Digital Signal Processing

Solved HW for Day 11

Question 1

Consider a discrete-time sequence $x[n]$ with DTFT $X(e^{j\omega})$ and the continuous-time interpolated signal,

$$x_0(t) = \sum_{n=-\infty}^{\infty} x[n]\text{rect}\,(t - n) \tag{1}$$

i.e. the signal interpolated with a zero-centered zero-order hold and $T_s = 1$ sec.

- Express $X_0(j\Omega)$ (the spectrum of $x_0(t)$) in terms of $X(e^{j\omega})$.
- Compare $X_0(j\Omega)$ to $X(j\Omega)$. We can look at $X(j\Omega)$ as the Fourier transform of the signal obtained from the sinc interpolation of $x[n]$ (always with $T_s = 1$):

$$x(t) = \sum_{n\in\mathbb{Z}} x[n]\mathrm{sinc}(t - n) \tag{2}$$

Comment on the result: you should point out two major problems.

Question 1, second part

- The signal $x(t)$ can be obtained from the zero-order hold interpolation $x_0(t)$ as $x(t) = x_0(t) * g(t)$ for some filter $g(t)$. Sketch the frequency response of $g(t)$
- Propose two solutions (one in the continuous-time domain, and another in the discrete-time domain) to eliminate or attenuate the distortion due to the zero-order hold. Discuss the advantages and disadvantages of each.

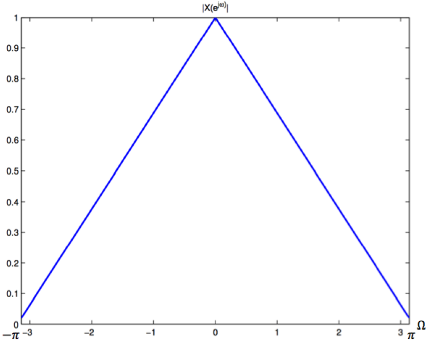
Solution of question 1

Q: Express $X_0(j\Omega)$ (the spectrum of $x_0(t)$) in terms of $X(e^{j\omega})$.

$$
\begin{aligned}
X_0(j\Omega) &= \int_{-\infty}^{\infty} x_0(t) e^{-j\Omega t} dt \\
&= \int_{-\infty}^{\infty} \sum_{n=-\infty}^{\infty} x[n] \mathrm{rect}\,(t-n) e^{-j\Omega t} dt \\
&= \sum_{n=-\infty}^{\infty} x[n] \int_{-\infty}^{\infty} \mathrm{rect}\,(t-n) e^{-j\Omega t} dt \\
&= \sum_{n=-\infty}^{\infty} x[n] e^{-j\Omega n} \int_{-1/2}^{1/2} e^{-j\Omega \tau} d\tau = \frac{\sin(\Omega/2)}{\Omega/2} X(e^{j\Omega}) \\
&= \mathrm{sinc}(\Omega/2\pi)\, X(e^{j\Omega}).
\end{aligned}
$$

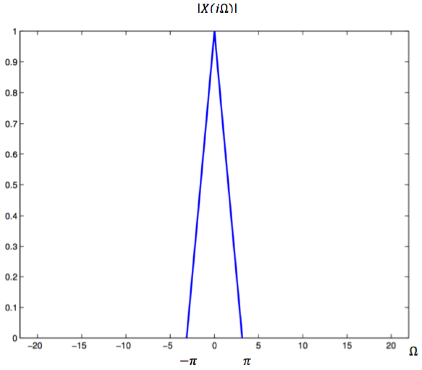
Q: Compare $X_0(j\Omega)$ to $X(j\Omega)$ and comment the result

Take for instance a discrete-time signal with a triangular spectrum such as



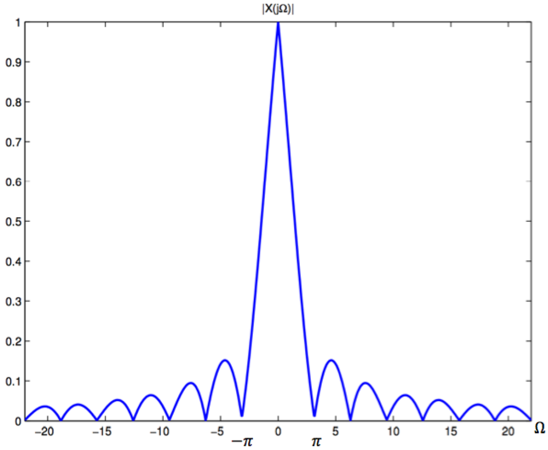
Solution of question 1

We know that the sinc interpolation will give us a continuous-time signal which is strictly bandlimited to the $[-\Omega_N, \Omega_N]$ interval (with $\Omega_N = \pi/T_s = \pi$) and whose shape is exactly triangular, like this one:



