Digital Signal Processing

Module 9: Digital Communication Systems

Module Overview:

- Module 9.1: The analog channel
- Module 9.2: Meeting the bandwidth constraint
- Module 9.3: Meeting the power constraint
- Module 9.4: Modulation and demodulation
- Module 9.5: Receiver design
- Module 9.6: ADSL

Digital Signal Processing

Module 9.1: Digital Communication Systems

Overview:

- The many incarnations of a signal
- Analog channel constraints
- Satisfying the constraints

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Digital data throughputs

- Transatlantic cable:
  - 1866: 8 words per minute ($\approx$5 bps)
  - 1956: AT&T, coax, 48 voice channels ($\approx$3Mbps)
  - 2005: Alcatel Tera10, fiber, 8.4 Tbps (8.4×10^{12} bps)
  - 2012: fiber, 60 Tbps
- Voiceband modems
  - 1950s: Bell 202, 1200 bps
  - 1990s: V90, 56Kbps
  - 2008: ADSL2+, 24Mbps

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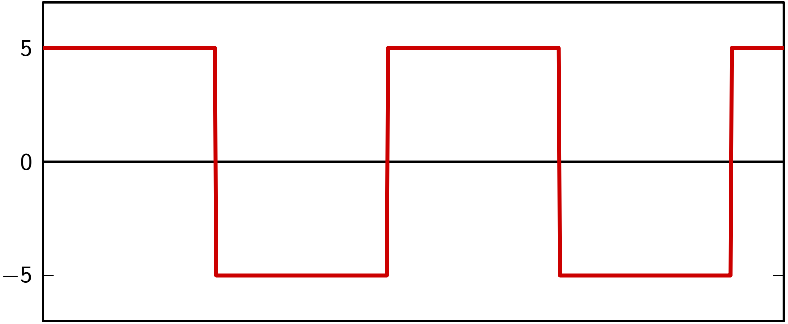
Success factors for digital communications

1) power of the DSP paradigm:

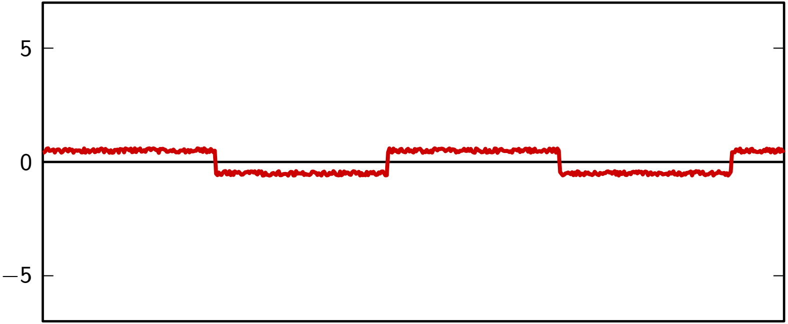
- integers are "easy" to regenerate
- good phase control
- adaptive algorithms

Regenerating signals



$x(t)$

Regenerating signals



$$x(t)/G + \sigma(t)$$

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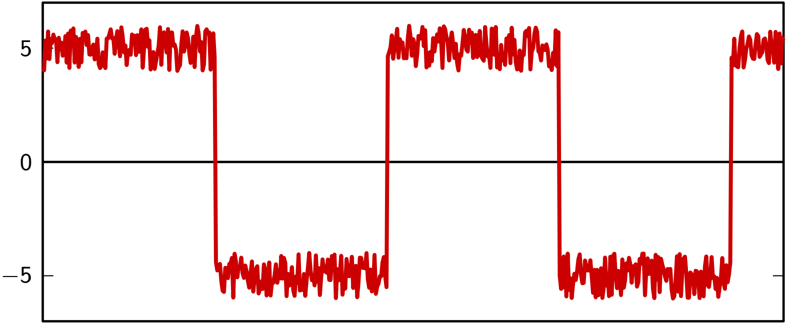

$$G[x(t)/G + \sigma(t)] = x(t) + G\sigma(t)$$

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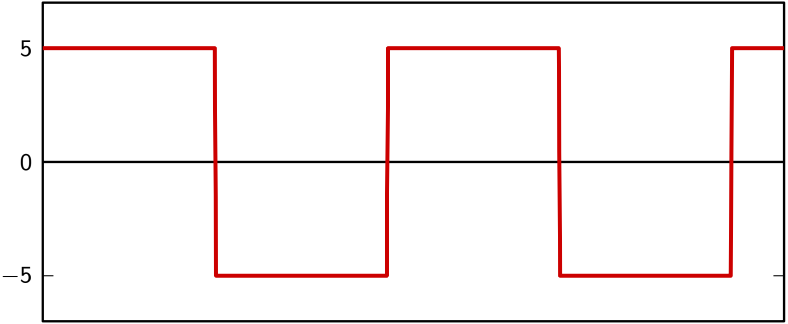


$$\hat{x}_1(t) = G\,\mathrm{sgn}[x(t) + \sigma(t)]$$

Success factors for digital communications

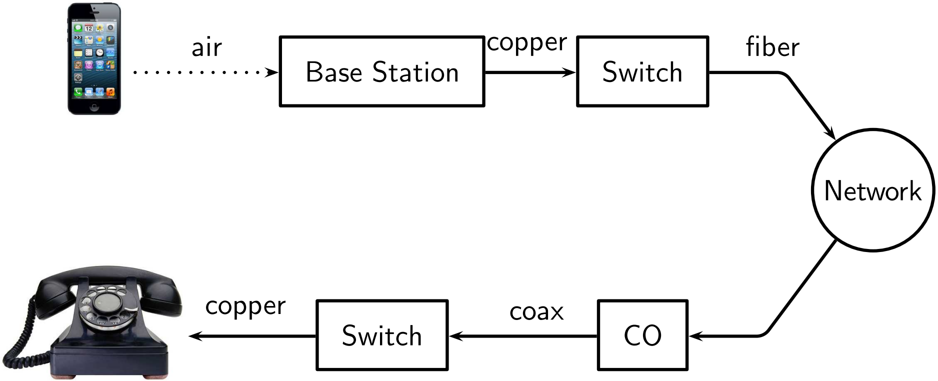
2) algorithmic nature of DSP is a perfect match with information theory:

- JPEG's entropy coding
- CD's and DVD's error correction
- trellis-coded modulation and Viterbi decoding

3) hardware advancement

- miniaturization
- general-purpose platforms
- power efficiency

The many incarnations of a conversation



The analog channel

unescapable "limits" of physical channels:

- bandwidth constraint
- power constraint

both constraints will affect the final *capacity* of the channel

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The analog channel's capacity

maximum amount of information that can be reliably delivered over a channel
(bits per second)

Bandwidth vs capacity

simple thought experiment:

- we want to transmit information encoded as a sequence of digital samples over a continuous-time channel
- we interpolate the sequence of samples with a period T_s
- if we make T_s small we can send more info per unit of time...
- ... but the bandwidth of the signal will grow as $1/T_s$

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Power and capacity

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- all channels introduce noise; at the receiver we have to “guess” what was transmitted
- suppose noise variance is 1
- suppose we are transmitting integers between 1 and 10: lots of guessing errors
- transmit only odd numbers: fewer errors but less information

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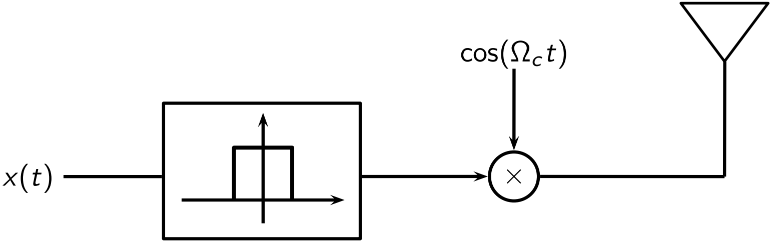

$x(t)$
$\cos(\Omega_c t)$

Example: the AM radio channel

- from 530kHz to 1.7MHz
- each channel is 8KHz
- power limited by law:
  - daytime/nighttime
  - interference
  - health hazards

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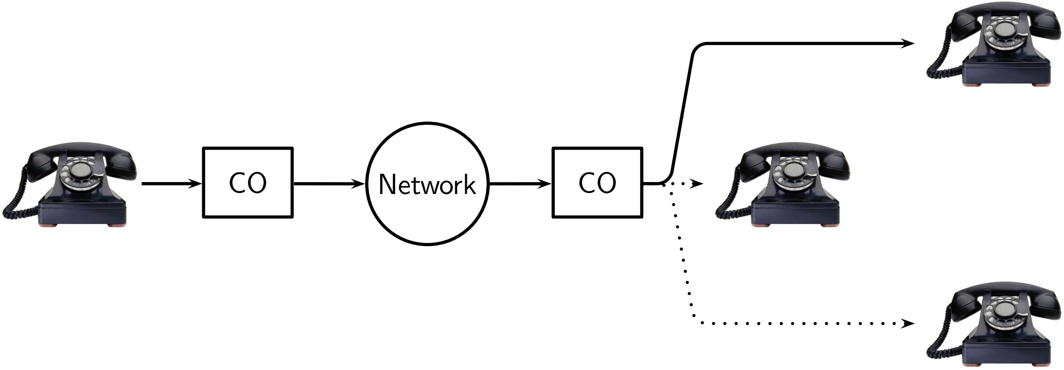

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- one channel from around 300Hz to around 3000Hz
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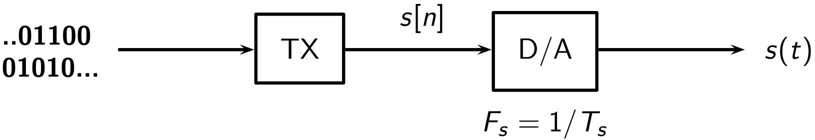
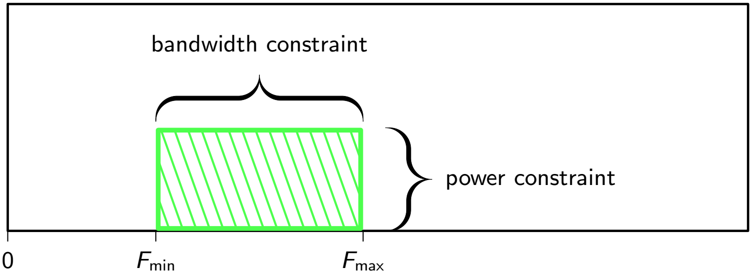
The all-digital paradigm

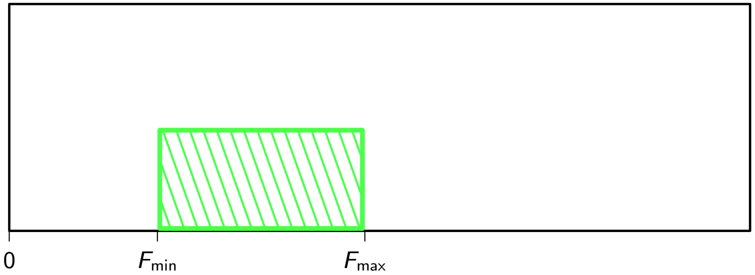
keep everything digital until we hit the physical channel



Let's look at the channel constraints



Converting the specs to a digital design



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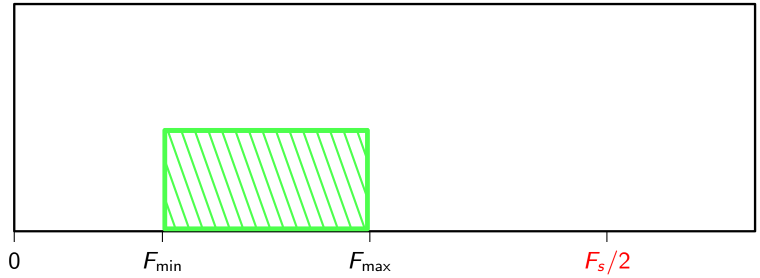

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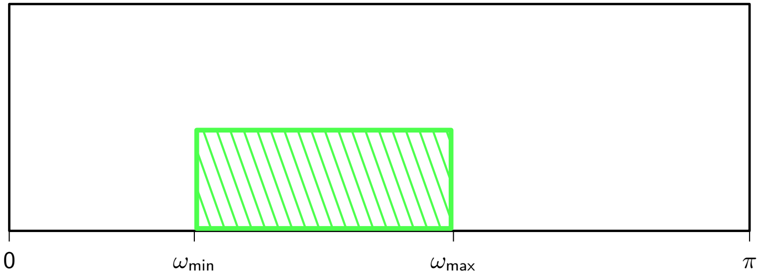

Transmitter design

some working hypotheses:

- convert the bitstream into a sequence of symbols $a[n]$ via a mapper
- model $a[n]$ as a white random sequence (add a scrambler on the bitstream to make sure)
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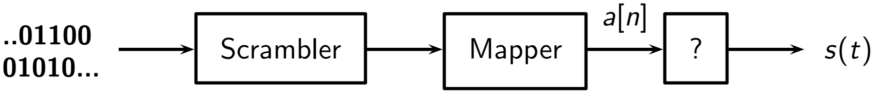
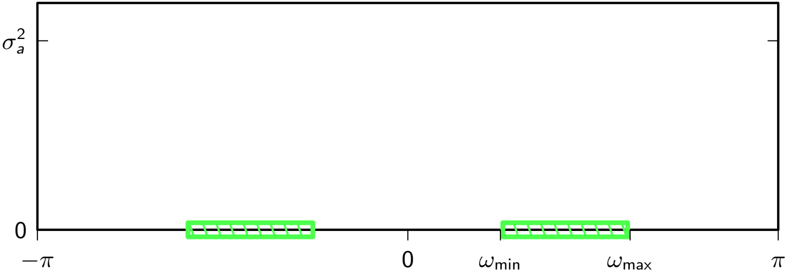


First problem: the bandwidth constraint

$$P_a(e^{j\omega}) = \sigma_a^2$$



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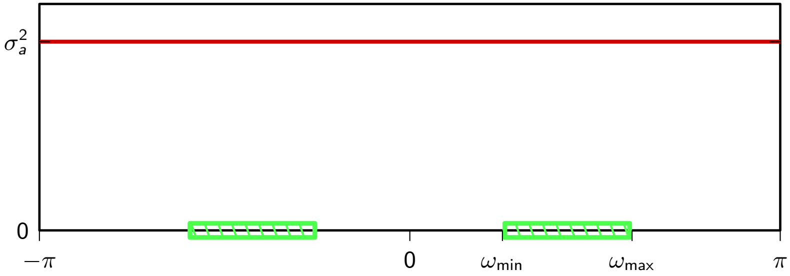

END OF MODULE 9.1

Digital Signal Processing

Module 9.2: Controlling the Bandwidth

Overview:

- Upsampling
- Fitting the transmitter's spectrum

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Shaping the bandwidth

Our problem:

- bandwidth constraint requires us to control the spectral support of a signal
- we need to be able to "shrink" the support of a full-band signal
- the answer is *multirate* techniques

