Digital Signal Processing

Module 2: Discrete-time signals

Video Introduction

Module Overview:

- Module 2.1: discrete-time signals and operators
- Module 2.2: the discrete-time complex exponential
- Module 2.3: the Karplus-Strong algorithm

Digital Signal Processing

Module 2.1: Discrete-time signals

Overview:

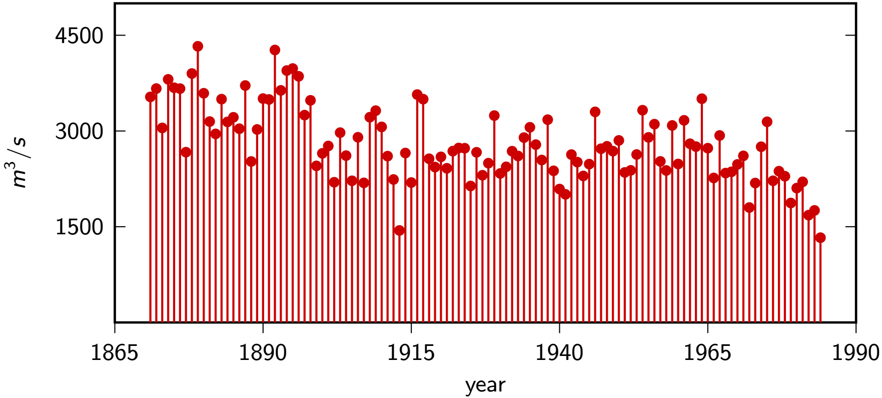
- discrete-time signals
- signal classes
- elementary operators
- shifts
- energy and power

Discrete-time signals have a long tradition...

Meteorology (limnology): the floods of the Nile

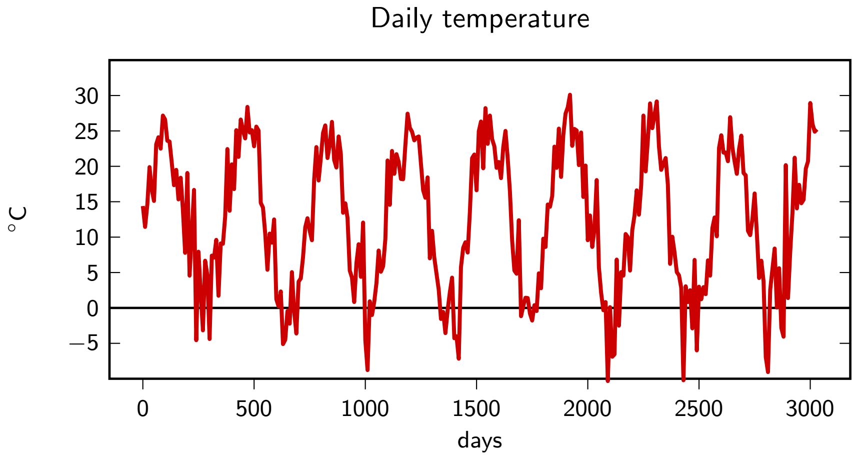
Representations of flood data: circa 2500 BC

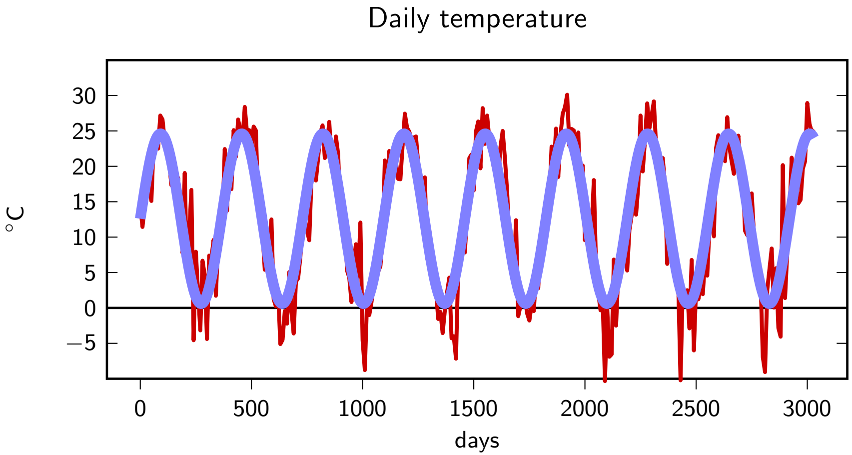
Discrete-time signals have a long tradition...

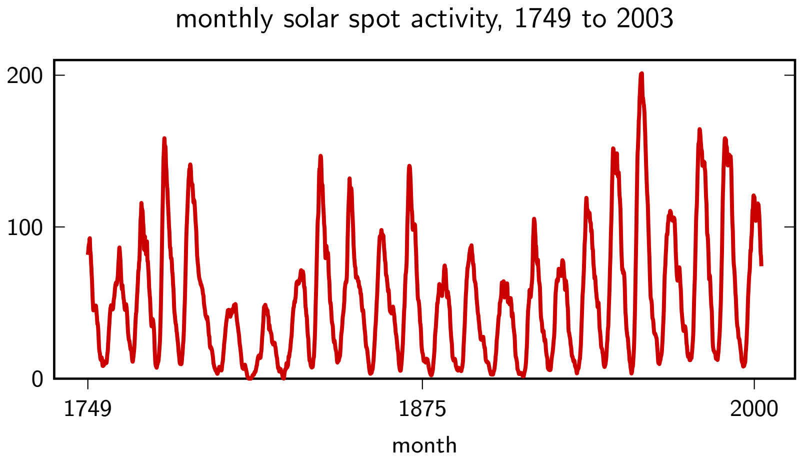


Representations of flood data: circa AD 2000

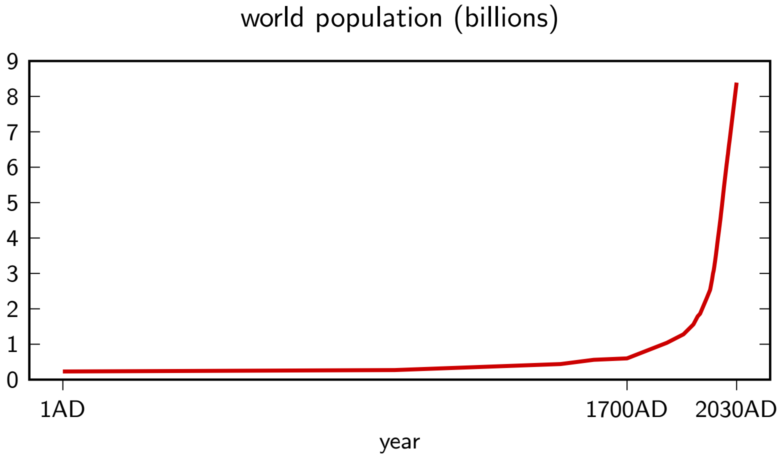
Probably your first scientific experiment...



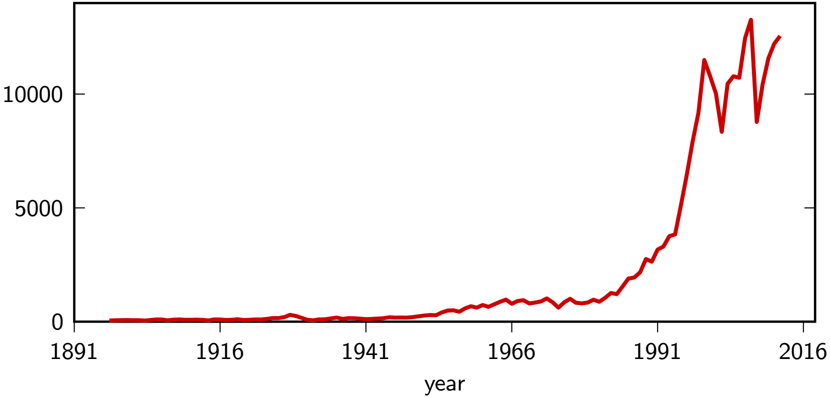

Daily temperature
°C
30
25
20
15
10
5
0
−5
0
500
1000
1500
2000
2500
3000
days


monthly solar spot activity, 1749 to 2003
200
100
0
1749
1875
2000
month

History and sociology



a purely man-made signal: the Dow Jones industrial average



More formally...

Discrete-time signal: a sequence of **complex** numbers

- one dimension (for now)
- notation: $x[n]$
- two-sided sequences: $x : \mathbb{Z} \to \mathbb{C}$
- n is dimension-less "time"
- analysis: periodic measurement
- synthesis: stream of generated samples

More formally...

Discrete-time signal: a sequence of **complex** numbers

- one dimension (for now)
- notation: $x[n]$
- two-sided sequences: $x : \mathbb{Z} \to \mathbb{C}$
- n is dimension-less "time"
- analysis: periodic measurement
- synthesis: stream of generated samples

More formally...

Discrete-time signal: a sequence of **complex** numbers

- one dimension (for now)
- notation: $x[n]$
- two-sided sequences: $x : \mathbb{Z} \to \mathbb{C}$
- n is dimension-less "time"
- analysis: periodic measurement
- synthesis: stream of generated samples

More formally...

Discrete-time signal: a sequence of **complex** numbers

- one dimension (for now)
- notation: $x[n]$
- two-sided sequences: $x : \mathbb{Z} \to \mathbb{C}$
- n is dimension-less "time"
- analysis: periodic measurement
- synthesis: stream of generated samples

More formally...

Discrete-time signal: a sequence of **complex** numbers

- one dimension (for now)
- notation: $x[n]$
- two-sided sequences: $x : \mathbb{Z} \to \mathbb{C}$
- n is dimension-less "time"
- analysis: periodic measurement
- synthesis: stream of generated samples

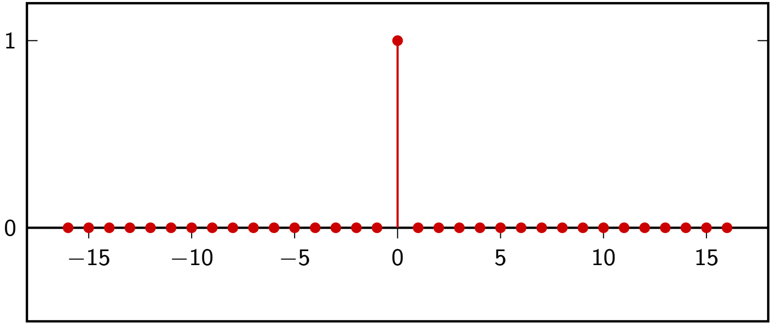
More formally...

Discrete-time signal: a sequence of **complex** numbers

- one dimension (for now)
- notation: $x[n]$
- two-sided sequences: $x : \mathbb{Z} \to \mathbb{C}$
- n is dimension-less "time"
- analysis: periodic measurement
- synthesis: stream of generated samples

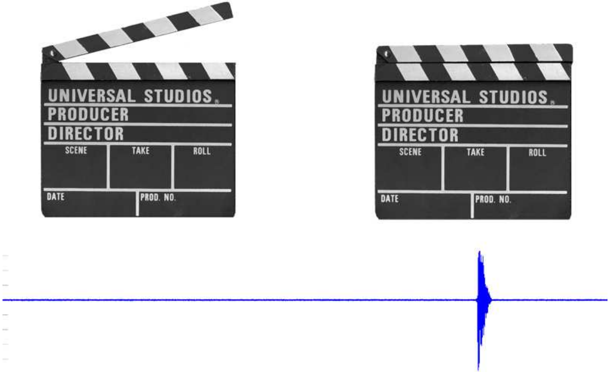
The delta signal

$$x[n] = \delta[n]$$

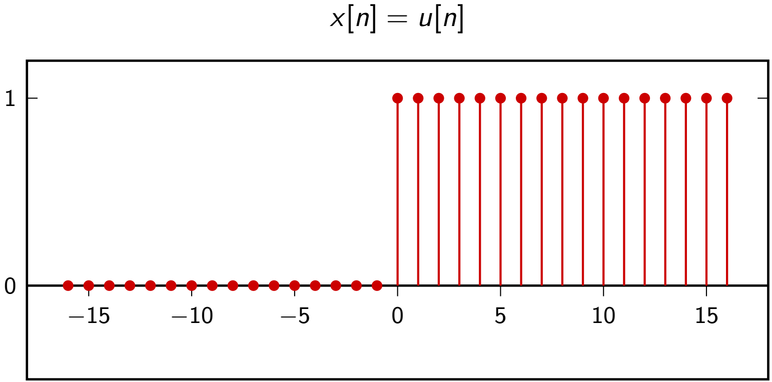


How do you synchronize audio and video...

How do you synchronize audio and video...



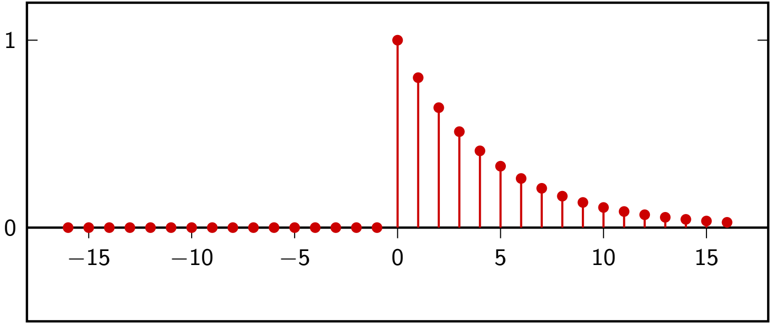
The unit step



The Frankenstein switch...

The exponential decay

$$x[n] = |a|^n \, u[n], \qquad |a| < 1$$



How fast does your coffee get cold...

How fast does your coffee get cold...

Newton's law of cooling:

$$\frac{dT}{dt} = -c(T - T_{\text{env}})$$

$$T(t) = T_{\text{env}} + (T_0 - T_{\text{env}})e^{-ct}$$

In practice:

- must have convection only
- must have large conductivity

How fast does your coffee get cold...

Newton's law of cooling:

$$\frac{dT}{dt} = -c(T - T_{\text{env}})$$

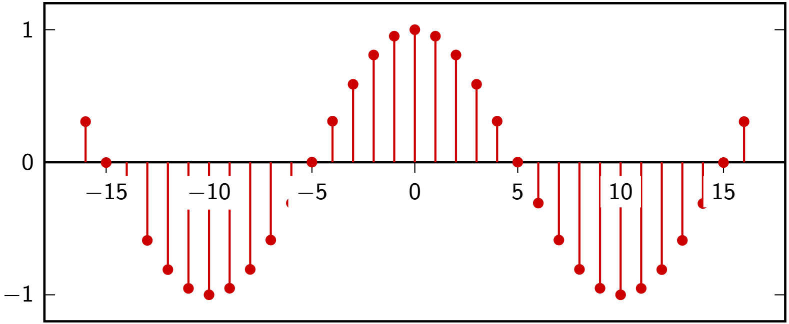
$$T(t) = T_{\text{env}} + (T_0 - T_{\text{env}})e^{-ct}$$

In practice:

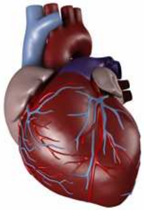
- must have convection only
- must have large conductivity

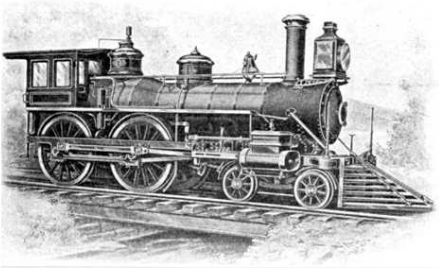
The sinusoid

$$x[n] = \sin(\omega_0 n + \theta)$$



Oscillations are everywhere!





