

Investigation of Duffing Oscillator

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The Duffing oscillator is a drive damped oscillator described by the following equation. $\ddot{x} + \delta\dot{x} + x + \gamma x^3 = f \cos(\omega t)$

Duffing oscillator is a periodically forced oscillator with a nonlinear elasticity. It is a simple model that can lead to chaos. [1]

QUESTION 1A

Duffing Oscillators can be chaotic systems.

Write a code that solves this equation for the case $\delta = f = 0$.

The code has been attached to the back of this document.

For the case of $\delta = f = 0$ our equation has been reduced to:

$$\ddot{x} + x + \gamma x^3 = 0 \quad (1)$$

Using this code, study how the period of the undamped undriven oscillator depends on the initial conditions for fixed γ . For simplicity, assume that the initial position is zero in this study. Discuss your results.

In this case for $\gamma > 0$, this spring is called a hardening spring. As can be observed in figure 7, as you increase the absolute value of the initial velocity, the period goes down, as the spring hardens from the increased initial velocity.

For $\gamma > 0$; the system yields a single potential well, in this case, the system is guaranteed to have a stable period regardless of initial velocity, as the object cannot escape the potential well. See figure 5 for the figure of a single potential well.

While for when $\gamma < 0$, it is called a softening spring (only valid for small values of x). As can be observed in figure 6, as you increase the absolute value of the initial velocity for the system, the period goes up, eventually to infinity, as the spring softens from the increased initial velocity.

For $\gamma < 0$; the potential is a double potential well. This means that for small initial velocity, the system's position oscillates back and forth in the inner well depending on initial velocity, yielding a stable period. And for large enough initial velocity, the object that the system describes can escape the potential well and can travel infinitely far away, therefore in this case, the object does not oscillate and does not have a period (period is infinity). See figure 4.

Note the for the double potential well, the period increases with increase in initial velocity until it reaches the escape velocity, then the period goes to infinity; it is no longer periodic.

It should be noted that for part A, the restoring force is as follows:

$$F = -x - \gamma x^3 \quad (2)$$

Therefore the potential of the system is as follows:

$$U = \frac{1}{2}x^2 + \frac{1}{4}\gamma x^4 \quad (3)$$

Where U is the potential.

We will now find the escape velocity for $\gamma < 0$.

For $\gamma < 0$:

Find the x values of the twin peaks of the double potential well by setting $\frac{dU}{dx} = 0$.

$$dU/dx = -F \quad (4)$$

$$x(1 + \gamma x^2) = 0 \quad (5)$$

$$x^2 = \frac{-1}{\gamma} \quad (6)$$

Subbing x back into the conservation of energy equation yields:

$$v = +/\sqrt{\frac{-1}{2m\gamma}} \quad (7)$$

Where $m = 1$ in our case. So for $\gamma < 0$, when v is exceeds the above value, the object escapes the double hump potential well.

Is the system always oscillatory for all initial velocities?

In the case of $\gamma > 0$, the system is oscillatory for all initial velocities except for initial velocity of 0. In the case of $\gamma < 0$, the system is oscillatory for all initial velocities except for initial velocity of 0 and if the absolute value of initial velocity is greater than the escape velocity that we just calculated.

The following figures would show varying initial velocity while all other parameters are constant.

Q1a i) Initial velocity is set to 1, figure shows oscillating behavior. The settings for the following figure are as follows:

This is a figure for $v_{initial} = 1.0$. See figure 1.

This is a figure for $v_{initial} = 100.0$ See figure 2.

For further information, please see figure 1; and see figure 3.

QUESTION 1B

Modify your code to solve this equation for the case of nonzero damping. For simplicity, assume that the initial position is zero in this study. For fixed, initial velocity, find γ that results in critical damping.

For an example of varying damping constant until the system becomes critically damping. Please see figure 8. In that figure, damping occurs at damping constant $\delta = 12.0$ and initial velocity is fixed at 0.1.

How does this value change as initial velocity changes?
 Afterwards, we plot critical damping constant versus initial velocity, here we can observe that, the critical damping constant δ has a quadratic relationship with initial velocity. Damping constant has a quadratic relationship with initial velocity with all other variables constant as shown in the following figure. See figure 9. Here $a*x^2 + b*x + c$ fits the function and the coefficients respectively are: $a=-0.0004$ $b=0.1810$ $c=10.5053$.