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Multirate signal processing

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- increase or decrease the number of samples in a discrete-time signal
- equivalent to going to continuous time and resampling
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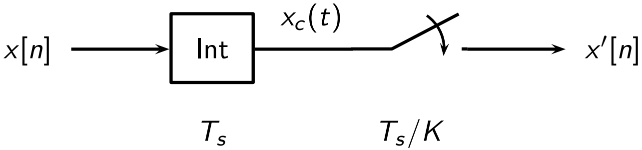
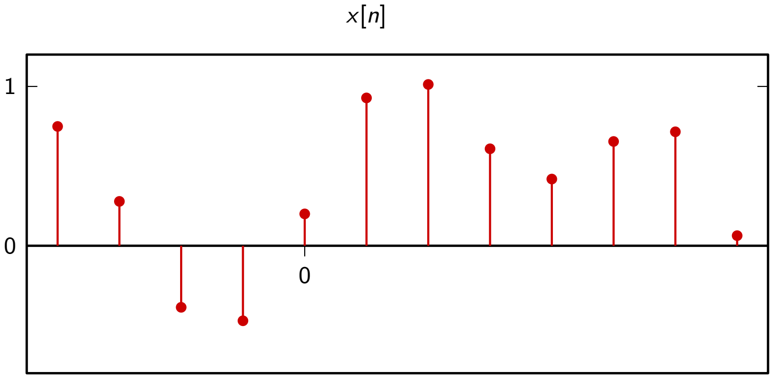
Upsampling via continuous time



Upsampling ($K = 3$)



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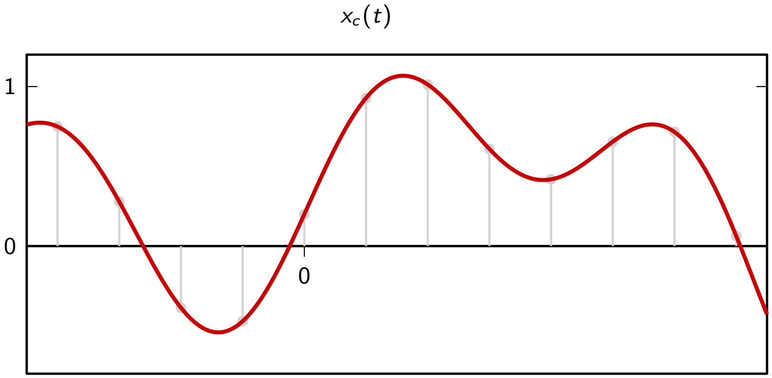

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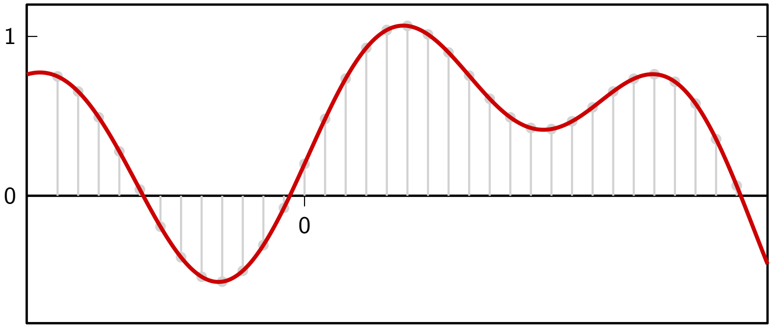

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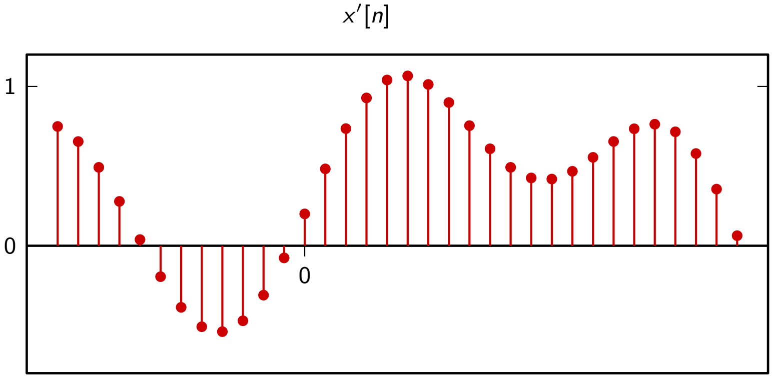

Upsampling

As per usual, we can choose $T_s = 1$...

$$x_c(t) = \sum_{m=-\infty}^{\infty} x[m]\,\mathrm{sinc}(t-m)$$

$$x'[n] = x_c(n/K)$$

$$= \sum_{m=-\infty}^{\infty} x[m]\,\mathrm{sinc}\left(\frac{n}{K} - m\right)$$

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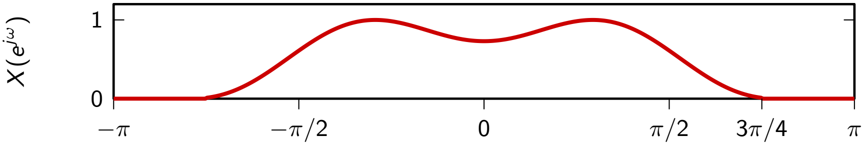
Upsampling (frequency domain)



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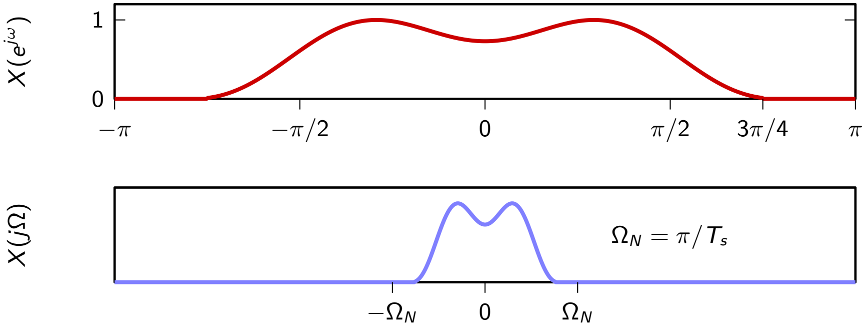

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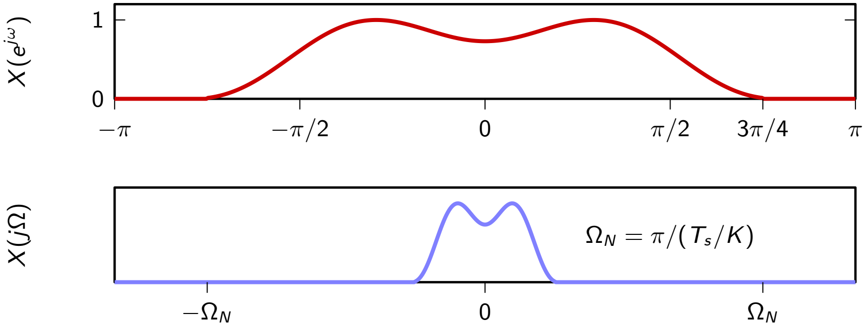

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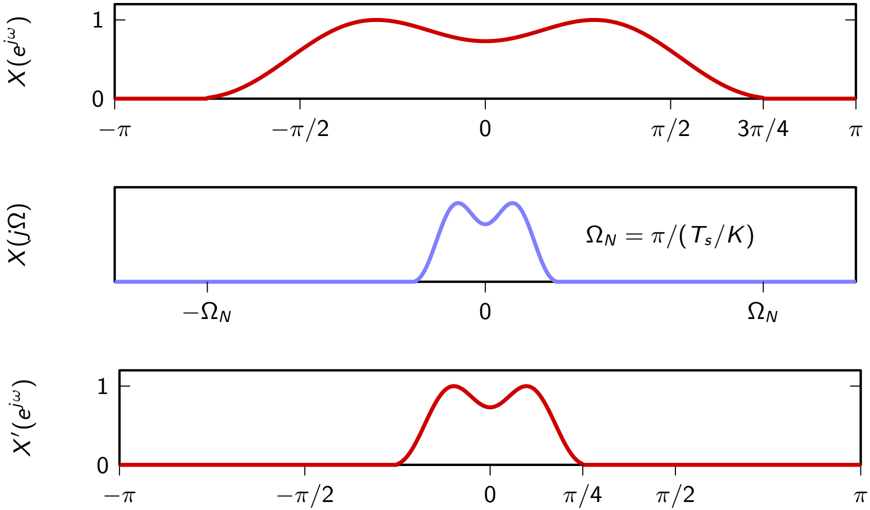

Upsampling in the digital domain

what can we do purely digitally?

- we need to "increase" the number of samples by K
- obviously $x_U[m] = x[n]$ when m multiple of K
- for lack of a better strategy, put zeros elsewhere
- example for $K = 3$:

$$x_U[m] = \ldots x[0], 0, 0, x[1], 0, 0, x[2], 0, 0, \ldots$$

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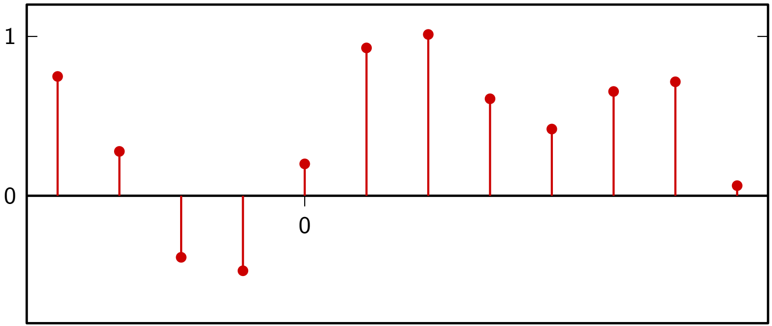
Upsampling in the Time Domain



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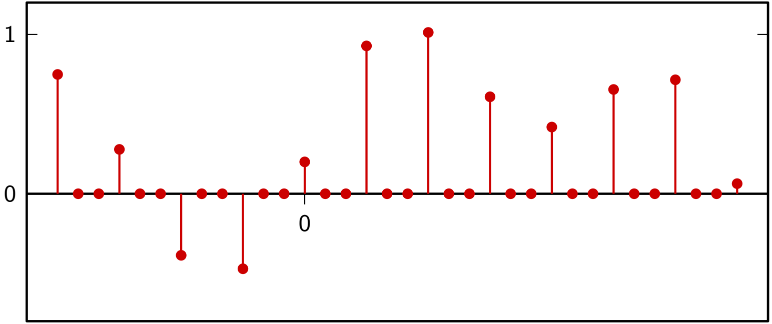

Upsampling in the digital domain

in the frequency domain

$$X_U(e^{j\omega}) = \sum_{m=-\infty}^{\infty} x_U[m]e^{-j\omega m}$$

$$= \sum_{n=-\infty}^{\infty} x[n]e^{-j\omega nK}$$

$$= X(e^{j\omega K})$$

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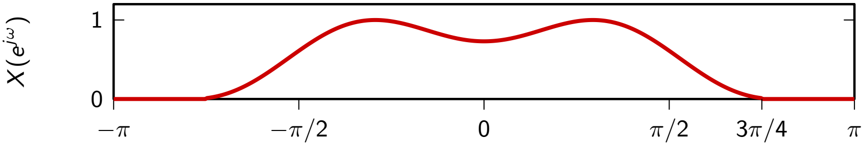

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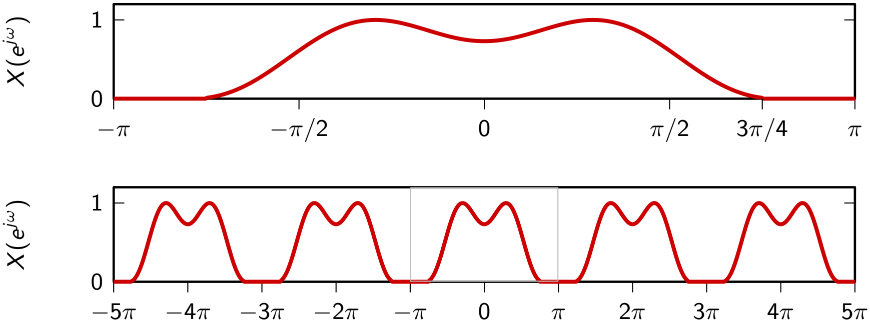

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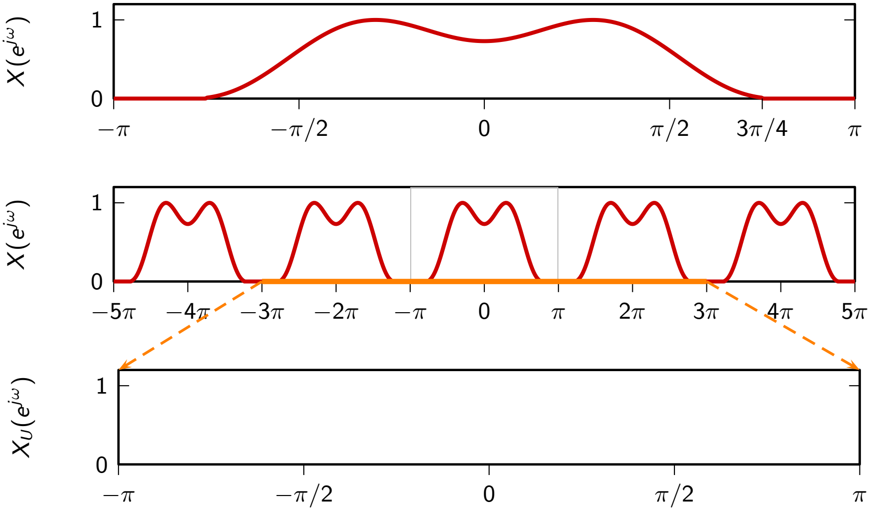

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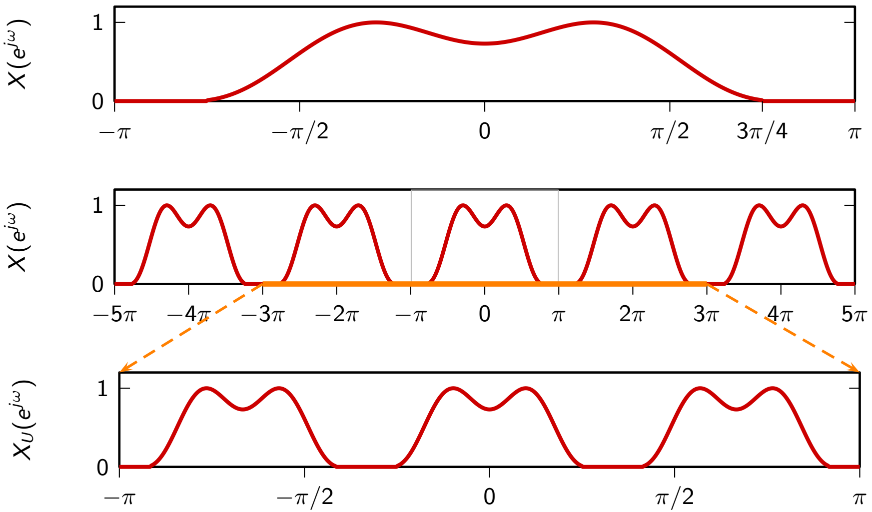

