QF 605 Fixed Income Securities

Group Project Report

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Part I (Bootstrapping Swap Curves)

Question 1: Bootstrap the OIS discount factor $D_0(0,T)$ and plot the discount curve for $T \in [0,30]$

Based on available information, we started by calculating the fed fund rate f_0 and OIS discount rate for the first year using below formula:

$$OIS_{1y} = \left(1+\frac{f_0}{360}\right)^{(1*360)} - 1 \qquad\qquad D_o\,(0,1\text{y}) = \frac{1}{\left(1+\frac{f_0}{360}\right)^{(1*360)}}$$

For the fed fund rate and OIS discount of the following years, it can be calculated using below relationship:

$$D_o(0,T_i) = D_o(0,T_{i-1}) * \frac{1}{\left(1+\frac{f_i}{360}\right)^{360}} \qquad ①$$

$$IRS\text{ rate} * \sum_{i=0.5}^{n} D_o(0,T_i) = \sum_{i=0.5}^{n} D_o(0,T_i) * \left(\left(1+\frac{f_i}{360}\right)^{360} - 1\right) \qquad ②$$

To perform bootstrapping, we can use equation ① to substitute $D_o(0,T_i)$ in equation ② so that equation ② has only one unknown parameter f_i. When the OIS rate of certain tenors is unknown, we can use linear interpolation on OIS discount factor to substitute the unknowns:

$$D_o(0,T_n) = D_o(0,T_{n+i}) + \frac{i}{i+j} * \left(D_o\left(0,T_{n-j}\right) - D_o(0,T_{n+i})\right), where\ D_o\left(0,T_{n-j}\right)\ is\ known.$$

Based on our bootstrapping algorithm, the OIS discount factor table is shown as below:

Tenor	OIS_DF	Tenor	OIS_DF	Tenor	OIS_DF	Tenor	OIS_DF	Tenor	OIS_DF	Tenor	OIS_DF
0.5	0.99875	5.5	0.97974	10.5	0.95314	15.5	0.92486	20.5	0.89744	25.5	0.87111
1.0	0.99701	6.0	0.97729	11.0	0.95030	16.0	0.92210	21.0	0.89481	26.0	0.86847
1.5	0.99527	6.5	0.97485	11.5	0.94747	16.5	0.91935	21.5	0.89217	26.5	0.86584
2.0	0.99353	7.0	0.97240	12.0	0.94463	17.0	0.91660	22.0	0.88954	27.0	0.86321
2.5	0.99177	7.5	0.96967	12.5	0.94179	17.5	0.91384	22.5	0.88691	27.5	0.86057
3.0	0.99002	8.0	0.96693	13.0	0.93896	18.0	0.91109	23.0	0.88427	28.0	0.85794
3.5	0.98807	8.5	0.96419	13.5	0.93612	18.5	0.90834	23.5	0.88164	28.5	0.85531
4.0	0.98612	9.0	0.96145	14.0	0.93328	19.0	0.90558	24.0	0.87901	29.0	0.85267
4.5	0.98415	9.5	0.95871	14.5	0.93045	19.5	0.90283	24.5	0.87637	29.5	0.85004
5.0	0.98218	10.0	0.95598	15.0	0.92761	20.0	0.90007	25.0	0.87374	30.0	0.84740

Question 2: Bootstrap the LIBOR discount factor $D_L(0,T)$ and plot the discount curve for $T \in [0,30]$

The approach of Bootstrap the LIBOR discount factor similar to bootstrapping OIS discount factor

For the first tenor (0,0.5y) can be derived using below formula:

$$D_L(0,0.5y) = \frac{1}{1 + 0.5 * L(0,0.5y)}$$

For the next tenors we have below relationship:

$$L(T_{i-1}, T_i) = \frac{D_L(0,T_{i-1}) - D_L(0,T_i)}{\Delta * D_L(0,T_i)} \quad ①$$

$$0.5 * IRS\ rate * \sum_{i=0.5}^{n} D_o(0,T_i) = 0.5 * \sum_{i=0.5}^{n} D_o(0,T_i) * L(T_{i-0.5}, T_i) \quad ②$$