QUESTION 1C

Now modify your code to solve this equation for the case of damped driven motion.

Code is attached in email.

Study the behavior of the oscillator for $0 < f < 5$ and $0 < \omega < 4$. Use Poincar sections to help interpret the results.

For Poincare sections, please see figure 11 for the case of negative gamma and figure 10 for the case of positive gamma.

I have included a wide range of figures to cover all the values for $0 < f < 5$ and $0 < \omega < 4$. For the other figures, please see figures 12, 13, 14, 15, 16. Each figure represents one value of the driving frequency ω , on each figure there are 6 lines representing 6 values of f , the driving amplitude. The system frequency spirals into match the driving frequency of the system. These figures are all poincare sections meaning that I only sample one point per period of the driving period.

Do you see evidence of period doubling?
 For the cases and parameters that I chose, there were no signs of period doubling.

Describe any other interesting phenomena that you observe.

The system frequency converges to the driving frequency over time due to damping. I have chosen my sampling frequency to be the same as the driving frequency to make the Poincare section. Basically, the explanation for the figures, at the initial time, you are on the outside of the spiral pattern, as time goes on, the system period converges to the driving period which is constant. You travel

from the outside of the spiral to the inside as time goes on. Eventually, all the points converge to a single point on the Poincar section, this is because when the system's period matches the driving period it you sample the system at the same state every sampling period. The number of spokes of the spiral pattern is proportional to ω , the driving frequency, and is also the frequency at which I sample at. For example for $\omega = 5.0$, I am sampling at 5.0 times the initial frequency of the system which is why there are five spokes on the diagram, because I am sampling at five points per period of the system. Later damping causes the system to lose energy until the frequency of the system matches the driving frequency ω . For large time values, I am then sampling once per period, and the Poincare section shows convergence around a single point on the phase plot.

NUMERICAL METHODS AND ERROR ANALYSIS

I am using the Runge Kutta 4 Method or RK4 for short. It was chosen for its relatively small error for a specific step size, and computational efficiency. Runge Kutta is an iterative method used in discretization for approximating the solutions of differential equations. In our case, $h = 0.005$ was used for question c, and this is the optimal step size.

Let σ be total error.
 $\text{machine precision} = \epsilon = 10^{-15}$ for double.

$$|\sigma| \leq \frac{\epsilon}{h^2} + O(h^4) \quad (8)$$

$$\frac{d\sigma}{dh} = 0 \quad (9)$$

$$\frac{d\sigma}{dh} = \frac{-2\epsilon}{h^3} + 4O(h^3) \quad (10)$$

$$h^6 \approx \epsilon \quad (11)$$

$$h \approx 10^{-2.5} \approx 0.003 \quad (12)$$

Therefore 0.003 is approximately the ideal step size for minimizing error. Therefore 0.005 was chosen as the step size for most of the calculations, as 0.005 is close to the ideal step size but a number ending in 5 is more convenient and easier to keep track of as 10 is a multiple of 5.

I have also included a diagram of the residual energy vs time plot.

$$\text{residualenergy} = \text{energy}(t) - \text{energy}_{\text{initial}} \quad (13)$$

Please see figure 17. This figure is based on the equation from question 1a where there is no damping or driving so the energy of the system should be constant. This graph of the energy residuals demonstrates error fluctuations on the order of 10^{-7} which is insignificant when compared to the order of the position values which is 1. Furthermore, the mean of the fluctuations are centered on the same y value for the time span that I plotted and show no drift. In conclusion, the RK4 method combined with my optimal step size is suitable for this project.

cluding linux. <http://www.codeblocks.org/>

ACKNOWLEDGMENTS

Thank you and Happy Holidays! 18

REFERENCES

Replace ? with underscore to get links to work. My latex compiler won't accept underscores for some reason.