Four signal classes

- finite-length
- infinite-length
- periodic
- finite-support

Four signal classes

- finite-length
- infinite-length
- periodic
- finite-support

Four signal classes

- finite-length
- infinite-length
- periodic
- finite-support

Four signal classes

- finite-length
- infinite-length
- periodic
- finite-support

Finite-length signals

- sequence notation: $x[n], \quad n = 0, 1, \ldots, N-1$
- vector notation: $\mathbf{x} = [x_0 \, x_1 \ldots x_{N-1}]^T$
- practical entities, good for numerical packages (Matlab and the like)

Finite-length signals

- sequence notation: $x[n], \quad n = 0, 1, \ldots, N-1$
- vector notation: $\mathbf{x} = [x_0\ x_1\ \ldots\ x_{N-1}]^T$
- practical entities, good for numerical packages (Matlab and the like)

Finite-length signals

- sequence notation: $x[n], \quad n = 0, 1, \ldots, N-1$
- vector notation: $\mathbf{x} = [x_0 \, x_1 \ldots x_{N-1}]^T$
- practical entities, good for numerical packages (Matlab and the like)

Infinite-length signals

- sequence notation: $x[n], \quad n \in \mathbb{Z}$
- abstraction, good for theorems

Infinite-length signals

- sequence notation: $x[n], \quad n \in \mathbb{Z}$
- abstraction, good for theorems

Periodic signals

- N-periodic sequence: $\tilde{x}[n] = \tilde{x}[n + kN], \quad n, k, N \in \mathbb{Z}$
- same information as finite-length of length N
- "natural" bridge between finite and infinite lengths

Periodic signals

- N-periodic sequence: $\tilde{x}[n] = \tilde{x}[n + kN], \quad n, k, N \in \mathbb{Z}$
- same information as finite-length of length N
- "natural" bridge between finite and infinite lengths

Periodic signals

- N-periodic sequence: $\tilde{x}[n] = \tilde{x}[n + kN], \quad n, k, N \in \mathbb{Z}$
- same information as finite-length of length N
- "natural" bridge between finite and infinite lengths

Finite-support signals

- Finite-support sequence:

$$\bar{x}[n] = \begin{cases} x[n] & \text{if } 0 \leq n < N \\ 0 & \text{otherwise} \end{cases} \qquad n \in \mathbb{Z}$$

- same information as finite-length of length N
- another bridge between finite and infinite lengths

Finite-support signals

- Finite-support sequence:

$$\bar{x}[n] = \begin{cases} x[n] & \text{if } 0 \le n < N \\ 0 & \text{otherwise} \end{cases} \qquad n \in \mathbb{Z}$$

- same information as finite-length of length N
- another bridge between finite and infinite lengths

Finite-support signals

- Finite-support sequence:

$$\bar{x}[n] = \begin{cases} x[n] & \text{if } 0 \leq n < N \\ 0 & \text{otherwise} \end{cases} \qquad n \in \mathbb{Z}$$

- same information as finite-length of length N
- another bridge between finite and infinite lengths

Elementary operators

- scaling:

$$y[n] = \alpha x[n]$$

- sum:

$$y[n] = x[n] + z[n]$$

- product:

$$y[n] = x[n] \cdot z[n]$$

- shift by k (delay):

$$y[n] = x[n - k]$$

Elementary operators

- scaling:

$$y[n] = \alpha x[n]$$

- sum:

$$y[n] = x[n] + z[n]$$

- product:

$$y[n] = x[n] \cdot z[n]$$

- shift by k (delay):

$$y[n] = x[n-k]$$

Elementary operators

- scaling:

$$y[n] = \alpha x[n]$$

- sum:

$$y[n] = x[n] + z[n]$$

- product:

$$y[n] = x[n] \cdot z[n]$$

- shift by k (delay):

$$y[n] = x[n - k]$$

Elementary operators

- scaling:

$$y[n] = \alpha x[n]$$

- sum:

$$y[n] = x[n] + z[n]$$

- product:

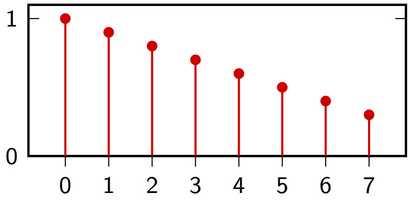
$$y[n] = x[n] \cdot z[n]$$

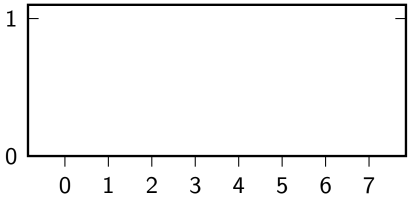
- shift by k (delay):

$$y[n] = x[n-k]$$

Shift of a finite-length: finite-support

$$[x_0 \quad x_1 \quad x_2 \quad x_3 \quad x_4 \quad x_5 \quad x_6 \quad x_7]$$





Shift of a finite-length: finite-support

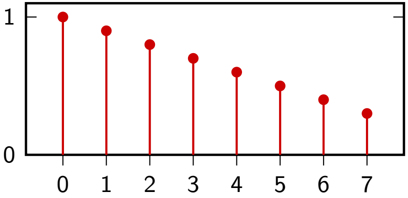
$$x[n]$$

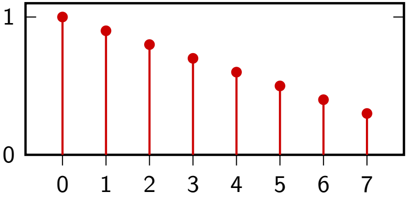
$$\ldots \; \boxed{x_0 \quad x_1 \quad x_2 \quad x_3 \quad x_4 \quad x_5 \quad x_6 \quad x_7} \; \ldots$$

Shift of a finite-length: finite-support

$$\bar{x}[n]$$

$$\ldots \quad 0 \quad 0 \quad 0 \quad \boxed{x_0 \quad x_1 \quad x_2 \quad x_3 \quad x_4 \quad x_5 \quad x_6 \quad x_7} \quad 0 \quad 0 \quad 0 \quad \ldots$$





Shift of a finite-length: finite-support

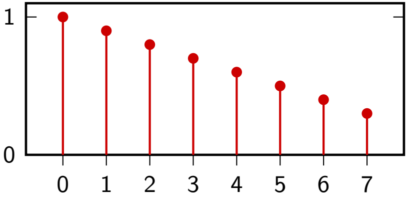
$$\bar{x}[n-1]$$

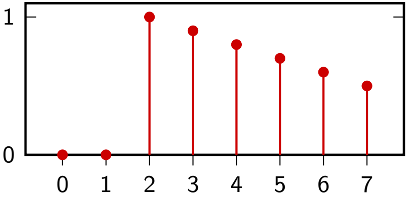
$$\dots \quad 0 \quad 0 \quad 0 \quad \boxed{0 \quad x_0 \quad x_1 \quad x_2 \quad x_3 \quad x_4 \quad x_5 \quad x_6} \quad x_7 \quad 0 \quad 0 \quad \dots$$

Shift of a finite-length: finite-support

$$\bar{x}[n-2]$$

$$\ldots \quad 0 \quad 0 \quad 0 \quad \boxed{0 \quad 0 \quad x_0 \quad x_1 \quad x_2 \quad x_3 \quad x_4 \quad x_5} \quad x_6 \quad x_7 \quad 0 \quad \ldots$$

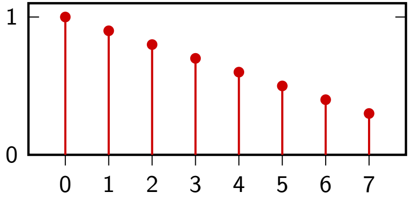


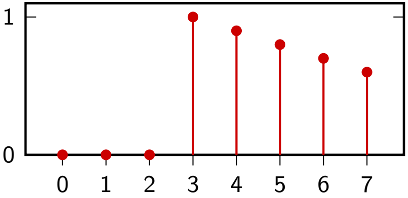


Shift of a finite-length: finite-support

$$\bar{x}[n-3]$$

$$\ldots \quad 0 \quad 0 \quad 0 \quad \boxed{0 \quad 0 \quad 0 \quad x_0 \quad x_1 \quad x_2 \quad x_3 \quad x_4} \quad x_5 \quad x_6 \quad x_7 \quad \ldots$$





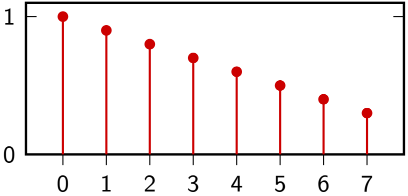
Shift of a finite-length: finite-support

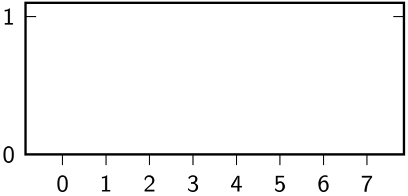
$$\bar{x}[n-4]$$

$$\ldots \quad 0 \quad 0 \quad 0 \quad \boxed{0 \quad 0 \quad 0 \quad 0 \quad x_0 \quad x_1 \quad x_2 \quad x_3} \quad x_4 \quad x_5 \quad x_6 \quad \ldots$$

Shift of a finite-length: periodic extension

$$[x_0 \quad x_1 \quad x_2 \quad x_3 \quad x_4 \quad x_5 \quad x_6 \quad x_7]$$

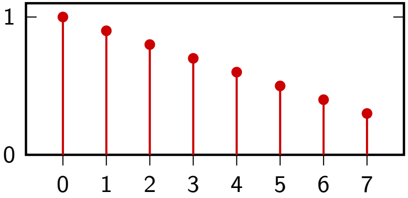


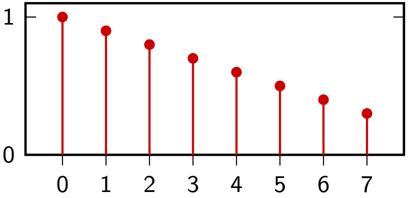


Shift of a finite-length: periodic extension

$$x[n]$$

$$\ldots \boxed{x_0 \quad x_1 \quad x_2 \quad x_3 \quad x_4 \quad x_5 \quad x_6 \quad x_7} \ldots$$

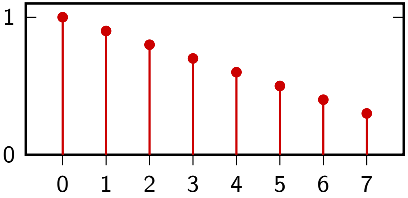


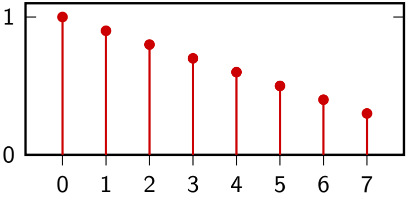


Shift of a finite-length: periodic extension

$$\tilde{x}[n]$$

$$\ldots \quad x_5 \quad x_6 \quad x_7 \quad \boxed{x_0 \quad x_1 \quad x_2 \quad x_3 \quad x_4 \quad x_5 \quad x_6 \quad x_7} \quad x_0 \quad x_1 \quad x_2 \quad \ldots$$

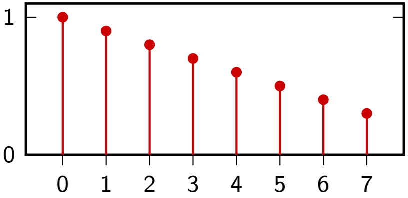


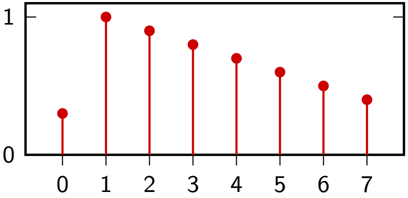


Shift of a finite-length: periodic extension

$$\tilde{x}[n-1]$$

$$\ldots \quad x_4 \quad x_5 \quad x_6 \quad \boxed{x_7 \quad x_0 \quad x_1 \quad x_2 \quad x_3 \quad x_4 \quad x_5 \quad x_6} \quad x_7 \quad x_0 \quad x_1 \quad \ldots$$

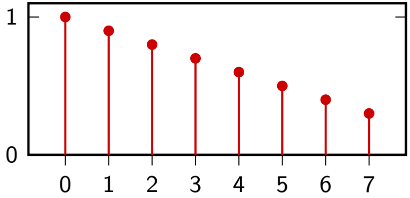


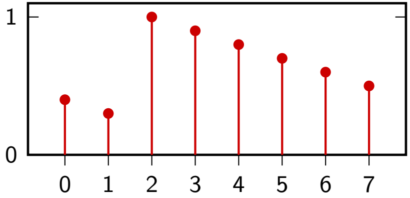


Shift of a finite-length: periodic extension

$$\tilde{x}[n-2]$$

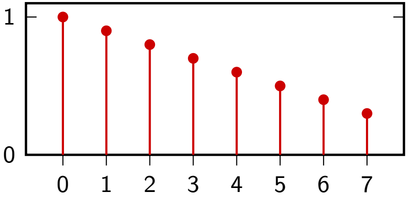
$$\ldots \quad x_3 \quad x_4 \quad x_5 \quad \boxed{x_6 \quad x_7 \quad x_0 \quad x_1 \quad x_2 \quad x_3 \quad x_4 \quad x_5} \quad x_6 \quad x_7 \quad x_0 \quad \ldots$$

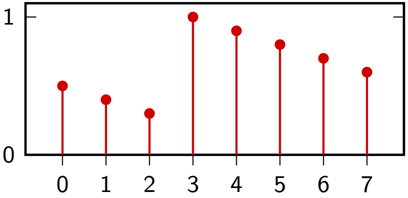




$$\tilde{x}[n-3]$$

$$\ldots \quad x_2 \quad x_3 \quad x_4 \quad \boxed{x_5 \quad x_6 \quad x_7 \quad x_0 \quad x_1 \quad x_2 \quad x_3 \quad x_4} \quad x_5 \quad x_6 \quad x_7 \quad \ldots$$

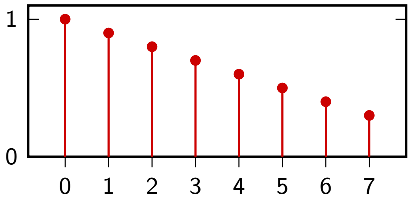


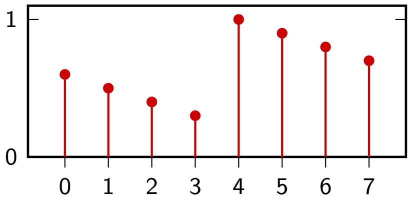


Shift of a finite-length: periodic extension

$$\tilde{x}[n-4]$$

$$\ldots \quad x_1 \quad x_2 \quad x_3 \quad \boxed{x_4 \quad x_5 \quad x_6 \quad x_7 \quad x_0 \quad x_1 \quad x_2 \quad x_3} \quad x_4 \quad x_5 \quad x_6 \quad \ldots$$





Energy and power

$$E_x = \sum_{n=-\infty}^{\infty} |x[n]|^2$$

$$P_x = \lim_{N\to\infty} \frac{1}{2N+1} \sum_{n=-N}^{N} |x[n]|^2$$

$$E_x = \sum_{n=-\infty}^{\infty} |x[n]|^2$$

$$P_x = \lim_{N\to\infty} \frac{1}{2N+1} \sum_{n=-N}^{N} |x[n]|^2$$

Energy and power: periodic signals

$$E_{\tilde{x}} = \infty$$

$$P_{\tilde{x}} \equiv \frac{1}{N} \sum_{n=0}^{N-1} |\tilde{x}[n]|^2$$

Energy and power: periodic signals

$$E_{\tilde{x}} = \infty$$

$$P_{\tilde{x}} \equiv \frac{1}{N} \sum_{n=0}^{N-1} |\tilde{x}[n]|^2$$

END OF MODULE 2.1

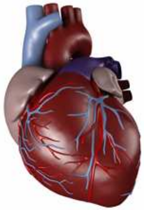
Digital Signal Processing

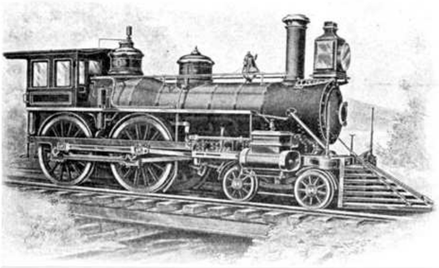
Module 2.2: the complex exponential

Overview:

- the complex exponential
- periodicity
- wagonwheel effect and maximum “speed”
- digital and real-world frequency

Oscillations are everywhere





Ingredients:

- a frequency ω (units: radians)
- an initial phase ϕ (units: radians)
- an amplitude A (units depending on underlying measurement)
- a trigonometric function

e.g. $x[n] = A\cos(\omega n + \phi)$

The oscillatory heartbeat

Ingredients:

- a frequency ω (units: radians)
- an initial phase ϕ (units: radians)
- an amplitude A (units depending on underlying measurement)
- a trigonometric function

e.g. $x[n] = A\cos(\omega n + \phi)$

The oscillatory heartbeat

Ingredients:

- a frequency ω (units: radians)
- an initial phase ϕ (units: radians)
- an amplitude A (units depending on underlying measurement)
- a trigonometric function

e.g. $x[n] = A\cos(\omega n + \phi)$

The oscillatory heartbeat

Ingredients:

- a frequency ω (units: radians)
- an initial phase ϕ (units: radians)
- an amplitude A (units depending on underlying measurement)
- a trigonometric function

e.g. $x[n] = A\cos(\omega n + \phi)$

Ingredients:

- a frequency ω (units: radians)
- an initial phase ϕ (units: radians)
- an amplitude A (units depending on underlying measurement)
- a trigonometric function

$$\text{e.g. } x[n] = A\cos(\omega n + \phi)$$

The complex exponential

the trigonometric function of choice in DSP is the complex exponential:

$$
\begin{aligned}
x[n] &= Ae^{j(\omega n+\phi)} \\
&= A[\cos(\omega n + \phi) + j\sin(\omega n + \phi)]
\end{aligned}
$$

Why complex exponentials?

- makes sense: sines and cosines always go together
- simpler math: trigonometry becomes algebra
- we can use complex numbers in digital systems

Why complex exponentials?

- makes sense: sines and cosines always go together
- simpler math: trigonometry becomes algebra
- we can use complex numbers in digital systems

Why complex exponentials?

