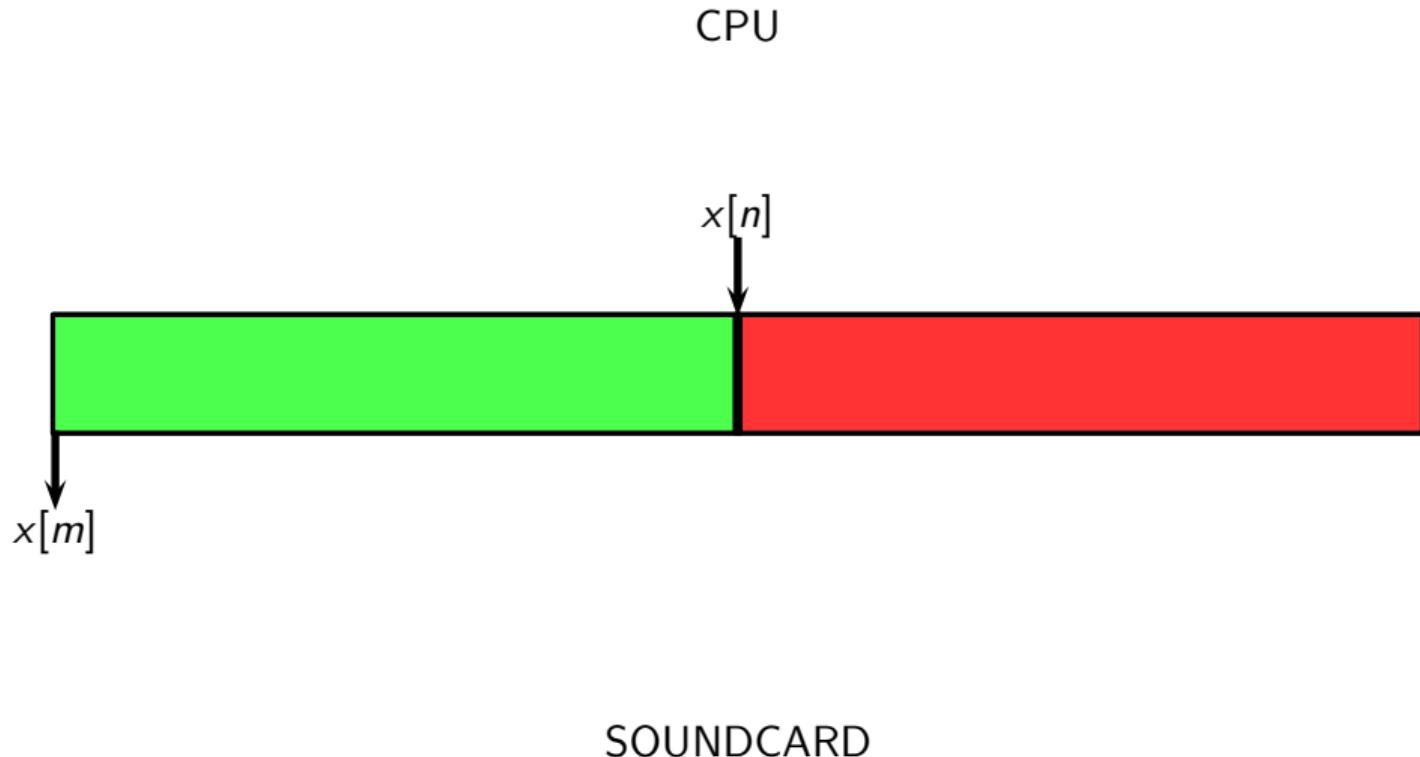
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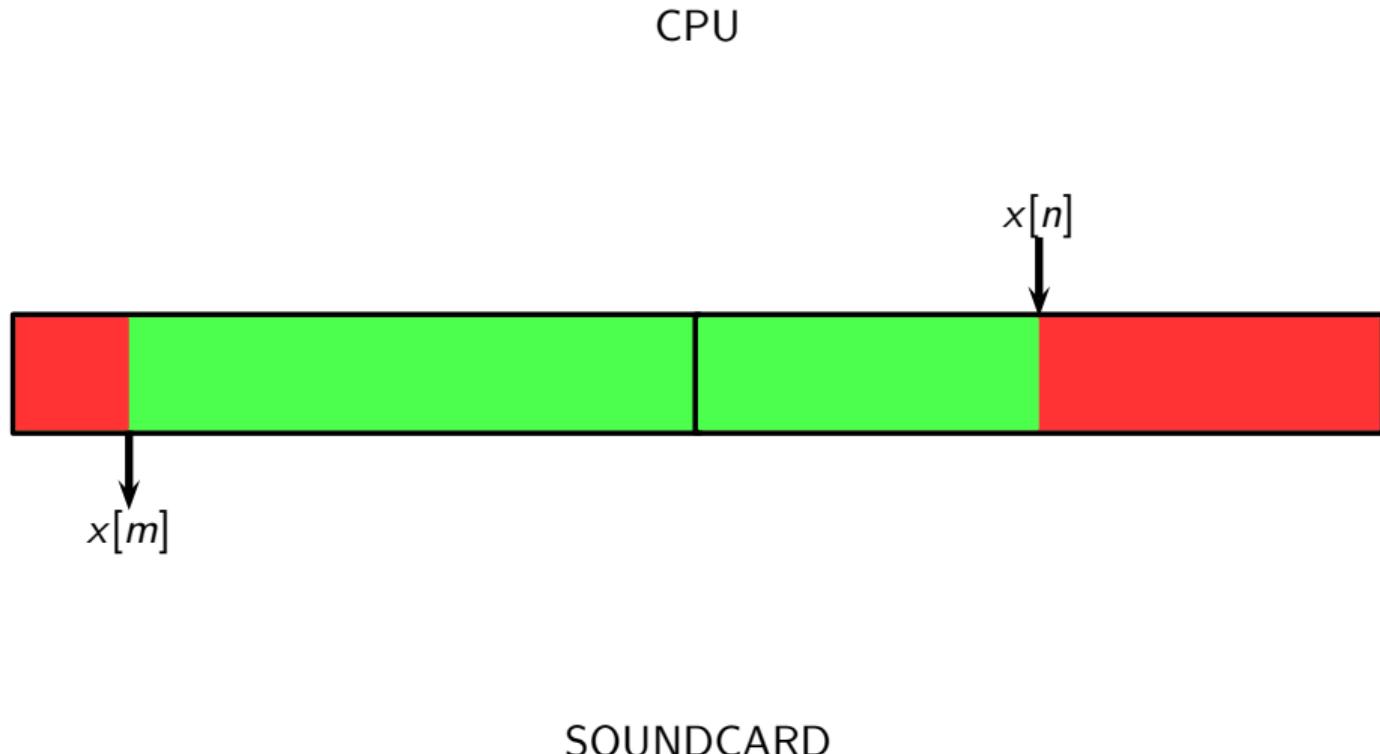
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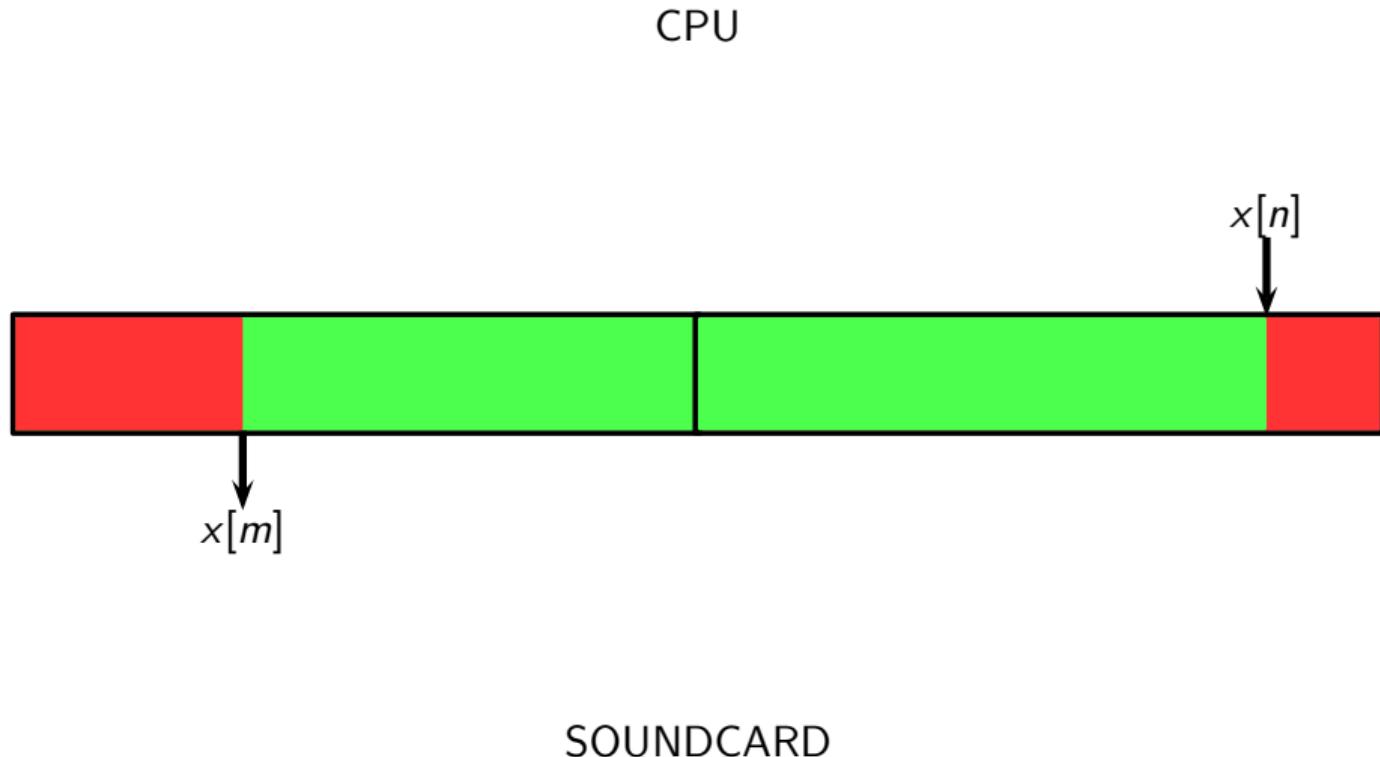
## Example: double buffering (output)



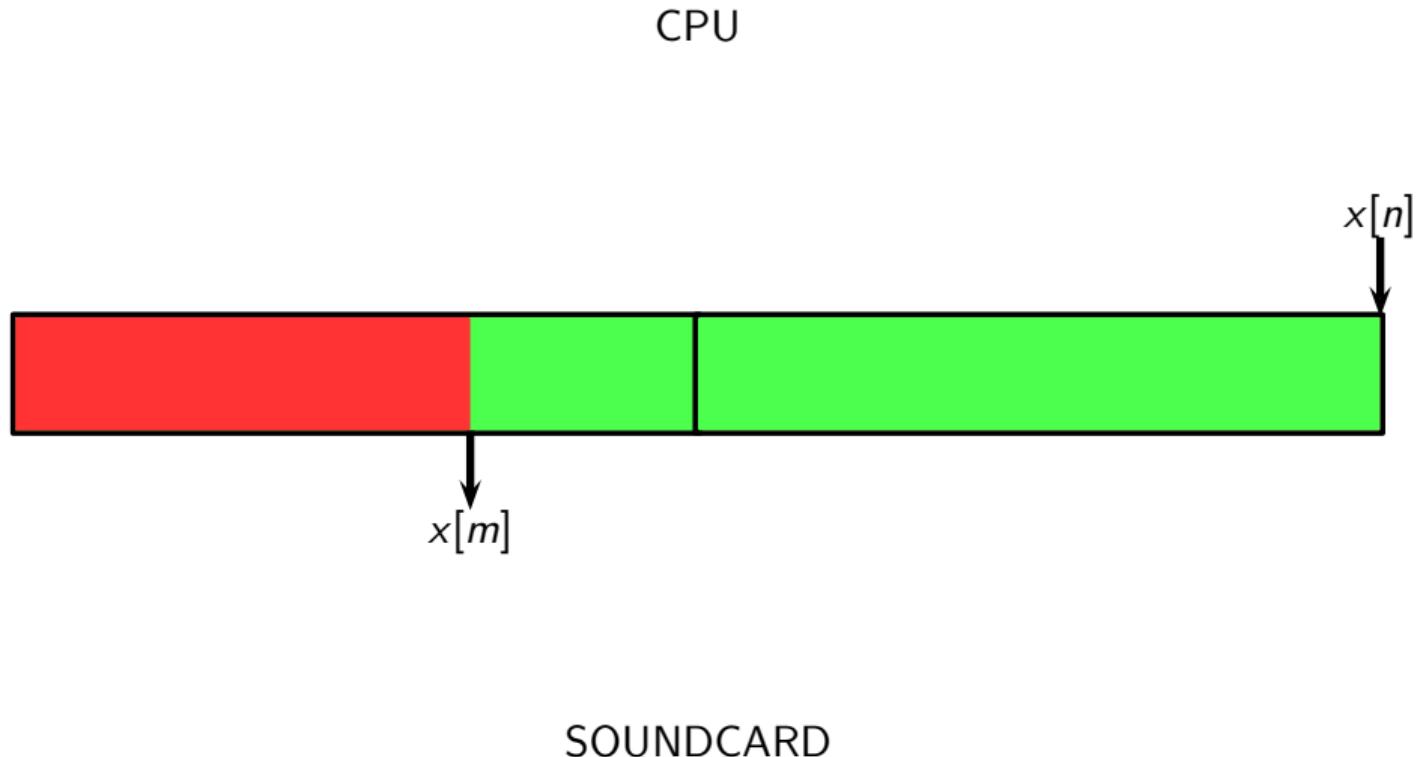
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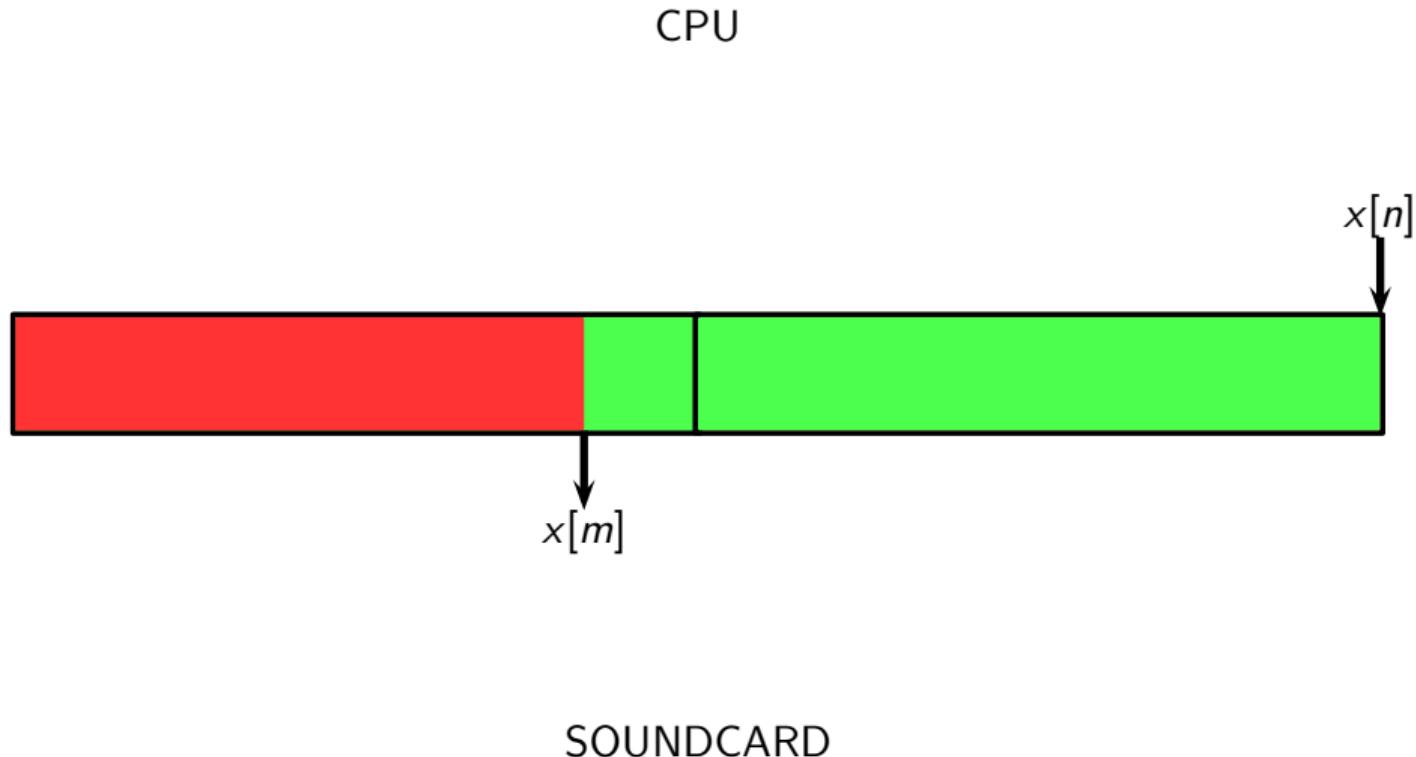
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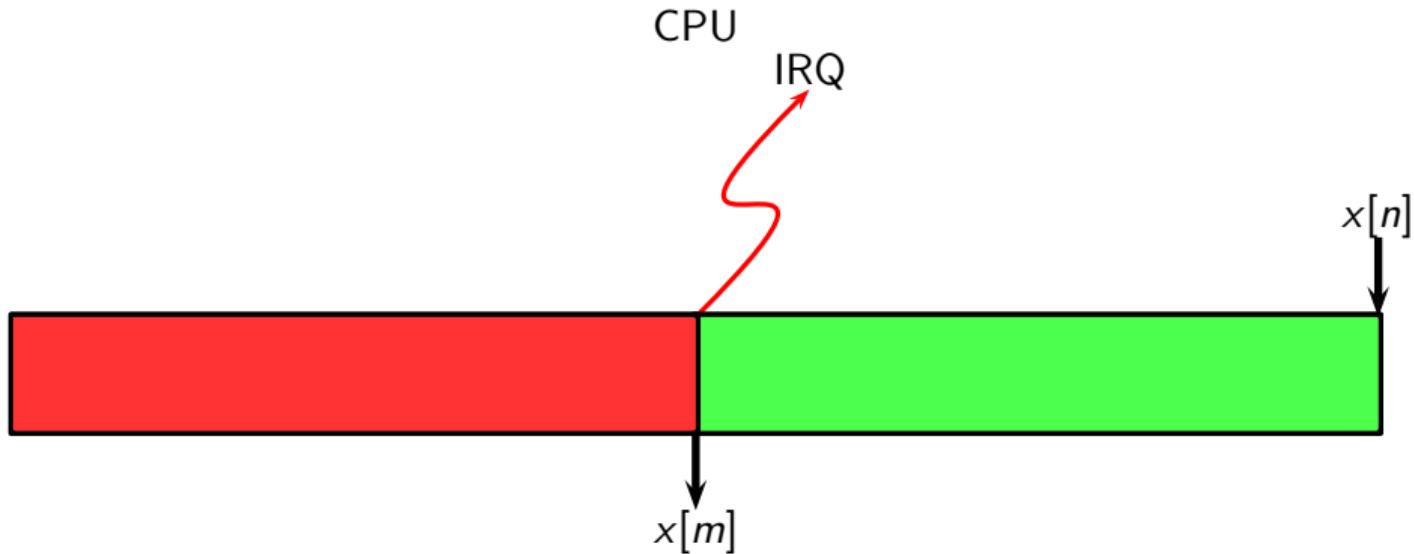
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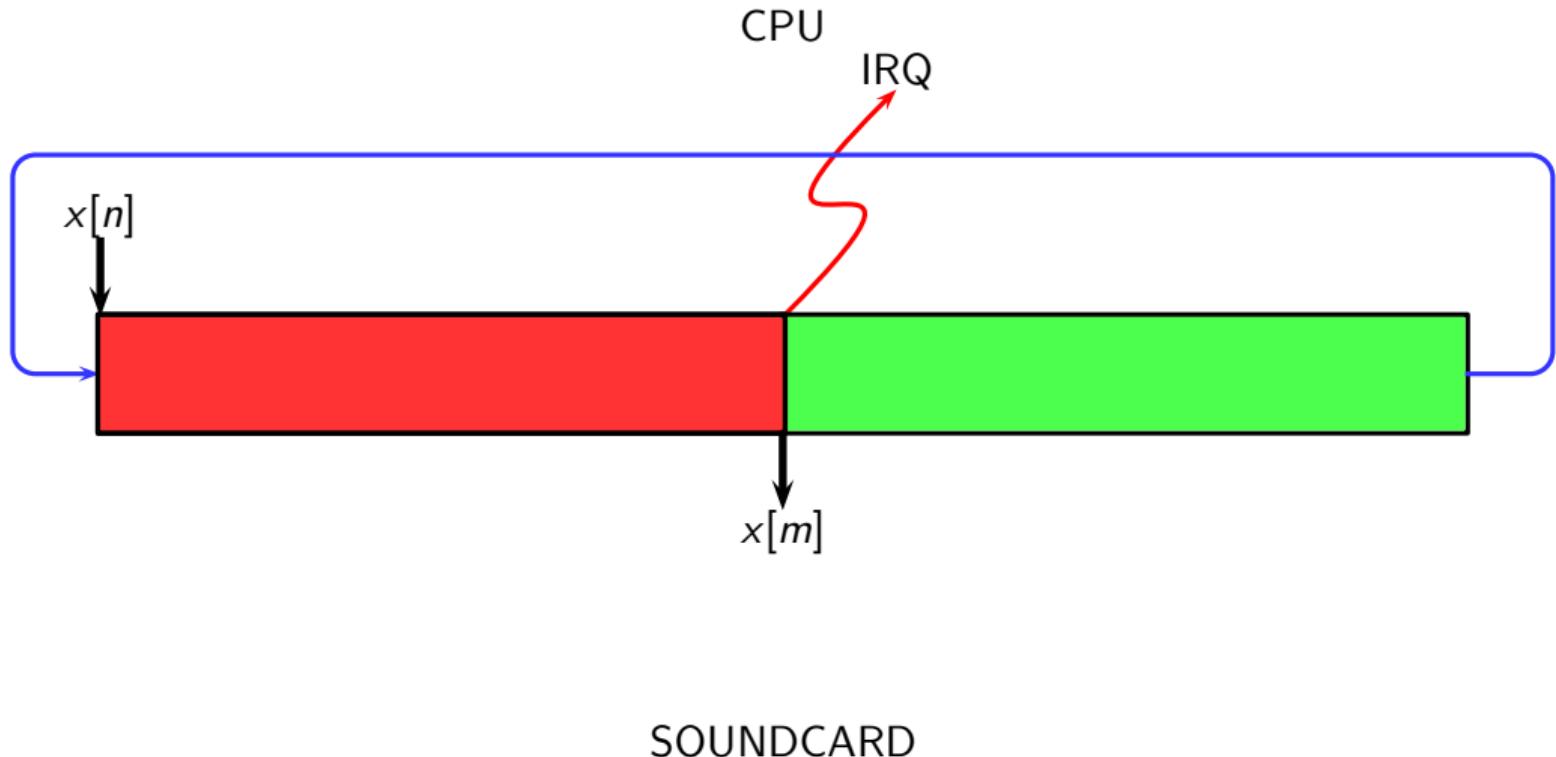