CODE

Some code based on code from course website at [2]. Code has been or will be sent by email. Code uses standard libraries and was compiled and tested on Code::Blocks. Code blocks was built to be cross platform and should work consistently across all os platforms in-

- * Electronic address: eduvchiu@gmail.com
- [1] scholarpedia, *Duffing oscillator* (2013), URL <http://www.scholarpedia.org/article/Duffing%3Foscillator>.
- [2] K. Schleich, *Physics 410 course website* (2013), URL <http://www.phas.ubc.ca/~phys410/Physics410/Lecture%3FSlides.html>.

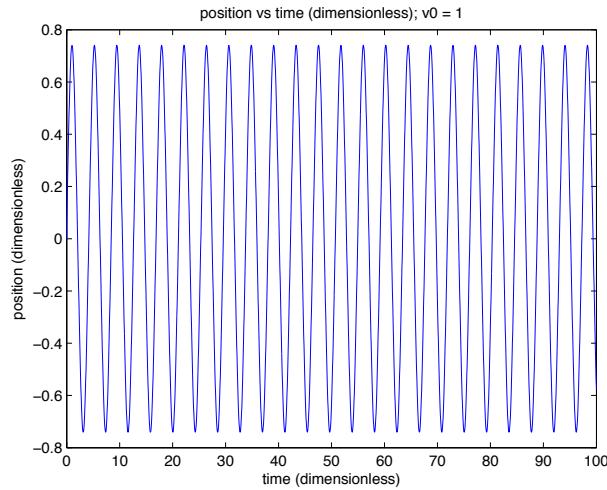


FIG. 1: q1a: oscillator

Parameters:

$v_{initial} = 1.0;$

All time steps are of 5.00000E-002.

Using $\gamma = 3.0$;

$period_{initial} = 10.0;$

$t_{final} = period_{initial} * 10.0$

$h = 0.05$

$numberofsteps = t_{final}/h;$

$x_{initial} = 0.0;$

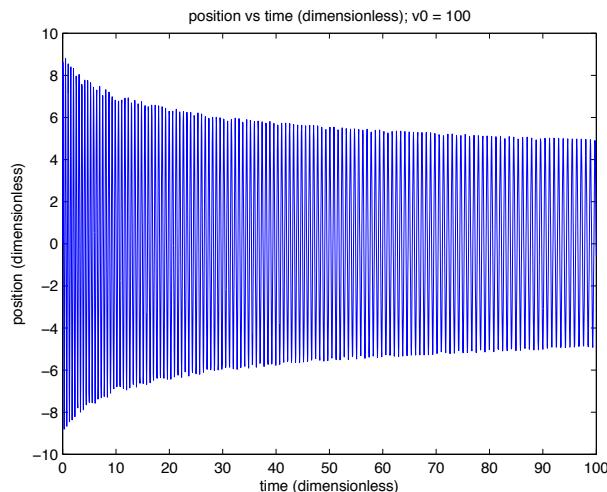


FIG. 2: q1a: oscillator

Parameters:

$v_{initial} = 100.0;$

All time steps are of 5.00000E-002.

Using $\gamma = 3.0$;

$period_{initial} = 10.0;$

$t_{final} = period_{initial} * 10.0$

$h = 0.05$

$numberofsteps = t_{final}/h;$

$x_{initial} = 0.0;$

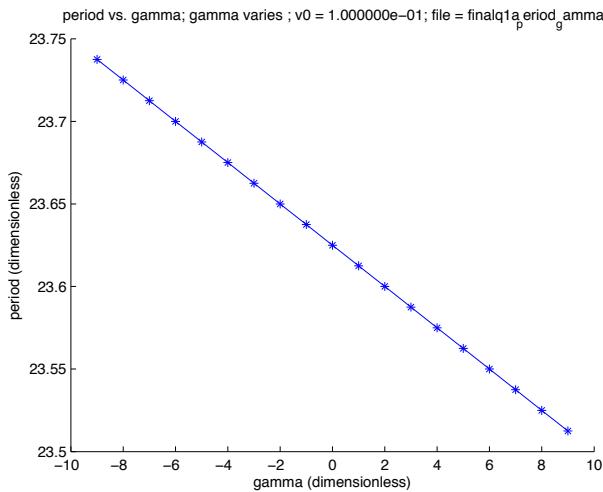


FIG. 3: period versus gamma

period has a linear relationship with gamma for small initial velocities Parameters:

$v_{initial} = 0.1$;

All time steps are of 5.00000E-002.

