Digital Signal Processing

Solved HW for Day 7

Question 1: LTI systems

Consider the transformation $\mathcal{H}\{x[n]\} = nx[n]$. Does $\mathcal{H}$ define an LTI system?

Solution of question 1

- The system is not time-invariant. To see this consider the following signals:

$$x[n] = \delta[n]$$
$$y[n] = \delta[n-1]$$

We have $\mathcal{H}\{x[n]\} = w[n] = 0$ and, clearly, it is $y[n] = x[n-1]$.
However,

$$\mathcal{H}\{y[n]\} = \delta[n-1] \neq w[n-1] = 0.$$

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Question 2: Convolution

Let $x[n]$ be a discrete-time sequence defined as

$$x[n] = \begin{cases} M - n & 0 \leq n \leq M, \\ M + n & -M \leq n \leq 0, \\ 0 & \text{otherwise.} \end{cases}$$

for some odd integer M.

- Show that $x[n]$ can be expressed as the convolution of two discrete-time sequences $x_1[n]$ and $x_2[n]$.
- Using the results found in (a), compute the DTFT of $x[n]$.

Solution of question 2

Q: Show that $x[n]$ can be expressed as the convolution of two discrete-time sequences $x_1[n]$ and $x_2[n]$.

- $x[n]$ can be written as the convolution of $x_1[n]$ and $x_2[n]$ defined as

$$x_1[n] = x_2[n] = \begin{cases} 1 & -(M-1)/2 \leq n \leq (M-1)/2 \\ 0 & \text{otherwise.} \end{cases}$$

$$= u[n + (M-1)/2] - u[n - (M+1)/2].$$

Solution of question 2

Then,

$$
\begin{aligned}
x_1[n] * x_2[n] &= \Sigma_k x_1[k] x_2[n-k] \\
&\overset{(1)}{=} \Sigma_k x_1[k] x_1[k-n] \\
&\overset{(2)}{=} x[n]
\end{aligned}
$$

- (1) follows from the fact that $x_1[n] = x_2[n]$ and the symmetry of $x_1[n]$.
- (2) follows by noticing that the sum corresponds to the size of the overlapping area between $x_1[k]$ and its n-shifted version $x_1[k-n]$.

Solution of question 2

Q: Using the previous result, compute the DTFT of $x[n]$

$$
\begin{aligned}
X_1(e^{j\omega}) &\overset{(1)}{=} \left(\frac{1}{1-e^{-j\omega}} + \frac{1}{2}\tilde{\delta}(\omega)\right)\left(e^{j\omega(M-1)/2} - e^{-j\omega(M+1)/2}\right) \\
&\overset{(2)}{=} \frac{e^{j\omega(M-1)/2} - e^{-j\omega(M+1)/2}}{1-e^{-j\omega}} = \frac{e^{-j\omega/2}(e^{j\omega M/2} - e^{-j\omega M/2})}{e^{-j\omega/2}(e^{j\omega/2} - e^{-j\omega/2})} \\
&= \frac{\sin(\omega M/2)}{\sin(\omega/2)}
\end{aligned}
$$

- (1) follows from the DTFT of $u[n]$
- (2) follows from $e^{jw(M-1)/2}\tilde{\delta}(w) = e^{-jw(M+1)/2}\tilde{\delta}(w) = \tilde{\delta}(w)$.

Solution of question 2

- Now, using the convolution theorem, we can write

$$\begin{aligned} X(e^{jw}) &= X_1(e^{jw})X_2(e^{jw}) \\ &= X_1(e^{jw})X_1(e^{jw}) \\ &= \left(\frac{sin(\omega M/2)}{sin(\omega/2)}\right)^2 . \end{aligned}$$

Question 3: Impulse response, part a

The impulse response of an LTI system is shown in the following figure:

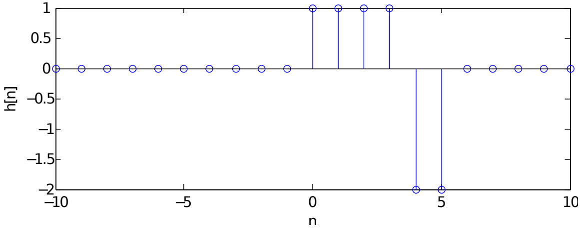
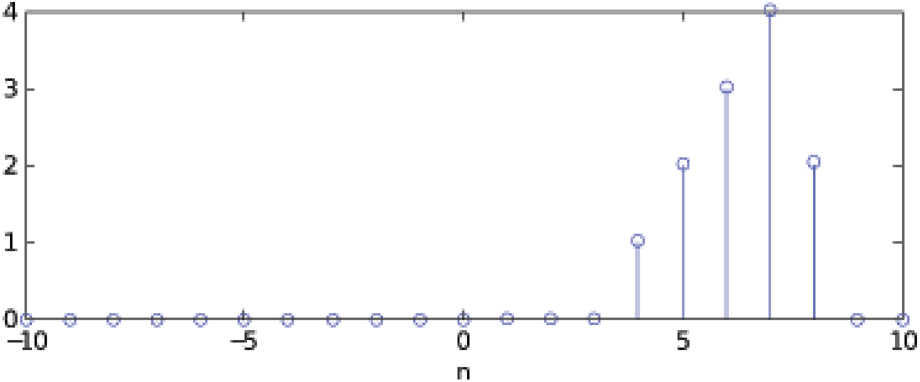


