# 时间序列复习

## 题型分析

| 题型 | 分值 | 考点 |
| --- | --- | --- |
| 填空题 | 24分/8题 | 上课笔记 |
| 选择题 | 18分/6题 | 上课笔记 |
| 简答题 | 8分/1题 | 建模流程图 |
| 计算题 | 30分/2题 | 作业 |
| 综合题 | 20分/1题 | 作业、课后习题 |

## 作业题详解

### 第一题

若序列长度为 100, 前 12 个样本自税关系数如下. :
$\rho_1=0.02, \rho_2=0.05, \rho_3=0.10, \rho_4=-0.02, \rho_5=0.05, \rho_6=0.01$
$\rho_7=0.12, \rho_8=-0.06, \rho_9=0.08, \rho_{10}=-0.05, \rho_{11}=0.02, \rho_{12}=-0.05$ . 该序列能否视为纯随机序列 $(\alpha=0.05)$?

**解答**

计算该序列各阶延迟的Q统计量及相应P值。由于延迟1-12阶Q统计量的p值均显著大于0.05，所以该序列为纯随机序列。

| 阶数 | Q | p | 阶数 | Q | p |
| --- | --- | --- | --- | --- | --- |
| 1 | 0.04 | 0.8414806 | 7 | 3.03 | 0.8822132 |
| 2 | 0.29 | 0.8650223 | 8 | 3.39 | 0.9075571 |
| 3 | 1.29 | 0.731509 | 9 | 4.03 | 0.9094269 |
| 4 | 1.33 | 0.8562654 | 10 | 4.28 | 0.9338329 |
| 5 | 1.58 | 0.9036573 | 11 | 4.32 | 0.9596046 |
| 6 | 1.59 | 0.9532936 | 12 | 4.57 | 0.9708238 |

### 第二题

已知某 $\mathrm{AR}(2)$ 模型为: $x_t=\phi_1 x_{t-1}+\phi_2 x_{t-2}+\varepsilon_t, \varepsilon_t \sim \mathrm{WN}\left(0, \sigma_{\varepsilon}^2\right)$, 且 $\rho_1=0.5, \rho_2=0.3$, 求 $\phi_1, \phi_2$ 的值。

**解答**

$AR(2)$ 模型有:

$$\begin{cases}\rho_1 = \frac{\phi_1}{1-\phi_2} \\ \rho_2 = \phi_1\rho_1 + \phi_2\end{cases} \Rightarrow \begin{cases}0.5 = \frac{\phi_1}{1-\phi_2} \\ 0.3 = 0.5\phi_1 + \phi_2\end{cases} \Rightarrow \begin{cases}\phi_1 = \frac{7}{15}, \phi_2 = \frac{1}{15} \\ \phi_2 = \frac{1}{15}\end{cases}$$

## 第三题

已知某 $AR(2)$ 模型为 : $(1-0.5B)(1-0.3B)x_t = \varepsilon_t, \varepsilon_t \sim \mathrm{WN}(0,1)$, 求 $E\left(x_t\right)$, $\operatorname{Var}\left(x_t\right), \rho_k, \phi_{kk}$, 其中 $k=1,2,3$ 。

**解答**

$$\begin{aligned}
&(1)\ (1-0.5B)(1-0.3B)x_t = \varepsilon_t \Leftrightarrow x_t = 0.8x_{t-1} - 0.15x_{t-2} + \varepsilon_t \\
&\quad E\left(x_t\right) = \frac{\phi_0}{1-\phi_1-\phi_2} = 0 \\
&(2)\ \operatorname{Var}\left(x_t\right) = \frac{1-\phi_2}{\left(1+\phi_2\right)\left(1-\phi_1-\phi_2\right)\left(1+\phi_1-\phi_2\right)} \\
&\qquad = \frac{1+0.15}{(1-0.15)(1-0.8+0.15)(1+0.8+0.15)} \\
&\qquad = 1.98 \\
&(3)\ \rho_1 = \frac{\phi_1}{1-\phi_2} = \frac{0.8}{1+0.15} = 0.70 \\
&\quad \rho_2 = \phi_1\rho_1 + \phi_2 = 0.8 \times 0.7 - 0.15 = 0.41 \\
&\quad \rho_3 = \phi_1\rho_2 + \phi_2\rho_1 = 0.8 \times 0.41 - 0.15 \times 0.7 = 0.22 \\
&(4)\ \phi_{11} = \rho_1 = 0.7 \\
&\quad \phi_{22} = \phi_2 = -0.15 \\
&\quad \phi_{33} = 0
\end{aligned}$$

## 第四题

某 $\mathrm{ARMA}(2,2)$ 模型为: $\Phi(B)x_t = 3 + \Theta(B)\varepsilon_t$, 求 $E\left(x_t\right)$ 。其中:
$\varepsilon_t \sim \mathrm{WN}\left(0, \sigma_{\varepsilon}^2\right), \Phi(\mathrm{B}) = (1-0.5B)^2$

**解答**

$$\Theta(B) = (1-0.5B)^2 \Rightarrow \phi_1 = 0.5, \quad \phi_2 = -0.25$$
$$E\left(x_t\right) = \frac{\phi_0}{1-\phi_1-\phi_2} = \frac{3}{1-0.5+0.25} = 4$$