Using $\gamma = varying$;

$period_{initial} = 10.0$;

$t_{final} = period_{initial} * 10.0$

$h = 0.05$

$number of steps = t_{final}/h$;

$x_{initial} = 0.0$;

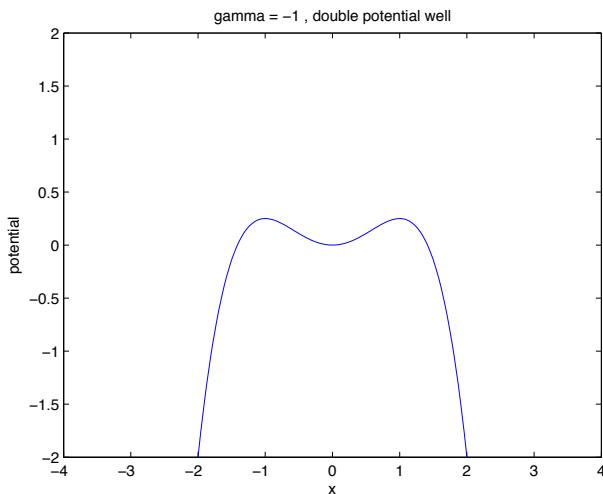


FIG. 4: double potential well $x_{initial} = 0.0$;

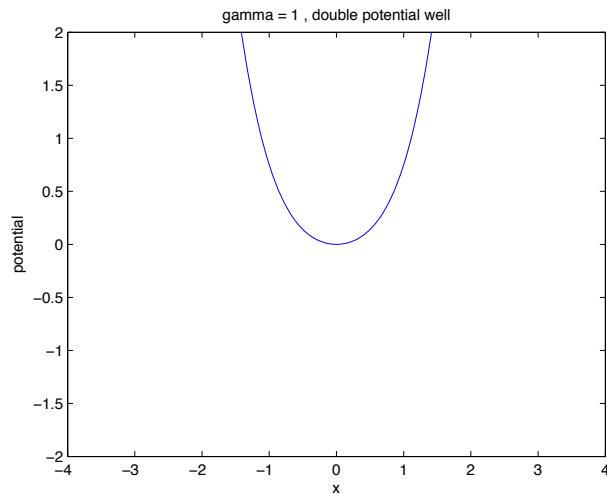


FIG. 5: single potential well $x_{initial} = 0.0$;

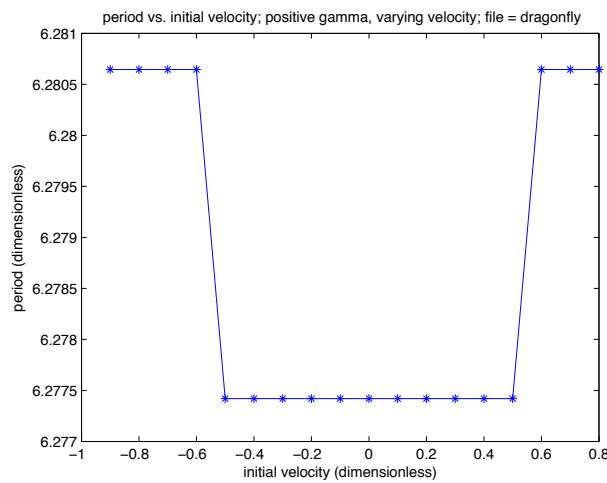


FIG. 6: period vs initial velocity for double potential well; fixed gamma = -0.1; note that the period constantly increases up until a certain v then the object has enough energy to escape the well and the period becomes infinite. $x_{initial} = 0.0$;

