Digital Signal Processing

Solved HW for Day 1

Question 1: Elementary operators and their properties

An operator S is a transformation of a given signal and is indicated by the notation:

$$y[n] = S\{x[n]\}.$$

For instance, the delay operator D is indicated as $D\{x[n]\} = x[n-1]$, and the differentiation operator is indicated as $\Delta\{x[n]\} = x[n] - D\{x[n]\} = x[n] - x[n-1]$.
A linear operator is one for which the following holds:

$$\begin{cases} S\{\alpha x[n]\} = \alpha S\{x[n]\} \\ S\{x[n] + y[n]\} = S\{x[n]\} + S\{y[n]\}. \end{cases}$$

Solution of Question 1

Q: Show that the delay operator D is linear.

$$D\{\alpha x[n]\} = \alpha x[n-1] = \alpha D\{x[n]\}$$
$$D\{x[n] + y[n]\} = x[n-1] + y[n-1] = D\{x[n]\} + D\{y[n]\}.$$

Solution of Question 1

Q: Show that the differentiation operator Δ is linear.

Δ is a *linear combination* of the identify operator with the linear operator D, therefore it is also linear.

Q: Show that the squaring operator $S\{x[n]\} = x^2[n]$ is not linear

$$S\{\alpha x[n]\} = \alpha^2 x^2[n] = \alpha^2 S\{x[n]\} \neq \alpha S\{x[n]\}.$$

Question 2: Follow-up of Question 1

In $\mathbb{C}^N$, any linear operator on a vector $\mathbf{x}$ can be expressed as a matrix-vector multiplication for a suitable matrix $\mathbf{A}$. Define the delay operator as the right circular shift of a vector:
$D\{\mathbf{x}\} = [x_{N-1} x_0 x_1 \dots x_{N-2}]^T$.

Assume $N = 4$ for convenience; it is easy to see that: $D\{\mathbf{x}\} = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \mathbf{x} = \mathbf{D}\mathbf{x}$.

Solution of Question 2

Q: Using the same definition of differentiation operator as in Question 1, write out the matrix form of the differentiation operator in $\mathbb{C}^4$.

$$\mathbf{\Delta} = \mathbf{I} - \mathbf{D} = \begin{bmatrix} 1 & 0 & 0 & -1 \\ -1 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & -1 & 1 \end{bmatrix}.$$

Solution of Question 2

Q: Consider the following matrix: $\mathbf{A} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 \end{bmatrix}$.

Which operator do you think it corresponds to?

The matrix realizes an integration operation over a vector in $\mathbb{C}^4$.

Question 3: Bases

Let $\{x(k)\}_{k=0,\ldots,N-1}$ be a basis for a subspace S.
Prove that any vector $z \in S$ is uniquely represented in this basis.

Hint: prove by contradiction.

Solution of Question 3

- Suppose by contradiction that the vector $\mathbf{z} \in S$ admits two distinct representations in the basis $\{\mathbf{x}^{(k)}\}_{k=0,\ldots,N-1}$:

$$\exists\{\alpha_0, \ldots, \alpha_{N-1}\}, \{\beta_0, \ldots, \beta_{N-1}\} \text{ such that:}$$

$$(\alpha_0, \ldots, \alpha_{N-1}) \neq (\beta_0, \ldots, \beta_{N-1})$$

$$\mathbf{z} = \Sigma_{k=0}^{N-1} \alpha_k \mathbf{x}^{(k)}, \mathbf{z} = \Sigma_{k=0}^{N-1} \beta_k \mathbf{x}^{(k)}.$$

- Thus, $\Sigma_{k=0}^{N-1} \alpha_k \mathbf{x}^{(k)} = \Sigma_{k=0}^{N-1} \beta_k \mathbf{x}^{(k)}$ or, equivalently, $\Sigma_{k=0}^{N-1} (\alpha_k - \beta_k) \mathbf{x}^{(k)} = 0$.
- Since $\{\mathbf{x}^{(k)}\}_{k=0,\ldots,N-1}$ is a basis, it is a set of independent vectors and, by definition, the above equation admits only the trivial solution $\alpha_k - \beta_k = 0$, $\forall k = 0, \ldots, N-1$.
- Thus, $\alpha_k = \beta_k$, $\forall k = 0, \ldots, N-1$ which concludes the proof.