Solution of question 1

Conversely, the spectrum of the continuous-time signal interpolated by the zero-order hold looks like this:



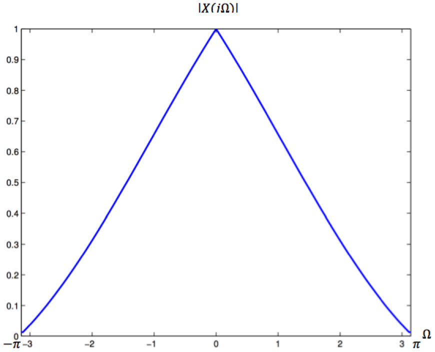
Solution of question 1

There are two main problems in the zero-order hold interpolation as compared to the sinc interpolation:

- The zero-order hold interpolation is NOT bandlimited: the 2π-periodic replicas of the digital spectrum leak through in the continuous-time signal as high frequency components. This is due to the sidelobes of the interpolation function in the frequency domain (rect in time $\leftrightarrow$ sinc in frequency) and it represents an undesirable high-frequency content which is typical of all local interpolation schemes.
- There is a distortion in the main portion of the spectrum (that between $-\Omega_N$ and Ω_N, with $\Omega_N = \pi$) due to the non-flat frequency response of the interpolation function. It can be seen in the zoom in version of the main portion of the spectrum in the next slide.

Solution of question 1

Q: The signal $x(t)$ can be obtained from the zero-order hold interpolation $x_0(t)$ as $x(t) = x_0(t) * g(t)$ for some filter $g(t)$. Sketch the frequency response of $g(t)$.



Solution of question 1

- Observe that $X(j\Omega)$ can be expressed as

$$X(j\Omega) = \begin{cases} X(e^{j\Omega}) & \text{if } \Omega \in [-\pi, \pi] \\ 0 & \text{otherwise,} \end{cases}$$

  where $X(e^{j\Omega})$ is the DTFT of the sequence $x[n]$ evaluated at $\omega = \Omega$.

- So

$$X(j\Omega) = X(e^{j\Omega})\mathrm{rect}\left(\frac{\Omega}{2\pi}\right) = X_0(j\Omega)\mathrm{sinc}^{-1}\left(\frac{\Omega}{2\pi}\right)\mathrm{rect}\left(\frac{\Omega}{2\pi}\right).$$

- Hence

$$G(j\Omega) = \mathrm{sinc}^{-1}\left(\frac{\Omega}{2\pi}\right)\mathrm{rect}\left(\frac{\Omega}{2\pi}\right),$$

Solution of question 1

Q: Propose two solutions (one in the continuous-time domain, and another in the discrete-time domain) to eliminate or attenuate the distortion due to the zero-order hold. Discuss the advantages and disadvantages of each.

- A first solution is to compensate for the distortion introduced by $G(j\Omega)$ in the discrete-time domain. This is equivalent to pre-filtering $x[n]$ with a discrete-time filter of magnitude $1/G(e^{j\Omega})$. The advantages of this method is that digital filters such as this one are very easy to design and that the filtering can be done in the discrete-time domain. The disadvantage is that this approach does not eliminate or attenuate the high frequency leakage outside of the baseband.
- Alternatively, one can cascade the interpolator with an analog lowpass filter to eliminate the leakage. The disadvantage is that it is hard to design an analog lowpass which can also compensate for the in-band distortion introduced by $G(j\Omega)$; such a filter will also introduce unavoidable phase distortion (no analog filter has linear phase).

Question 2: A bizarre interpolator

Consider the local interpolation scheme of the previous exercise but assume that the characteristic of the interpolator is the following:

$$I(t) = \begin{cases} 1 - 2|t| & \text{for} \quad |t| \leq 1/2 \\ 0 & \quad\quad \text{otherwise} \end{cases} \tag{3}$$

This is a triangular characteristic with the same support as the zero-order hold. If we pick an interpolation interval T_s and interpolate a given discrete-time signal $x[n]$ with $I(t)$, we obtain a continuous-time signal:

$$x(t) = \sum_n x[n] I\left(\frac{t - nT_s}{T_s}\right) \tag{4}$$

Assume that the spectrum of x[n] between $-\pi$ and π is

$$X(e^{j\omega}) = \begin{cases} 1 & \text{for} \quad |\omega| \leq 2\pi/3 \\ 0 & \quad\quad \text{otherwise} \end{cases} \tag{5}$$

(with the obvious 2π-periodicity over the entire frequency axis).