Figure: Impulse response for exercise 3-a

Q: Determine and carefully sketch the response of this system to the input $x[n] = u[n-4]$.

Solution of question 3, part a



Question 3: Impulse response, part b

Calculate the impulse response of the following system.

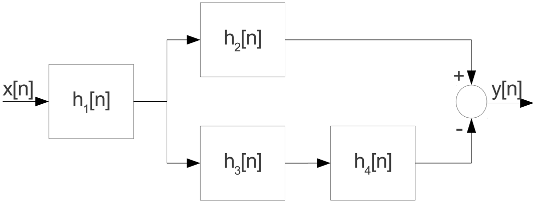


Figure: Impulse response block schema for exercise 3-b

Question 3: Impulse response, part b

The impulse responses of the separate blocks are:

- $h_1[n] = 3(-1)^n(\frac{1}{4})^n u[n-2]$
- $h_2[n] = h_3[n] = u[n+2]$
- $h_4[n] = \delta[n-1]$

Determine system's BIBO stability and causality.

Solution of question 3, part b

Q: Calculate the impulse response of the system.

- We have:

$$
\begin{aligned}
h[n] &= h_1[n] * (h_2[n] - h_3[n] * h_4[n]) \\
&= h_1[n] * (u[n+2] - u[n+2] * \delta[n-1]) \\
&= h_1[n] * (u[n+2] - u[n+1]) \\
&= h_1[n] * \delta[n+2] \\
&= h_1[n+2] \\
&= 3(-1)^n (\frac{1}{4})^{n+2} u[n].
\end{aligned}
$$

Solution of question 3, part b

Q: Determine system's BIBO stability and causality.

- A discrete system is BIBO stable if the impulse response is absolutely summable. We have:

$$\sum_{n=-\infty}^{\infty} |h[n]| = 3\frac{1}{1-1/4} - 3\left(1+\frac{1}{4}\right) = \frac{1}{4},$$

which means the system is BIBO stable.

- The system is causal because $h[n] = 0$ for $n < 0$.

Question 4: System properties

Let $x[n]$ be a signal. Consider the following systems with output $y[n]$.

- $y[n] = x[-n]$,
- $y[n] = e^{-j\omega n}x[n]$,
- $y[n] = \sum_{k=n-n_0}^{n+n_0} x[k]$,
- $y[n] = ny[n-1] + x[n]$, such that if $x[n] = 0$ for $n < n_0$, then $y[n] = 0$ for $n < n_0$.

Determine if such systems are: linear, time invariant, stable (BIBO) and causal or anti-causal. Also, compute the impulse response of each system.

Solution of question 4

First of all note that you can always compute the impulse response of a system, even if the system is not LTI. In the latter case, however, the impulse response will NOT characterize the system, i.e. the output to a generic input will have to be computed explicitly and not as the convolution of the input with the impulse response.

Q: $y[n] = x[-n]$

$\mathcal{H}$ reverses the time axis, i.e. it flips the values of the input sequence across $n = 0$.

- $\mathcal{H}$ is linear:

$$\mathcal{H}\{ax_1[n] + bx_2[n]\} = ax_1[-n] + bx_2[-n] = a\mathcal{H}\{x_1[n]\} + b\mathcal{H}\{x_2[n]\}.$$

Solution of question 4

- $\mathcal{H}$ is NOT time invariant: the easier way to see that is to transform the impulse $\delta[n]$, which leaves the input unchanged. Now shift the impulse by one to the RIGHT and transform $\delta[n-1]$: $\mathcal{H}\{\delta[n-1]\} = \delta[-n-1] = \delta[n+1]$; time-reversing this signal therefore moves the single nonzero value from $n = 1$ to $n = -1$ which is equivalent to shifting the previous output by one to the LEFT! More formally:

$$\mathcal{H}\{x[n-n_0]\} = x[-n-n_0] \neq y[n-n_0].$$

- $\mathcal{H}$ is BIBO stable since it does not change the values of the input.
- $\mathcal{H}$ is not causal. Again, consider transforming $\delta[n-1]$: this creates nonzero values in the output for $n < 0$.
- As we saw before $\mathcal{H}\{\delta[n]\} = h[n] = \delta[n]$.