- makes sense: sines and cosines always go together
- simpler math: trigonometry becomes algebra
- we can use complex numbers in digital systems

The advantages of complex exponentials

Example: change the phase of a pure cosine

$$\cos(\omega n + \phi) = a \cos(\omega n) + b \sin(\omega n), \qquad a = \cos \phi, \;\; b = -\sin \phi$$

- each sinusoid is always a sum of sine and cosine
- we have to remember complex trigonometric formulas
- we have to carry more terms in our equations

The advantages of complex exponentials

Example: change the phase of a pure cosine

$$\cos(\omega n + \phi) = a\cos(\omega n) + b\sin(\omega n), \qquad a = \cos\phi, \;\; b = -\sin\phi$$

- each sinusoid is always a sum of sine and cosine
- we have to remember complex trigonometric formulas
- we have to carry more terms in our equations

The advantages of complex exponentials

Example: change the phase of a pure cosine

$$\cos(\omega n + \phi) = a\cos(\omega n) + b\sin(\omega n), \qquad a = \cos\phi, \;\; b = -\sin\phi$$

- each sinusoid is always a sum of sine and cosine
- we have to remember complex trigonometric formulas
- we have to carry more terms in our equations

The advantages of complex exponentials

Example: change the phase of a pure cosine

$$\cos(\omega n + \phi) = a\cos(\omega n) + b\sin(\omega n), \qquad a = \cos\phi, \;\; b = -\sin\phi$$

- each sinusoid is always a sum of sine and cosine
- we have to remember complex trigonometric formulas
- we have to carry more terms in our equations

The advantages of complex exponentials

Example: change the phase of a pure cosine

$$\mathrm{Re}\{e^{j(\omega n+\phi)}\} = \mathrm{Re}\{e^{j\omega n}\, e^{j\phi}\}$$

- sine and cosine “live” together
- phase shift is simple multiplication
- notation is simpler

The advantages of complex exponentials

Example: change the phase of a pure cosine

$$\mathrm{Re}\{e^{j(\omega n+\phi)}\} = \mathrm{Re}\{e^{j\omega n}\, e^{j\phi}\}$$

- sine and cosine “live” together
- phase shift is simple multiplication
- notation is simpler

The advantages of complex exponentials

Example: change the phase of a pure cosine

$$\mathrm{Re}\{e^{j(\omega n+\phi)}\} = \mathrm{Re}\{e^{j\omega n}\, e^{j\phi}\}$$

- sine and cosine "live" together
- phase shift is simple multiplication
- notation is simpler

The advantages of complex exponentials

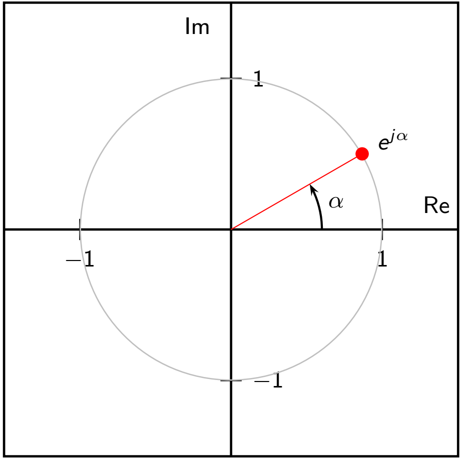
Example: change the phase of a pure cosine

$$\mathrm{Re}\{e^{j(\omega n+\phi)}\} = \mathrm{Re}\{e^{j\omega n}\, e^{j\phi}\}$$

- sine and cosine “live” together
- phase shift is simple multiplication
- notation is simpler

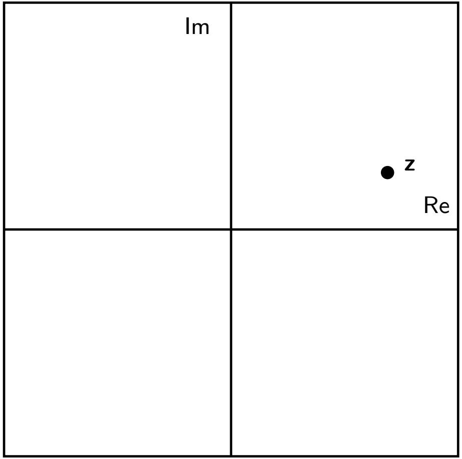
The complex exponential

$$e^{j\alpha} = \cos\alpha + j\sin\alpha$$



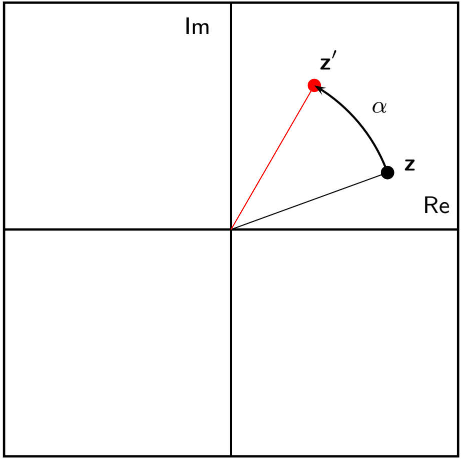
The complex exponential

$\mathbf{z}$: point on the complex plane



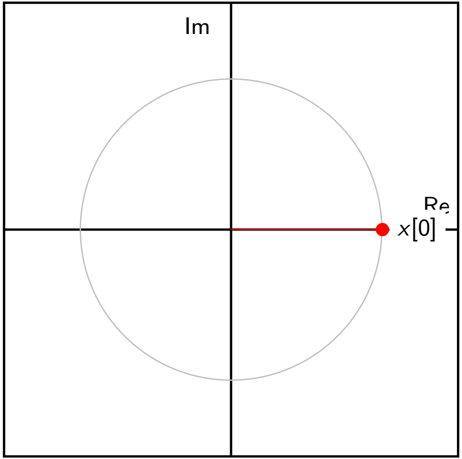
The complex exponential

rotation: $\mathbf{z}' = \mathbf{z}\, e^{j\alpha}$



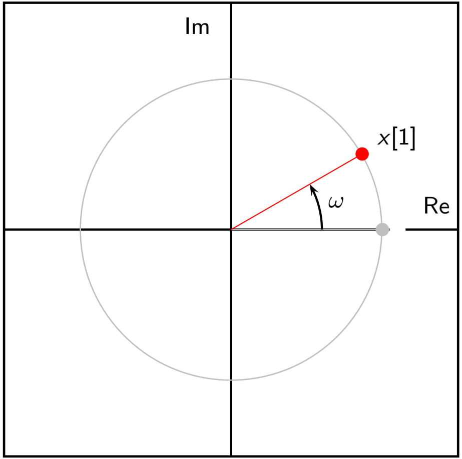
The complex exponential generating machine

$$x[n] = e^{j\omega n}; \qquad x[n+1] = e^{j\omega} x[n]$$



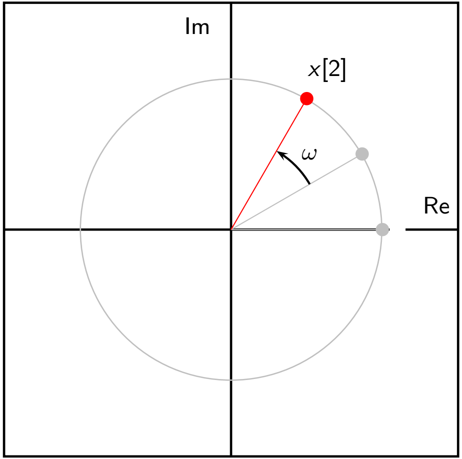
The complex exponential generating machine

$$x[n] = e^{j\omega n}; \qquad x[n+1] = e^{j\omega} x[n]$$

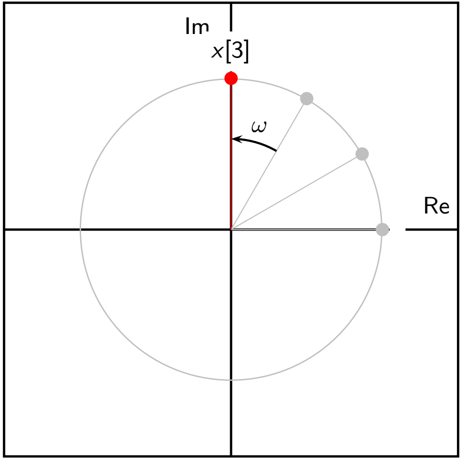


The complex exponential generating machine

$$x[n] = e^{j\omega n}; \qquad x[n+1] = e^{j\omega} x[n]$$

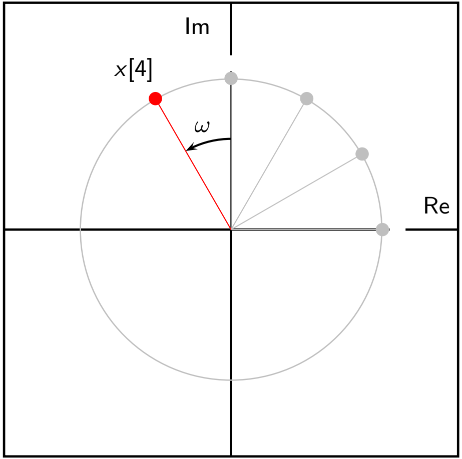


$$x[n] = e^{j\omega n}; \qquad x[n+1] = e^{j\omega} x[n]$$



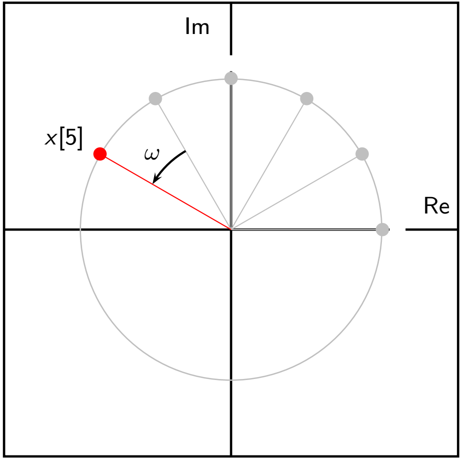
The complex exponential generating machine

$$x[n] = e^{j\omega n}; \qquad x[n+1] = e^{j\omega} x[n]$$

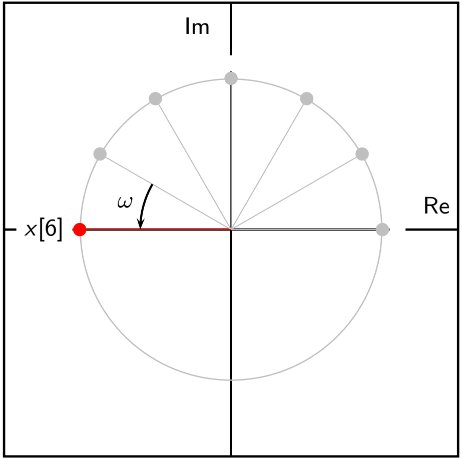


The complex exponential generating machine

$$x[n] = e^{j\omega n}; \qquad x[n+1] = e^{j\omega} x[n]$$

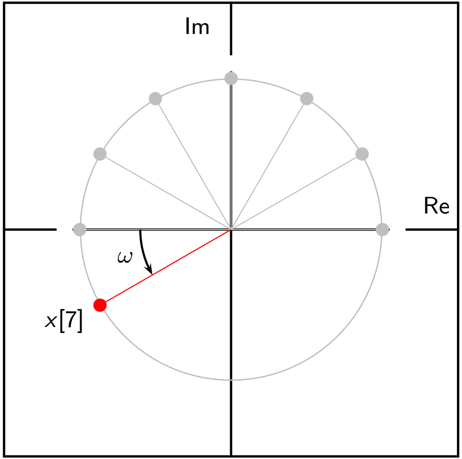


$$x[n] = e^{j\omega n}; \qquad x[n+1] = e^{j\omega} x[n]$$



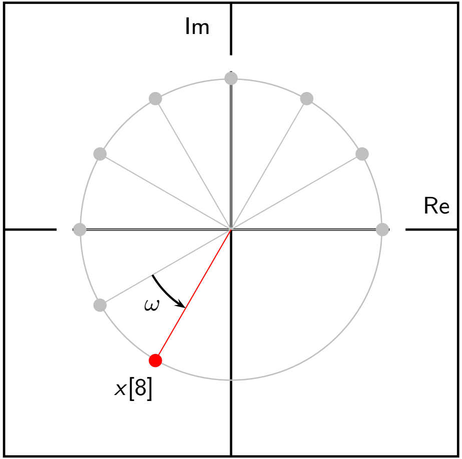
The complex exponential generating machine

$$x[n] = e^{j\omega n}; \qquad x[n+1] = e^{j\omega} x[n]$$



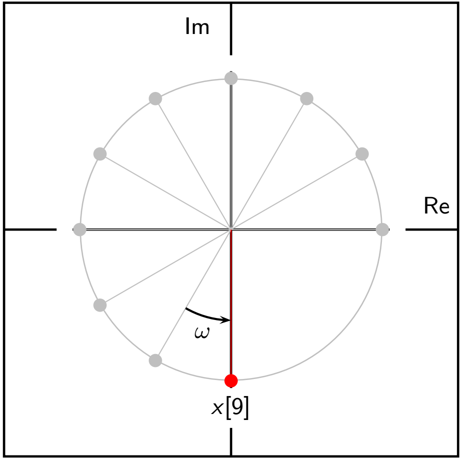
The complex exponential generating machine

$$x[n] = e^{j\omega n}; \qquad x[n+1] = e^{j\omega} x[n]$$



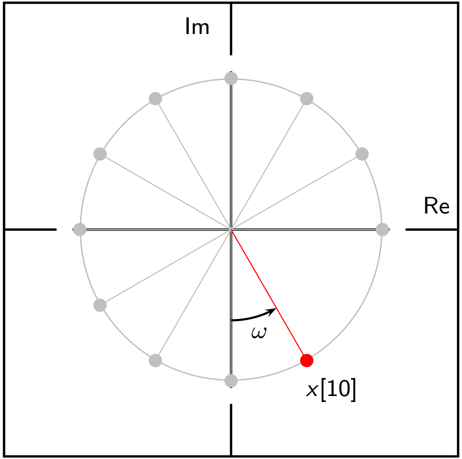
The complex exponential generating machine

$$x[n] = e^{j\omega n}; \qquad x[n+1] = e^{j\omega} x[n]$$



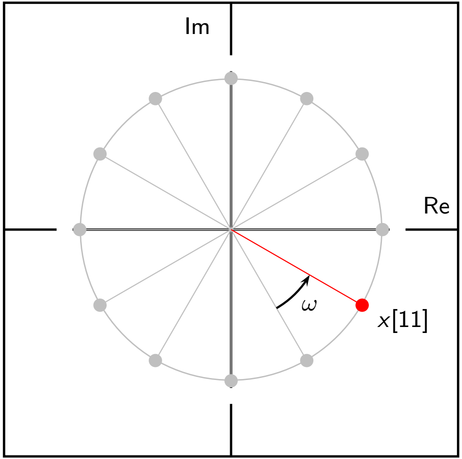
The complex exponential generating machine

$$x[n] = e^{j\omega n}; \qquad x[n+1] = e^{j\omega}x[n]$$



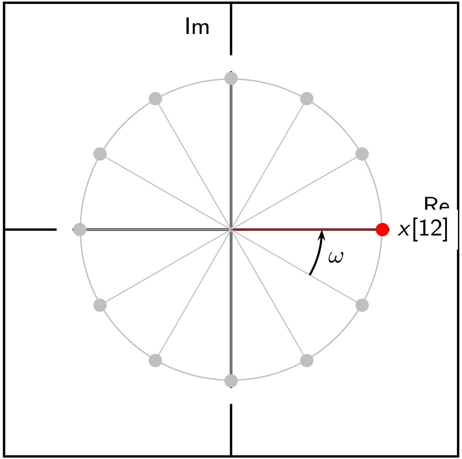
The complex exponential generating machine

$$x[n] = e^{j\omega n}; \qquad x[n+1] = e^{j\omega}x[n]$$



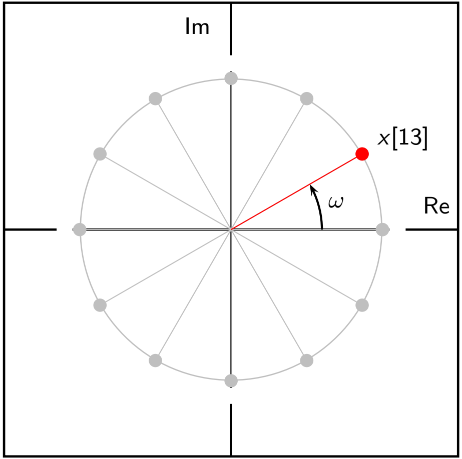
The complex exponential generating machine

$$x[n] = e^{j\omega n}; \qquad x[n+1] = e^{j\omega} x[n]$$



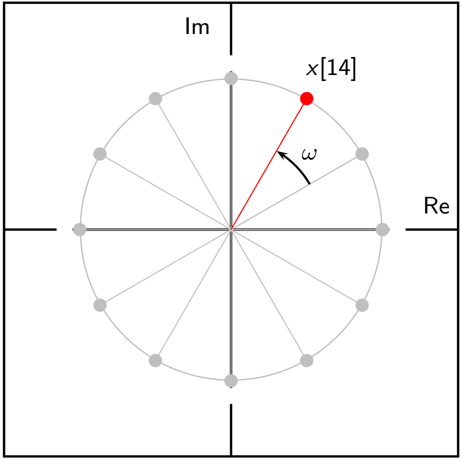
The complex exponential generating machine

$$x[n] = e^{j\omega n}; \qquad x[n+1] = e^{j\omega} x[n]$$



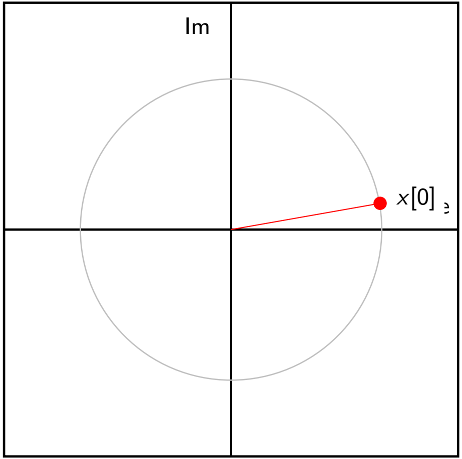
The complex exponential generating machine

$$x[n] = e^{j\omega n}; \qquad x[n+1] = e^{j\omega} x[n]$$



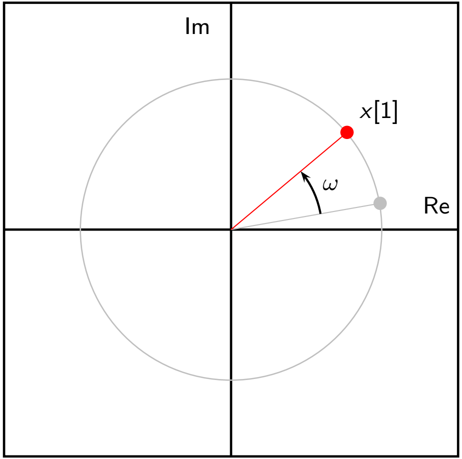
Initial phase

$$x[n] = e^{j(\omega n + \phi)}; \qquad x[n+1] = e^{j\omega} x[n], \quad x[0] = e^{j\phi}$$



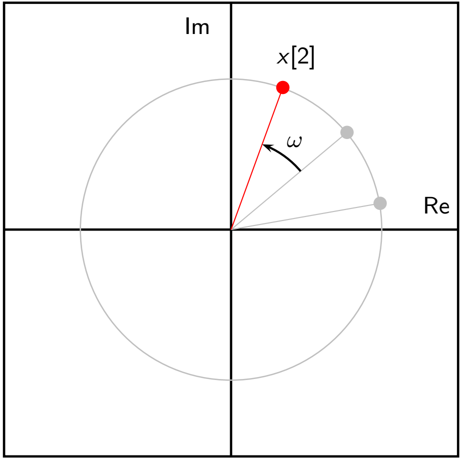
Initial phase

$$x[n] = e^{j(\omega n + \phi)}; \qquad x[n+1] = e^{j\omega} x[n], \quad x[0] = e^{j\phi}$$



