Digital Signal Processing

Solved HW for Day 12

Question 1: Another view of Sampling

One of the standard ways of describing the sampling operation relies on the concept of "modulation by a pulse train". Choose a sampling interval T_s and define a continuous-time pulse train $p(t)$ as:

$$p(t) = \sum_{k=-\infty}^{\infty} \delta(t - kT_s).$$

The Fourier Transform of the pulse train is

$$P(j\Omega) = \frac{2\pi}{T_s} \sum_{k=-\infty}^{\infty} \delta(\Omega - k\frac{2\pi}{T_s})$$

This is tricky to show, so just take the result as is. The "sampled" signal is simply the modulation of an arbitrary-continuous time signal $x(t)$ by the pulse train:

$$x_s(t) = p(t)\,x(t).$$

Derive the Fourier transform of $x_s(t)$ and show that if $x(t)$ is bandlimited to π/T_s then we can reconstruct $x(t)$ from $x_s(t)$.

Solution of question 1

By using the modulation theorem, we have

$$X_s(j\Omega) = X(j\Omega)P(j\Omega)$$
$$= \int_{\mathbb{R}} X(j\tilde{\Omega})P(j(\Omega - \tilde{\Omega}))d\tilde{\Omega} = \frac{2\pi}{T_s} \int_{\mathbb{R}} X(j\tilde{\Omega}) \sum_{k\in\mathbb{Z}} \delta\left(\Omega - \tilde{\Omega} - k\frac{2\pi}{T_s}\right) d\tilde{\Omega}$$

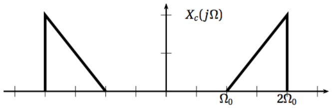
$$X_s(j\Omega) = X(j\Omega) * P(j\Omega)$$
$$= \int_{\mathbb{R}} X(j\tilde{\Omega})P(j(\Omega - \tilde{\Omega}))d\tilde{\Omega} = \frac{2\pi}{T_s} \int_{\mathbb{R}} X(j\tilde{\Omega}) \sum_{k\in\mathbb{Z}} \delta\left(\Omega - \tilde{\Omega} - k\frac{2\pi}{T_s}\right) d\tilde{\Omega}$$
$$= \frac{2\pi}{T_s} \sum_{k\in\mathbb{Z}} \int_{\mathbb{R}} X(j\tilde{\Omega})\delta\left(\Omega - \tilde{\Omega} - k\frac{2\pi}{T_s}\right) d\tilde{\Omega} = \frac{2\pi}{T_s} \sum_{k\in\mathbb{Z}} X\left(j\left(\Omega - k\frac{2\pi}{T_s}\right)\right).$$

Solution of question 1

In other words, the spectrum of the delta-modulated signal is just the periodic repetition (with period $(2\pi/T_s)$) of the original spectrum. If the latter is bandlimited to (π/T_s) there will be no overlap and therefore $x(t)$ can be obtained simply by lowpass filtering $x_s(t)$ (in the continuous-time domain).

Question 2: Aliasing can be good! Part-1

Consider a signal $x_c(t)$ with the following spectrum:



- What is the bandwidth of the signal? What is the minimum sampling period in order to satisfy the sampling theorem?
- Take a sampling period $T_s = \frac{\pi}{\Omega_0}$; clearly, with this sampling period, there will be aliasing. Plot the DTFT of the discrete-time signal $x_a[n] = x_c(nT_s)$.
- Suggest a block diagram to reconstruct $x_c(t)$ from $x_a[n]$.
- Therefore, we can exploit aliasing to reduce the sampling frequency necessary to sample a bandpass signal. What is the minimum sampling frequency to be able to reconstruct with the above strategy a real signal whose frequency support on the positive axis is $[\Omega_0, \Omega_1]$?

Solution of question 2

- The highest nonzero frequency is $2\Omega_0$ and therefore $x_c(t)$ is $2\Omega_0$-bandlimited for a total bandwidth of $4\Omega_0$.
- The maximum sampling period (i.e. the inverse of the *minimum* sampling frequency) which satisfies the sampling theorem is $T_s = \pi/(2\Omega_0)$.
- Note however that the total support over which the (positive) spectrum is nonzero is the interval $[\Omega_0, 2\Omega_0]$ so that one could say that the total *effective* positive bandwidth of the signal is just Ω_0.

Solution of question 2

Q: Plot the DTFT of the discrete-time signal $x_a[n] = x_c(nT_s)$.

- The digital spectrum will be the rescaled version of the periodized continuous-time spectrum

$$\tilde{X}_c(j\Omega) = \sum_{k=-\infty}^{\infty} X_c(j(\Omega - 2k\Omega_0)).$$

- The general term $X_c(j\Omega - j2k\Omega_0)$ is nonzero for $\Omega_0 \leq |\Omega - 2k\Omega_0| \leq 2\Omega_0 \quad \text{for } k \in \mathbb{Z}$. or equivalently

$$\begin{aligned} (2k+1)\Omega_0 &\leq \Omega \leq (2k+2)\Omega_0 \\ (2k-2)\Omega_0 &\leq \Omega \leq (2k-1)\Omega_0 \end{aligned}$$

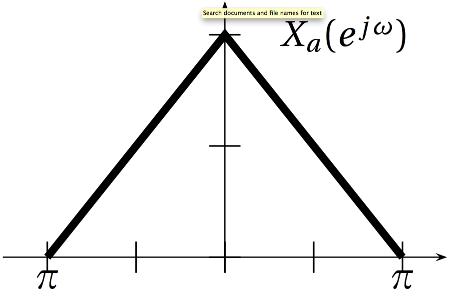
- Non-overlapping intervals, therefore, no disruptive superpositions of the copies of the spectrum!