Next, we can use equation ① to substitute the $L(T_{i-1}, T_i)$ in equation ② so that equation ② has only one unknown $D_L(0,T_i)$. When the IRS rate of some tenor is unknown, we still use linear interpolation on LIBOR discount factor:

$$D_L(0,T_n) = D_L(0,T_{n+i}) + \frac{i}{i+j} * \left(D_L\left(0,T_{n-j}\right) - D_L(0,T_{n+i})\right),\ D_L\left(0,T_{n-j}\right)\ is\ known.$$

The LIBOR discount curve is shown as below:

Tenor	LIBOR_DF	Tenor	LIBOR_DF	Tenor	LIBOR_DF	Tenor	LIBOR_DF	Tenor	LIBOR_DF	Tenor	LIBOR_DF
0.5	0.98765	5.5	0.83280	10.5	0.67855	15.5	0.53679	20.5	0.39899	25.5	0.30671
1.0	0.97258	6.0	0.81660	11.0	0.66439	16.0	0.52251	21.0	0.38976	26.0	0.29748
1.5	0.95738	6.5	0.80041	11.5	0.65022	16.5	0.50822	21.5	0.38053	26.5	0.28826
2.0	0.94218	7.0	0.78422	12.0	0.63606	17.0	0.49394	22.0	0.37131	27.0	0.27903
2.5	0.92633	7.5	0.76897	12.5	0.62190	17.5	0.47965	22.5	0.36208	27.5	0.26980
3.0	0.91048	8.0	0.75371	13.0	0.60773	18.0	0.46536	23.0	0.35285	28.0	0.26057
3.5	0.89473	8.5	0.73846	13.5	0.59357	18.5	0.45108	23.5	0.34362	28.5	0.25135
4.0	0.87898	9.0	0.72321	14.0	0.57941	19.0	0.43679	24.0	0.33439	29.0	0.24212
4.5	0.86399	9.5	0.70796	14.5	0.56524	19.5	0.42250	24.5	0.32517	29.5	0.23289
5.0	0.84899	10.0	0.69271	15.0	0.55108	20.0	0.40822	25.0	0.31594	30.0	0.22366

Question 3: Calculate the forward swap rate.

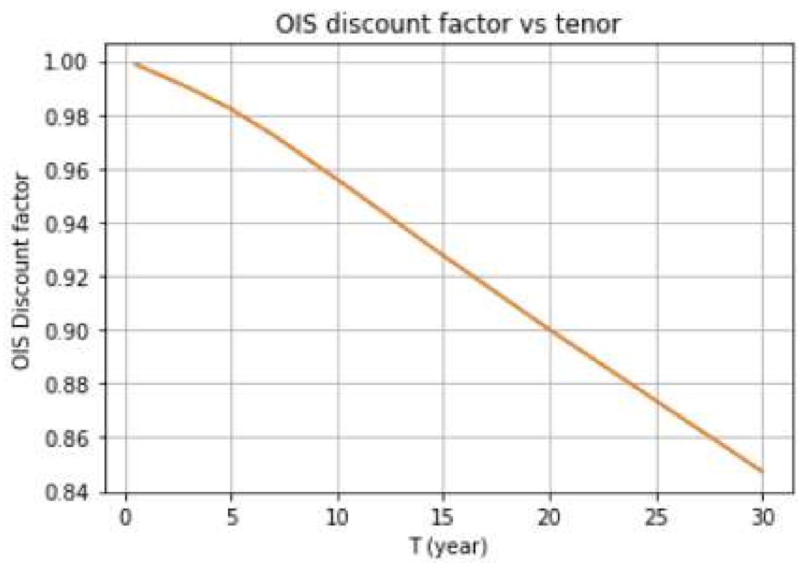
Forward Swap Rate can be calculated using below formula:

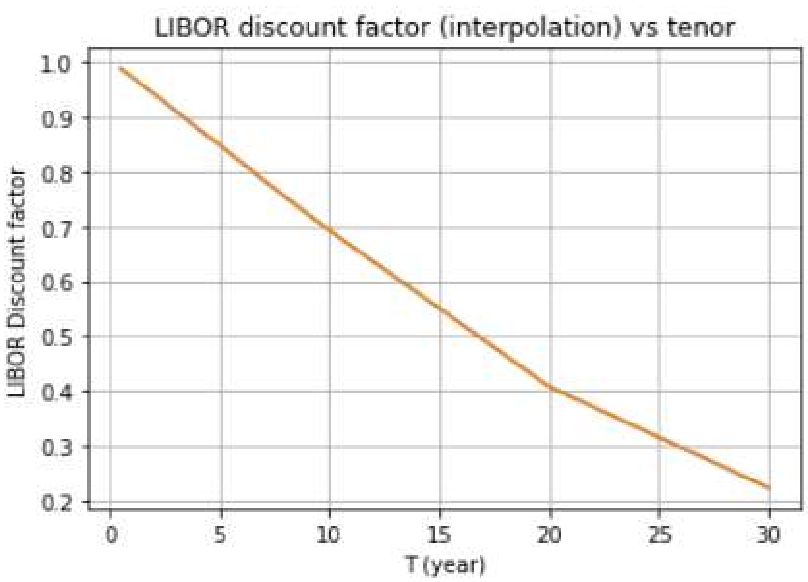
$$S(T_i, T_{i+m}) = \frac{0.5 * \sum_{n=i+1}^{i+m} D_o(0, T_n) * L(T_{n-0.5}, T_n)}{0.5 * \sum_{n=i+1}^{i+m} D_o(0, T_n)}$$

The forward swap rate is calculated as below:

Forward Swap Rate					
Expiry/Tenor	**1Y**	**2Y**	**3Y**	**5Y**	**10Y**
1Y	0.032007	0.033259	0.034011	0.035255	0.038428
5Y	0.039274	0.040075	0.040072	0.041093	0.043634
10Y	0.042189	0.043116	0.044097	0.046249	0.053458

<u>OIS discount curve and LIBOR discount curve are shown as follow:</u>





Part II (Swaption Calibration)

Question 1: Calibrating Displaced-Diffusion Model

Under the Swaption tab of IR Data.xlsm, swaption implied volatilities are provided. We can use the par forward swap rate calculated in Part I to derive corresponding strike prices for different implied volatilities. Based on strikes and implied volatility, the payer swaption and receiver swaption price can then be calculated using Black 76 Formula

The Displaced-diffusion model (DD Model) is essentially the same as the Black 76 model, except that it models the movements of (F + Shift) as the underlying asset, instead of F. To better fit the market, the model can be thought of as a "weighted average" between a normal and a lognormal model, with β weight given to the normal model. Then the Displaced-diffusion Model states the price for a Swap is given by:

$$C^{DD} = C^{B76}\left(\frac{F}{\beta}, K + \frac{1-\beta}{\beta}F, \sigma\beta, T\right) = PVBP[F'N(d1) - KN(d2)]$$

$$P^{DD} = P^{B76}\left(\frac{F}{\beta}, K + \frac{1-\beta}{\beta}F, \sigma\beta, T\right) = PVBP[KN(d1) - F'N(d2)]$$

Displaced-Diffusion Model is a model that combine the normal distribution and lognormal distribution, where beta is the weight of lognormal behaviour and (1-beta) is the weight of normal behaviour. The think we need to do here is to use Displaced-Diffusion Model to fit the market data.

We choose to assign more weight on ATM swaption to make the optimal solution to be stable instead of giving us only the local optimal solution. By doing so, we can obtain the calibration Displaced-diffusion model parameters. The calibrated parameters are showing as follow:

Calibrated Displaced-Diffusion Parameters:

Sigma					
Expiry/Tenor	**1Y**	**2Y**	**3Y**	**5Y**	**10Y**
1Y	0.2250	0.2872	0.2978	0.2607	0.2447
5Y	0.2726	0.2983	0.2998	0.2660	0.2451
10Y	0.2854	0.2928	0.2940	0.2674	0.2437

Beta					
Expiry/Tenor	**1Y**	**2Y**	**3Y**	**5Y**	**10Y**
1Y	0.000001	0.000001	0.000001	0.000001	0.000023
5Y	0.000001	0.000001	0.000002	0.000017	0.058060
10Y	0.000001	0.000003	0.000015	0.000066	0.005126

(Please refer to **Appendix A** for plotted Displaced Diffusion Model)

Question 2: Calibrating SABR Model

Similar to what we did in Displaced Diffusion Model, we first use the given SABR function to get σ_{sabr}, then use SABR calibration function to find the error term between the Market data and σ_{sabr}, After getting the calibration function, we then search best fit parameters using least squares.