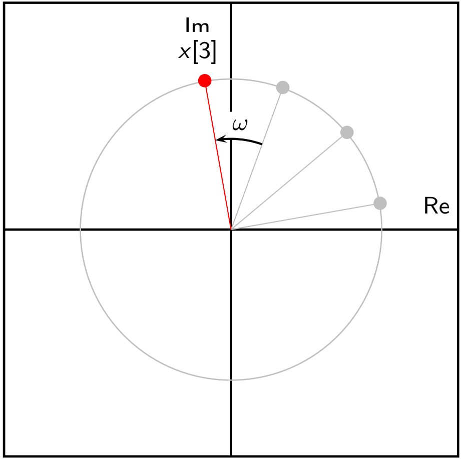
Initial phase

$$x[n] = e^{j(\omega n + \phi)}; \qquad x[n+1] = e^{j\omega} x[n], \quad x[0] = e^{j\phi}$$



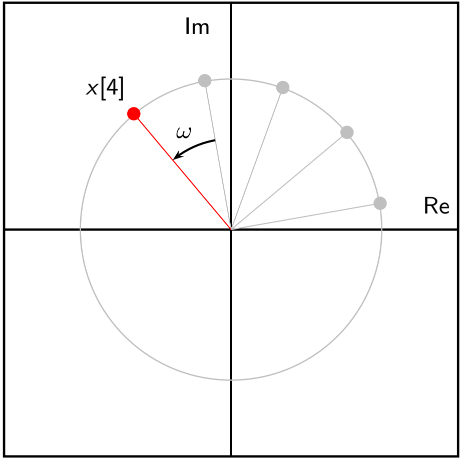
Initial phase

$$x[n] = e^{j(\omega n+\phi)}; \qquad x[n+1] = e^{j\omega}x[n], \quad x[0] = e^{j\phi}$$



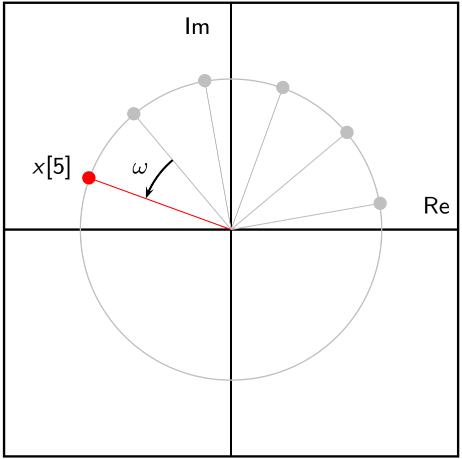
Initial phase

$$x[n] = e^{j(\omega n + \phi)}; \qquad x[n+1] = e^{j\omega} x[n], \quad x[0] = e^{j\phi}$$



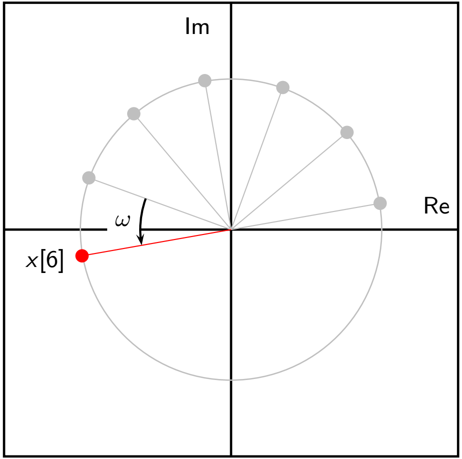
Initial phase

$$x[n] = e^{j(\omega n + \phi)}; \qquad x[n+1] = e^{j\omega} x[n], \quad x[0] = e^{j\phi}$$



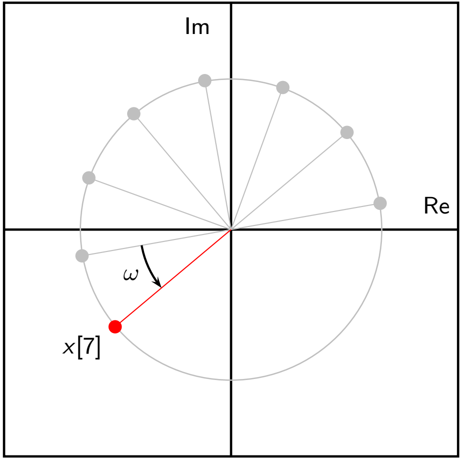
Initial phase

$$x[n] = e^{j(\omega n + \phi)}; \qquad x[n+1] = e^{j\omega} x[n], \quad x[0] = e^{j\phi}$$



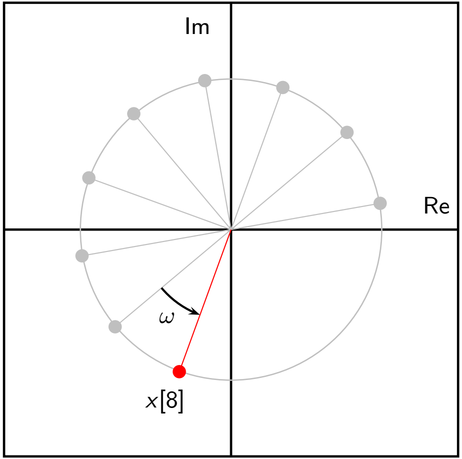
Initial phase

$$x[n] = e^{j(\omega n+\phi)}; \qquad x[n+1] = e^{j\omega}x[n], \quad x[0] = e^{j\phi}$$



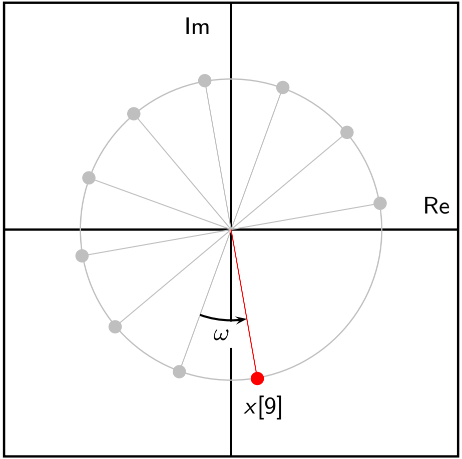
Initial phase

$$x[n] = e^{j(\omega n+\phi)}; \qquad x[n+1] = e^{j\omega}x[n], \quad x[0] = e^{j\phi}$$



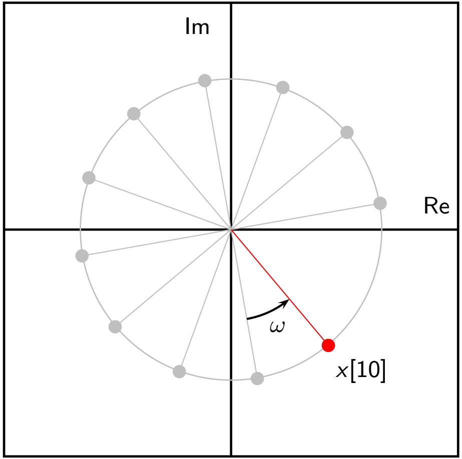
Initial phase

$$x[n] = e^{j(\omega n + \phi)}; \qquad x[n+1] = e^{j\omega} x[n], \quad x[0] = e^{j\phi}$$



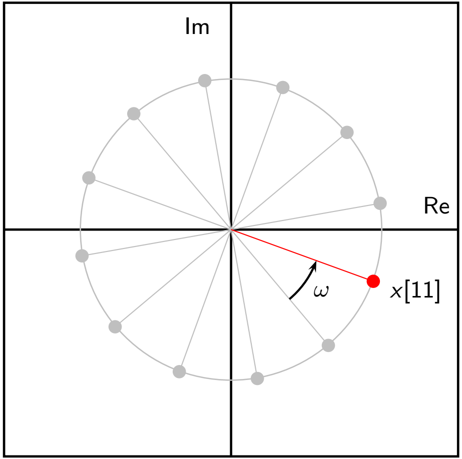
Initial phase

$$x[n] = e^{j(\omega n + \phi)}; \qquad x[n+1] = e^{j\omega} x[n], \quad x[0] = e^{j\phi}$$



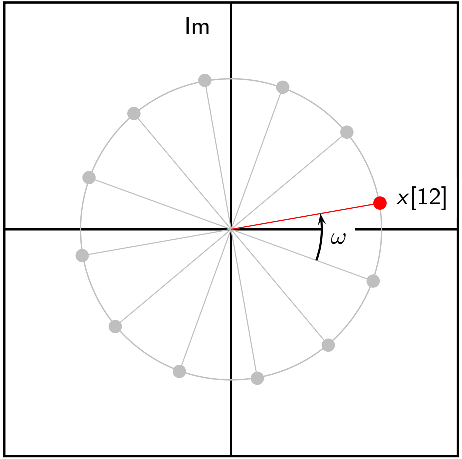
Initial phase

$$x[n] = e^{j(\omega n + \phi)}; \qquad x[n+1] = e^{j\omega} x[n], \quad x[0] = e^{j\phi}$$



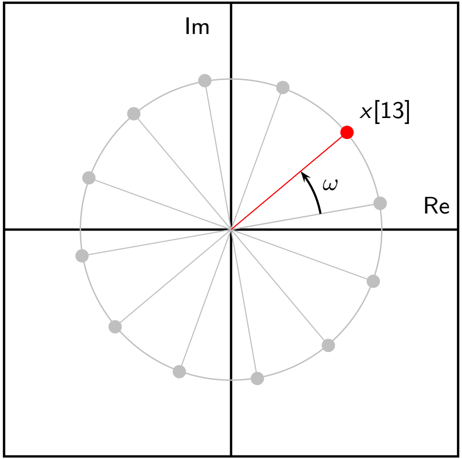
Initial phase

$$x[n] = e^{j(\omega n + \phi)}; \qquad x[n+1] = e^{j\omega} x[n], \quad x[0] = e^{j\phi}$$



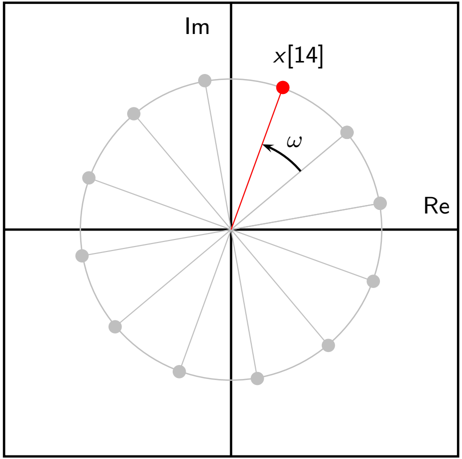
Initial phase

$$x[n] = e^{j(\omega n + \phi)}; \qquad x[n+1] = e^{j\omega} x[n], \quad x[0] = e^{j\phi}$$



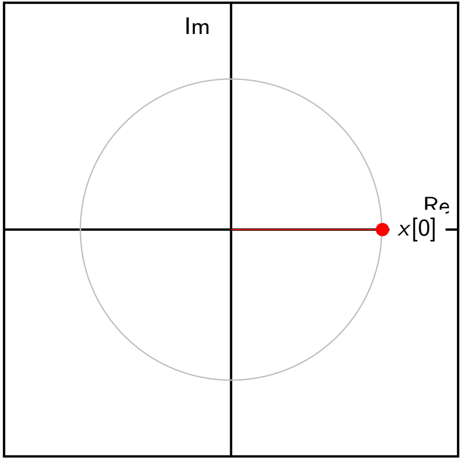
Initial phase

$$x[n] = e^{j(\omega n + \phi)}; \qquad x[n+1] = e^{j\omega} x[n], \quad x[0] = e^{j\phi}$$



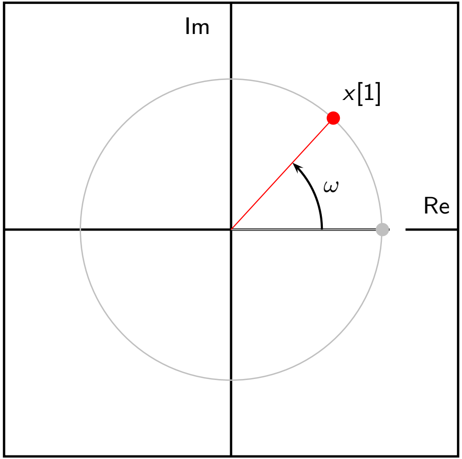
Careful: not every sinusoid is periodic in discrete time

$$x[n] = e^{j\omega n}; \qquad x[n+1] = e^{j\omega} x[n]$$



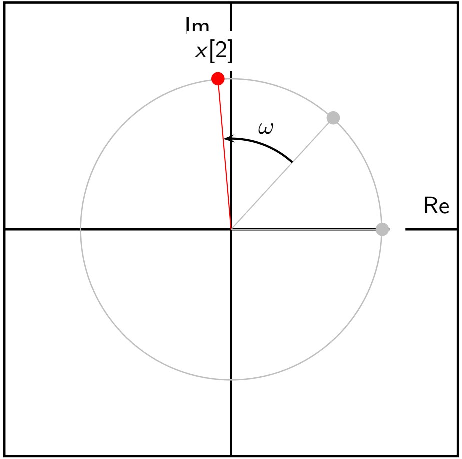
Careful: not every sinusoid is periodic in discrete time

$$x[n] = e^{j\omega n}; \qquad x[n+1] = e^{j\omega} x[n]$$



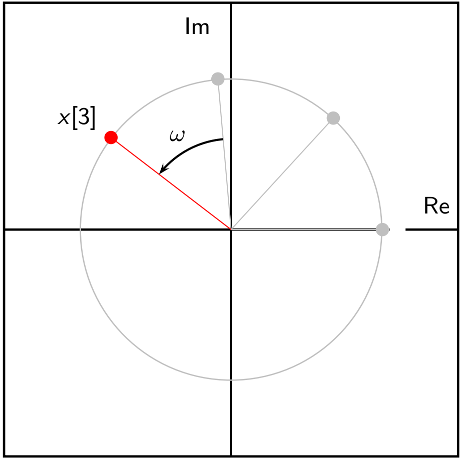
Careful: not every sinusoid is periodic in discrete time

$$x[n] = e^{j\omega n}; \qquad x[n+1] = e^{j\omega}x[n]$$



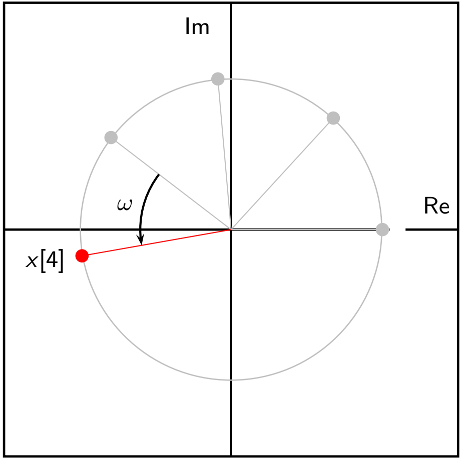
Careful: not every sinusoid is periodic in discrete time

$$x[n] = e^{j\omega n}; \qquad x[n+1] = e^{j\omega} x[n]$$



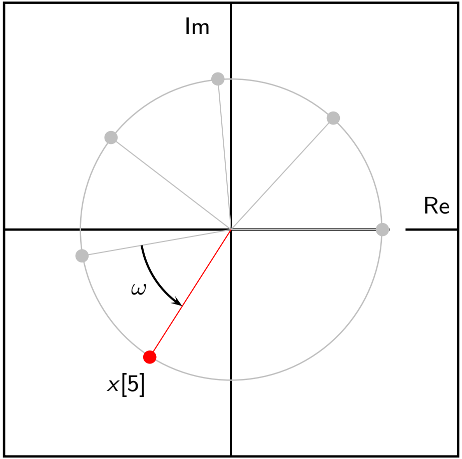
Careful: not every sinusoid is periodic in discrete time

$$x[n] = e^{j\omega n}; \qquad x[n+1] = e^{j\omega} x[n]$$



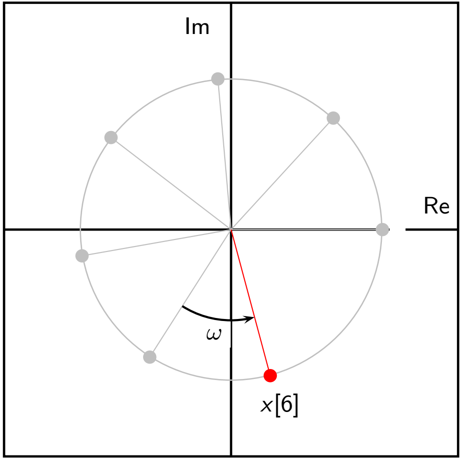
Careful: not every sinusoid is periodic in discrete time

$$x[n] = e^{j\omega n}; \qquad x[n+1] = e^{j\omega}x[n]$$



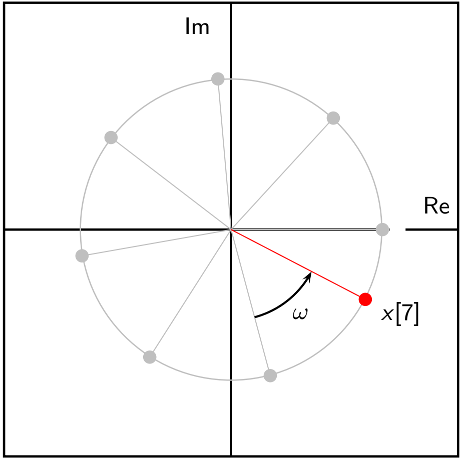
Careful: not every sinusoid is periodic in discrete time

$$x[n] = e^{j\omega n}; \qquad x[n+1] = e^{j\omega} x[n]$$



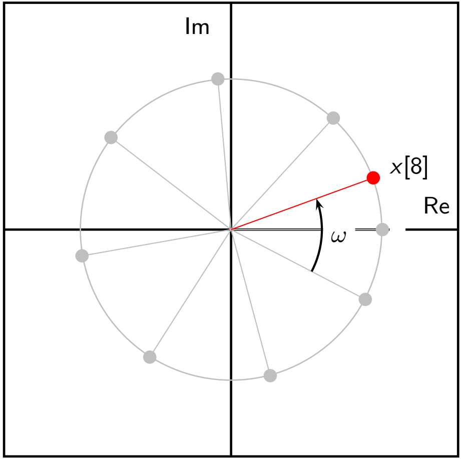
Careful: not every sinusoid is periodic in discrete time