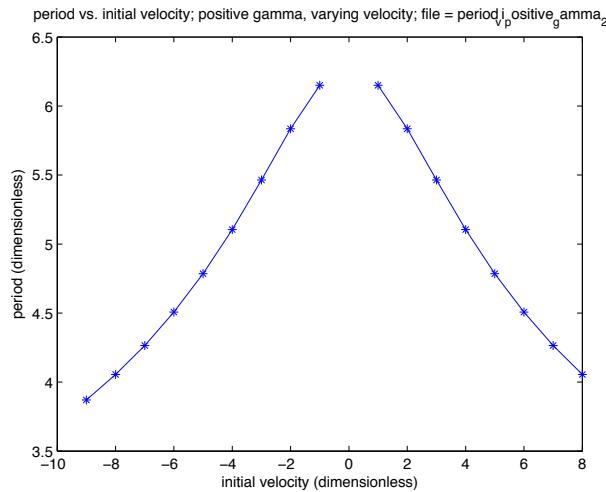


FIG. 7: period vs initial velocity for single potential well; fixed gamma = 0.1; note that theoretically the period constantly increases for initial velocity increase but I suspect that some numerical value is decreasing period after a certain point for some reason. $x_{initial} = 0.0$;

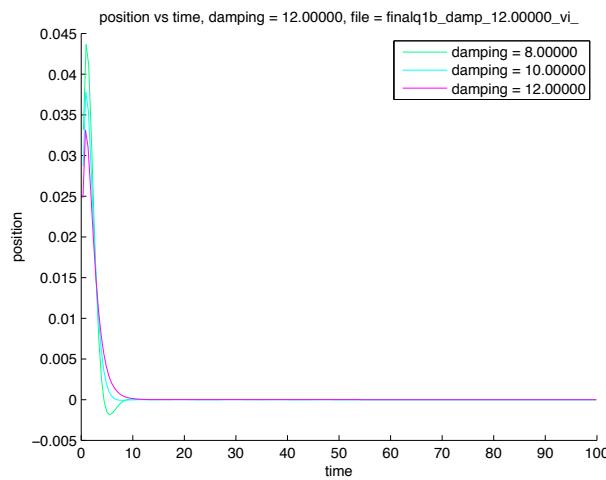


FIG. 8: Critical damping at around damping constant = 11.5.
 $v_{initial} = 0.1$;

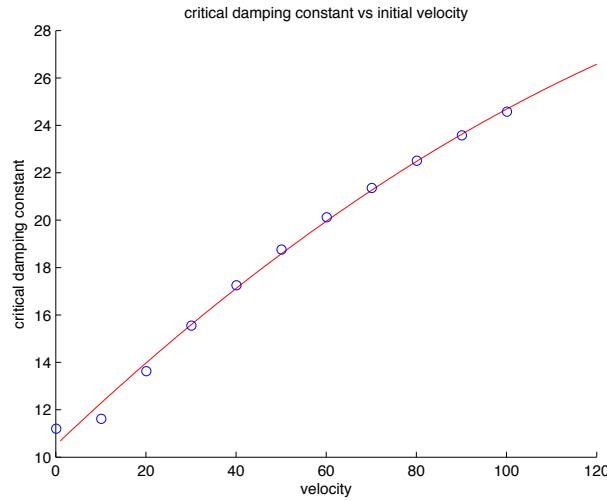


FIG. 9: Critical damping at around damping constant = 11.5.
 $v_{initial} = 0.1$;

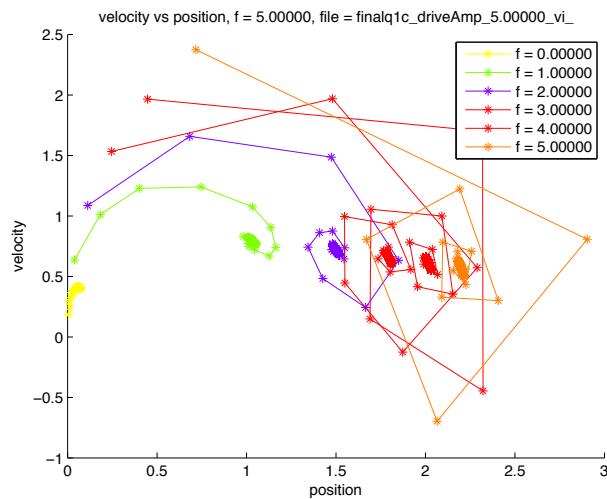


FIG. 10: velocity vs position; poincare section, one point per period T
positive gamma