Calibrated SABR Parameters:

Alpha					
Expiry/Tenor	**1Y**	**2Y**	**3Y**	**5Y**	**10Y**
1Y	0.139074	0.184632	0.196838	0.177968	0.169925
5Y	0.166617	0.199548	0.210349	0.190266	0.175077
10Y	0.178247	0.196311	0.208236	0.202485	0.179410

Rho					
Expiry/Tenor	**1Y**	**2Y**	**3Y**	**5Y**	**10Y**
1Y	-0.633226	-0.525124	-0.482828	-0.414034	-0.254155
5Y	-0.585660	-0.547118	-0.549822	-0.506963	-0.420208
10Y	-0.548311	-0.547527	-0.553546	-0.566329	-0.504683

Nu					
Expiry/Tenor	**1Y**	**2Y**	**3Y**	**5Y**	**10Y**
1Y	2.049444	1.677547	1.438248	1.065701	0.792740
5Y	1.340419	1.062109	0.936808	0.674500	0.511096
10Y	1.010145	0.928629	0.871643	0.723756	0.582162

(Please refer to **Appendix A** for plotted SABR Model)

Question 3: Price the following swaptions using the calibrated displaced-diffusion and SABR model:

- **payer 2y * 10y K = 1%, 2%, 3%, 4%, 5%, 6%, 7%, 8%**
- **receiver 8y * 10y K = 1%, 2%, 3%, 4%, 5%, 6%, 7%, 8%**

According to the Displaced Diffusion model and SABR model, we need to know the following parameters F, K, T, sigma, alpha, beta, rho, and nu. Based on parameters' data we calculated in earlier, we set the interpolation function using cubic spline method to acquire calibrated sigma, alpha, beta, rho and nu for 2y x 10y and 8y x10y respectively.

The value of forward swap rate (F) is calculated using the same equation $S(T_i, T_{i+m})$ from question 3 of part I. After we put the calibrated parameters into our Displaced Diffusion model and SABR model, we obtained the following price of the swaptions:

K	1%	2%	3%	4%	5%	6%	7%	8%
Pay 2y*10y DD	0.288132	0.194914	0.112309	0.051363	0.017410	0.004139	0.000664	0.000070
Pay 2y*10y SABR	0.289527	0.198105	0.114841	0.051804	0.021194	0.010654	0.006569	0.004603
Rec 8y*10y DD	0.018651	0.033584	0.056418	0.088905	0.132164	0.186423	0.250987	0.324417
Rec 8y*10y SABR	0.017246	0.035660	0.058261	0.087842	0.128579	0.184592	0.255452	0.336176

Based on the result above, we can observe that as strike goes larger, the price of payer becomes lower, but the price of receiver becomes higher, and both models' results are very close to each other. However, since SABR does a better job in fitting market implied volatility, the result from SABR model is more accurate than Displaced Diffusion Model (please refer to **Appendix A** for visualized comparison).

Part III (Convexity Correction)

Question 1:

- **Calculate PV of a leg receiving CMS10y semi-annually over the next 5 years**
- **Calculate PV of a leg receiving CMS2y quarterly over the next 10 years**

A CMS leg is a collection of CMS rates paid over a period. To calculate the PV, we will use the following formula:

$$CMS\ leg\ PV = \sum_{i=0.5}^{N} D_0(0,T_i) * Delta * CMS\left(S_{n,N}(T_i)\right)$$

And $CMS\left(S_{n,N}(T)\right)$ is given by:

$$E^T\left[S_{n,N}(T)\right] = g(F) + \frac{1}{D_0(0,T)}\left[\int_0^F h''(K)V^{rec}(K)dK + \int_F^\infty h''(K)V^{pay}(K)dK\right]$$

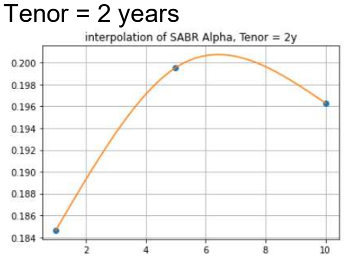
IRR-settled option pricer (V^{pay}or V^{rec}) is given by:

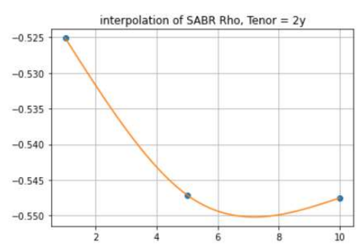
$$V_{n,N}(0) = D_0(0,T_n) * IRR\left(S_{n,N}(0)\right) * Black\ 76\left(S_{n,N}(0), K, \sigma_{sabr}, T\right)$$