## Example: double buffering (output)



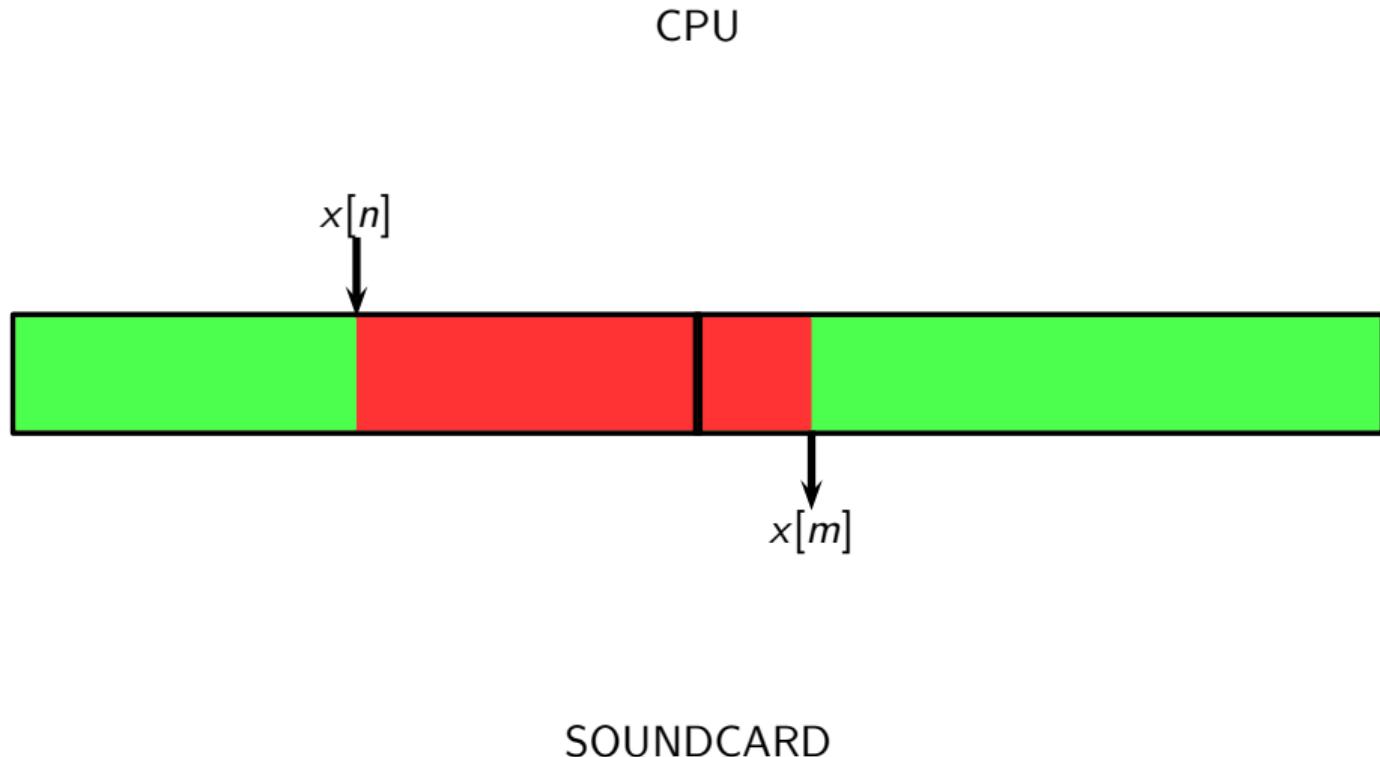
SOUNDCARD

## Example: double buffering (output)

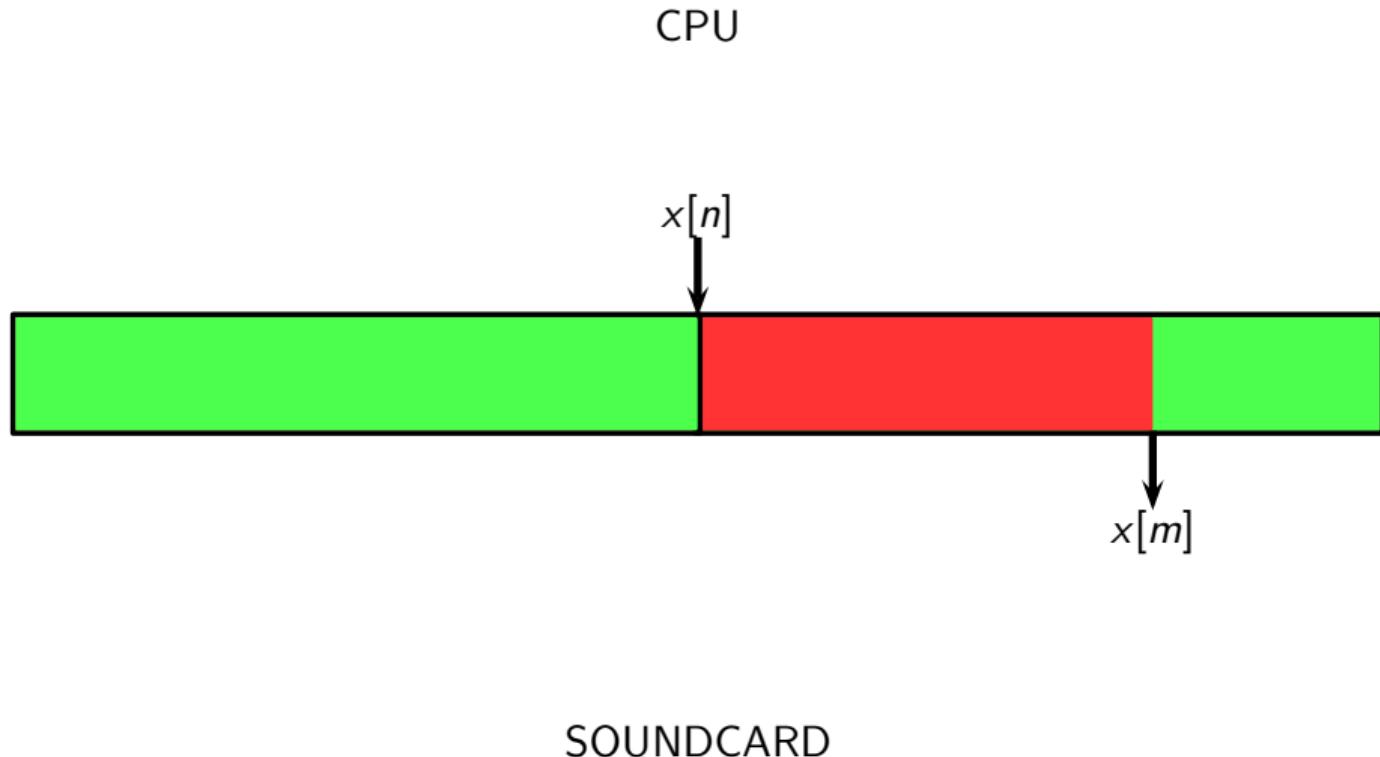


SOUNDCARD

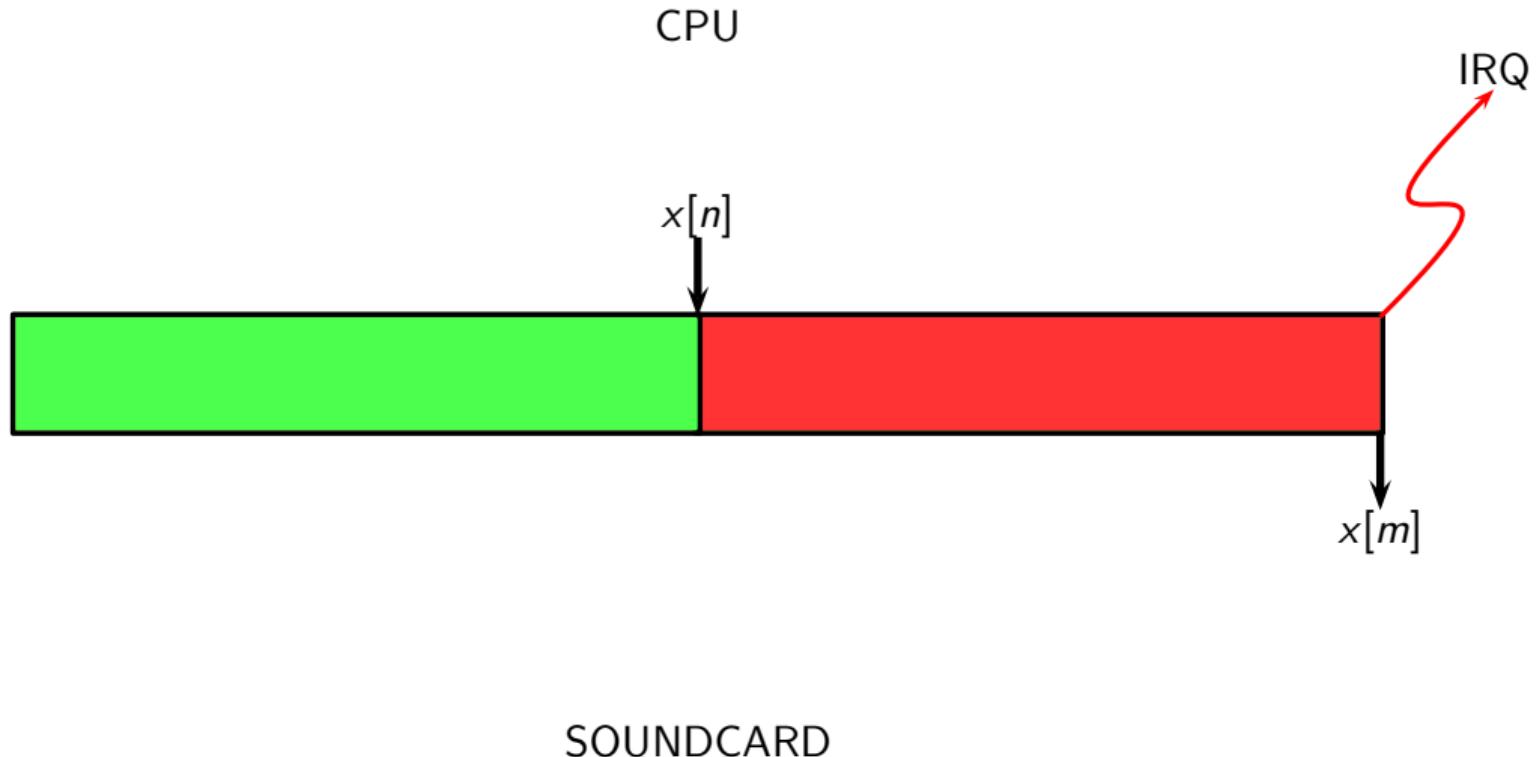
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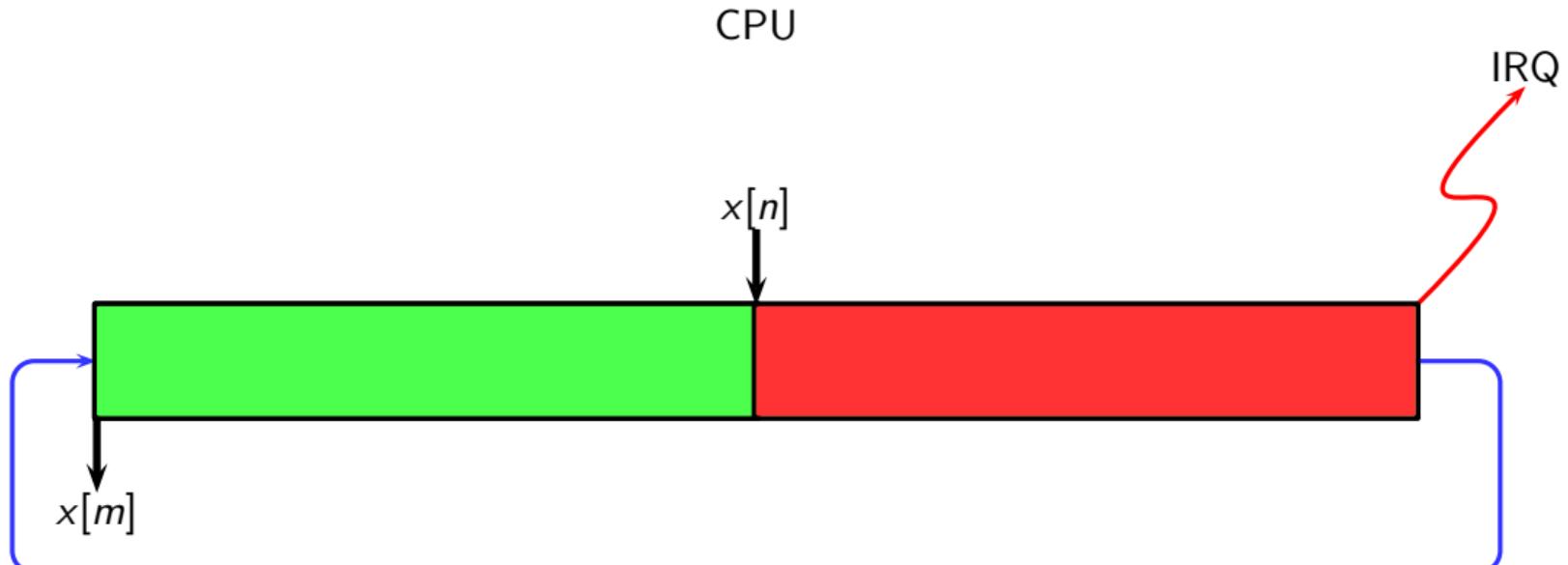
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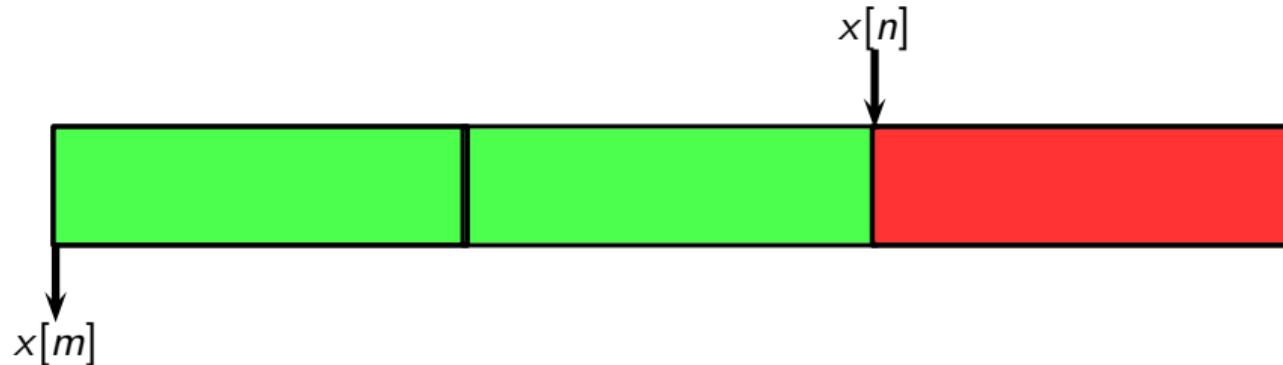