$$x[n] = e^{j\omega n}; \qquad x[n+1] = e^{j\omega} x[n]$$



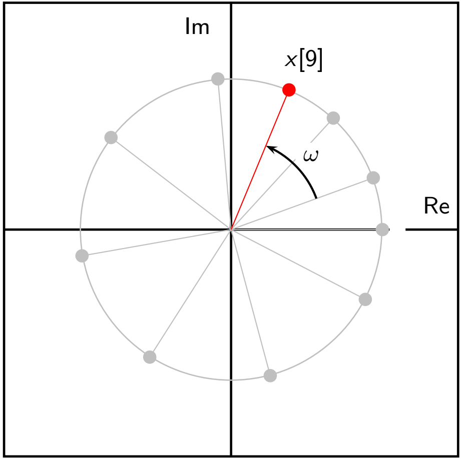
Careful: not every sinusoid is periodic in discrete time

$$x[n] = e^{j\omega n}; \qquad x[n+1] = e^{j\omega}x[n]$$



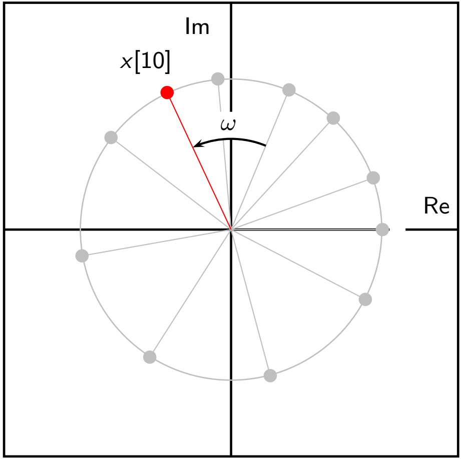
Careful: not every sinusoid is periodic in discrete time

$$x[n] = e^{j\omega n}; \qquad x[n+1] = e^{j\omega} x[n]$$



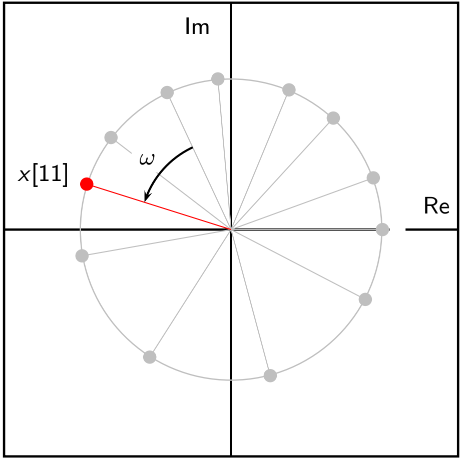
Careful: not every sinusoid is periodic in discrete time

$$x[n] = e^{j\omega n}; \qquad x[n+1] = e^{j\omega} x[n]$$



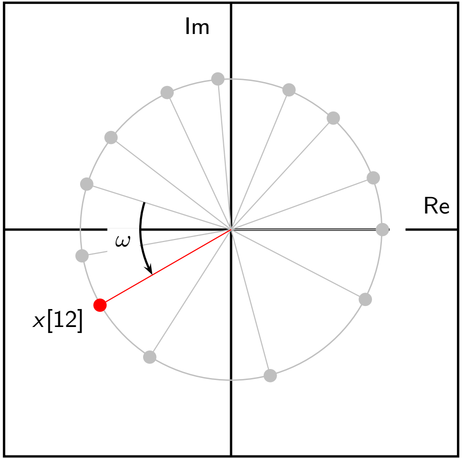
Careful: not every sinusoid is periodic in discrete time

$$x[n] = e^{j\omega n}; \qquad x[n+1] = e^{j\omega} x[n]$$



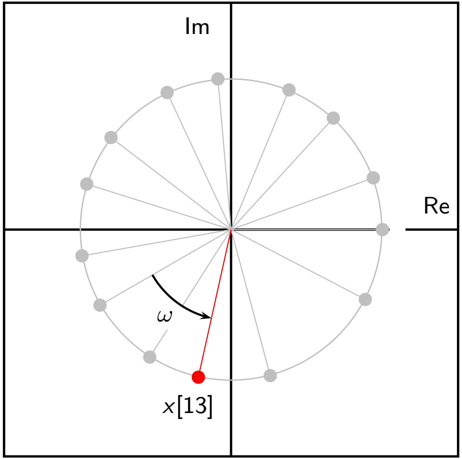
Careful: not every sinusoid is periodic in discrete time

$$x[n] = e^{j\omega n}; \qquad x[n+1] = e^{j\omega} x[n]$$



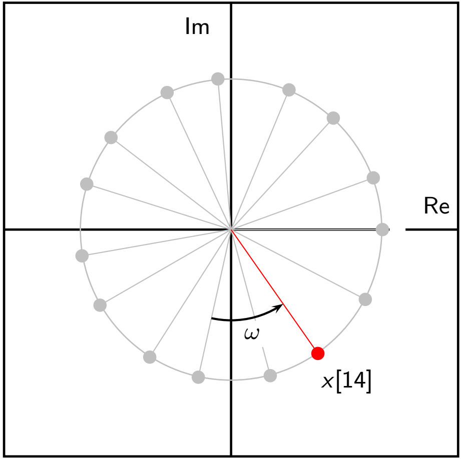
Careful: not every sinusoid is periodic in discrete time

$$x[n] = e^{j\omega n}; \qquad x[n+1] = e^{j\omega} x[n]$$



Careful: not every sinusoid is periodic in discrete time

$$x[n] = e^{j\omega n}; \qquad x[n+1] = e^{j\omega}x[n]$$



$$e^{j\omega n} \text{ periodic } \iff \omega = \frac{M}{N}2\pi, M, N \in \mathbb{N}$$

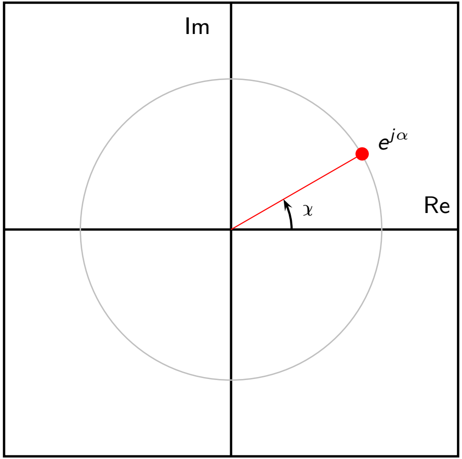
$$e^{j\omega} = e^{j(\omega+2k\pi)} \quad \forall k \in \mathbb{N}$$

Periodicity

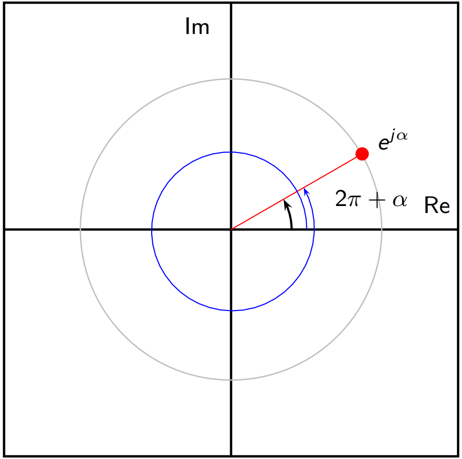
$$e^{j\omega n} \text{ periodic} \iff \omega = \frac{M}{N}2\pi, M, N \in \mathbb{N}$$

$$e^{j\omega} = e^{j(\omega+2k\pi)} \quad \forall k \in \mathbb{N}$$

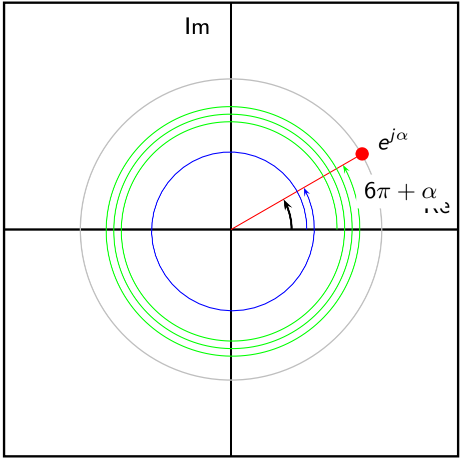
One point, many names



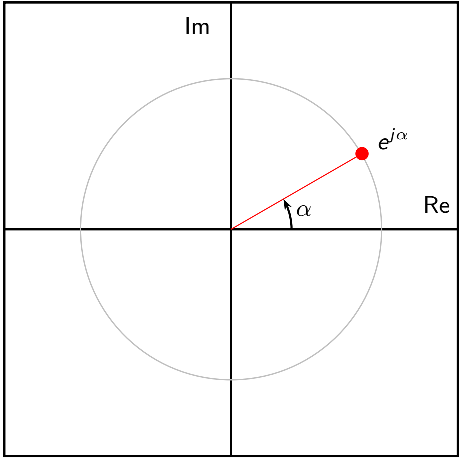
One point, many names



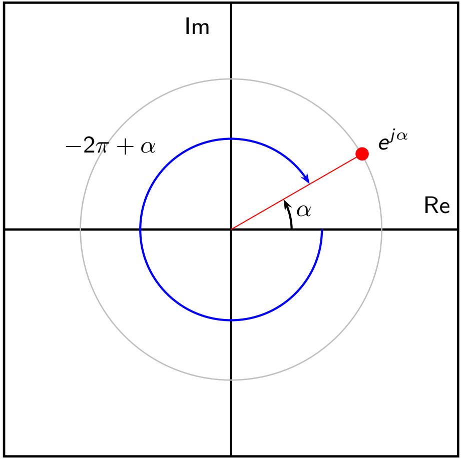
One point, many names



One point, many names



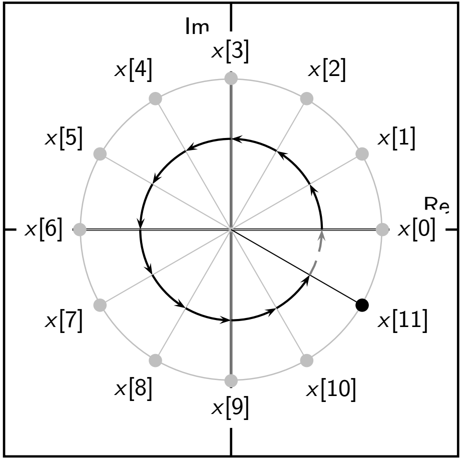
One point, many names



How “fast” can we go?

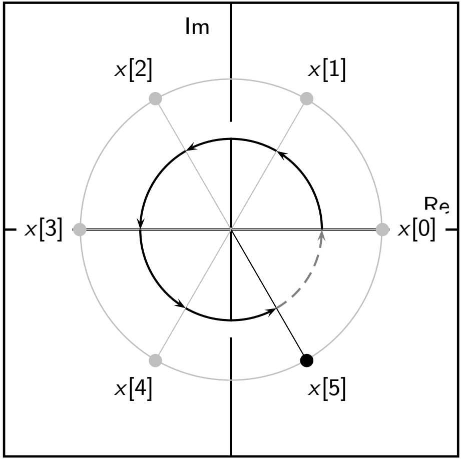
How "fast" can we go?

$$\omega = 2\pi/12$$



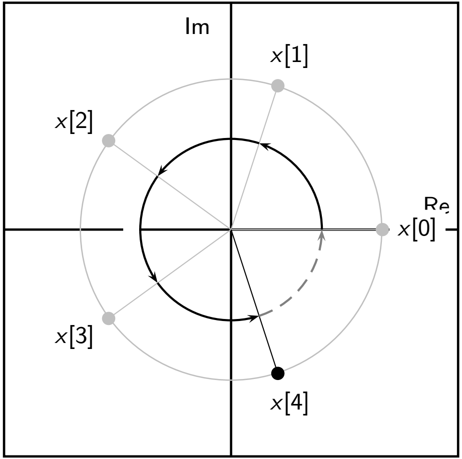
How "fast" can we go?

$$\omega = 2\pi/6$$



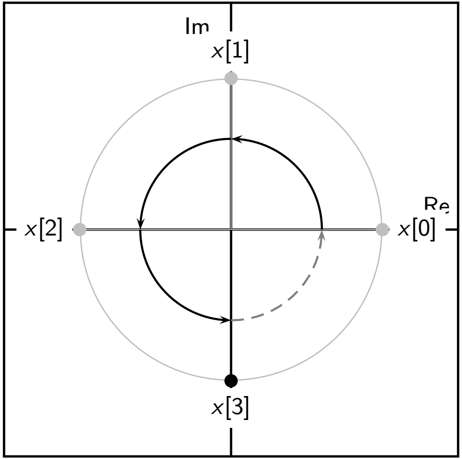
How "fast" can we go?

$$\omega = 2\pi/5$$



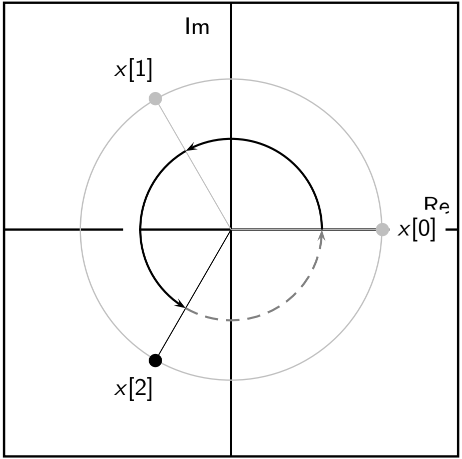
How "fast" can we go?

$$\omega = 2\pi/4$$

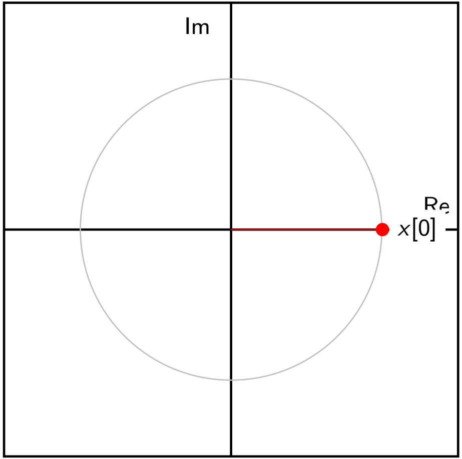


How “fast” can we go?

$$\omega = 2\pi/3$$

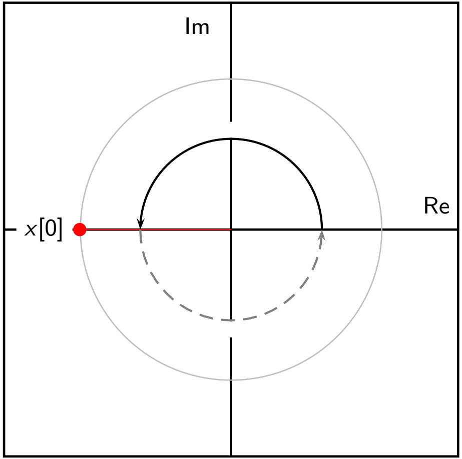


$$\omega = 2\pi/2 = \pi$$



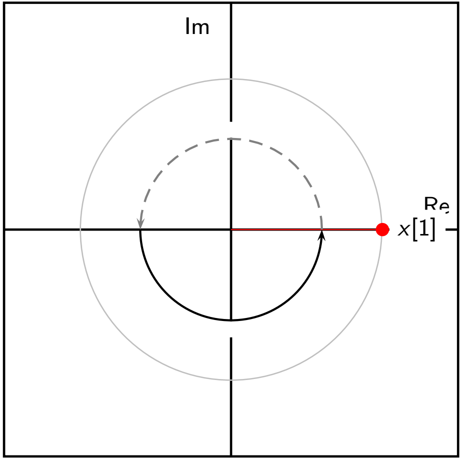
How "fast" can we go?

$$\omega = 2\pi/2 = \pi$$



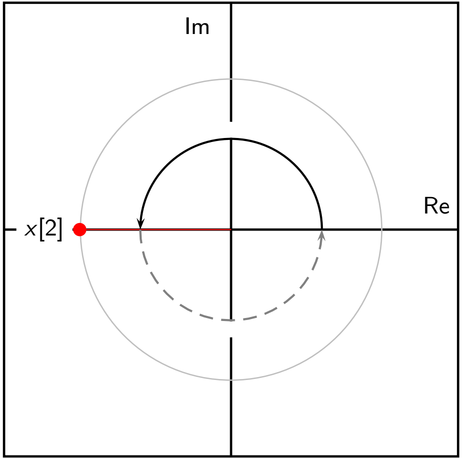
How “fast” can we go?

$$\omega = 2\pi/2 = \pi$$



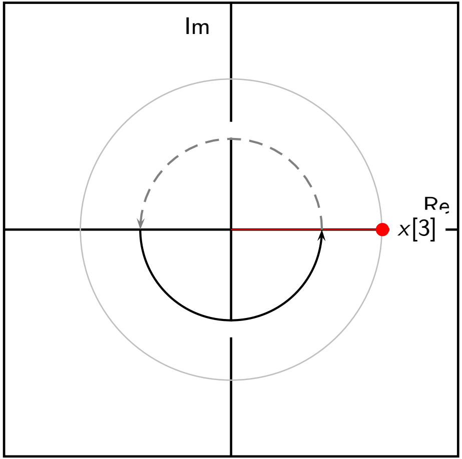
How “fast” can we go?