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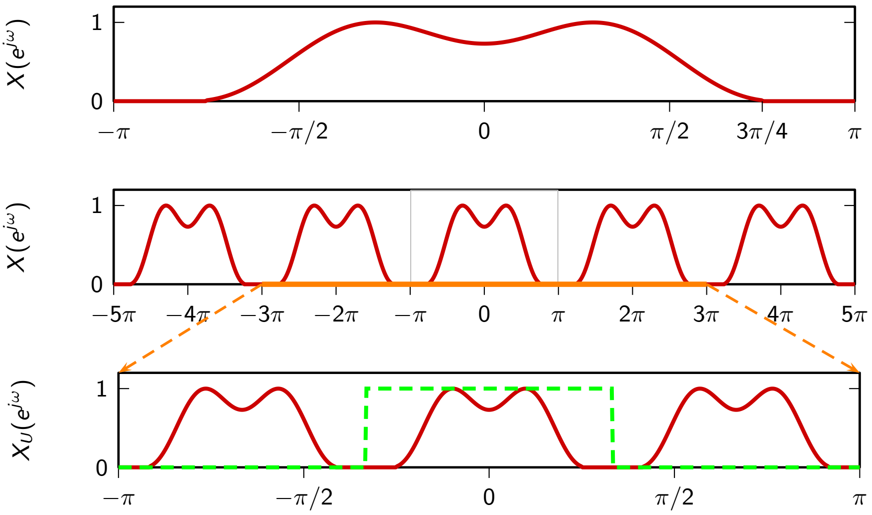

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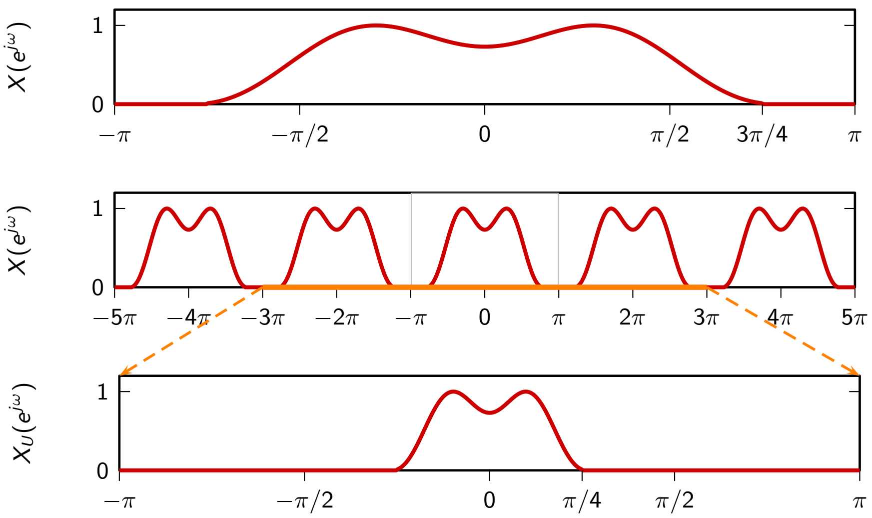

Upsampling in the digital domain

back in time domain...

- insert $K-1$ zeros after every sample
- ideal lowpass filtering with $\omega_c = \pi/K$

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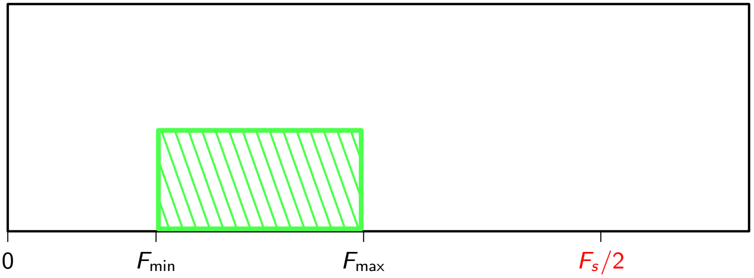
Remember the bandwidth constraint?



Here's a neat trick

let $W = F_{\max} - F_{\min}$; pick F_s so that:

- $F_s > 2F_{\max}$ (obviously)
- $F_s = KW,\ K \in \mathbb{N}$
- $\omega_{\max} - \omega_{\min} = 2\pi \dfrac{W}{F_s} = \dfrac{2\pi}{K}$
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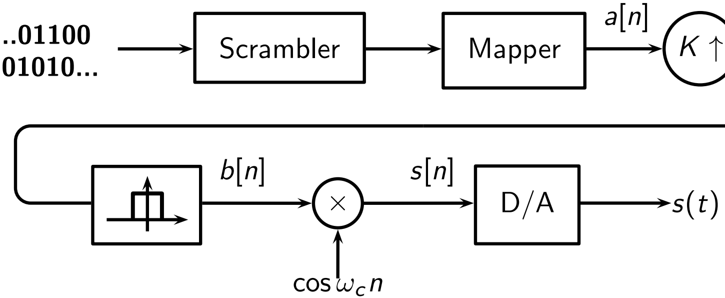
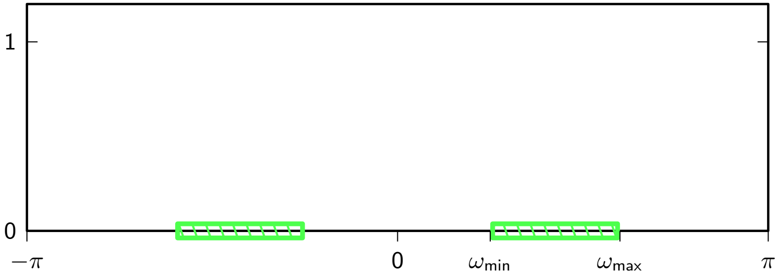
Transmitter design, continued



Meeting the bandwidth constraint



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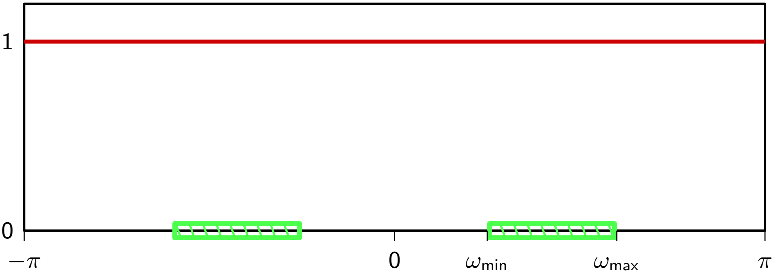

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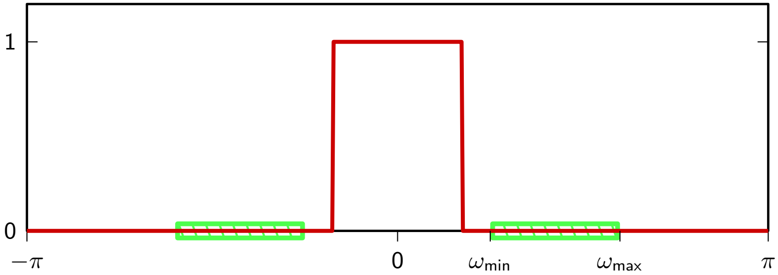

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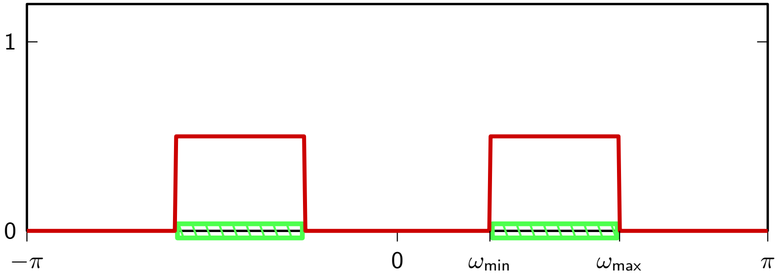
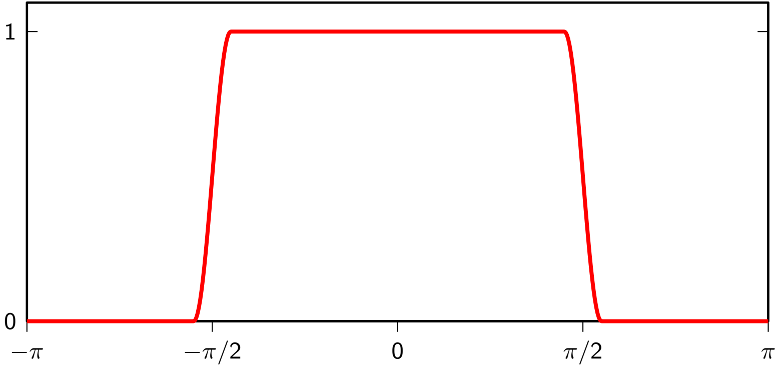


Raised Cosine



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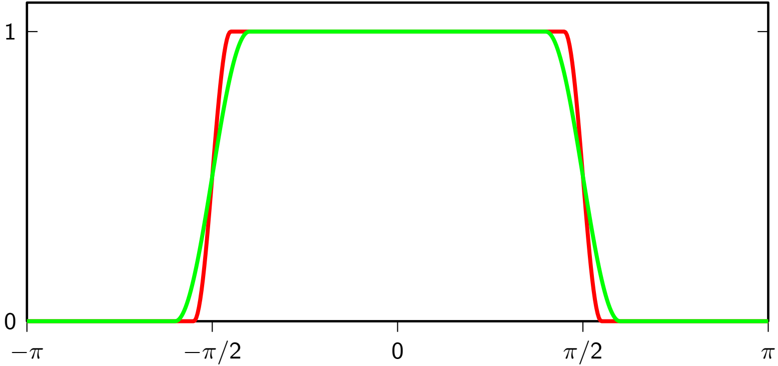

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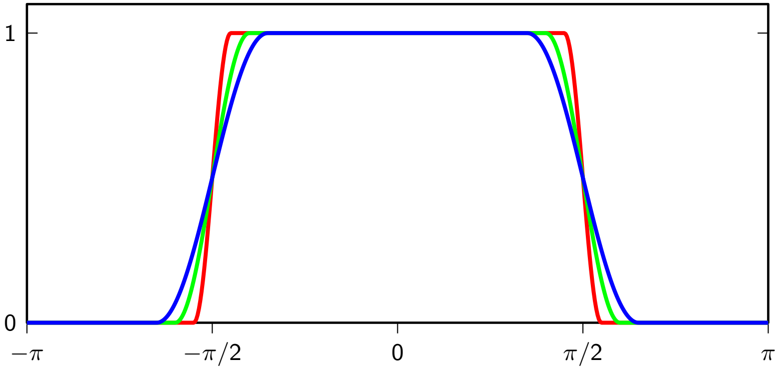

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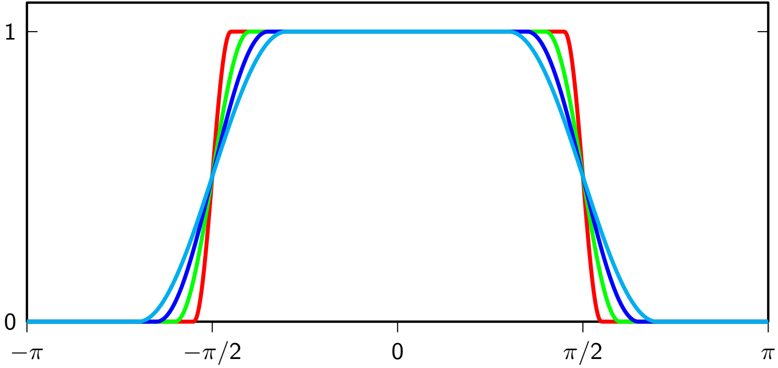

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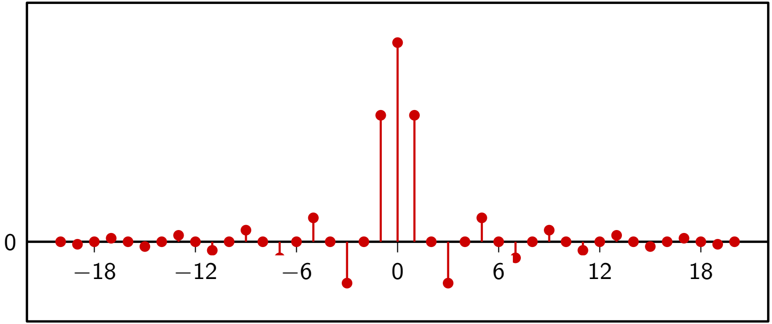
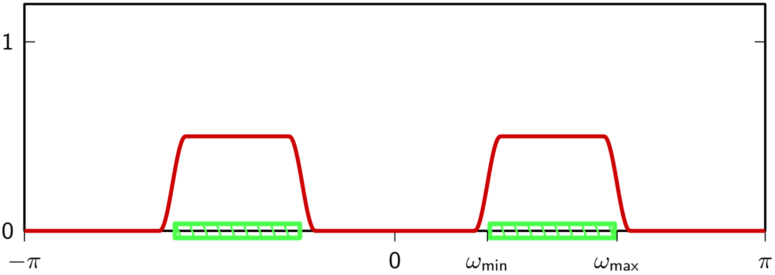


Spectral shaping with raised cosine



END OF MODULE 9.2

Digital Signal Processing

Module 9.3: Controlling the Power

Overview:

- Noise and probability of error
- Signaling alphabet and power
- QAM signaling

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- transmitter sends a sequence of symbols $a[n]$
- receiver obtaines a sequence $\hat{a}[n]$
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- when noise is large, we make an error

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depends on:

- power of the noise wrt power of the signal
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- we have a (randomized) bitstream coming in
- we want to send some upsampled and interpolated samples over the channel
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Mappers and slicers

mapper:

- split incoming bitstream into chunks
- assign a symbol $a[n]$ from a finite alphabet $\mathcal{A}$ to each chunk

slicer:

- receive a value $\hat{a}[n]$
- decide which symbol from $\mathcal{A}$ is "closest" to $\hat{a}[n]$
- piece back together the corresponding bitstream

Example: two-level signaling

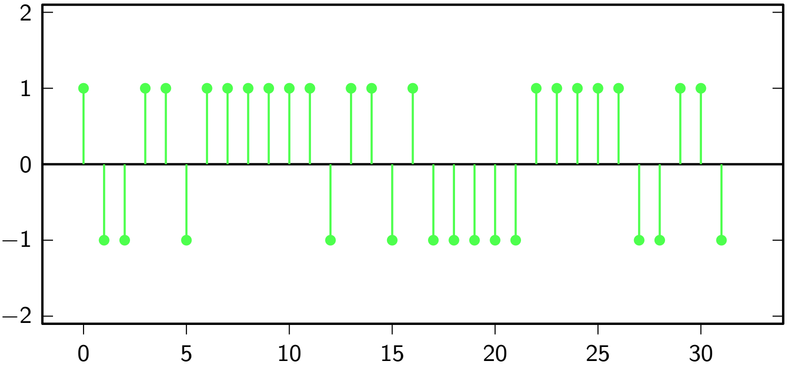
mapper:

- split incoming bitstream into single bits
- $a[n] = G$ if the bit is 1, $a[n] = -G$ if the bit is 0

slicer:

- n-th bit $= \begin{cases} 1 & \text{if } \hat{a}[n] > 0 \\ 0 & \text{otherwise} \end{cases}$

Example: two-level signaling



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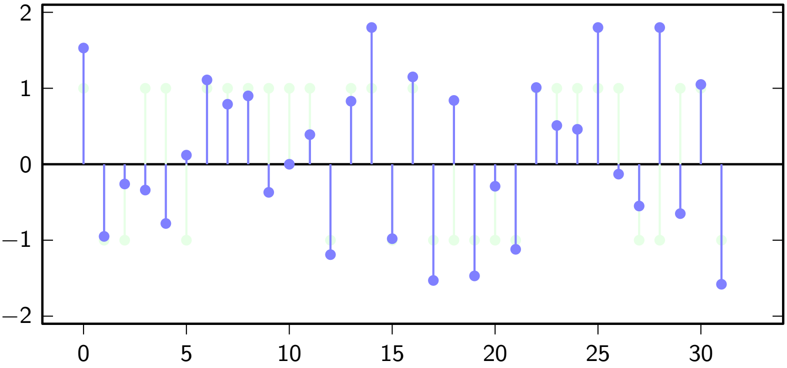

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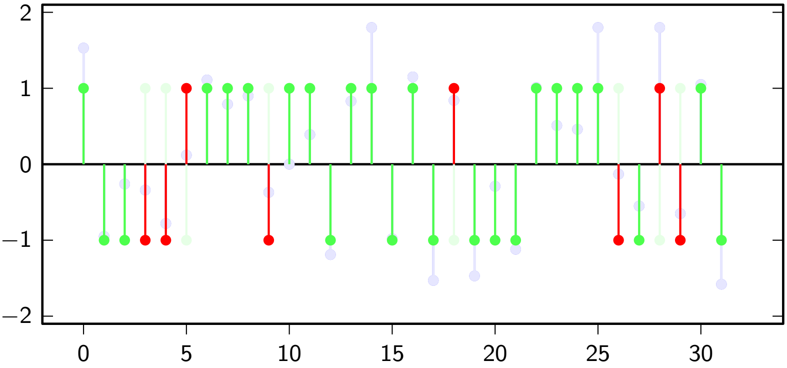

Example: two-level signaling

let's look at the probability of error after making some hypotheses:

- $\hat{a}[n] = a[n] + \eta[n]$
- bits in bitstream are equiprobable
- noise and signal are independent
- noise is additive white Gaussian noise with zero mean and variance σ_0

Example: two-level signaling

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\begin{aligned}
P_{\text{err}} = P[\ \eta[n] < -G \ \mid \ n\text{-th bit is } 1\ ]\, P[\ n\text{-th bit is } 1\ ] + \\
P[\ \eta[n] > G \ \mid \ n\text{-th bit is } 0\ ]\, P[\ n\text{-th bit is } 0\ ] \\
= (P[\ \eta[n] < -G\ ] + P[\ \eta[n] > G\ ])/2 \\
= P[\ \eta[n] > G\ ] \\
= \int_G^{\infty} \frac{1}{\sqrt{2\pi\sigma_0^2}} e^{-\frac{\tau^2}{2\sigma_0^2}}\, d\tau \\
= Q(G/\sigma_0) = \frac{1}{2}\operatorname{erfc}((G/\sigma_0)/\sqrt{2})
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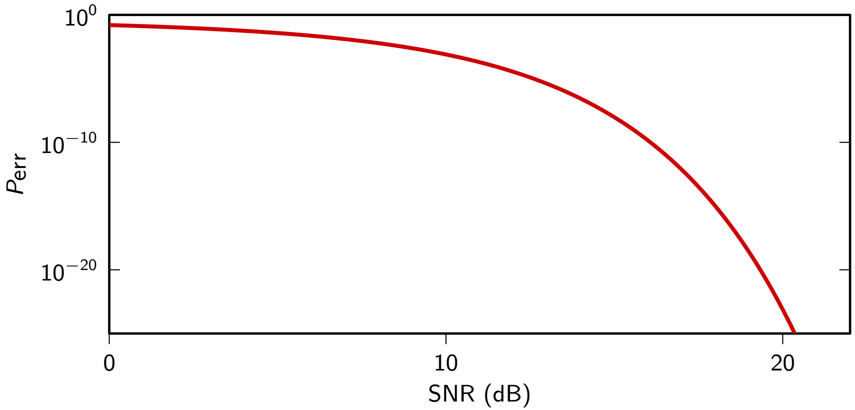

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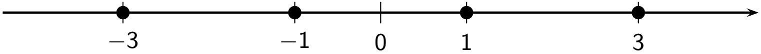
mapper:

- split incoming bitstream into chunks of M bits
- chunks define a sequence of integers $k[n] \in \{0, 1, \ldots, 2^M - 1\}$
- $a[n] = G((-2^M + 1) + 2k[n])$ (odd integers around zero)

slicer:

- $a'[n] = \arg\min_{a \in \mathcal{A}} [|\hat{a}[n] - a|]$

PAM, $M = 2, G = 1$



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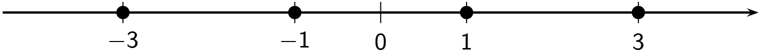

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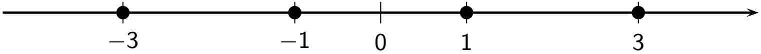

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From PAM to QAM

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- can we increase the throughput even further?
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mapper:

- split incoming bitstream into chunks of M bits, M even
- use $M/2$ bits to define a PAM sequence $a_r[n]$
- use the remaining $M/2$ bits to define an independent PAM sequence $a_i[n]$
- $a[n] = G(a_r[n] + ja_i[n])$

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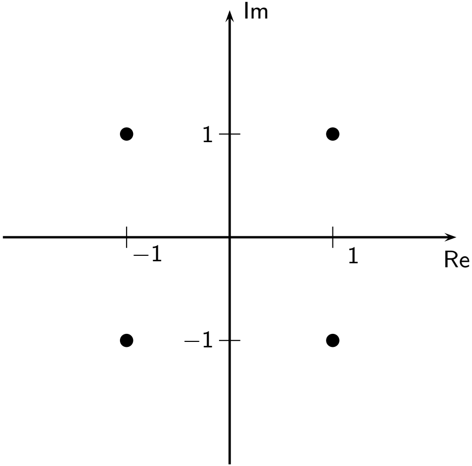
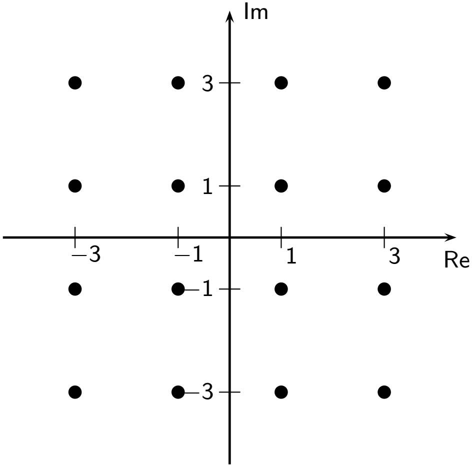
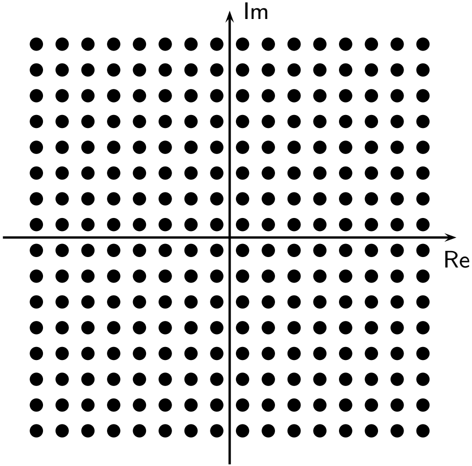
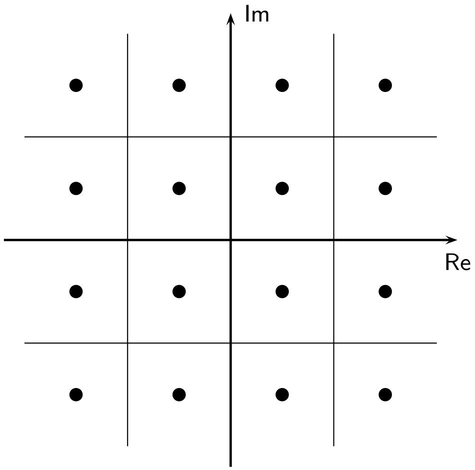
QAM, $M = 2$, $G = 1$




Im
3
1
−3
−1
1
3
Re
−1
−3

QAM, $M = 8$, $G = 1$

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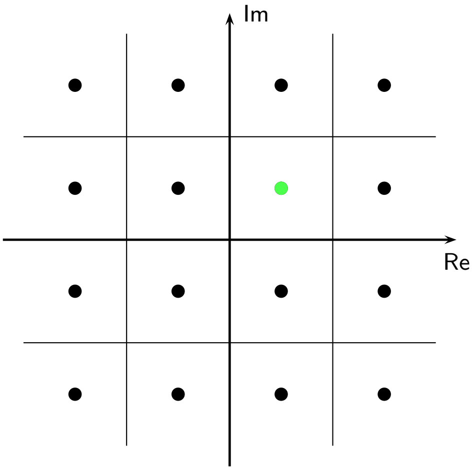

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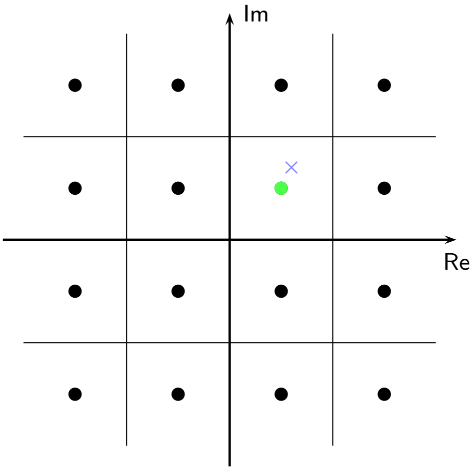

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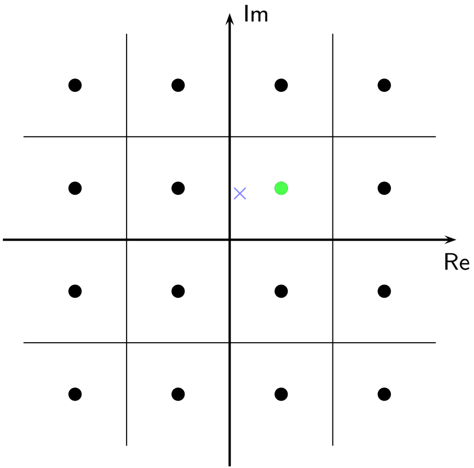

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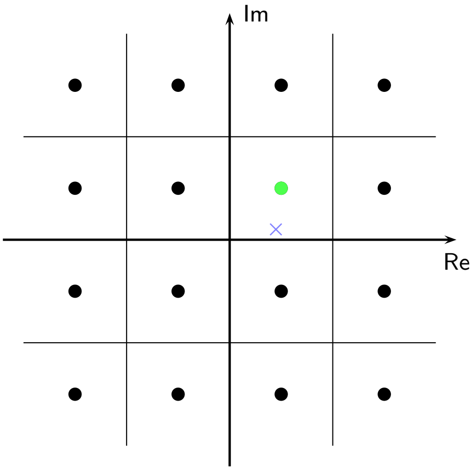

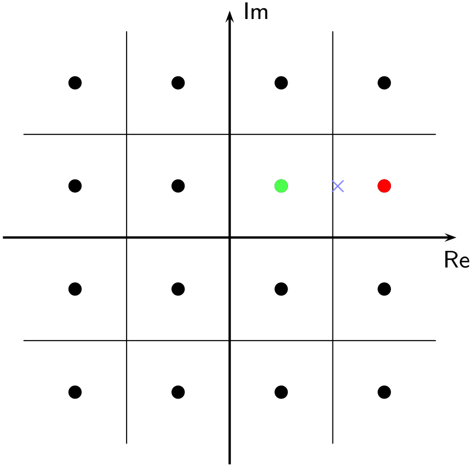
Im
Re

QAM, probability of error

$$P_{\text{err}} = 1 - P[|\operatorname{Re}(\eta[n])| < G \wedge |\operatorname{Im}(\eta[n])| < G]$$

$$= 1 - \int_D f_\eta(z)\, dz$$

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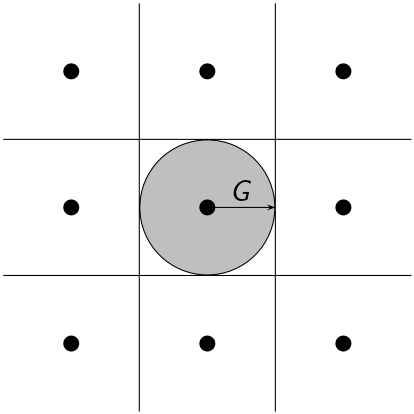

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transmitted power (all symbols equiprobable and independent):

$$\sigma_s^2 = G^2 \frac{1}{2^M} \sum_{a \in \mathcal{A}} |a|^2$$

$$= G^2 \frac{2}{3}(2^M - 1)$$

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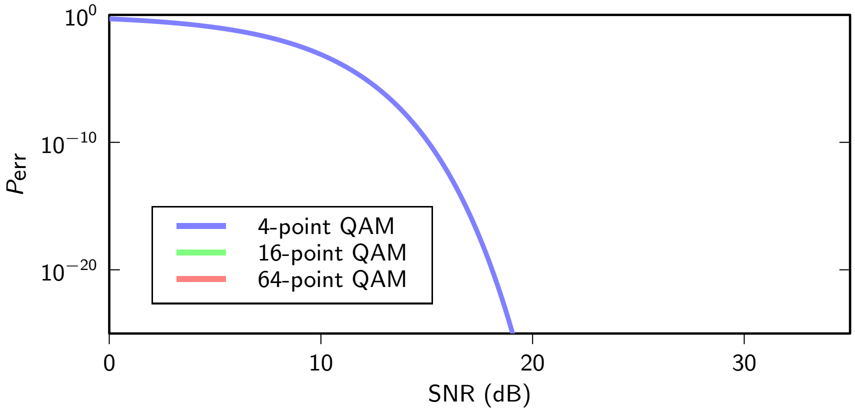