Solution of question 4

Q: $y[n] = e^{-j\omega n} x[n]$

- $\mathcal{H}$ is linear:

$$\mathcal{H}\{ax_1[n] + bx_2[n]\} = e^{-j\omega n}(ax_1[n] + bx_2[n]) = a\mathcal{H}\{x_1[n]\} + b\mathcal{H}\{x_2[n]\}.$$

- $\mathcal{H}$ is NOT time invariant except in the trivial case when $\omega = 0$. (In general, when a function of the index n appears in the coefficients of the difference equation describing the system, you no longer have a CCDE and therefore the system in time-variant.)

$$\mathcal{H}\{x[n - n_0]\} = e^{-j\omega n} x[n - n_0] = e^{j\omega n_0} y[n - n_0].$$

- $\mathcal{H}$ is BIBO stable:

$$|x[n]| \leq M \Rightarrow |\mathcal{H}\{x[n]\}| = |x[n]| \leq M.$$

- $\mathcal{H}$ is causal.
- $\mathcal{H}\{\delta[n]\} = h[n] = \delta[n]$.

Solution of question 4

Q: $y[n] = \sum_{k=n-n_0}^{n+n_0} x[k]$

- $\mathcal{H}$ is linear:

$$\mathcal{H}\{ax_1[n] + bx_2[n]\} = \sum_{k=n-n_0}^{n+n_0} (ax_1[k] + bx_2[k]) = a\mathcal{H}\{x_1[n]\} + b\mathcal{H}\{x_2[n]\}.$$

- $\mathcal{H}$ is time invariant:

$$\mathcal{H}\{x[n-n_0]\} = \sum_{k=n-n_0}^{n+n_0} x[k-n_0] = \sum_{k=n-2n_0}^{n} x[k] = y[n-n_0].$$

Solution of question 4

- $\mathcal{H}$ is BIBO stable:

$$|x[n]| \leq M \Rightarrow \mathcal{H}\{x[n]\} \leq |2n_0 + 1|M.$$

- $\mathcal{H}$ is not causal.
- The system impulse response is

$$h[n] = \begin{cases} 1 & \text{if } |n| \leq |n_0|, \\ 0 & \text{otherwise.} \end{cases}$$

Solution of question 4

Q: $y[n] = ny[n-1] + x[n]$ with zero initial conditions.

- Linearity can be established simply by noticing that the output is a *linear* combination of the input and previous outputs. If we want a more formal demonstration we can invoke the zero initial conditions (all is zero for $n < 0$) and write for a composite input:

$$
\begin{aligned}
y[0] &= ax_1[0] + bx_2[0] = ay_1[0] + by_2[0] \\
y[1] &= y[0] + ax_1[1] + bx_2[1] = ay_1[0] + by_2[0] + ax_1[1] + bx_2[1] \\
&= a(y_1[0] + x_1[1]) + b(y_2[0] + x_2[1]) = ay_1[1] + by_2[1] \\
y[2] &= 2y[1] + ax_1[2] + bx_2[2] = 2(ay_1[1] + by_2[1]) + ax_1[2] + bx_2[2] \\
&= a(2y_1[1] + x_2[1]) + b(2y_2[2] + x_2[2]) = ay_1[2] + by_2[2] \\
y[3] &= \ldots
\end{aligned}
$$

Solution of question 4

- the system is NOT time invariant, as we could guess by noticing that the difference equation describing it is not constant-coefficient. For a formal proof, consider the impulse response:

$$h[n] = 0 \text{ for } n < 0$$

$$h[0] = \delta[n] = 1$$

$$h[1] = 1 \cdot 1 + 0 = 1$$

$$h[2] = 2 \cdot 1 + 0 = 2$$

$$h[3] = 3 \cdot 2 + 0 = 6 \ldots$$

$$h[n] = n!u[n]$$

Solution of question 4

Now consider the input to $x[n] = \delta[n-1]$:

$$y[n] = 0 \text{ for } n < 0$$

$$y[0] = 0$$

$$y[1] = 1 \cdot 0 + \delta[0] = 1$$

$$y[2] = 2 \cdot 1 + 0 = 2$$

$$y[3] = 3 \cdot 2 + 0 = 6 \ldots$$

$$y[n] = n!u[n] - \delta[n] \neq h[n-1]$$

Solution of question 4

- $\mathcal{H}$ is time invariant: it is easy to check that $\mathcal{H}\{\delta[n-1]\} = h[n-1]$.
- The system is clearly unstable, since the response to the delta sequence is a factorially growing sequence.
- $\mathcal{H}$ is causal.