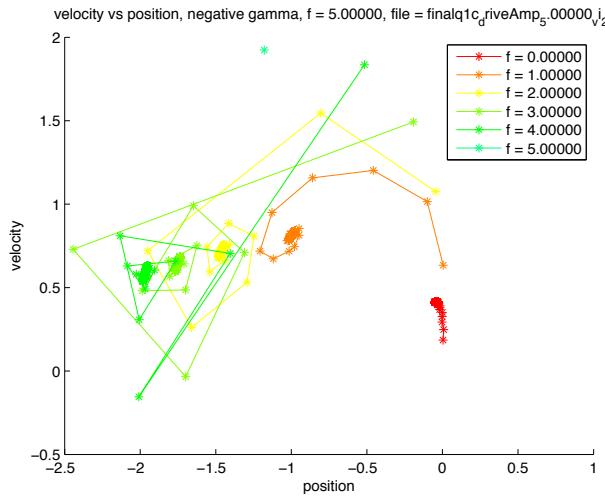


FIG. 11: velocity vs position; poincare section, one point per period T
negative gamma

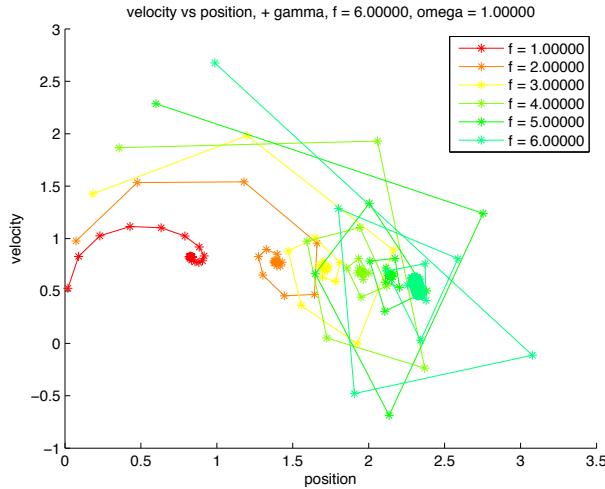


FIG. 12: poincare section: velocity vs position

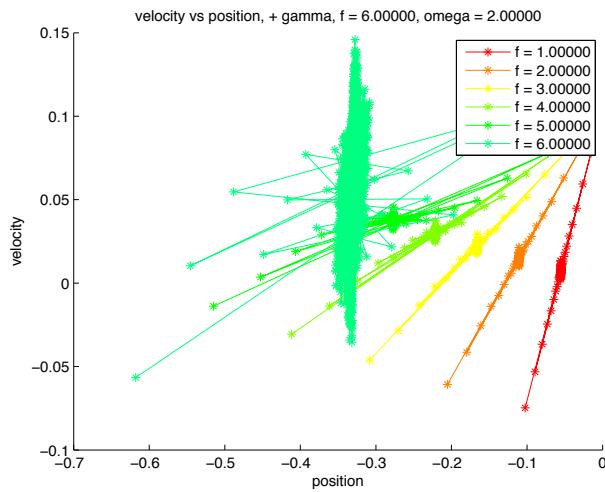


FIG. 13: poincare section: velocity vs position

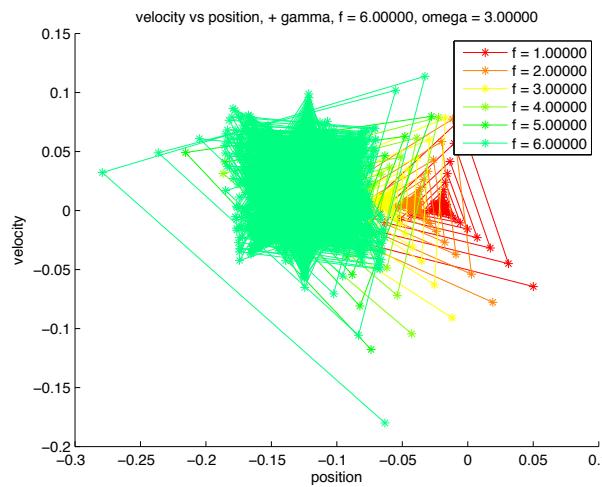