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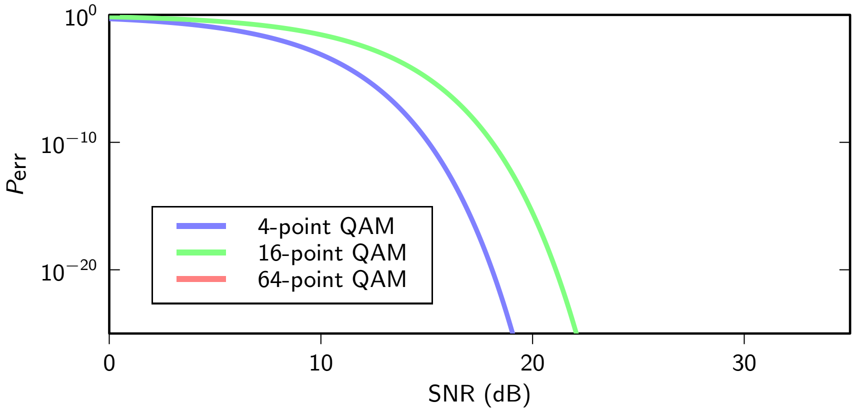

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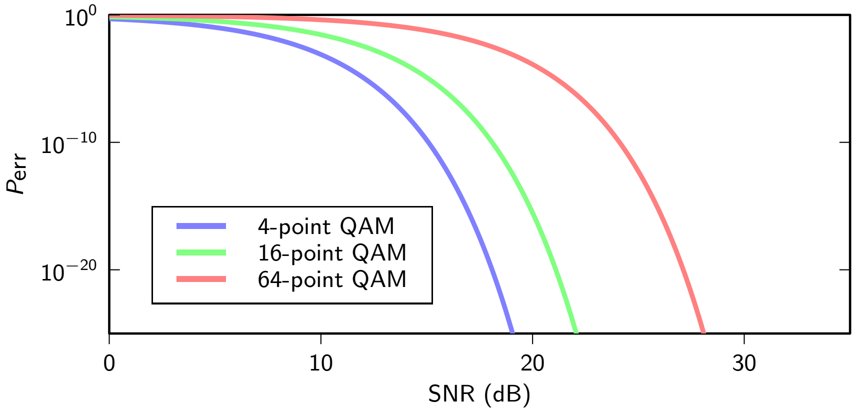

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- find out the SNR imposed by the channel's power constraint
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but how do we transmit *complex* symbols over a real channel?

where we stand:

- we know how to fit the bandwidth constraint
- with QAM, we know how many bits per symbol we can use given the power constraint
- we know the theoretical throughput of the transmitter

but how do we transmit *complex* symbols over a real channel?

END OF MODULE 9.3

Digital Signal Processing

Module 9.4: Modulation and Demodulation

Overview:

- Trasmitting and recovering the complex passband signal
- Design example
- Channel capacity

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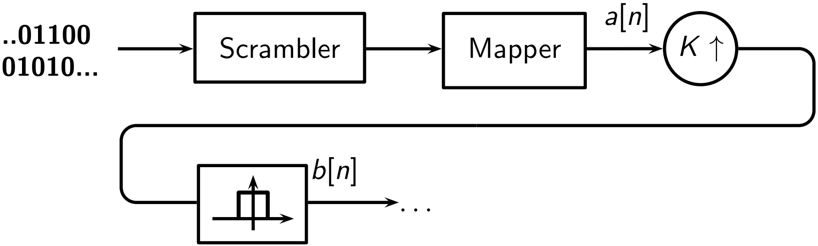
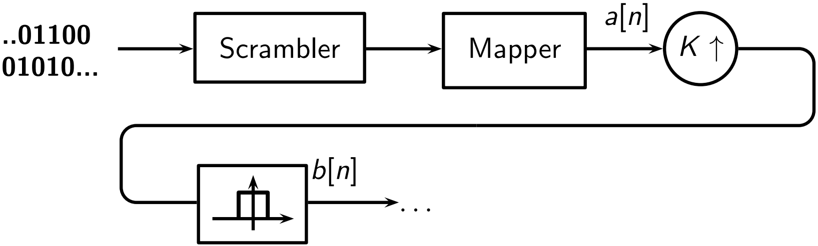


$b[n] = b_r[n] + jb_i[n]$ is a complex-valued baseband signal

QAM transmitter design



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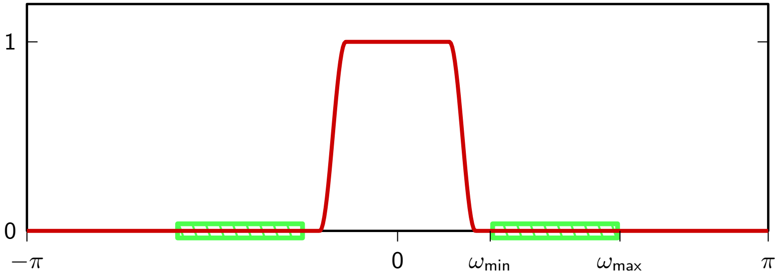
Complex baseband signal



The passband signal

$$
\begin{aligned}
s[n] &= \mathrm{Re}\{b[n]\, e^{j\omega_c n}\} \\
&= \mathrm{Re}\{(b_r[n] + jb_i[n])(\cos\omega_c n + j\sin\omega_c n)\} \\
&= b_r[n]\cos\omega_c n - b_i[n]\sin\omega_c n
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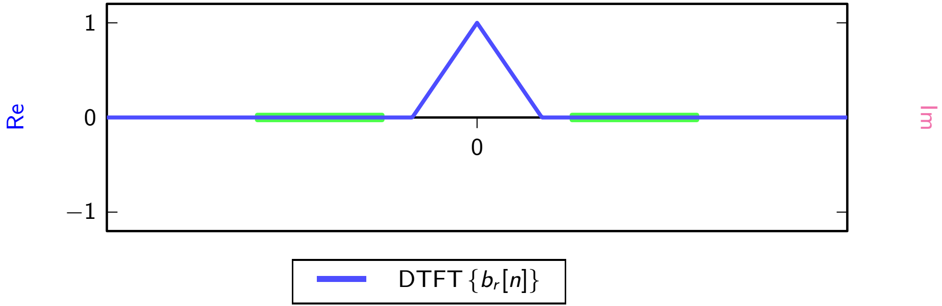

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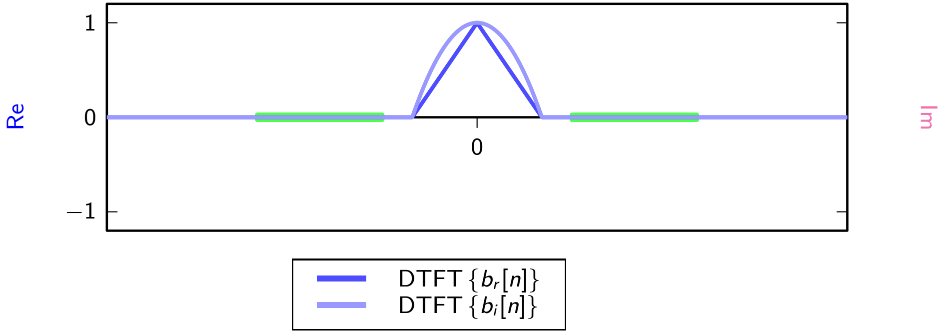

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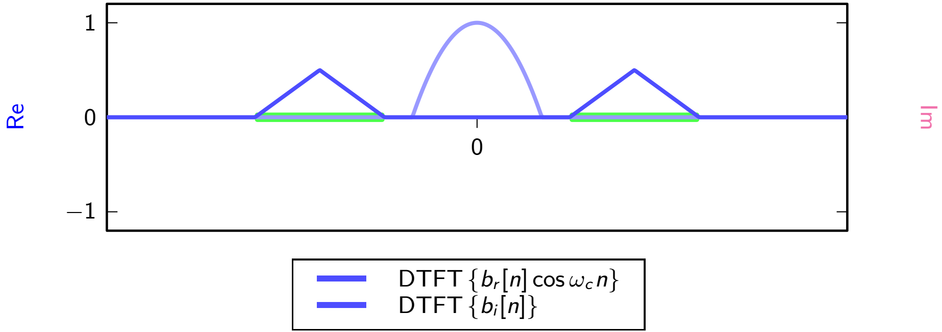

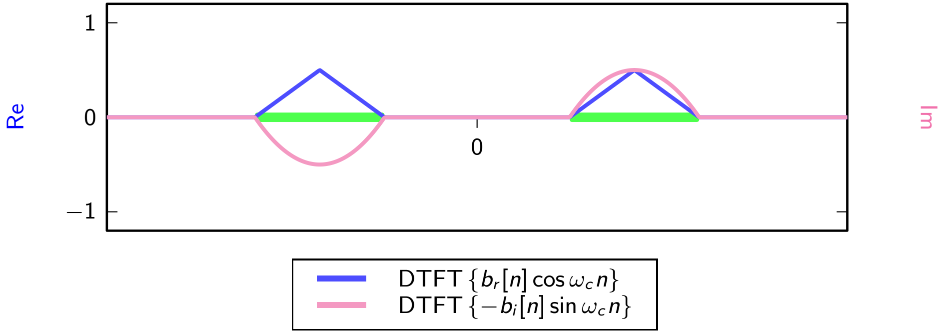

1
0
−1
0
Re
Im
DTFT $\{b_r[n]\cos\omega_c n\}$
DTFT $\{-b_i[n]\sin\omega_c n\}$

let's try the usual method (multiplying by the carrier, see Module 5.5):

$$
\begin{aligned}
s[n]\cos\omega_c n &= b_r[n]\cos^2\omega_c n - b_i[n]\sin\omega_c n\cos\omega_c n \\
&= b_r[n]\frac{1+\cos 2\omega_c n}{2} - b_i[n]\frac{\sin 2\omega_c n}{2} \\
&= \frac{1}{2}b_r[n] + \frac{1}{2}(b_r[n]\cos 2\omega_c n - b_i[n]\sin 2\omega_c n)
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$$

Recovering the baseband signal

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Complex baseband signal

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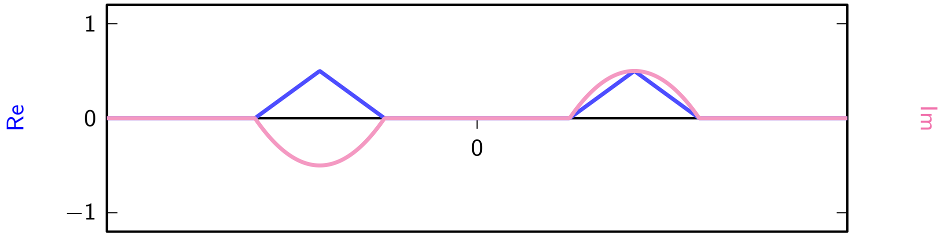

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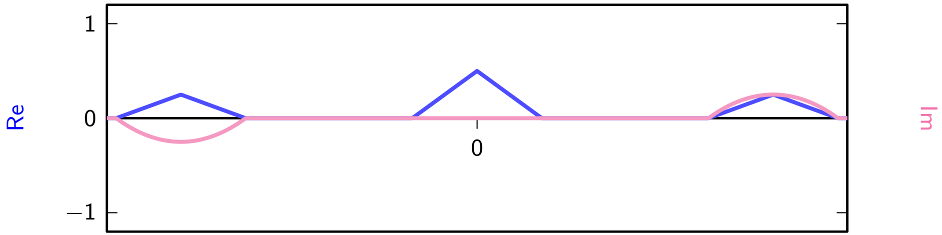

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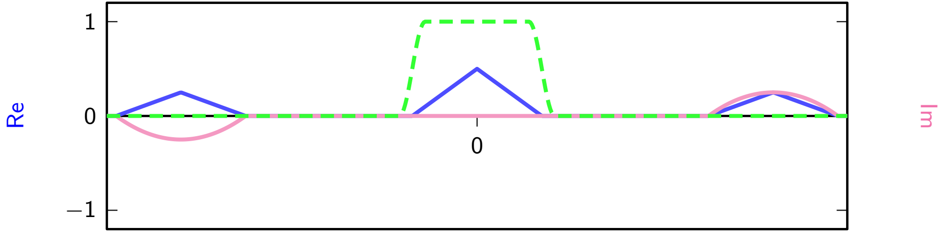

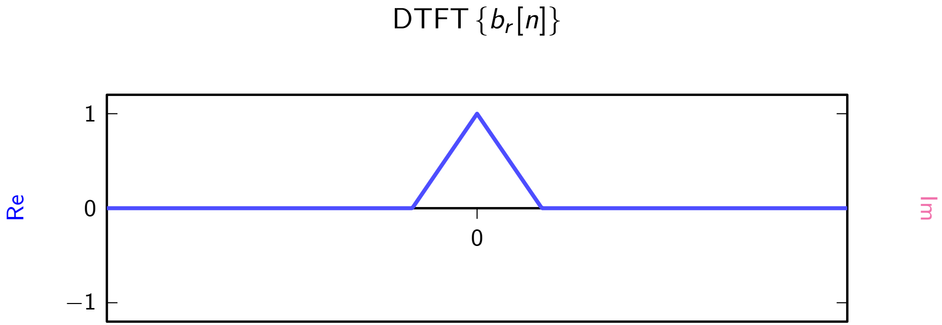
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Recovering the baseband signal

- as a lowpass filter, you can use the same filter used in upsampling
- matched filter technique

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similarly:

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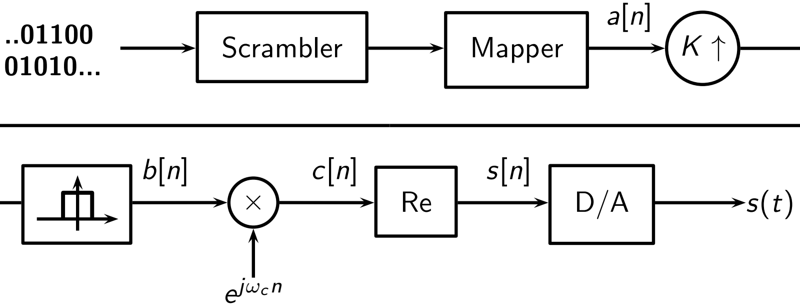
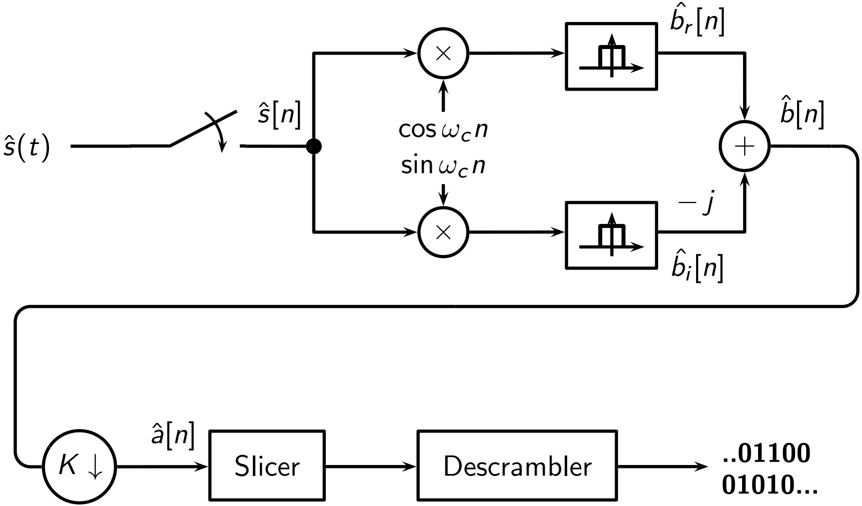
QAM transmitter, final design