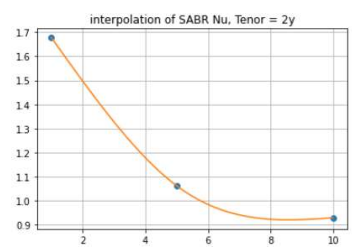
We used the calibrated SABR to derive the sigma for Black 76 Model in the calculation of IRR settled option. Since the SABR parameter value of Alpha, Rho and Nu with same tenor across different expiry is not linear, we used cubic-splined interpolation in the estimation of SABR parameters:

Alpha **Rho** **Nu**

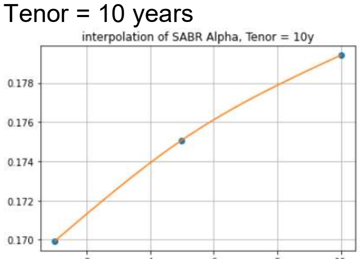
Tenor = 2 years

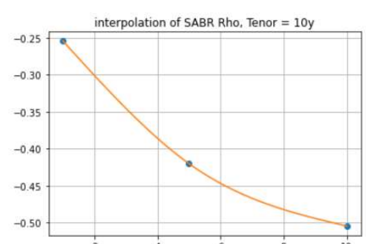


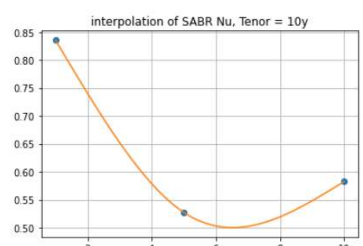




Tenor = 10 years







Based on our calculation:

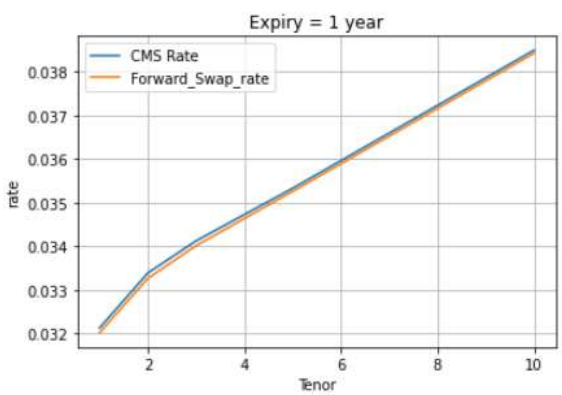
- The PV of a leg receiving CMS10y semi-annually over the next 5 years is 0.20211.
- The PV of a leg receiving CMS2y quarterly over the next 10 years is 0.38106.

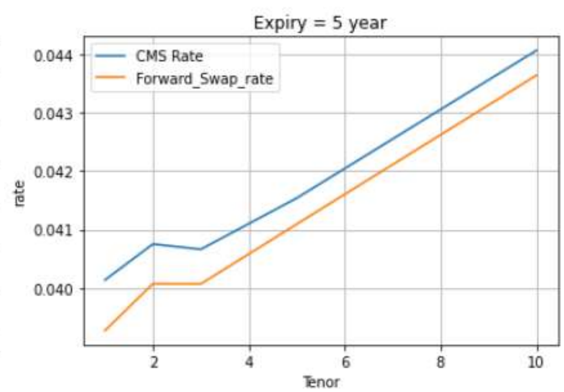
Question 2: Compare the forward swap rates with the CMS rate:

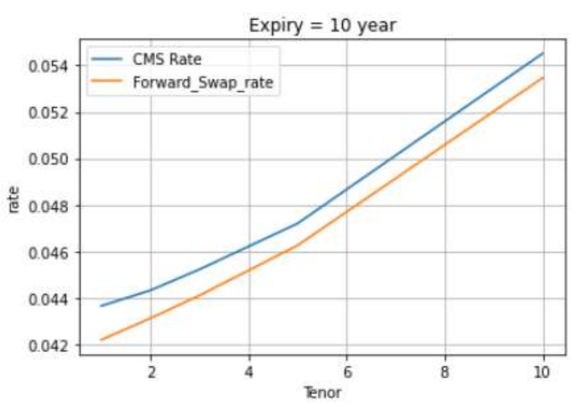
- **1y x 1y, 1y x 2y, 1y x 3y, 1y x 5y, 1y x 10y**
- **5y x 1y, 5y x 2y, 5y x 3y, 5y x 5y, 5y x 10y**
- **10y x 1y, 10y x 2y, 10y x 3y, 10y x 5y, 10y x 10y**