$$\omega = 2\pi/2 = \pi$$



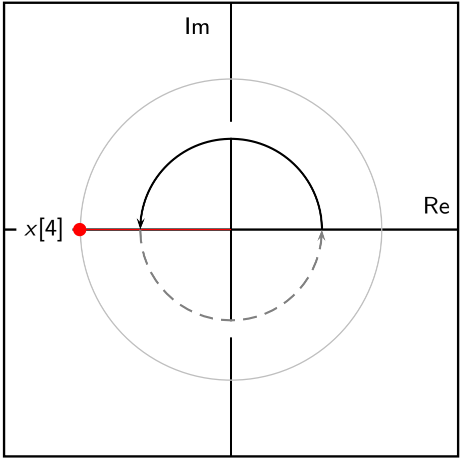
How “fast” can we go?

$$\omega = 2\pi/2 = \pi$$



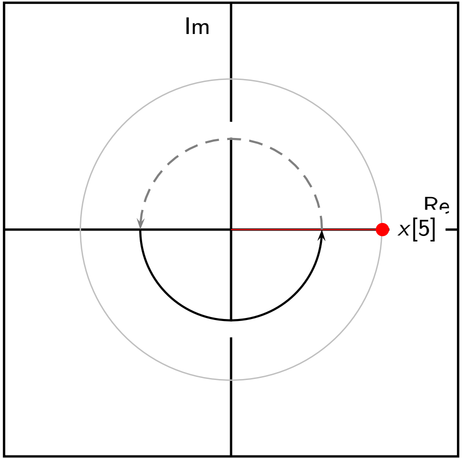
How "fast" can we go?

$$\omega = 2\pi/2 = \pi$$



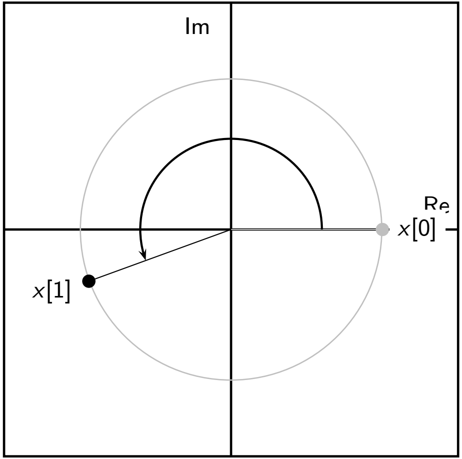
How "fast" can we go?

$$\omega = 2\pi/2 = \pi$$



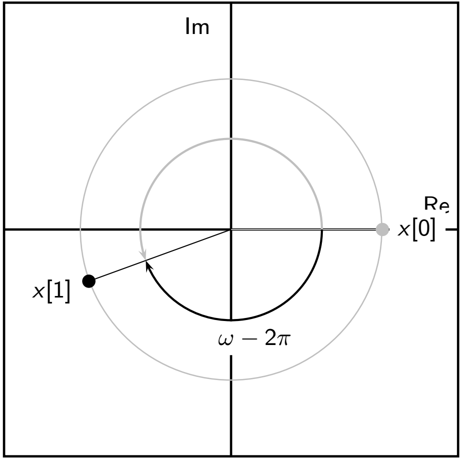
What if we go “faster”?

$$\pi < \omega < 2\pi$$



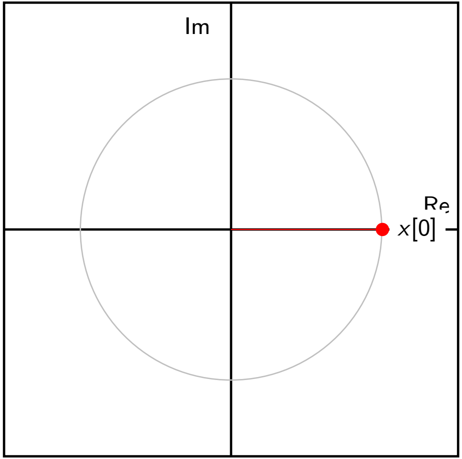
What if we go “faster”?

$$\pi < \omega < 2\pi$$



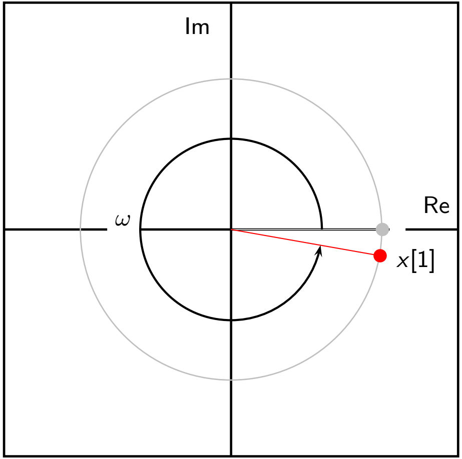
Let's go really too fast

$$\omega = 2\pi - \alpha, \quad \alpha \text{ small}$$



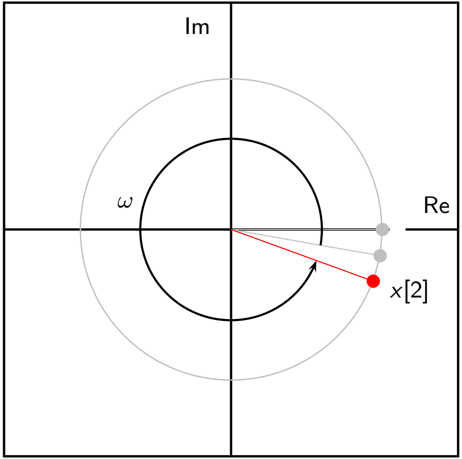
Let's go really too fast

$$\omega = 2\pi - \alpha, \quad \alpha \text{ small}$$



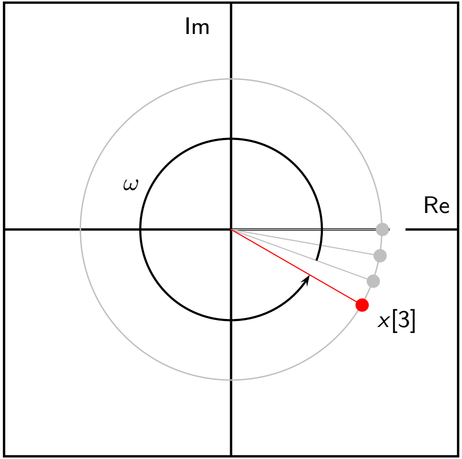
Let's go really too fast

$$\omega = 2\pi - \alpha, \quad \alpha \text{ small}$$



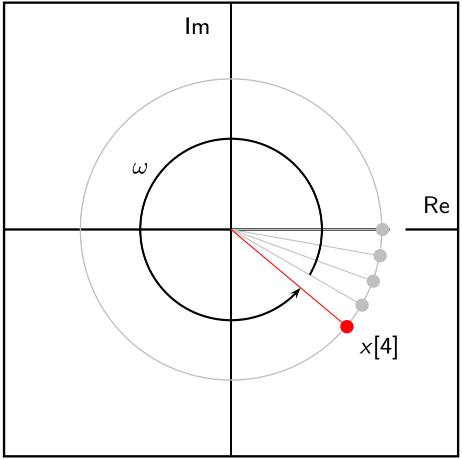
Let's go really too fast

$$\omega = 2\pi - \alpha, \quad \alpha \text{ small}$$



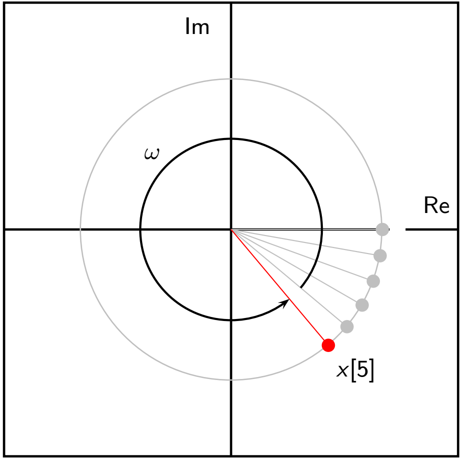
Let's go really too fast

$$\omega = 2\pi - \alpha, \quad \alpha \text{ small}$$



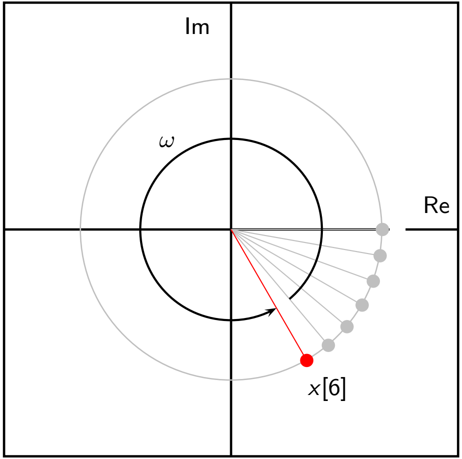
Let's go really too fast

$$\omega = 2\pi - \alpha, \quad \alpha \text{ small}$$



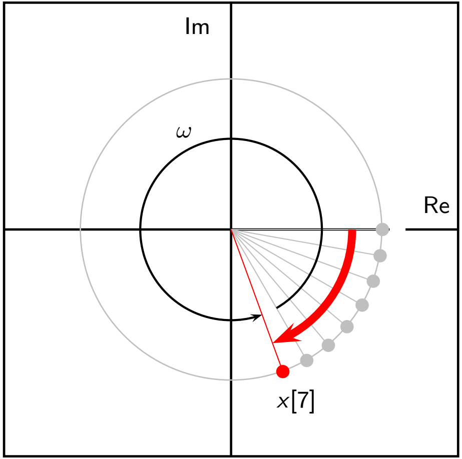
Let's go really too fast

$$\omega = 2\pi - \alpha, \quad \alpha \text{ small}$$



Let's go really too fast

$$\omega = 2\pi - \alpha, \quad \alpha \text{ small}$$



The wagonwheel effect

Digital vs physical frequency

- Discrete time:
  - n: no physical dimension (just a counter)
  - periodicity: how many samples before pattern repeats
- "Real world":
  - periodicity: how many *seconds* before pattern repeats
  - frequency measured in Hz (s^{-1})

Digital vs physical frequency

- Discrete time:
  - n: no physical dimension (just a counter)
  - periodicity: how many samples before pattern repeats
- "Real world":
  - periodicity: how many *seconds* before pattern repeats
  - frequency measured in Hz (s^{-1})

Digital vs physical frequency

- Discrete time:
  - n: no physical dimension (just a counter)
  - periodicity: how many samples before pattern repeats
- "Real world":
  - periodicity: how many *seconds* before pattern repeats
  - frequency measured in Hz (s^{-1})

Digital vs physical frequency

- Discrete time:
  - n: no physical dimension (just a counter)
  - periodicity: how many samples before pattern repeats
- "Real world":
  - periodicity: how many *seconds* before pattern repeats
  - frequency measured in Hz (s^{-1})

Digital vs physical frequency

- Discrete time:
  - n: no physical dimension (just a counter)
  - periodicity: how many samples before pattern repeats
- "Real world":
  - periodicity: how many *seconds* before pattern repeats
  - frequency measured in Hz (s^{-1})

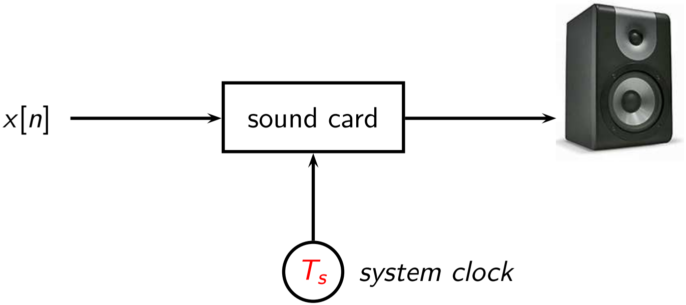
Digital vs physical frequency

- Discrete time:
  - n: no physical dimension (just a counter)
  - periodicity: how many samples before pattern repeats
- "Real world":
  - periodicity: how many *seconds* before pattern repeats
  - frequency measured in Hz (s^{-1})

How your PC plays sounds

$$x[n] \longrightarrow \boxed{\text{sound card}} \longrightarrow$$

How your PC plays sounds



Digital vs physical frequency

- set T_s, time in seconds between samples
- periodicity of M samples $\longrightarrow$ periodicity of MT_s seconds
- real world frequency:

$$f = \frac{1}{MT_s}$$

Digital vs physical frequency

- set T_s, time in seconds between samples
- periodicity of M samples $\longrightarrow$ periodicity of MT_s seconds
- real world frequency:

$$f = \frac{1}{MT_s}$$

Digital vs physical frequency

- set T_s, time in seconds between samples
- periodicity of M samples $\longrightarrow$ periodicity of MT_s seconds

- real world frequency:

$$f = \frac{1}{MT_s}$$

END OF MODULE 2.2

Digital Signal Processing

Module 2.3: the Karplus-Strong algorithm

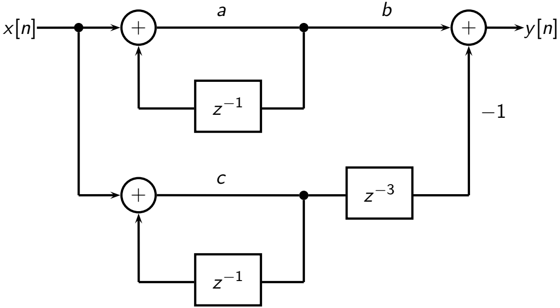
Overview:

- DSP building blocks
- moving averages and simple feedback loops
- a sound synthesizer

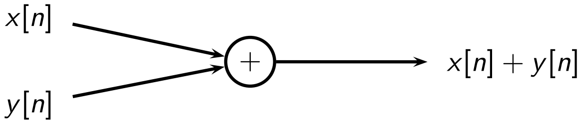
Overview:

- DSP as Lego: The fundamental building blocks
- Averages and moving averages
- Recursion: Revisiting your bank account
- Building a simple recursive synthesizer
- Examples of sounds

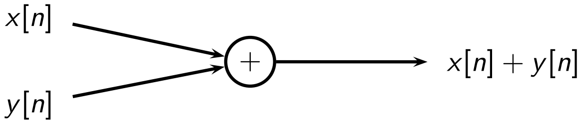
DSP as Lego

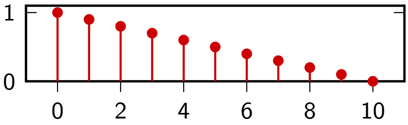


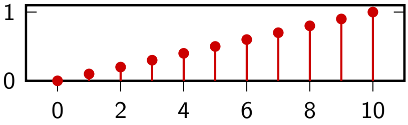
Building Blocks: Adder



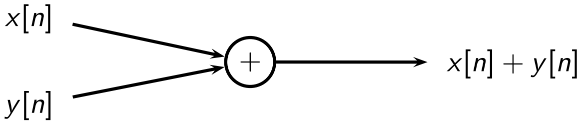
Building Blocks: Adder

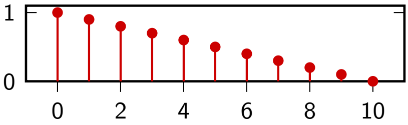


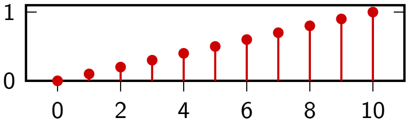


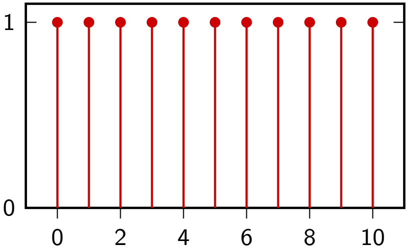


Building Blocks: Adder







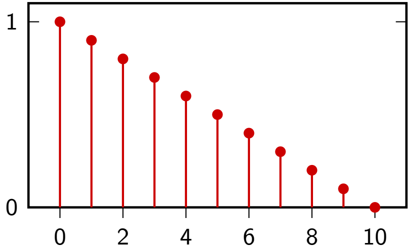


Building Blocks: Multiplier

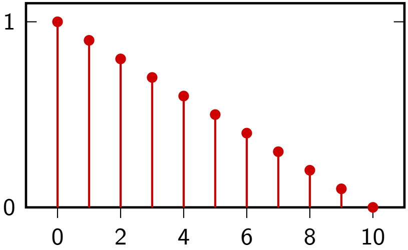
$$x[n] \xrightarrow{\quad\alpha\quad} \alpha x[n]$$

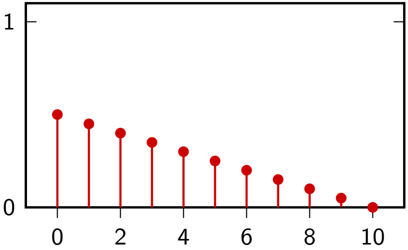
Building Blocks: Multiplier

$$x[n] \xrightarrow{\quad\alpha\quad} \alpha x[n]$$



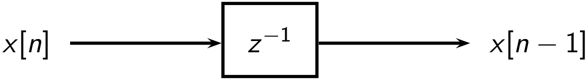
$$x[n] \xrightarrow{\alpha} \alpha x[n]$$



