Digital Signal Processing

Solved HW for Day 9

Question 1: Pole-zero plot and stability

Consider a causal LTI system with the following transfer function:

$$H(z) = \frac{3 + 4.5z^{-1}}{1 + 1.5z^{-1}} - \frac{2}{1 - 0.5z^{-1}}$$

Sketch the pole-zero plot of the transfer function and specify its region of convergence. Is the system stable?

Solution of question 1

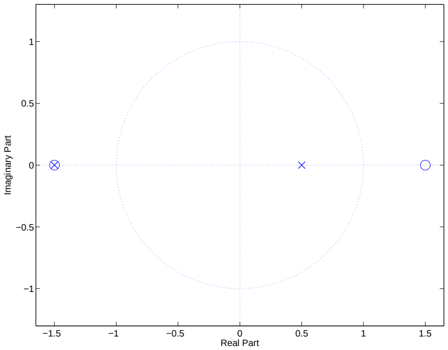
We have to factorize the filter expression to make explicit the numerator and denominator roots:

$$H(z) = \frac{3 + 4.5z^{-1}}{1 + 1.5z^{-1}} - \frac{2}{1 - 0.5z^{-1}} = \frac{(z - 1.5)(z + 1.5)}{(z - 0.5)(z + 1.5)}$$

In this way we see that the zeros of this system are in $z_{01} = 1.5$ and $z_{02} = -1.5$ and the poles in $z_{p1} = 0.5$ and $z_{p2} = -1.5$.

Question 1: Pole-zero plot and stability-2

Now we can draw the pole-zero plot.:



As our system is causal, the ROC extends outward from the outermost pole. In this case, the unit circle is not include in the ROC, so the system is not stable.

Question 2: Stability

Consider a causal discrete system represented by the following difference equation:

$$y[n] - 3.25y[n-1] + 0.75y[n-2] = x[n-1] + 3x[n-2].$$

- Compute the transfer function and check the stability of this system both analytically and graphically.
- If the input signal is $x[n] = \delta[n] - 3\delta[n-1]$, compute the z-transform of the output signal and discuss the stability.
- Take an arbitrary input signal that does not cancel the unstable pole of the transfer function. Compute the z-transform of the output signal and discuss the stability.

Solution of question 2

Q: Compute the transfer function and check the stability of this system both analytically and graphically.

The transfer function of the system is given by:

$$Y(z)(1 - 3.25z^{-1} + 0.75z^{-2}) = X(z)(z^{-1} + 3z^{-2}),$$

and we obtain

$$H(z) = \frac{Y(z)}{X(z)} = \frac{z^{-1} + 3z^{-2}}{1 - 3.25z^{-1} + 0.75z^{-2}} =$$

$$= \frac{z^{-1}(1 + 3z^{-1})}{(1 - 0.25z^{-1})(1 - 3z^{-1})} = \frac{z + 3}{(z - 0.25)(z - 3)}.$$

Solution of question 2

Since the system is causal, the convergence region is $|z| > 3$. We can see that there is the pole $z = 3$ that is out of the unit circle and therefore the system is unstable (Figure 6).

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>> zplane ([0 1 3], [1 -3.25 0.75])
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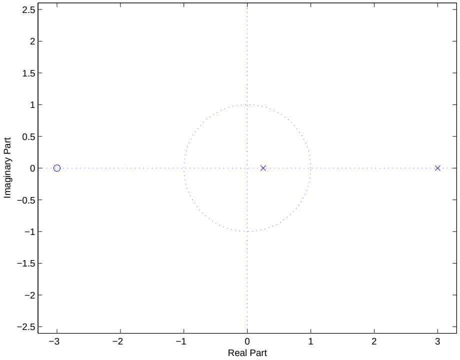


Solution of question 2

Q: If the input signal is $x[n] = \delta[n] - 3\delta[n-1]$, compute the z-transform of the output signal and discuss the stability.

Z-transform of the output signal is:

$$Y(z) \quad = \quad H(z)X(z) = \frac{z^{-1}(1+3z^{-1})}{(1-0.25z^{-1})(1-3z^{-1})}(1-3z^{-1})$$

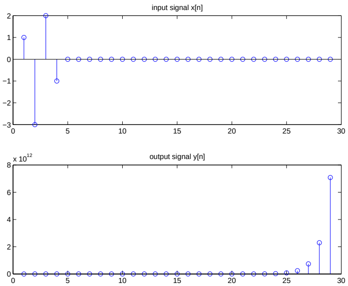
$$= \frac{z^{-1}+3z^{-2}}{1-0.25z^{-1}}.$$

From $Y(z)$ we can see that the unstable pole $z=3$ is canceled and only the pole $z=0.25$ of $Y(z)$ is left. Since the system is causal, even from the unstable system we can get the stable output if the unstable pole is canceled by the input signal.