Based on our calculation, we have derived the following results:

Terms	CMS rate (A)	Forward Swap rate (B)	Difference (A-B)
1y x 1y	0.032120	0.032007	0.000113
1y x 2y	0.033381	0.033259	0.000122
1y x 3y	0.034120	0.034011	0.000109
1y x 5y	0.035326	0.035255	0.000071
1y x 10y	0.038496	0.038428	0.000068
5y x 1y	0.040129	0.039274	0.000855
5y x 2y	0.040756	0.040075	0.000681
5y x 3y	0.040664	0.040072	0.000592
5y x 5y	0.041534	0.041093	0.000441
5y x 10y	0.044057	0.043634	0.000423
10y x 1y	0.043633	0.042189	0.001444
10y x 2y	0.044320	0.043116	0.001204
10y x 3y	0.045223	0.044097	0.001126
10y x 5y	0.047208	0.046249	0.000959
10y x 10y	0.054529	0.053458	0.001071







Conclusion:

From above result, we can abserve that, CMS rate is always higher than the forward swap rate, and the difference can be explained by convexity adjustment. We can also find that the convexity adjustment become larger as the maturity gets longer. In contrast, Tenor has little impact on the convexity adjustment.

Part IV (Decompounded Options)

Question 1: Use static replication to value the PV of payoff : $\boldsymbol{CMS\ 10y^{\frac{1}{4}} - 0.04^{\frac{1}{2}}\ at\ time\ T = 5y}$

To calculate the PV of payoff function, we can the Leibniz's Rule on the IRR payer and receiver swaption formulas to derive the contract valuation formula stated below:

$$V_0 = D_0(0,T)g(F) + \int_0^F h''(K)V^{rec}(K)dK + \int_F^\infty h''(K)V^{pay}(K)dK$$

For the payoff of $CMS\ 10y^{\frac{1}{4}} - 0.04^{\frac{1}{2}}$ at time T = 5y, we can use 5y x 10y forward swap rate (F) and OIS discount factor of D_0 (0, 5) calculated from part I and substitute the following into our contract valuation formula:

$$g(F) = F^{\frac{1}{4}} - 0.2 \qquad g'(F) = \frac{1}{4}F^{-\frac{3}{4}} \qquad g''(F) = -\frac{3}{16}F^{-\frac{7}{4}}$$

$$h(K) = \frac{g(K)}{IRR(K)}$$

$$h'(K) = \frac{IRR(K)g'(K) - g(K)IRR'(K)}{IRR(K)^2}$$

$$h''(K) = \frac{IRR(K)g''(K) - IRR''(K)g(K) - 2 * IRR'(K)g'(K)}{IRR(K)^2} + \frac{2 * IRR'(K)^2 g(K)}{IRR(K)^3}$$

Based on the valuation formula above and calibrated SABR parameters, we can obtain the present value of the payoff at T = 5 is 0.249753.

Question 2: Use static replication to value the PV of payoff: $\boldsymbol{(CMS\ 10y^{\frac{1}{4}} - 0.04^{\frac{1}{2}})^+\ at\ time\ T = 5y}$

Since the payoff is $(CMS\ 10y^{\frac{1}{4}} - 0.04^{\frac{1}{2}})^+$, we can take the contract as CMS caplet.

For the payoff, $(CMS\ 10y^{\frac{1}{4}} - 0.04^{\frac{1}{2}})^+$,to be positive:

$$F^{\frac{1}{4}} > 0.2$$
$$F > 0.2^4$$
$$F > 0.0016 = L$$

We can treat the payoff as CMS Caplet strikes at L = 0.0016. Hence,

$$\begin{aligned} CMS\ Caplet &= D(0,T)\int_L^\infty g(K)f(K)dK \\ &= \int_L^\infty h(K)\frac{\partial^2 V^{pay}(K)}{\partial K^2}dK \\ &= h'(L)V^{pay}(L) + \int_L^\infty h''(K)V^{pay}(K)dK \end{aligned}$$

Using above valuation formula, above and interpolated SABR parameters from part II, we can obtain the present value of this payoff is 0.031198.

Appendix A:

<u>Displaced-Diffusion Model VS SABR Model VS Market Volatilities:</u>