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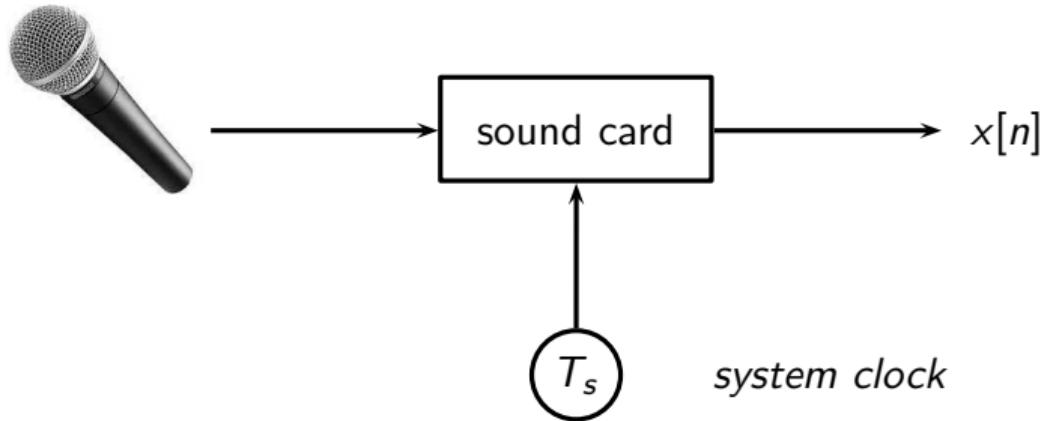
- ▶ double buffering introduces a delay  $d = T_s \times \frac{L}{2}$  seconds
- ▶ if CPU doesn't fill the buffer fast enough: **underflow**

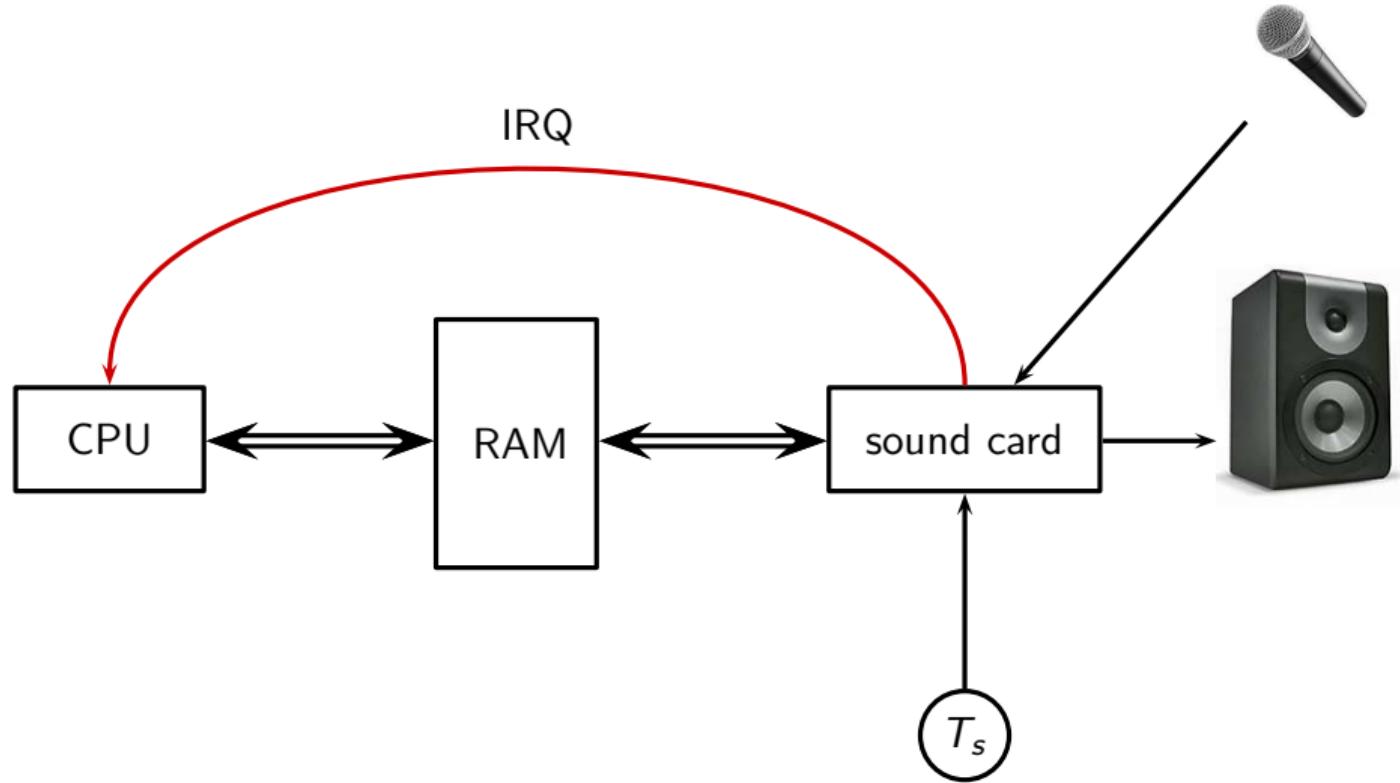
- ▶ double buffering introduces a delay  $d = T_s \times \frac{L}{2}$  seconds
- ▶ if CPU doesn't fill the buffer fast enough: **underflow**



- ▶ call the CPU more often (balance load)
- ▶ keep reasonable underflow protection

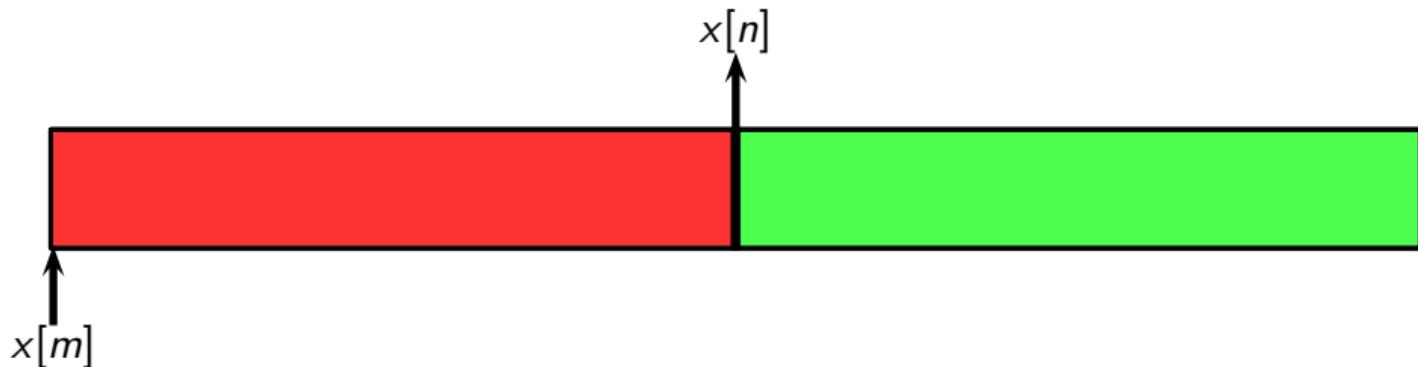
# What about the input?





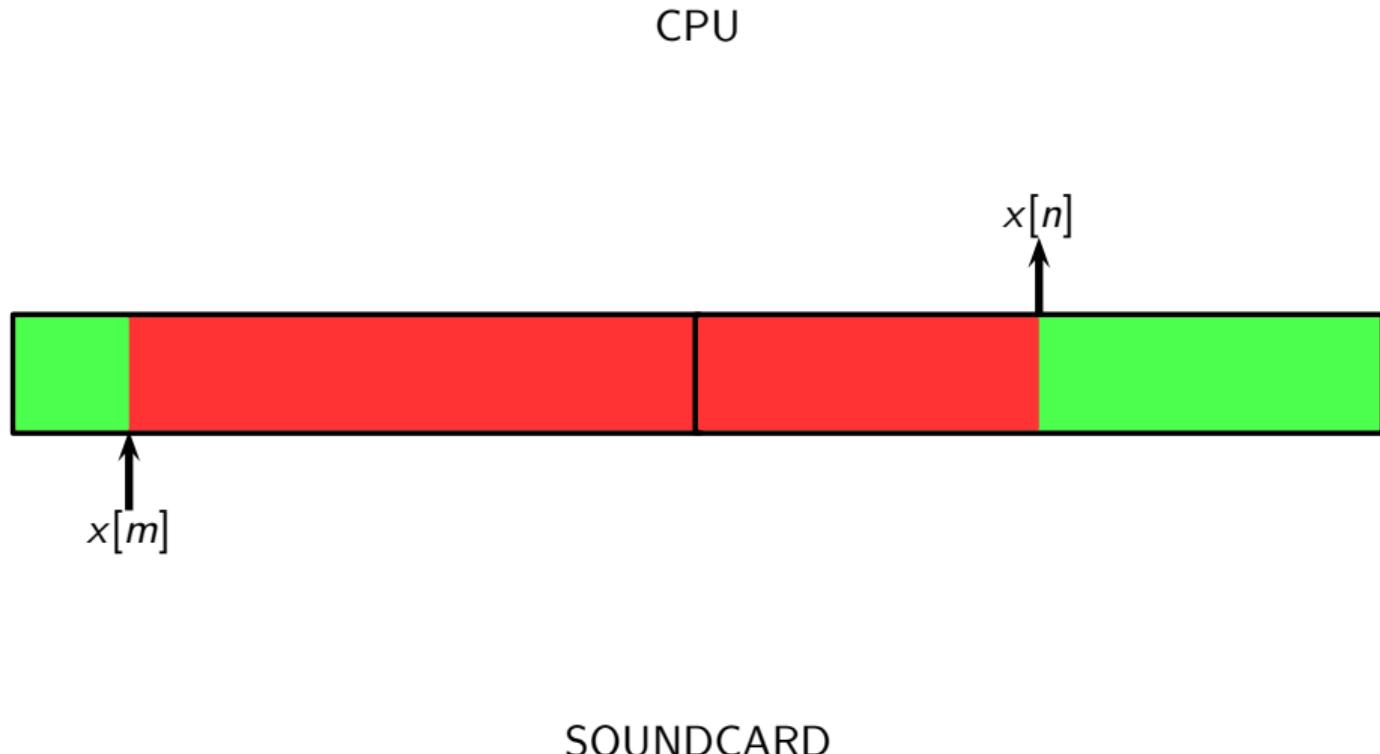
## Example: double buffering (input)

CPU

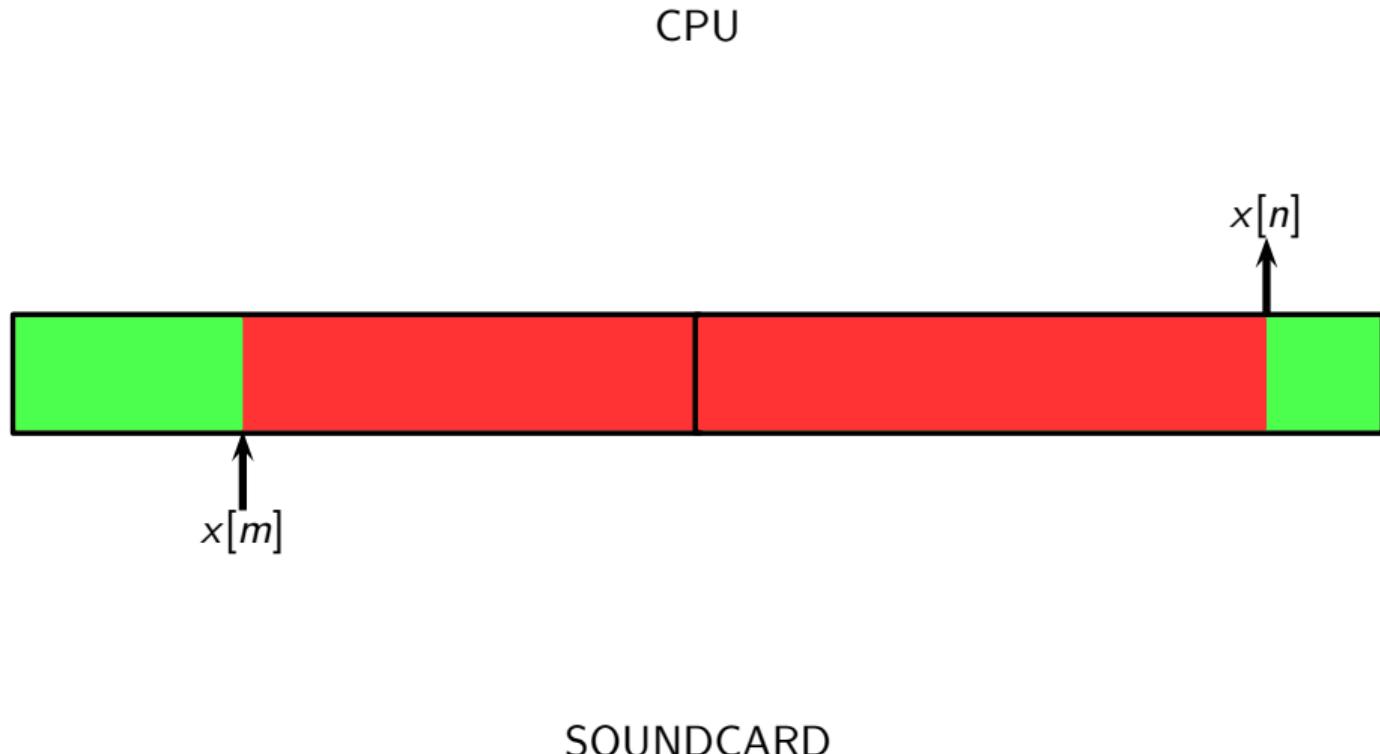


SOUNDCARD

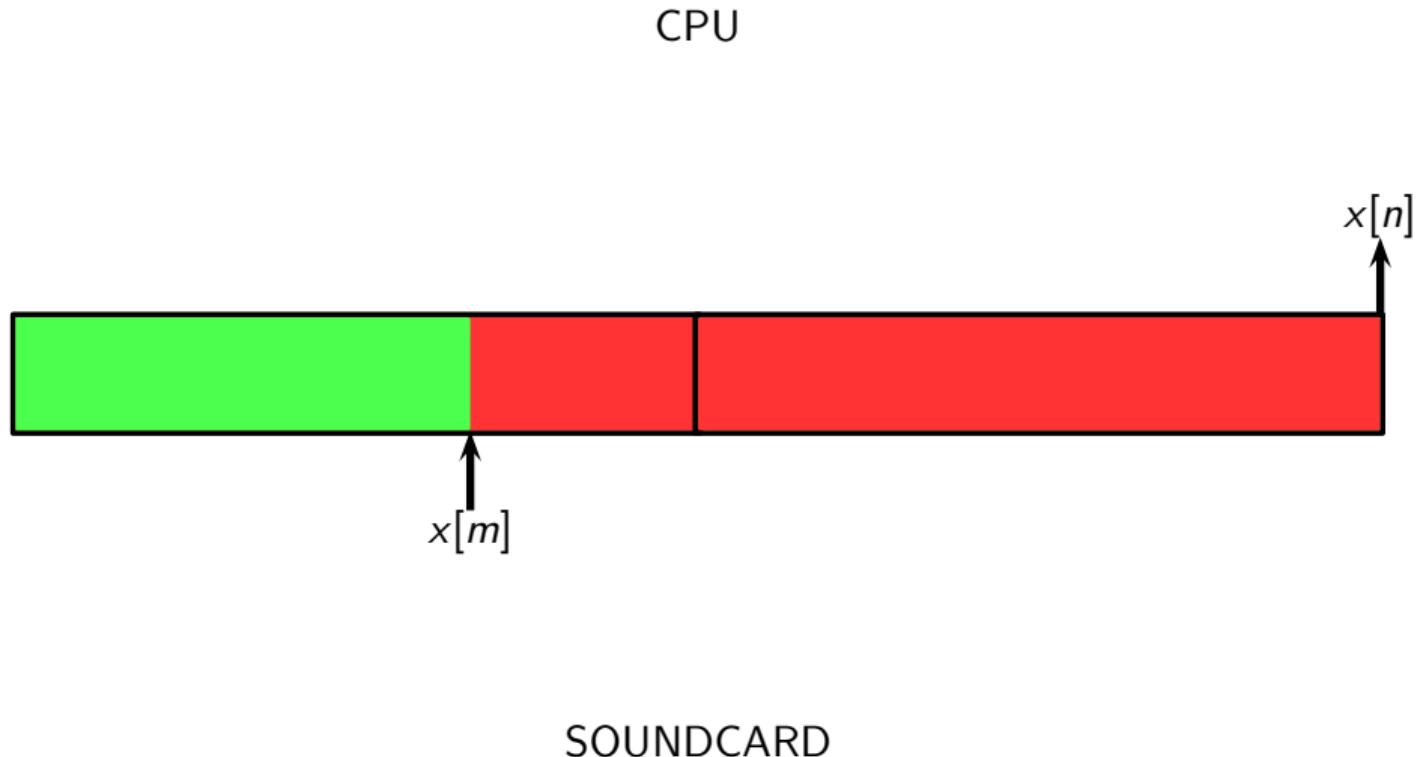
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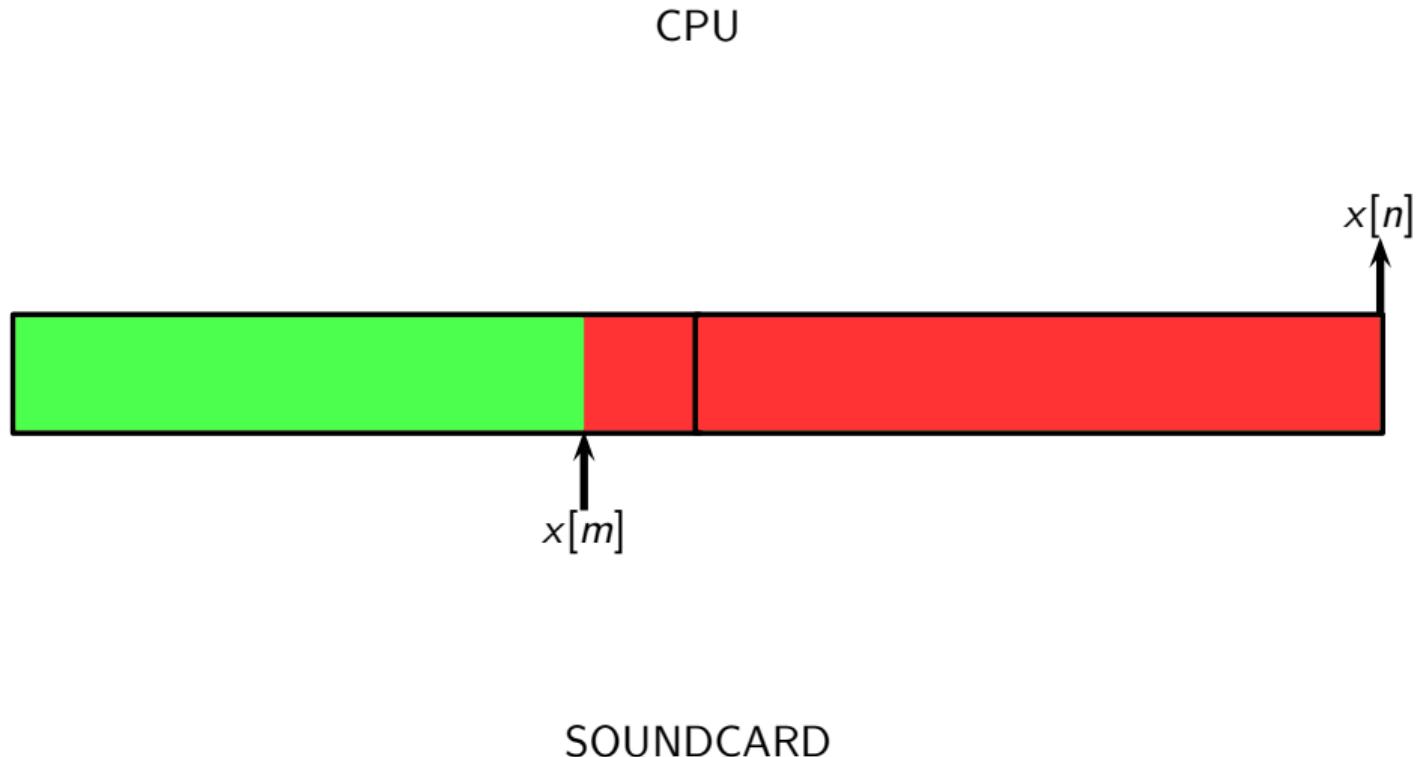
## Example: double buffering (input)