QAM receiver, idealized design



Example: the V.32 voiceband modem

- analog telephone channel: $F_{\text{min}} = 450\text{Hz}$, $F_{\text{max}} = 2850\text{Hz}$
- usable bandwidth: $W = 2400\text{Hz}$, center frequency $F_c = 1650\text{Hz}$
- pick $F_s = 3 \cdot 2400 = 7200\text{Hz}$, so that $K = 3$
- $\omega_c = 0.458\pi$

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- maximum SNR: 22dB
- pick $P_{\text{err}} = 10^{-6}$
- using QAM, we find

$$M = \log_2\left(1 - \frac{3}{2}\frac{10^{22/10}}{\ln(10^{-6})}\right) \approx 4.1865$$

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Theoretical channel capacity

- we used very specific design choices to derive the throughput
- what is the best one can do?
- Shannon's capacity formula is the upper bound

$$C = W \log_2(1 + SNR)$$

- for instance, for the previous example $C \approx 17500$ bps
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END OF MODULE 9.4

Digital Signal Processing

Module 9.5: Receiver Design

Overview:

- Adaptive equalization
- Timing recovery

Overview:

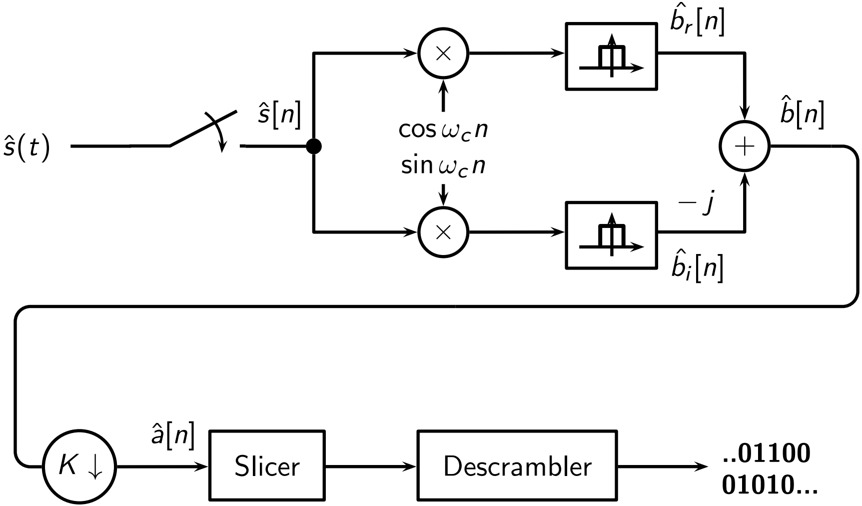
- Adaptive equalization
- Timing recovery

A blast from the past

A blast from the past

- a sound familiar to anyone who's used a modem or a fax machine
- what's going on here?

Graphically



if $\hat{s}[n] = \cos((\omega_c + \omega_0)n)$:

$$\begin{aligned}
\hat{b}[n] &= \mathcal{H}\{\cos((\omega_c + \omega_0)n)\cos(\omega_c n) - j\cos((\omega_c + \omega_0)n)\sin(\omega_c n)\} \\
&= \mathcal{H}\{\cos(\omega_0 n) + \cos((2\omega_c + \omega_0)n) - j\sin((2\omega_c + \omega_0)n) + j\sin(\omega_0 n)\} \\
&= \cos(\omega_0 n) + j\sin(\omega_0 n) \\
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\end{aligned}$$

Pilot tones

if $\hat{s}[n] = \cos((\omega_c + \omega_0)n)$:

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\end{aligned}$$

In slow motion

It's a dirty job...

but a receiver has to do it:

- interference
- propagation delay
- linear distortion
- clock drifts

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- interference $\rightarrow$ handshake and line probing
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It's a dirty job...

but a receiver has to do it:

- interference $\rightarrow$ handshake and line probing
- propagation delay $\rightarrow$ delay estimation
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It's a dirty job...

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- interference → handshake and line probing
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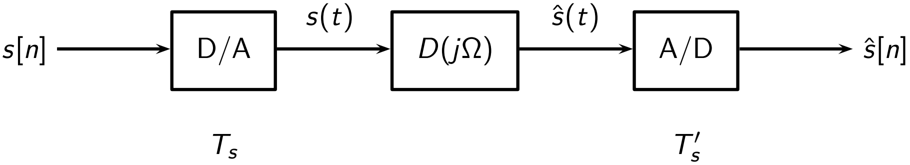
- linear distortion → adaptive equalization
- clock drifts → timing recovery

The two main problems



- channel distortion $D(j\Omega)$
- (time-varying) discrepancies in clocks $T_s' = T_s$

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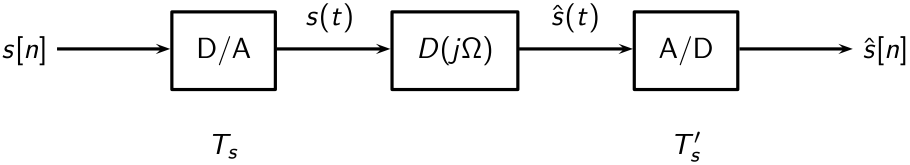

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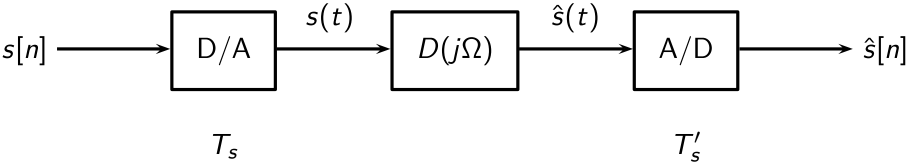

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Delay compensation

Assume the channel is a simple delay: $\hat{s}(t) = s(t-d) \Rightarrow D(j\Omega) = e^{-j\Omega d}$

- channel introduces a delay of d seconds
- we can write $d = (b+\tau)T_s$ with $b \in \mathbb{N}$ and $|\tau| < 1/2$
- b is called the *bulk delay*
- τ is the fractional delay

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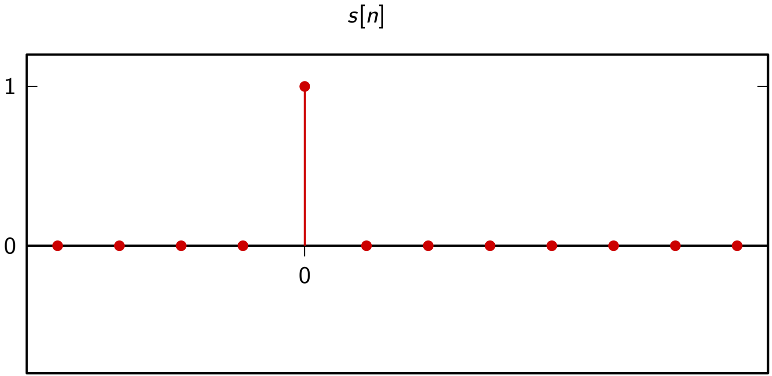
Offsetting the bulk delay ($T_s = 1$)



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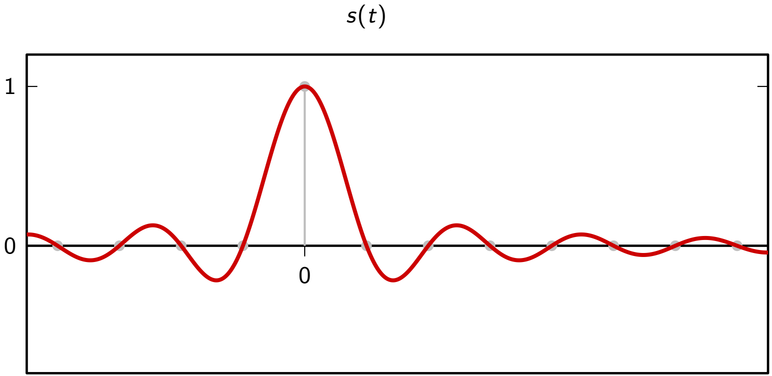

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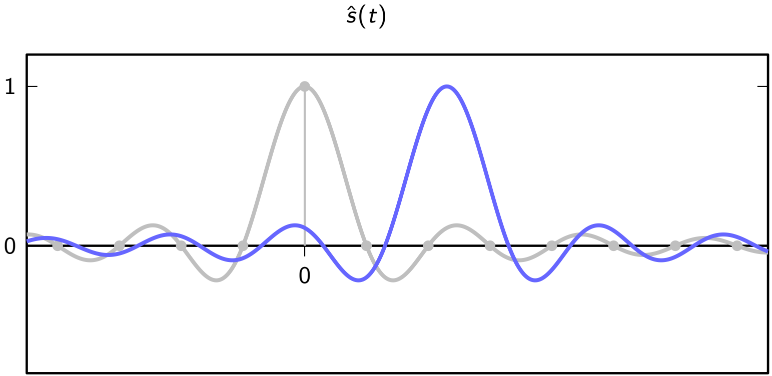

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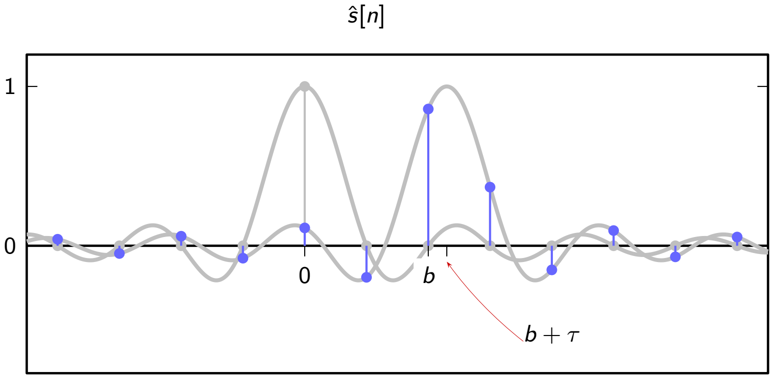

Estimating the fractional delay

- transmit $b[n] = e^{j\omega_0 n}$ (i.e. $s[n] = \cos((\omega_c + \omega_0)n)$)
- receive $\hat{s}[n] = \cos((\omega_c + \omega_0)(n - b - \tau))$
- after demodulation and bulk delay offset:

$$\hat{b}[n] = e^{j\omega_0(n-\tau)}$$

- multiply by known frequency

$$\hat{b}[n]\, e^{-j\omega_0 n} = e^{-j\omega_0 \tau}$$

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- after demodulation and bulk delay offset:

$$\hat{b}[n] = e^{j\omega_0(n-\tau)}$$

- multiply by known frequency

$$\hat{b}[n]\, e^{-j\omega_0 n} = e^{-j\omega_0 \tau}$$

Compensating for the fractional delay

- $\hat{s}[n] = s(n - \tau)T_s$ (after offsetting bulk delay)
- we need to compute subsample values
- in theory, compensate with a sinc fractional delay $h[n] = \text{sinc}(n + \tau)$
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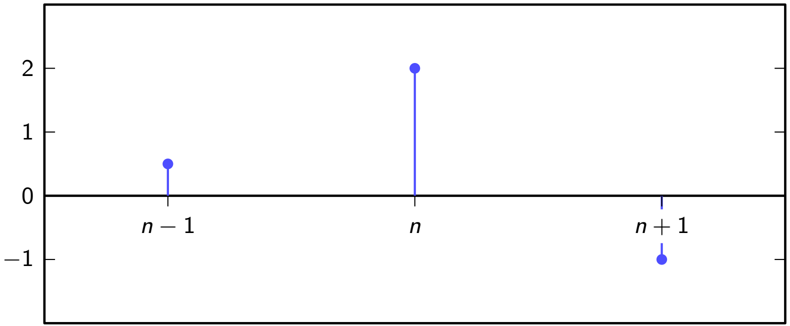

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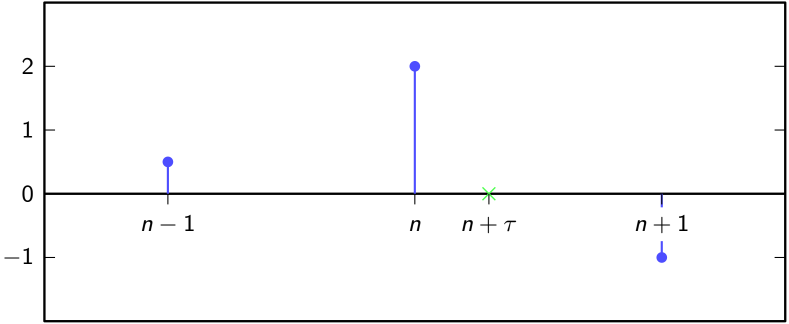

Lagrange approximation (see Module 6.2)

as per usual, choose $T_s = 1$

- we want to compute $x(n+\tau)$, with $|\tau| < 1/2$
- local Lagrange approximation around n

$$x_L(n;t) = \sum_{k=-N}^{N} x[n-k] L_k^{(N)}(t)$$

$$L_k^{(N)}(t) = \prod_{\substack{i=-N \\ i \neq k}}^{N} \frac{t-i}{k-i} \qquad\qquad k = -N, \ldots, N$$

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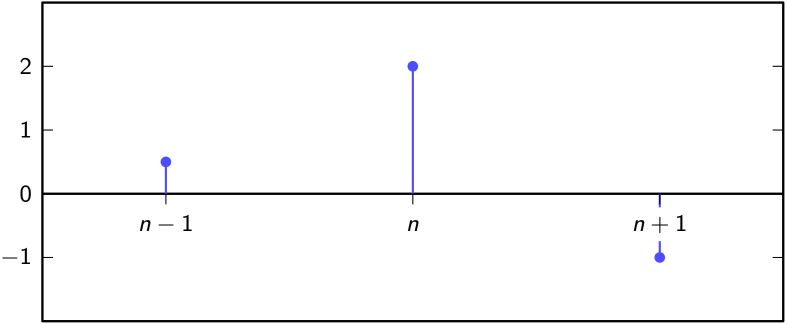

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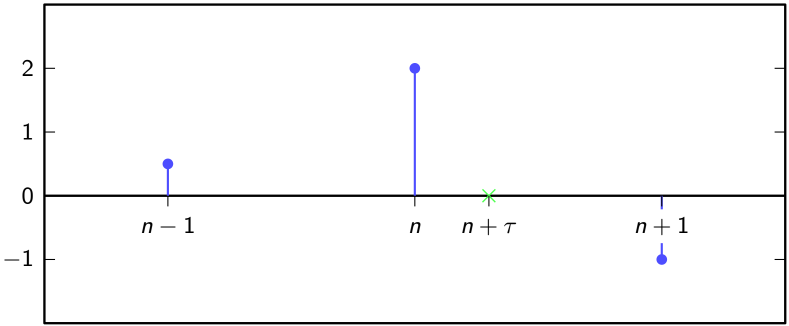