Solution of question 2

- The digital spectrum is

$$X(e^{j\omega}) = \frac{1}{T_s} \sum_{k=-\infty}^{\infty} X_c(j\frac{\omega}{T_s} - j\frac{2\pi k}{T_s})$$

which looks like this (with 2π-periodicity, of course):



Solution of question 2

Q: Suggest a block diagram to reconstruct $x_c(t)$ from $x_a[n]$.

Here's a possible scheme (verify that it works):

- Sinc-interpolate $x_a[n]$ with period T_s to obtain $x_b(t)$
- Multiply $x_b(t)$ by $\cos(2\Omega_0 t)$ in the continuous time domain to obtain $x_p(t)$ (i.e. modulate by a carrier at frequency (Ω_0/π) Hz).
- Bandpass filter $x_p(t)$ with an ideal bandpass filter with (positive) passband equal to $[\Omega_0, 2\Omega_0]$ to obtain $x_c(t)$.

Solution of question 2

Q: What is the minimum sampling frequency to be able to reconstruct with the above strategy a real signal whose frequency support on the positive axis is $[\Omega_0, \Omega_1]$?

- The effective *positive* bandwidth of such a signal is $\Omega_\Delta = (\Omega_1 - \Omega_0)$.
- The sampling frequency must be at least equal to the effective total bandwidth, so a first condition on the maximum allowable sampling period: $T_{\max} < \pi/\Omega_\Delta$.
- Case 1: assume Ω_1 is a multiple of the bandwidth, i.e. $\Omega_1 = M\Omega_\Delta$ for some integer M (in the previous case, it was $M = 2$).
  In this case, the argument we made in the previous point can be easily generalized: if we pick $T_s = \pi/\Omega_\Delta$ and sample we have that

$$\tilde{X}_c(j\Omega) = \sum_{k=-\infty}^{\infty} X_c(j(\Omega - 2k\Omega_\Delta)).$$

Solution of question 2

- The general term $X_c(j\Omega - j2k\Omega_\Delta)$ is nonzero only for

$$\Omega_0 \leq |\Omega - 2k\Omega_\Delta| \leq \Omega_1 \quad \text{for } k \in \mathbb{Z}.$$

- Since $\Omega_0 = \Omega_1 - \Omega_\Delta = (M-1)\Omega_\Delta$, this translates to

$$\begin{aligned} (2k+M-1)\Omega_\Delta \quad &\leq \Omega \leq \quad (2k+M)\Omega_\Delta \\ (2k-M)\Omega_\Delta \quad &\leq \Omega \leq \quad (2k-M+1)\Omega_\Delta \end{aligned}$$

again non-overlapping intervals!

Solution of question 2

- Case 2: Ω_1 is *not* a multiple of the bandwidth.
  The easiest thing to do is to change the lower frequency Ω_0 to a new frequency Ω_0' so that the new bandwidth $\Omega_1 - \Omega_0'$ divides Ω_1 exactly. In other words we set a new lower frequency Ω_0' so that it will be $\Omega_1 = M(\Omega_1 - \Omega_0')$ for some integer M; it is easy to see that

$$M = \left\lfloor \frac{\Omega_1}{\Omega_1 - \Omega_0} \right\rfloor.$$

  since this is the maximum number of copies of the Ω_Δ-wide spectrum which fit *with no overlap* in the $[0, \Omega_0]$ interval.

- If $\Omega_\Delta > \Omega_0$ we cannot hope to reduce the sampling frequency and we have to use normal sampling. This artificial change of frequency will leave a small empty "gap" in the new bandwidth $[\Omega_0', \Omega_1]$, but that's no problem.

Solution of question 2

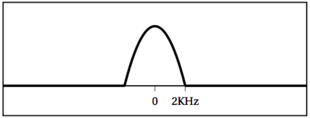
- Now we use the previous result and sample with $T_s = \pi/(\Omega_1 - \Omega_0')$ with no overlap. Since $(\Omega_1 - \Omega_0') = \Omega_1/M$, we have that, in conclusion, the maximum sampling period is

$$T_{\max} = \frac{\pi}{\Omega_1}\left\lfloor \frac{\Omega_1}{\Omega_1 - \Omega_0} \right\rfloor$$

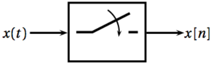
i.e. we obtain a sampling frequency reduction factor of $\lfloor \Omega_1/(\Omega_1 - \Omega_0) \rfloor$.

Question 3

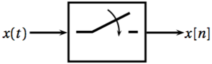
Consider a bandlimited continuous-time signal $x(t)$ with a spectrum $X(j\Omega)$ as sketched in the following figure:



Sketch the DTFT of the output signal for each of the following systems, where the frequency of the raw sampler is indicated on each line:

- 

$T_s = 4\text{KHz}$

- 

$T_s = 2\text{KHz}$

Solution of question 3