## 第五题

对于 $\mathrm{AR}(1)$ 模型: $x_t - \mu = \phi_1\left(x_{t-1} - \mu\right) + \varepsilon_t$, 根据 $t$ 个历史观察值数据: $\cdots, 10.1, 9.6$, 已求出 $\hat{\mu} = 10, \hat{\phi}_1 = 0.3, \hat{\sigma}_{\epsilon}^2 = 9$ 。
(1) 求 $x_{t+3}$ 的 $95\%$ 的置信区间。
(2) 假定新获得观察值数据 $x_{t+1} = 10.5$, 用更新数据求 $x_{t+3}$ 的 $95\%$ 的置信区间。

**解答**

(1)

$$\begin{aligned}
&\phi_0 = \mu\left(1-\phi_1\right) = 10 \times (1-0.3) = 7 \\
&\hat{x}_{t+1} = 7 + 0.3 \times 9.6 = 9.88 \\
&\hat{x}_{t+2} = 7 + 0.3 \times 9.88 = 9.964 \\
&\hat{x}_{t+3} = 7 + 0.3 \times 9.964 = 9.9892 \\
&\operatorname{Var}\left(\hat{x}_{t+3}\right) = \left(G_0^2 + G_1^2 + G_2^2\right)\sigma_{\varepsilon}^2 = \left(1 + 0.3^2 + 0.3^4\right) \times 9 = 9.8829
\end{aligned}$$

所以 $\hat{x}_{t+3}$ 的 $95\%$ 的置信区间等于 $9.9892 \pm 1.96\sqrt{9.8829}$, 即 $(3.83, 16.15)$
(2) 更新数据后

$$\hat{x}_{t+2} = 7 + 0.3 \times 10.5 = 10.15$$
$$\hat{x}_{t+3} = 7 + 0.3 \times 10.15 = 10.045$$
$$\mathrm{Var}\left(\hat{x}_{t+3}\right) = \left(G_0^2 + G_1^2\right)\sigma_\varepsilon^2 = \left(1 + 0.3^2\right) \times 9 = 9.81$$

所以 $\hat{x}_{t+3}$ 的 $95\%$ 的置信区间等于 $10.045 \pm 1.96\sqrt{9.819}$, 即 $(3.91, 16.18)$

## 第六题

判断下列模型是否平稳

1. $x_t = 0.4 + 0.6x_{t-1} + \epsilon_t$
2. $x_t = 0.56 + 0.5x_{t-1} - 0.06x_{t-2} + \epsilon_t$
3. $x_t = 2x_{t-1} - x_{t-2} + \epsilon_t$

**解答**

1. (特征根判别法) 先中心化，即$\mu = \frac{\phi_0}{1-\phi_1} = 1$,令$y_t = x_t - 1$,原式变为$y_t = 0.6y_{t-1} + \epsilon_t$,设$y_t = \lambda^t$为$y_t - 0.6y_{t-1} = 0$的解，则$\lambda^t - 0.6\lambda^{t-1} = 0 \Rightarrow \lambda = 0.6$,由于$|\lambda| < 1$,可知模型平稳
2. (平稳域判别法) 先中心化，即$\mu = \frac{\phi_0}{1-\phi_1-\phi_2} = 1$,令$y_t = x_t - 1$,原式变为$y_t = 0.5y_{t-1} - 0.06y_{t-2} + \epsilon_t$,可以知道$\phi_1 = 0.5, \phi_2 = -0.06$,于是有
$$\begin{cases} |\phi_2| < 1 \\ \phi_2 + \phi_1 < 1 \\ \phi_2 - \phi_1 < 1 \end{cases}$$
故模型平稳
3. (Green函数判别法)设模型的传递函数为$x_t = \sum\limits_{j=1}^{\infty} G_j\epsilon_{t-j}$,则有
$(1 - \alpha B + B^2)(G_0 + G_1B + \cdots)\epsilon_j = \epsilon_j$
$$\begin{cases} G_0 = 1 \\ G_1 - 2G_0 = 0 \\ G_2 - 2G_1 + G_0 = 0 \\ \qquad \cdots \\ G_j - 2G_{j-1} + G_{j-2} = 0 (j \geq 2) \end{cases} \Rightarrow \begin{cases} G_0 = 1 \\ G_1 = 2 \\ G_j = 1 + G_{j-1} \\ G_j = \frac{(-1)^{j+1}}{2} \end{cases}$$
由于$\lim\limits_{j\to\infty} G_j = 0$,因此该模型平稳。

## 第七题

判断下列模型是否可逆

1. $x_t = \epsilon_t - 0.5\epsilon_{t-1}$
2. $x_t = \epsilon_t - 0.6\epsilon_{t-1} + 0.09\epsilon_{t-2}$