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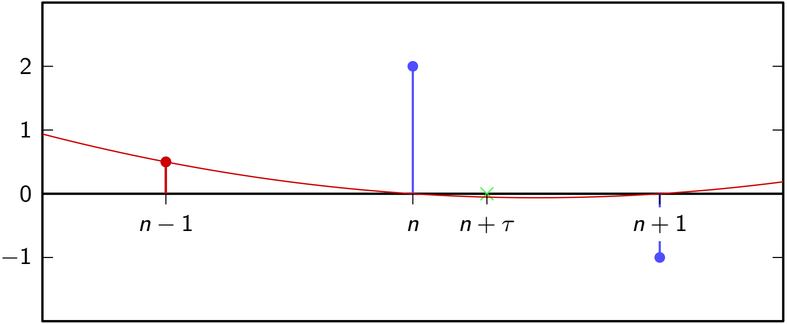

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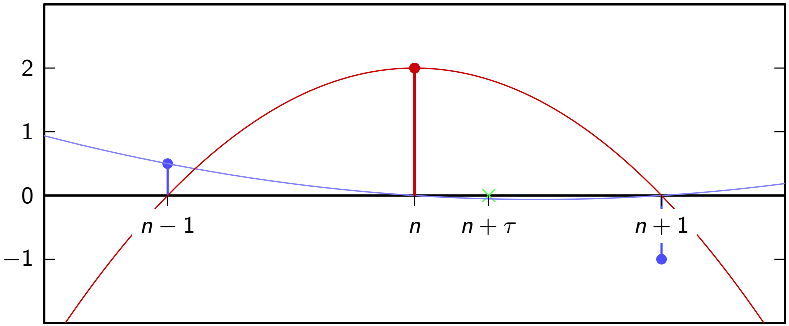

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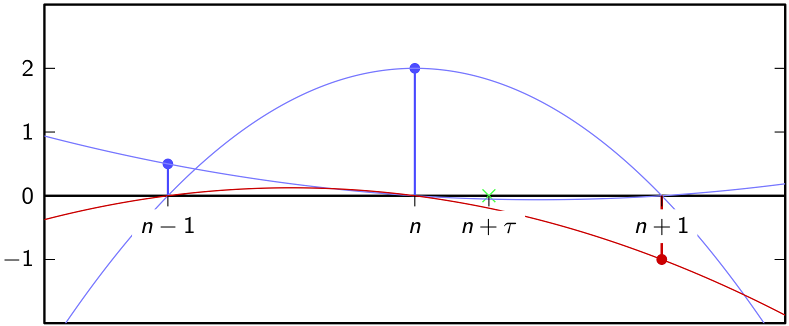

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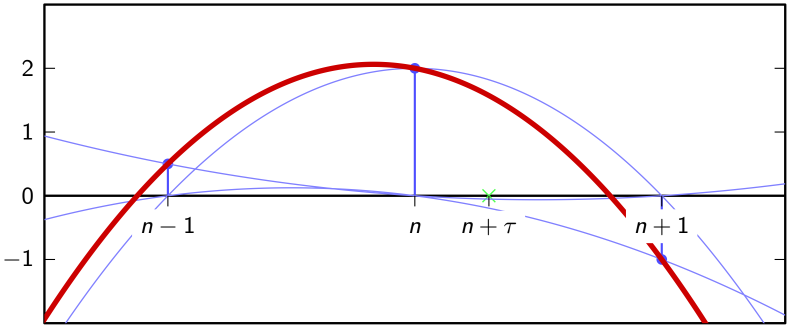

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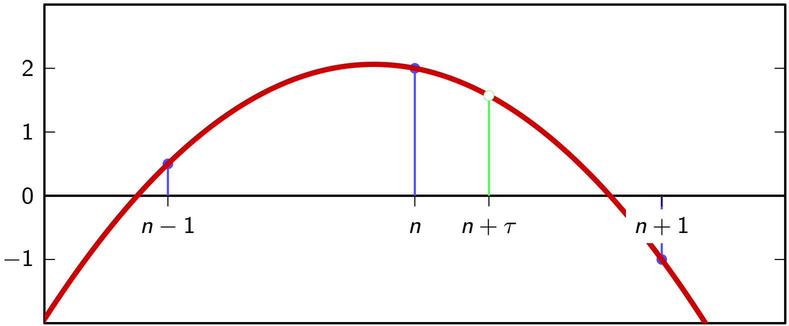

Lagrange interpolation as an FIR

- $x(n+\tau) \approx x_L(n; \tau)$
- define $d_\tau[k] = L_k^{(N)}(\tau), \; k = -N, \ldots, N$
- $d_\tau[k]$ form a $(2N+1)$-tap FIR
- $x_L(n; \tau) = (x * d_\tau)[n]$

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Example ($N = 1$, second order approximation)

$$L^{(1)}_{-1}(t) = t\frac{t-1}{2}$$

$$L^{(1)}_{0}(t) = (1-t)(1+t)$$

$$L^{(1)}_{1}(t) = t\frac{t+1}{2}$$

Example ($N = 1$, second order approximation)

$$d_{0.2}[n] = \begin{cases} -0.08 & n = -1 \\ 0.96 & n = 0 \\ 0.12 & n = 1 \\ 0 & \text{otherwise} \end{cases}$$

Delay compensation algorithm

- estimate the delay τ
- compute the $2N+1$ Lagrangian coefficients
- filter with the resulting FIR

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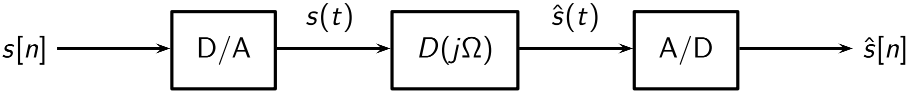
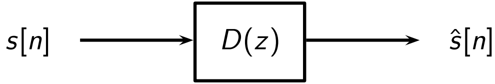
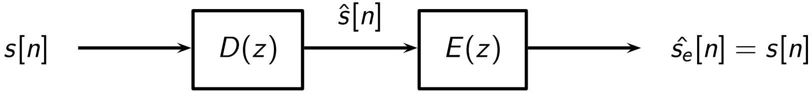
Compensating for the distortion




$s[n]$
$D(z)$
$\hat{s}[n]$

Example: adaptive equalization



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- in theory, $E(z) = 1/D(z)$
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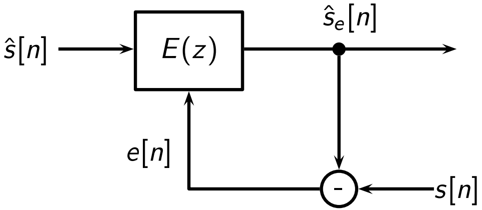
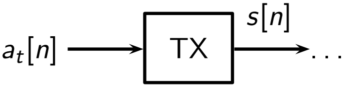


Adaptive equalization: bootstrapping via a training sequence



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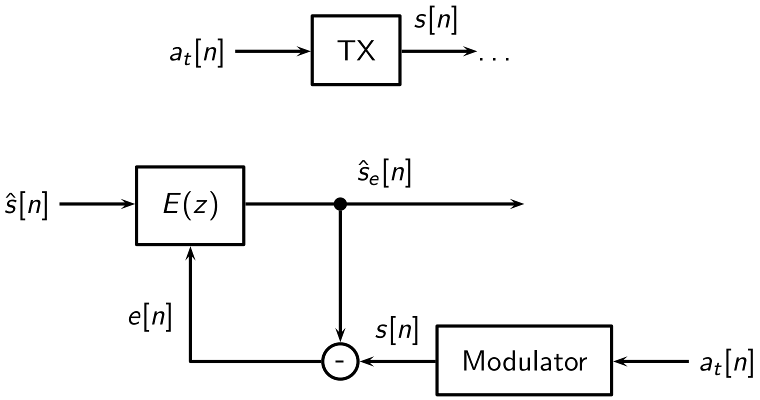
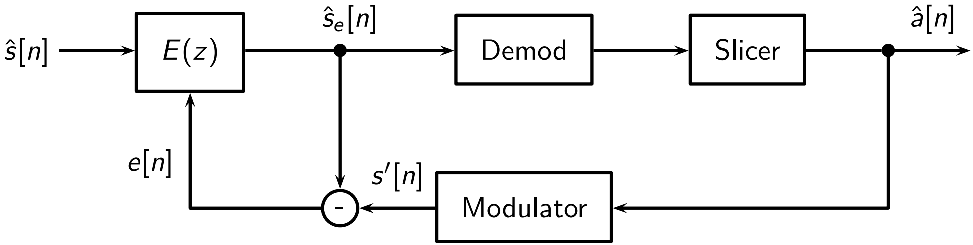


Adaptive equalization: online mode



So much more to do...

- how do we perform the adaptation of the coefficients?
- how do we compensate for differences in clocks?
- how do we recover from interference?
- how do we improve resilience to noise?

adaptive signal processing

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adaptive signal processing

END OF MODULE 9.5

Digital Signal Processing

Module 9.6: ADSL

Overview:

- Channel
- Signaling strategy
- Discrete Multitone Modulation (DMT)

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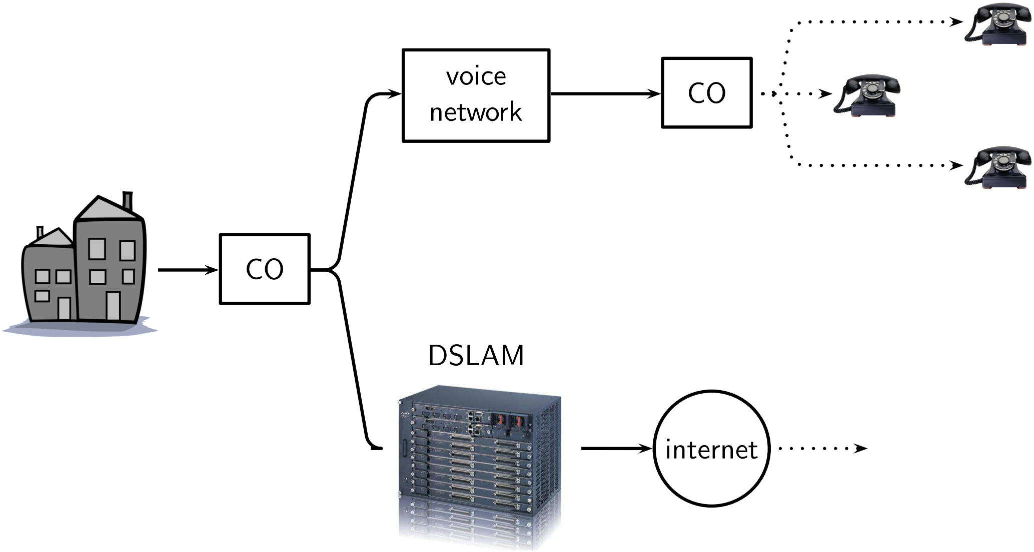
The telephone network today



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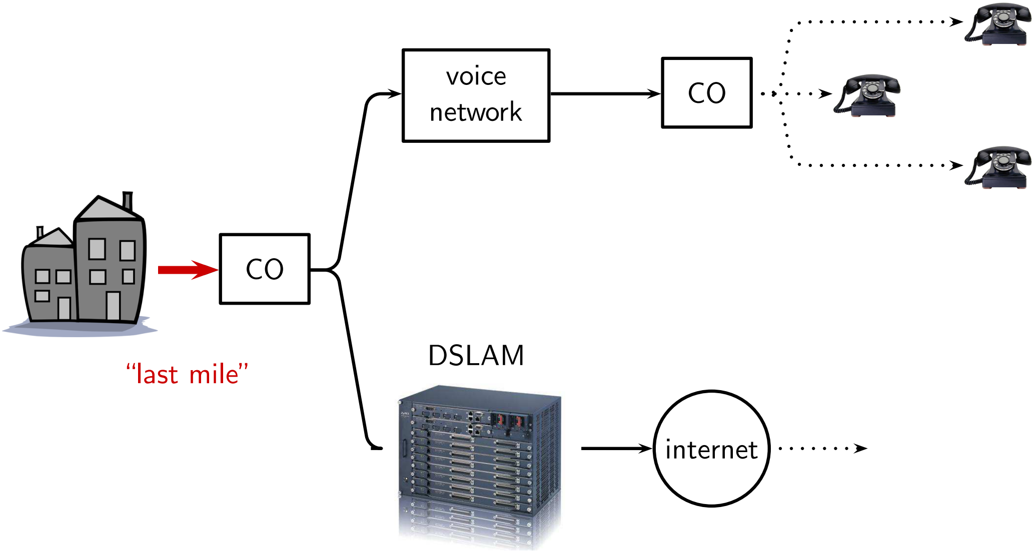

The last mile

- copper wire (twisted pair) between home and nearest CO
- very large bandwidth (well over 1MHz)
- very uneven spectrum: noise, attenuation, interference, etc.