Question 4: Review of complex numbers

Q: Let $s[n] := \frac{1}{2^n} + j\frac{1}{3^n}$. Compute $\sum_{n=1}^{\infty} s[n]$.

Question 4: Review of complex numbers

Recall that

$$\sum_{i=0}^{N} z^k = \begin{cases} \frac{1-z^{N+1}}{1-z} & \text{for } z \neq 1 \\ N+1 & \text{for } z = 1\,. \end{cases}$$

Proof for $z \neq 0$ (for $z = 1$ is trivial)

$$s = 1 + z + z^2 + \ldots + z^N\,,$$

$$-zs = -z - z^2 - \ldots - z^N - z^{N+1}\,.$$

Summing the above two equations gives

$$(1-z)s = 1 - z^{N+1} \Rightarrow s = \frac{1-z^{N+1}}{1-z}\,.$$

Similarly

$$\sum_{k=N_1}^{N_2} z^k = z^{N_1} \sum_{k=0}^{N_2-N_1} z^k = \frac{z^{N_1} - z^{N_2+1}}{1-z}\,.$$

Question 4: Review of complex numbers

Q: Let $s[n] := \frac{1}{2^n} + j\frac{1}{3^n}$. Compute $\sum_{n=1}^{\infty} s[n]$.

We have

$$\sum_{n=1}^{N} s[n] = \sum_{n=1}^{N} 2^{-n} + j\sum_{n=1}^{N} 3^{-n}$$

$$= \frac{1}{2} \cdot \frac{1-2^{-N}}{1-2^{-1}} + j\frac{1}{3} \cdot \frac{1-3^{-N}}{1-3^{-1}} = \left(1-2^{-N}\right) + j\frac{1}{2}\left(1-3^{-N}\right).$$

Now,

$$\lim_{N\to\infty} 2^{-N} = \lim_{N\to\infty} 3^{-N} = 0\,.$$

Therefore,

$$\sum_{n=1}^{\infty} s[n] = 1 + \frac{1}{2}j\,.$$

Q: Same question with $s[n] := \left(\frac{j}{3}\right)^n$.

We can write

$$\sum_{k=1}^{N} s[k] = \frac{j}{3} \cdot \frac{1-(j/3)^N}{1-j/3}\,.$$

Since $|\frac{j}{3}| = \frac{1}{3} < 1$, we have $\lim_{N\to\infty}(j/3)^N = 0$. Therefore,

$$\sum_{k=1}^{\infty} s[k] = \frac{j}{3-j} = \frac{j(3+j)}{10} = -\frac{1}{10} + j \cdot \frac{3}{10}\,.$$

Question 4: Review of complex numbers

Q: Characterize the set of complex numbers satisfying $z^* = z^{-1}$.

From $z^* = z^{-1}$ with $z \in \mathbb{C}$, we have

$$zz^* = 1, \quad \forall\, z \neq 0.$$

Therefore, $|z|^2 = 1$ and, consequently, $|z| = 1$. It follows that all the z such that $z^* = z^{-1}$ describe the unit circle.

Question 4: Review of complex numbers

Q: Find 3 complex numbers $\{z_0, z_1, z_2\}$ which satisfy $z_i^3 = 1, i = 1, 2, 3.$

Remark that $e^{j2k\pi} = 1$, for all $k \in \mathbb{Z}$. Therefore, $z_k = e^{j\frac{2k\pi}{3}}$ is such that $z_k^3 = 1$. Now z_k is periodic of period 3, i.e. $z_k = z_{k+3l}$, for all $l \in \mathbb{Z}$. Therefore the (only) three different complex numbers are

$$z_0 = 1, \quad z_1 = e^{j\frac{2\pi}{3}} \quad \text{and} \quad z_2 = e^{j\frac{4\pi}{3}}.$$

Question 4: Review of complex numbers

Q: What is the following infinite product $\prod_{n=1}^{\infty} e^{j\pi/2^n}$?