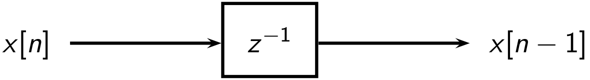


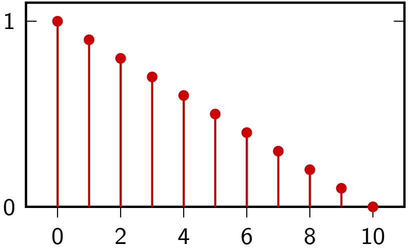
$\alpha = 0.5$

Building Blocks: Unit Delay

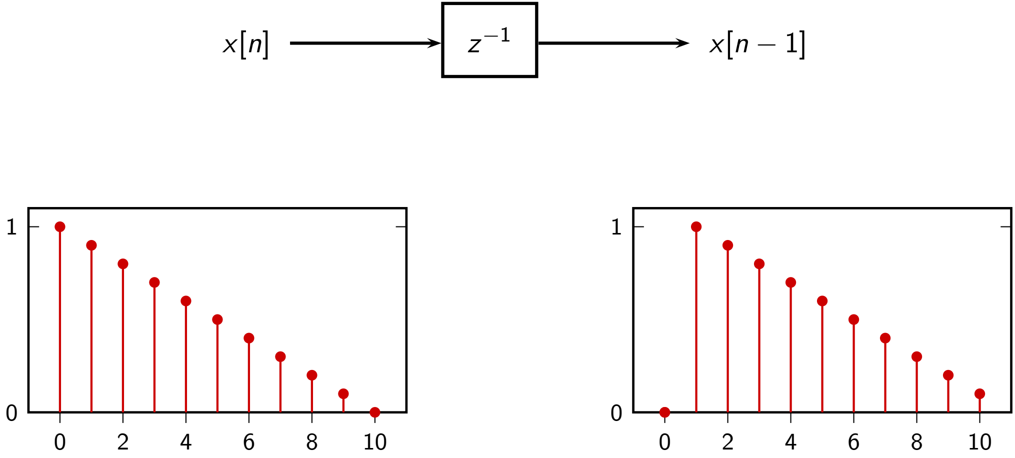


Building Blocks: Unit Delay





Building Blocks: Unit Delay

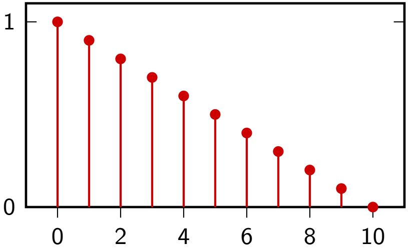


Building Blocks: Arbitrary Delay

$$x[n] \longrightarrow \boxed{z^{-N}} \longrightarrow x[n-N]$$

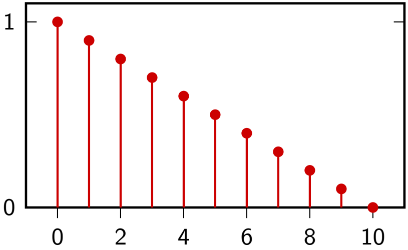
Building Blocks: Arbitrary Delay

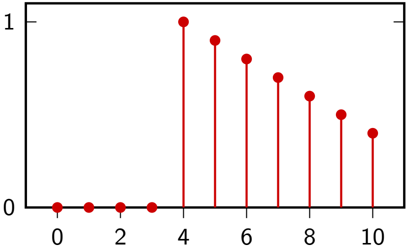
$$x[n] \longrightarrow \boxed{z^{-N}} \longrightarrow x[n-N]$$



Building Blocks: Arbitrary Delay

$$x[n] \longrightarrow \boxed{z^{-N}} \longrightarrow x[n-N]$$





$N = 4$

The 2-point Moving Average

- simple average:

$$m = \frac{a+b}{2}$$

- moving average: take a "local" average

$$y[n] = \frac{x[n] + x[n-1]}{2}$$

The 2-point Moving Average

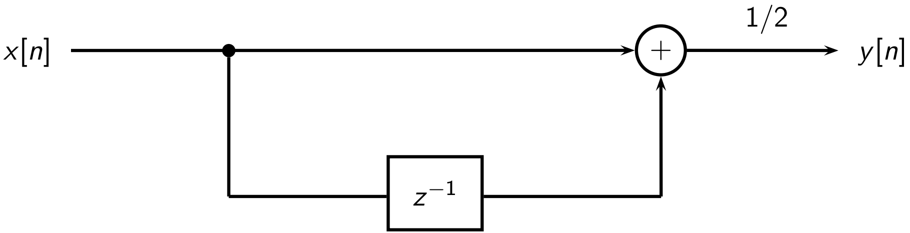
- simple average:

$$m = \frac{a+b}{2}$$

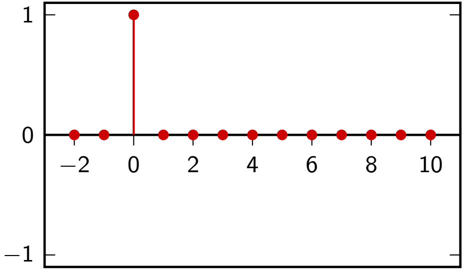
- moving average: take a "local" average

$$y[n] = \frac{x[n] + x[n-1]}{2}$$

The 2-point Moving Average Using Lego

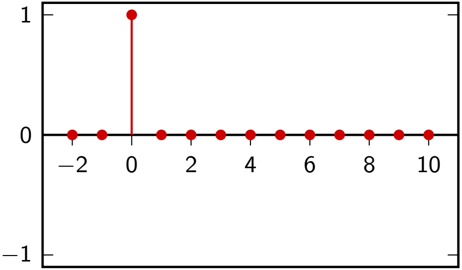


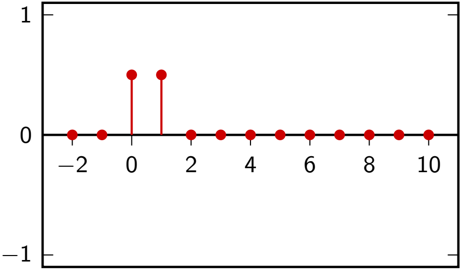
$x[n] = \delta[n]$



Let's average...

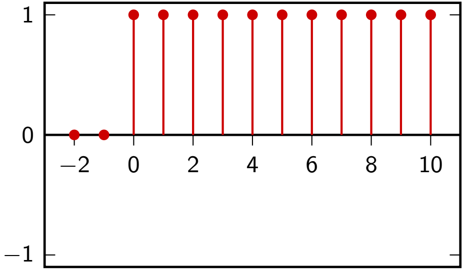
$x[n] = \delta[n]$





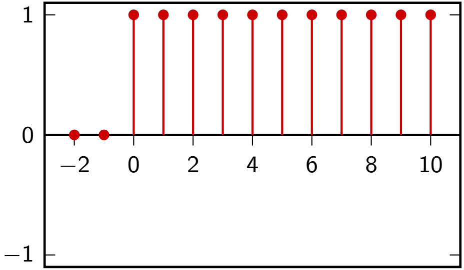
Let's average...

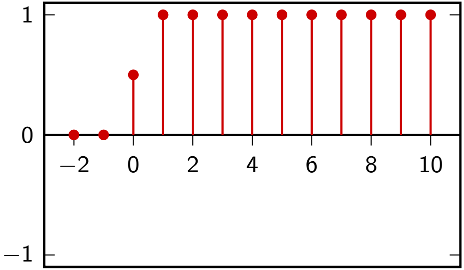
$x[n] = u[n]$



Let's average...

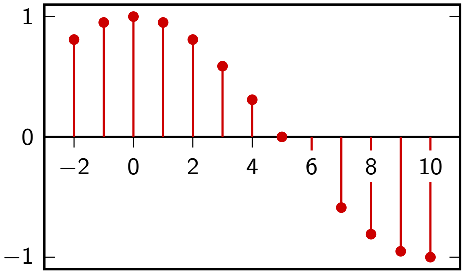
$x[n] = u[n]$



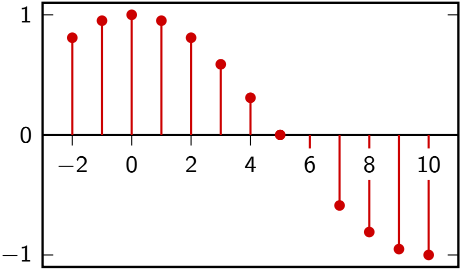


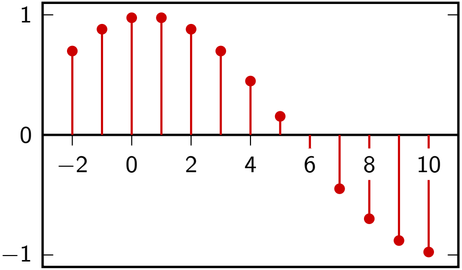
Let's average...

$$x[n] = \cos(\omega n), \quad \omega = \pi/10$$



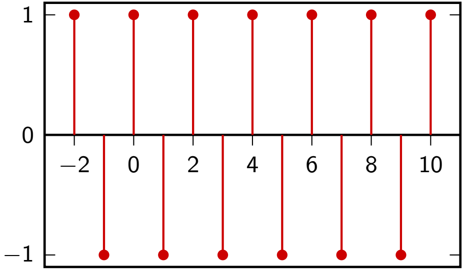
$$x[n] = \cos(\omega n), \quad \omega = \pi/10$$



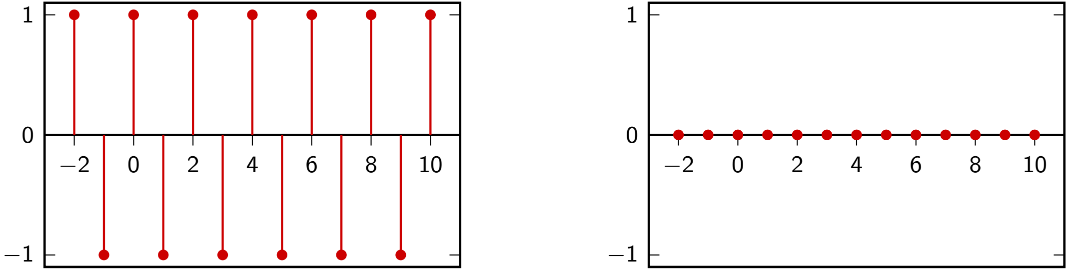


Let's average...

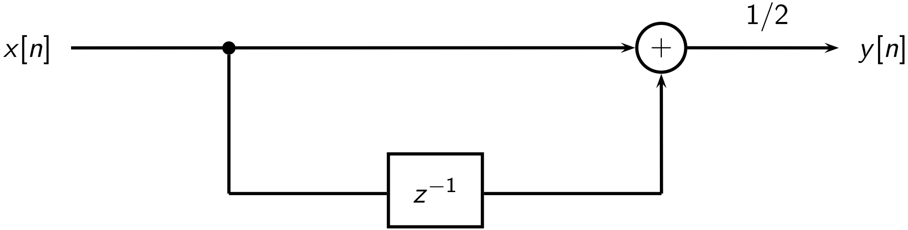
$$x[n] = \cos(\omega n), \quad \omega = \pi$$



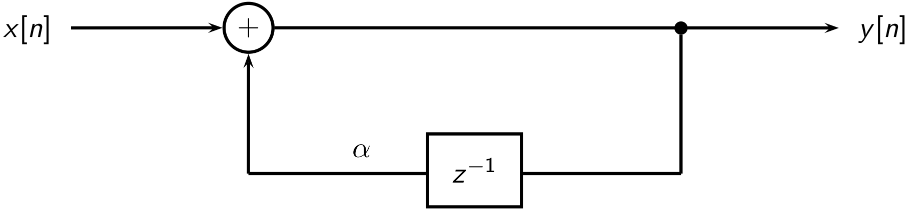
$$x[n] = \cos(\omega n), \quad \omega = \pi$$



What if we reverse the loop?



What if we reverse the loop?



A simple model for banking

A simple equation to describe compound interest:

- constant interest/borrowing rate of 5% per year
- interest accrues on Dec 31
- deposits/withdrawals during year n: $x[n]$
- balance at year n:

$$y[n] = 1.05\, y[n-1] + x[n]$$

A simple model for banking

A simple equation to describe compound interest:

- constant interest/borrowing rate of 5% per year
- interest accrues on Dec 31
- deposits/withdrawals during year n: $x[n]$
- balance at year n:

$$y[n] = 1.05\, y[n-1] + x[n]$$

A simple model for banking

A simple equation to describe compound interest:

- constant interest/borrowing rate of 5% per year
- interest accrues on Dec 31
- deposits/withdrawals during year n: $x[n]$
- balance at year n:

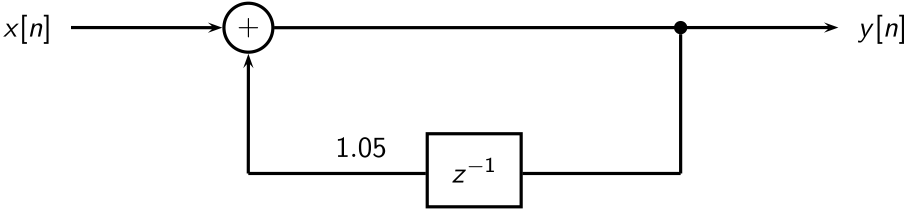
$$y[n] = 1.05\, y[n-1] + x[n]$$

A simple model for banking

A simple equation to describe compound interest:

- constant interest/borrowing rate of 5% per year
- interest accrues on Dec 31
- deposits/withdrawals during year n: $x[n]$
- balance at year n:

$$y[n] = 1.05\, y[n-1] + x[n]$$

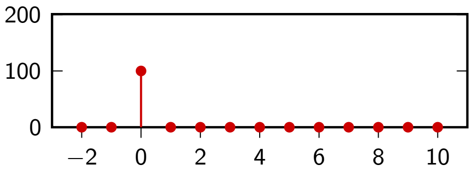


$$y[n] = 1.05\, y[n-1] + x[n]$$

Example: the one-time investment

$x[n] = 100\,\delta[n]$

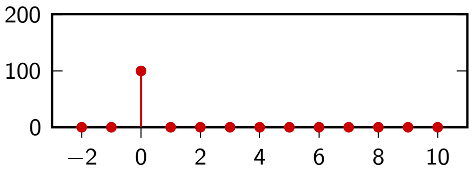
- $y[0] = 100$
- $y[1] = 105$
- $y[2] = 110.25$, $y[3] = 115.7625$ etc.
- In general: $y[n] = (1.05)^n 100\, u[n]$



Example: the one-time investment

$x[n] = 100\,\delta[n]$

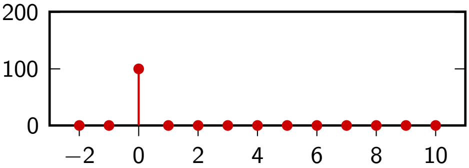
- $y[0] = 100$
- $y[1] = 105$
- $y[2] = 110.25, \ y[3] = 115.7625$ etc.
- In general: $y[n] = (1.05)^n 100\, u[n]$



Example: the one-time investment

$x[n] = 100\,\delta[n]$

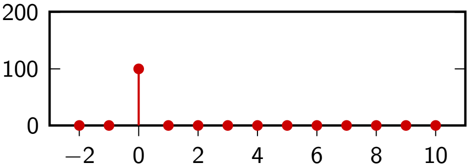
- $y[0] = 100$
- $y[1] = 105$
- $y[2] = 110.25$, $y[3] = 115.7625$ etc.
- In general: $y[n] = (1.05)^n 100\, u[n]$



Example: the one-time investment

$x[n] = 100\,\delta[n]$

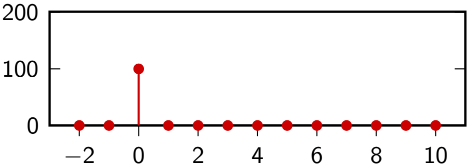
- $y[0] = 100$
- $y[1] = 105$
- $y[2] = 110.25,\ y[3] = 115.7625$ etc.
- In general: $y[n] = (1.05)^n 100\, u[n]$

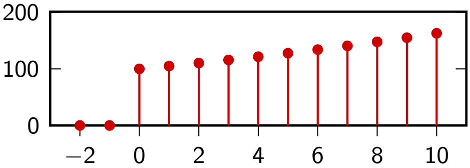


Example: the one-time investment

$x[n] = 100\,\delta[n]$

- $y[0] = 100$
- $y[1] = 105$
- $y[2] = 110.25,\ y[3] = 115.7625$ etc.
- In general: $y[n] = (1.05)^n 100\, u[n]$

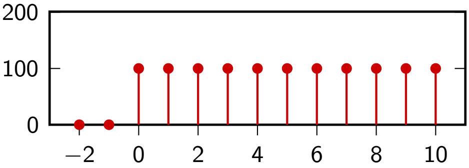




Example: the saver

$x[n] = 100\, u[n]$

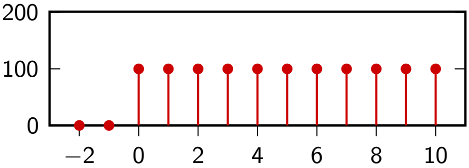
- $y[0] = 100$
- $y[1] = 205$
- $y[2] = 315.25,\ y[3] = 431.0125$ etc.
- In general: $y[n] = 2000\left((1.05)^{n+1} - 1\right) u[n]$



Example: the saver

$x[n] = 100\, u[n]$

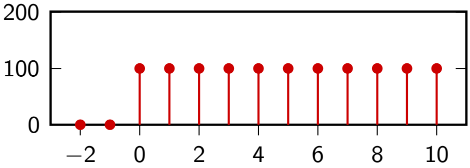
- $y[0] = 100$
- $y[1] = 205$
- $y[2] = 315.25$, $y[3] = 431.0125$ etc.
- In general: $y[n] = 2000\left((1.05)^{n+1} - 1\right) u[n]$



Example: the saver

$x[n] = 100\,u[n]$

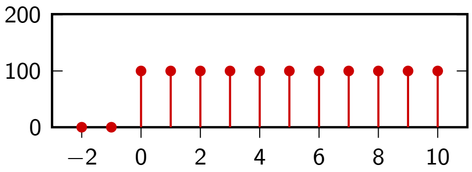
- $y[0] = 100$
- $y[1] = 205$
- $y[2] = 315.25, \; y[3] = 431.0125$ etc.
- In general: $y[n] = 2000\left((1.05)^{n+1} - 1\right) u[n]$



Example: the saver

$x[n] = 100\, u[n]$

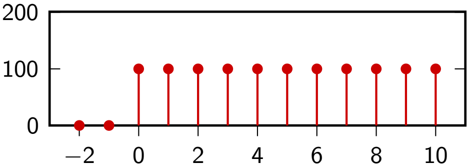
- $y[0] = 100$
- $y[1] = 205$
- $y[2] = 315.25, \; y[3] = 431.0125$ etc.
- In general: $y[n] = 2000\left((1.05)^{n+1} - 1\right) u[n]$