FIG. 14: poincare section: velocity vs position

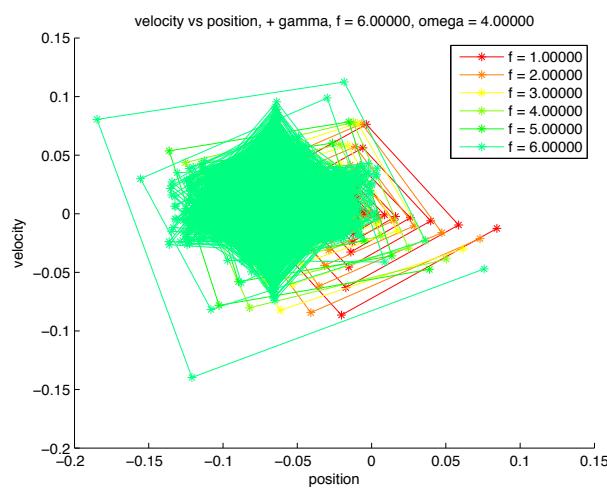


FIG. 15: poincare section: velocity vs position

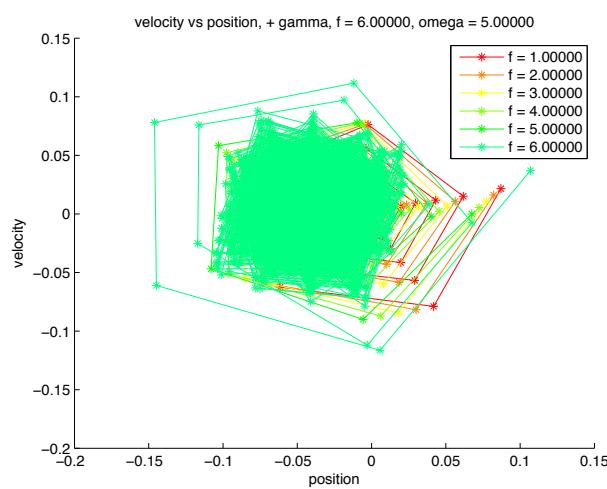


FIG. 16: poincare section: velocity vs position

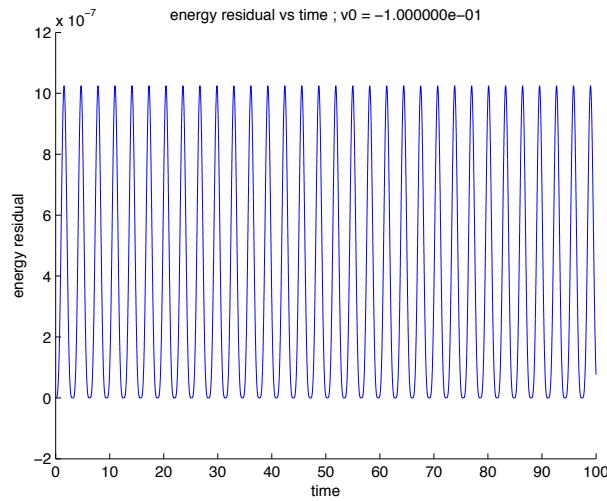


FIG. 17: energy residual vs time for q1a

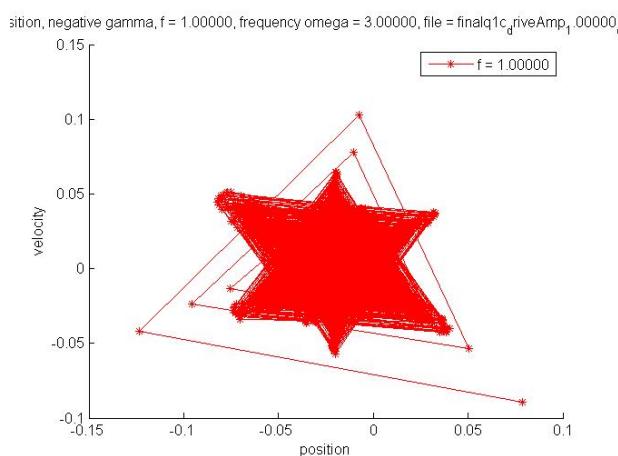


FIG. 18: Happy Holidays! (not a real figure)