We have

$$\prod_{n=1}^{N} e^{j\frac{\pi}{2^n}} = e^{j\pi \sum_{n=1}^{N} 2^{-n}} = e^{j\pi \frac{1}{2} \cdot \frac{1-2^{-N}}{1-1/2}}.$$

Since $\lim_{N\to\infty} 2^{-N} = 0$,

$$\prod_{n=1}^{\infty} e^{j\frac{\pi}{2^n}} = e^{j\pi} = -1.$$

Q: Find the the eigenvalues and eigenvectors of $\mathbf{A} = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}$

$$\mathbf{A} = \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}$$

$$c(t) = \det\left(\lambda \boldsymbol{I} - \boldsymbol{A}\right) = \det\begin{pmatrix} \lambda - 1 & -2 \\ -2 & \lambda - 1 \end{pmatrix} = (\lambda - 1)^2 - 4 = (\lambda - 3)(\lambda + 1) \; .$$

Therefore, the eigenvalues are -1 and 3.

Question 5: Review of eigenvalues and eigenvectors

The eigenvector $\boldsymbol{x} = [x_1, x_2]$ corresponding to the eigenvalue λ is computes as the solution of the equation

$$\boldsymbol{A}\boldsymbol{x} = \lambda \boldsymbol{x} \,.$$

The eigenvector corresponding to $\lambda = -1$ is given by

$$\boldsymbol{A}\boldsymbol{x} = -\boldsymbol{x} \,.$$

We get $x_2 = -x_1$. Choosing $x_1 = 1$ and $x_2 = -1$, after normalization we obtain

$$\boldsymbol{x} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -1 \end{pmatrix} \,.$$

Following the same approach, the eigenvector corresponding to $\lambda = 3$ is

$$\boldsymbol{x} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix} .$$

Question 5: Review of eigenvalues and eigenvectors

Q: Find the the eigenvalues and eigenvectors of $\mathbf{B} = \begin{bmatrix} \alpha & \beta \\ \beta & \alpha \end{bmatrix}$.

$$\det(\lambda \boldsymbol{I} - \boldsymbol{B}) = (\lambda - (\alpha + \beta))(\lambda - (\alpha - \beta)) \; .$$

The eigenvalues are $\alpha - \beta$ and $\alpha + \beta$. The eigenvectors are then

$$\boldsymbol{x} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -1 \end{pmatrix} \qquad \text{for } \lambda = \alpha - \beta$$

$$\boldsymbol{x} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix} \qquad \text{for } \lambda = \alpha + \beta.$$

Remark that if $\beta = 0$, there is only an eigenvalue with a corresponding eigenvector

$$\boldsymbol{x} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix} \qquad \text{for } \lambda = \alpha.$$

Question 5: Review of eigenvalues and eigenvectors

Q: Show that $\mathbf{A} = \mathbf{V}\mathbf{D}\mathbf{V}^T$ where the columns of $\mathbf{V}$ correspond to the eigenvectors of $\mathbf{A}$ and $\mathbf{D}$ is a diagonal matrix whose main diagonal corresponds to the eigenvalues of $\mathbf{A}$.

$$\boldsymbol{V} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}$$

$$\boldsymbol{V}\boldsymbol{D}\boldsymbol{V}' = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix} \cdot \begin{pmatrix} -1 & 0 \\ 0 & 3 \end{pmatrix} \cdot \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} = \frac{1}{2}\begin{pmatrix} 2 & 4 \\ 4 & 2 \end{pmatrix} = \boldsymbol{A}$$

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