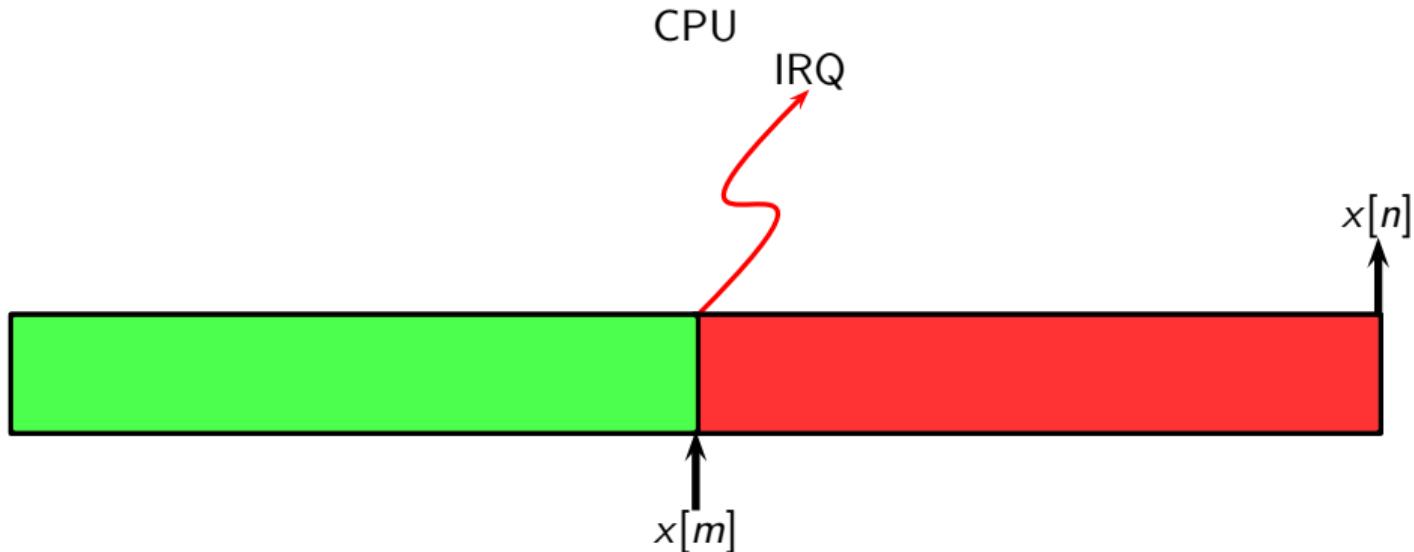
## Example: double buffering (input)



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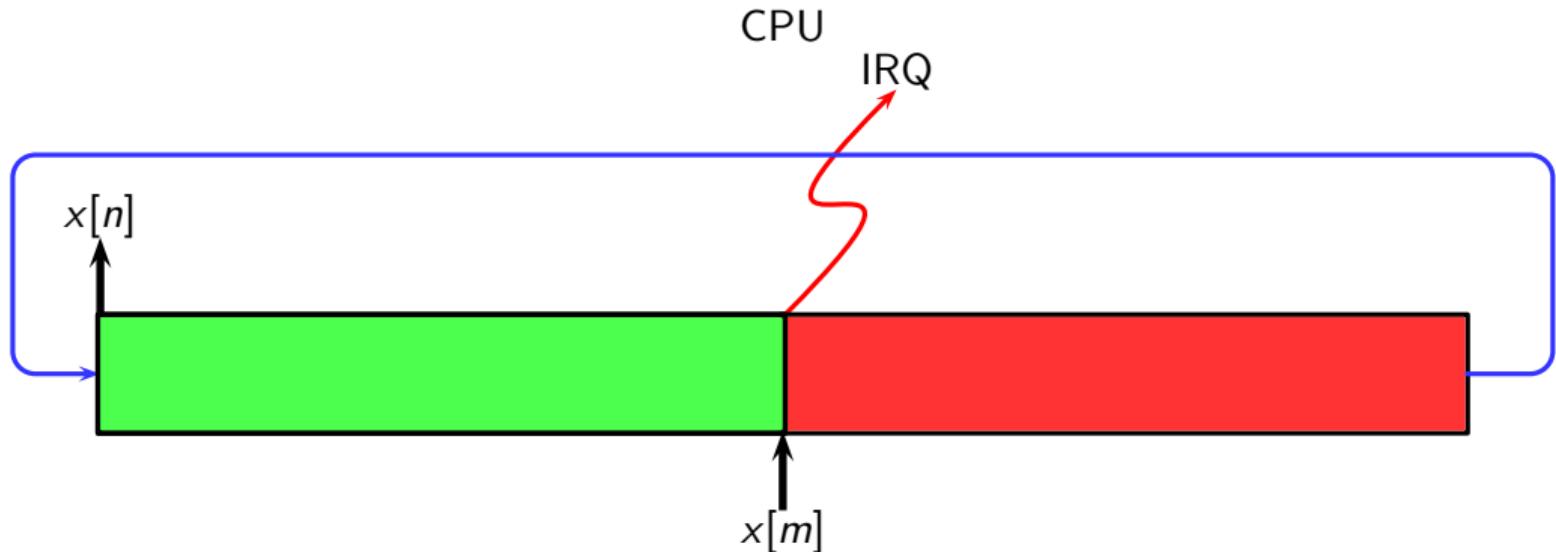


## Example: double buffering (input)



SOUNDCARD

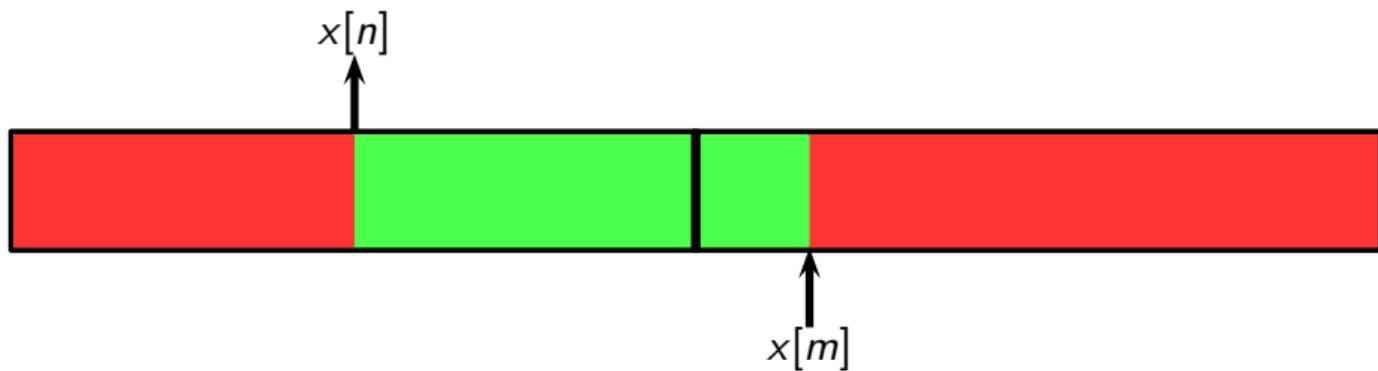
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SOUNDCARD

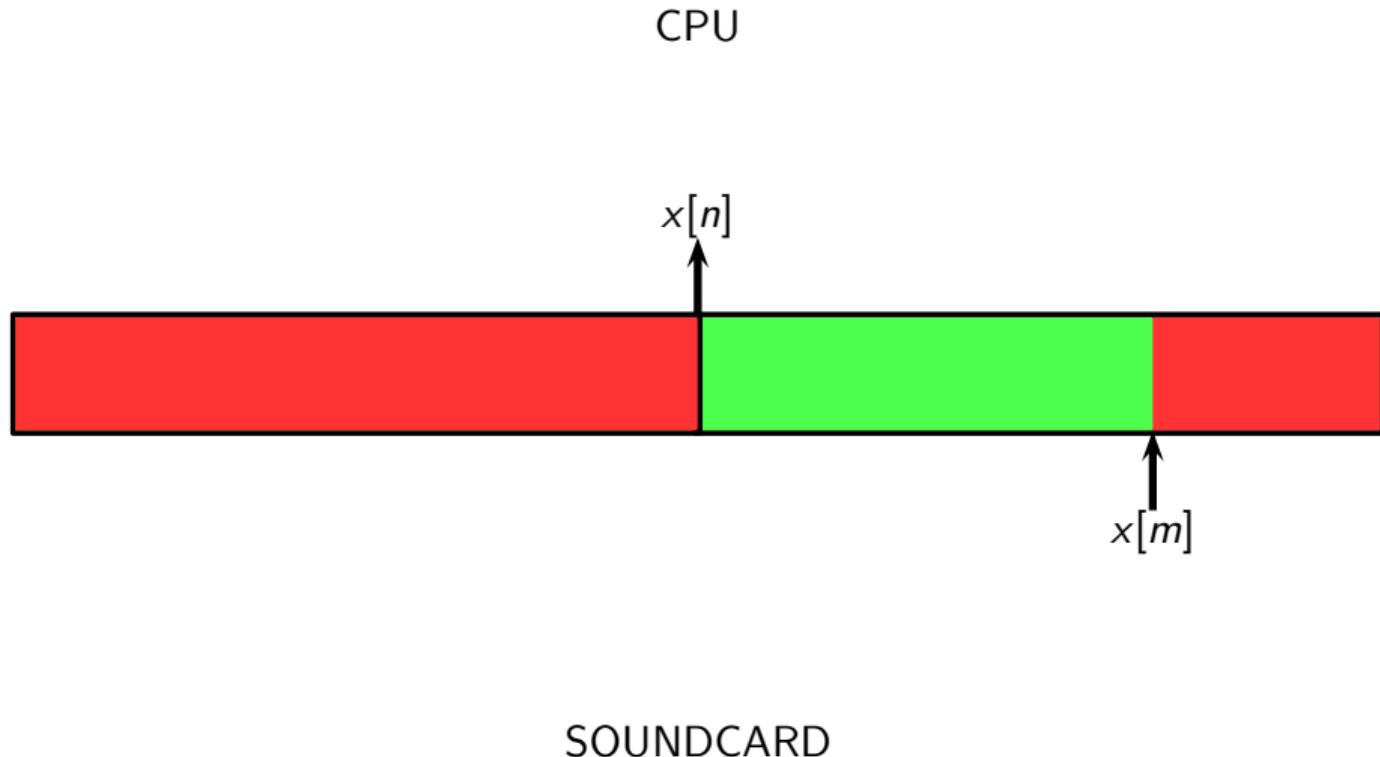
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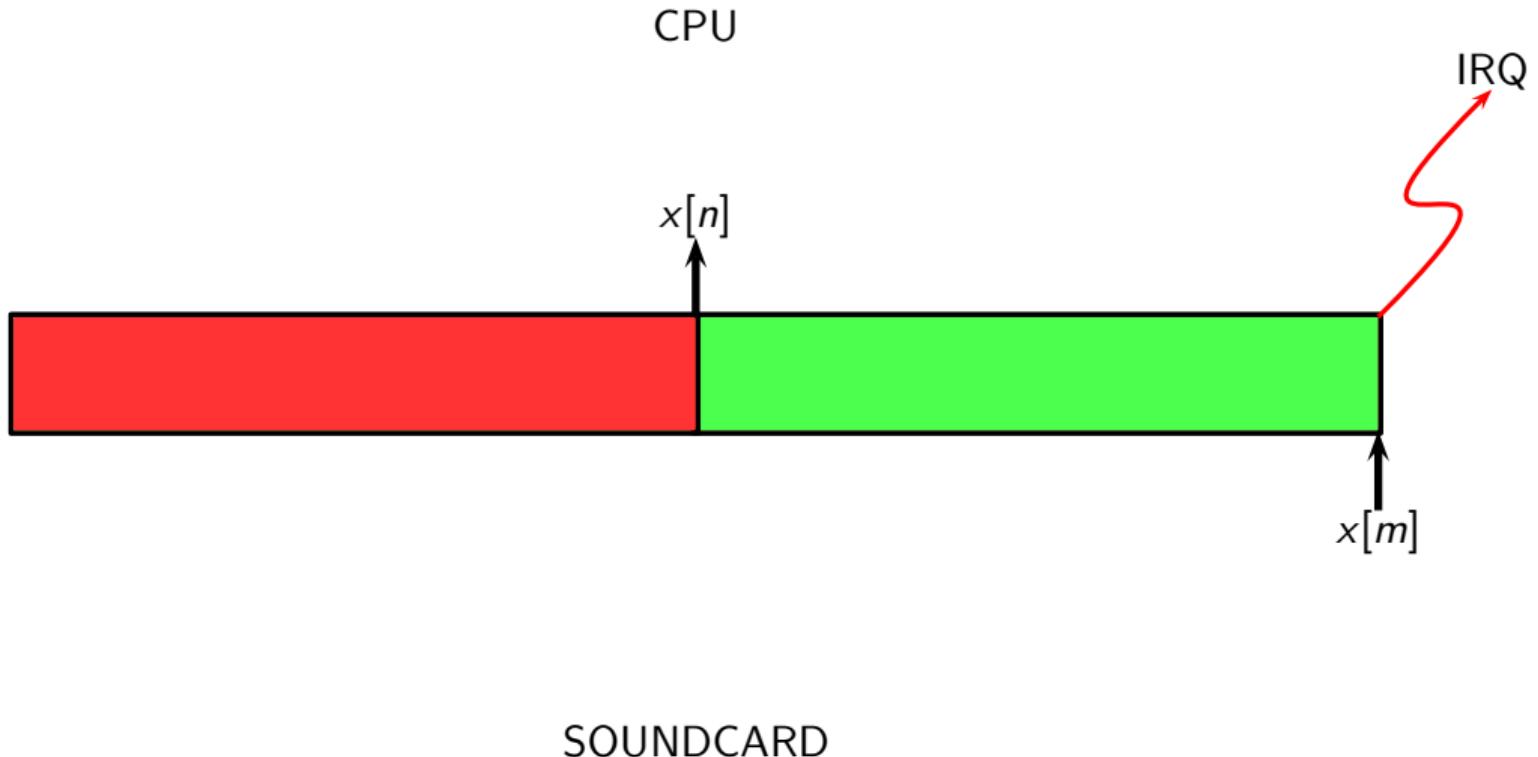