Exercise 2: A bizarre interpolator, second part

- Compute and sketch the Fourier transform $I(j\Omega)$ of the interpolating function $I(t)$ – Recall that the triangular function can be expresses as the convolution of rect(2t) with itself.
- Sketch the Fourier transform $X(j\Omega)$ of the interpolated signal $x(t)$; in particular, clearly mark the Nyquist frequency $\Omega_N = \pi/T_s$.
- The use of I(t) instead of a sinc interpolator introduces two types of errors: briefly describe them.
- To eliminate the error in the baseband $[-\Omega_N, \Omega_N]$ we can pre-filter the signal $x[n]$ before interpolating with $I(t)$. Write the frequency response of the discrete-time filter $H(e^{j\omega})$.

Solution of question 2

Q: Compute and sketch the Fourier transform $I(j\Omega)$ of the interpolating function $I(t)$ – Recall that the triangular function can be expresses as the convolution of rect(2t) with itself.

- From last exercise, we know that the Fourier Transform of rect(t) is

$$\frac{\sin(\Omega/2)}{\Omega/2} = \mathrm{sinc}(\Omega/2\pi)$$

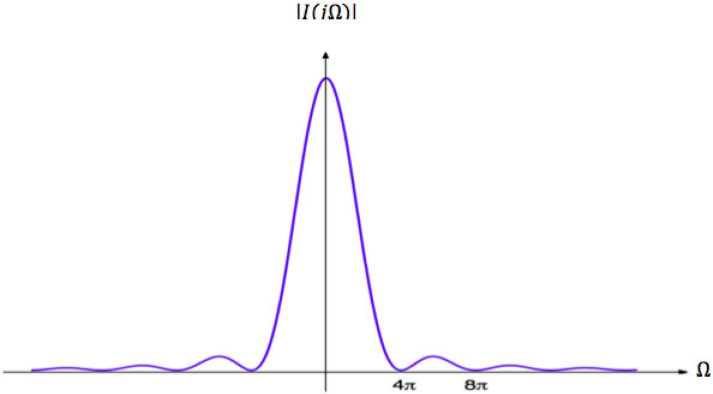
- In our case,

$$I(t) = 2\ \mathrm{rect}(2t) * \mathrm{rect}(2t)$$

- so

$$I(j\Omega) = \left(2 \cdot \frac{1}{2}\frac{\sin(\Omega/4)}{\Omega/4}\right)^2 = \frac{1}{2}\mathrm{sinc}^2(\Omega/4\pi)$$

Solution of question 2



Solution of question 2

Q: Sketch the Fourier transform $X(j\Omega)$ of the interpolated signal $x(t)$; in particular, clearly mark the Nyquist frequency $\Omega_N = \pi/T_s$.

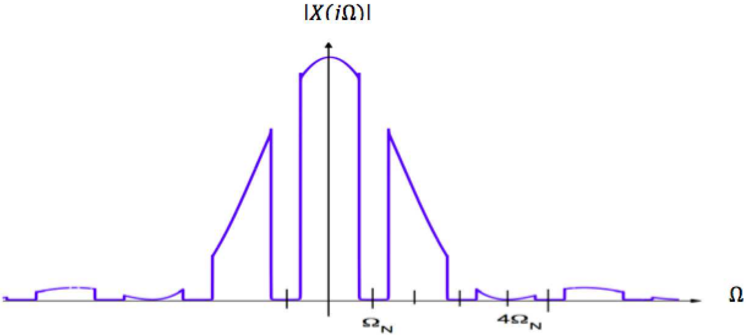
$$x(t) = \sum_n x[n] I(\frac{t - nT_s}{T_s})$$

Then,

$$\begin{aligned} X(j\Omega) = & \quad \textstyle\sum_n x[n] \mathrm{FT}\{I(\frac{t-nT_s}{T_s})\} = \\ & \textstyle\sum_n x[n] e^{-j\Omega n} \frac{T_s}{4} \mathrm{sinc}^2(\frac{\Omega T_s}{4\pi}) = \\ & \frac{T_s}{4} X(e^{j\Omega}) \mathrm{sinc}^2(\frac{\Omega T_s}{4\pi}) \end{aligned}$$

So the Fourier transform of the interpolated signal is composed of the products of two parts (recall that, as usual, $\Omega_N = \pi / T_S$):

- The $2\Omega_N$-periodic spectrum $X(e^(j\pi\Omega/\Omega_N))$
- The Fourier transform of the interpolating function.