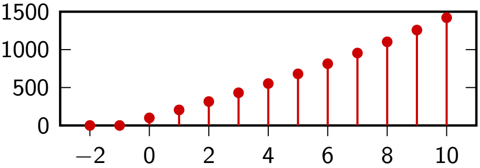


Example: the saver

$x[n] = 100\, u[n]$

- $y[0] = 100$
- $y[1] = 205$
- $y[2] = 315.25, \; y[3] = 431.0125$ etc.
- In general: $y[n] = 2000\, ((1.05)^{n+1} - 1)\, u[n]$

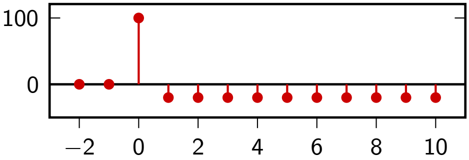




Example: The independently wealthy

$x[n] = 100\,\delta[n] - 5\,u[n-1]$

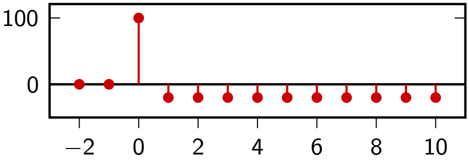
- $y[0] = 100$
- $y[1] = 100$
- $y[2] = 100$, $y[3] = 100$ etc.
- In general: $y[n] = 100\,u[n]$



Example: The independently wealthy

$x[n] = 100\,\delta[n] - 5\,u[n-1]$

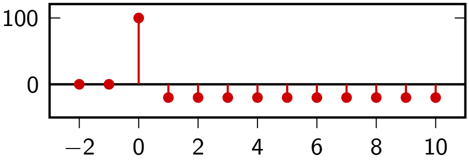
- $y[0] = 100$
- $y[1] = 100$
- $y[2] = 100, \; y[3] = 100$ etc.
- In general: $y[n] = 100\,u[n]$



Example: The independently wealthy

$x[n] = 100\,\delta[n] - 5\,u[n-1]$

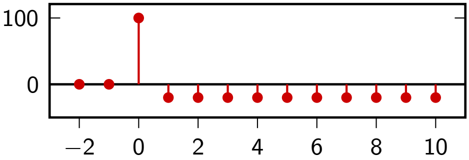
- $y[0] = 100$
- $y[1] = 100$
- $y[2] = 100,\ y[3] = 100$ etc.
- In general: $y[n] = 100\,u[n]$



Example: The independently wealthy

$x[n] = 100\,\delta[n] - 5\,u[n-1]$

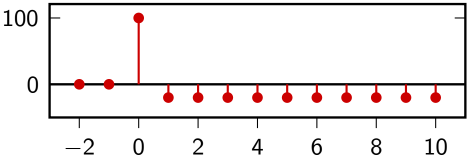
- $y[0] = 100$
- $y[1] = 100$
- $y[2] = 100,\ y[3] = 100$ etc.
- In general: $y[n] = 100\,u[n]$

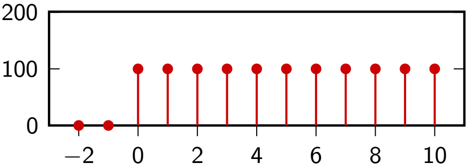


Example: The independently wealthy

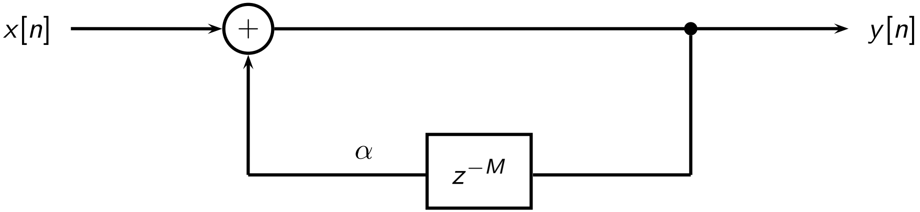
$x[n] = 100\,\delta[n] - 5\,u[n-1]$

- $y[0] = 100$
- $y[1] = 100$
- $y[2] = 100,\ y[3] = 100$ etc.
- In general: $y[n] = 100\,u[n]$





A simple generalization

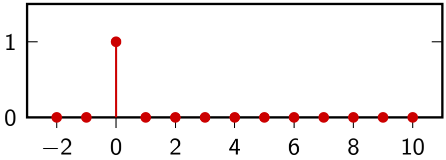


$$y[n] = \alpha\, y[n-M] + x[n]$$

Example

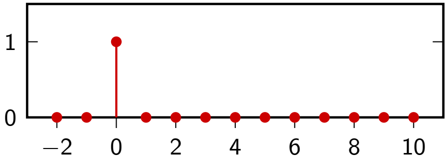
$M = 3,\ \alpha = 0.7,\ x[n] = \delta[n]$

- $y[0] = 1,\ y[1] = 0,\ y[2] = 0$
- $y[3] = 0.7,\ y[4] = 0,\ y[5] = 0$
- $y[6] = 0.7^2,\ y[7] = 0,\ y[8] = 0$, etc.



$M = 3,\ \alpha = 0.7,\ x[n] = \delta[n]$

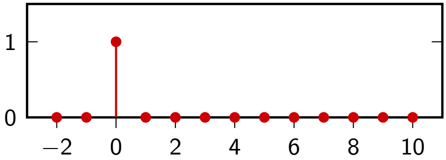
- $y[0] = 1,\ y[1] = 0,\ y[2] = 0$
- $y[3] = 0.7,\ y[4] = 0,\ y[5] = 0$
- $y[6] = 0.7^2,\ y[7] = 0,\ y[8] = 0$, etc.



Example

$M = 3,\ \alpha = 0.7,\ x[n] = \delta[n]$

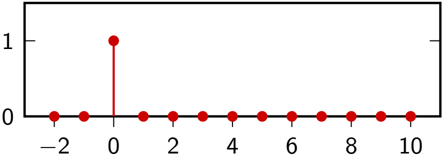
- $y[0] = 1,\ y[1] = 0,\ y[2] = 0$
- $y[3] = 0.7,\ y[4] = 0,\ y[5] = 0$
- $y[6] = 0.7^2,\ y[7] = 0,\ y[8] = 0$, etc.

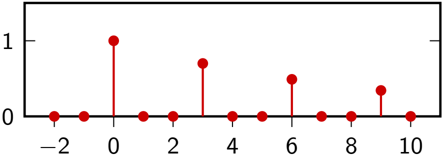


Example

$M = 3,\ \alpha = 0.7,\ x[n] = \delta[n]$

- $y[0] = 1,\ y[1] = 0,\ y[2] = 0$
- $y[3] = 0.7,\ y[4] = 0,\ y[5] = 0$
- $y[6] = 0.7^2,\ y[7] = 0,\ y[8] = 0$, etc.

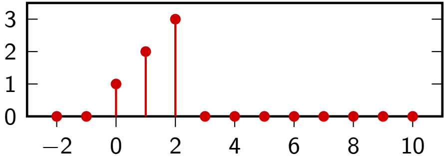


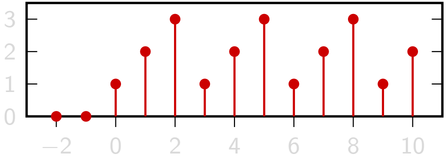


Example

$M = 3,\ \alpha = 1,\ x[n] = \delta[n] + 2\,\delta[n-1] + 3\,\delta[n-2]$

- $y[0] = 1,\ y[1] = 2,\ y[2] = 3$
- $y[3] = 1,\ y[4] = 2,\ y[5] = 3$
- $y[6] = 1,\ y[7] = 2,\ y[8] = 3$, etc.

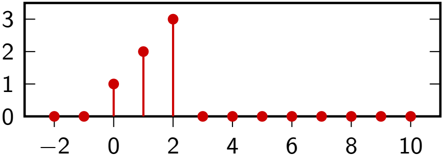


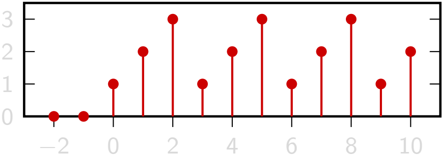


Example

$M = 3,\ \alpha = 1,\ x[n] = \delta[n] + 2\,\delta[n-1] + 3\,\delta[n-2]$

- $y[0] = 1,\ y[1] = 2,\ y[2] = 3$
- $y[3] = 1,\ y[4] = 2,\ y[5] = 3$
- $y[6] = 1,\ y[7] = 2,\ y[8] = 3$, etc.

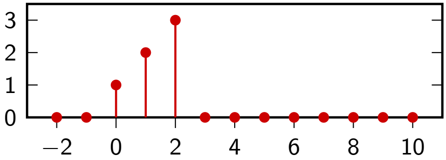


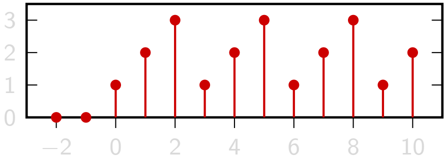


Example

$M = 3,\ \alpha = 1,\ x[n] = \delta[n] + 2\,\delta[n-1] + 3\,\delta[n-2]$

- $y[0] = 1,\ y[1] = 2,\ y[2] = 3$
- $y[3] = 1,\ y[4] = 2,\ y[5] = 3$
- $y[6] = 1,\ y[7] = 2,\ y[8] = 3$, etc.





Example

$M = 3,\ \alpha = 1,\ x[n] = \delta[n] + 2\,\delta[n-1] + 3\,\delta[n-2]$

- $y[0] = 1,\ y[1] = 2,\ y[2] = 3$
- $y[3] = 1,\ y[4] = 2,\ y[5] = 3$
- $y[6] = 1,\ y[7] = 2,\ y[8] = 3$, etc.

We can make music with that!

- build a recursion loop with a delay of M
- choose a signal $\bar{x}[n]$ that is nonzero only for $0 \leq n < M$
- choose a decay factor
- input $\bar{x}[n]$ to the system
- play the output

We can make music with that!

- build a recursion loop with a delay of M
- choose a signal $\bar{x}[n]$ that is nonzero only for $0 \leq n < M$
- choose a decay factor
- input $\bar{x}[n]$ to the system
- play the output

We can make music with that!

- build a recursion loop with a delay of M
- choose a signal $\bar{x}[n]$ that is nonzero only for $0 \leq n < M$
- choose a decay factor
- input $\bar{x}[n]$ to the system
- play the output

We can make music with that!

- build a recursion loop with a delay of M
- choose a signal $\bar{x}[n]$ that is nonzero only for $0 \leq n < M$
- choose a decay factor
- input $\bar{x}[n]$ to the system
- play the output

We can make music with that!

- build a recursion loop with a delay of M
- choose a signal $\bar{x}[n]$ that is nonzero only for $0 \leq n < M$
- choose a decay factor
- input $\bar{x}[n]$ to the system
- play the output

How do we “play” it, really?

- M-tap delay $\longrightarrow$ M-sample “periodicity”
- associate time T to sample interval
- periodic signal of frequency

$$f = \frac{1}{MT}\text{Hz}$$

- example: $T = 22.7\mu s$, $M = 100$

$$f \approx 440\text{Hz}$$

How do we "play" it, really?

- M-tap delay $\longrightarrow$ M-sample "periodicity"
- associate time T to sample interval
- periodic signal of frequency

$$f = \frac{1}{MT}\text{Hz}$$

- example: $T = 22.7\mu s$, $M = 100$

$$f \approx 440\text{Hz}$$

How do we "play" it, really?

- M-tap delay $\longrightarrow$ M-sample "periodicity"
- associate time T to sample interval
- periodic signal of frequency

$$f = \frac{1}{MT}\text{Hz}$$

- example: $T = 22.7\mu s$, $M = 100$

$$f \approx 440\text{Hz}$$

How do we "play" it, really?

- M-tap delay $\longrightarrow$ M-sample "periodicity"
- associate time T to sample interval
- periodic signal of frequency

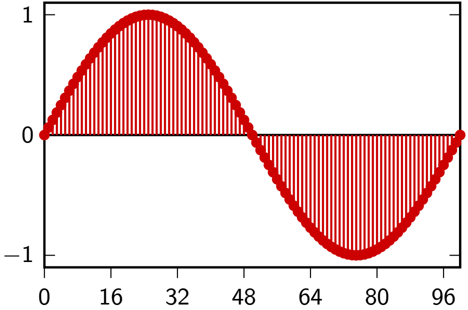
$$f = \frac{1}{MT}\text{Hz}$$

- example: $T = 22.7\mu s$, $M = 100$

$$f \approx 440\text{Hz}$$

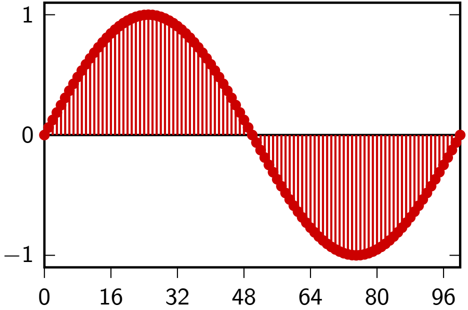
Playing a sine wave

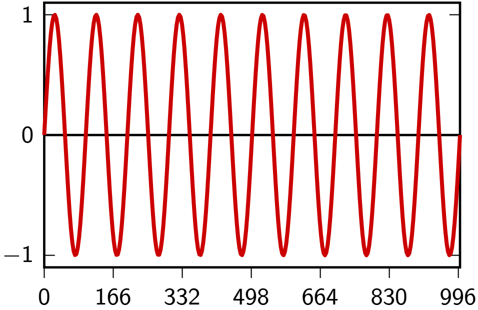
$M = 100$, $\alpha = 1$, $\bar{x}[n] = \sin(2\pi\, n/100)$ for $0 \leq n < 100$ and zero elsewhere



Playing a sine wave

$M = 100$, $\alpha = 1$, $\bar{x}[n] = \sin(2\pi\, n/100)$ for $0 \leq n < 100$ and zero elsewhere





Introducing some realism

- M controls frequency (pitch)
- α controls envelope (decay)
- $\bar{x}[n]$ controls color (timbre)

Introducing some realism

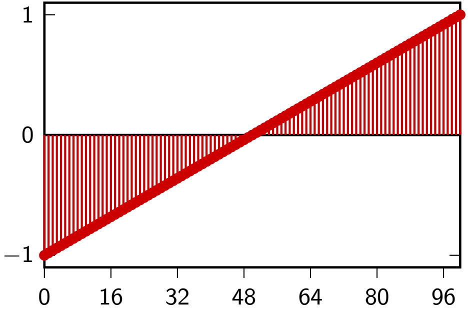
- M controls frequency (pitch)
- α controls envelope (decay)
- $\bar{x}[n]$ controls color (timbre)

Introducing some realism

- M controls frequency (pitch)
- α controls envelope (decay)
- $\bar{x}[n]$ controls color (timbre)

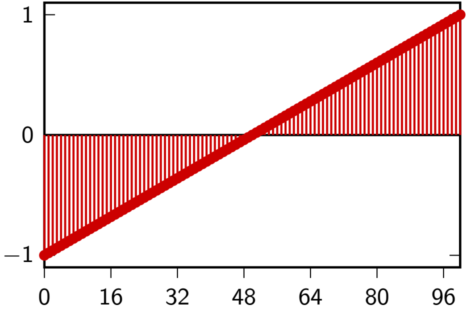
A proto-violin

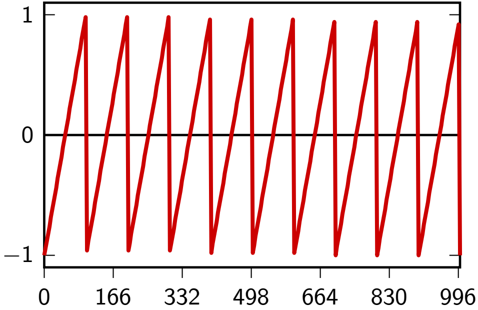
$M = 100$, $\alpha = 0.95$, $\bar{x}[n]$: zero-mean sawtooth wave between 0 and 99, zero elsewhere



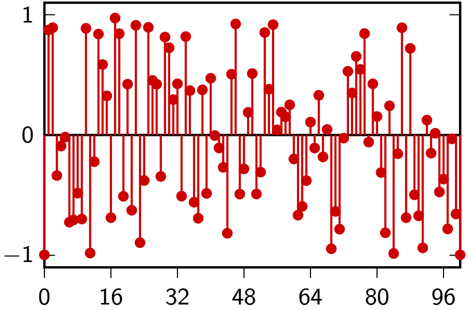
A proto-violin

$M = 100$, $\alpha = 0.95$, $\bar{x}[n]$: zero-mean sawtooth wave between 0 and 99, zero elsewhere

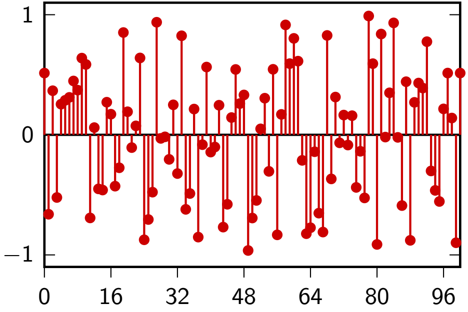


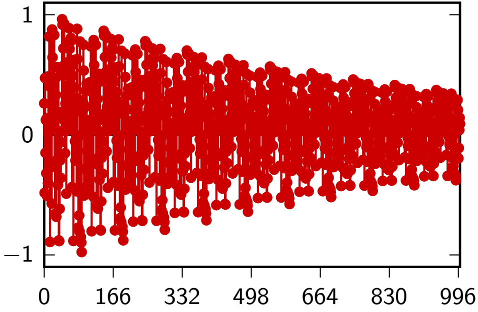


$M = 100$, $\alpha = 0.9$, $\bar{x}[n]$: 100 random values between 0 and 99, zero elsewhere



$M = 100$, $\alpha = 0.9$, $\bar{x}[n]$: 100 random values between 0 and 99, zero elsewhere





Recap

- We have seen basic elements:
  - adders
  - multipliers
  - delays
- We have seen two systems
  - moving averages
  - recursive systems
- We were able to build simple systems with interesting properties
- to understand all of this in more details we need a mathematical framework!

END OF MODULE 2.3

END OF MODULE 2