SOUNDCARD

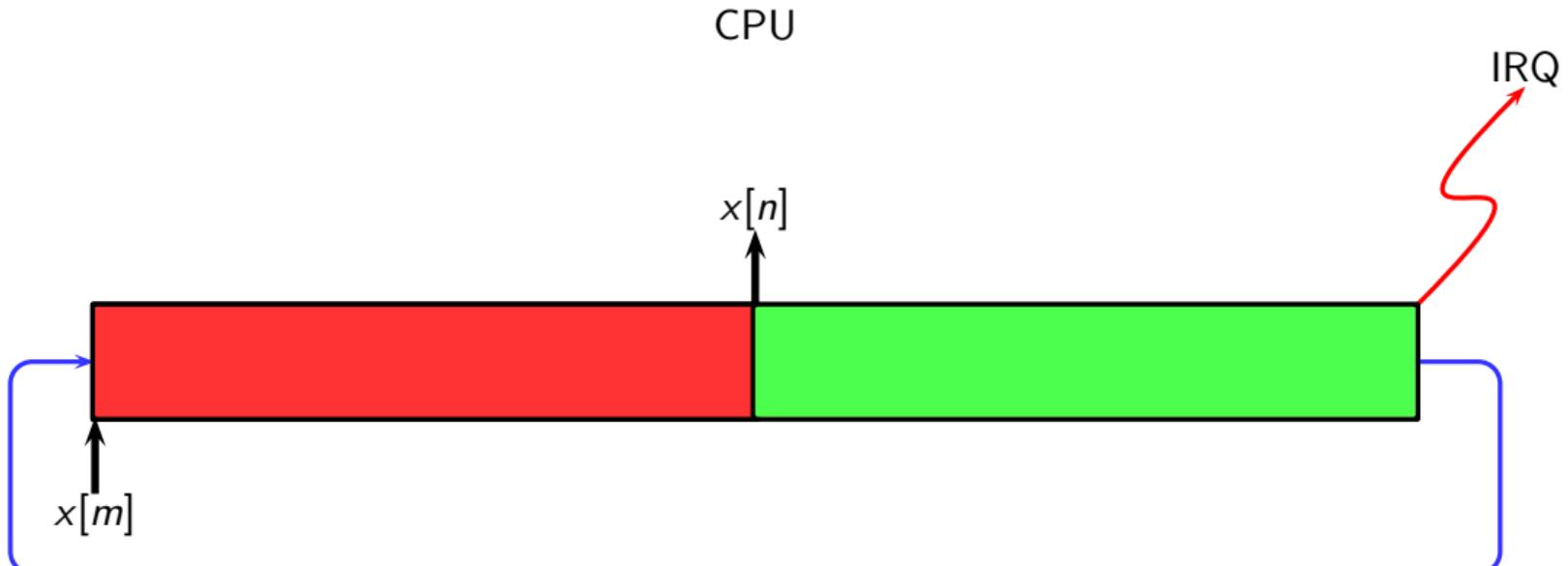
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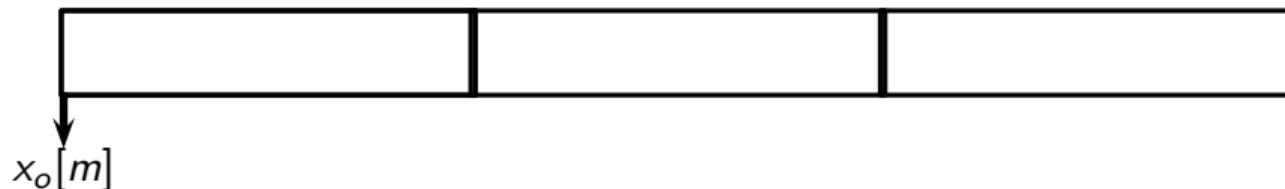
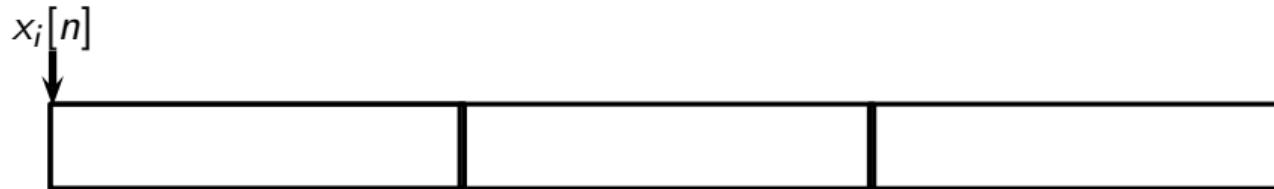
SOUNDCARD

- ▶ multiple input buffers and output buffers
- ▶ equal chunk sizes
- ▶ input IRQ drives processing

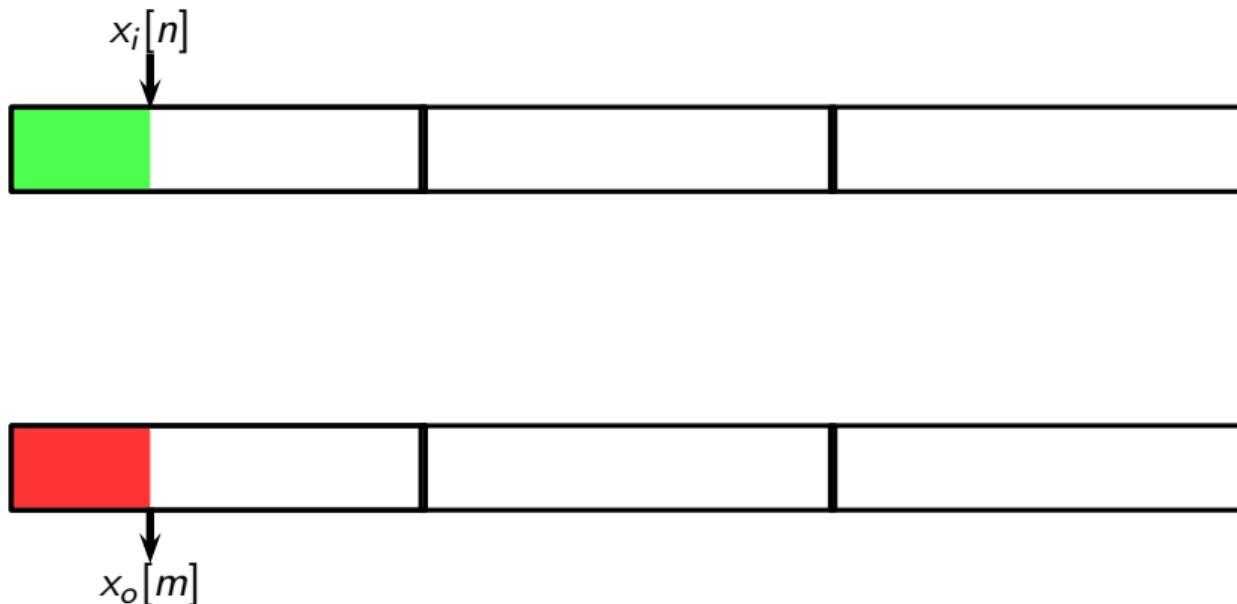
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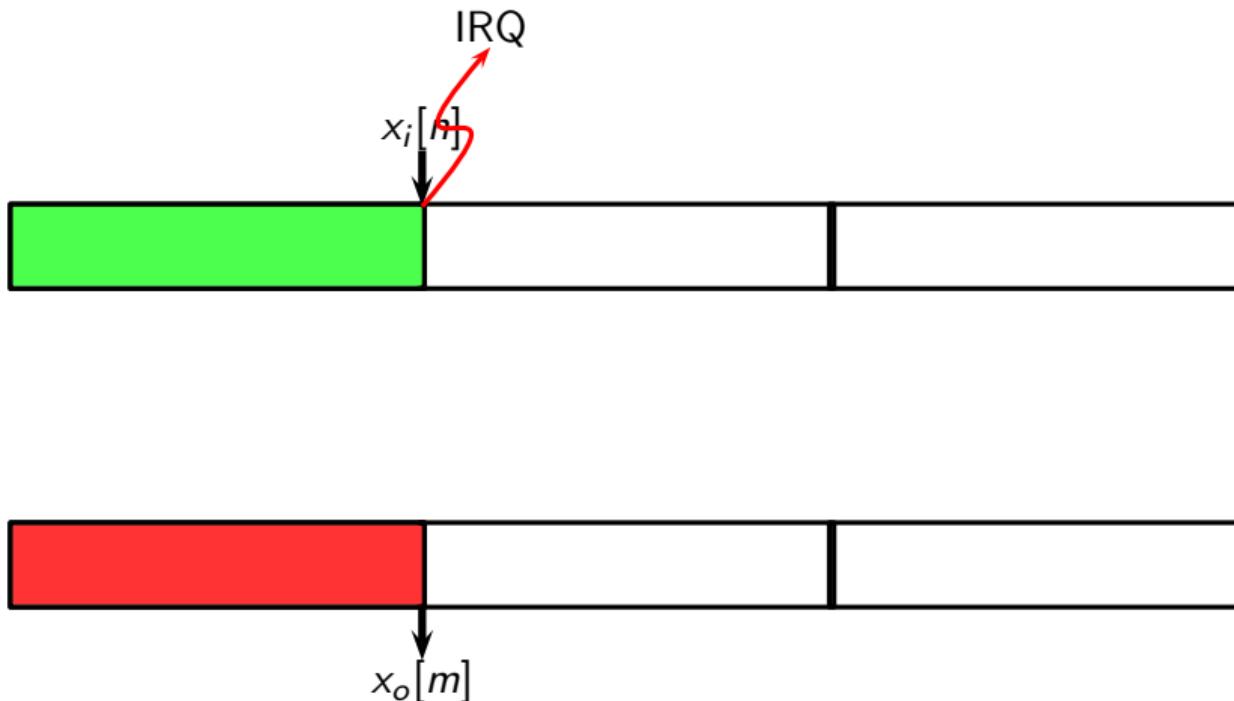
# Real-time I/O processing with multiple buffering



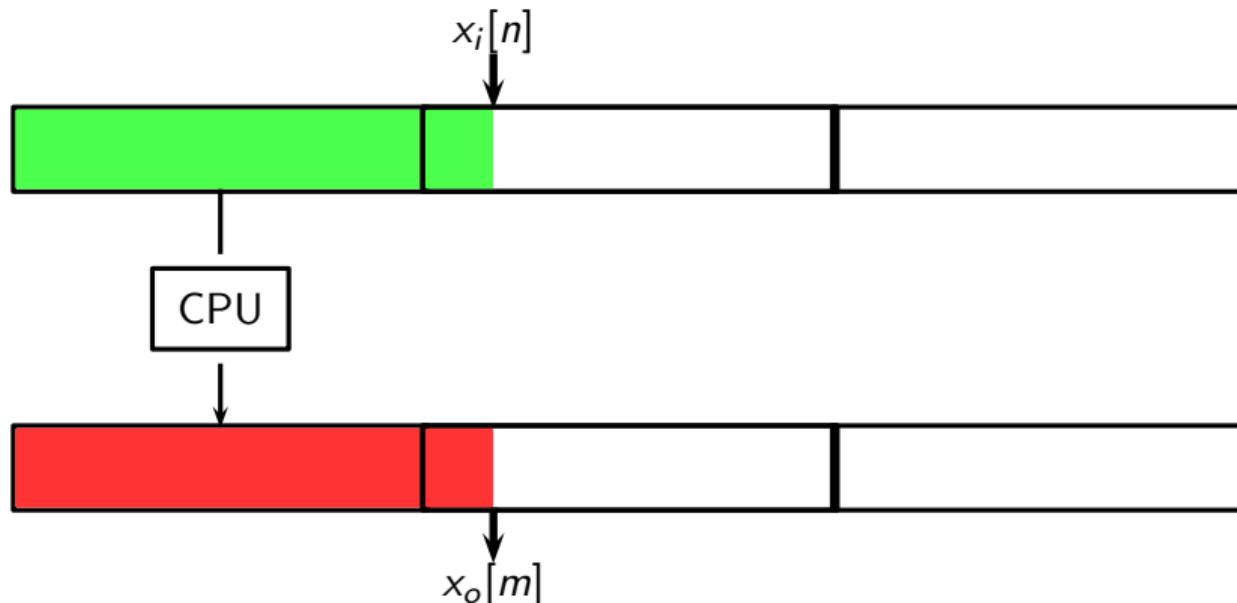
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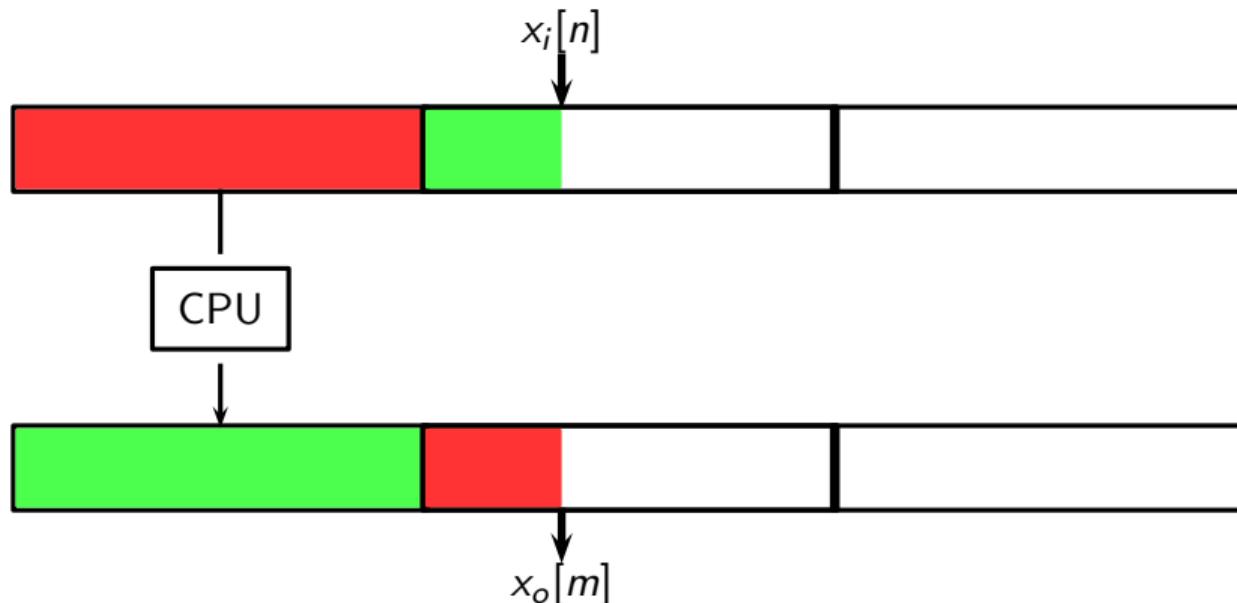
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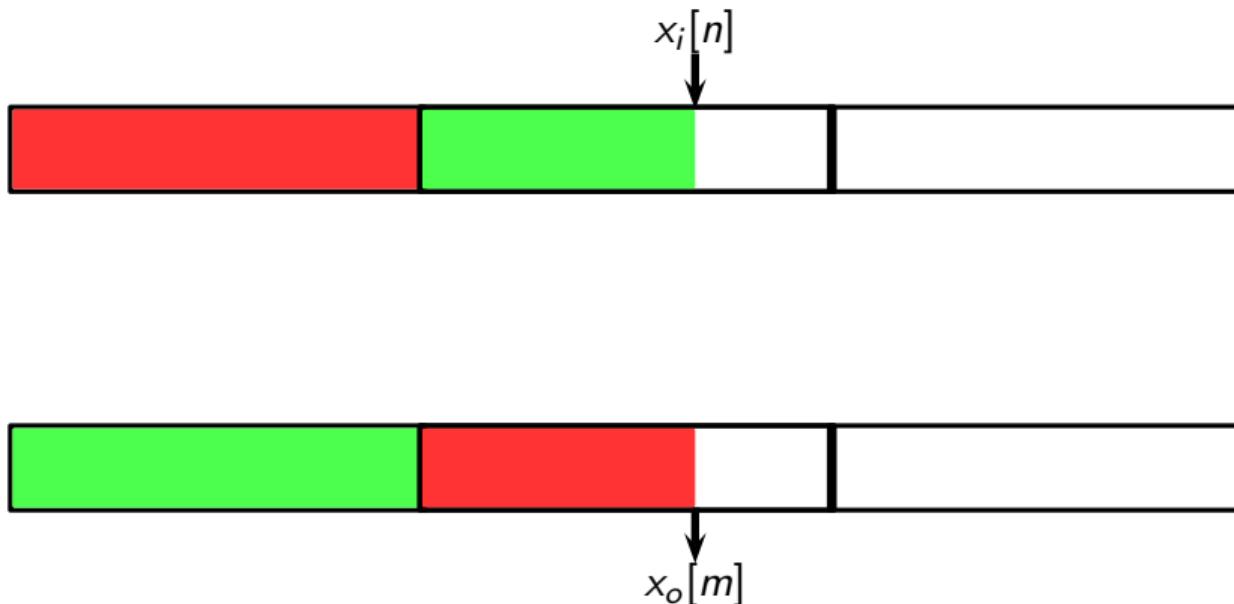


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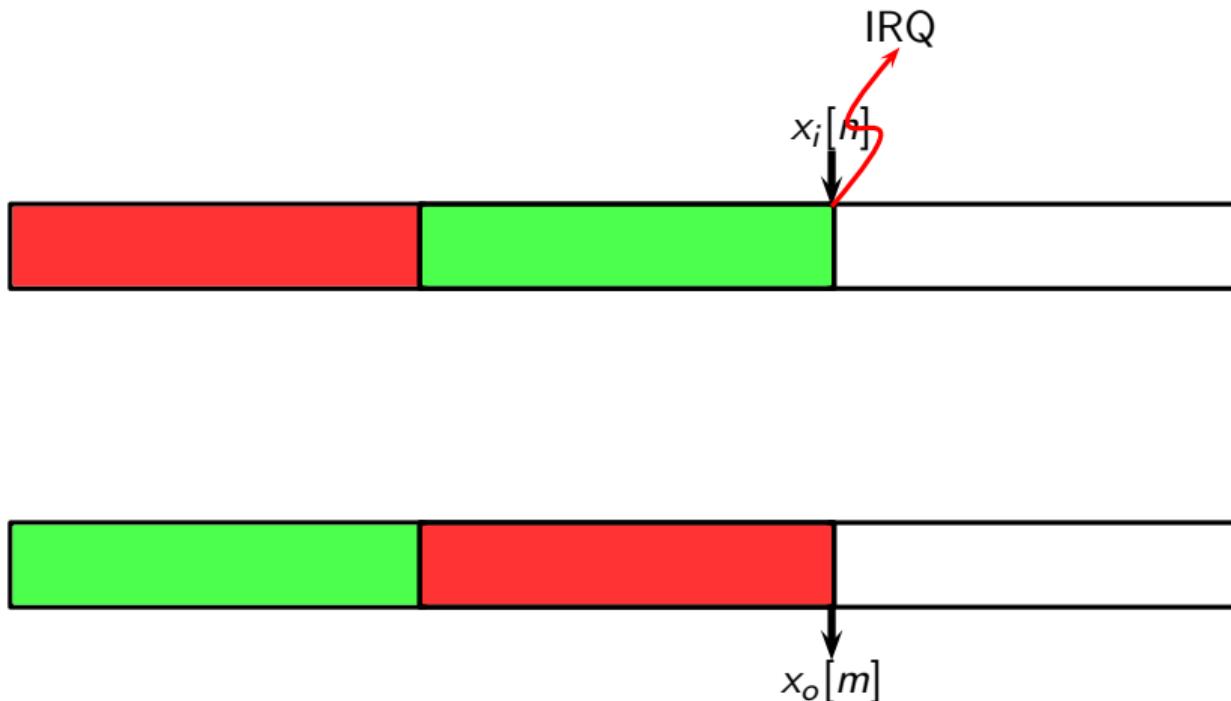