Solution of question 2

Q: The use of I(t) instead of a sinc interpolator introduces two types of errors: briefly describe them.

There are two types of error, in-band and out-of-band:

- **In – band**: The spectrum between $[-\Omega_N, \Omega_N]$ (the baseband) is distorted by the non-flat response of the interpolating function over the baseband.
- **Out – of – band**:The periodic copies of $X(e^(j\pi\Omega/\Omega_N))$ outside of $[-\Omega_N, \Omega_N]$ are not eliminated by the interpolation filter, since it is not an ideal lowpass.

Q: Write the frequency response of the discrete-time filter $H(e^{j\omega})$.

We need to undo the linear distortion introduced by the nonflat response of the interpolation filter in the baseband. The idea is to have a modified spectrum $H(e^{j\omega})X(e^{j\omega})$ so that, in the $[-\Omega_N, \Omega_N]$, we have

$$X(j\Omega) = X(e^{j\Omega T_s}).$$

If we use $H(e^{j\omega})X(e^{j\omega})$ in the interpolation formula, we have

$$X(j\Omega) = \frac{T_s}{4} H(e^{j\Omega T_s}) X(e^{j\Omega T_s}) \,\mathrm{sinc}^2\left(\frac{\Omega T_s}{4\pi}\right)$$

so that

$$H(e^{j\Omega T_s}) = [\frac{T_s}{4}\mathrm{sinc}^2\left(\frac{\Omega T_s}{4\pi}\right)]^{-1}.$$

Solution of question 2

Therefore, the frequency response of the digital filter will be

$$H(e^{j\omega}) = \frac{4}{T_s}\text{sinc}^{-2}\left(\frac{\omega}{4\pi}\right), \qquad -\pi \leq \omega \leq \pi$$

prolonged by 2π-periodicity over the entire frequency axis.

Question 3: Sampling and Interpolation for Bandlimited Vectors

In the lecture we saw how to sample and interpolate π-bandlimited functions. In this problem we will do the same but, instead, with band limited vectors.

- A vector $\mathbf{x} \in \mathbb{C}^M$ is called bandlimited when there exists $k_0 \in \{0, 1, \ldots, M-1\}$ such that its DFT coefficient sequence X satisfies $X_k = 0$ for all k with $|k - \frac{M}{2}| > \frac{k_0 - 1}{2}$. The smallest such k_0 is called the bandwidth of $\mathbf{x}$.
- A vector in $\mathbb{C}^M$ that is not bandlimited is called full band. The set of vectors in $\mathbb{C}^M$ with bandwidth at most k_0 is a subspace.
- For $\mathbf{x}$ in such a bandlimited subspace, find a linear mapping $\Phi : \mathbb{C}^{k_0} \to \mathbb{C}^M$ (i.e., a basis) so that the system described by $\Phi\Phi^*$ achieves perfect recovery $\hat{\mathbf{x}} = \mathbf{x}$ (i.e. $\Phi\Phi^*$ equals the identity matrix).

Solution of question 3

- Call our bandlimited subspace S. If we find a orthogonal basis Φ such that $S = \mathcal{R}(\Phi)$, sampling followed by interpolation $\Phi\Phi^*$ will lead to perfect recovery.

- Since all the $X_k = 0$ for $|k - \frac{M}{2}| < \frac{(K_0-1)}{2}$, we build our Φ^* by taking the DFT matrix W and removing the rows whose indices satisfy $|k - \frac{M}{2}| < \frac{(K_0-1)}{2}$. We call this matrix $\widehat{W}$,

$$\Phi^* = \widehat{W}.$$

Solution of question 3

- For the interpolation we use the IDFT matrix with the columns pruned according to the indices satisfying the aforementioned relationship,

$$\Phi = \widehat{W}^{-1} = \widehat{W}^*.$$

- Note now that we can also multiply Φ^* by any unitary matrix to get a plausible sampling operator. That is $U\Phi^*$ and ΦU^* are still ideally matched since $UU^* = U^*U = I$.