**解答**

1. 设模型的逆转形式为$\epsilon_t = \sum\limits_{j=0}^{\infty} I_jx_{t-j}$,即$x_t = (1 - 0.5B)(I_0 + I_1B + \cdots)x_t$,于是有

$$\begin{cases} I_0 = 1 \\ I_1 - 0.5I_0 = 0 \\ \quad \cdots \\ I_j - 0.5I_{j-1} = 0 (j \geq 1) \end{cases} \Rightarrow \begin{cases} I_0 = 1 \\ I_j = 0.5^j \end{cases}$$

由于$\lim\limits_{j\to\infty} I_j = 0$,因此该模型可逆。

2. 设模型的逆转形式为$\epsilon_t = \sum\limits_{j=0}^{\infty} I_j x_{t-j}$,即$x_t = (1 - 0.6B + 0.09B^2)(I_0 + I_1 B + \cdots)x_t$,于是有

$$\begin{cases} I_0 = 1 \\ I_1 - 0.6I_0 = 0 \\ I_2 - 0.6I_1 + 0.09I_0 = 0 \\ \quad \cdots \\ I_j - 0.6I_{j-1} + 0.09I_{j-2} = 0 (j \geq 2) \end{cases} \Rightarrow \begin{cases} I_0 = 1 \\ I_j - 0.3I_{j-1} = 0.3(I_{j-1} - 0.3I_{j-2}) \\ I_j = (j+1) \times 0.3^j \end{cases}$$

由于$\lim\limits_{j\to\infty} I_j = 0$,因此该模型可逆。

## 第八题

判断下列模型的平稳性与可逆性

1. $x_t = x_{t-1} - 0.5x_{t-2} + \epsilon_t - \frac{4}{5}\epsilon_{t-1} + \frac{16}{25}\epsilon_{t-2}$
2. $x_t = 0.5x_{t-1} - 0.06x_{t-2} + \epsilon_t - 0.8\epsilon_{t-1} + 0.16\epsilon_{t-2}$

**解答**

1. (平稳性) 令$\lambda^t$代入$x_t - x_{t-1} + 0.5x_{t-2} = 0$得$\lambda^t - \lambda^{t-1} + 0.5\lambda^{t-2} = 0$,可以解得$\lambda_1 = \frac{1+i}{2}, \lambda_2 = \frac{1-i}{2}$,由于$|\lambda_1| < 1, |\lambda_2| < 1$,因此该模型平稳

   (可逆性) 令$\epsilon_t = \lambda^t$代入$\epsilon_t - \frac{4}{5}\epsilon_{t-1} + \frac{16}{25}\epsilon_{t-2} = 0$得$\lambda^t - \frac{4}{5}\lambda^{t-2} = 0$，可以解得$\lambda_1 = \frac{2}{5} + \frac{2\sqrt{3}}{5}i, \lambda_2 = \frac{2}{5} - \frac{2\sqrt{3}}{5}i$,由于$|\lambda_1| = |\lambda_2| = \frac{4}{5} < 1$,因此该模型可逆
   【特征根判别法】

2. (平稳性) 设模型传递形式为$x_i = \sum\limits_{j=0}^{\infty} G_j \epsilon_{t-j}$,即
   $(1 - 0.5B + 0.06B^2)(G_0 + G_1 B + \cdots)\epsilon_t = (1 - 0.8B + 0.16B^2)\epsilon_t$,于是有

$$\begin{cases} G_0 = 1 \\ G_1 - 0.5G_0 = -0.8 \\ G_2 - 0.5G_1 + 0.06G_0 = 0.16 \\ G_3 - 0.5G_2 + 0.06G_1 = 0 \\ \quad \cdots \\ G_j - 0.5G_{j-1} + 0.06G_{j-2} = 0 (j \geq 3) \end{cases} \Rightarrow \begin{cases} G_0 = 1 \\ G_1 = -0.3 \\ G_2 = -0.05 \\ G_j - 0.3G_{j-1} = 0.2^j \\ G_j = \sum\limits_{n=0}^{j-1} 0.3^n \times 0.2^{j-n} + 0.3^j \end{cases}$$

   由于$\lim\limits_{j\to\infty} G_j = 0$,因此该模型平稳。

   (可逆性) 设模型传递形式为$\epsilon_i = \sum\limits_{j=0}^{\infty} I_j x_{t-j}$,即
   $(1 - 0.5B + 0.06B^2)x_t = (1 - 0.8B + 0.16B^2)(I_0 + I_1 B + \cdots)x_t$,于是有

$$\begin{cases} I_0 = 1 \\ I_1 - 0.8I_0 = -0.5 \\ I_2 - 0.8I_1 + 0.16I_0 = 0.06 \\ I_3 - 0.8I_2 + 0.16I_1 = 0 \\ \quad \cdots \\ I_j - 0.8G_{j-1} + 0.16I_{j-2} = 0 (j \geq 3) \end{cases} \Rightarrow \begin{cases} I_0 = 1 \\ I_1 = 0.3 \\ I_2 = 0.14 \\ I_j - 0.4I_{j-1} = 0.02 \times 0.4^{j-2} \\ I_j = 0.02 \times (j-1)0.4^{j-2} + 0.3 \times 0.4^{j-1} \end{cases}$$

由于$\lim_{j \to \infty} I_j = 0$,因此该模型可逆。【Green函数判别法】