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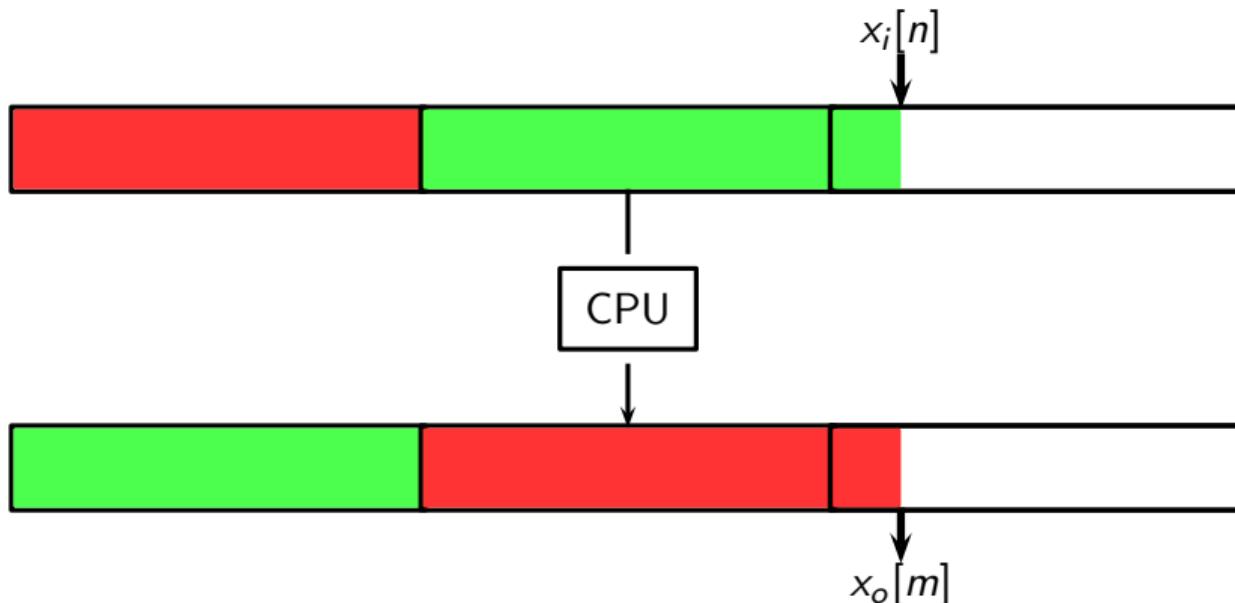




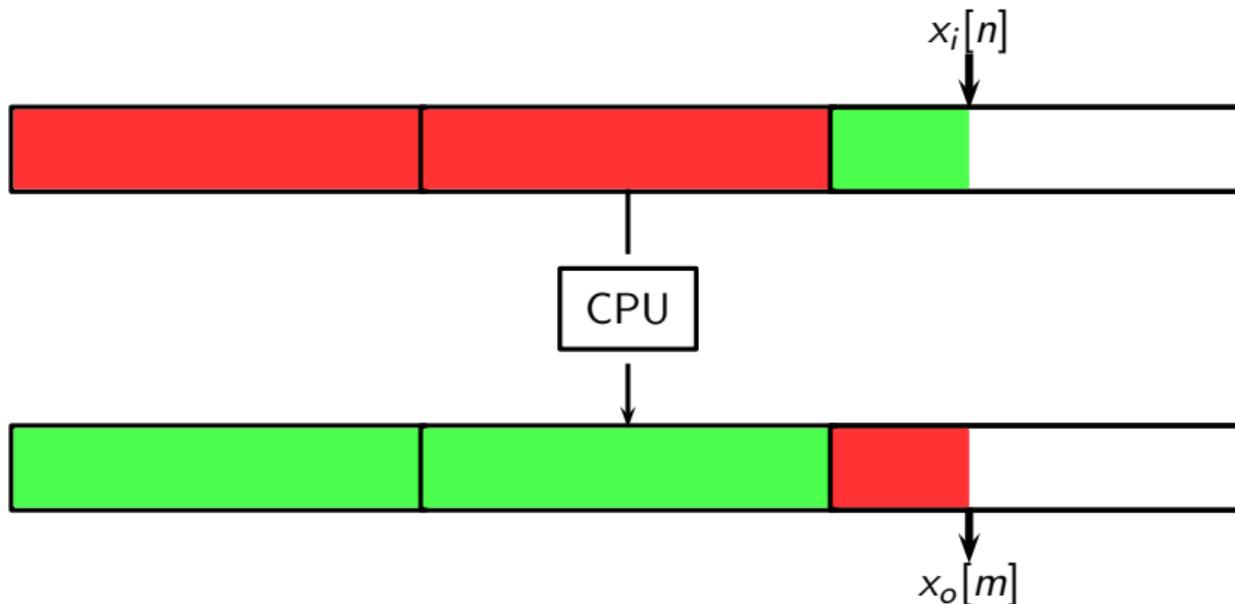
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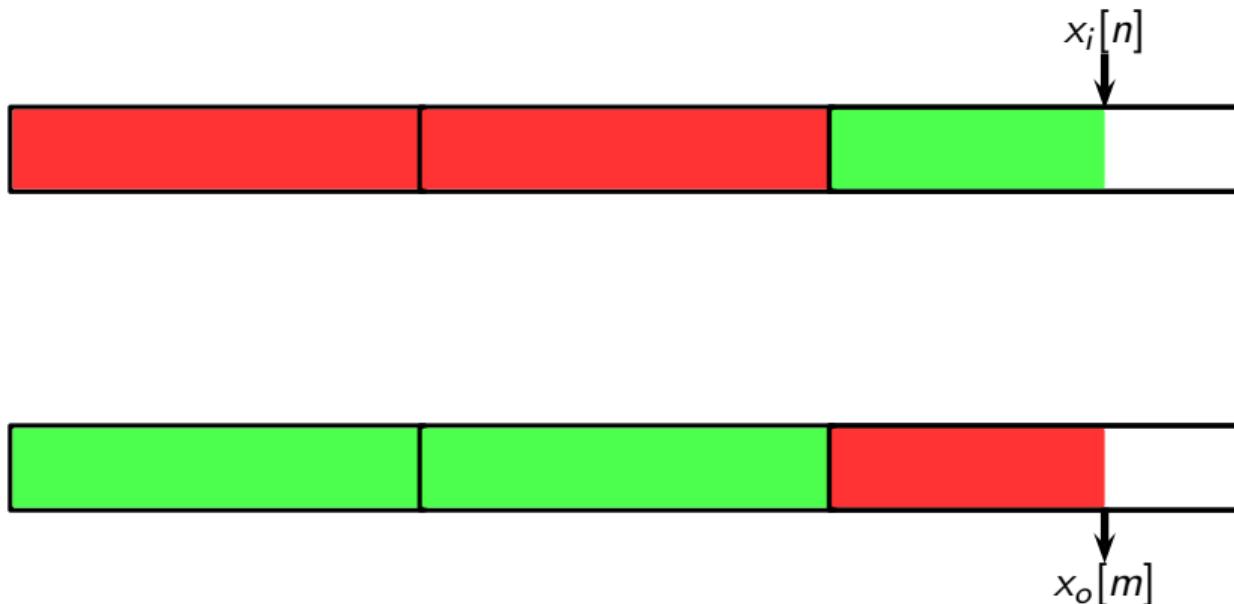
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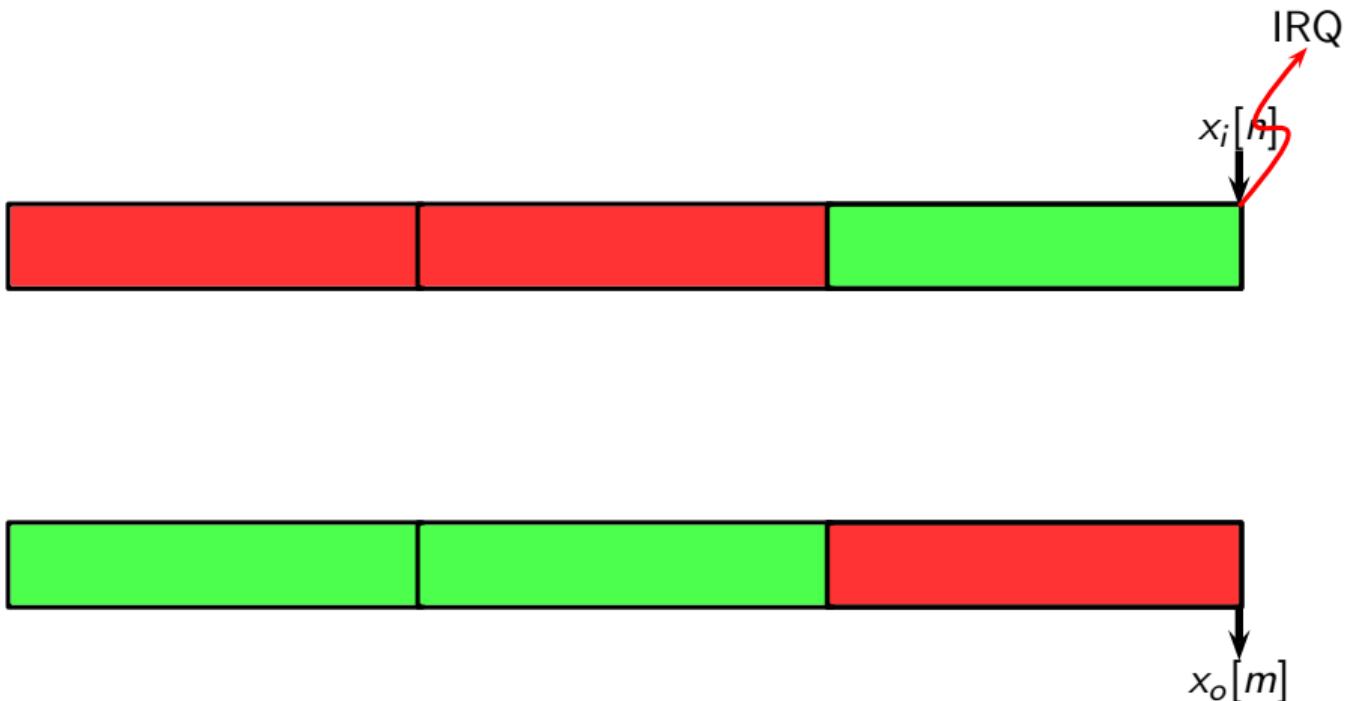
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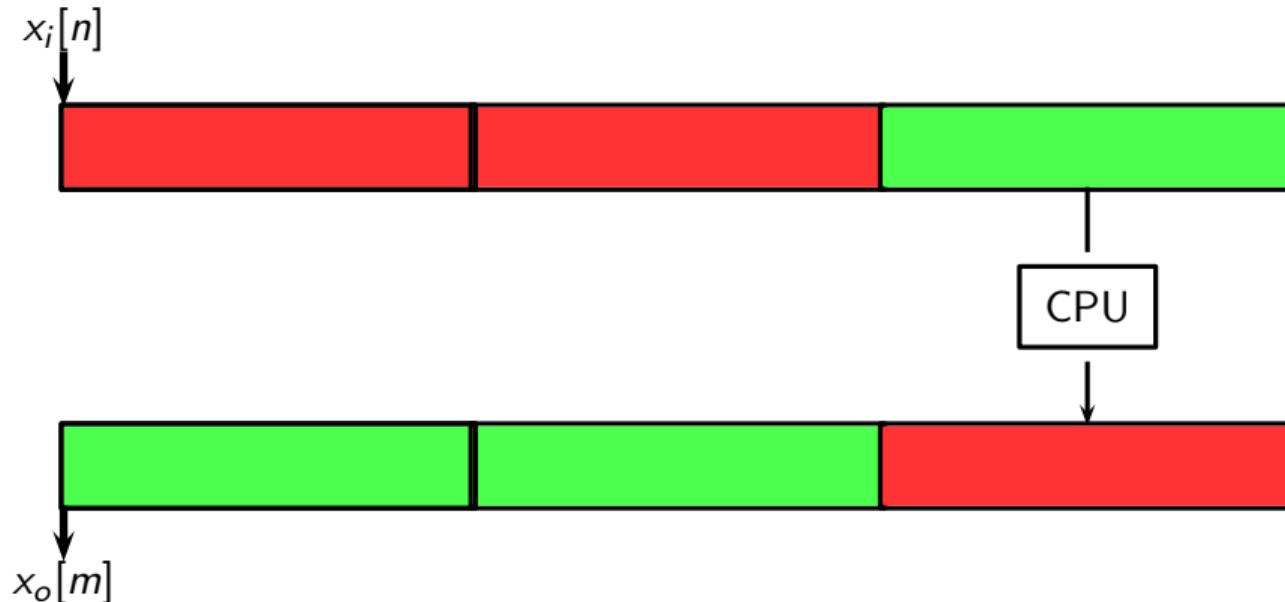
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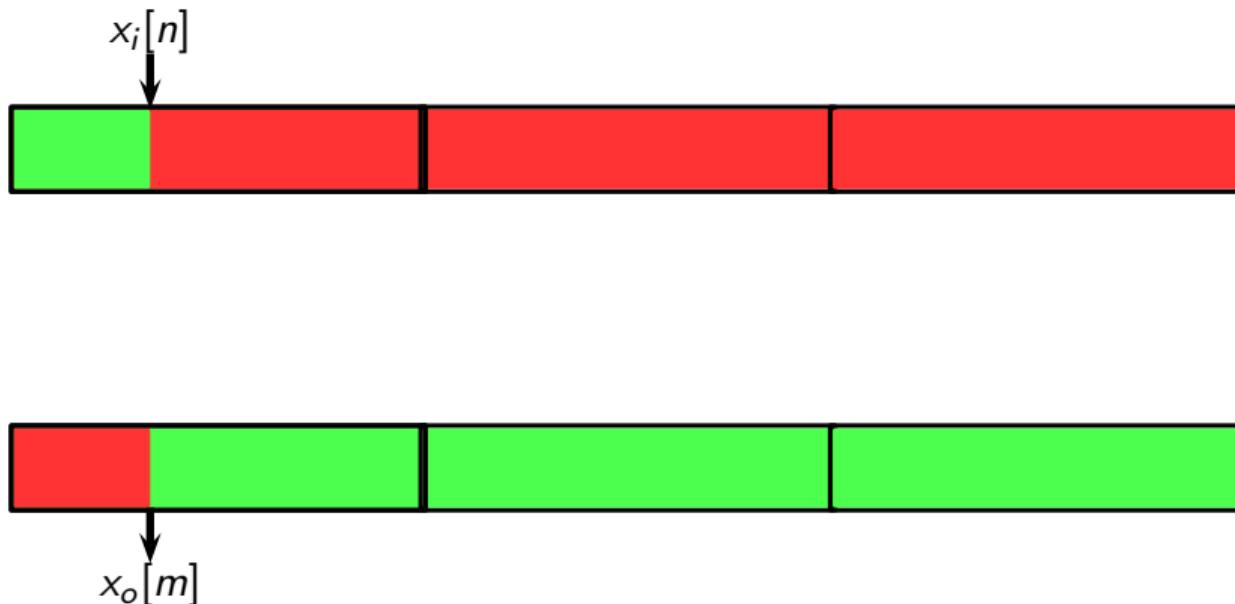
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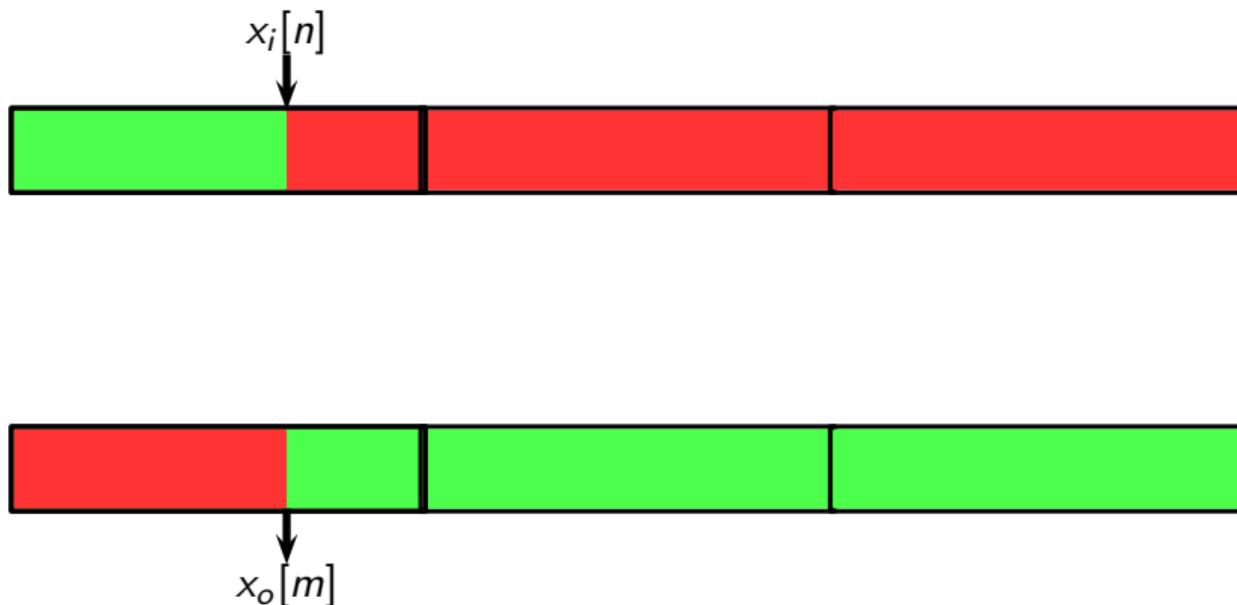
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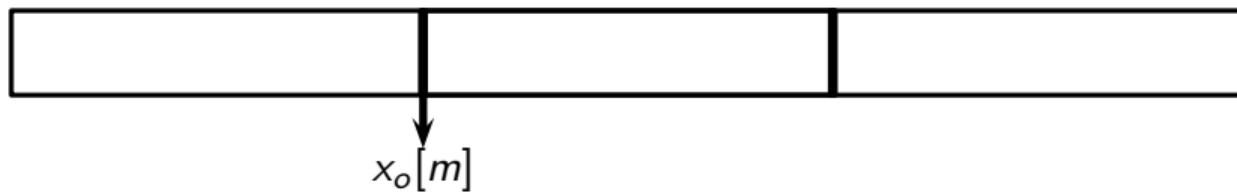
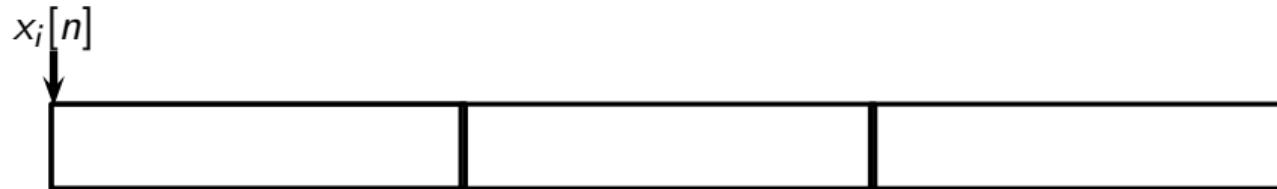
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- ▶ usually start output process first
- ▶ buffers can be collapsed

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► low level:

- study soundcard data sheet (each one is different)
- write code to program soundcard via writes to IO ports
- write an interrupt handler
- write the code to handle the data

► high level:

- choose a good API
- write a callback function to handle the data

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## Implementation

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- write a callback function to handle the data

```
int Callback( const void *input,  
              void *output,  
              unsigned long samples,  
              ...);
```

## Callback example

```
int Callback( const void *input,
              void *output,
              unsigned long samples)
{
    float* pIn  = (float *)input;
    float* pOut = (float *)output;

    for (int n = 0; n < samples; n++)
        *pOut++ = Process(*pIn++);
}
```