Solution of question 2

Q: Take an arbitrary input signal that does not cancel the unstable pole of the transfer function. Compute the z-transform of the output signal and discuss the stability.

```
>> x=[1 -3 2 -1 zeros(1,25)];
>> y=filter( [0 1 3], [1 -3.25 0.5], x);
>> subplot(211), stem(x), subplot(212), stem(y);
```



Question 3: Properties of the z-transform

Let $x[n]$ be a discrete-time sequence and $X(z)$ its corresponding z-transform with appropriate ROC.

- Prove that the following relation holds:

$$nx[n] \stackrel{Z}{\longleftrightarrow} -z\frac{d}{dz}X(z).$$

- Show that

$$(n+1)\alpha^n u[n] \stackrel{Z}{\longleftrightarrow} \frac{1}{(1-\alpha z^{-1})^2}, \qquad |z| > |\alpha|.$$

- Suppose that the above expression corresponds to the impulse response of an LTI system. What can you say about the causality of such a system? About its stability?

- Let $\alpha = 0.8$, what is the spectral behavior of the corresponding filter? What if $\alpha = -0.8$?

Solution of question 3

Q: Prove that the following relation holds:

$$nx[n] \overset{Z}{\longleftrightarrow} -z\frac{d}{dz}X(z).$$

Let $X(z) = \Sigma_n x[n]z^{-n}$. We have that

$$\begin{aligned}\frac{d}{dz}X(z) &= \frac{d}{dz}\left(\Sigma_n x[n]z^{-n}\right) \\ &= \Sigma_n(-n)x[n]z^{-n-1} \\ &= -z^{-1}\Sigma_n nx[n]z^{-n}\end{aligned}$$

and the relation follows directly.

Solution of question 3

Q: Show that

$$(n+1)\alpha^n u[n] \stackrel{Z}{\longleftrightarrow} \frac{1}{(1-\alpha z^{-1})^2}, \qquad |z| > |\alpha|.$$

We have that

$$\alpha^n u[n] \stackrel{Z}{\longleftrightarrow} \frac{1}{1-\alpha z^{-1}}.$$

Using the previous result, we find

$$n\alpha^n u[n] \stackrel{Z}{\longleftrightarrow} -z\frac{d}{dz}\left(\frac{1}{1-\alpha z^{-1}}\right) = \frac{\alpha z^{-1}}{(1-\alpha z^{-1})^2}.$$

Solution of question 3

Thus,

$$(n+1)\alpha^{n+1}u[n+1] \stackrel{\mathcal{Z}}{\longleftrightarrow} z\frac{\alpha z^{-1}}{(1-\alpha z^{-1})^2}$$

and

$$(n+1)\alpha^{n}u[n+1] \stackrel{\mathcal{Z}}{\longleftrightarrow} \frac{1}{(1-\alpha z^{-1})^2}.$$

The relation follows by noticing that

$$(n+1)\alpha^{n}u[n+1] = (n+1)\alpha^{n}u[n]$$

since when $n=-1$ both sides are equal to zero.

Question 3: Properties of the z-transform-3

Q: Suppose that the above expression corresponds to the impulse response of an LTI system. What can you say about the causality of such a system? About its stability?

The system is causal since the ROC corresponds to the outside of a circle of radius α (or equivalently since the impulse response is zero when $n < 0$). The system is stable when the unit circle lies inside the ROC, i.e. when $|\alpha| \leq 1$.

Q: Let $\alpha = 0.8$, what is the spectral behavior of the corresponding filter? What if $\alpha = -0.8$?

When $\alpha = 0.8$, the angular frequency of the pole is $\omega = 0$. Thus the filter is lowpass. When $\alpha = -0.8$, $\omega = \pi$ and the filter is highpass.

Question 4: Interleaving sequences

Consider two two-sided sequences $h[n]$ and $g[n]$ and consider a third sequence $x[n]$ which is built by interleaving the values of $h[n]$ and $g[n]$:

$$x[n] = \ldots, h[-3], g[-3], h[-2], g[-2], h[-1], g[-1], h[0], g[0], h[1], g[1], h[2], g[2], h[3], g[3], \ldots$$

with $x[0] = h[0]$.

