Digital Signal Processing

Solved HW for Day 3

Question 1

Q: Derive a simple expression for the DFT of the time-reversed signal $\mathbf{x}_r = [x[N-1]\ x[N-2]\ x[1]\ x[0]]^T$ in terms of the DFT $\mathbf{X}$ of the signal $\mathbf{x}$.

Hint: you may find it useful to remark that $W_N^k = W_N^{-(N-k)}$.

Solution of question 1

- Recall the DFT (analysis) formula

$$X[k] = \sum_{n=0}^{N-1} x[n] W_N^{nk}.$$

- The DFT of the time-reversed signal can be written as

$$X_r[k] = \sum_{n=0}^{N-1} x_r[n] W_N^{nk} = \sum_{n=0}^{N-1} x[N-1-n] W_N^{nk}$$

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Solution of question 1

- By replacing W_N^n with $W_N^{-(N-n)}$, we get

$$X_r[k] = \sum_{n=0}^{N-1} x[N-1-n] W_N^{-(N-n)k} = \sum_{n=0}^{N-1} x[n] W_N^{-(n+1)k}$$

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$$= W_N^{-k} \sum_{n=0}^{N-1} x[n] W_N^{-nk} = W_N^{-k} \sum_{n=0}^{N-1} x[n] W_N^{n(N-k)}$$

Solution of question 1

- By replacing W_N^n with $W_N^{-(N-n)}$, we get

$$
\begin{aligned}
X_r[k] &= \sum_{n=0}^{N-1} x[N-1-n] W_N^{-(N-n)k} = \sum_{n=0}^{N-1} x[n] W_N^{-(n+1)k} \\
&= W_N^{-k} \sum_{n=0}^{N-1} x[n] W_N^{-nk} = W_N^{-k} \sum_{n=0}^{N-1} x[n] W_N^{n(N-k)} \\
&= W_N^{-k} X[N-k].
\end{aligned}
$$

Question 2: DFT manipulation

Consider a length-N signal $\mathbf{x} = [x[0]\ x[1]\ \dots\ x[N-1]]^T$ and the corresponding vector of DFT coefficients $\mathbf{X} = [X[0]\ X[1]\ \dots\ X[N-1]]^T$.

Consider now the length-$2N$ signal obtained by interleaving the values of $\mathbf{x}$ with zeros:
$\mathbf{x}_2 = [x[0]\ \ 0\ \ x[1]\ \ 0\ \ x[2]\ \ 0\ \ \dots\ \ x[N-1]\ \ 0]^T$.

Q: Express $\mathbf{X}_2$ (the $2N$-point DFT of $\mathbf{x}_2$) in terms of $\mathbf{X}$.

Solution of question 2

- Then, knowing that

$$W_{2N}^{2nk} = e^{-j\frac{2\pi}{2N}2nk} = e^{-j\frac{2\pi}{N}nk} = \begin{cases} W_N^{nk}, & 0 \leq k < N \\ W_N^{n(k-N)}, & N \leq k < 2N \end{cases}$$

we get

$$X_2[k] = \begin{cases} \sum_{n=0}^{N-1} x[n] W_N^{nk} & = X[k], & 0 \leq k < N \\ \sum_{n=0}^{N-1} x[n] W_N^{n(k-N)} & = X[(k-N)], & N \leq k < 2N \end{cases}$$

Question 3

Compute the DFT of the length-4 real signal $\mathbf{x} = [a, b, c, d]^T$.

Q: For which values of $a, b, c, d \in \mathbb{R}$ is the DFT real?

Solution of question 3

- The DFT of the length-4 real signal $\mathbf{x} = [a, b, c, d]^T$ is

$$X[k] = a + be^{-j\frac{2\pi}{4}k} + ce^{-j\frac{2\pi}{4}2k} + de^{-j\frac{2\pi}{4}3k}$$

Solution of question 3

- The DFT of the length-4 real signal $\mathbf{x} = [a, b, c, d]^T$ is

$$X[k] = a + be^{-j\frac{2\pi}{4}k} + ce^{-j\frac{2\pi}{4}2k} + de^{-j\frac{2\pi}{4}3k}$$
$$= a + b(-j)^k + c(-j)^{2k} + d(-j)^{3k}$$

Solution of question 3

- The DFT of the length-4 real signal $\mathbf{x} = [a, b, c, d]^T$ is

$$
\begin{aligned}
X[k] &= a + be^{-j\frac{2\pi}{4}k} + ce^{-j\frac{2\pi}{4}2k} + de^{-j\frac{2\pi}{4}3k} \\
&= a + b(-j)^k + c(-j)^{2k} + d(-j)^{3k} \\
&= \begin{cases} a+b+c+d, & k=0 \\ a-c-j(b-d), & k=1 \\ a-b+c-d, & k=2 \\ a-c+j(b-d), & k=3 \end{cases}
\end{aligned}
$$

Therefore, the DFT vector $\mathbf{X}$ is real iff $b = d$.