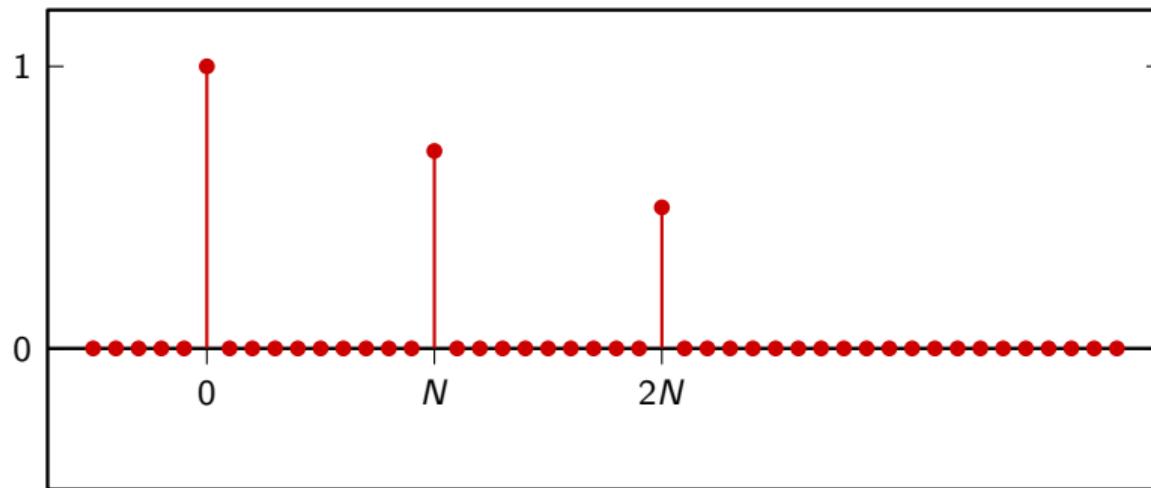
```
// 10 sec @ 24KHz
enum {BUF_LEN = 0x10000};
enum {BUF_MASK = BUF_LEN -1};

float m_pY[BUF_LEN];
float m_pX[BUF_LEN];
int m_Ix;
int m_Iy;
```

```
float Process(float Sample)
{
    m_pX[m_Ix] = Sample;
    float y = Effect();
    m_pY[m_Iy] = y;
    m_Ix = (m_Ix + 1) & BUF_MASK;
    m_Iy = (m_Iy + 1) & BUF_MASK;

    return y;
}
```

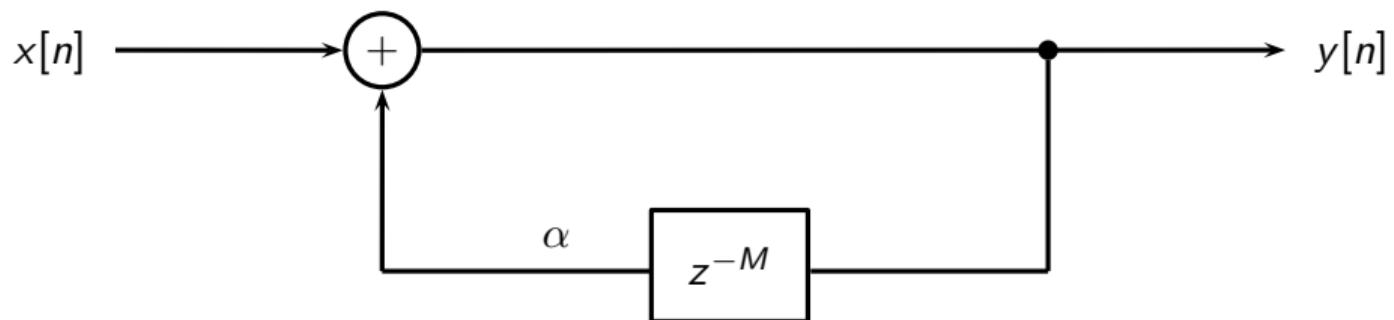
$$y[n] = \frac{a x[n] + b x[n - N] + c x[n - 2N]}{a + b + c}$$



```
float Echo()
{
    static float a = 1;
    static float b = 0.7f;
    static float c = 0.5f;
    static float norm = 1.0f / (a+b+c);
    static int N = (int)(0.3 * m_SR);

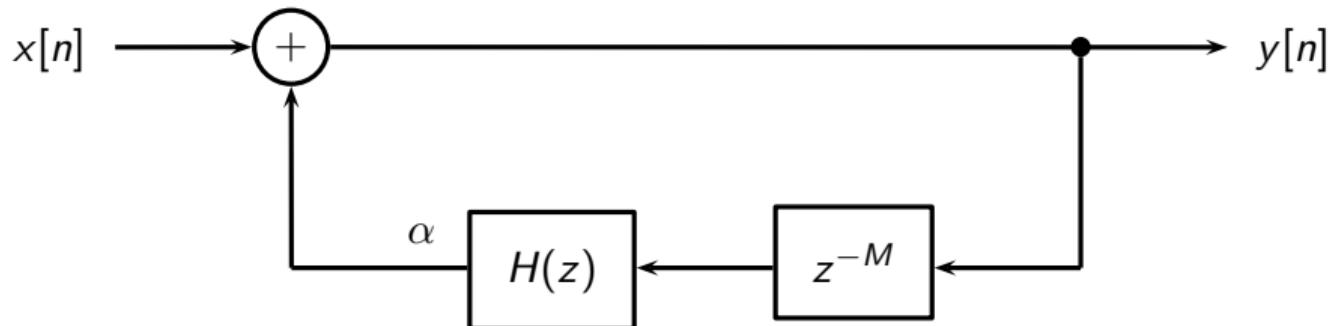
    return norm * (    a * m_pX[m_Ix]
                    + b * m_pX[(m_Ix + BUF_LEN - N)    & BUF_MASK]
                    + c * m_pX[(m_Ix + BUF_LEN - 2*N) & BUF_MASK]);
}
```

remember the KS algorithm? it's a sort of IIR echo



$$y[n] = \alpha y[n - M] + x[n]$$

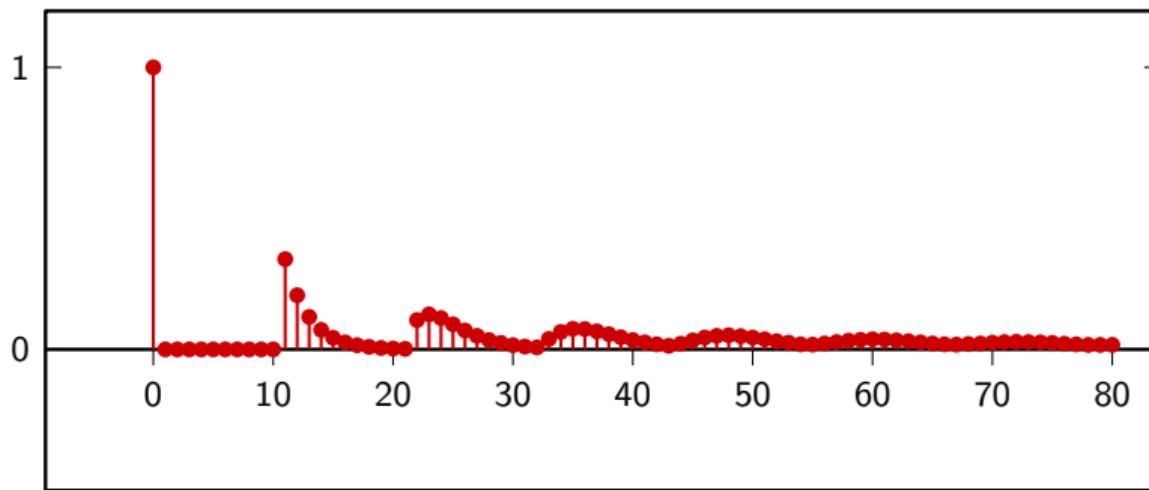
a natural echo has a lowpass characteristic



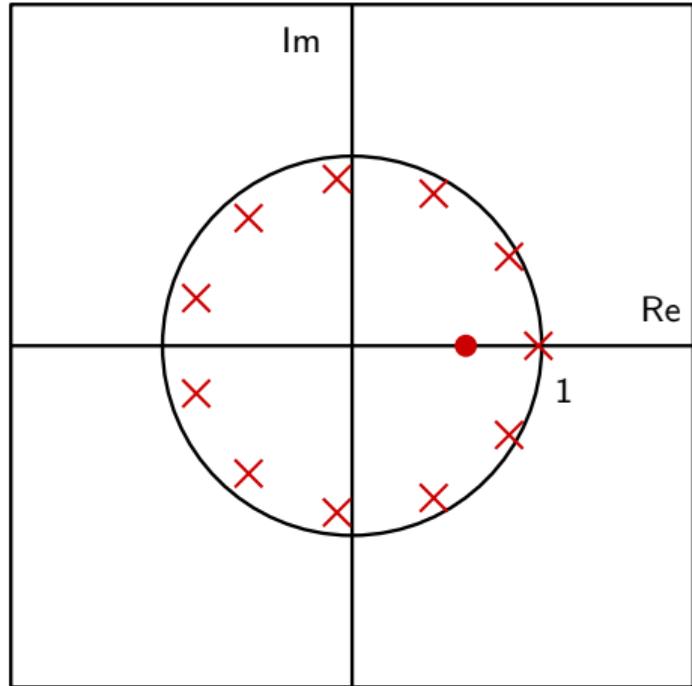
$$y[n] = \alpha(h * y)[n - M] + x[n]$$

Choose for instance  $H(z) = \text{leaky integrator}$ :

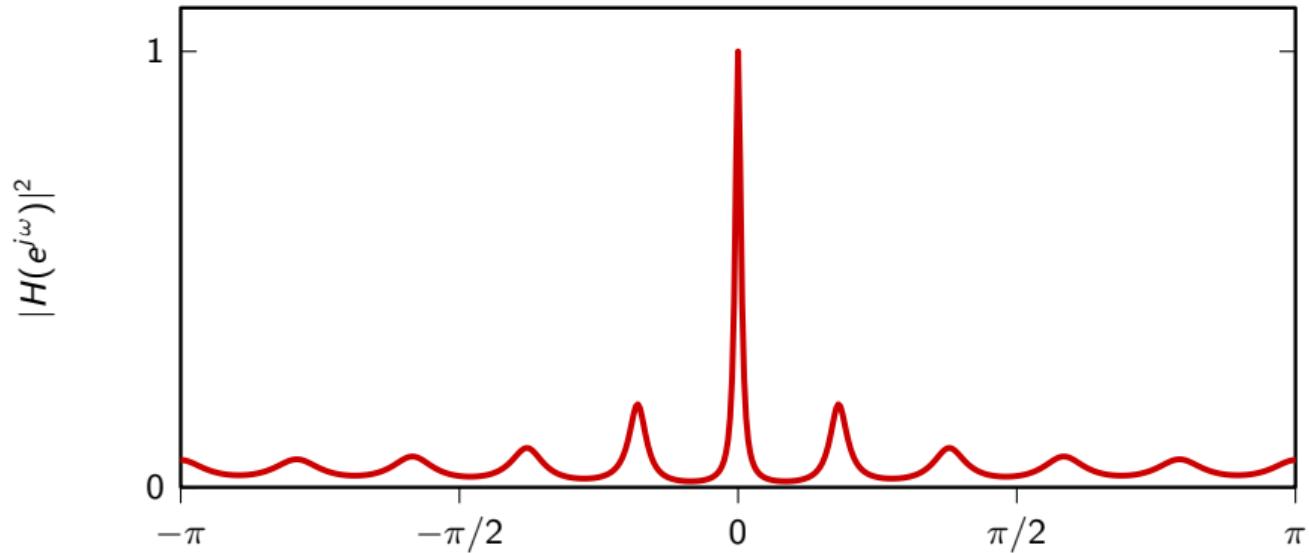
$$y[n] = x[n] - \lambda x[n-1] + \lambda y[n-1] + \alpha(1-\lambda)y[n-N]$$



$$N = 10, \lambda = 0.6, \alpha = 0.8$$



# A better echo



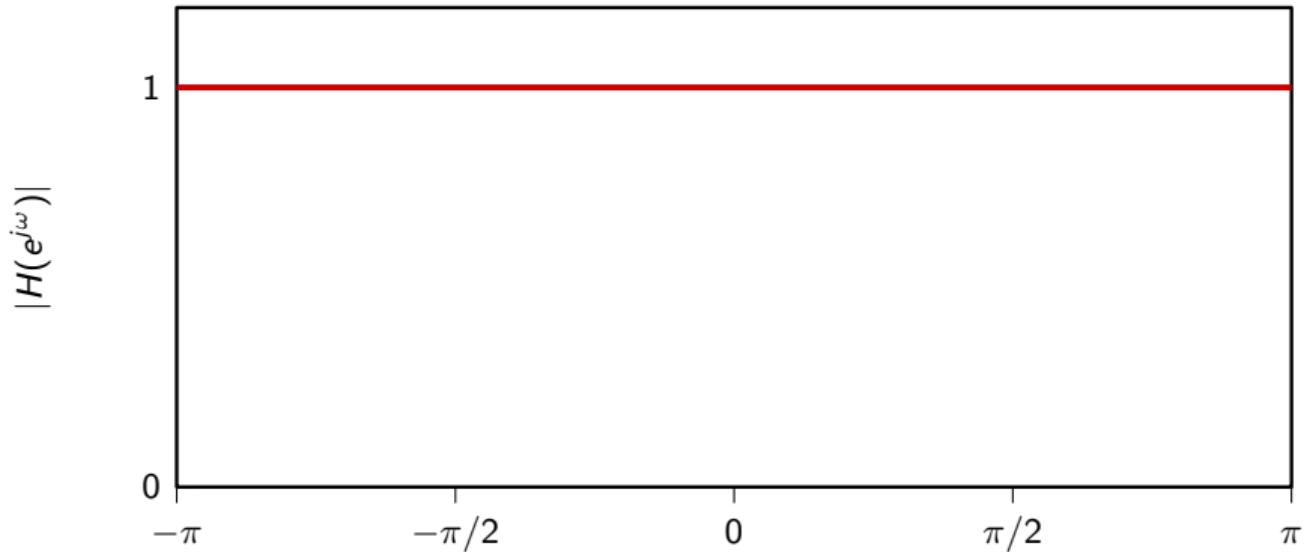
```
float NaturalEcho()
{
    static float a = 0.7;
    static float L = 0.6;
    static float norm = 1.0f / (1+a);
    static int N = (int)(0.3 * m_SR);

    return norm * (
                    m_pX[m_Ix]
                    - L *      m_pX[(m_Ix + BUF_LEN - 1) & BUF_MASK]
                    + L *      m_pY[(m_Iy + BUF_LEN - 1) & BUF_MASK]
                    + a*(1-L) * m_pY[(m_Iy + BUF_LEN - N) & BUF_MASK]);
}
```

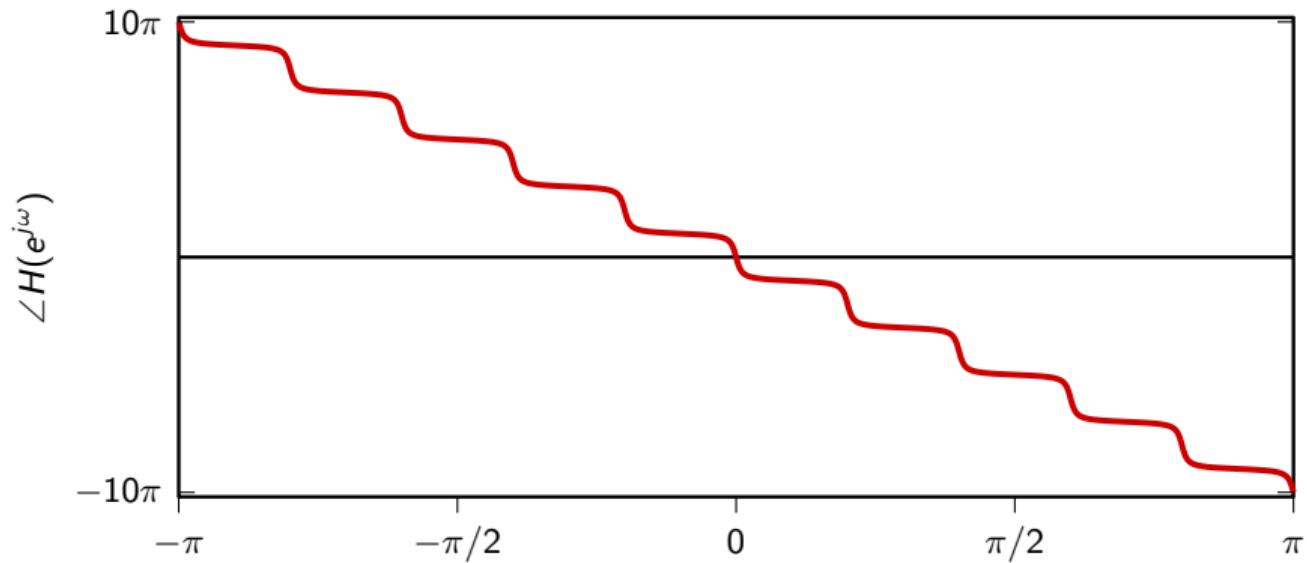
- ▶ reverb is given by the superposition of many many echos with different delays and magnitudes
- ▶ many ways to simulate, always rather costly
- ▶ a cheap alternative is to use an allpass filter

$$H(z) = \frac{-\alpha + z^{-N}}{1 - \alpha z^{-N}}$$

## Reverb, magnitude response



## Reverb, phase response



```
float Reverb()
{
    static float a = 0.8;
    static int N = (int)(0.006 * m_SR);

    return ( - a * m_pX[m_Ix]
            +      m_pX[(m_Ix + BUF_LEN - N) & BUF_MASK]
            + a * m_pY[(m_Iy + BUF_LEN - N) & BUF_MASK]);
}
```

- ▶ distortion: clip the signal

$$y[n] = \text{trunc}(ax[n])/a$$

- ▶ tremolo: sinusoidal amplitude modulation

$$y[n] = (1 + \cos(\omega_0 n)/G)x[n]$$

- ▶ flanger: sinusoidal delay

$$y[n] = x[n] + x[n - \lfloor d(1 + \cos(\omega_0 n)) \rfloor]$$

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$$y[n] = x[n] + x[n - \lfloor d(1 + \cos(\omega_0 n)) \rfloor]$$

```
float Fuzz()
{
    static float limit = 0.005;
    static float G = 5;

    float y = m_pX[m_Ix];
    if (y > limit)
        y = limit;
    if (y < -limit)
        y = -limit;
    return G*y;
}
```

```
float Tremolo()
{
    static double phi = 5 * 2*PI / m_SR; // 5Hz LFO
    static double omega = 0;

    omega = omega + phi;
    return (1 + cos(omega)/4) * m_pX[m_Ix];
}
```

```
float Flanger()
{
    static int N = (int)(0.002 * m_SR);    // max delay
    static double phi = 0.1 * 2*PI / m_SR;  // 0.1Hz LFO
    static double omega = 0;

    int d = (int)(N * (1 + cos(omega)));
    omega = omega + phi;
    return 0.5f + (m_pX[m_Ix] + m_pX[(m_Ix + BUF_LEN - d) & BUF_MASK]);
}
```