- Express the z-transform of $x[n]$ in terms of the z-transforms of $h[n]$ and $g[n]$.
- Assume that the ROC of $H(z)$ is $0.64 < |z| < 4$ and that the ROC of $G(z)$ is $0.25 < |z| < 9$. What is the ROC of $X(z)$?

Solution of question 4

Q: Express the z-transform of $x[n]$ in terms of the z-transforms of $h[n]$ and $g[n]$.

We have that:

$$\begin{aligned} X(z) &= \sum_{n=-\infty}^{\infty} h[n]z^{-2n} + g[n]z^{-(2n+1)} \\ &= H(z^2) + z^{-1}G(z^2) \end{aligned}$$

Solution of question 4

Q: Assume that the ROC of $H(z)$ is $0.64 < |z| < 4$ and that the ROC of $G(z)$ is $0.25 < |z| < 9$. What is the ROC of $X(z)$?

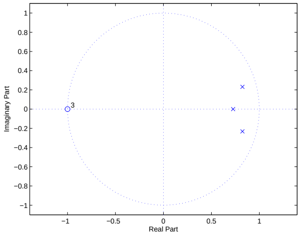
- The ROC is determined by the zeros of the transform.
- Since the sequence is two sided, the ROC is a ring bounded by two poles z_L and z_R such that $|z_L| < |z_R|$ and no other pole has magnitude between $|z_L|$ and $|z_R|$.
- Consider $H(z)$; if z_0 is a pole of $H(z)$, $H(z^2)$ will have two poles at $\pm z_0^{1/2}$; however, the square root preserves the monotonicity of the magnitude and therefore no new poles will appear between the circles $|z| = \sqrt{|z_L|}$ and $|z| = \sqrt{|z_R|}$.
- Therefore the ROC for $H(z^2)$ is the ring $\sqrt{|z_L|} < |z| < \sqrt{|z_R|}$. The ROC of the sum $H(z^2) + z^{-1}G(z^2)$ is the intersection of the ROCs, and so

$$\text{ROC } = 0.8 < |z| < 2.$$

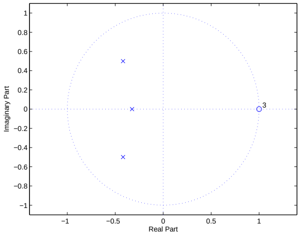
Question 5: Transfer function, zeros and poles

The following figure shows the zeros and poles of three different filters with the unit circle for reference. Each zero is represented with a 'o' and each pole with a 'x' on the plot. Multiple zeros and poles are indicated by the multiplicity number shown to the upper right of the zero or pole.

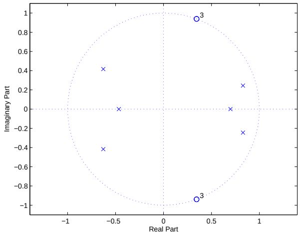
Sketch the magnitude of each frequency response and determine the type of filter.



(a) Diagram 1



(b) Diagram 2



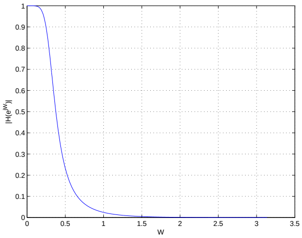
(c) Diagram 3

Figure : Zeros and Poles Diagrams

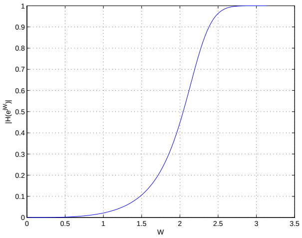
Solution of question 5

Q: Sketch the magnitude of each frequency response

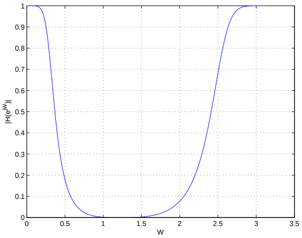
To obtain the frequency response of a filter, we analyze the z-transform on the unit circle, that is, in $z = e^{j\omega}$. Figure 18 shows the exact magnitude of each frequency response:



(a) Diagram 1



(b) Diagram 2



(c) Diagram 3

Q: Determine the type of each filter

1. The first filter is a low-pass filter. Note that there are three poles located in low frequency (near $\omega = 0$), while there is a zero located in high frequency ($\omega = \pi$).
2. The second filter is just the opposite. The zero is located in low frequency, while the influence of the three poles is maximum in high frequency ($\omega = \pi$). Therefore, it is a high-pass filter.
3. In the third system, there are poles which affect low and high frequency and two zeros close to $w = \pi/2$. Therefore, this system is a stop-band filter.