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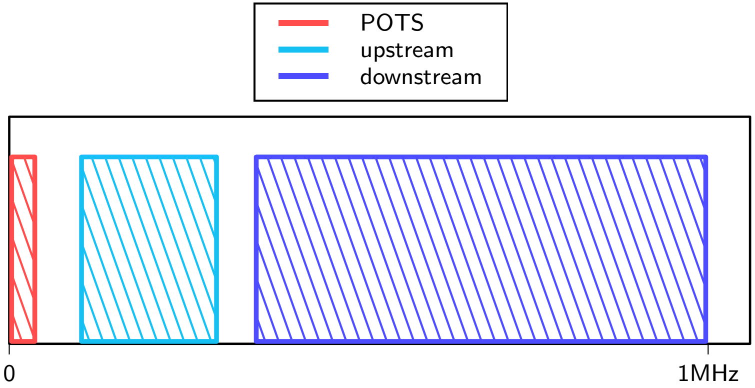

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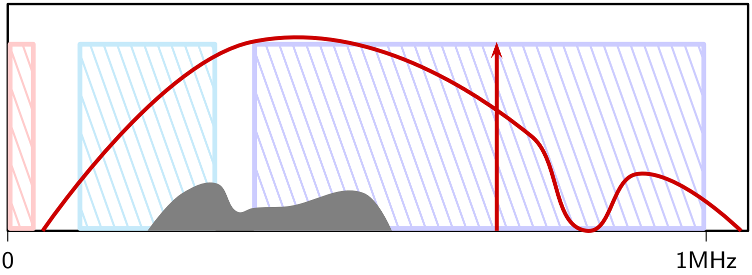
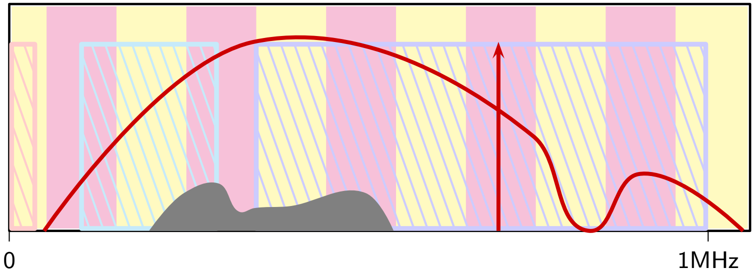


Idea: split the band into independent subchannels



Subchannel structure

- allocate N subchannels over the total positive bandwidth
- equal subchannel bandwidth $F_{\max}/N$
- equally spaced subchannels with center frequency $kF_{\max}/N$, $k = 0, \ldots, N-1$

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The digital design

- pick $F_s = 2F_{\max}$ ($F_{\max}$ is high now!)
- center frequency for each subchannel $\omega_k = 2\pi \frac{kF_{\max}/N}{F_s} = \frac{2\pi}{2N}k$
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- to send symbols over a subchannel: upsampling factor $K \geq 2N$

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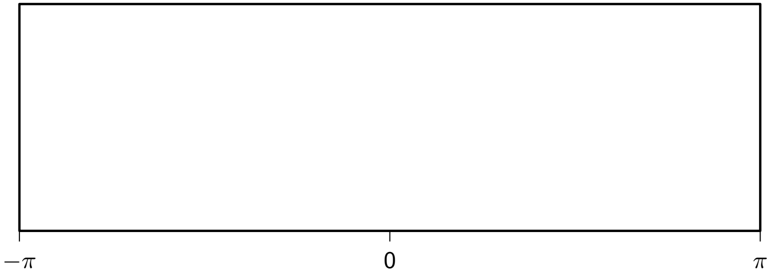

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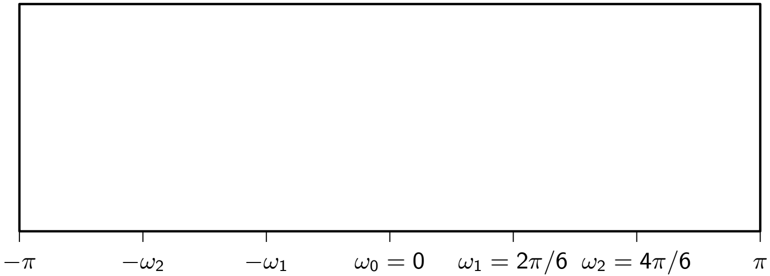

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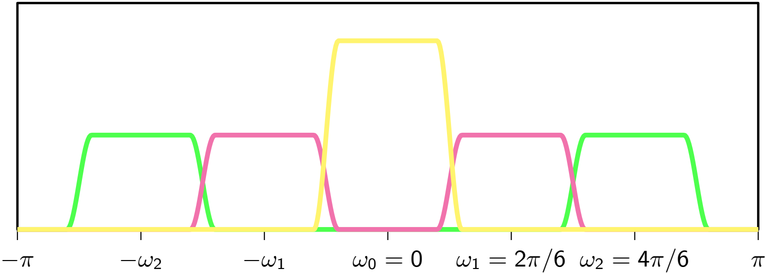

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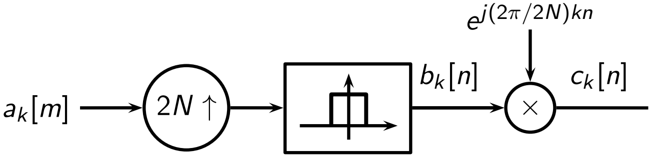
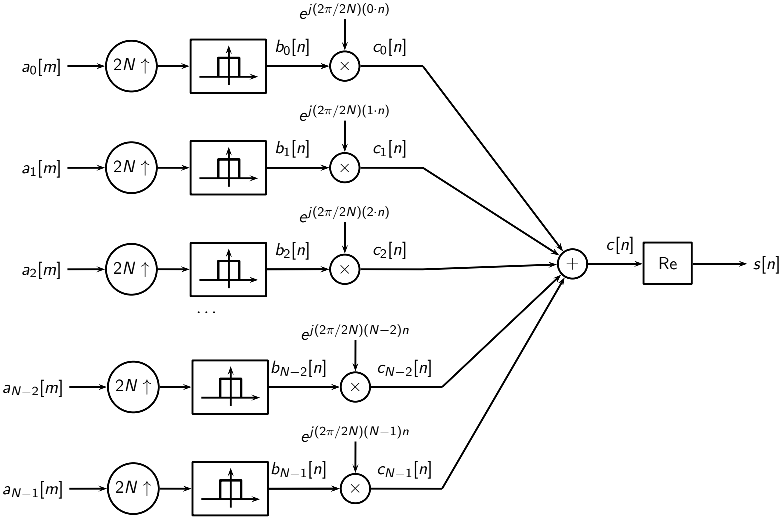
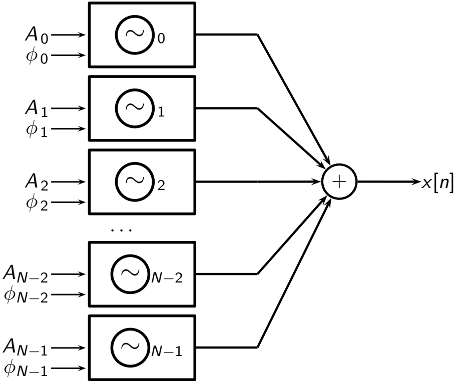


The bank of modems



If it looks familiar...

check back Module 4.3, the DFT reconstruction formula:



DMT via IFFT

- we will show that transmission can be implemented efficiently via an IFFT
- Discrete Multitone Modulation

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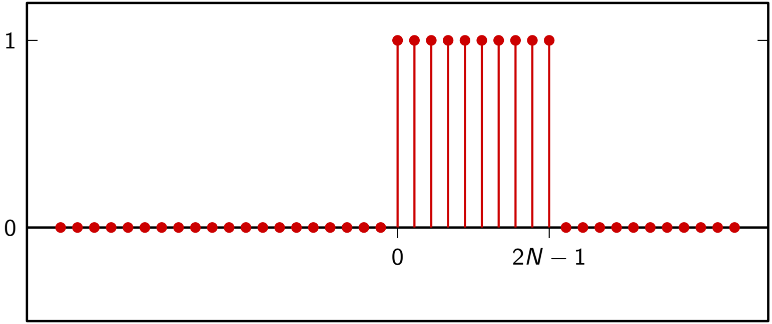
The great ADSL trick

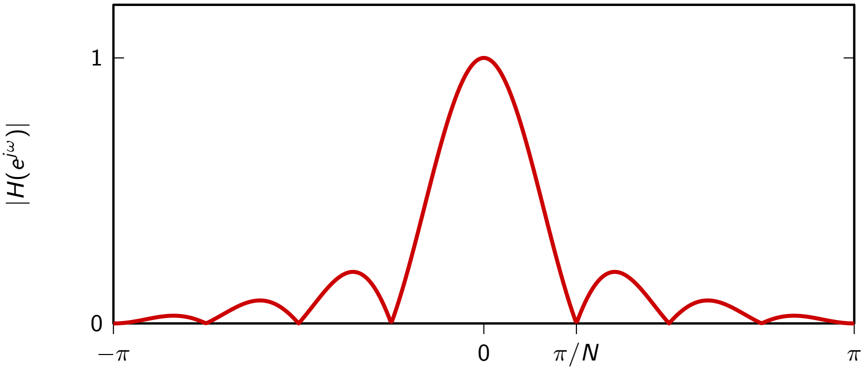
instead of using a good lowpass filter, use the $2N$-tap interval indicator:

$$h[n] = \begin{cases} 1 & \text{for } 0 \leq n < 2N \\ 0 & \text{otherwise} \end{cases}$$

Interval indicator signal (Module 4.7)



DTFT of interval signal (Module 4.7)



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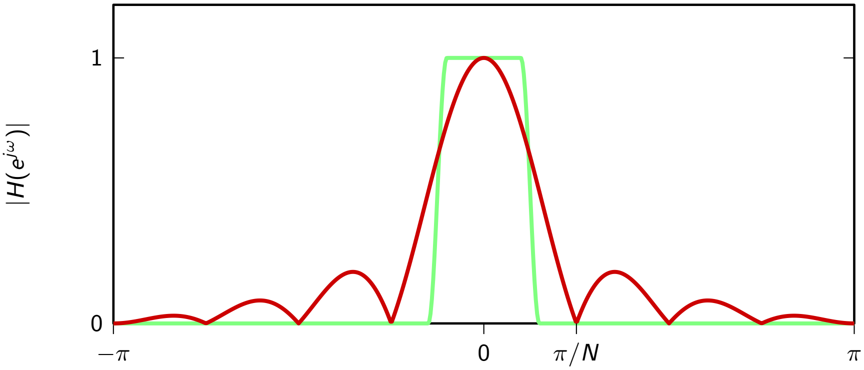
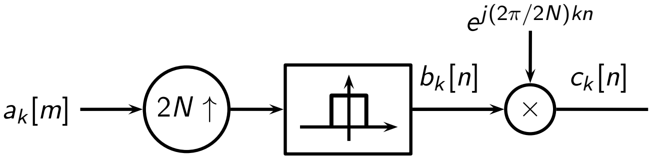


Back to the subchannel modem



rate: B symbols/sec

$2NB$ samples/sec

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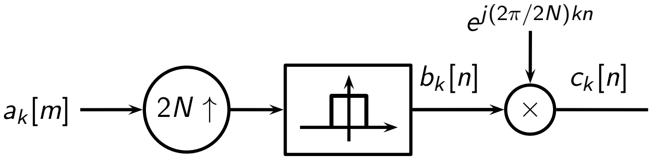

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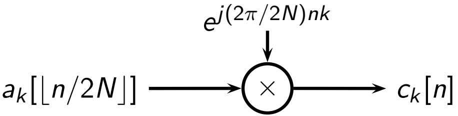
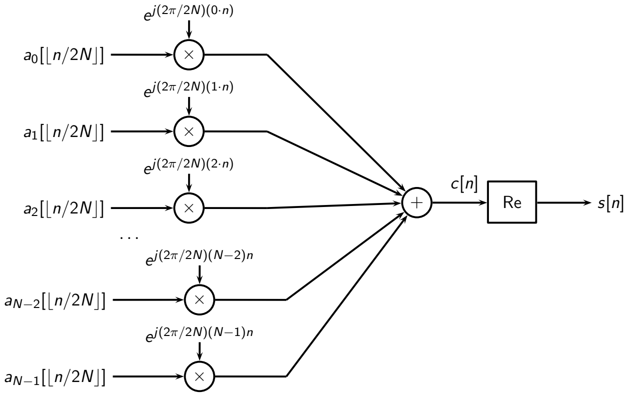
by using the indicator function as a lowpass:



The bank of modems, revisited



The complex output signal

$$c[n] = \sum_{k=0}^{N-1} a_k[\lfloor n/2N \rfloor] e^{j\frac{2\pi}{2N}nk}$$

$$= 2N \cdot \mathrm{IDFT}_{2N}\left\{\begin{bmatrix} a_0[m] & a_1[m] & \ldots & a_{N-1}[m] & 0 & 0 & \ldots & 0 \end{bmatrix}\right\}[n]$$

$$(m = \lfloor n/2N \rfloor)$$

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We can do even better!

- we are interested in $s[n] = \mathrm{Re}\{c[n]\} = (c[n] + c^*[n])/2$
- it is easy to prove (exercise) that:

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$$\mathrm{IDFT}\left\{\begin{bmatrix} x_0 & x_1 & x_2 & \ldots & x_{N-2} & x_{N-1}\end{bmatrix}\right\}^* = \mathrm{IDFT}\left\{\begin{bmatrix} x_0 & x_{N-1} & x_{N-2} & \ldots & x_2 & x_1\end{bmatrix}^*\right\}$$

- $c[n] = 2N \cdot \mathrm{IDFT}\left\{\begin{bmatrix} a_0[m] & a_1[m] & \ldots & a_{N-1}[m] & 0 & 0 & \ldots & 0\end{bmatrix}\right\}[n]$
- therefore

$$s[n] = N \cdot \mathrm{IDFT}\left\{\begin{bmatrix} 2a_0[m] & a_1[m] & \ldots & a_{N-1}[m] & a^*_{N-1}[m] & a^*_{N-2}[m] & \ldots & a^*_1[m]\end{bmatrix}\right\}[n]$$

We can do even better!

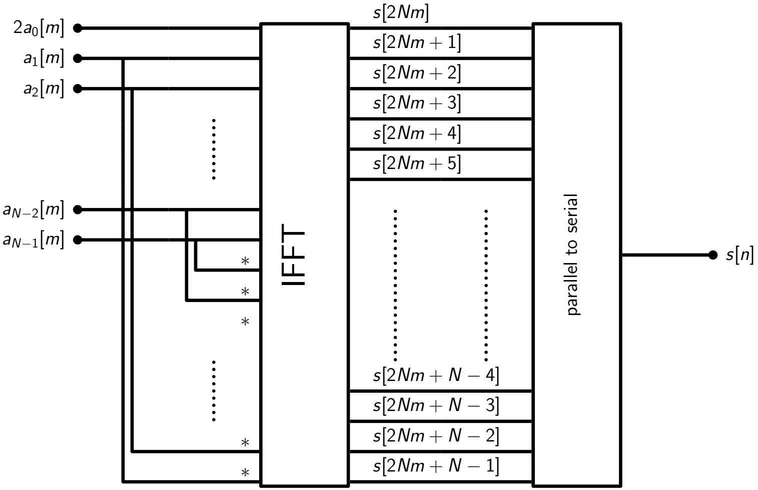
- we are interested in $s[n] = \mathrm{Re}\{c[n]\} = (c[n] + c^*[n])/2$
- it is easy to prove (exercise) that:

$$\mathrm{IDFT}\left\{\begin{bmatrix} x_0 & x_1 & x_2 & \ldots & x_{N-2} & x_{N-1} \end{bmatrix}\right\}^* = \mathrm{IDFT}\left\{\begin{bmatrix} x_0 & x_{N-1} & x_{N-2} & \ldots & x_2 & x_1 \end{bmatrix}^*\right\}$$

- $c[n] = 2N \cdot \mathrm{IDFT}\left\{\begin{bmatrix} a_0[m] & a_1[m] & \ldots & a_{N-1}[m] & 0 & 0 & \ldots & 0 \end{bmatrix}\right\}[n]$
- therefore

$$s[n] = N \cdot \mathrm{IDFT}\left\{\begin{bmatrix} 2a_0[m] & a_1[m] & \ldots & a_{N-1}[m] & a^*_{N-1}[m] & a^*_{N-2}[m] & \ldots & a^*_1[m] \end{bmatrix}\right\}[n]$$

ADSL transmitter



ADSL specs

- $F_{\text{max}} = 1104\text{KHz}$
- $N = 256$
- each QAM can send from 0 to 15 bits per symbol
- forbidden channels: 0 to 7 (voice)
- channels 7 to 31: upstream data
- max theoretical throughput: 14.9Mbps (downstream)

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END OF MODULE 9.6

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