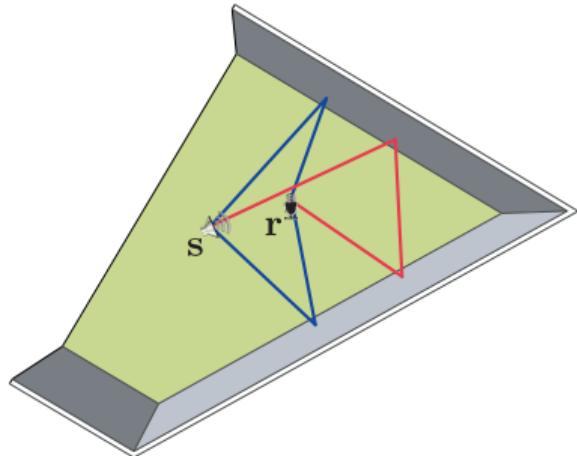
**END OF MODULE 5.11**

# Digital Signal Processing

Module 5.12: Dereverberation and Echo Cancelation

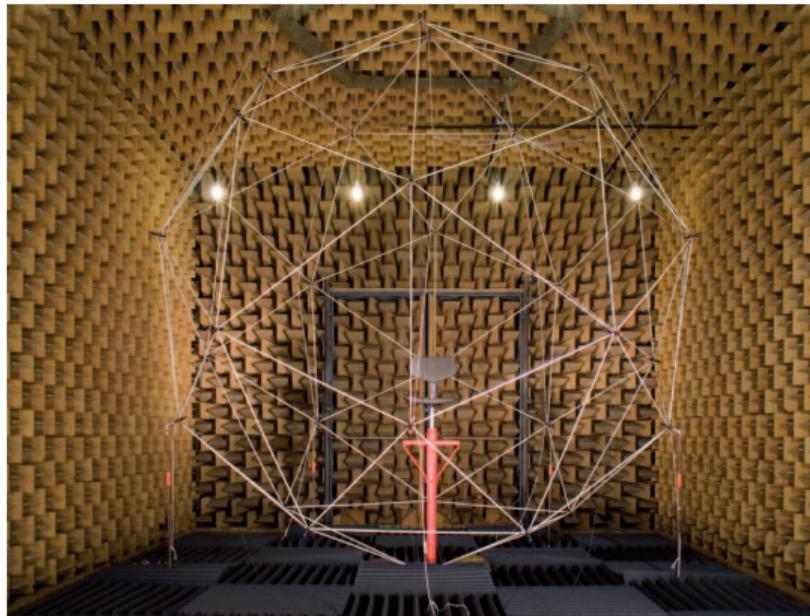
Guest lecture by  
Ivan Dokmanić

- ▶ Free space propagation, sound pressure  $\sim \frac{1}{\text{distance}}$
- ▶ Every reflection attenuates the sound by  $\alpha \in [0, 1]$



- ▶ Room is a linear filter, it acts by convolution
- ▶  $y_{\text{listener}}[n] = x_{\text{source}}[n] * h[n; \text{positions}]$
- ▶ We will suppress the dependence on source and listener positions for simplicity

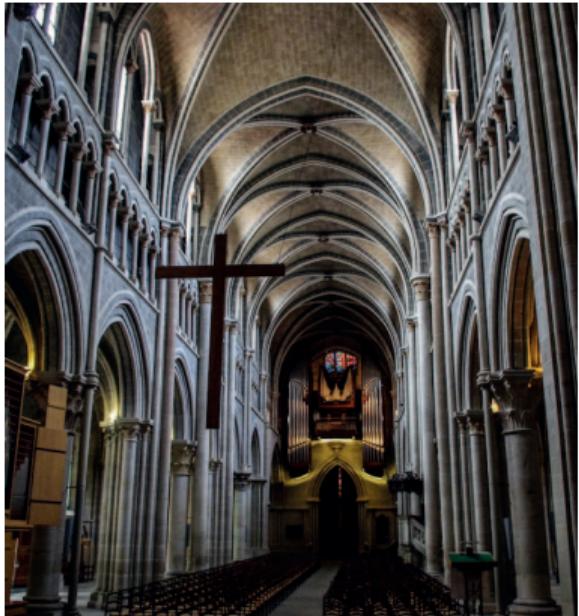
# Anechoic Chamber



► Sound in this room

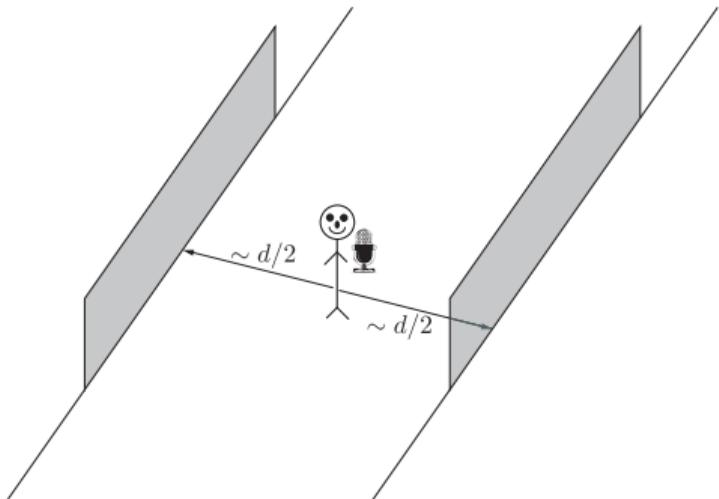


- ▶ Impulse response
- ▶ Sound in this room



- ▶ Impulse response
- ▶ Sound in this room

- ▶ Consider (almost) colocated speaker and microphone between two walls at distance  $d$



- ▶ The room impulse response is

$$h[n] = \sum_{k=0}^{\infty} \frac{\alpha^k}{B(k + \epsilon)T} \delta[n - kN]$$

where

$$N = \text{round}(T \times F_s) = \text{round}\left(\frac{d}{c}F_s\right)$$

and  $c \approx 340\text{m/s}$  is the speed of sound

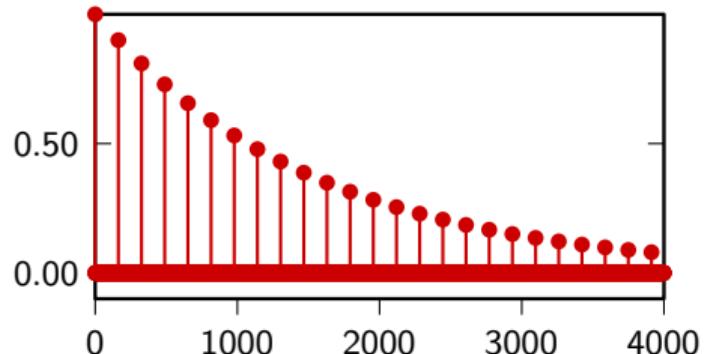
- ▶  $\frac{1}{k}$  term is difficult to handle in the  $z$ -domain, so we'll simplify
- ▶ Assume first that the dominant attenuation is due to reflections

$$h_a[n] = \sum_{k=0}^{\infty} \alpha^k \delta[n - kN]$$

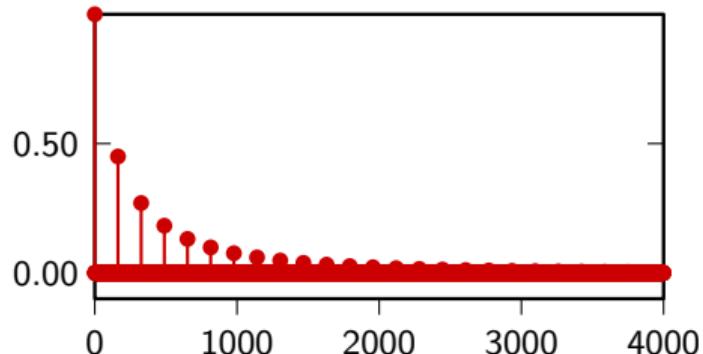
- ▶ This has a simple  $z$ -transform!

# How do These Sound and Look?

- ▶ Simplified and realistic impulse responses for  $F_s = 44100$  Hz,  $d = 7$  m,  $\alpha = 0.9$
- ▶ Original sound



Simplified room impulse response



Realistic room impulse response

- ▶ Reverberated sound is  $y[n] = (h_a * x)[n]$
- ▶ We want to design a filter  $h_i[n]$  such that  $(h_i * y)[n] = x[n]$ , that is

$$x = h_i * y = h_i * (h_a * x) = (h_i * h_a) * x.$$

so that  $(h_i * h_a)[n] = \delta[n]$ , because  $(\delta * x)[n] = x[n]$

- ▶ In the  $z$ -domain this becomes  $H_i(z)H_a(z) = 1$  and we get that  $H_i(z) = \frac{1}{H_a(z)}$

- ▶ Let's get rid of the room! The transfer function is computed as

$$\begin{aligned} H_a(z) &= \sum_{n=0}^{\infty} h_a[n] z^{-n} \\ &= \sum_{n=0}^{\infty} \left( \sum_{k=0}^{\infty} \alpha^k \delta[n - kN] \right) z^{-n} \\ &= \sum_{n=0}^{\infty} \alpha^n z^{-nN} = \frac{1}{1 - \alpha z^{-N}} \end{aligned}$$

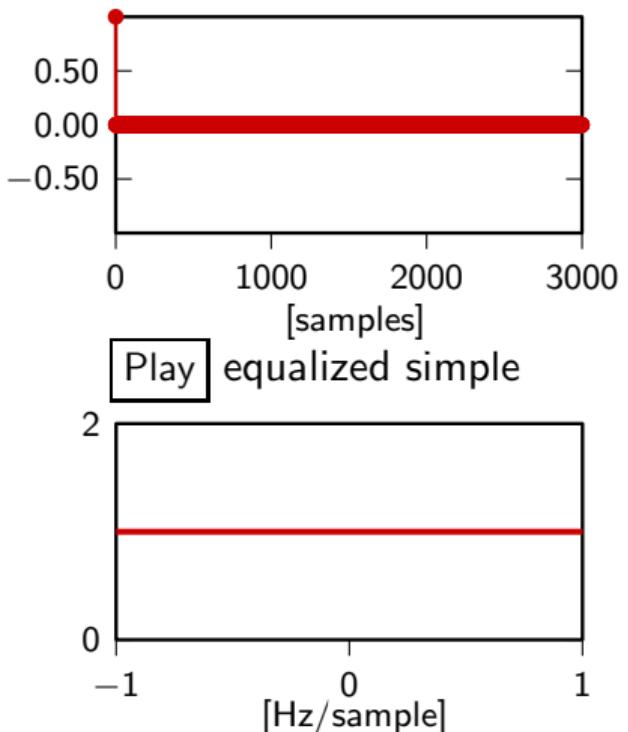
- ▶ The inverse is

$$H_i(z) = H_a(z)^{-1} = 1 - \alpha z^{-N} \leftrightarrow h_i[n] = \delta[n] - \alpha \delta[n - N]$$

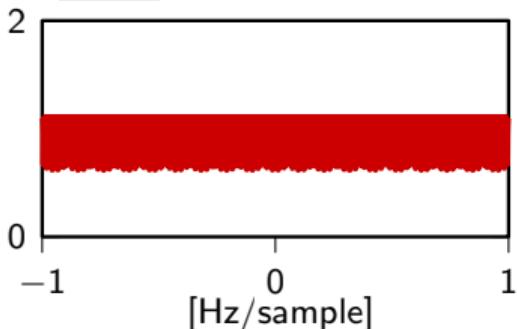
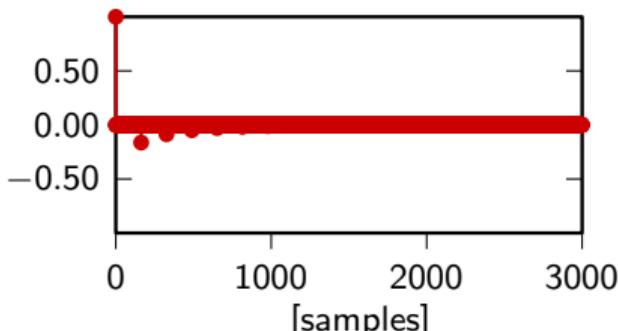
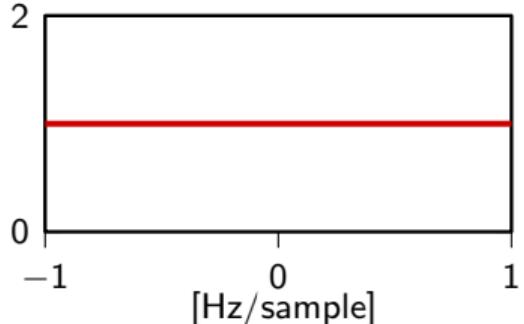
- ▶ But this is a simple FIR filter!
- ▶ For motivated students: observation that exponentials are canceled by simple FIR filters is essential for the so-called Finite Rate of Innovation sampling

# Equalized Room, Sounds and Plots

Equalized room  
impulse responses

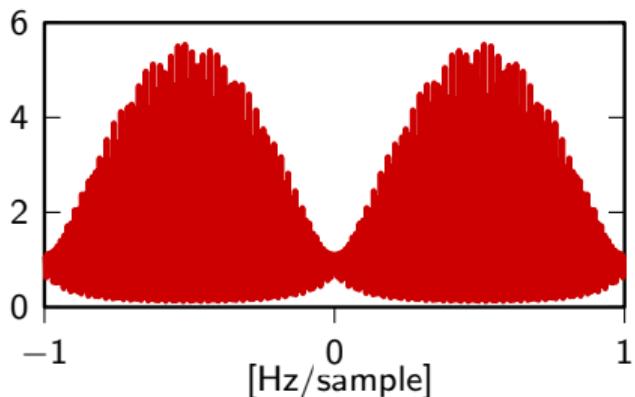
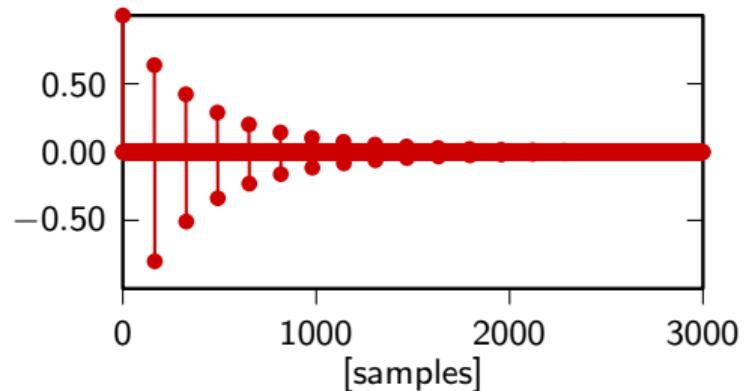


Magnitude of  
equalized room  
transfer functions



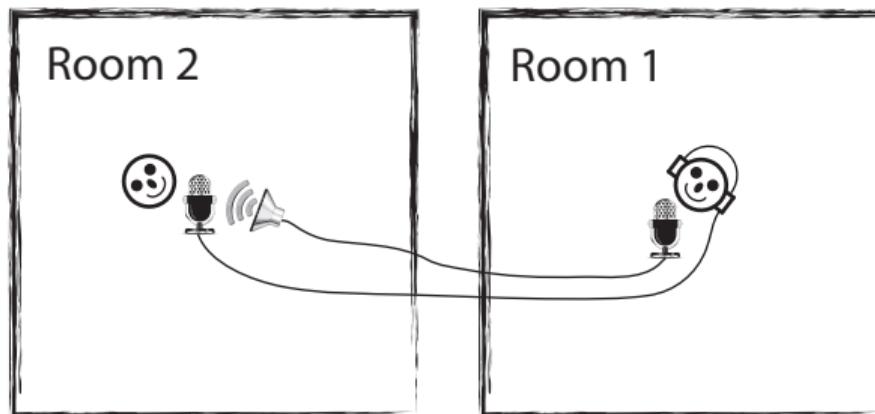
# Equalized Room, Model Mismatch

- ▶ Let's hear a different kind of model mismatch: 1% error in room size
- ▶  original sound
- ▶  “equalized”, 1% error in  $d$

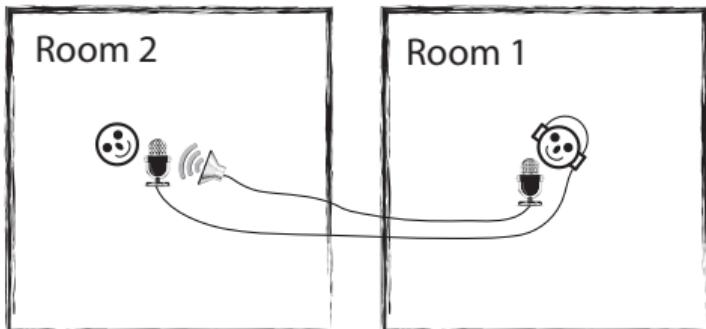


# A Teleconference Call

- ▶ Consider now two people talking over a phone, as shown in the figure

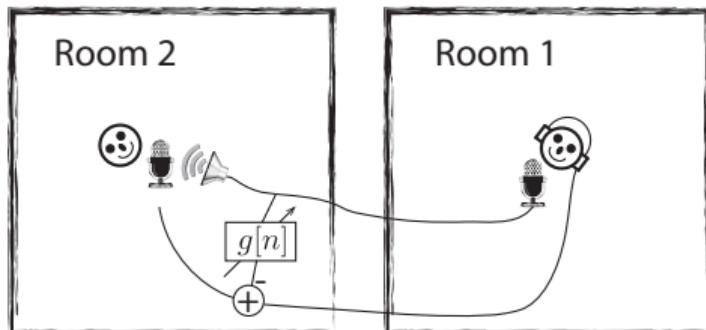


# A Teleconference Call



- ▶ Person in room one talks into the microphone
- ▶ Signal gets transmitted into another room with a delay and reproduced over a loudspeaker
- ▶ This gets convolved with the room 2, transmitted back, and reproduced over headphones,
- ▶ Room 1 and Room 2 equal,  $d = 7$  m,  $\alpha = 0.8$ , delay = 0.3 s

- ▶ Must estimate and update the filter  $g[n]$  that models Room 2, the loudspeaker and the microphone (not trivial!)



- ▶ Only subtracting the incoming sound (that is, setting  $g[n] = \delta[n]$ ) is not enough,
- ▶ With ideal knowledge of  $g[n]$ , we get the expected result, only Room 1 remains

# Is it Really This Simple?



- ▶ What if you are not halfway between the walls?

**END OF MODULE 5.12**

END OF MODULE 5